Equivariant Moving Frames, Lie Pseudo-Groups, and Local Equivalence Problems

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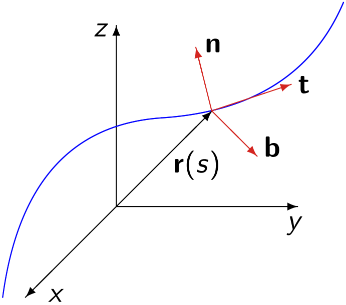
Moving frames

- Moving trihedrons:
  - curves:
    - Martin Bartels (1769–1836)
    - Frédéric Frenet (1816–1900)
    - Joseph Serret (1819–1855)
  - surfaces:
    - Gaston Darboux (1842–1917)
- Moving frames (submanifolds):
  - Élie Cartan (1869–1951)
    - Shiing-Shen Chern (1911–2004)
    - Robert Gardner (1939–1998)
    - Joseph Landsberg
    - Niky Kamran
    - Thomas Ivey
    - Phillip Griffiths
    - Robert Bryant



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$$\frac{d\mathbf{n}}{ds} = -\kappa\,\mathbf{t} + \tau\,\mathbf{b}$$

$$\frac{d\mathbf{b}}{ds} = -\tau\,\mathbf{n}$$

κ — curvature, τ — torsion

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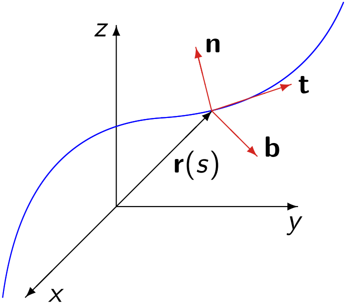

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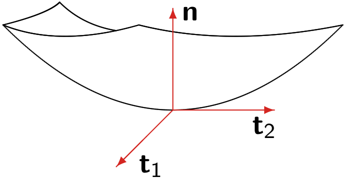

- H – Mean curvature
- K – Gauss curvature

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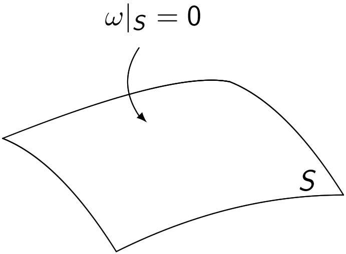

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- $d\omega = \theta \wedge \omega + T(\omega \wedge \omega)$

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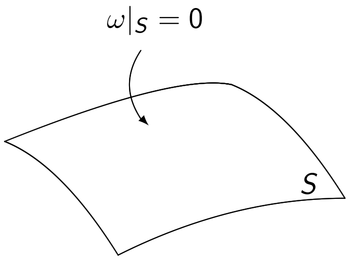

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Equivariant moving frames

Moving frames $\neq$ frames!

New theoretical foundation of Cartan's moving frame method (1999 –):

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- Mark Fels
- Irina Kogan
- Juha Pohjanpelto
- Mirelle Boutin
- Pilwon Kim
- Jeongoo Cheh
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- Francis
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(Equivariant) moving frames

Powerful tool for studying geometric properties of submanifolds and their invariants under the action of a group of transformations:

- Differential geometry (Riemannian, Kähler, ...)
- Equivalence problems, symmetry
- Differential invariants
- Integrability
- Characteristic cohomology & conservation laws of differential eqns.
- (Invariant) variational bicomplex
  - null Lagrangians, Helmholtz conditions
  - G-invariant Lagrangian $\rightarrow$ G-invariant E–L equations
- Geometric control theory
- Invariant finite difference numerical schemes
- Computer vision
- Structure theory of ∞-dimensional Lie pseudo-groups
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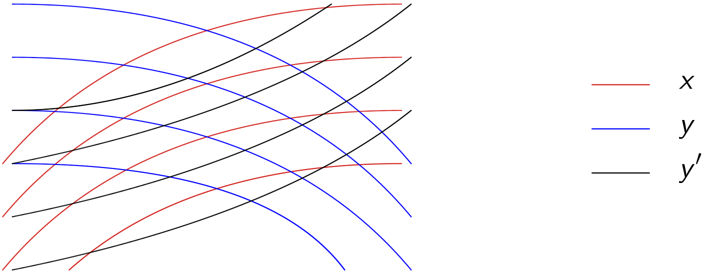
Disclaimer

- All constructions and results are local.
- We work in the analytic category.

3-web equivalence [Blaschke–Chern]

- When are 3-webs in $\mathbb{R}^2$ equivalent?



- When is $y' = f(x,y)$ equivalent to $Y' = F(X,Y)$ up to

$$X = \alpha(x), \qquad Y = \beta(y), \qquad U = \frac{\beta_y u}{\alpha_x} \qquad (u = y' \neq 0)?$$

(with $\alpha_x, \beta_y \neq 0$) ← ∞-dimensional Lie pseudo-group

Pseudo-groups

Definition

M – analytic manifold. A pseudo-group $\mathcal{G}$ is a collection of local analytic diffeomorphisms ϕ: dom $\phi \subset M \to M$ such that

- Identity: $\mathbb{1}_M \in \mathcal{G}$
- Inverses: $\phi^{-1} \in \mathcal{G}$
- Restriction: $U \subset \text{dom}\ \phi \quad \Rightarrow \quad \phi|_U \in \mathcal{G}$
- Continuation: dom $\phi = \cup U_\kappa$ and $\phi|_{U_\kappa} \in \mathcal{G} \quad \Rightarrow \quad \phi \in \mathcal{G}$
- Composition: im $\phi \subset \text{dom}\ \psi \quad \Rightarrow \quad \psi \circ \phi \in \mathcal{G}$

Example

$\mathcal{D} = \mathcal{D}(M)$ – pseudo-group of all local analytic diffeomorphisms $Z = \phi(z)$

$$z = (z^1, \ldots, z^m) \quad - \quad \text{source coordinates}$$
$$Z = (Z^1, \ldots, Z^m) \quad - \quad \text{target coordinates}$$

Jets of diffeomorphisms [Ehresmann, 1953]

Let $0 \leqslant n \leqslant \infty$:

