Group Foliation of PDE Using Moving Frames

(ISM Graduate Students Seminar)

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Outline

Some Notation

Let

$$x = (x^1, \ldots, x^p) \in \mathcal{X} \simeq \mathbb{R}^p$$

denote the independent variables and

$$u = (u^1, \ldots, u^q) \in \mathcal{U} \simeq \mathbb{R}^q$$

denote the dependent variables. We use

$$u^{(n)} = (u^1, \ldots, u^q, u^1_{x^1}, u^1_{x^2}, \ldots) \in \mathcal{U}^{(n)}$$

to denote the derivatives of u with respect to x up to order n. Finally we write

$$\Delta(x, u^{(n)}) = \Delta(z^{(n)}) = 0$$

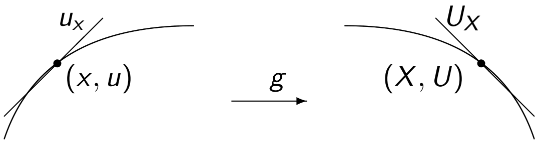
to denote a system of partial differential equations.

Prolonged Action

Let G be an r-dimensional Lie group acting on $\mathcal{X} \times \mathcal{U}$:

$$X = X(x,u) = g \cdot x, \qquad U = U(x,u) = g \cdot u.$$

This induces an action on the derivatives $u^1_{x^1}$, $u^1_{x^1x^1}$, $\ldots$.



The expressions of the prolonged action are found by the chain rule. This can be done as follows:

- Let
$$d_H X^j = \sum_{i=1}^{p} D_{x^i} X^j dx^i, \qquad i = 1, \ldots, p.$$
- The corresponding dual total differential operators are
$$D_{X^i} = \sum_{j=1}^{p} W_i^j D_{x^j}, \qquad \text{where} \qquad (W_i^j) = (D_{x^j} X^i)^{-1}.$$
- Then
$$U^\alpha_{X^J} = D_X^J(U^\alpha), \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0.$$

Running Example

As our running example we consider the following $GL(2)$ action

$$X = \alpha x + \beta y, \qquad Y = \gamma x + \delta y, \qquad U = \lambda u, \qquad \lambda = \alpha\delta - \beta\gamma \neq 0.$$

Then

$$d_H X = \alpha dx + \beta dy, \qquad d_H Y = \gamma dx + \delta dy,$$

and the lifted total differential operators are

$$D_X = \frac{1}{\lambda}(\delta D_x - \gamma D_y), \qquad D_Y = \frac{1}{\lambda}(-\beta D_x + \alpha D_y).$$

Thus

$$U_X = \delta u_x - \gamma u_y, \qquad U_Y = -\beta u_x + \alpha u_y,$$

$$U_{XX} = \frac{\delta^2 u_{xx} - 2\gamma\delta u_{xy} + \gamma^2 u_{yy}}{\lambda}, \qquad U_{YY} = \frac{\beta^2 u_{xx} - 2\alpha\beta u_{xy} + \alpha^2 u_{yy}}{\lambda},$$

$$U_{XY} = \frac{-\beta\delta u_{xx} + (\alpha\delta + \beta\gamma)u_{xy} - \alpha\gamma u_{yy}}{\lambda}, \qquad \ldots$$

Symmetry of a PDE system

Definition

Let

$$\mathcal{S}_\Delta = \{(x, u^{(n)}) : \Delta(x, u^{(n)}) = 0\}.$$

A Lie group G is a symmetry group of $\Delta(x, u^{(n)}) = 0$ if and only if

$$g(\mathcal{S}_\Delta) \subset \mathcal{S}_\Delta, \qquad \forall g \in G.$$

Example

The nonlinear PDE

$$u_t = u_{xx} - \frac{u_x^2}{u}$$

admits the symmetry subgroup

$$G: \quad X = \alpha x + a, \quad T = \alpha^2 t + b, \quad U = \lambda u, \quad \alpha, \lambda \in \mathbb{R}^+, \quad a, b \in \mathbb{R}.$$

