# Solving Local Equivalence Problems with the Equivariant Moving Frame Method

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**Abstract**

In this paper we show how to solve local equivalence problems of submanifolds using the theory of equivariant moving frames. The relations between Cartan's equivalence method of coframes and the equivariant moving frame method are discussed and illustrated with many examples. In contrast to Cartan's approach, the equivariant moving frame solution is not based on the theory of $G$-structures. It thus offers an alternative method for tackling equivalence problems.

## 1 Introduction

First introduced by the Estonian mathematician Martin Bartels and primarily developed by Élie Cartan, [1], the theory of moving frames is a powerful tool for studying geometric properties of submanifolds under the action of a (pseudo-) group of transformations. Already in Cartan's original work, [4, 5, 6, 7], the theory found many important applications. It was used to solve local equivalence problems, it was at the foundation of his structure theory of infinite-dimensional Lie pseudo-groups, and became a tool to study Riemannian geometry, conformal geometry and geometric properties of differential equations. Modern treatments of Cartan's moving frame method can be found, for example, in [3, 15, 18, 19, 27, 40].

Recently, Fels and Olver proposed in [12, 13] a new theoretical foundation to the method of moving frames. For a Lie group $G$ acting on $n$-th order jets $\mathrm{J}^n(M,p)$ of $p$-dimensional submanifolds of $M$, a moving frame is a $G$-equivariant section of the trivial bundle $\mathrm{J}^n(M,p)\times G \to \mathrm{J}^n(M,p)$. This new approach to moving frames is now referred as the *equivariant moving frame method*. It offers many interesting features. First, the implementation of the method is completely algorithmic, and once an equivariant

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moving frame is obtained it induces an invariantization map that projects differential functions and differential forms to their invariant counterparts yielding a complete collection of local differential invariants and an invariant coframe on $\mathrm{J}^\infty(M,p)$. Thus far, one of the most important results of the theory is the *universal recurrence formula* for the exterior differential of invariant differential forms. With this formula, Kogan and Olver were able to obtained, for the first time, a general explicit group-invariant formula for the Euler–Lagrange equations associated with an invariant variational problem, [23]. In [42] the same formula was used to show that the cohomology of the invariant Euler–Lagrange complex is isomorphic to the Lie algebra cohomology of the group action. But probably more importantly, the universal recurrence formula is the key that unveils the structure of the algebra of differential invariants of a group action, [29, 30, 31]. This last fact will play an important role in our solution of local equivalence problems.

In [33, 34] the theory of equivariant moving frames was successfully extended to infinite-dimensional Lie pseudo-group actions opening the way to many new applications. Some of the first applications can be found in [8, 9, 46] where some algebras of differential invariants of infinite-dimensional symmetry groups of partial differential equations have been completely characterized and their Maurer–Cartan structure equations computed. An application to the classification of Laplace invariants and the factorization of linear partial differential operator can be found in [41]. The purpose of this paper is to apply the equivariant moving frame theory to solve the general problem of local equivalence of submanifolds under a Lie pseudo-group action. As stated in [34], the solution to this problem is one of the motivations for the development of the equivariant moving frame method.

In the same spirit as Cartan, the strategy is to construct sufficiently many invariants to distinguish inequivalent submanifolds. Thanks to the invariantization map, those invariants are easily found with the equivariant moving frame method. Since the construction of an equivariant moving frame does not rely on the theory of exterior differential systems and the theory of $G$-structures, solving an equivalence problem with the equivariant moving frame method avoids some of the technical features of Cartan's solution in terms of coframes. For example, there is no need to absorb non-essential torsion as we do not work with the structure equations, [15, 19, 27]. Furthermore, with the universal recurrence formula and the structure equations of the equivalence pseudo-group all computations can be performed symbolically.

The paper is structured as follows. Following [33, 34, 35], we begin with a review of some of the important aspects of the theory of equivariant moving frames. First, we explain how to obtain the structure equations of a Lie pseudo-group from its infinitesimal determining system in Section 2. Then given a Lie pseudo-group action on submanifolds, the construction of an equivariant moving frame is discussed in Section 3. A moving frame exists in the neighborhood of a submanifold jet provided the pseudo-group action is free and regular. In typical equivalence problem, the equivalence pseudo-group action admits singular submanifolds where the action is not free and where no moving frame can be defined. To take care of these singular submanifolds we introduce the notion of a partial moving frame. In Section 4 we explain how to apply the equivariant moving frame method to solve local equivalence problems of submanifolds. We are particularly interested in comparing our results with Cartan's equivalence method of coframes. To illustrate many aspects of the theory, the equivalence of first order scalar Lagrangian on the line will be our running example. We chose this problem as it is one of the simplest equivalence problems leading to a standard moving frame. For examples of equivalence problems admitting singular submanifolds, we consider the local equivalence of second order ordinary differential equations under fiber preserving transformations and contact transformations. Finally, we also consider the simultaneous equivalence of a nonvanishing two-form and a non-zero vector field in $\mathbb{R}^3$. This problem is interesting as it is one of the rare known equivalence problems admitting involutive structure equations with an essential invariant. The equivalence problems considered in our examples have already been solved in [16, 20, 21, 22, 27]. The examples were chosen because they span all possible situations that can occur when solving an equivalence problem. We also chose simple examples so that one can concentrate on understanding how the equivariant moving frame method applies and compare our computations with Cartan's equivalence method. Applications of the equivariant moving frame method to new equivalence problems will be considered elsewhere.

**Remark 1.1.** The theory of infinite-dimensional Lie pseudo-groups relies on the Cartan–Kähler Theorem, [3, 27], which requires analyticity. For this reason, all our constructions and results hold in the analytic category. Thus, implicitly, all manifolds, maps, differential forms and vector fields are assumed to be analytic. For Lie pseudo-groups of finite-type, i.e. pseudo-groups that come from Lie group actions, analyticity can be replaced by smoothness instead.

# 2 Structure Equations

In this section we review the derivation of the structure equations of a Lie pseudo-group action. We mainly follow [33]. As we will show in Section 4.3, the structure equations of an equivalence pseudo-group action can be used to derive the structure equations obtained with Cartan's equivalence method symbolically.

The first step to derive the structure equations of a Lie pseudo-group $\mathcal{G}$ acting on a manifold $M$ is to obtain the structure equations of the pseudo-group of all local diffeomorphisms of $M$.

## 2.1 Diffeomorphism Pseudo-Group

Let $M$ be an analytic $m$-dimensional manifold. We denote by $\mathcal{D} = \mathcal{D}(M)$ the pseudo-group of all local analytic diffeomorphisms of $M$. For each integer $0 \leq n \leq \infty$, let $\mathcal{D}^{(n)}$ denote the bundle formed by their $n$-th order jets. For $k \geq n$, let $\pi^k_n\colon \mathcal{D}^{(k)} \to \mathcal{D}^{(n)}$ denote the standard projection. The local coordinates of the $n$ jet of a local diffeomorphism $\varphi$ are given by $\mathrm{j}_n\varphi = (z, Z^{(n)})$, where $z = (z^1, \ldots, z^m)$ are the *source coordinates* on $M$, $Z = (Z^1, \ldots, Z^m)$ the *target coordinates* also on $M$, and the corresponding *jet coordinates* $Z^a_B$ representing the partial derivatives $\partial^k\varphi^a(z)/\partial z^{b^1} \cdots \partial z^{b^k}$ of the local diffeomorphism $Z = \varphi(z)$, with $1 \leq a, b^1, \ldots, b^k \leq m$ and $1 \leq k = \#A \leq n$. Following Cartan [6, 7] and the recent work of Olver and Pohjanpelto [33, 34] we systematically use lower case letters, $z$, $x$, $u$, ... for the source coordinates $\boldsymbol{\sigma}(\varphi) = z$ and the corresponding upper case letters $Z$, $X$, $U$, ... for the target coordinates $\boldsymbol{\tau}(\varphi) = Z$ of local diffeomorphisms $Z = \varphi(z)$.

The diffeomorphism jet bundle $\mathcal{D}^{(\infty)}$ has the structure of a groupoid, [25]. The groupoid multiplication follows from the composition of local diffeomorphism. Given, $g^{(\infty)} = \mathrm{j}_\infty\varphi|_z$, $h^{(\infty)} = \mathrm{j}_\infty\psi|_Z$, with $Z = \boldsymbol{\tau}(\mathrm{j}_\infty\varphi|_z) = \boldsymbol{\sigma}(\mathrm{j}_\infty\psi|_Z)$, we write $h^{(\infty)} \cdot g^{(\infty)} =$

$\mathrm{j}_\infty(\psi\circ\varphi)|_z$. Throughout the paper, local diffeomorphisms $\psi\in\mathcal{D}$ act on $\mathcal{D}^{(\infty)}$ by right multiplication:

$$R_\psi(\mathrm{j}_\infty\varphi|_z)=\mathrm{j}_\infty(\varphi\circ\psi^{-1})|_{\psi(z)}. \tag{2.1}$$

Let $T^*\mathcal{D}^{(\infty)}$ be the cotangent space of the infinite diffeormorphism jet bundle $\mathcal{D}^{(\infty)}$ defined pointwise as the direct limit of the direct system $(\pi^{n+1}_n)^*\colon T^*\mathcal{D}^{(n)}\to T^*\mathcal{D}^{(n+1)}$, $n\geq 0$, [2]. The cotangent space $T^*\mathcal{D}^{(\infty)}$ naturally splits into horizontal and vertical (contact) components. In terms of the local coordinates $z^a$, $Z^b_A$, the *horizontal subbundle* is spanned by the one-forms $dz^a=d_M z^a$, $a=1,\ldots,m$, while the *vertical subbundle* is spanned by the basic *contact one-forms*

$$\Upsilon^b_A=dZ^b_A-\sum_{a=1}^m Z^b_{A,a}dz^a,\qquad b=1,\ldots,m,\qquad \#A\geq 0.$$

The decomposition of $T^*\mathcal{D}^{(\infty)}$ accordingly splits the differential in two components

$$d=d_M+d_G,$$

where the subscript on the vertical differential $d_G$ refers to the groupoid structure of $\mathcal{D}^{(\infty)}$. In particular, if $F(z,Z^{(n)})$ is any differential function, then

$$d_MF=\sum_{a=1}^m(\mathbb{D}_{z^a}F)dz^a\qquad\text{and}\qquad d_GF=\sum_{b=1}^m\sum_{\#A\geq 0}\frac{\partial F}{\partial Z^b_A}\Upsilon^b_A,$$

where

$$\mathbb{D}_{z^a}=\frac{\partial}{\partial z^a}+\sum_{b=1}^m\sum_{\#A\geq 0}Z^b_{A,a}\frac{\partial}{\partial Z^b_A},\qquad a=1,\ldots,m, \tag{2.2}$$

are the *total derivative operators* on $\mathcal{D}^{(\infty)}$.

Since the target coordinate functions $Z^a\colon\mathcal{D}^{(\infty)}\to\mathbb{R}$ are invariant under the right action (2.1), so are their differentials $dZ^a$. The splitting of the differential into horizontal and contact components is also right-invariant. This implies that the one-forms

$$\sigma^a=d_MZ^a=\sum_{b=1}^m Z^a_b dz^b,\qquad a=1,\ldots,m, \tag{2.3}$$

form an invariant horizontal coframe, while

$$\mu^a=\Upsilon^a=d_GZ^a=dZ^a-\sum_{b=1}^m Z^a_b dz^b,\qquad a=1,\ldots,m, \tag{2.4}$$

are the zero-th order invariant contact forms. Writing the horizontal component of the exterior differential of a differential function $F\colon\mathcal{D}^{(\infty)}\to\mathbb{R}$ in terms of the invariant horizontal coframe (2.3)

$$d_MF=\sum_{a=1}^m(\mathbb{D}_{Z^a}F)\sigma^a$$

serves to define the dual invariant total differential operators

$$\mathbb{D}_{Z^a}=\sum_{b=1}^m w^b_a\,\mathbb{D}_{z^b},\qquad a=1,\ldots,m,\qquad\text{where}\qquad \left(\,w^b_a(z,Z^{(1)})\,\right)=\left(\frac{\partial Z^b}{\partial z^a}\right)^{-1} \tag{2.5}$$

denotes the inverse Jacobian matrix. Higher-order right-invariant contact forms are obtained by repeatedly Lie differentiating the zero-th order invariant contact forms (2.4) with respect to the invariant differential operators (2.5):

$$\mu^b_A = \mathbb{D}^A_Z \mu^b = \mathbb{D}_{Z^{a_1}} \cdots \mathbb{D}_{Z^{a_k}} \mu^b, \qquad b = 1, \ldots, m, \qquad \#A \geq 0. \tag{2.6}$$

The differential operators $\mathbb{D}_{Z^1}, \ldots, \mathbb{D}_{Z^m}$ commute, so the order of differentiation in (2.6) is immaterial. The right-invariant contact forms $\mu^{(\infty)} = (\ \ldots\ \mu^a_A\ \ldots\ )$ define the *Maurer–Cartan forms* for the diffeomorphism pseudo-group $\mathcal{D}$, and they, together with the horizontal forms (2.3) provide a right-invariant coframe on $\mathcal{D}^{(\infty)}$.