Definition

For $Z = \phi(z) \in \mathcal{D}(M)$ let $\phi^{(n)}|_z$ denote its n-jet at $z \in M$:

$$\phi^{(n)}|_z \sim \text{coefficients of the } n^{\text{th}} \text{ order Taylor polynomial centered at } z.$$

Example

$$X = \phi(x) \in \mathcal{D}(\mathbb{R}) \Rightarrow \phi^{(2)}|_{x_0} \sim \phi(x_0) + \phi'(x_0)(x - x_0) + \frac{\phi''(x_0)}{2}(x - x_0)^2.$$

Definition

$\mathcal{D}^{(n)} \longrightarrow M$ is the n^{th} order diffeomorphism jet bundle, whose points are jets $\phi^{(n)}|_z$. Local coordinates are given by

$$(z, Z^{(n)}) = (\ldots z^a \ldots Z^b \ldots Z^b_A \ldots), \qquad Z^b_A = \frac{\partial^k Z^b}{\partial z^{a_1} \cdots \partial z^{a_k}}.$$

Lie pseudo-groups [Cartan]

Definition

A Lie pseudo-group $\mathcal{G}$ is a pseudo-group whose transformations are the solutions to an involutive system of partial differential equations

$$F^{(n)}(z, Z^{(n)}) = 0 \qquad (\star)$$

called the determining system of $\mathcal{G}$.

Definition

- $\mathfrak{g} = \{\text{infinitesimal generators}\}$
- $\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} \in \mathfrak{g}$ if and only if it is a solution to the infinitesimal determining system

$$L^{(n)}(z, \zeta^{(n)}) = 0$$

obtained by linearizing ($\star$) at $\mathbb{1}_z^{(n)}$.

Example – continuation

Example

$$\mathcal{G}: \qquad X = \alpha(x), \qquad Y = \beta(y), \qquad U = \frac{\beta_y u}{\alpha_x}, \qquad (\phi_x, \beta_u \neq 0)$$

- Determining system:

$$X_y = X_u = Y_x = Y_u = 0, \qquad U = \frac{Y_y u}{X_x}.$$

- $\mathcal{G}^{(\infty)}|_{(x,y,u)} \simeq \{(\alpha, \alpha_x, \alpha_{xx}, \ldots, \beta, \beta_y, \beta_{yy}, \ldots)\}.$
- Infinitesimal generators:

$$\mathbf{v} = \xi(x)\frac{\partial}{\partial x} + \eta(y)\frac{\partial}{\partial y} + u(\eta_y - \xi_x)\frac{\partial}{\partial u}.$$

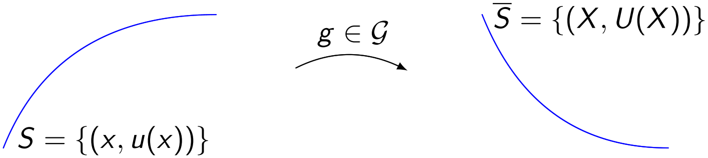
- Infinitesimal determining system:

$$\xi_y = \xi_u = \eta_x = \eta_u = 0, \qquad \phi = u(\eta_y - \xi_x).$$

- $\mathfrak{g}^{(\infty)}|_{(x,y,u)} \simeq \{(\xi, \xi_x, \xi_{xx}, \ldots, \eta, \eta_y, \eta_{yy}, \ldots)\}.$

Action of Lie pseudo-groups on submanifolds

- $\mathcal{G}$ acts on M
- We are now interested in the induced action of a Lie pseudo-group $\mathcal{G}$ on p-dimensional submanifolds $S \subset M$:



We assume

$$S = \{(x, u(x)) = (x^1, \ldots, x^p, u^1, \ldots, u^q)\}.$$

Submanifold jets

Definition

Let $\mathrm{J}^n \to M$ be the n^{th} order submanifold jet bundle. Local coordinates:

$$z^{(n)} = (x, u^{(n)}) = (\dots x^i \dots u^\alpha_J \dots), \qquad \#J \leq n.$$

$\mathcal{G}^{(n)}$ acts on $z^{(n)}$ (n^{th} prolonged action):

$$(x, u^{(n)}) \mapsto (X, U^{(n)}) = g^{(n)} \cdot (x, u^{(n)}).$$

The local coordinates

$$U^\alpha_J = F^\alpha_J(x, u^{(n)}, g^{(n)})$$

are obtained by implicit differentiation.

Prolonged action – example

The prolonged action of $X = \alpha(x)$, $Y = \beta(y)$, $U = \dfrac{\beta_y u}{\alpha_x}$ on surfaces $u = f(x,y)$ is obtained by applying

$$\mathcal{D}_x = \frac{1}{\alpha_x} D_x, \qquad \mathcal{D}_y = \frac{1}{\beta_y} D_y$$

to U:

$$U_X = \frac{u_x \beta_y}{\alpha_x^2} - \frac{u \beta_y \alpha_{xx}}{\alpha_x^3}, \qquad U_Y = \frac{u_y}{\alpha_x} + \frac{u \beta_{yy}}{\beta_y \alpha_x},$$

$$U_{XX} = \frac{u_{xx}\beta_y}{\alpha_x^3} - 3\frac{u_x \beta_y \alpha_{xx}}{\alpha_x^4} - \frac{u\beta_y\alpha_{xxx}}{\alpha_x^4} + 3\frac{u\beta_y\alpha_{xx}^2}{\alpha_x^5},$$

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Equivariant moving frames [Olver–Pohjanpelto, 2008]

Definition

Let $\mathcal{E}^{(n)} \to \mathrm{J}^n$ be the n^{th} order lifted bundle defined as

$$\mathcal{E}^{(n)} = \mathrm{J}^n \times_M \mathcal{G}^{(n)} \simeq \{(z^{(n)}, g^{(n)}) : z = \pi_0^n(z^{(n)}) = \widetilde{\pi}_0^n(g^{(n)})\}.$$

Definition

A right moving frame of order n is a right $\mathcal{G}$-equivariant section $\rho^{(n)} : \mathcal{V}^n \to \mathcal{E}^{(n)}$ defined on an open subset $\mathcal{V}^n \subset \mathrm{J}^n$.