Infinitesimal Generators

Proposition

Let $\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\partial_{x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\partial_{u^\alpha}$ be an infinitesimal generator of the group action, i.e.,

$$\mathbf{v} = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} g_\epsilon \cdot (x,u),$$

then the n-th order prolongation of the infinitesimal generator is given by

$$\mathbf{v}^{(n)} = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} g_\epsilon \cdot (x, u^{(n)}) = \sum_{i=1}^{p} \xi^i(x,u)\partial_{x^i} + \sum_{\alpha=1}^{q} \sum_{0\leq \# J\leq n} \phi_\alpha^J(x, u^{(\# J)})\partial_{u_J^\alpha},$$

where

$$\phi_\alpha^{J,j} = D_{x^j}(\phi_\alpha^J) - \sum_{i=1}^{p} D_{x^j}(\xi^i)\cdot u_{J,i}^\alpha .$$

Illustration

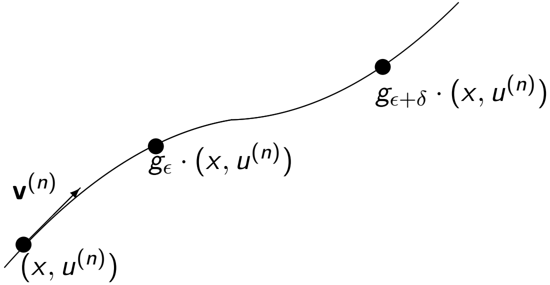


Figure: Infinitesimal Generator.

Example Continued

The infinitesimal generators of

$$X = \alpha x + \beta y, \qquad Y = \gamma x + \delta y, \qquad U = \lambda u, \qquad \lambda = \alpha\delta - \beta\gamma.$$

are

$$\begin{aligned}
\mathbf{v}_\alpha &= \frac{d}{d\alpha}\bigg|_{\alpha,\delta=1,\beta,\gamma=0} g\cdot(x,t,u) = x\partial_x + u\partial_u,\\
\mathbf{v}_\beta &= \frac{d}{d\beta}\bigg|_{\alpha,\delta=1,\beta,\gamma=0} g\cdot(x,t,u) = y\partial_x,\\
\mathbf{v}_\gamma &= \frac{d}{d\gamma}\bigg|_{\alpha,\delta=1,\beta,\gamma=0} g\cdot(x,t,u) = x\partial_y,\\
\mathbf{v}_\delta &= \frac{d}{d\delta}\bigg|_{\alpha,\delta=1,\beta,\gamma=0} g\cdot(x,t,u) = y\partial_y + u\partial_u.
\end{aligned}$$

The components of second order prolongation of $\mathbf{v}_\alpha = x\partial_x + u\partial_u$ are

$$\phi_\alpha^x = u_x - u_x = 0, \qquad \phi_\alpha^y = u_y$$
$$\phi_\alpha^{xx} = -u_{xx}, \qquad \phi_\alpha^{xy} = u_{xy} - u_{xy} = 0 \qquad \phi_\alpha^{yy} = u_{yy}.$$

Thus

$$\mathbf{v}_\alpha^{(2)} = x\partial_x + u\partial_u + u_y\partial_{u_y} - u_{xx}\partial_{u_{xx}} + u_{yy}\partial_{u_{yy}}.$$

Similarly,

$$\begin{aligned}
\mathbf{v}_\beta^{(2)} &= y\partial_x - u_x\partial_{u_y} - u_{xx}\partial_{u_{xy}} - 2u_{xy}\partial_{u_{yy}}, \\
\mathbf{v}_\gamma^{(2)} &= x\partial_y - u_y\partial_{u_x} - 2u_{xy}\partial_{u_{xx}} - u_{yy}\partial_{u_{xy}}, \\
\mathbf{v}_\delta^{(2)} &= y\partial_y + u\partial_u + u_x\partial_{u_x} + u_{xx}\partial_{u_{xx}} - u_{yy}\partial_{u_{yy}}.
\end{aligned}$$

Infinitesimal Symmetry Criterion

Proposition

A connected group of transformations G is a symmetry group of a (fully regular) system of differential equations $\Delta(x, u^{(n)}) = 0$ if and only if

$$\mathbf{v}^{(n)}(\Delta)\Big|_{\Delta=0} = 0, \qquad \forall\, \mathbf{v} \in \mathfrak{g}.$$

- In practice the infinitesimal generators of symmetry are found, and integrated to obtain the group transformations.
- As we'll see shortly, the infinitesimal generators play an important role in structure of the algebra of differential invariants.

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- In practice the infinitesimal generators of symmetry are found, and integrated to obtain the group transformations.
- As we'll see shortly, the infinitesimal generators play an important role in structure of the algebra of differential invariants.