To concisely express the structure equations of the invariant coframe $\{\sigma, \mu^{(\infty)}\}$ the vector-valued Maurer–Cartan formal power series $\mu[\![H]\!] = (\mu^1[\![H]\!], \ldots, \mu^m[\![H]\!])^T$, with components

$$\mu^b[\![H]\!] = \sum_{\#A \geq 0} \frac{1}{A!}\, \mu^b_A\, H^A, \qquad b = 1 \ldots, m, \tag{2.7}$$

is introduced. Here $H = (H^1, \ldots, H^m)$ are formal power series parameters, while $A! = i_1!\, i_2! \cdots i_m!$, where $i_l$ stands for the number of occurrences of the integer $1 \leq l \leq m$ in $A$. The structure equations for the right-invariant forms $\mu^a_A$ are obtained by comparing the coefficients of the various powers of $H$ in the power series identity

$$d\mu[\![H]\!] = \nabla\mu[\![H]\!] \wedge (\mu[\![H]\!] - dZ), \tag{2.8a}$$

where $dZ = (dZ^1, \ldots, dZ^m)^T$ and $\nabla\mu[\![H]\!] = \big(\partial\mu^a[\![H]\!]/\partial H^b\big)$ denotes the $m \times m$ Jacobian matrix obtained by formal differentiation of the power series (2.7) with respect to the parameters $H$. The structure equations for the invariant horizontal forms $\sigma = (\sigma^1, \ldots, \sigma^m)^T$ are

$$d\sigma = \nabla\mu[\![0]\!] \wedge \sigma. \tag{2.8b}$$

**Theorem 2.1.** The *structure equations* of the diffeomorphism pseudo-group $\mathcal{D}$ are given by the equations (2.8).

**Example 2.2.** To illustrate some of the above formulas we consider the planar diffeomorphism pseudo-group $\mathcal{D}(\mathbb{R}^2)$. Let $(x, u)$ be coordinates on $\mathbb{R}^2$, then a local diffeomorphism in $\mathcal{D}(\mathbb{R}^2)$ is denoted by

$$X = \phi(x, u), \qquad U = \beta(x, u), \qquad \text{where} \qquad \phi_x\beta_u - \phi_u\beta_x \neq 0.$$

The local coordinates of $\mathcal{D}^{(2)}(\mathbb{R}^2)$, for example, are

$$(x, u, X_x, X_u, U_x, U_u, X_{xx}, X_{xu}, X_{uu}, U_{xx}, U_{xu}, U_{uu}).$$

The invariant horizontal coframe (2.3) is spanned by the two differential one-forms

$$\sigma^x = d_M X = X_x dx + X_u du, \qquad \sigma^u = d_M U = U_x dx + U_u du,$$

and the corresponding dual total differential operators on $\mathcal{D}^{(\infty)}(\mathbb{R}^2)$ are

$$\mathbb{D}_X = \frac{U_u \mathbb{D}_x - U_x \mathbb{D}_u}{X_x U_u - X_u U_x}, \qquad \mathbb{D}_U = \frac{-X_u \mathbb{D}_x + X_x \mathbb{D}_u}{X_x U_u - X_u U_x}. \tag{2.9}$$

The zero-th order Maurer–Cartan forms (2.4) for the diffeomorphism pseudo-group $\mathcal{D}(\mathbb{R}^2)$ are the zero-th order contact forms

$$\begin{aligned} \mu^x &= \Upsilon^x = d_G X = dX - X_x dx - X_u du, \\ \mu^u &= \Upsilon^u = d_G U = dU - U_x dx - U_u du. \end{aligned} \tag{2.10}$$

The higher order Maurer–Cartan forms are obtained by Lie differentiating (2.10) with respect to the differential operators (2.9). For example,

$$\begin{aligned} \mu^x_X &= \frac{U_u \Upsilon^x_x - U_x \Upsilon^x_u}{X_x U_u - X_u U_x}, & \mu^x_U &= \frac{X_x \Upsilon^x_u - X_u \Upsilon^x_x}{X_x U_u - X_u U_x}, \\ \mu^u_X &= \frac{U_u \Upsilon^u_x - U_x \Upsilon^u_u}{X_x U_u - X_u U_x}, & \mu^x_U &= \frac{X_x \Upsilon^u_u - X_u \Upsilon^u_x}{X_x U_u - X_u U_x}, \end{aligned}$$

where

$$\begin{aligned} \Upsilon^x_x &= dX_x - X_{xx} dx - X_{xu} du, & \Upsilon^x_u &= dX_u - X_{xu} dx - X_{uu} du, \\ \Upsilon^u_x &= dU_x - U_{xx} dx - U_{xu} du, & \Upsilon^x_u &= dU_u - U_{xu} dx - U_{uu} du. \end{aligned}$$

are the first order contact forms on $\mathcal{D}^{(\infty)}(\mathbb{R}^2)$. Let

$$\begin{aligned} \mu^x[\![H,K]\!] &= \sum_{j,k\geq 0} \frac{\mu^x_{j,k}}{j!k!} H^j K^k, & \mu^x[\![H,K]\!] - dX &= -\sigma^x + \sum_{\substack{j+k\geq 1 \\ j,k\geq 0}} \frac{\mu^x_{j,k}}{j!k!} H^j K^k, \\ \mu^u[\![H,K]\!] &= \sum_{j,k\geq 0} \frac{\mu^u_{j,k}}{j!k!} H^j K^k, & \mu^u[\![H,K]\!] - dU &= -\sigma^u + \sum_{\substack{j+k\geq 1 \\ j,k\geq 0}} \frac{\mu^u_{j,k}}{j!k!} H^j K^k, \end{aligned}$$

be the Maurer–Cartan formal power series (2.7), where $\mu^x_{j,k} = \mathbb{D}^j_X \mathbb{D}^k_U \mu^x$, $\mu^u_{j,k} = \mathbb{D}^j_X \mathbb{D}^k_U \mu^u$. Then the structure equations (2.8) are

$$\begin{pmatrix} d\sigma^x \\ d\sigma^u \end{pmatrix} = \begin{pmatrix} \mu^x_X & \mu^x_U \\ \mu^u_X & \mu^u_U \end{pmatrix} \wedge \begin{pmatrix} \sigma^x \\ \sigma^u \end{pmatrix},$$

$$\begin{pmatrix} d\mu^x[\![H,K]\!] \\ d\mu^u[\![H,K]\!] \end{pmatrix} = \begin{pmatrix} \mu^x_H[\![H,K]\!] & \mu^x_K[\![H,K]\!] \\ \mu^u_H[\![H,K]\!] & \mu^u_K[\![H,K]\!] \end{pmatrix} \wedge \begin{pmatrix} \mu^x[\![H,K]\!] - dX \\ \mu^u[\![H,K]\!] - dU \end{pmatrix}.$$

Equating the powers of $H$ and $K$ we obtain the structure equations:

$$\begin{aligned} d\sigma^x &= -d\mu^x = \mu^x_X \wedge \sigma^x + \mu^x_U \wedge \sigma^u, \\ d\sigma^u &= -d\mu^u = \mu^u_X \wedge \sigma^x + \mu^u_U \wedge \sigma^u, \\ d\mu^x_X &= -\mu^x_{XX} \wedge \sigma^x - \mu^x_{XU} \wedge \sigma^u + \mu^x_U \wedge \mu^u_X, \\ d\mu^u_X &= -\mu^u_{XX} \wedge \sigma^x - \mu^u_{XU} \wedge \sigma^u + \mu^u_X \wedge (\mu^x_X - \mu^u_U), \\ d\mu^x_U &= -\mu^x_{XU} \wedge \sigma^x - \mu^x_{UU} \wedge \sigma^u - \mu^x_U \wedge (\mu^x_X - \mu^u_U), \\ d\mu^u_U &= -\mu^u_{XU} \wedge \sigma^x - \mu^u_{UU} \wedge \sigma^u + \mu^u_X \wedge \mu^x_U, \end{aligned} \tag{2.11}$$

and so on.

## 2.2 Lie Pseudo-Groups

With the structure equations of the diffeomorphism pseudo-group at hand it is a fairly straightforward task to obtain the structure equations of any Lie pseudo-group from its infinitesimal data. Several definitions of Lie pseudo-groups can be found in the literature depending on the technical hypotheses made by the authors. In the analytic category, Lie pseudo-groups can be defined as follows.

**Definition 2.3.** A pseudo-group $\mathcal{G} \subset \mathcal{D}$ is called *regular* of order $n^\star \geq 1$ if, for all finite $n \geq n^\star$, the pseudo-group jets form an embedded subbundle $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ and the projection $\pi_n^{n+1}\colon \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a fibration.

The Lie requirement on the pseudo-group is encapsulated in the following definition.

**Definition 2.4.** An analytic pseudo-group $\mathcal{G} \subset \mathcal{D}$ is called a *Lie pseudo-group* if $\mathcal{G}$ is regular of order $n^\star \geq 1$ and, moreover, every local diffeomorphism $\varphi$ of $\mathcal{D}$ satisfying $\mathrm{j}_{n^\star}\varphi \subset \mathcal{G}^{(n^\star)}$ belongs to the pseudo-group, i.e. $\varphi \in \mathcal{G}$.

In local coordinates, for $n \geq n^\star$ the pseudo-group jet subbundle $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ is characterized by an involutive system, [39], of $n$-th order partial differential equations

$$F^{(n)}(z, Z^{(n)}) = 0, \tag{2.12}$$

called the *determining system* for the pseudo-group. When $n \geq n^\star$, the determining system of $\mathcal{G}^{(n+1)}$ is obtained by differentiating the determining equations (2.12) for $\mathcal{G}^{(n)}$ with respect to the total differential operators (2.2).

Once restricted to a Lie pseudo-group $\mathcal{G} \subset \mathcal{D}$, the Maurer–Cartan forms (2.6) are no longer linearly independent. As we now explain, the coordinate expressions for the Maurer–Cartan forms are not required to determine the linear dependencies. This can be done symbolically from the infinitesimal data of the Lie pseudo-group.

Let $\mathfrak{g}$ denote the Lie algebra of infinitesimal generators of the Lie pseudo-group $\mathcal{G}$. By definition $\mathfrak{g}$ consists of the locally defined vector fields

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} \tag{2.13}$$

whose flows belong to $\mathcal{G}$. for $0 \leq n \leq \infty$, let $\mathrm{J}^n TM$, denote the bundle of $n$-th order jets of sections of $TM$. Local coordinates on $\mathrm{J}^n TM$ are given by

$$(z, \zeta^{(n)}) = (\ldots, z^a, \ldots, \zeta^b_A, \ldots), \qquad b = 1, \ldots, m, \qquad 0 \leq \#A \leq n,$$

where $\zeta^a_A$ represents the partial derivative $\partial^{\#A}\zeta^b/\partial z^A$. By our regularity assumption on the pseudo-group $\mathcal{G}$, for each $n \geq n^\star$, $\mathrm{J}^n\mathfrak{g}$ is a subbundle of $\mathrm{J}^n TM$ which locally is prescribed by a system of linear partial differential equations

$$L^{(n)}(z, \zeta^{(n)}) = \sum_{b=1}^{m} \sum_{\#A \leq n} h^A_{b;\upsilon}(z)\zeta^b_A = 0, \qquad \upsilon = 1, \ldots, k, \tag{2.14}$$

known at the *infinitesimal determining system.* The system (2.14) is obtained by linearization of the determining system (2.12) at the identity jet $\mathbb{1}^{(n)}$.

**Theorem 2.5.** For each $n \geq n^\star$, the homogeneous linear algebraic system

$$L^{(n)}(Z, \mu^{(n)}) = 0, \tag{2.15}$$

obtained from the infinitesimal determining equations (2.14) by formally replacing[1] the source coordinates $z^a$ by the corresponding target coordinates $Z^a$, and the vector field jet coordinates $\zeta_A^a$ by the corresponding Maurer–Cartan forms $\mu_A^a$, serves to define the complete set of linear dependencies among the Maurer–Cartan forms $\mu^{(n)}$ once restricted to the Lie pseudo-group $\mathcal{G}$.

**Corollary 2.6.** The structure equations of a Lie pseudo-group $\mathcal{G}$ are obtained by restricting the diffeomorphism structure equations (2.8) to the solution space of (2.15).

**Remark 2.7.** The structure equations of a Lie pseudo-group are different from its *Maurer–Cartan structures*. We refer the reader to [37, 45, 46] for a discussion on this aspect.

For future reference we introduce Cartan's notion of isomorphic prolongation of a pseudo-group, [6, 39, 44].

**Definition 2.8.** Let $\pi \colon M \to N$ be a fiber bundle and $\mathcal{H}$, $\mathcal{G}$ two pseudo-group actions on $M$ and $N$ respectively. The pseudo-group $\mathcal{G}$ is an *isomorphic prolongation* of $\mathcal{H}$ if there is a one-to-one correspondence between elements $\varphi \in \mathcal{G}$ and $\phi \in \mathcal{H}$ satisfying $\pi \circ \varphi = \phi \circ \pi$.