Let

$$\rho^{(n)}(z^{(n)}) = (z^{(n)}, \rho^{(n)}(z^{(n)})).$$

Right-equivariance:

$$(g^{(n)} \cdot z^{(n)}, \rho^{(n)}(g^{(n)} \cdot z^{(n)})) = (g^{(n)} \cdot z^{(n)}, \rho^{(n)}(z^{(n)}) \cdot (g^{(n)})^{-1})$$

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Existence of moving frames

Proposition

A moving frame of order n exists if and only if $\mathcal{G}^{(n)}$ acts freely and regularly on $\mathcal{V}^n \subset \mathrm{J}^n$.

- Regularity:

- Freeness:

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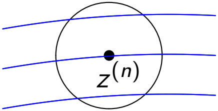
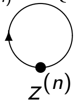


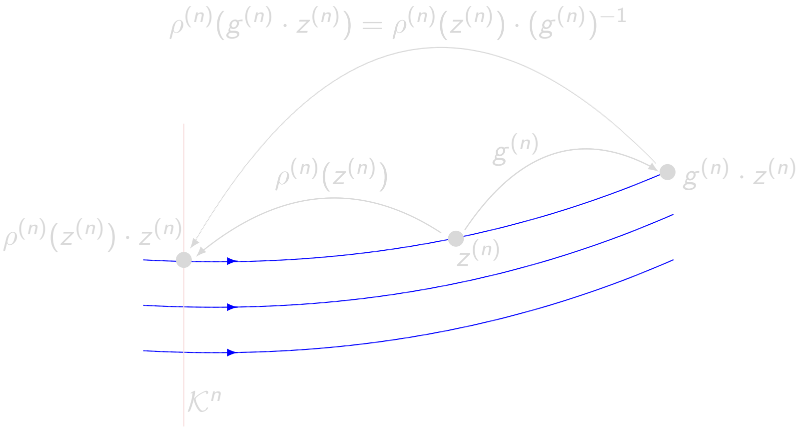
- Freeness:

$$\mathcal{G}^{(n)}_{z^{(n)}} = \{\mathbb{1}^{(n)}_z\}$$



Moving frame construction – illustration

Illustration



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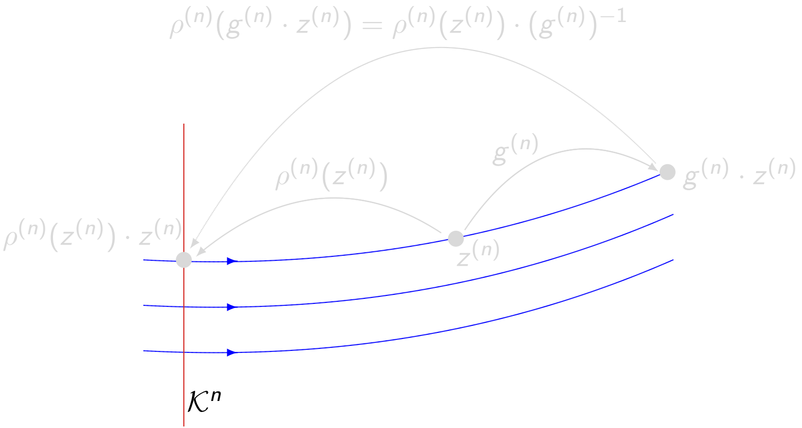

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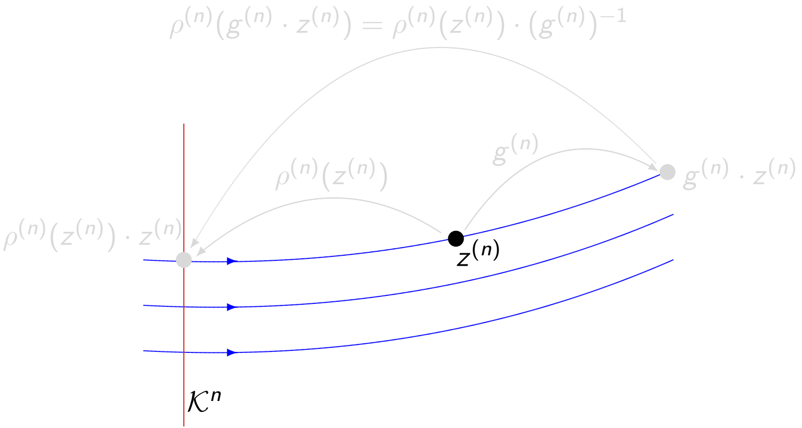

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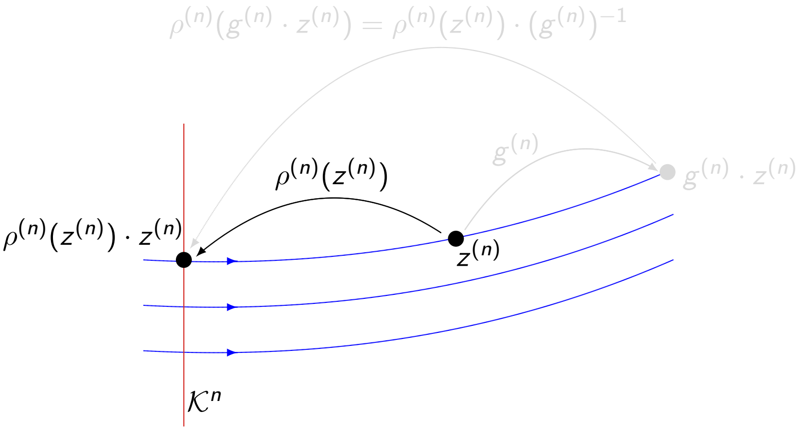

