Moving frames for Lie pseudo-groups (Junior Colloquium)

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September 11, 2007

Outline

Why study moving frames?

Because it's COOL! (My answer)

Why study moving frames? (Peter's answer)

- Very general: Lie groups and Lie pseudo-groups,
- explicit expressions for the invariants of a group action and the structure of the "algebra" of invariants.
- structure equations,
- equivalence problems,
- invariant variational problems,
- geometric integration: invariant numerical schemes,
- computer vision (please direct your questions to Joe Kenney),
- ...

Fiber bundle

Definition

A Fiber bundle is a triple (E, π, B) where E and B are manifolds and $\pi : E \longrightarrow B$ is a surjective submersion such that $\forall b \in B$ there exists a neighborhood $W_b \subset B$ and a diffeomorphism

$$\phi_b : \pi^{-1}(W_b) \longrightarrow W_b \times F_b.$$

Example

- The cylinder: $(S^1 \times \mathbb{R}, \mathrm{pr}_1, S^1)$,
- The Mobius band E=$\{(x, y) \in [0, 1] \times \mathbb{R} \mid (0, y) \sim (1, -y)\}$: $(E, \mathrm{pr}_1, [0, 1])$. For all $x \in [0, 1]$ there exist a diffeomorphism ϕ_x such that

$$\phi_x(\pi^{-1}((x - 1/2, x + 1/2))) = (x - 1/2, x + 1/2) \times \mathbb{R}.$$

Jet space

Definition

Two submanifolds $S, \widetilde{S} \subset M$ of dimension p have n^{th} order contact at $z_0 \in S \cap \widetilde{S}$ if there exists a local coordinate chart W containing $z_0 = (x_0, u_0)$ such that $S \cap W$ and $\widetilde{S} \cap W$ coincide with the graph of $u = f(x)$, $u = \widetilde{f}(x)$ respectively and

$$\sum_{0 \leq \#J \leq n} \partial_J f(x_0)/J! = \sum_{0 \leq \#J \leq n} \partial_J \widetilde{f}(x_0)/J!.$$

Definition

The n-th extended jet bundle $J^{(n)} = J^{(n)}(M, p)$ is defined as the equivalence classes of p-dimensional submanifolds $S \subset M$ under the equivalence relation of n^{th} order contact. For $M = X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$

$$j_n S|_{(x,u)} = (x, u^{(n)}) = (x^1, \ldots, x^p, \ldots, u^\alpha_J, \ldots), \quad 1 \leq \alpha \leq q, 0 \leq \#J \leq n.$$

$J^{(n)}$ is a fiber bundle: $J^{(n)} \xrightarrow{\pi^n_0} M$, $\pi^n_0(x, u^{(n)}) = (x, u)$.

Example

Let $M = \mathbb{R}^2$, $S = \{(x, x^2) : x \in \mathbb{R}\}$ and $\widetilde{S} = \{(x, x^3) : x \in \mathbb{R}\}$. Since

$$j_2 S = (x, x^2, 2x, 2) \qquad \text{and} \qquad j_2\widetilde{S} = (x, x^3, 3x^2, 6x),$$

we have that

$$j_1 S|_{(0,0)} = j_1\widetilde{S}|_{(0,0)} = (0, 0, 0),$$

but

$$j_2 S|_{(0,0)} = (0, 0, 0, 2) \neq j_2\widetilde{S}|_{(0,0)} = (0, 0, 0, 0).$$

So S and $\widetilde{S}$ have first order contact at $(0, 0)$.

Bicomplex structure of $\Lambda^*(J^{(\infty)})$

- Basis of horizontal forms:

$$dx^i, \qquad i = 1, \ldots, p,$$

- Basis of vertical (contact) forms:

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{n} u^\alpha_{J,i} dx^i$$

The differential of a differential function $F : J^{(\infty)} \to \mathbb{R}$ can then be written

$$dF = d_J F = d_H F + d_V F = \sum_{i=1}^{n} D_{x^i} F \, dx^i + \sum_{\alpha=1}^{q} \sum_{\#J \geq 0} \frac{\partial F}{\partial u^\alpha_J} \theta^\alpha_J.$$

Example

For $F : J^{(1)}(\mathbb{R}^2, 1) \to \mathbb{R}$, $(x, u, u_x) \mapsto F(x, u, u_x)$,

$$
\begin{aligned}
dF &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial u} du + \frac{\partial F}{\partial u_x} du_x \\
&= \left(\frac{\partial F}{\partial x} + u_x \frac{\partial F}{\partial u} + u_{xx} \frac{\partial F}{\partial u_x} \right) dx + \frac{\partial F}{\partial u}(du - u_x dx) + \frac{\partial F}{\partial u_x}(du_x - u_{xx} dx) \\
&= (D_x F) dx + \frac{\partial F}{\partial u}\theta + \frac{\partial F}{\partial u_x}\theta_x.
\end{aligned}
$$

n-th diffeomorphism jet bundle

$\mathcal{D} = \mathcal{D}(M) =$ pseudo-group of all local diffeomorphisms.

Definition

The n-th diffeomorphism jet bundle $\mathcal{D}^{(n)}$ consists of the equivalence classes of diffeomorphisms under the equivalence relation

$$\phi(z_0) \sim \psi(z_0) \iff \sum_{0 \leq \#J \leq n} \frac{1}{J!} \frac{\partial^{\#J}\phi}{\partial z^J}\bigg|_{z_0} (z - z_0)^J = \sum_{0 \leq \#J \leq n} \frac{1}{J!} \frac{\partial^{\#J}\psi}{\partial z^J}\bigg|_{z_0} (z - z_0)^J.$$

The local coordinates on $\mathcal{D}^{(n)}$ are indicated by $(z, Z^{(n)})$.