**Example 2.9.** In this example we compute the structure equations for the fiber preserving equivalence pseudo-group of first order scalar variational problems

$$\mathcal{L}[u] = \int l(x,u,p)dx, \qquad \text{where} \qquad p = u_x \quad \text{and} \quad l \neq 0, \tag{2.16}$$

given by[2]

$$\mathcal{G}: \qquad X = \phi(x), \qquad U = \beta(x,u), \qquad P = \frac{p\beta_u + \beta_x}{\phi_x}, \qquad L = \frac{l}{\phi_x}, \tag{2.17}$$

where $\phi_x \neq 0$ and $\beta_u \neq 0$. The determining equations for the pseudo-group action (2.17) are

$$X_u = X_p = X_l = 0, \qquad U_p = U_l = 0, \qquad P = \frac{pU_u + U_x}{X_x}, \qquad L = \frac{l}{X_x}. \tag{2.18}$$

Clearly, all diffeomorphisms satisfying the system of equations (2.18) are of the form (2.17). To obtain the infinitesimal determining equations of the pseudo-group we linearize (2.18) at the identity jet $\mathbb{1}^{(1)}$. If we denote an infinitesimal generator of the pseudo-group action by

$$\mathbf{v} = \xi(x,u,p,l)\frac{\partial}{\partial x} + \eta(x,u,p,l)\frac{\partial}{\partial u} + \alpha(x,u,p,l)\frac{\partial}{\partial p} + \gamma(x,u,p,l)\frac{\partial}{\partial l}, \tag{2.19}$$

[1]This formal replacement is called the lift map in the theory equivariant moving frames. See Definitions 3.8 and 3.11.

[2]The pseudo-group action 2.17 corresponds to the "standard equivalence problem" which requires two functionals to have the same values on functions: $\mathcal{L}[u] = \overline{\mathcal{L}}[\overline{u}]$. The more general "divergence equivalence problem" which requires the Euler–Lagrange to match up requires the Lagrangian to transform according to the rule $L = (l + D_x A(x,u))/\phi_x$, where $A(x,u)$ is an arbitrary differential function, [21, 27].

then the infinitesimal determining system is given by

$$\xi_u = \xi_p = \xi_l = 0, \qquad \eta_p = \eta_l = 0, \qquad \alpha = p(\eta_u - \xi_x) + \eta_x, \qquad \gamma = -l\xi_x. \tag{2.20}$$

Under the replacement

$$\xi_A \to \mu^x_A, \qquad \eta_A \to \mu^u_A, \qquad \alpha_A \to \mu^p_A, \qquad \gamma_A \to \mu^l_A, \qquad (x,u,p,l) \to (X,U,P,L),$$

the infinitesimal determining equations (2.20) give the linear dependencies

$$\mu^x_U = \mu^x_P = \mu^x_L = 0, \quad \mu^u_P = \mu^u_L = 0, \quad \mu^p = P(\mu^u_U - \mu^x_X) + \mu^u_X, \quad \mu^l = -L\mu^x_X \tag{2.21}$$

among the Maurer–Cartan forms of order $\leq 1$ of $\mathcal{D}(\mathbb{R}^4)$. The linear dependencies among higher order Maurer–Cartan forms are obtained by differentiating the system of equations (2.21) with respect to the invariant total differential operators $\mathbb{D}_X$, $\mathbb{D}_U$, $\mathbb{D}_P$, $\mathbb{D}_L$ defined in (3.3). It follows from (2.21) and its prolongations that

$$\mu^x_{X^j}, \qquad \mu^u_{X^jU^k}, \qquad j,k \geq 0, \tag{2.22}$$

is a basis of Maurer–Cartan forms for the pseudo-group (2.17). The coordinate expressions of the Maurer–Cartan forms (2.22) are identical to the formulas for the Maurer–Cartan forms of $\mathcal{D}(\mathbb{R}^2)$ of Example 2.2 with the difference that $X_u$ and its derivatives with respect to the variables $x$ and $u$ are equal to zero.

The horizontal forms of the pseudo-group action (2.17) are given by

$$\sigma^x = X_x dx, \qquad \sigma^u = U_x dx + U_u du, \qquad \sigma^l = \frac{1}{X_x}\left[-\frac{lX_{xx}}{X_x}dx + dl\right],$$
$$\sigma^p = \frac{1}{X_x}\left[\left(pU_{xu} + U_{xx} - \frac{X_{xx}}{X_x}\right)dx + (pU_{uu} + U_{ux})du + U_u dp\right].$$

Their structure equations are obtained by restricting the equations (2.8b) to the kernel of (2.21) and its prolongations. This leads to

$$\begin{aligned}
d\sigma^x =&\mu^x_X \wedge \sigma^x,\\
d\sigma^u =&\mu^u_X \wedge \sigma^x + \mu^u_U \wedge \sigma^u,\\
d\sigma^p =&\mu^p_X \wedge \sigma^x + \mu^p_U \wedge \sigma^u + \mu^p_P \wedge \sigma^p\\
=&[P(\mu^u_{UX} - \mu^x_{XX}) + \mu^u_{XX}] \wedge \sigma^x + [P\mu^u_{UU} + \mu^u_{UX}] \wedge \sigma^u + (\mu^u_U - \mu^x_X) \wedge \sigma^p,\\
d\sigma^l =&\mu^l_X \wedge \sigma^x + \mu^l_U \wedge \sigma^u + \mu^l_P \wedge \sigma^p + \mu^l_L \wedge \sigma^l\\
=& - L\mu^x_{XX} \wedge \sigma^x - \mu^x_X \wedge \sigma^l.
\end{aligned} \tag{2.23a}$$

The structure equations for the basis of Maurer–Cartan forms (2.22) are obtained from the structure equations (2.11) by setting $\mu^x_{X^jU^{k+1}} = 0$, $j,k \geq 0$:

$$\begin{aligned}
d\mu^x =& - \mu^x_X \wedge \sigma^x,\\
d\mu^u =& - \mu^u_X \wedge \sigma^x - \mu^u_U \wedge \sigma^u,\\
d\mu^x_X =& - \mu^x_{XX} \wedge \sigma^x,\\
d\mu^u_X =& - \mu^u_{XX} \wedge \sigma^x - \mu^u_{XU} \wedge \sigma^u + \mu^u_X \wedge (\mu^x_X - \mu^u_U),\\
d\mu^u_U =& - \mu^u_{XU} \wedge \sigma^x - \mu^u_{UU} \wedge \sigma^u,
\end{aligned} \tag{2.23b}$$

and so on.

The pseudo-group action (2.17) is a good example of isomorphic prolongation. Clearly, it is an isomorphic prolongation of the pseudo-group

$$X = \phi(x), \qquad U = \beta(x,u), \tag{2.24}$$

acting of the plane.

# 3 Equivariant Moving Frames

For infinite-dimensional Lie pseudo-group actions, the equivariant moving frame construction was first laid out in [34]. In Section 3.1 we review this construction and in Section 3.2 we introduce the notion of a partial equivariant moving frame.

As in the previous section, let $\mathcal{G}$ be a Lie pseudo-group action on an $m$-dimensional manifold $M$. We are now interested in the induced action of $\mathcal{G}$ on $p$-dimensional submanifolds $S \subset M$. For each integer $0 \leq n \leq \infty$, let $\mathrm{J}^n = \mathrm{J}^n(M,p)$ denote the $n$-th order *submanifold jet bundle*, defined as the set of equivalence classes under the equivalence relation of $n$-th order contact at a single point; see [27, 47] for more details . For $k \geq n$, we use $\pi^k_n \colon \mathrm{J}^k \to \mathrm{J}^n$ to denote the canonical projection. Locally, the coordinates on $M$ can be written as $z = (x,u)$ where $x = (x^1,\ldots,x^p)$ are considered to be the independent variables parametrizing a submanifold $S \subset M$ and $u = (u^1,\ldots,u^q)$, $q = m-p$, the dependent variables. The induced coordinates on $\mathrm{J}^n$ are denoted by $z^{(n)} = (x,u^{(n)})$, where $u^{(n)}$ denotes the derivatives $u^\alpha_J = \partial^{\#J}u^\alpha/\partial x^J$ of the $u$'s with respect to the $x$'s of order $0 \leq \#J \leq n$.

Let $\mathcal{E}^{(n)} \to \mathrm{J}^n$ be the *lifted bundle* obtained by taking the pull-back bundle of $\mathcal{G}^{(n)} \to M$ via the projection $\pi^n_0 : \mathrm{J}^n \to M$. Local coordinates on $\mathcal{E}^{(n)}$ are given by $(z^{(n)},g^{(n)})$, where the base coordinates $z^{(n)} = (x,u^{(n)}) \in \mathrm{J}^n$ are the submanifold jet coordinates and the fiber coordinates $g^{(n)}$ parametrize the pseudo-group jets. The bundle $\mathcal{E}^{(n)}$ carries the structure of a groupoid, with *source map* $\boldsymbol{\sigma}(z^{(n)},g^{(n)}) = z^{(n)}$ and *target map* $\boldsymbol{\tau}(z^{(n)},g^{(n)}) = Z^{(n)} = g^{(n)}\cdot z^{(n)}$ given by the *prolonged action.* The local coordinate expressions for the prolonged action $Z^{(n)}$ are obtained by implementing the chain rule. Let

$$d_H X^i = \sum_{j=1}^p (D_{x^j}X^i)dx^j, \qquad i = 1,\ldots,p, \tag{3.1}$$

be the *lifted horizontal coframe* on $\mathcal{E}^{(\infty)}$, where

$$D_{x^j} = \frac{\partial}{\partial x^j} + \sum_{\alpha=1}^q \sum_{\#J\geq 0} u^\alpha_{J,j}\frac{\partial}{\partial u^\alpha_J}, \qquad j = 1,\ldots,p, \tag{3.2}$$

are the *total derivative operators* on the submanifold jet bundle $\mathrm{J}^\infty$. The *lifted total differential operators* are defined by the formula

$$d_H F(z^{(n)}) = \sum_{i=1}^p (D_{x^i}F)dx^i = \sum_{i=1}^p (D_{X^i}F)d_H X^i.$$

More explicitly

$$D_{X^i} = \sum_{j=1}^p W^j_i D_{x^j}, \qquad \text{where} \qquad (W^j_i) = (D_{x^i}X^j)^{-1} \tag{3.3}$$

is the inverse of the total Jacobian matrix. By differentiating the target dependent variables $U^\alpha = g \cdot u^\alpha$ with respect to the lifted total differential operators (3.3) we obtain the explicit expressions for the prolonged action[3]:

$$X^i, \qquad \widehat{U}^\alpha_J = D^J_X U^\alpha = D_{X^{j^1}} \cdots D_{X^{j^k}} U^\alpha, \qquad k = \#J \geq 0. \tag{3.4}$$

A local diffeomorphism $\varphi \in \mathcal{G}$ acts on the set

$$\{(z^{(n)}, g^{(n)}) \in \mathcal{E}^{(n)} |\ \pi^n_0(z^{(n)}) \in \text{dom}\ \varphi\}$$

by

$$\varphi \cdot (z^{(n)}, g^{(n)}) = (\mathrm{j}_n\varphi|_z \cdot z^{(n)}, g^{(n)} \cdot \mathrm{j}_n\varphi^{-1}|_{\varphi(z)}), \tag{3.5}$$

where $\pi^n_0(z^{(n)}) = z$. The action (3.5) is the concatenation of the prolonged action on submanifold jets with the right action (2.1) on $\mathcal{G}^{(n)}$. It is called the $n$-th order *lifted action* of $\mathcal{G}$ on $\mathcal{E}^{(n)}$.

## 3.1 Regular Submanifold Jets

**Definition 3.1.** Let $\mathcal{G}$ be a regular Lie pseudo-group acting on $M$. A *moving frame* of *order* $n$ is a $\mathcal{G}$-equivariant local section $\rho^{(n)} \colon \mathrm{J}^n \to \mathcal{E}^{(n)}$.

In a system of local coordinates we use the notation

$$\rho^{(n)}(z^{(n)}) = (z^{(n)}, \widetilde{\rho}^{(n)}(z^{(n)}))$$

to denote a moving frame. One can always define left and right moving frames. For a right moving frame the $\mathcal{G}$-equivariance means that

$$\begin{aligned} \varphi \cdot \rho^{(n)}(z^{(n)}) =& \varphi \cdot (z^{(n)}, \widetilde{\rho}^{(n)}(z^{(n)})) = (\mathrm{j}_n\varphi|_z \cdot z^{(n)}, \widetilde{\rho}^{(n)}(\mathrm{j}_n\varphi|_z \cdot z^{(n)})) \\ =& (\mathrm{j}_n\varphi|_z \cdot z^{(n)}, \widetilde{\rho}^{(n)}(z^{(n)}) \cdot \mathrm{j}_n\varphi^{-1}|_{\varphi(z)}), \end{aligned}$$

for $\varphi \in \mathcal{G}$ and $\boldsymbol{\sigma}(\mathrm{j}_n\varphi|_z) = z^{(n)}$. The existence of a moving frame requires the prolonged pseudo-group action be free and regular. Recall that the action is regular if all the orbits of the pseudo-group action have the same dimension and that each point $z^{(n)} \in \mathrm{J}^n$ has arbitrarily small neighborhoods whose intersection with each orbit is a connected subset thereof.

**Definition 3.2.** The *jet isotropy subgroup* of a submanifold jet $z^{(n)} \in \mathrm{J}^n$ is defined as $\mathcal{G}_{z^{(n)}} = \boldsymbol{\tau}^{-1}\{z^{(n)}\} \cap \boldsymbol{\sigma}^{-1}\{z^{(n)}\} \subset \mathcal{E}^{(n)}|_z$. The pseudo-group is said to act *freely* at $z^{(n)}$ if $\mathcal{G}_{z^{(n)}} = \{(z^{(n)}, \mathbb{1}^{(n)})\}$. The pseudo-group acts *locally freely* at $z^{(n)}$ if $\mathcal{G}_{z^{(n)}}$ is discrete. The pseudo-group $\mathcal{G}$ is said to act *(locally) freely at order* $n$ if it acts (locally) freely on an open subset $\mathcal{V}^n \subset \mathrm{J}^n$, called the set of *regular n-jets.*

**Remark 3.3.** As explained and illustrated in [34] it is important to notice that the above definition of freeness for Lie pseudo-group actions is slightly different from the standard definition of freeness for Lie group actions, [13].