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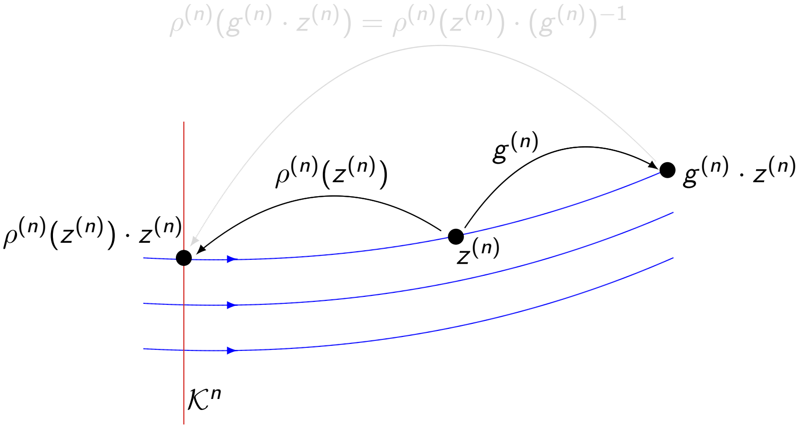

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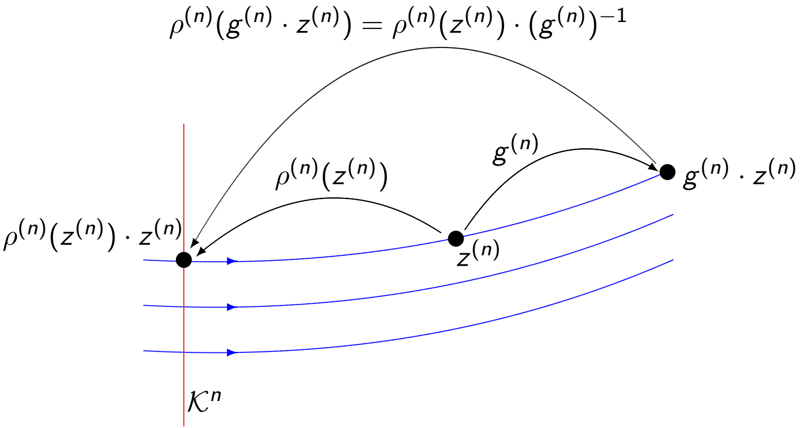
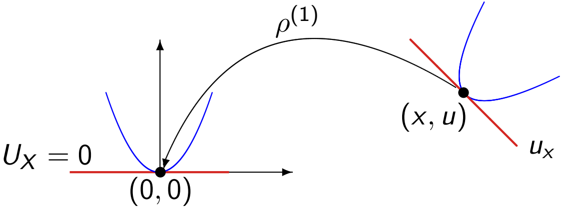


Second illustration

Cross-section $\longleftrightarrow$ normal form

Illustration

Euclidean group action: translation + rotation



$$\mathcal{K}^{(1)}: \qquad x = 0, \quad u = 0, \quad u_x = 0$$

Moving frame construction – example

The prolonged action of

$$X = \alpha(x), \qquad Y = \beta(y), \qquad U = \frac{\beta_y u}{\alpha_x}$$

on J^2 is

$$U_X = \frac{u_x \beta_y}{\alpha_x^2} - \frac{u \beta_y \alpha_{xx}}{\alpha_x^3}, \qquad U_Y = \frac{u_y}{\alpha_x} + \frac{u \beta_{yy}}{\beta_y \alpha_x},$$

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Provided $0 \neq (\ln|u|)_{xy}$ ($\sim$ Blaschke–Chern curvature)

$$\mathcal{K}^\infty = \{u = u_{xy} = 1,\, x = y = u_{x^k} = u_{y^k} = 0 : k \geqslant 1\}.$$

Solving the normalization equations

$$\overset{0}{\overset{\|}{X}} = \alpha, \qquad \overset{0}{\overset{\|}{Y}} = \beta, \qquad \overset{1}{\overset{\|}{U}} = \frac{\beta_y u}{\alpha_x}, \qquad \overset{0}{\overset{\|}{U_X}} = \frac{u_x\beta_y}{\alpha_x^2} - \frac{u\beta_y\alpha_{xx}}{\alpha_x^3},$$

$$\overset{0}{\overset{\|}{U_Y}} = \frac{u_y}{\alpha_x} + \frac{u\beta_{yy}}{\beta_y\alpha_x}, \qquad \overset{0}{\overset{\|}{U_{XX}}} = \frac{u_{xx}\beta_y}{\alpha_x^3} - 3\frac{u_x\beta_y\alpha_{xx}}{\alpha_x^4} - \frac{u\beta_y\alpha_{xxx}}{\alpha_x^4} + 3\frac{u\beta_y\alpha_{xx}^2}{\alpha_x^5},$$

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Solving the normalization equations

$$0 = \alpha, \qquad 0 = \beta, \qquad 1 = \frac{\beta_y u}{\alpha_x}, \qquad 0 = \frac{u_x \beta_y}{\alpha_x^2} - \frac{u \beta_y \alpha_{xx}}{\alpha_x^3},$$

$$0 = \frac{u_y}{\alpha_x} + \frac{u \beta_{yy}}{\beta_y \alpha_x}, \qquad 0 = \frac{u_{xx}\beta_y}{\alpha_x^3} - 3\frac{u_x \beta_y \alpha_{xx}}{\alpha_x^4} - \frac{u \beta_y \alpha_{xxx}}{\alpha_x^4} + 3\frac{u \beta_y \alpha_{xx}^2}{\alpha_x^5},$$

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Solving the normalization equations

$$0 = \alpha, \qquad 0 = \beta, \qquad 1 = \frac{\beta_y u}{\alpha_x}, \qquad 0 = \frac{u_x \beta_y}{\alpha_x^2} - \frac{u \beta_y \alpha_{xx}}{\alpha_x^3},$$

$$0 = \frac{u_y}{\alpha_x} + \frac{u \beta_{yy}}{\beta_y \alpha_x}, \qquad 0 = \frac{u_{xx}\beta_y}{\alpha_x^3} - 3\frac{u_x \beta_y \alpha_{xx}}{\alpha_x^4} - \frac{u \beta_y \alpha_{xxx}}{\alpha_x^4} + 3\frac{u \beta_y \alpha_{xx}^2}{\alpha_x^5},$$

$$0 = \frac{u_{yy}}{\beta_y \alpha_x} + \frac{u_y \beta_{yy}}{\beta_y^2 \alpha_x} + \frac{u \beta_{yyy}}{\beta_y^2 \alpha_x} - \frac{u \beta_{yy}^2}{\beta_y^3 \alpha_x}, \quad 1 = \frac{u_{xy}}{\alpha_x^2} + \frac{u_x \beta_{yy}}{\beta_y \alpha_x^2} - \frac{u_y \alpha_{xx}}{\alpha_x^3} - \frac{u \beta_{yy} \alpha_{xx}}{\beta_y \alpha_x^3}, \quad \ldots$$

we obtain ρ:

$$\alpha = 0, \quad \beta = 0, \quad \alpha_x = \sqrt{u(\ln|u|)_{xy}}, \quad \beta_y = \frac{1}{u}\alpha_x, \quad \alpha_{xx} = \frac{u_x}{u}\alpha_x,$$