- Source map: $z = \boldsymbol{\sigma}^{(n)}(z, Z^{(n)})$.
- Target map: $Z = (X, U) = \boldsymbol{\tau}^{(n)}(z, Z^{(n)})$.

Lie pseudo-groups

Definition

$\mathcal{G} \subset \mathcal{D}$ is called a Lie pseudo-group if $\exists\ n^\star \geq 1$ such that $\forall$ finite $n \geq n^\star$:

- $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ forms a smooth, embedded subbundle,
- $\pi_n^{n+1} : \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a bundle map,
- every smooth local solution $Z = \phi(z)$ to the determining system $\mathcal{G}^{(n)}$ belongs to $\mathcal{G}$,
- $\mathcal{G}^{(n)} = \mathrm{pr}^{(n-n^\star)}\mathcal{G}^{(n^\star)}$ is obtained by prolongation.

The minimal value of $n^\star$ is called the *order* of the Lie pseudo-group.

$\mathbf{g}^{(n)} = (z, g^{(n)})$ indicates the local coordinates of a jet $\mathbf{g}^{(n)} \in \mathcal{G}^{(n)}$, with the pseudo-group parameters $g^{(n)} = (g_1, \ldots, g_{r_n})$.

Note about Lie pseudo-groups

In contrast to the finite dimensional theory of Lie groups, there is still no generally accepted abstract object to play the role of an infinite dimensional Lie group. Lie pseudo-groups only arise through their action on a manifold.

Remember, a Lie group G is a manifold with a group structure such that

$$G \times G \to G, \qquad (g,h) \mapsto gh^{-1}$$

is smooth.

Example

Let $M = \{(x,u) : u \neq 0\} \subset \mathbb{R}^2$. Consider the Lie pseudo-group of transformations

$$X = f(x), \qquad U = \frac{u}{f'(x)},$$

where $f(x) \in \mathcal{D}(\mathbb{R})$. The pseudo-group jets are given by

$$X = f, \quad U = \frac{u}{f_x}, \quad X_x = f_x, \quad X_u = 0, \quad U_x = -\frac{u f_{xx}}{f_x^2}, \quad U_u = \frac{1}{f_x}, \ldots .$$

The involutive system characterizing $\mathcal{G}^{(n)}$ is given by the $(n-1)$ prolongation of the involutive first order system

$$X_x = \frac{u}{U}, \qquad X_u = 0, \qquad U_u = \frac{U}{u},$$

obtained by recursively applying

$$\mathbb{D}_x = \frac{\partial}{\partial x} + X_x \frac{\partial}{\partial X} + U_x \frac{\partial}{\partial U} + X_{xx} \frac{\partial}{\partial X_x} + X_{xu} \frac{\partial}{\partial X_u} + U_{xx} \frac{\partial}{\partial U_x} + U_{xu} \frac{\partial}{\partial U_u} + \cdots,$$
$$\mathbb{D}_u = \frac{\partial}{\partial u} + X_u \frac{\partial}{\partial X} + U_u \frac{\partial}{\partial U} + X_{xu} \frac{\partial}{\partial X_x} + X_{uu} \frac{\partial}{\partial X_u} + U_{xu} \frac{\partial}{\partial U_x} + U_{uu} \frac{\partial}{\partial U_u} + \cdots .$$

Infinitesimal generators and prolongation

Given a pseudo-group $\mathcal{G}$ we denote by $\mathfrak{g}$ the local Lie algebra of infinitesimal generators, i.e. the set of locally defined vector fields whose flows belong to the pseudo-group. Let

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{n} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x,u) \frac{\partial}{\partial u^\alpha} \in \mathfrak{g},$$

then the *n*-th prolongation of $\mathbf{v}$ is given by

$$\mathbf{v}^{(n)} = \mathbf{v} + \sum_{\alpha=1}^{q} \sum_{\#J=1}^{n} \phi_J^\alpha(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha},$$

where

$$\phi_{J,j}^\alpha = D_{x^j} \phi_J^\alpha - \sum_{i=1}^{p} D_{x^j} \xi^i \cdot u_{J,i}^\alpha, \quad 1 \leq \alpha \leq q,\ 1 \leq j \leq p.$$

The coefficients of $\mathbf{v}^{(n)}$ is the solution to the linearized determining equations of $\mathcal{G}^{(n)}$ at the identity jet $\mathbb{1}_z^{(n)}$.

Example

Consider the Lie pseudo-group of transformations

$$X = f(x), \qquad U = \frac{u}{f'(x)}.$$

The infinitesimal generator of the action is

$$\mathbf{v} = \xi(x)\frac{\partial}{\partial x} - u\xi_x(x)\frac{\partial}{\partial u},$$

$\xi(x) = f'(x)$. The first prolongation of $\mathbf{v}$ is

$$\mathbf{v}^{(1)} = \xi\frac{\partial}{\partial x} - u\xi_x\frac{\partial}{\partial u} - (u\xi_{xx} + 2u_x\xi_x)\frac{\partial}{\partial u_x}.$$

The coefficients of $\mathbf{v} = \xi(x,u)\partial_x + \phi(x,u)\partial_u$ is the solution to the linearized determining equations

$$\phi = -u\xi_x, \qquad \xi_u = 0, \qquad \phi_u = \frac{\phi}{u}.$$

n-th regularized jet bundle (n-th lifted bundle)

Definition

The local coordinates of the n-th regularized jet bundle $\mathcal{H}^{(n)}$ are $(z^{(n)}, g^{(n)})$, where $\pi_0^n(z^n) = \boldsymbol{\sigma}^{(n)}(g^{(n)})$.