[3]Hats are added over the transformed jet coordinates to avoid confusion with the diffeomorphism jet coordinates $U^\alpha_A$, $A = (a^1, \ldots, a^k)$, $1 \leq a^l \leq m$.

**Theorem 3.4.** Suppose $\mathcal{G}^{(n)}$ acts freely on $\mathcal{V}^n \subset \mathrm{J}^n$ with its orbits forming a regular foliation, then an $n$-th order moving frame exists in the neighborhood of every $z^{(n)} \in \mathcal{V}^n$.

In light of Theorem 3.4, the domain of definition of a moving frame in Definition 3.1 should strictly speaking be restricted to set of regular jets $\mathcal{V}^n \subset \mathrm{J}^n$. An important result in the theory of equivariant moving frames is the *persistence of freeness* proved in [35, 36].

**Theorem 3.5.** For $n > 0$, if the pseudo-group $\mathcal{G}$ acts (locally) freely at $z^{(n)}$ then it acts (locally) freely at any $z^{(k)} \in \mathrm{J}^k$, $k > n$, with $\pi^k_n(z^{(k)}) = z^{(n)}$.

A pseudo-group $\mathcal{G}$ is said to act *eventually freely* if, for some $n > 0$, it acts freely on an open subset $\mathcal{V}^n \subset \mathrm{J}^n$, and hence on the open subsets $\mathcal{V}^k = (\pi^k_n)^{-1}\mathcal{V}^n \subset \mathrm{J}^k$ for $k \geq n$. The minimal such $n$ is called the *order of freeness*, and denoted by $n_\star$.

**Definition 3.6.** A submanifold $S \subset M$ is said to be *regular* if there exists $n \geq n_\star$ such that $\mathrm{j}_n S|_z \in \mathcal{V}^n$ for all $z \in S$.

A moving frame is constructed through a normalization procedure based on a choice of cross-section $\mathcal{K}^n \subset \mathcal{V}^n$ to the pseudo-group orbits, that is, a transversal submanifold of the complementary dimension. Assuming freeness, the associated (locally defined) right moving frame section $\rho^{(n)}\colon \mathcal{V}^n \to \mathcal{E}^{(n)}$ is uniquely characterized by the condition that $\boldsymbol{\tau}(\rho^{(n)}(z^{(n)})) \in \mathcal{K}^n$. Let $z_{i_1}, \ldots, z_{i_{r_n}}$ be the $r_n = \dim \mathcal{G}^{(n)}$ submanifold jet components determining the cross-section $\mathcal{K}^n$. Then the moving frame $\rho^{(n)}$ is obtained by solving the *normalization equations*

$$Z_{i_1}(x, u^{(n)}, g^{(n)}) = c_1, \qquad \ldots \qquad Z_{i_{r_n}}(x, u^{(n)}, g^{(n)}) = c_{r_n}, \tag{3.6}$$

for the pseudo-group parameters $g^{(n)} = \widetilde{\rho}^{(n)}(x, u^{(n)})$. The invariants appearing on the left-hand side of the normalization equations (3.6) are called *phantom invariants.*

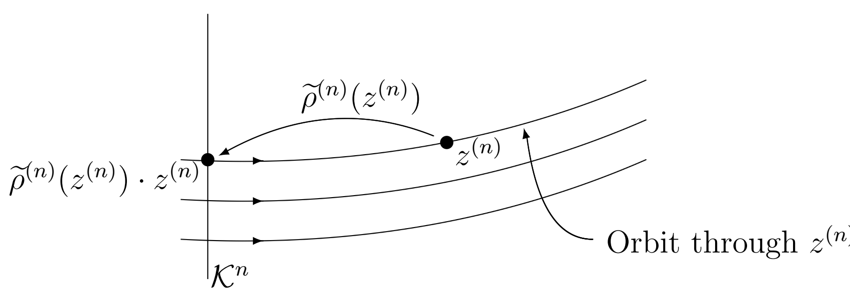


Figure 1: $n$-th order (right) moving frame $\rho^{(n)}(z^{(n)}) = (z^{(n)}, \widetilde{\rho}^{(n)}(z^{(n)}))$.

For each order above the order of freenes, a new cross-section must be selected. We require compatibility of the cross-sections, in the sense that $\pi^k_n(\mathcal{K}^k) = \mathcal{K}^n$ for all $k \geq n \geq n_\star$, which implies compatibility of the resulting moving frames: $\pi^k_n(\rho^{(n)}(z^{(k)})) = \rho^{(n)}(\pi^k_n(z^{(k)}))$. A compatible sequence of moving frames is simply called a moving frame and denoted by $\rho\colon \mathcal{V}^\infty \to \mathcal{E}^{(\infty)}$.

**Example 3.7.** In this example we construct a moving frame for the equivalence pseudo-group of first order scalar variational problems introduced in Example 2.9. The independent variables for this problem are $(x, u, p)$ and the Lagrangian $l = l(x, u, p)$ is the dependent variable. To obtain the prolonged action of (2.17) we first compute the lifted total differential operators (3.3). Since the lifted horizontal coframe corresponding to the pseudo-group action (2.17) is

$$d_H X = \phi_x dx, \quad d_H U = \beta_x dx + \beta_u du, \quad d_H P = \frac{\psi_x - \gamma\psi}{\phi_x} dx + \frac{\psi_u}{\phi_x} du + \frac{\beta_u}{\phi_x} dp, \tag{3.7}$$

where $\gamma(x) = \phi_{xx}/\phi_x$, and $\psi(x, u, p) = p\beta_u + \beta_x$, the lifted total differential operators are

$$D_P = \frac{\phi_x}{\beta_u} D_p, \quad D_U = \frac{1}{\beta_u}\left[D_u - \frac{\psi_u}{\phi_x} D_P\right], \quad D_X = \frac{1}{\phi_x}\left[D_x - \beta_x D_U - \frac{\psi_x - \psi\gamma}{\phi_x} D_P\right]. \tag{3.8}$$

Iterated applications of the lifted total differential operators (3.8) to the invariant $L = l/\phi_x$ yields the higher order lifted invariants

$$\begin{aligned}
L_P =& \frac{l_p}{\beta_u}, \qquad L_U = \frac{l_u - \psi_u L_P}{\beta_u \phi_x}, \\
L_X =& \frac{1}{\phi_x^2}\left[-\phi_{xx} L + l_x - \frac{\beta_x}{\beta_u}(l_u - \psi_u L_P) - \frac{\psi_x - \psi\gamma}{\beta_u} l_p\right], \\
L_{PP} =& \frac{\phi_x}{\beta_u^2} l_{pp}, \qquad L_{PU} = \frac{1}{\beta_u}\left[-\frac{\beta_{uu}}{\beta_u} L_P + \frac{l_{pu}}{\beta_u} - \frac{\psi_u}{\phi_x} L_{PP}\right], \\
L_{PX} =& \frac{1}{\phi_x}\left[-\frac{\beta_{ux}}{\beta_u} L_P + \frac{l_{px}}{\beta_u} - \beta_x L_{PU} - \frac{\psi_x - \psi\gamma}{\phi_x} L_{PP}\right], \\
L_{UU} =& \frac{1}{\beta_u}\left[-\frac{\beta_{uu}}{\beta_u} L_U + \frac{l_{uu} - \psi_{uu} L_P}{\beta_u \phi_x} - \psi_u \left(\frac{l_{pu} - \beta_{uu} L_P}{\beta_u^2 \phi_x}\right) - \frac{\psi_u}{\phi_x} L_{PU}\right], \\
L_{UX} =& \frac{1}{\phi_x}\left[-\left(\frac{\beta_{ux}}{\beta_u} + \gamma\right) L_U + \frac{l_{ux} - \psi_{ux} L_P}{\beta_u \phi_x} - \psi_u \left(\frac{l_{px} - \beta_{ux} L_P}{\beta_u^2 \phi_x}\right) - \beta_x L_{UU} \right. \\
& \left. - \frac{\psi_x - \psi\gamma}{\phi_x} L_{PU}\right], \\
L_{XX} =& \frac{1}{\phi_x}\left[D_x(L_X) - \beta_x L_{UX} - \frac{\psi_x - \psi\gamma}{\phi_x} L_{PX}\right], \\
L_{PPP} =& \frac{\phi_x^2}{\beta_u^3} l_{ppp}, \qquad L_{PPU} = \frac{1}{\beta_u}\left[-2\frac{\beta_{uu}}{\beta_u} L_{PP} + \frac{\phi_x}{\beta_u^2} l_{ppu} - \frac{\psi_u}{\phi_x} L_{PPP}\right], \\
L_{PPX} =& \frac{1}{\phi_x}\left[-2\frac{\beta_{ux}}{\beta_u} L_{PP} + \frac{1}{\beta_u^2}(\phi_{xx} l_{pp} + \phi_x l_{ppx}) - \beta_x L_{PPU} - \frac{\psi_x - \psi\gamma}{\phi_x} L_{PPP}\right], \\
L_{PUU} =& \frac{1}{\beta_u}\left[-\frac{\beta_{uu}}{\beta_u} L_{PU} + \frac{1}{\beta_u}\left(\left(2\frac{\beta_{uu}^2}{\beta_u^3} - \frac{\beta_{uuu}}{\beta_u^2}\right) l_p - 2\frac{\beta_{uu}}{\beta_u^2} l_{pu} + \frac{l_{puu}}{\beta_u} - \frac{\psi_u}{\beta_u^2} l_{ppu} \right.\right. \\
& \left.\left. + \left(2\frac{\psi_u \beta_{uu}}{\beta_u^3} - \frac{\psi_{uu}}{\beta_u^2}\right) l_{pp}\right) - \frac{\psi_u}{\phi_x} L_{PPU}\right], \\
L_{PUX} =& \frac{1}{\phi_x}\left[-\frac{\beta_{ux}}{\beta_u} L_{PU} + \frac{1}{\beta_u}\left(\left(2\frac{\beta_{uu}\beta_{ux}}{\beta_u^3} - \frac{\beta_{uux}}{\beta_u^2}\right) l_p - \frac{\beta_{uu}}{\beta_u^2} l_{px} + \frac{l_{pux}}{\beta_u} - \frac{\beta_{ux}}{\beta_u^2} l_{pu} \right.\right. \\
& \left.\left. - \frac{\psi_u}{\beta_u^2} l_{ppx} + \left(2\frac{\psi_u \beta_{ux}}{\beta_u^3} - \frac{\psi_{ux}}{\beta_u^2}\right) l_{pp}\right) - \beta_x L_{PUU} - \frac{\psi_x - \psi\gamma}{\phi_x} L_{PPU}\right],
\end{aligned} \tag{3.9}$$

and so on. In the process of solving for the pseudo-group parameters we find that the action is free on

$$\mathcal{V}^\infty = \mathrm{J}^\infty \setminus \{l_p \equiv 0,\ l_{pp} \equiv 0\}. \tag{3.10}$$

From the coordinate expressions for the lifted invariants (3.9) we see that on the set of regular jets (3.10) the invariants $L_P$ and $L_{PP}$ do not vanish. Also, the lifted invariant $L$ is not zero as we assume $l \neq 0$. On (3.10) a cross-section to the pseudo-group action is given by

$$\begin{gathered} X = U = P = 0, \qquad L = L_P = 1, \\ L_{U^i X^j} = 0, \qquad i+j \geq 1, \qquad L_{PX^k} = L_{PU^k} = 0, \qquad k \geq 1. \end{gathered} \tag{3.11}$$

Solving the normalization equations (3.11) for the first few pseudo-group parameters we obtain

$$\begin{gathered} \phi = 0, \qquad \beta = 0, \qquad \phi_x = l, \qquad \beta_x = -p\, l_p, \qquad \beta_u = l_p, \\ \psi_u = l_u, \qquad \psi_x = -\frac{l_p}{l_{pp}} \widetilde{E}(l) - p\, l_u, \qquad \beta_{uu} = \frac{1}{l_p} \frac{D(l, l_p)}{D(p, u)}, \qquad \phi_{xx} = \widehat{D}(l), \\ \psi_{uu} = l_{uu} - \frac{l_u^2}{l_p^2} l_{pp}, \qquad \psi_{ux} = l_{ux} + \frac{l_u}{l_p} \widetilde{E}(l) + \frac{p l_u^2}{l_p^2} l_{pp}, \\ \beta_{uuu} = \frac{1}{l_p} \frac{D(l_{pu}, l)}{D(u, p)} + l_{uu} + \frac{l_u^2 l_{pp}^2}{l_p^3} - \frac{l_u l_p^2}{l} L_{PPU}, \end{gathered} \tag{3.12}$$

where

$$\widetilde{E}(l) = l_u - l_{xp} - p\, l_{up}$$

is the truncated Euler operator, obtained by omitting the second derivative term from the Euler operator

$$E(l) = l_u - l_{xp} - p\, l_{up} - q\, l_{pp}, \qquad (q = u_{xx}).$$

The operator

$$\widehat{D} = D_x + pD_u + \frac{\widetilde{E}(l)}{l_{pp}} D_p$$

is the adapted total derivative which coincides with the total derivative in $x$ when applied to solutions of the Euler–Lagrange equation $E(l) = 0$ and the dependent variable $u$ is considered to be a function of $x$. Finally,

$$\frac{D(l, l_p)}{D(p, u)} = D_p(l) \cdot D_u(l_p) - D_u(l) \cdot D_p(l_p) = l_p l_{up} - l_u l_{pp}$$

denotes the determinant of the total Jacobian matrix of $l$ and $l_p$ with respect to the variables $p$ and $u$. Note that the invariant $L_{PPU}$ appearing in the normalization of $\beta_{uuu}$ is a well-defined expression of the Lagrangian and its derivatives as $L_{PPU}$ depends on the normalized pseudo-group parameters (3.12) (Note $L_{PPU}$ does not depend on $\beta_{uuu}$). The normalized pseudo-group parameters (3.12) constitute part of the (right) moving frame associated to the cross-section (3.11). Further normalization of the pseudo-group parameters can easily be achieved using a symbolic software like Mathematica or Maple.