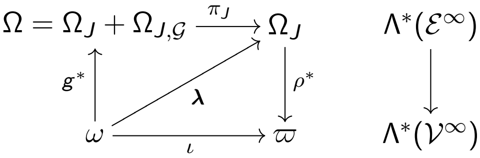
$$\beta_{yy} = -\frac{u_y}{u^2}\alpha_x, \quad \alpha_{xxx} = \frac{u_{xx}}{u}\alpha_x, \quad \beta_{yyy} = \frac{2u_y^2 - u u_{yy}}{u^3}\alpha_x, \quad \ldots$$

Invariantization

Definition

Let $\rho\colon \mathcal{V}^\infty \to \mathcal{E}^{(\infty)}$ be a right moving frame and $\omega \in \Lambda^*(\mathcal{V}^\infty)$. The invariantization map $\iota\colon \Lambda^*(\mathcal{V}^\infty) \to \Lambda^*(\mathcal{V}^\infty)$ is

1. lift: $\boldsymbol{\lambda}(\omega) = \pi_J[g^*\omega]$.
2. $\iota(\omega) = \rho^*(\boldsymbol{\lambda}(\omega))$.



- Differential functions $\Rightarrow$ differential invariants

$$\{\boldsymbol{\lambda}(x^i) = X^i, \quad \boldsymbol{\lambda}(u^\alpha_J) = U^\alpha_J\} \quad \xrightarrow{\rho^*} \quad \{\iota(x^i) = X^i, \quad \iota(u^\alpha_J) = U^\alpha_J\}$$

- Differential forms $\Rightarrow$ invariant differential forms

$$\{\boldsymbol{\lambda}(dx^i) = \varpi^i, \quad \boldsymbol{\lambda}(\theta^\alpha_J) = \vartheta^\alpha_J\} \quad \xrightarrow{\rho^*} \quad \{\iota(dx^i) = \varpi^i, \quad \iota(\theta^\alpha_J) = \vartheta^\alpha_J\}$$

- Differential operators $\Rightarrow$ invariant differential operators

$$\boldsymbol{\lambda}(D_{x^i}) = \mathcal{D}_i \quad \xrightarrow{\rho^*} \quad \iota(D_{x^i}) = \mathcal{D}_i$$

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1. lift: $\boldsymbol{\lambda}(\omega) = \pi_J[g^*\omega]$.
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$$\begin{array}{ccc} \Omega = \Omega_J + \Omega_{J,\mathcal{G}} & \xrightarrow{\pi_J} & \Omega_J \\ \uparrow g^* & \nearrow \boldsymbol{\lambda} & \downarrow \rho^* \\ \omega & \xrightarrow[\iota]{} & \varpi \end{array} \qquad \begin{array}{c} \Lambda^*(\mathcal{E}^\infty) \\ \downarrow \\ \Lambda^*(\mathcal{V}^\infty) \end{array}$$

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$$\{\boldsymbol{\lambda}(x^i) = X^i, \quad \boldsymbol{\lambda}(u^\alpha_J) = U^\alpha_J\} \quad \xrightarrow{\rho^*} \quad \{\iota(x^i) = X^i, \quad \iota(u^\alpha_J) = U^\alpha_J\}$$

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Example – invariantization of u_{xxy}, u_{yyx}, dx, dy, D_x, D_y

Substituting: $\alpha = 0, \quad \beta = 0, \quad \alpha_x = \sqrt{u(\ln|u|)_{xy}}, \quad \beta_y = \dfrac{1}{u}\alpha_x, \quad \alpha_{xx} = \dfrac{u_x}{u}\alpha_x,$

$$\beta_{yy} = -\frac{u_y}{u^2}\alpha_x, \quad \alpha_{xxx} = \frac{u_{xx}}{u}\alpha_x, \quad \beta_{yyy} = \frac{2u_y^2 - uu_{yy}}{u^3}\alpha_x, \quad \ldots$$

---

into
$$\boldsymbol{\lambda}(u_{xxy}) = U_{XXY} = \frac{u_{xxy}}{\alpha_x^3} + \frac{u_{xx}\beta_{yy}}{\beta_y\alpha_x^3} - 3\frac{u_{xy}\alpha_{xx}}{\alpha_x^4} - 3\frac{u_x\beta_{yy}\alpha_{xx}}{\beta_y\alpha_x^4} - \frac{u_y\alpha_{xxx}}{\alpha_x^4} - \frac{u\beta_{yy}\alpha_{xxx}}{\beta_y\alpha_x^4} + 3\frac{u_y\alpha_{xx}^2}{\alpha_x^5} + 3\frac{u\beta_{yy}\alpha_{xx}^2}{\beta_y\alpha_x^5},$$

$$\boldsymbol{\lambda}(u_{yyx}) = U_{YYX} = \frac{u_{yyx}}{\beta_y\alpha_x^2} - \frac{u_{yy}\alpha_{xx}}{\beta_y\alpha_x^3} + \frac{u_{yx}\beta_{yy}}{\beta_y^2\alpha_x^2} - \frac{u_y\beta_{yy}\alpha_{xx}}{\beta_y^2\alpha_x^3} + \frac{u_x\beta_{yyy}}{\beta_y^2\alpha_x^2} - \frac{u\beta_{yyy}\alpha_{xx}}{\beta_y^2\alpha_x^3} - \frac{u_x\beta_{yy}^2}{\beta_y^3\alpha_x^2} + \frac{u\beta_{yy}^2\alpha_{xx}}{\beta_y^3\alpha_x^3},$$