Moving Frames

Definition

Let G be a Lie group acting on M. A right (left) moving frame is a G-equivariant map $\rho : M \to G$, i.e.

$$\rho_r(g \cdot z) = \rho_r(z) \cdot g^{-1} \qquad \text{(for a right moving frame)},$$

$$\rho_l(g \cdot z) = g \cdot \rho_l(z) \qquad \text{(for a left moving frame)}.$$

Left and right moving frames are related by

$$\rho_l(z) = (\rho_r(z))^{-1}.$$

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Definition

A Lie group G acts freely on a manifold M if $G_z = \{e\}$ for all $z \in M$.

Definition

A Lie group G acts regularly on a manifold M if the orbits of G have constant dimension and for all $z \in M$ there exists a neighborhood $\mathcal{N}_z$ such that $\mathcal{N}_z \cap \mathcal{O}_z$ is connected.

Theorem

If G acts freely and regularly at $z \in M$ then there exists a moving frame in a neighborhood of z.

Theorem

Let G act freely and regularly on M. Let $\mathcal{K}$ be a cross-section to the group orbits. For $z \in M$, let $g = \rho(z)$ be the unique group element that maps z to the cross-section: $g \cdot z = \rho(z) \cdot z \in \mathcal{K}$. Then $\rho : M \to G$ is a right equivariant moving frame for the group action.

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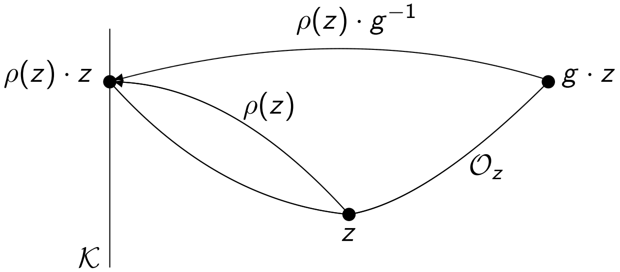
Moving Frame in Action



For us $M = \mathcal{X} \times \mathcal{U}^{(n)}$ and $z = z^{(n)}$.

How to Obtain a Moving Frame?

1. Compute the explicit local coordinate formulas for the prolonged group transformations

$$w^{(n)}(g, z^{(n)}) = Z^{(n)} = g^{(n)} \cdot z^{(n)}, \qquad z^{(n)} = (x, u^{(n)}). \tag{1}$$

2. Choose (typically) a coordinate cross-section $\mathcal{K}^n = \{z_1 = c_1, \ldots, z_r = c_r\}$ obtained by setting $r = \dim G$ of the components of $z^{(n)}$ equal to constants.

3. Solve the normalization equations

$$w_1(g, z^{(n)}) = c_1, \qquad w_r(g, z^{(n)}) = c_r, \tag{2}$$

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z^{(n)}$.

Example Continued

For the $GL(2)$ action

$$X = \alpha x + \beta y, \qquad Y = \gamma x + \delta y, \qquad U = \lambda u, \qquad \lambda = \alpha\delta - \beta\gamma,$$
$$U_X = \delta u_x - \gamma u_y, \qquad U_Y = -\beta u_x + \alpha u_y,$$

we can choose the cross-section

$$X = 1, \qquad Y = 0, \qquad U_X = 1, \qquad U_Y = 0.$$

Solving for the group parameters we obtain

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{xu_x + yu_y} \begin{pmatrix} u_x & u_y \\ -y & x \end{pmatrix},$$

provided $xu_x + yu_y \neq 0$.

Invariantization

Once a moving frame is known we can invariantize differential functions (and differential forms). For differential functions

$$\iota(F(z^{(n)})) = \rho^*(F(g \cdot z^{(n)})) = F(\rho^{(n)}(z^{(n)}) \cdot z^{(n)}) = I(z^{(n)}).$$

$(g \cdot I(z^{(n)}) = F(\rho^{(n)}(g \cdot z^{(n)}) \cdot g \cdot z^{(n)}) = F(\rho^{(n)}(z^{(n)}) \cdot g^{-1} \cdot g \cdot z^{(n)}))$

In particular we can invariantize $x^1, \ldots, x^p, u^1, \ldots, u^q, u_{x^1}, \ldots$.

Theorem

Let ρ be a moving frame, then

$$H^i = \iota(x^i), \quad I^\alpha_J = \iota(u^\alpha_J), \quad \#J \geq 0$$

constitutes a complete set of differential invariants.