Once a moving frame is obtained it is possible to systematically invariantize differential functions, differential forms and differential operators. The space of differential forms on $\mathcal{E}^{(\infty)}$ splits into

$$\mathbf{\Omega}^* = \bigoplus_{k,l} \mathbf{\Omega}^{k,l} = \bigoplus_{i,j,l} \mathbf{\Omega}^{i,j,l},$$

where $l$ indicates the number of Maurer–Cartan forms (2.6), $k = i + j$ the number of *jet forms*, with $i$ indicating the number of horizontal forms $dx^i$, $1 \leq i \leq p$, and $j$ the number of basic contact forms

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i} dx^i, \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0, \tag{3.13}$$

on $\mathrm{J}^\infty$. Let

$$\mathbf{\Omega}^*_J = \bigoplus_k \mathbf{\Omega}^{k,0} = \bigoplus_{i,j} \mathbf{\Omega}^{i,j,0} \tag{3.14}$$

denote the subspace of jet forms consisting of those differential forms containing no Maurer–Cartan forms. Let $\pi_J \colon \mathbf{\Omega}^* \to \mathbf{\Omega}^*_J$ be the natural projection that takes a differential form $\Omega$ on $\mathcal{E}^{(\infty)}$ to its jet component $\pi_J(\Omega)$ obtained by annihilating all Maurer–Cartan forms in $\Omega$.

**Definition 3.8.** The *lift* of a differential form $\omega$ on $\mathrm{J}^\infty$ is the jet form

$$\Omega = \boldsymbol{\lambda}(\omega) = \pi_J[\boldsymbol{\tau}^*(\omega)]. \tag{3.15}$$

Since the pull-back $\boldsymbol{\tau}^*(\omega)$ of a differential form $\omega$ on $\mathrm{J}^\infty$ is invariant and the lifted action (3.5) sends jet forms to jet forms, the lift $\boldsymbol{\lambda}(\omega)$ is an invariant jet form defined on $\mathcal{E}^{(\infty)}$. In particular, the lift of the submanifold jet coordinates $z^{(n)}$ coincides with the prolonged action (3.4), i.e. $\boldsymbol{\lambda}(z^{(n)}) = Z^{(n)}$. Also we note that the lifts

$$\Omega^i = \boldsymbol{\lambda}(dx^i) = \sum_{i=1}^{p} (D_{x^j} X^i) dx^j + \sum_{\alpha=1}^{q} X^i_{u^\alpha} \theta^\alpha = d_H X^i + \sum_{\alpha=1}^{q} X^i_{u^\alpha} \theta^\alpha, \tag{3.16}$$

of the horizontal forms $dx^i$ are invariant horizontal forms on $\mathcal{E}^{(\infty)}$ if and only if the pseudo-group action is projectable, that is $X^i_{u^\alpha} = 0$ for $i = 1, \ldots, p$ and $\alpha = 1, \ldots, q$. On the other hand the lift of a contact form $\Theta^\alpha_J = \boldsymbol{\lambda}(\theta^\alpha_J)$ is always a contact form.

**Definition 3.9.** Let $\rho \colon \mathcal{V}^\infty \to \mathcal{E}^{(\infty)}$ be a moving frame. If $\omega$ is a differential form on $\mathcal{V}^\infty$, then its *invariantization* is the invariant differential form

$$\iota(\Omega) = \rho^*[\boldsymbol{\lambda}(\omega)]. \tag{3.17}$$

In particular, the invariantization map (3.17) can be applied to the submanifold jet coordinates $(x, u^{(\infty)})$.

**Proposition 3.10.** Let $\rho \colon \mathcal{V}^\infty \to \mathcal{E}^{(\infty)}$ be a moving frame, then the invariants

$$(H, I^{(n)}) = \iota(x, u^{(n)}), \tag{3.18}$$

called *normalized differential invariants*, form a complete set of functionally independent differential invariants of order less or equal to $n$.

In the following we use the notation

$$\varpi^i = \rho^*(\Omega^i) = \iota(dx^i), \qquad \vartheta^\alpha_J = \rho^*(\Theta^\alpha_J) = \iota(\theta^\alpha_J)$$

to denote the invariantization of the jet forms $dx^i$, $\theta^\alpha_J$. If the pseudo-group acts non-projectably the "invariant horizontal forms"

$$\varpi^i = \omega^i + \eta^i, \qquad i = 1, \ldots, p, \tag{3.19}$$

are not purely horizontal in the usual bi-grade $\boldsymbol{\Omega}^{i,j}(\mathrm{J}^\infty)$ by virtue of (3.16). The differential forms (3.19) are in fact the sum of a contact invariant horizontal one-form $\omega^i$, [27], with a contact correction one-form $\eta^i$ making $\varpi^i$ invariant. The local coordinate expressions of the contact invariant one-forms $\omega^i$ are obtained by taking the pull-back of the lifted horizontal forms (3.1) with respect to the moving frame $\rho$:

$$\omega^i = \rho^*(d_H X^i) = \sum_{j=1}^{p} \rho^*(D_{x^j} X^i) dx^j, \qquad i = 1, \dots, p.$$

The contact invariant one-forms $\omega^i$ serve to define the invariant total differential operators $\mathcal{D}_i$ via the identity

$$d_H F(z^{(n)}) = \sum_{i=1}^{p} (D_{x^i} F) dx^i = \sum_{i=1}^{p} (\mathcal{D}_i F) \omega^i.$$

The local coordinate expressions of the invariant differential operators $\mathcal{D}_i$ are obtained from (3.3) by taking the pull-backs of $W_i^j$ with respect to the moving frame $\rho$:

$$\mathcal{D}_i = \sum_{j=1}^{p} \rho^*(W_i^j) D_{x^j}, \qquad \text{where} \qquad (W_i^j) = (D_{x^i} X^j)^{-1}.$$

One of the most important results in the theory of equivariant moving frames is the *recurrence formula* for lifted differential forms, [34]. This formula requires to extend the lift map $\boldsymbol{\lambda}$ to vector field jet coordinates.

**Definition 3.11.** The lift of a vector jet coordinate $\zeta_A^b$ is defined to be the Maurer–Cartan form $\mu_A^b$:

$$\boldsymbol{\lambda}(\zeta_A^b) = \mu_A^b, \qquad \text{for} \qquad b = 1, \dots, m, \qquad \#A \geq 0.$$

More generally, the lift of any finite linear combination of vector field jet coordinates

$$\sum_{b=1}^{m} \sum_{\#A \geq 0} P_b^A(z^{(n)}) \zeta_A^b$$

is defined to be the invariant group one-form

$$\boldsymbol{\lambda}\left[\sum_{b=1}^{m} \sum_{\#A \geq 0} P_b^A(z^{(n)}) \zeta_A^b\right] = \sum_{b=1}^{m} \sum_{\#A \geq 0} P_b^A(Z^{(n)}) \mu_A^b.$$

**Theorem 3.12.** Let $\omega$ be a differential form on $\mathrm{J}^\infty$. Then

$$d[\boldsymbol{\lambda}(\omega)] = \boldsymbol{\lambda}[d\omega + \mathbf{v}^{(\infty)}(\omega)], \tag{3.20}$$

where $\mathbf{v}^{(\infty)}$ is the prolongation of the vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x,u) \frac{\partial}{\partial u^\alpha} \;\in\; \mathfrak{g} \tag{3.21}$$

given by

$$\mathbf{v}^{(\infty)} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_{k=\#J\geq 0} \phi^{\alpha;J}(x,u^{(k)})\frac{\partial}{\partial u^\alpha_J} \;\in\; \mathfrak{g}^{(\infty)}, \tag{3.22}$$

with $\phi^{\alpha;J}$ defined recursively by the prolongation formula, [26],

$$\phi^{\alpha;J,j} = D_{x^j}\phi^{\alpha;J} - \sum_{i=1}^{p}(D_{x^j}\xi^i)u^\alpha_{J,i}. \tag{3.23}$$

In particular, the identity (3.20) applies to the lifted differential invariants $X^i$, $\widehat{U}^\alpha_J$ giving

$$\begin{aligned} dX^i =&\Omega^i + \mu^i, && i = 1,\dots,p,\\ d\widehat{U}^\alpha_J =&\sum_{j=1}^{p}\widehat{U}^\alpha_{J,j}\Omega^j + \Theta^\alpha_J + \widehat{\phi}^{\alpha;J}, && \alpha = 1,\dots,q, \quad \#J\geq 0, \end{aligned} \tag{3.24}$$

where $\Omega^i = \boldsymbol{\lambda}(dx^i)$, $\Theta^\alpha_J = \boldsymbol{\lambda}(\theta^\alpha_J)$ are the lifts of the jet forms, and $\widehat{\phi}^{\alpha;J} = \boldsymbol{\lambda}(\phi^{\alpha;J})$ are *correction terms* obtained by lifting the prolonged vector field coefficients (3.23).

**Corollary 3.13.** Let $\rho\colon \mathcal{V}^\infty \to \mathcal{E}^{(\infty)}$ be a moving frame and $\omega$ a differential form on $\mathcal{V}^\infty$, then

$$d\iota(\omega) = \iota[d\omega + \mathbf{v}^{(\infty)}(\omega)]. \tag{3.25}$$

Of particular interest to us is when $\omega$ is one of the submanifold jet coordinate functions $x^i$, $u^\alpha_J$. We introduce the notation $\nu^{(n)} = \rho^*(\mu^{(n)})$ to denote the pull-back of the Maurer–Cartan forms $\mu^{(n)}$ via the moving frame $\rho$. Also recall our notation convention: $\varpi^i = \iota(dx^i)$, $\vartheta^\alpha_J = \iota(\theta^\alpha_J)$ and $H^i = \iota(x^i)$, $I^\alpha_J = \iota(u^\alpha_J)$. Then the identity (3.25) applied to $x^i$ and $u^\alpha_J$ yields the recurrence formulas

$$\begin{aligned} dH^i =&\varpi^i + \nu^i, && i = 1,\dots,p,\\ dI^\alpha_J =&\sum_{i=1}^{p} I^\alpha_{J,i}\varpi^i + \vartheta^\alpha_J + \widehat{\psi}^{\alpha;J}, && \alpha = 1,\dots,q, \quad \#J\geq 0, \end{aligned} \tag{3.26}$$

where the correction terms are obtained by invariantizating the prolonged vector field coefficients (3.22)

$$\nu^i = \rho^*(\mu^i) = \iota(\xi^i), \qquad \widehat{\psi}^{\alpha;J}(H, I^{(n)}, \nu^{(n)}) = \rho^*(\widehat{\phi}^{\alpha;J}) = \iota(\phi^{\alpha;J}), \qquad \#J = n.$$

The differential in (3.26) splits into invariant horizontal and vertical components. Let $\pi_{\mathcal{H}}$ denote the invariant horizontal projection onto the differential forms $\{\varpi^i\}$ and $\pi_{\mathcal{V}}$ the projection onto the invariant vertical (contact) differential forms $\{\vartheta^\alpha_J\}$. Since

$$d_{\mathcal{H}}F(z^{(n)}) = \pi_{\mathcal{H}}\circ dF = \sum_{i=1}^{p}(\mathcal{D}_i F)\varpi^i$$

for any differential function $F(z^{(n)})$, the recurrence relations (3.26) yield the identities

$$\mathcal{D}_j H^i = \sum_{j=1}^{p}\delta^i_j + \pi_{\mathcal{H}}(\nu^i), \qquad \mathcal{D}_j I^\alpha_J = I^\alpha_{J,j} + M^\alpha_{J,j}, \tag{3.27a}$$

$$d_{\mathcal{V}}H^i = \pi_{\mathcal{V}}\circ dH^i = \pi_{\mathcal{V}}(\nu^i), \qquad d_{\mathcal{V}}I^\alpha_J = \pi_{\mathcal{V}}\circ dI^\alpha_J = \vartheta^\alpha_J + \pi_{\mathcal{V}}(\widehat{\psi}^{\alpha;J}), \tag{3.27b}$$

where $M^\alpha_{J,j} = M^\alpha_{J,j}(H, I^{(n)})$ is the $\varpi^j$ component of $\pi_\mathcal{H}(\widehat{\psi}^{\alpha;J})$. The equalities (3.27a) give explicit relations between the normalized invariants and their invariants total derivatives. The commutation relations among the invariant total differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are also direct consequences of the universal recurrence relation (3.25). First replacing $\omega$ by the horizontal forms $dx^i$ in (3.25) we obtain the equations

$$d_\mathcal{H}\varpi^i = \pi_\mathcal{H} \circ d\varpi^i = -\sum_{1\le j<k\le p} T^i_{jk}\varpi^j \wedge \varpi^k, \qquad i = 1, \ldots, p.$$

Since the operators $\mathcal{D}_i$ are dual to the invariant horizontal forms $\varpi^i$ it follows that

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^{p} T^k_{ij}\mathcal{D}_k, \qquad 1 \le i, j \le p. \tag{3.28}$$

The functions $T^i_{jk}$ are differential invariants of the pseudo-group action and are known as the *commutator invariants*, [35]. The recurrence relations (3.27a) together with the commutation relations (3.28) contain all the information on the structure of the algebra of differential invariants of a pseudo-group action, [9, 34, 35, 46].