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and
$$\boldsymbol{\lambda}(dx) = \varpi^x = \alpha_x\,dx, \qquad \boldsymbol{\lambda}(dy) = \varpi^y = \beta_y\,dy,$$
$$\boldsymbol{\lambda}(D_x) = \mathcal{D}_x = \frac{1}{\alpha_x}D_x, \qquad \boldsymbol{\lambda}(D_y) = \mathcal{D}_y = \frac{1}{\beta_y}D_y.$$

Example – invariantization of u_{xxy}, u_{yyx}, dx, dy, D_x, D_y

yields

- the differential invariants

$$\iota(u_{xxy}) = U_{XXY} = -2D_x\left(\frac{1}{\sqrt{u(\ln|u|)_{xy}}}\right) - 2\frac{u_x(\ln|u|)_{xy}}{(u(\ln|u|)_{xy})^{3/2}},$$

$$\iota(u_{yyx}) = U_{XYY} = -2uD_y\left(\frac{1}{\sqrt{u(\ln|u|)_{xy}}}\right),$$

- the invariant horizontal forms

$$\iota(dx) = \varpi^x = \sqrt{u(\ln|u|)_{xy}}\,dx, \qquad \iota(dy) = \varpi^y = \frac{\sqrt{u(\ln|u|)_{xy}}}{u}\,dy$$

- the invariant differential operators

$$\iota(D_x) = \mathcal{D}_x = \frac{1}{\sqrt{u(\ln|u|)_{xy}}}D_x, \qquad \iota(D_y) = \mathcal{D}_y = \frac{u}{\sqrt{u(\ln|u|)_{xy}}}D_y.$$

Algebra of differential invariants

Proposition

$\{X^i = \iota(x^i), U^\alpha_J = \iota(u^\alpha_J)\}$ – complete system of functionally independent differential invariants.

Theorem (Lie–Tresse)

The differential invariant algebra is locally generated by a finite number of differential invariants

$$I_1, \ldots, I_\ell$$

and $p = \dim S$ invariant differential operators

$$\mathcal{D}_1, \ldots, \mathcal{D}_p.$$

Question: How do we find $I_1, \ldots, I_\ell$?

Universal recurrence relation

Theorem

Let $F : \mathrm{J}^n \to \mathbb{R}$

$$d\boldsymbol{\lambda}[F(z^{(n)})] = \boldsymbol{\lambda}[dF(z^{(n)})] + \boldsymbol{\lambda}[\mathbf{v}^{(n)}(F(z^{(n)}))], \qquad \mathbf{v} \in \mathfrak{g}$$

where

- $\mathbf{v}^{(n)} = n^{th}$ *order prolongation of* $\mathbf{v}$
- $\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} \quad \Rightarrow \quad \mathbf{v}^{(n)}$ *depends on* ζ^a_B
- $\boldsymbol{\lambda}(\zeta^a_B) = \mu^a_B \quad - \quad$ *Maurer–Cartan forms of* $\mathcal{G}$

Corollary

$$d\iota[F(z^{(n)})] = \iota[dF(z^{(n)})] + \iota[\mathbf{v}^{(n)}(F(z^{(n)}))], \qquad \mathbf{v} \in \mathfrak{g}$$

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Example – continuation: $X=\alpha(x), Y=\beta(y), U=u\beta_y/\alpha_x$

Let $\mathbf{v}=\xi(x)\partial_x+\eta(y)\partial_y+u(\eta_y-\xi_x)\partial_u$ and $\boldsymbol{\lambda}(\xi_{x^k})=\mu_k$, $\boldsymbol{\lambda}(\eta_{y^k})=\nu_k$, $U_{X^iY^j}=U_{i,j}$:

Applying the universal recurrence relation to x we have

$$d\boldsymbol{\lambda}(x)=\boldsymbol{\lambda}[dx+\mathbf{v}^{(n)}(x)]$$

$$\Downarrow$$

$$d\boldsymbol{\lambda}(x)=\boldsymbol{\lambda}[dx]+\boldsymbol{\lambda}[\xi]$$

$$\Downarrow$$

$$dX=\varpi^x+\mu$$

Applied to y, u, u_x, u_y, ..., we obtain

Example – continuation: $X = \alpha(x), Y = \beta(y), U = u\beta_y/\alpha_x$

Let $\mathbf{v} = \xi(x)\partial_x + \eta(y)\partial_y + u(\eta_y - \xi_x)\partial_u$ and $\boldsymbol{\lambda}(\xi_{x^k}) = \mu_k$, $\boldsymbol{\lambda}(\eta_{y^k}) = \nu_k$, $U_{X^iY^j} = U_{i,j}$:

$$
\begin{aligned}
dX &\equiv \varpi^x + \mu,\\
dY &\equiv \varpi^y + \nu,\\
dU &\equiv U_{1,0}\varpi^x + U_{0,1}\varpi^y + U(\nu_1 - \mu_1),\\
dU_{1,0} &\equiv U_{2,0}\varpi^x + U_{1,1}\varpi^y - U\mu_2 + U_{1,0}(\nu_1 - 2\mu_1),\\
dU_{0,1} &\equiv U_{1,1}\varpi^x + U_{0,2}\varpi^y + U\nu_2 - U_{0,1}\mu_1,\\
dU_{2,0} &\equiv U_{3,0}\varpi^x + U_{2,1}\varpi^y - U\mu_3 - 3U_{1,0}\nu_1 + U_{2,0}(\nu_1 - 3\mu_1),\\
dU_{0,2} &\equiv U_{1,2}\varpi^x + U_{0,3}\varpi^y + U_{0,1}\nu_2 + U\nu_3 - U_{0,2}(\mu_1 + \nu_1),\\
dU_{11} &\equiv U_{2,1}\varpi^x + U_{1,2}\varpi^y + U_{1,0}\nu_2 - U_{0,1}\mu_2 - 2U_{1,1}\mu_1,\\
dU_{2,1} &\equiv U_{3,1}\varpi^x + U_{2,2}\varpi^y - U_{0,1}\mu_3 - 3U_{1,1}\mu_2 + U_{2,0}\nu_2 - 3U_{2,1}\mu_1,\\
dU_{1,2} &\equiv U_{2,2}\varpi^x + U_{1,3}\varpi^y + U_{1,1}\nu_2 + U_{1,0}\nu_3 - U_{0,2}\mu_2 - U_{1,2}(\nu_1 + 2\mu_1),\\
&\vdots
\end{aligned}
$$