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Example Continued

For the $GL(2)$ action the moving frame found was

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{xu_x + yu_y} \begin{pmatrix} u_x & u_y \\ -y & x \end{pmatrix}, \qquad \lambda = \frac{1}{xu_x + yu_y}.$$

By definition of the cross-section

$$\iota(x) = 1, \qquad \iota(y) = 0, \qquad \iota(u_x) = 1, \qquad \iota(u_y) = 0,$$

and the invariants are

$$\begin{aligned} \iota(u) =& \rho^*(\lambda u) = \frac{u}{xu_x + yu_y}, \\ \iota(u_{xx}) =& \rho^* \left(\frac{\delta^2 u_{xx} - 2\gamma\delta u_{xy} + \gamma^2 u_{yy}}{\lambda} \right) = \frac{x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy}}{xu_x + yu_y}, \end{aligned}$$

and so on.

Invariant Differential Operators

Definition

A differential operator $\mathcal{D}$ is said to be an invariant differential operator if for all differential invariant $I(z^{(n)})$, $\mathcal{D}I$ is also a differential invariant.

Proposition

If there is p independent variables then there is p independent invariant differential operators.

Once a moving frame is known it is straightforward to obtain p independent invariant differential operators:

$$\mathcal{D}_i = \sum_{j=1}^{p} \rho^*(W_i^j) D_{x^j}, \qquad \text{where} \qquad (W_i^j) = (D_{x^j} X^i)^{-1},$$

$i = 1, \ldots, p$.

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$i = 1, \ldots, p.$

Example Continued

For our $GL(2)$ running example we found the lifted total differential operators

$$\mathcal{D}_X = \frac{1}{\lambda}(\delta D_x - \gamma D_y), \qquad \mathcal{D}_Y = \frac{1}{\lambda}(-\beta D_x + \alpha D_y).$$

and the moving frame

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{xu_x + yu_y}\begin{pmatrix} u_x & u_y \\ -y & x \end{pmatrix}, \qquad \lambda = \frac{1}{xu_x + yu_y}.$$

Then

$$\mathcal{D}_1 = xD_x + yD_y, \qquad \mathcal{D}_2 = -u_y D_x + u_x D_y.$$

Recurrence Relations

Theorem

Let $\mu^1, \ldots, \mu^r \in \mathfrak{g}^*$ be the Maurer–Cartan forms dual to $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathfrak{g}$. For any differential function $F(z^{(n)})$

$$\sum_{i=1}^{p} \mathcal{D}_i \iota(F)\omega^i = \sum_{i=1}^{p} \iota(D_{x^i} F)\omega^i + \sum_{\kappa=1}^{r} \iota[\mathbf{v}_\kappa^{(\infty)}(F)] \cdot \nu^\kappa, \tag{3}$$

where $\nu^\kappa = \pi_{\mathcal{H}} \circ \rho^*(\mu^\kappa)$ is the horizontal component of the pull-back of the Maurer–Cartan form μ^κ via the moving frame and

$$\omega^i = \rho^*[\sum_{j=1}^{p} (D_{x^j} X^i) dx^j].$$

The differential forms ν^κ can be deduced directly from (3).

Example Continued

For our $GL(2)$ example we have

$$H^1 = \iota(x) = 1, \quad H^2 = \iota(y) = 0 \quad I_1 = \iota(u_x) = 1, \quad I_2 = \iota(u_y) = 0,$$

and the corresponding recurrence relations imply

$$\begin{aligned} 0 &= \omega^1 + \nu^\alpha, & 0 &= \omega^2 + \nu^\gamma, \\ 0 &= I_{11}\omega^1 + I_{12}\omega^2 + \nu^\delta, & 0 &= I_{12}\omega^2 + I_{22}\omega^2 - \nu^\beta. \end{aligned}$$

Thus

$$\begin{aligned} \nu^\alpha &= -\omega^1, & \nu^\gamma &= -\omega^2, \\ \nu^\delta &= -(I_{11}\omega^1 + I_{12}\omega^2), & \nu^\beta &= I_{12}\omega^1 + I_{22}\omega^2. \end{aligned}$$