$\mathcal{G}$ acts on $\mathcal{H}^{(n)}$ in the following way:

$$g \cdot (z^{(n)}, h^{(n)}) = (g^{(n)} z^{(n)}, h^{(n)} (g^{-1})^{(n)}) = ((g \cdot z)^{(n)}, (hg^{-1})^{(n)}).$$

Definition

A function $I(z^{(n)}, g^{(n)})$ is said to be a lifted differential invariant if

$$I(h^{(n)} \cdot (z^{(n)}, g^{(n)})) = I(z^{(n)}, g^{(n)}), \qquad \forall h \in \mathcal{G}.$$

$\widetilde{\boldsymbol{\tau}}^{(n)}(z^{(n)}, g^{(n)}) = (g \cdot z)^{(n)}$ is a lifted invariant:

$$h^{(n)} \cdot (g^{(n)} \cdot z^{(n)}) = g^{(n)} (h^{-1})^{(n)} \cdot h^{(n)} z^{(n)} = g^{(n)} \cdot z^{(n)}.$$

Tricomplex structure of $\Lambda^*(\mathcal{H}^{(\infty)})$

- Jet forms:

$$dx^i, \quad i = 1, \ldots, p, \qquad \theta^\alpha_J, \quad \alpha = 1, \ldots, q, \ \#J \geq 0,$$

- group forms:

$$\Upsilon^b_A = \mathbb{D}^A_z \Upsilon^b = dZ^b_A - \sum_{a=1}^{m} Z^b_{A,a} dz^a, \qquad b = 1 \ldots, m, \ \#A \geq 0.$$

This defines a contact structure on $\Lambda^*(\mathcal{H}^{(\infty)})$:

$$\Lambda^*(\mathcal{H}^{(\infty)}) = \Lambda^*(X) \oplus \mathcal{C}.$$

Let $F : \mathcal{H}^{(\infty)} \to \mathbb{R}$, then

$$dF = d_J F + d_G F = d_H F + d_V F + d_G F$$

where

$$d_H F = \sum_{i=1}^{p} (D_{x^i} F) dx^i, \qquad d_V F = \sum_{\alpha=1}^{q} \left[(\mathbb{D}_{u^\alpha} F)\theta^\alpha + \sum_{\#J \geq 1} \frac{\partial F}{\partial u_J^\alpha} \theta_J^\alpha \right],$$

$$d_G F = \sum_{b=1}^{m} \sum_{\#A \geq 0} \frac{\partial F}{\partial Z_A^b} \gamma_A^b.$$

and

$$\mathbb{D}_{z^a} = \frac{\partial}{\partial z^a} + \sum_{b=1}^{m} \sum_{\#A \geq 0} Z_{A,a}^b \frac{\partial}{\partial Z_A^b}, \qquad a = 1, \ldots, m,$$

$$D_{x^i} = \mathbb{D}_{x^i} + \sum_{\alpha=1}^{q} \left[\mathbb{D}_{u^\alpha} + \sum_{\#J \geq 1} u_{J,i}^\alpha \frac{\partial}{\partial u_J^\alpha} \right], \qquad i = 1, \ldots, p.$$

Example

Consider the Lie pseudo-group $X = f(x)$, $U = u/f_x(x)$, then

$$
\begin{aligned}
\Upsilon^x &= dX - X_x dx - X_u du = df - f_x dx, \\
\Upsilon^x_x &= df_x - f_{xx} dx, \qquad \Upsilon^x_{xx} = df_{xx} - f_{xxx} dx, \qquad \ldots, \\
\Upsilon^x_u &= \Upsilon^x_{uu} = \ldots = 0, \\
\Upsilon^u &= dU - U_x dx - U_u du = d\left(\frac{u}{f_x}\right) + \frac{u f_{xx}}{f_x^2} dx - \frac{1}{f_x} du \\
&= -\frac{u df_x}{f_x^2} + \frac{u f_{xx}}{f_x^2} dx = -\frac{u}{f_x^2} \Upsilon^x_x, \\
\Upsilon^u_u &= -\frac{1}{f_x^2} \Upsilon^x_x, \\
\Upsilon^u_x &= u\left(\frac{2 f_{xx}}{f_x^3} \Upsilon^x_x - \frac{1}{f_x^2} \Upsilon^x_{xx}\right), \\
&\vdots
\end{aligned}
$$

Invariant differential operators

Definition

Let $\pi : E \to M$ be a fiber manifold with a contact structure, i.e.

$$\Lambda^*(E) = \Lambda^*(M) \oplus \mathcal{C},$$

$\omega \in \Lambda^*(M)$ is contact invariant if given a group of transformations G acting on E

$$g^*\omega = \omega + \theta_g, \qquad \text{where} \qquad \theta_g \in \mathcal{C}, \ \forall g \in G.$$

Definition

A differential operator D is said to be an invariant differential operator if

$$I \quad \text{invariant} \quad \Rightarrow \quad DI \quad \text{invariant}$$

Proposition

Let $\omega^1, \ldots, \omega^m$ be a horizontal contact invariant basis of $\Lambda^1(M)$ then the dual operators $D_1, \ldots, D_m$ form a complete set of invariant differential operators.

The operators D_a are defined by the identity

$$d_M F = \sum_{a=1}^{m} D_a F \, \omega^a.$$

Proof.