An important feature of the recurrence formula (3.25) (or (3.26)) is that the coordinate expressions for the differential invariants $(H, I^{(\infty)})$, the invariant differential forms $\varpi^i$, $\vartheta^\alpha_J$, the Maurer–Cartan forms $\mu^b_A$ and the moving frame $\rho$ are not required for the equations to be used. The only pieces of information required are the cross-section $\mathcal{K}^\infty$ and the infinitesimal generators $\mathbf{v}^{(\infty)}$ of the Lie pseudo-group $\mathcal{G}$. The key observation is that the unknown differential forms $\nu^{(n)} = \rho^*(\mu^{(n)})$ can be obtained directly from the recurrence relations of the phantom invariants. By construction, the invariantization of the jet coordinates defining the normalization equations (3.6) (which defines the cross-section $\mathcal{K}^n$) are constants, i.e. $\iota(z_{i_1}) = c_1, \ldots, \iota(z_{i_{r_n}}) = c_{r_n}$. Thus left-hand side of the recurrence relations (3.26) for the phantom invariants are zero and form a system of equations for the pulled-back Maurer–Cartan forms $\nu^{(n)}$. The freeness assumption on the pseudo-group action guarantees that the system as a unique solution in terms of the invariant horizontal forms $\varpi^i$ and the invariant contact forms $\vartheta^\alpha_J$. With the recurrence relations (3.27a) at hand and the commutation relations (3.28) it is now possible to study the structure of the algebra of differential invariants.

### 3.1.1 Algebra of Differential Invariants for Regular Submanifold Jets

**Definition 3.14.** A set of differential invariants $\mathcal{I} = \{I_\kappa\}$ is said to be a *generating set* for the algebra of differential invariants if every invariant can be locally expressed as a function of the invariants $I_\kappa \in \mathcal{I}$ and their invariant derivatives $\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1}\mathcal{D}_{j_2}\cdots\mathcal{D}_{j_k} I_\kappa$.

The *Basis Theorem* for differential invariants, first formulated by Lie [24], and extended by Tresse, [43], to infinite-dimensional Lie pseudo-group actions, states that the generating set $\mathcal{I}$ is finite. For Lie group actions, proofs based on the equivariant moving frame method can be found in [13, 17]. For infinite-dimensional Lie pseudo-groups, a proof based to the equivariant moving frame method also exits, [35], but it is much more subtle. The proof requires the introduction of two important modules associated with the prolonged pseudo-group action. We now briefly review the constructions appearing in [35] as in the next section we will explain how those constructions extend to partial moving frames.

Let $\mathbb{R}[t,T]$ be the algebra of real polynomials in the variables $t = (t_1, \ldots, t_m)$, $T = (T^1, \ldots, T^m)$ (recall that $m$ is the dimension of the manifold $M$ on which a Lie pseudo-group $\mathcal{G}$ acts) and let

$$\mathcal{T} = \left\{ \eta(t,T) = \sum_{a=1}^{m} \eta_a(t) T^a \right\} \simeq \mathbb{R}[t] \otimes \mathbb{R}^m \subset \mathbb{R}[t,T]$$

be the $\mathbb{R}[t]$ module consisting of homogeneous linear polynomials in the variable $T$. Let $\mathcal{T}^n \subset \mathcal{T}$ be the subspace of homogeneous polynomials of degree $n$ in $t$. Then there is a natural grading on $\mathcal{T} = \oplus_{n \geq 0} \mathcal{T}^n$. The notations $\mathcal{T}^{\leq n} = \oplus_{k=0}^{n} \mathcal{T}^k$ and $\mathcal{T}^{\geq n} = \oplus_{k=n}^{\infty} \mathcal{T}^k$ are used to denote the space of polynomials of degree $\leq n$ and $\geq n$ respectively. Let $\mathbf{H} : \mathcal{T} \to \mathcal{T}$ be the *highest order terms* operator such that for $0 \neq \eta \in \mathcal{T}^{\leq n}$ the equality $\eta = \mathbf{H}(\eta) + \lambda$ holds, where $0 \neq \mathbf{H}(\eta) \in \mathcal{T}^n$ and $\lambda \in \mathcal{T}^{\leq n-1}$. Locally, $(\mathrm{J}^\infty TM)^* \simeq M \times \mathcal{T}$ via the pairing $\langle \mathrm{j}_\infty \mathbf{v}; t_A T^b \rangle = \zeta_A^b$. Under this isomorphism the infinitesimal linear determining equations (2.14) are identified to the polynomials

$$\eta_\upsilon(z;t,T) = \sum_{b=1}^{m} \sum_{\# A \leq n} h^A_{b;\upsilon}(z) t_A T^b, \qquad \upsilon = 1, \ldots, k. \tag{3.29}$$

**Definition 3.15.** The *symbol* $\mathbf{\Sigma}(L^{(n)})$ of the linear differential equations (2.14) consists of the highest order terms of its defining polynomial (3.29):

$$\mathbf{\Sigma}[L^{(n)}(z,\zeta^{(n)})] = \left\{ \mathbf{H}[\eta_\upsilon(z;t,T)] = \sum_{b=1}^{m} \sum_{\# A = n} h^A_{b;\upsilon}(z) t_A T^b : \ \upsilon = 1, \ldots, k \right\}.$$

Let $\mathcal{L} = (\mathrm{J}^\infty \mathfrak{g})^\perp \subset (\mathrm{J}^\infty TM)^*$ denote the *annihilator subbundle* of the infinitesimal generator jet bundle. Let

$$\mathcal{I} = \mathbf{H}(\mathcal{L}) \tag{3.30}$$

be the span of the highest order terms of the annihilating polynomials at each $z \in M$. At the symbol level, total differentiation with respect to the operators (2.2) corresponds to multiplication:

$$\mathbf{H}(\mathbb{D}_{z^a} L^{(n)}) = t_a \mathbf{H}(L^{(n)}), \qquad a = 1, \ldots, m. \tag{3.31}$$

Since formal integrability requires that the linear determining system (2.14) be closed under the application of the total derivative operators (2.2), it follows from (3.31) that at each point $z \in M$, the fiber $\mathcal{I}|_z$ forms a graded submodule of $\mathcal{T}$. This submodule is known as the *symbol module* of the pseudo-group at the point $z$.

We now introduce the *prolonged symbol* algebra for the prolonged infinitesimal generators (3.23) of a pseudo-group action. Let $s = (s_1 \ldots, s_p)$, $S = (S^1, \ldots, S^q)$ and consider the $\mathbb{R}[s]$ module

$$\widehat{\mathcal{S}} = \left\{ \widehat{\sigma}(s,S) = \sum_{\alpha=1}^{q} \widehat{\sigma}_\alpha(s) S^\alpha \right\} \simeq \mathbb{R}[s] \otimes \mathbb{R}^q.$$

of polynomials that are linear in $S$. Let

$$\mathcal{S} = \mathbb{R}^p \oplus \widehat{\mathcal{S}} = \sum_{n=-1}^{\infty} \mathcal{S}^n,$$

where

$$\mathcal{S}^{-1} = \{c \cdot \widetilde{s} = c_1 \widetilde{s}_1 + \cdots + c_p \widetilde{s}_p\} \simeq \mathbb{R}^p,$$

and $\widetilde{s} = (\widetilde{s}_1, \ldots, \widetilde{s}_p) \in \mathbb{R}^p$ are extra variables. The space $\mathcal{S}$ is endowed with the structure of an $\mathbb{R}[s]$ module by taking the usual module structure on $\widehat{\mathcal{S}}$ and setting

$$\tau(s)\widetilde{s}_i = \tau(0)\widetilde{s}_i \qquad \text{for any polynomial} \qquad \tau(s) \in \mathbb{R}[s].$$

The *highest order term map* $\mathbf{H} : \mathcal{S} \to \mathcal{S}$ is defined so that

$$\mathbf{H}[\sigma(\widetilde{s}, s, S)] = \mathbf{H}[\widehat{\sigma}(s, S)], \qquad \text{where} \qquad \sigma(\widetilde{s}, s, S) = c \cdot \widetilde{s} + \widehat{\sigma}(s, S).$$

The cotangent bundle $T^*\mathrm{J}^\infty$ is identified with $\mathrm{J}^\infty \times \mathcal{S}$ via the pairing

$$\langle \mathbf{V}; \widetilde{s}_i \rangle = \xi^i, \qquad \langle \mathbf{V}; S^\alpha \rangle = Q^\alpha = \phi^\alpha - \sum_{i=1}^p u_i^\alpha \xi^i,$$
$$\langle\, s_J S^\alpha \rangle = \phi^{\alpha;J}, \qquad \text{for} \quad n = \#J \geq 1,$$

whenever

$$\mathbf{V} = \sum_{i=1}^p \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_{\#J \geq 0} \phi^{\alpha;J} \frac{\partial}{\partial u_J^\alpha} \quad \in \quad T\mathrm{J}^\infty|_{z^{(\infty)}}. \tag{3.32}$$

Note that the coefficients of the vector field (3.32) are arbitrary and do not have to be the coefficients of the prolonged vector field (3.22). Given $z^{(n)} \in \mathrm{J}^n|_z$, the prolongation formula (3.22) defines the *prolongation map*

$$\mathbf{p}^{(n)} = \mathbf{p}^{(n)}_{z^{(n)}} : \mathrm{J}^n TM|_z \to T\mathrm{J}^n|_{z^{(n)}}, \qquad \mathbf{p}^{(n)}(\mathrm{j}_n \mathbf{v}|_z) = \mathbf{v}^{(n)}|_{z^{(n)}}.$$

In the projective limit, let $\mathbf{p} = \mathbf{p}^{(\infty)} : \mathrm{J}^\infty TM|_z \to T\mathrm{J}^\infty|_{z^{(\infty)}}$. Fix $z^{(\infty)} \in \mathrm{J}^\infty$ with $\pi_0^\infty(z^{(\infty)}) = z$, then the projection map $\mathbf{p}$ induces the *dual prolongation map* $\mathbf{p}^* : \mathcal{S} \to \mathcal{T}$ defined by

$$\langle \mathrm{j}_\infty \mathbf{v}; \mathbf{p}^*(\sigma) \rangle = \langle \mathbf{p}(\mathrm{j}_\infty \mathbf{v}); \sigma \rangle = \langle \mathbf{v}^{(\infty)}; \sigma \rangle \quad \text{for all} \quad \mathrm{j}_\infty \mathbf{v} \in \mathrm{J}^\infty TM|_z, \quad \sigma \in \mathcal{S}.$$

Next consider the particular polynomials

$$\begin{aligned} \beta_i(t) =& t_i + \sum_{\alpha=1}^q u_i^\alpha t_{p+\alpha}, \qquad & i = 1, \ldots, p, \\ B^\alpha(T) =& T^{p+\alpha} - \sum_{i=1}^p u_i^\alpha T^i, \qquad & \alpha = 1, \ldots, q. \end{aligned} \tag{3.33}$$

Geometrically, the polynomial $B^\alpha(T)$ is the symbol of $Q^\alpha$, the $\alpha$-th component of the characteristic of $\mathbf{v}$ while $\beta_i(t)$ represents the symbol of the $i$-th total derivative operator (3.2):

$$\mathbf{\Sigma}(D_{x^i} L^{(n)}) = \beta_i(t) \mathbf{\Sigma}(L^{(n)}),$$

where $L^{(n)}$ is the linear differential equations (2.14). For fixed first order jet coordinates $u_i^\alpha$, the functions (3.33) define the linear map

$$\boldsymbol{\beta} : \mathbb{R}^{2m} \to \mathbb{R}^m, \quad \text{given by} \quad s_i = \beta_i(t), \quad S^\alpha = B^\alpha(T). \tag{3.34}$$

Then for $\widehat{\sigma} \in \widehat{\mathcal{S}} \subset \mathcal{S}$ we have the equality

$$\mathbf{H}[\mathbf{p}^*(\widehat{\sigma})] = \boldsymbol{\beta}^*[\mathbf{H}(\widehat{\sigma})]. \tag{3.35}$$

Consider the *prolonged annihilator subbundle*

$$\mathcal{Z} = (\mathfrak{g}^{(\infty)})^{\perp} = (\mathbf{p}^*)^{-1}\mathcal{L} \subset \mathcal{S}, \tag{3.36}$$

and define the subspace

$$\mathcal{U} = \mathbf{H}(\mathcal{Z}) \subset \mathcal{S}$$

corresponding to span of the highest order terms of the prolonged annihilators. In general $\mathcal{U}$ is not a submodule.