In some more details . . . . Recall that

$$
\begin{aligned}
\mathbf{v}_\alpha^{(2)} =& x\partial_x + u\partial_u + u_y\partial_{u_y} - u_{xx}\partial_{u_{xx}} + u_{yy}\partial_{u_{yy}},\\
\mathbf{v}_\beta^{(2)} =& y\partial_x - u_x\partial_{u_y} - u_{xx}\partial_{u_{xy}} - 2u_{xy}\partial_{u_{yy}},\\
\mathbf{v}_\gamma^{(2)} =& x\partial_y - u_y\partial_{u_x} - 2u_{xy}\partial_{u_{xx}} - u_{yy}\partial_{u_{xy}},\\
\mathbf{v}_\delta^{(2)} =& y\partial_y + u\partial_u + u_x\partial_{u_x} + u_{xx}\partial_{u_{xx}} - u_{yy}\partial_{u_{yy}}.
\end{aligned}
$$

Then the recurrence relation for u_x gives

$$
\begin{aligned}
\mathcal{D}_1\iota(u_x)\omega^1 + \mathcal{D}_2\iota(u_x)\omega^2 =& \iota(D_x u_x)\omega^1 + \iota(D_y u_x)\omega^2 + \iota[\mathbf{v}_\alpha^{(1)}(u_x)]\nu^\alpha\\
& + \iota[\mathbf{v}_\beta^{(1)}(u_x)]\nu^\beta + \iota[\mathbf{v}_\gamma^{(1)}(u_x)]\nu^\gamma + \iota[\mathbf{v}_\delta^{(1)}(u_x)]\nu^\delta\\
\mathcal{D}_1(1)\omega^1 + \mathcal{D}_2(1)\omega^2 =& \iota(u_{xx})\omega^1 + \iota(u_{xy}) + \iota(0)\nu^\alpha + \iota(0)\nu^\beta\\
& + \iota(-u_y)\nu^\gamma + \iota(u_x)\nu^\delta\\
0 =& I_{11}\omega^1 + I_{12}\omega^2 + \nu^\delta
\end{aligned}
$$

$\iota(u_x) = 1,\ \iota(u_y) = 0.$

The recurrence relation for u gives

$$\begin{aligned}\mathcal{D}_1 I\, \omega^1 + \mathcal{D}_2 I\, \omega^2 =& \iota(D_x u)\omega^1 + \iota(D_y u)\omega^2 + \iota(u)\nu^\alpha + \iota(u)\nu^\delta \\ =& \omega^1 - I\, \omega^1 - I(I_{11}\omega^1 + I_{12}\omega^2)\end{aligned}$$

Hence

$$\mathcal{D}_1 I = 1 - I(1 + I_{11}), \qquad \mathcal{D}_2 I = I \cdot I_{12},$$

Substituting the functions/variables u_{xx}, u_{xy}, u_{yy}, ... into the recurrence relations gives recurrence relations for the higher order differential invariants.

Finiteness Theorem

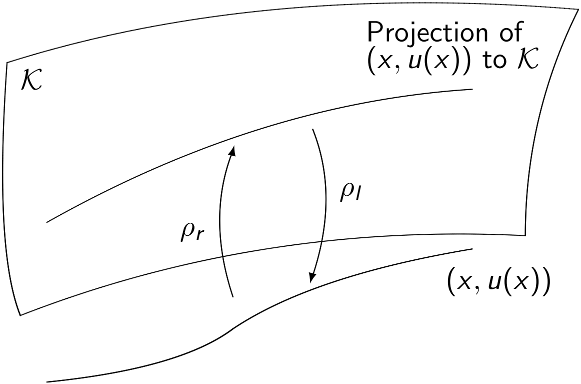
Theorem

Let G be a Lie group acting locally effectively on $\mathcal{X} \times \mathcal{U}$. Then the algebra of differential invariants is finitely generated.

- The Theorem says that there exists a finite set of differential invariants
$$\{J^1, \ldots, J^k\} \tag{4}$$
such that all other differential invariants can be obtain by taking certain combinations of (4) and their invariant derivations $\mathcal{D}_1 I^1, \ldots, \mathcal{D}_p I^1, \ldots, \mathcal{D}_1 J^k, \ldots, \mathcal{D}_p J^k, \mathcal{D}_1^2 I_1, \ldots$.
- The proof of the theorem relies on the recurrence relations for the differential invariants.