Let $I : E \to \mathbb{R}$ be an invariant then

$$\sum_{a=1}^{m} D_a I \, \omega^a + \theta = dI = g^* dI = \sum_{a=1}^{m} g^*(D_a I) \, \omega^a + \theta_g$$

□

Lifted invariant differential operators

On $\mathcal{H}^{(\infty)}$

$$\omega^i = d_H X^i = \sum_{j=1}^{p} D_{x^j} X^i dx^j, \qquad i = 1, \ldots, p,$$

is a contact invariant horizontal coframe on the open dense set where $\det(D_{x^i} X^j) \neq 0$. This follows from the fact that X^i is a lifted invariant!

For simplicity consider $M = \mathbb{R}^2$. Let $(X, U) = \phi(x, u)$ be a group transformation, and $I(x, u^{(n)})$ an invariant, then

$$\begin{aligned} \phi^* d_H I(x, u^{(n)}) &= (D_x I)|_{\phi^{(n)}(x, u^{(n)})} dX = D_X I(X, U^{(n)}) D_x X dx + \theta_\phi \\ &= D_x(I \circ \phi^{(n)}(x, u^{(n)})) dx + \theta_\phi = D_x I(x, u^{(n)}) dx + \theta_\phi, \end{aligned}$$

where θ_ϕ is a contact form depending on ϕ.

The associated invariant differential operators are

$$D_{X^i} = \sum_{j=1}^{p} W^i_j D_{x^j}, \qquad (W^i_j) = (D_{x^i} X^j)^{-1},$$

i.e. for $F : \mathcal{H}^{(\infty)} \longrightarrow \mathbb{R}$,

$$d_H F = \sum_{i=1}^{p} D_{X^i} F \; d_H X^i.$$

So

$$\widehat{U}^\alpha_J = g^{(n)} \cdot u^\alpha_J = D^J_X U^\alpha = D_{X^{j^1}} \cdots D_{X^{j^{\#J}}} U^\alpha, \qquad \alpha = 1, \ldots, q,$$

are lifted differential invariants.

Some definitions

Definition

A Lie pseudo-group $\mathcal{G}$ acting on M is said to be regular if all its orbits have the same dimension and each point $z \in M$ has arbitrarily small neighborhoods whose intersection with each orbit is a connected subset.

Definition

The pseudo-group $\mathcal{G}$ acts freely at $z^{(n)} \in J^{(n)}$ if $\mathcal{G}^{(n)}_{z^{(n)}} = \{\mathbb{1}^{(n)}_z\}$, and locally freely at $z^{(n)}$ if $\mathcal{G}^{(n)}_{z^{(n)}}$ is a discrete subgroup of $\mathcal{G}^{(n)}_z$. The pseudo-group $\mathcal{G}$ is said to act *(locally) freely at order n* if it acts (locally) freely on an open subset $\mathcal{V}^{(n)} \subset J^{(n)}$, called the set of *regular n-jets*.

Definition

A cross-section to the pseudo-group orbits is a transverse submanifold to the orbits of complementary dimension.

Definition

A section of a fiber bundle (E, π, B) is a smooth map $s : B \longrightarrow E$ such that $\pi \circ s = id|_B$.

Example

A section of the trivial fiber bundle $(X \times U, \mathrm{pr}_1, X)$ is given by

$$s(x) = (x, f(x)), \quad \text{where} \quad f : X \to U \quad \text{is a smooth function.}$$

Moving frame

Definition

A right moving frame $\rho^{(n)}$ of order n is a right $\mathcal{G}^{(n)}$-equivariant local section of the bundle $\mathcal{H}^{(n)} \to J^{(n)}$, i.e. $\rho^{(n)} : J^{(n)} \to \mathcal{H}^{(n)}$ satisfies

$$\sigma^{(n)}(\rho^{(n)}(z^{(n)})) = z^{(n)}, \qquad \rho^{(n)}(\mathbf{g}^{(n)} \cdot z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot (\mathbf{g}^{(n)})^{-1},$$

for all $\mathbf{g}^{(n)} \in \mathcal{G}^{(n)}|_z$, with $z = \pi_0^n(z^{(n)})$, and groupoid inverse $(\mathbf{g}^{(n)})^{-1} \in \mathcal{G}^{(n)}|_{\boldsymbol{\tau}^{(n)}(\mathbf{g}^{(n)})}$, such that $z^{(n)}$ and $\mathbf{g}^{(n)} \cdot z^{(n)}$ lie in the domain of definition of $\rho^{(n)}$.

A left moving frame can be obtained by group inversion

$$\rho_l^{(n)}(z^{(n)}) = (\rho_r^{(n)}(z^{(n)}))^{-1}.$$

Theorem

Suppose $\mathcal{G}^{(n)}$ acts freely and regularly on $\mathcal{V}^{(n)} \subset J^{(n)}$. Let $\mathcal{K}^{(n)} \subset \mathcal{V}^{(n)}$ be a (local) cross-section to the pseudo-group orbits. Given $z^{(n)} \in \mathcal{V}^{(n)}$, define $\rho^{(n)}(z^{(n)}) \in \mathcal{H}^{(n)}$ to be the unique groupoid jet such that $\widetilde{\boldsymbol{\tau}}^{(n)}(\rho^{(n)}(z^{(n)})) \in \mathcal{K}^{(n)}$. Then $\rho^{(n)} : J^{(n)} \to \mathcal{H}^{(n)}$ is a (right) moving frame for $\mathcal{G}$ defined on an open subset of $\mathcal{V}^{(n)}$.