**Definition 3.16.** The *prolonged symbol submodule* is defined as the inverse image of the symbol module (3.30) under the polynomial pull-back morphism (3.34):

$$\mathcal{J} = (\boldsymbol{\beta}^*)^{-1}(\mathcal{I}) = \{\widehat{\sigma}(s, S) : \boldsymbol{\beta}^*(\widehat{\sigma})(s, S) = \widehat{\sigma}(\beta(t), B(T)) \in \mathcal{I}\} \subset \widehat{S}. \tag{3.37}$$

**Proposition 3.17.** A pseudo-group action $\mathcal{G}$ acts locally freely at a submanifold jet $z^{(n)}$ if and only if the prolongation map $\mathbf{p}^{(n)} : \mathrm{J}^n\mathfrak{g}|_z \to \mathfrak{g}^{(n)}|_{z^{(n)}}$ is a linear isomorphism or equivalently

$$\mathbf{p}^*(\mathcal{S}^{\leq n}) + \mathcal{L}^{\leq n}|_z = \mathcal{T}^{\leq n}.$$

From (3.35) and (3.36) the containment $\mathcal{U} \subset \mathcal{J}$ always holds. When the action is locally free the containment becomes an equality and brings algebraic structure into the problem.

**Lemma 3.18.** Suppose $\mathcal{G}$ acts locally freely at $z^{(n)} \in \mathrm{J}^n$, then $\mathcal{U}^k|_{z^{(k)}} = \mathcal{J}^k|_{z^{(k)}}$ for all $k > n$ and all $z^{(k)} \in \mathrm{J}^k$ with $\pi^k_n(z^{(k)}) = z^{(n)}$.

The equality $\mathcal{U}^k|_{z^{(k)}} = \mathcal{J}^k|_{z^{(k)}}$ is key to proving the Basis Theorem, [35]. Since $\mathcal{J}^{>n^\star} = \mathcal{J} \cap \widehat{\mathcal{S}}^{>n^\star}$ is a polynomial ideal it has a *Gröbner basis*, [10]. For completeness, we now recall the definition of a Gröbner basis in our particular framework. First, given a degree compatible ordering on $\widehat{S}$, let $\mathrm{lt}(\sigma)$ be the *leading term* of $\sigma$. For an ideal $I \subset \widehat{S}$, $\mathrm{lt}(I)$ be the set of leading terms of elements of $I$. Then, given a subset $E \subset \widehat{S}$ finite- or infinite-dimensional we denote the monomial ideal generated by $E$ (in $\widehat{S}$) by $\langle E \rangle$.

**Definition 3.19.** For a fix degree compatible ordering, a finite number of elements $\{\widehat{\sigma}_1, \ldots, \widehat{\sigma}_l\}$ in $\mathcal{J}^{>n^\star}$ is said to be a Gröbner basis of $\mathcal{J}^{>n^\star}$ if

$$\langle \mathrm{lt}(\widehat{\sigma}_1), \ldots, \mathrm{lt}(\widehat{\sigma}_l) \rangle = \langle \mathrm{lt}(\mathcal{J}^{>n^\star}) \rangle.$$

**Proposition 3.20.** Given a degree compatible ordering on $\widehat{S}$, the ideal $\mathcal{J}^{>n^\star}$ has a Gröbner basis $\{\widehat{\sigma}_1, \ldots, \widehat{\sigma}_l\}$ and this Gröbner basis is a basis of $\mathcal{J}^{>n^\star}$.

Now given a moving frame, the invariantization map (3.17) is used to invariantize the preceding algebraic constructions. Let

$$\eta(x, u; t, T) = \sum_{b=1}^{m} \sum_{\#A \leq n} h^A_b(x, u) t_A T^b$$

be a section of the annihilator bundle $\mathcal{L}$, its invariantization is the polynomial

$$\widetilde{\eta}(H,I;t,T)=\sum_{b=1}^{m}\sum_{\#A\leq n}h_b^A(H,I)t_AT^b$$

obtained by replacing the coordinates on $M$ by their invariantizations. Similarly let

$$\widehat{\sigma}(x,u^{(k)};s,S)=\sum_{\alpha=1}^{q}\sum_{\#J\leq n}h_\alpha^J(x,u^{(k)})s_JS^\alpha \quad \in \widehat{\mathcal{S}}^{\leq n}$$

be a prolonged symbol polynomial, then its invariatization is the polynomial

$$\widetilde{\sigma}(H,I^{(k)},s,S)=\iota(\widehat{\sigma}(x,u^{(k)};s,S))=\sum_{\alpha=1}^{q}\sum_{\#J\leq n}h_\alpha^J(H,I^{(k)})s_JS^\alpha. \tag{3.38}$$

Let $\widetilde{\mathcal{L}}=\iota(\mathcal{L})$ denote the *invariantized annihilator bundle*, $\widetilde{\mathcal{I}}=\iota(\mathcal{I})$ the *invariantized symbol submodule* and $\widetilde{\mathcal{J}}=\iota(\mathcal{J})$ the *invariantized prolonged symbol module*. Identifying the polynomial (3.38) with the differential invariant

$$I_{\widetilde{\sigma}}=\sum_{\alpha=1}^{q}\sum_{\#J\geq 0}h_\alpha^J(H,I^{(k)})I_J^\alpha$$

we come to the following important result proved in [35].

**Theorem 3.21.** Let $\mathcal{G}$ be a Lie pseudo-group that acts freely on an open subset of the submanifold bundle at order $n^\star$. Then a finite generating system for its differential invariant algebra consists of

- the differential invariants $I_\nu=I_{\widetilde{\sigma}_\nu}$, where $\widetilde{\sigma}_1,\ldots,\widetilde{\sigma}_l$ form a Gröbner basis for the invariantized prolonged symbol submodule $\widetilde{\mathcal{J}}^{>n^\star}$, and, possibly,
- a finite number of additional differential invariants of order $\leq n^\star$.

**Example 3.22.** We continue Example 3.7. Given the moving frame (3.12) we can invariantize the submanifold jet coordinates to obtain a complete set of differential invariants. This is achieved by substituting the expressions for the normalized pseudo-group parameters (3.12) into the lifted invariants (3.9):

$$\begin{aligned}
I_{pp}=\iota(l_{pp})=\frac{ll_{pp}}{l_p^2},\qquad I_{ppp}=\iota(l_{ppp})=\frac{l^2l_{ppp}}{l_p^3},\\
I_{ppu}=\iota(l_{ppu})=\frac{l}{l_p^5}\left[-2l_{pp}\frac{D(l_u,l_p)}{D(p,u)}+l_p\frac{D(l_{pp},l)}{D(u,p)}\right],\\
I_{ppx}=\iota(l_{ppx})=\frac{1}{l}\left[-2\left(l_ul_p-p\frac{D(l,l_p)}{D(p,u)}\right)\frac{L_{PP}}{l_p^2}+\frac{1}{l_p^2}(\widehat{D}(l)l_{pp}+ll_{ppx})\right.\\
\left.+pl_pl_{ppu}+\left(\frac{l_p}{l_{pp}}\widetilde{E}(l)-pl_u\right)\frac{I_{ppp}}{l}\right].
\end{aligned} \tag{3.39}$$

The coordinate expression of the invariant $I_{pux}=\iota(l_{pux})$ is a little bit too long to write down but it is a simple exercise of substituting the expressions (3.12) into $L_{PUX}$ given in (3.9).

As the pseudo-group (2.17) acts projectably, the invariantization of the horizontal coframe $\{dx, du, dp\}$ is simply obtained by pulling-back the lifted horizontal coframe (3.7) via the moving frame (3.12):

$$\begin{aligned} \varpi^x &= \omega^x = \iota(dx) = l dx, \qquad \varpi^u = \omega^u = \iota(du) = l_p(du - pdx), \\ \varpi^p &= \omega^p = \iota(dp) = \frac{l_u}{l}(du - pdx) + \frac{l_p}{l}\left(dp - \frac{\widetilde{E}(l)}{l_{pp}}dx\right). \end{aligned} \tag{3.40}$$

The corresponding dual invariant total differential operators to (3.40) are

$$\mathcal{D}_p = \frac{l}{l_p}D_p, \qquad \mathcal{D}_u = \frac{1}{l_p^2}(l_p D_u - l_u D_p), \qquad \mathcal{D}_x = \frac{1}{l}\left(D_x + pD_u + \frac{\widetilde{E}(l)}{l_{pp}}D_p\right). \tag{3.41}$$

We can also systematically invariantize the basic contact forms (3.13). For example,

$$\begin{aligned} \vartheta =& \iota(\theta) = \iota(dl - l_x dx - l_u du - l_p dp) = \iota(dl) - \iota(dp) \\ =& \rho^*\left(\pi_J\left(d\left(\frac{l}{\phi_x}\right)\right)\right) - \varpi^p = \rho^*\left(\frac{\theta + l_x dx + l_u du + l_p dp}{\phi_x} - \frac{l\phi_{xx}}{\phi_x^2}dx\right) - \varpi^p = \frac{\theta}{l}. \end{aligned}$$

Similarly, the invariantization of the first order contact forms are

$$\begin{aligned} \vartheta_p = \iota(\theta_p) = \frac{\theta_p}{l_p}, \qquad \vartheta_u = \iota(\theta_u) = \frac{1}{l l_p}\left[\theta_u - \frac{l_u}{l_p}\theta_p\right], \\ \vartheta_x = \iota(\theta_x) = \frac{1}{l^2}\left[-\frac{\widehat{D}(l)}{l}\theta + \theta_x + p\theta_u + \frac{\widetilde{E}(l)}{l_{pp}}\theta_p\right]. \end{aligned}$$

We now show how to use the recurrence relations (3.24). First, to compute the correction terms $\widehat{\phi}^{\alpha;J}$ we need the coordinate expressions for the infinitesimal generators of the pseudo-group action (2.17). From the infinitesimal determining system (2.20), those are seen to be given by

$$\mathbf{v} = \xi(x)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + [p(\eta_u - \xi_x) + \eta_x]\frac{\partial}{\partial p} - l\xi_x\frac{\partial}{\partial l}, \tag{3.42}$$

where $\xi(x)$ and $\eta(x, u)$ are two arbitrary differentiable functions. Using the formula (3.23) to compute the prolongation of the vector field (3.42) the recurrence relations (3.24) reduce to

$$\begin{aligned} dX =& \Omega^x + \mu^x, \qquad dU = \Omega^u + \mu^u, \qquad dP = \Omega^p + P(\mu^u_U - \mu^x_X) + \mu^u_X, \\ dL =& L_X\Omega^x + L_U\Omega^u + L_P\Omega^p + \Theta - L\mu^x_X, \\ dL_P =& L_{PX}\Omega^x + L_{PU}\Omega^u + L_{PP}\Omega^p + \Theta_P - L_P\mu^u_U, \\ dL_U =& L_{UX}\Omega^x + L_{UU}\Omega^u + L_{PU}\Omega^p + \Theta_U - L_U(\mu^x_X + \mu^u_U) - L_P(P\mu^u_{UU} + \mu^u_{UX}), \\ dL_X =& L_{XX}\Omega^x + L_{UX}\Omega^u + L_{PX}\Omega^p + \Theta_X - 2L_X\mu^x_X - L\mu^x_{XX} - L_U\mu^u_X \\ & - L_P[P(\mu^u_{UX} - \mu^x_{XX}) + \mu^u_{XX}], \\ dL_{PP} =& L_{PPX}\Omega^x + L_{PPU}\Omega^u + L_{PPP}\Omega^p + \Theta_{PP} - L_{PP}(2\mu^u_U - \mu^x_X), \\ dL_{PU} =& L_{PUX}\Omega^x + L_{PUU}\Omega^u + L_{PPU}\Omega^p + \Theta_{PU} - L_P\mu^u_{UU} - 2L_{PU}\mu^u_U \\ & - L_{PP}(P\mu^u_{UU} + \mu^u_{UX}), \end{aligned}$$

$$
\begin{aligned}
dL_{PX} =& L_{PXX}\Omega^x + L_{PUX}\Omega^u + L_{PPX}\Omega^p + \Theta_{PX} - L_P\mu^u_{UX} - L_{PX}(\mu^u_U + \mu^x_X)\\
& - L_{PU}\mu^u_X - L_{PP}[P(\mu^u_{UX} - \mu^x_{XX}) + \mu^u_{XX}],\\
dL_{UU} =& L_{UUX}\Omega^x + L_{UUU}\Omega^u + L_{PUU}\Omega^p + \Theta_{UU} - L_U\mu^u_{UU} - L_{UU}(\mu^x_X + 2\mu^u_U)\\
& - L_P(P\mu^u_{UUU} + \mu^u_{UUX}) - 2L_{PU}(P\mu^u_{UU} + \mu^u_{UX}),\\
dL_{UX} =& L_{UXX}\Omega^x + L_{UUX}\Omega^u + L_{PUX}\Omega^p + \Theta_{UX} - L_U(\mu^x_{XX} + \mu^u_{UX})\\
& - L_{UX}(2\mu^x_X + \mu^u_U) - L_P(P\mu^u_{UUX} + \mu^u_{UXX}) - L_{PX}(P\mu^u_{UU} + \mu^u_{UX})\\
& - L_{UU}\mu^u_X - L_{PU}[P(\mu^u_{UX} - \mu^x_{XX}) + \mu^u_{XX}],\\
dL_{XX} =& L_{XXX}\Omega^x + L_{UXX}\Omega^u + L_{PXX}\Omega^p + \Theta_{XX} - 3L_X\mu^x_{XX} - L\mu^x_{XXX}\\
& - 3L_{XX}\mu^x_X - L_U\mu^u_{XX} - 2L_{UX}\mu^u_X - 2L_{PX}[P(\mu^u_{UX} - \mu^x_{XX}) + \mu^u_{XX}]\\
& - L_P[P(\mu^u_{UXX} - \mu^x_{XXX}) + \mu^u_{XXX}],\\
dL_{PPP} =& L_{PPPX}\Omega^x + L_{PPPU}\Omega^u + L_{PPPP}\Omega^p + \Theta_{PPP} - L_{PPP}(3\mu^u_U - 2\mu^x_X),\\
dL_{PPU} =& L_{PPUX}\Omega^x + L_{PPUU}\Omega^u + L_{PPPU}\Omega^p + \Theta_{PPU} - 2L_{PP}\mu^u_{UU} + L_{PPU}(\mu^x_X - 3\mu^u_U)\\
& - L_{PPP}(P\mu^u_{UU} + \mu^u_{UX}),\\
dL_{PPX} =& L_{PPXX}\Omega^x + L_{PPUX}\Omega^u + L_{PPPX}\Omega^p + \Theta_{PPX} + L_{PP}(\mu^x_{XX} - 2\mu^u_{UX})\\
& - 2L_{PPX}\mu^u_U - L_{PPU}\mu^u_X - L_{PPP}[P(\mu^u_{UX} - \mu^x_{XX}) + \mu^u_{XX}],\\
dL_{PUX} =& L_{PUXX}\Omega^x + L_{PUUX}\Omega^u + L_{PPUX}\Omega^p + \Theta_{PUX} - L_P\mu^u_{UUX} - L_{PX}\mu^u_{UU}\\
& - L_{PU}\mu^u_{UX} - L_{PUX}(2\mu^u_U + \mu^x_X) - L_{PP}(P\mu^u_{UUX} + \mu^u_{UXX})\\
& - L_{PPX}(P\mu^u_{UU} + \mu^u_{UX}) - L_{PUU}\mu^u_X - L_{PPU}[P(\mu^u_{UX} - \mu^x_{XX}) + \mu^u_{XX}],
\end{aligned}
\tag{3.43}
$$