Group Foliation of PDE

Let $\Delta(x, u^{(n)}) = 0$ be a system of PDE with non-trivial symmetry group G. The goal is to obtain non-invariants solutions using moving frames. In picture this is done as follows:



Algorithm

- Let $\Delta(x, u^{(n)}) = 0$ be a system of PDE with non-trivial symmetry group G.
- We assume G acts freely and regularly on $\mathcal{X} \times \mathcal{U}^{(n)}$.
- We assume that there is at least $p+1$ independent differential invariants of order $\leq n$. (By introducing dummy variables and companion equations to the PDE system the assumption can always be satisfied [Mansfield, 10].)
- Choose p invariants, call them $J^1, \ldots, J^p$, to play the role of the independent variable.
- Then

$$\frac{d}{dJ^i} = \sum_{i=1}^{p} W_i^j \mathcal{D}_j, \qquad \text{where} \qquad W_i^j = (\mathcal{D}_j J^i)^{-1}.$$

$i = 1, \ldots, p$, are invariant differential operators.

- Use the recurrence relations to find a generating set of differential invariants $\{J^1, \dots, J^p, L^1, \dots, L^s\}$ with invariant differential operator $d/dJ^1, \dots, d/dJ^p$.
- Write the invariant PDE $\Delta(x, u^{(n)}) = 0$ as

$$\Delta(J, L^{(k)}) = 0,$$

- and add all syzygies.
- Solve the system of PDE. To obtain the solution to the original PDE we need to reconstruct the solution.

- Consider a faithful representation of the group $G \in M_{l\times l}$. Then the left moving frame satisfies the equation

$$d\rho_l(J) = -\rho_l(J)(d\rho_r\cdot\rho_r^{-1})(J). \tag{5}$$

The term in parenthesis corresponds to the pull-back of the right invariant Maurer-Cartan forms by the right moving frame ρ_r. Their expressions can be obtained from the recurrence relations. Equation (5) is invariant hence its evolution is known.

- Once (5) is solved, the original solution is obtained by computing

$$x(J) = \rho_l(J)\cdot H, \qquad u(J) = \rho_l(J)\cdot I.$$

An Application

We apply the group foliation method to the nonlinear PDE

$$u_t = u_{xx} - \frac{u_x^2}{u}. \tag{6}$$

The equation (6) is invariant under the transformation group

$$X = x, \qquad T = t, \qquad U = \lambda u, \qquad \lambda > 0.$$

A moving frame is obtained by choosing the cross-section $U = 1$. Solving for λ we obtain

$$\lambda = \frac{1}{u}, \qquad u > 0.$$

The first four differential invariants are given by

$$H^1 = \iota(x) = x, \qquad H^2 = \iota(t) = t,$$
$$I = \iota(u_x) = \rho^*\left(\frac{u_x}{\lambda}\right) = \frac{u_x}{u}, \qquad J = \iota(u_t) = \rho^*\left(\frac{u_t}{\lambda}\right) = \frac{u_t}{u}.$$

The algebra of differential invariants is generated by

$$x, \qquad t, \qquad I, \qquad J.$$

There is a syzygy between I and J:

$$D_t I = \frac{u_{xt}}{u} - \frac{u_x u_t}{u^2} = D_x J.$$

In terms of the invariants, the original nonlinear PDE ($u_t = u_{xx} - u_x^2/u$) reduces to

$$J = D_x I.$$

Thus we need to solve the system of PDE

$$D_t I = D_x J, \qquad J = D_x I.$$

It follows that I must satisfy the heat equation

$$D_t I = D_x^2 I.$$

We recover the original solution by the reconstruction process. A faithful representation of the dilation group is given by

$$\lambda(x,t) = e^{\alpha(x,t)}.$$

Then the equation $d\rho_l(x,t) = -\rho_l(x,t)(d\rho_r \cdot \rho_r^{-1})(x,t)$ gives

$$\rho_x\, dx + \rho_t\, dt = -\rho \cdot \nu^\lambda$$
$$\alpha_x\, dx + \alpha_t\, dt = I\, dx + J\, dt.$$

The right-hand side of the last equality is a consequence of the recurrence relation for the phantom invariant $U = 1$:

$$0 = I\, dx + J\, dt + \nu^\lambda \quad \Rightarrow \quad \nu^\lambda = -(I\, dx + J\, dt).$$

Thus

$$\sigma_x = I, \qquad \sigma_t = J = D_x I = \sigma_{xx}.$$

In conclusion, the solution to

$$u_t = u_{xx} - \frac{u_x^2}{u}$$

is given by

$$x = x, \qquad t = t, \qquad u = e^{\alpha(x,t)} \qquad \text{where} \qquad \alpha_t = \alpha_{xx}.$$

Remark

This result was first derived using infinitesimal methods.

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