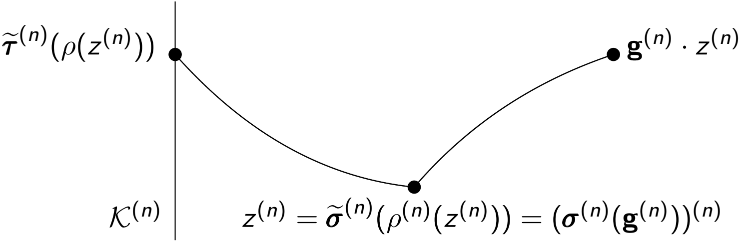

Figure: Moving frame.

Moving frame construction

- Compute

$$\widehat{U}^{\alpha}_{J} = D^{J}_{X} U^{\alpha}, \qquad D^{J}_{X} = D_{X^{j_1}} \cdots D_{X^{j_{\#J}}},$$

where $D_{X^i} = \sum_{j=1}^{p} W^i_j D_{x^j}, \qquad (W^i_j) = (D_{x^j} X^i)^{-1}.$

Moving frame construction

- Compute
$$\widehat{U}^\alpha_J = D^J_X U^\alpha, \qquad D^J_X = D_{X^{j_1}} \cdots D_{X^{j_{\#J}}},$$
where $D_{X^i} = \sum_{j=1}^p W^i_j D_{x^j}, \qquad (W^i_j) = (D_{x^j} X^i)^{-1}.$
- Fix r_n transformed coordinates $(X, \widehat{U}^{(n)}) = F^{(n)}(z^{(n)}, g^{(n)})$ to be constant
$$F_1(z^{(n)}, g^{(n)}) = c_1 \qquad \ldots \qquad F_{r_n}(z^{(n)}, g^{(n)}) = c_{r_n}.$$

Moving frame construction

- Compute
$$\widehat{U}^\alpha_J = D^J_X U^\alpha, \qquad D^J_X = D_{X^{j_1}} \cdots D_{X^{j_{\#J}}},$$
where $D_{X^i} = \sum_{j=1}^p W^i_j D_{x^j}, \qquad (W^i_j) = (D_{x^j} X^i)^{-1}$.
- Fix r_n transformed coordinates $(X, \widehat{U}^{(n)}) = F^{(n)}(z^{(n)}, g^{(n)})$ to be constant
$$F_1(z^{(n)}, g^{(n)}) = c_1 \qquad \ldots \qquad F_{r_n}(z^{(n)}, g^{(n)}) = c_{r_n}.$$
- Solve the normalization equations for the pseudo-group parameters $g^{(n)} = g^{(n)}(z^{(n)})$.

Invariantization

Definition

The lift of a differential form $\omega \in \Lambda^*(J^{(\infty)})$ is the jet form

$$\Omega = \boldsymbol{\lambda}(\omega) = \pi_J((\widetilde{\boldsymbol{\tau}}^{(\infty)})^*\omega).$$

Definition

Let $\rho^{(\infty)} : J^{(\infty)} \to \mathcal{H}^{(\infty)}$ be a complete moving frame. The invariantization of $\omega \in \Lambda^*(J^{(\infty)})$ is the invariant differential form

$$\iota(\omega) = (\rho^{(\infty)})^*[\boldsymbol{\lambda}(\omega)].$$

In particular

$$\iota(x^i, u^\alpha_J) = (H^i, I^\alpha_J), \qquad i = 1, \ldots, p \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0.$$

Example

Let $v = y' = f(x,y)$ and consider the Lie pseudo-group of transformations

$$(X,Y,V) = \left(\alpha(x), \beta(y), \frac{v\beta_y(y)}{\alpha_x(x)}\right) \quad \Rightarrow \quad D_X = \frac{1}{\alpha_x} D_x, \qquad D_Y = \frac{1}{\beta_y} D_y.$$

$$\widehat{V}_X = \frac{v_x\beta_y}{\alpha_x^2} - \frac{v\beta_y\alpha_{xx}}{\alpha_x^3},$$

$$\widehat{V}_Y = \frac{v_y}{\alpha_x} + \frac{v\beta_{yy}}{\beta_y\alpha_x},$$

$$\widehat{V}_{XX} = \frac{v_{xx}\beta_y}{\alpha_x^3} - \frac{3v_x\beta_y\alpha_{xx}}{\alpha_x^4} - \frac{v\beta_y\alpha_{xxx}}{\alpha_x^4} + \frac{3v\beta_y\alpha_{xx}^2}{\alpha_x^5},$$

$$\widehat{V}_{YY} = \frac{v_{yy}}{\beta_y\alpha_x} + \frac{v_y\beta_{yy}}{\beta_y^2\alpha_x} + \frac{v\beta_{yyy}}{\beta_y^2\alpha_x} - \frac{v\beta_{yy}^2}{\beta_y^3\alpha_x},$$

$$\widehat{V}_{XY} = \frac{v_{xy}}{\alpha_x^2} + \frac{v_x\beta_{yy}}{\beta_y\alpha_x^2} - \frac{v_y\alpha_{xx}}{\alpha_x^3} - \frac{v\beta_{yy}\alpha_{xx}}{\beta_y\alpha_x^3},$$

$$\widehat{V}_{XXX} = \frac{v_{xxx}\beta_y}{\alpha_x^4} - \frac{6v_{xx}\beta_y\alpha_{xx}}{\alpha_x^5} - \frac{4v_x\beta_y\alpha_{xxx}}{\alpha_x^5} + \frac{15v_x\beta_y\alpha_{xx}^2}{\alpha_x^6} - \frac{v\beta_y\alpha_{xxxx}}{\alpha_x^5} + \frac{10v\beta_y\alpha_{xxx}\alpha_{xx}}{\alpha_x^6} - \frac{15v\beta_y\alpha_{xx}^3}{\alpha_x^7},$$