Example – continuation: $X = \alpha(x), Y = \beta(y), U = u\beta_y/\alpha_x$

$$\mathcal{K}^\infty = \{U = U_{1,1} = 1, \quad X = Y = U_{1,0} = U_{0,1} = U_{2,0} = U_{0,2} = \cdots = 0\}$$

$$\begin{aligned}
dX &\equiv \varpi^x + \mu, \\
dY &\equiv \varpi^y + \nu, \\
dU &\equiv U_{1,0}\varpi^x + U_{0,1}\varpi^y + U(\nu_1 - \mu_1), \\
dU_{1,0} &\equiv U_{2,0}\varpi^x + U_{1,1}\varpi^y - U\mu_2 + U_{1,0}(\nu_1 - 2\mu_1), \\
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0 &\equiv \varpi^x + \mu, \\
0 &\equiv \varpi^y + \nu, \\
0 &\equiv \nu_1 - \mu_1, \\
0 &\equiv \varpi^y - \mu_2, \\
0 &\equiv \varpi^x + \nu_2, \\
0 &\equiv U_{2,1}\varpi^y - \mu_3, \\
0 &\equiv U_{1,2}\varpi^x + \nu_3, \\
0 &\equiv U_{2,1}\varpi^x + U_{1,2}\varpi^y - 2\mu_1, \\
dU_{2,1} &\equiv U_{3,1}\varpi^x + U_{2,2}\varpi^y - 3\mu_2 - 3U_{2,1}\mu_1, \\
dU_{1,2} &\equiv U_{2,2}\varpi^x + U_{1,3}\varpi^y + \nu_2 - U_{1,2}(\nu_1 + 2\mu_1), \\
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$$\begin{aligned}
\mu &\equiv -\varpi^x, \\
\nu &\equiv -\varpi^y, \\
\nu_1 &\equiv (U_{2,1}\varpi^x + U_{1,2}\varpi^y)/2, \\
\mu_2 &\equiv \varpi^y, \\
\nu_2 &\equiv -\varpi^x, \\
\mu_3 &\equiv U_{2,1}\varpi^y, \\
\nu_3 &\equiv -U_{1,2}\varpi^x, \\
\mu_1 &\equiv (U_{2,1}\varpi^x + U_{1,2}\varpi^y)/2, \\
dU_{2,1} &\equiv U_{3,1}\varpi^x + U_{2,2}\varpi^y - 3\mu_2 - 3U_{2,1}\mu_1, \\
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\nu_3 &\equiv -U_{1,2}\varpi^x, \\
\mu_1 &\equiv (U_{2,1}\varpi^x + U_{1,2}\varpi^y)/2, \\
dU_{2,1} &\equiv \left(U_{3,1} - \frac{3}{2}U_{2,1}^2\right)\varpi^x + \left(U_{2,2} - 3 - \frac{3}{2}U_{2,1}U_{1,2}\right)\varpi^y, \\
dU_{1,2} &\equiv \left(U_{2,2} - 1 - \frac{3}{2}U_{1,2}U_{2,1}\right)\varpi^x + \left(U_{1,3} - \frac{3}{2}U_{1,2}^2\right)\varpi^y,
\end{aligned}$$

Example – continuation: $X = \alpha(x), Y = \beta(y), U = u\beta_y/\alpha_x$

$$dU_{2,1} \equiv \left(U_{3,1} - \frac{3}{2}U_{2,1}^2\right)\varpi^x + \left(U_{2,2} - 3 - \frac{3}{2}U_{2,1}U_{1,2}\right)\varpi^y,$$
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Since

$$dU_{i,j} \equiv (D_x U_{i,j})dx + (D_y U_{i,j})dy = (\mathcal{D}_x U_{i,j})\varpi^x + (\mathcal{D}_y U_{i,j})\varpi^y$$

we conclude

$$\mathcal{D}_x U_{2,1} = U_{3,1} - \frac{3}{2}U_{2,1}^2, \qquad \mathcal{D}_y U_{2,1} = U_{2,2} - 3 - \frac{3}{2}U_{2,1}U_{1,2},$$
$$\mathcal{D}_x U_{1,2} = U_{2,2} - 1 - \frac{3}{2}U_{1,2}U_{2,1}, \qquad \mathcal{D}_y U_{1,2} = U_{1,3} - \frac{3}{2}U_{1,2}^2.$$

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Proposition

$U_{1,2}$, $U_{2,1}$ generate the algebra of differential invariants.

Syzygy: $$\mathcal{D}_x U_{1,2} - \mathcal{D}_y U_{2,1} = 2.$$

Signature

Definition

Let $\{I_1, \ldots, I_\ell\}$ be a generating set and S a submanifold. The n^{th} order signature map $\mathbf{I}_S^{(n)} \colon S \to \mathbb{R}^{d_n}$ is defined by

$$\mathcal{D}_J I_\kappa|_{z \in S}, \qquad \kappa = 1, \ldots, \ell, \qquad \#J \leq n.$$

Proposition

Let ϱ_n denote the rank of $\mathbf{I}_S^{(n)}$. Then $0 \leq \varrho_0 < \varrho_1 < \cdots < \varrho_s = \varrho_{s+1} = \cdots = r \leq p$. The smallest s for which $\varrho_s = \varrho_{s+1} = r$ is called the order of the signature map.

Definition

The image $\mathfrak{S}^{(n)}(\rho, S) = \{\mathbf{I}_S^{(n)}(z) : z \in S\}$ is called the n^{th} order signature manifold.