$$\widehat{V}_{YYY} = \frac{v_{yyy}}{\beta_y^2\alpha_x} + \frac{2v_y\beta_{yyy}}{\beta_y^3\alpha_x} - \frac{3v_y\beta_{yy}^2}{\beta_y^4\alpha_x} + \frac{v\beta_{yyyy}}{\beta_y^3\alpha_x} - \frac{4v\beta_{yy}\beta_{yyy}}{\beta_y^4\alpha_x} + \frac{3v\beta_{yy}^3}{\beta_y^5\alpha_x},$$

$$\widehat{V}_{XXY} = \frac{v_{xxy}}{\alpha_x^3} + \frac{v_{xx}\beta_{yy}}{\beta_y\alpha_x^3} - \frac{3v_{xy}\alpha_{xx}}{\alpha_x^4} - \frac{3v_x\beta_{yy}\alpha_{xx}}{\beta_y\alpha_x^4} - \frac{v_y\alpha_{xxx}}{\alpha_x^4} - \frac{v\beta_{yy}\alpha_{xxx}}{\beta_y\alpha_x^4} + \frac{3v_y\alpha_{xx}^2}{\alpha_x^5} + \frac{3v\beta_{yy}\alpha_{xx}^2}{\beta_y\alpha_x^5},$$

$$\widehat{V}_{YYX} = \frac{v_{yyx}}{\beta_y\alpha_x^2} - \frac{v_{yy}\alpha_{xx}}{\beta_y\alpha_x^3} + \frac{v_{yx}\beta_{yy}}{\beta_y^2\alpha_x^2} - \frac{v_y\beta_{yy}\alpha_{xx}}{\beta_y^2\alpha_x^3} + \frac{v_x\beta_{yyy}}{\beta_y^2\alpha_x^2} - \frac{v\beta_{yyy}\alpha_{xx}}{\beta_y^2\alpha_x^3} - \frac{v_x\beta_{yy}^2}{\beta_y^3\alpha_x^2} + \frac{v\beta_{yy}^2\alpha_{xx}}{\beta_y^3\alpha_x^3}.$$

Cross-section:

$$(X, Y, V, \widehat{V}_X, \widehat{V}_Y, \widehat{V}_{XX}, \widehat{V}_{YY}, \widehat{V}_{XY}, \ldots) = (0, 0, 1, 0, 0, 0, 0, 1, \ldots).$$

$$\alpha = 0, \qquad \beta = 0,$$

$$\alpha_x = \pm v \sqrt{\frac{v v_{xy} - v_x v_y}{v^3}}, \qquad \beta_y = \pm \sqrt{\frac{v v_{xy} - v_x v_y}{v^3}},$$

$$\alpha_{xx} = \pm v_x \sqrt{\frac{v v_{xy} - v_x v_y}{v^3}}, \qquad \beta_{yy} = \mp \frac{v_y}{v} \sqrt{\frac{v v_{xy} - v_x v_y}{v^3}},$$

$$\alpha_{xxx} = \pm v_{xx} \sqrt{\frac{v v_{xy} - v_x v_y}{v^3}}, \qquad \beta_{yyy} = \pm \frac{2v_y^2 - v v_{yy}}{v^2} \sqrt{\frac{v v_{xy} - v_x v_y}{v^3}}.$$

$$\iota(v_{xxy}) = I_{21} = \left(\frac{v}{v v_{xy} - v_x v_y} \right)^{3/2} \left(v_{xxy} + \frac{3 v_x^2 v_y}{v^2} - \frac{3 v_{xy} v_x}{v} - \frac{v_y v_{xx}}{v} \right),$$

$$\iota(v_{yyx}) = I_{12} = \left(\frac{v}{v v_{xy} - v_x v_y} \right)^{3/2} \left(v v_{yyx} + \frac{v_y^2 v_x}{v} - v_{yy} v_x - v_{yx} v_y \right).$$

Recurrence formulas

Theorem

Let $\omega \in \Lambda^*(J^{(\infty)})$, *then*

$$d[\iota(\omega)] = \iota[d\omega + \mathbf{v}^{(\infty)}(\omega)].$$

Definition

The lift of a vector field jet coordinate is

$$\boldsymbol{\lambda}(\zeta^a_J) = \mu^a_J, \qquad a = 1, \ldots, m, \qquad \#J \geq 0,$$

and more generally,

$$\boldsymbol{\lambda}\left(\sum_{a=1}^{m} \sum_{\#J \leq n} \zeta^a_J \omega^a_J\right) = \sum_{a=1}^{m} \sum_{\#J \leq n} \mu^a_J \wedge \boldsymbol{\lambda}(\omega^a_J).$$

Theorem

The *recurrence formulas* *between the differentiated invariants and the normalized invariants are*

$$\sum_{j=1}^{p}(\mathcal{D}_j H^i)\varpi^j = \pi_H(\iota(dx^i + \xi^i)) = \varpi^i + \pi_H(\iota(\xi^i)), \qquad i = 1, \ldots, p$$

$$\sum_{j=1}^{p}(\mathcal{D}_j I^\alpha_J)\varpi^j = \pi_H(\iota(\sum_{j=1}^{p} u^\alpha_{J,j} dx^j + \theta^\alpha_J + \phi^\alpha_J)) = \sum_{j=1}^{p} I^\alpha_{J,j}\varpi^j + \pi_H(\iota(\phi^\alpha_J)),$$

$\varpi^i = \iota(dx^i)$, $i = 1, \ldots, p$. The $\mathcal{D}_i$ are uniquely determined by

$$d_H F = \sum_{i=1}^{p} \mathcal{D}_i F\, \varpi^i.$$

Commutation relations

The commutation relations $[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^{p} C_{ij}^k \mathcal{D}_k$ for the invariant differential operators $\mathcal{D}_i$, $i = 1, \ldots, p$, can be obtained from the identity $d\iota(\omega) = \iota[d\omega + \mathbf{v}^{(\infty)}(\omega)]$:

$$- \sum_{1 \le i < j \le p} C_{ij}^k \, \varpi^i \wedge \varpi^j = d\omega^k = d\iota(dx^k) = \iota[d^2 x^k + \mathbf{v}^{(\infty)}(dx^k)] = \iota[d\xi^k(x,u)].$$

This follows from the fact that

$$0 = d_H^2 F(x) = d_H \sum_{i=1}^{p} \mathcal{D}_i F \, \varpi^i = \sum_{1 \le i < j \le p} (\mathcal{D}_i \mathcal{D}_j - \mathcal{D}_j \mathcal{D}_i) F \, \varpi^i \wedge \varpi^j + \sum_{i=1}^{p} \mathcal{D}_i F \, d\varpi^i$$

$$= \sum_{1 \le i < j \le p} \left([\mathcal{D}_i, \mathcal{D}_j] F + \sum_{k=1}^{p} -C_{ij}^k \mathcal{D}_k F \right) \varpi^i \wedge \varpi^j$$

Continuation of the previous example

- Invariant horizontal frame:

$$\begin{aligned}\varpi^1 = \iota(dx) &= (\rho^{(\infty)})^*(\pi_J((\widetilde{\boldsymbol{\tau}}^{(\infty)})^* dx)) = (\rho^{(\infty)})^*(\pi_J dX) \\ &= (\rho^{(\infty)})^*(d_J X) = (\rho^{(\infty)})^*(d_H X) \\ &= (\rho^{(\infty)})^*(X_x dx) = (\rho^{(\infty)})^*(\alpha_x dx) \\ &= v\sqrt{\frac{v v_{xy} - v_x v_y}{v^3}} dx,\end{aligned}$$

$$\varpi^2 = \iota(dy) = (\rho^{(\infty)})^*(\beta_y dy) = \sqrt{\frac{v v_{xy} - v_x v_y}{v^3}} dy.$$

- Invariant differential operators

$$\mathcal{D}_1 = \frac{1}{v}\sqrt{\frac{v^3}{v v_{xy} - v_x v_y}} D_x,$$

$$\mathcal{D}_2 = \sqrt{\frac{v^3}{v v_{xy} - v_x v_y}} dy.$$

The infinitesimal generator of the Lie pseudo-group action $(X, Y, V) = (\alpha(x), \beta(y), v\beta_y(y)/\alpha_x(x))$ is

$$\mathbf{v} = \xi(x)\frac{\partial}{\partial x} + \eta(y)\frac{\partial}{\partial y} + v(\eta_y - \xi_x)\frac{\partial}{\partial v}, \quad \text{where} \quad \xi(x) = \alpha_x, \quad \eta(x) = \beta_y.$$

The coefficients of the third prolongation, with $\phi(x, y, v) = v(\eta_y - \xi_x)$, are

$$\begin{aligned}
\phi^x &= -v\xi_{xx} + (\eta_y - 2\xi_x)v_x, \\
\phi^y &= v\eta_{yy} - \xi_x v_y, \\
\phi^{xx} &= -v\xi_{xxx} - 3v_x\xi_{xx} + (\eta_y - 3\xi_x)v_{xx}, \\
\phi^{yy} &= v_y\eta_{yy} + v\eta_{yyy} - \xi_x v_{yy} - \eta_y v_{yy}, \\
\phi^{xy} &= v_x\eta_{yy} - \xi_{xx}v_y - 2\xi_x v_{xy}. \\
\phi^{xxx} &= -v\xi_{xxxx} - 4v_x\xi_{xxx} - 6v_{xx}\xi_{xx} + (\eta_y - 4\xi_x)v_{xxx}, \\
\phi^{yyy} &= 2v_y\eta_{yyy} + v\eta_{yyyy} - (\xi_x + 2\eta_y)v_{yyy}, \\
\phi^{xxy} &= -v_y\xi_{xxx} - 3v_{xy}\xi_{xx} + \eta_{yy}v_{xx} - 3\xi_x v_{xxy}, \\
\phi^{yyx} &= \eta_{yy}v_{xy} + \eta_{yyy}v_x - \xi_{xx}v_{yy} - (\eta_y + 2\xi_x)v_{yyx},
\end{aligned}$$