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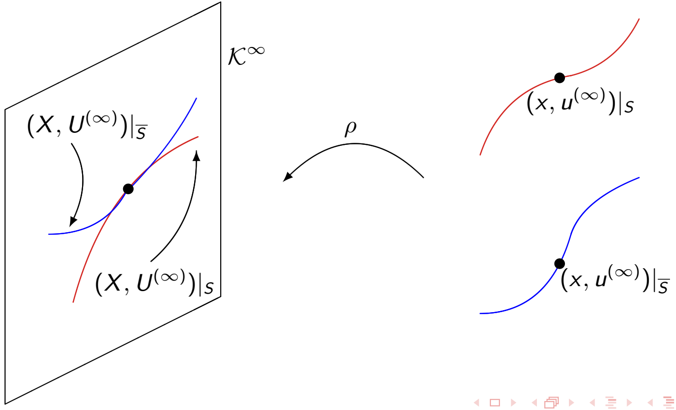
Equivalence – regular submanifolds

Theorem

Let $S, \overline{S} \subset M$. There exists $g \in \mathcal{G}$ such that $g \cdot S = \overline{S} \Leftrightarrow$

- $\mathbf{I}_S$, $\mathbf{I}_{\overline{S}}$ *have same order* $\overline{s} = s$
- $\mathfrak{S}^{(s+1)}(\rho, S)$, $\mathfrak{S}^{(s+1)}(\rho, \overline{S})$ *overlap*



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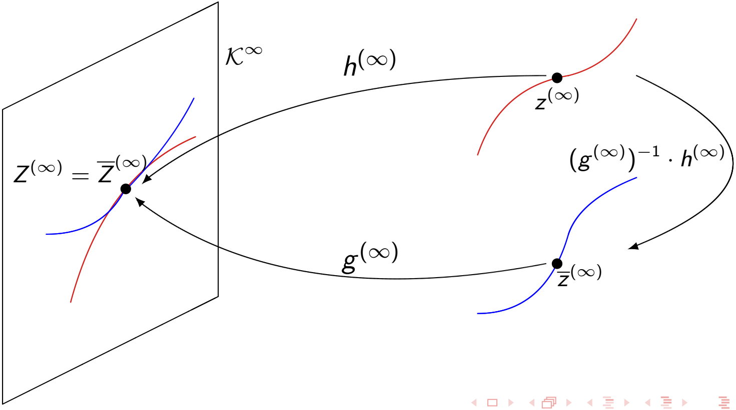
Theorem

Let $S, \overline{S} \subset M$. *There exists* $g \in \mathcal{G}$ *such that* $g \cdot S = \overline{S} \Leftrightarrow$

- $\mathbf{I}_S$, $\mathbf{I}_{\overline{S}}$ *have same order* $\overline{s} = s$
- $\mathfrak{S}^{(s+1)}(\rho, S)$, $\mathfrak{S}^{(s+1)}(\rho, \overline{S})$ *overlap*



Equivalence of (regular) 3-webs

Provided $(\ln|u|)_{xy} \neq 0$:

$$\mathbf{I}_S^{(n)} = (U_{1,2}, U_{2,1}, \mathcal{D}_x U_{1,2}, \mathcal{D}_y U_{1,2}, \ldots)|_S$$

where

$$U_{2,1} = -2D_x\left(\frac{1}{\sqrt{u(\ln|u|)_{xy}}}\right) - 2\frac{u_x(\ln|u|)_{xy}}{(u(\ln|u|)_{xy})^{3/2}},$$

$$U_{1,2} = -2uD_y\left(\frac{1}{\sqrt{u(\ln|u|)_{xy}}}\right),$$

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Partial moving frames (V-2010)

To deal with the case $(\ln|u|)_{xy} = 0$, we introduce

Partial moving frames

$$\Downarrow$$

equivariant moving frame theory for non-free actions

- The invariantization map ✓
- Universal recurrence relation ✓
- Algebra of differential invariants – finitely generated ✓
- Solution to local equivalence problems ✓

Equivalence – Cartan vs equivariant moving frames

Equivariant (partial) moving frame method

Lie pseudo-group $\downarrow$ structure equations (OP-2005)

Cartan moving frame method

Some highlights:

- Symbolic computations
- Universal recurrence relation $\longrightarrow$ syzygies (Gröbner basis)
- No EDS, no absorption of torsion, ...
- Solution to challenging equivalence problems?
  (V-2010) - Local equivalence of $u_{xx} = f(x, u, v, u_x, v_x)$ under

$$X = \phi(x), \quad U = \beta(x, u), \quad V = \alpha(x, u, v).$$

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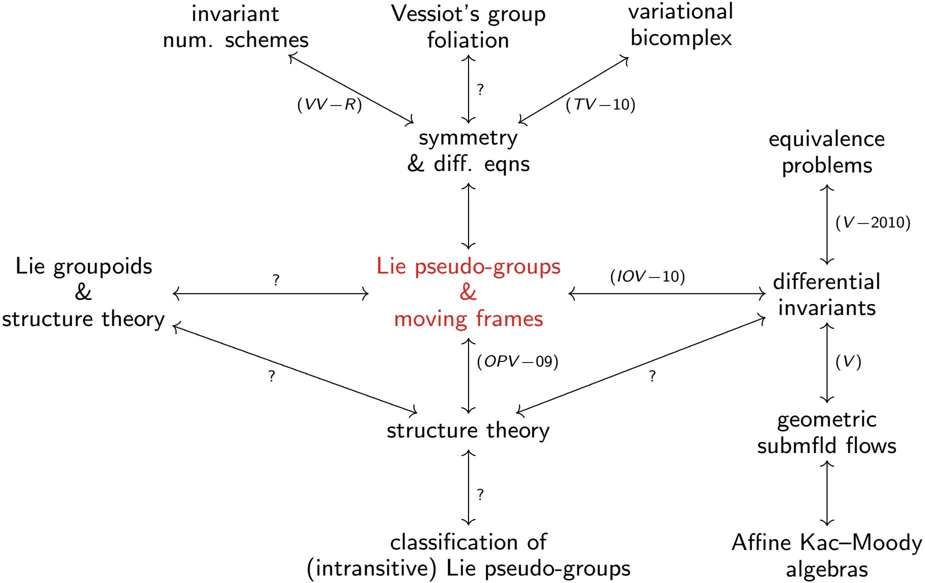
Perspectives



Important question

What is the complete abstract theory of ∞-dimensional Lie pseudo-groups???