$$(\mathcal{D}_1 H^1)\varpi^1 + (\mathcal{D}_2 H^1)\varpi^2 = \varpi^1 + \beta_0^1,$$
$$(\mathcal{D}_1 H^2)\varpi^1 + (\mathcal{D}_2 H^2)\varpi^2 = \varpi^2 + \beta_0^2,$$
$$(\mathcal{D}_1 I_{00})\varpi^1 + (\mathcal{D}_2 I_{00})\varpi^2 = I_{10}\varpi^1 + I_{01}\varpi^2 + I_{00}(\beta_1^2 - \beta_1^1),$$
$$(\mathcal{D}_1 I_{10})\varpi^1 + (\mathcal{D}_2 I_{10})\varpi^2 = I_{20}\varpi^1 + I_{11}\varpi^2 - I_{00}\beta_2^1 + I_{10}(\beta_1^2 - 2\beta_1^1),$$
$$(\mathcal{D}_1 I_{01})\varpi^1 + (\mathcal{D}_2 I_{01})\varpi^2 = I_{11}\varpi^1 + I_{02}\varpi^2 + I_{00}\beta_2^2 - I_{01}\beta_1^1,$$
$$(\mathcal{D}_1 I_{20})\varpi^1 + (\mathcal{D}_2 I_{20})\varpi^2 = I_{30}\varpi^1 + I_{21}\varpi^2 - I_{00}\beta_3^1 - 3I_{10}\beta_2^1 + I_{20}(\beta_1^2 - 3\beta_1^1),$$
$$(\mathcal{D}_1 I_{02})\varpi^1 + (\mathcal{D}_2 I_{02})\varpi^2 = I_{12}\varpi^1 + I_{03}\varpi^2 + I_{01}\beta_2^2 + I_{00}\beta_3^2 - I_{02}\beta_1^1 - I_{02}\beta_1^2,$$
$$(\mathcal{D}_1 I_{11})\varpi^1 + (\mathcal{D}_2 I_{11})\varpi^2 = I_{21}\varpi^1 + I_{12}\varpi^2 + I_{10}\beta_2^2 - I_{01}\beta_2^1 - 2I_{11}\beta_1^1,$$
$$(\mathcal{D}_1 I_{30})\varpi^1 + (\mathcal{D}_2 I_{30})\varpi^2 = I_{40}\varpi^1 + I_{31}\varpi^2 - I_{00}\beta_4^1 - 4I_{10}\beta_3^1 - 6I_{20}\beta_2^1 + I_{30}(\beta_1^2 - 4\beta_1^1),$$
$$(\mathcal{D}_1 I_{03})\varpi^1 + (\mathcal{D}_2 I_{03})\varpi^2 = I_{13}\varpi^1 + I_{04}\varpi^2 + 2I_{10}\beta_3^2 + I_{00}\beta_4^2 - I_{03}(\beta_1^1 + 2\beta_1^2),$$
$$(\mathcal{D}_1 I_{21})\varpi^1 + (\mathcal{D}_2 I_{21})\varpi^2 = I_{31}\varpi^1 + I_{22}\varpi^2 - I_{01}\beta_3^1 - 3I_{11}\beta_2^1 + I_{20}\beta_2^2 - 3I_{21}\beta_1^1,$$
$$(\mathcal{D}_1 I_{12})\varpi^1 + (\mathcal{D}_2 I_{12})\varpi^2 = I_{22}\varpi^1 + I_{13}\varpi^2 + I_{11}\beta_2^2 + I_{10}\beta_3^2 - I_{02}\beta_2^1 - I_{12}(\beta_1^2 + 2\beta_1^1),$$

Cross-section:

$$H^1 = 0, \quad H^2 = 0, \quad I_{00} = 1, \quad I_{10} = 0, \quad I_{01} = 0, \quad I_{20} = 0, \quad I_{02} = 0,$$
$$I_{11} = 1, \quad I_{30} = 0, \quad I_{03} = 0, \quad I_{40} = 0, \quad I_{04} = 0, \quad \ldots$$

$$-\beta_0^1 = \beta_2^2 = \varpi^1, \qquad -\beta_0^2 = \beta_2^1 = \varpi^2, \qquad \beta_1^1 = \beta_1^2 = \frac{I_{21}\varpi^1 + I_{12}\varpi^2}{2},$$
$$\beta_3^1 = I_{21}\varpi^2, \qquad \beta_3^2 = -I_{12}\varpi^1, \qquad \beta_4^1 = I_{31}\varpi^2, \qquad \beta_4^2 = -I_{13}\varpi^1.$$

$$\mathcal{D}_1 I_{21} = I_{31} - \frac{3}{2} I_{21}^2, \qquad \mathcal{D}_2 I_{21} = I_{22} - 3 - \frac{3}{2} I_{21} I_{12},$$
$$\mathcal{D}_1 I_{12} = I_{22} + 1 - \frac{3}{2} I_{12} I_{21}, \qquad \mathcal{D}_2 I_{12} = I_{13} - \frac{3}{2} I_{12}^2.$$

I_{12} and I_{21} generate the algebra of differential invariants.

Commutation relation

$$
\begin{aligned}
d\varpi^1 &= \iota(d\xi) = \iota(\xi_x dx) = \beta_1^1 \wedge \varpi^1 = \left(\frac{I_{21}\varpi^1 + I_{12}\varpi^2}{2}\right) \wedge \varpi^1 \\
&= -\frac{I_{12}}{2}\varpi^1 \wedge \varpi^2, \\
d\varpi^2 &= \iota(d\eta) = \iota(\eta_y dy) = \beta_1^2 \wedge \varpi_2 = \left(\frac{I_{21}\varpi^1 + I_{12}\varpi^2}{2}\right) \wedge \varpi^2 \\
&= \frac{I_{21}}{2}\varpi^1 \wedge \varpi^2.
\end{aligned}
$$

So

$$
[\mathcal{D}_1, \mathcal{D}_2] = \frac{I_{12}}{2}\mathcal{D}_1 - \frac{I_{21}}{2}\mathcal{D}_2.
$$

- If $\mathcal{D}_2 I_{12} \neq 0$, then

$$I_{21} = \frac{2[\mathcal{D}_2, \mathcal{D}_1] I_{12} + I_{12} \mathcal{D}_1 I_{12}}{\mathcal{D}_2 I_{12}},$$

- if $\mathcal{D}_1 I_{21} \neq 0$, then

$$I_{12} = \frac{2[\mathcal{D}_1, \mathcal{D}_2] I_{21} + I_{21} \mathcal{D}_2 I_{21}}{\mathcal{D}_1 I_{21}}.$$

Under the assumption $(\mathcal{D}_2 I_{12})^2 + (\mathcal{D}_1 I_{21})^2 \neq 0$, the algebra of differential invariants is generated by one invariant!

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