and so on. Pulling-back the recurrence relations (3.43) via the moving frame (3.12), we use the recurrence relations for the phantom invariants (3.11) to obtain the expressions for the pulled-back Maurer–Cartan forms:

$$
\begin{gathered}
\nu^x = -\varpi^x, \qquad \nu^u = -\varpi^u, \qquad \nu^u_X = -\varpi^p, \qquad \nu^x_X = \varpi^p + \vartheta,\\
\nu^u_U = I_{pp}\varpi^p + \vartheta_p, \qquad \nu^u_{XX} = \frac{1}{I_{pp}}(I_{pux}\varpi^u + I_{ppx}\varpi^p + \vartheta_{px} - \vartheta_u),\\
\nu^u_{XU} = \vartheta_u, \qquad \nu^x_{XX} = \vartheta_x - \frac{1}{I_{pp}}(I_{pux}\varpi^u + I_{ppx}\varpi^p + \vartheta_{px} - \vartheta_u),\\
\nu^u_{UU} = I_{pux}\varpi^x + I_{ppu}\varpi^p + \vartheta_{pu} - I_{pp}\vartheta_u, \qquad \nu^u_{UUX} = \vartheta_{uu},\\
\nu^u_{UXX} = I_{pux}\varpi^p + \vartheta_{ux}, \qquad \ldots.
\end{gathered}
\tag{3.44}
$$

Substituting the expressions (3.44) in the recurrence relation for $L_{PP}$, for example, yields

$$
dI_{pp} = I_{ppx}\varpi^x + I_{ppu}\varpi^u + I_{ppp}\varpi^p + \vartheta_{pp} + I_{pp}(\varpi^p - 2I_{pp}\varpi^p + \vartheta - 2\vartheta_p). \tag{3.45}
$$

Since the differential of $I_{pp}$ is $dI_{pp} = d_{\mathcal{H}}I_{pp} + d_{\mathcal{V}}I_{pp} = \mathcal{D}_xI_{pp}\varpi^x + \mathcal{D}_uI_{pp}\varpi^u + \mathcal{D}_pI_{pp}\varpi^p + d_{\mathcal{V}}I_{pp}$ we conclude from (3.45) that

$$
\begin{gathered}
\mathcal{D}_xI_{pp} = I_{ppx}, \qquad \mathcal{D}_uI_{pp} = I_{ppu}, \qquad \mathcal{D}_pI_{pp} = I_{ppp} + I_{pp}(1 - 2I_{pp}),\\
d_{\mathcal{V}}I_{pp} = \vartheta_{pp} + I_{pp}(\vartheta - 2\vartheta_p).
\end{gathered}
\tag{3.46}
$$

Similarly, the horizontal component of the recurrence relations for the differential invariants $I_{ppp}$, $I_{ppu}$, $I_{ppx}$ and $I_{pux}$ yields the relations

$$\begin{gathered}
\mathcal{D}_x I_{ppp} = I_{pppx}, \qquad \mathcal{D}_u I_{ppp} = I_{pppu}, \qquad \mathcal{D}_p I_{ppp} = I_{pppp} + I_{ppp}(2 - 3I_{pp}), \\
\mathcal{D}_x I_{ppu} = I_{ppux} - 2I_{pp}I_{pux}, \qquad \mathcal{D}_p I_{ppu} = I_{pppu} + I_{ppu}(1 - 5I_{pp}), \\
\mathcal{D}_u I_{ppu} = I_{ppuu}, \qquad \mathcal{D}_x I_{ppx} = I_{ppxx}, \qquad \mathcal{D}_u I_{ppx} = I_{ppux} - I_{pux}\left(\frac{I_{ppp}}{I_{pp}} + 1\right), \\
\mathcal{D}_p I_{ppx} = I_{pppx} - I_{ppx}\left(1 + 2I_{pp} + \frac{I_{ppp}}{I_{pp}}\right) + I_{ppu}, \qquad \mathcal{D}_x I_{pux} = I_{puxx}, \\
\mathcal{D}_u I_{pux} = I_{puux} - \frac{I_{ppu}I_{pux}}{I_{pp}}, \qquad \mathcal{D}_p I_{pux} = I_{ppux} - I_{pux}(3I_{pp} + 1) - \frac{I_{ppu}I_{ppx}}{I_{pp}}.
\end{gathered} \tag{3.47}$$

From (3.47) we conclude that all fourth order normalized invariants are expressible in terms of the invariants $I_{ppp}$, $I_{ppu}$, $I_{ppx}$, $I_{pux}$, $I_{pp}$ and their invariant derivatives. Combined with (3.46) the fourth order normalized invariants are in fact expressible solely in terms of $I_{pp}$, $I_{pux}$ and their invariant derivatives. Repeating the argument for the higher order normalized differential invariants we conclude that $\{I_{pp}, I_{pux}\}$ is a generating set for the algebra of differential invariants.

**Proposition 3.23.** The algebra of differential invariants of the pseudo-group action (2.17) is generated by $\{I_{pp}, I_{pux}\}$.

## 3.2 Singular Submanifold Jets

As stated in Theorem 3.4, a moving frame exists in a neighborhood of a submanifold jet $z^{(\infty)}$ provided the action is free and regular. But most infinite-dimensional pseudo-group actions admit submanifold jets where the action is not free.

**Example 3.24.** All pseudo-group actions that satisfy

$$r_n = \dim \mathcal{G}^{(n)}|_z > \dim \mathrm{J}^n|_z = (m-p)\binom{p+n}{p} \qquad \text{for all } n \geq 1 \tag{3.48}$$

can never be free. Indeed, the inequality (3.48) implies that for all $n \geq 1$ we have $\mathcal{G}_{z^{(n)}} \neq \{\mathbb{1}^{(n)}|_{z^{(n)}}\}$ since the dimension of the pseudo-group fiber is larger than the space on which it acts. For such pseudo-groups, all submanifold jet $z^{(\infty)}$ are singular.

Singular submanifold jets play an important role in the solution of equivalence problems and cannot be neglected. At those points, the next best thing that can be done is to introduce a *partial moving frame*.

**Definition 3.25.** Let $\mathcal{G}$ be a regular pseudo-group action on $M$. A submanifold jet $z^{(n)} \in \mathrm{J}^n$ is said to be *singular* if the pseudo-group does not act freely at $z^{(n)}$. The set of $n$-th order singular jets is be denoted by $\mathcal{S}^n \subset \mathrm{J}^n$.

By definition, $\mathcal{S}^n = \mathrm{J}^n \setminus \mathcal{V}^n$, and for all $n$ smaller than the order of freeness $n_\star$ the equality $\mathcal{S}^n = \mathrm{J}^n$ holds, except for $n=0$ as any pseudo-group action trivially satisfies the freeness condition of Definition 3.2 since $\mathcal{G}^{(0)}_z = \{\mathbb{1}|^{(0)}_z\}$. The $n$-th order singular subset is characterized by the infinitesimal condition

$$\mathcal{S}^n = \left\{ z^{(n)} \in \mathrm{J}^n \ : \ \dim \mathfrak{g}^{(n)}|_{z^{(n)}} < r_n = \dim \mathcal{G}^{(n)} \right\}, \qquad n \geq 1.$$

**Definition 3.26.** A submanifold jet $z^{(\infty)}$ is said to be (totally) *singular* if for all $n \geq 1$ its projection $\pi_n^\infty(z^{(\infty)}) \in \mathcal{S}^n$ is a singular submanifold jet. The set of singular submanifold jets $z^{(\infty)}$ is denoted by $\mathcal{S}^\infty \subset \mathrm{J}^\infty$.

We introduce the notation

$$\mathcal{S}^\infty_n = \pi_n^\infty(\mathcal{S}^\infty) \subset \mathcal{S}^n, \qquad n \geq 1,$$

to denote the set obtain by truncating the singular submanifold jets in $\mathcal{S}^\infty$ at order $n$. At order zero we have $\mathcal{S}^\infty_0 = M$.

**Definition 3.27.** A submanifold $S \subset M$ is singular at a point $z \in S$ if $\mathrm{j}_n S|_z \subset \mathcal{S}^n$ for all $n \geq 1$. A submanifold $S$ is said to be *singular* if for every $z \in S$ the submanifold is singular at $z$.

The space of singular jets $\mathcal{S}^\infty$ can be very complicated. We now make some regularity assumptions on $\mathcal{S}^\infty$ and its projections $\mathcal{S}^\infty_n$. We assume that there exists a finite $n_0 \geq 1$ such that $\mathcal{S}^\infty_{n_0}$ is a $\mathcal{G}$-invariant bundle of $\mathrm{J}^{n_0}$ (more precisely a Zariski open subset of $\mathrm{J}^{n_0}$) of the form

$$\mathcal{S}^\infty_{n_0} = \{(x, u^{(n_0)}) : E^{(n_0)}(x, u^{(n_0)}) \neq 0 \quad \text{and} \quad F^{(n_0)}(x, u^{(n_0)}) = 0\}$$

such that for all $n \geq n_0$ the subset $\mathcal{S}^\infty_n$ is obtained from $\mathcal{S}^\infty_{n_0}$ by "prolongation" in the following sense

$$\mathcal{S}^\infty_n = \{(x, u^{(n)}) : E^{(n_0)}(x, u^{(n_0)}) \neq 0 \quad \text{and} \quad (D^x_J F^{(n_0)})(x, u^{(n)}) = 0,\ 0 \leq \#J \leq n - n_0\}.$$

The $\mathcal{G}$-invariance means that for all $g^{(n_0)} \in \mathcal{G}^{(n_0)}$ and $z^{(n_0)} \in \mathcal{S}^\infty_{n_0}$ we have $g^{(n_0)} \cdot z^{(n_0)} \in \mathcal{S}^\infty_{n_0}$. The integer $n_0$ is called the *determining order of the singular submanifold jet bundle* $\mathcal{S}^\infty$. We also allows the possibility that $\mathcal{S}^\infty = \mathrm{J}^\infty$. As mentioned in Example 3.24, this occurs when the dimension of fibers of the pseudo-group is too large.

The different bundles $\mathcal{S}^\infty$ satisfying the above regularity assumptions naturally appear as one tries to normalize the pseudo-group parameters in the lifted invariants (3.4). The differential functions $E^{(n_0)}(x, u^{(n_0)})$ appear as one imposes non-degeneracy conditions on some lifted invariants (3.4) while the functions $F^{(n_0)}(x, u^{(n_0)})$ come from assuming that some lifted invariants are identically equal to zero. In a local equivalence problem, the different sets $\mathcal{S}^\infty$ (together with $\mathcal{V}^\infty$) correspond to the different branches in the solution. The examples in Section 4 will illustrate and clarify the regularity assumptions made on $\mathcal{S}^\infty$. Finally, though the pseudo-group action is no longer free on $\mathcal{S}^\infty$ we continue to assume that the pseudo-group action is regular.

**Definition 3.28.** For $n \geq 1$, let

$$\mathcal{G}_{\mathcal{S}^\infty_n} = \bigcup_{z^{(\infty)} \in \mathcal{S}^\infty} \mathcal{G}_{\pi_n^\infty(z^{(\infty)})},$$

be the collection of isotropy groups of the submanifold jets $z^{(n)} \in \mathcal{S}^\infty_n$. The limit

$$\mathcal{G}_{\mathcal{S}^\infty} = \bigcup_{z^{(\infty)} \in \mathcal{S}^\infty} \mathcal{G}_{z^{(\infty)}} \tag{3.49}$$

is called *isotropy pseudo-group* of $\mathcal{S}^\infty$.

Two regularity assumptions are made on the isotropy pseudo-group $\mathcal{G}_{\mathcal{S}^{(\infty)}}$. We assume there exists a finite $n^\star \geq n_0 \geq 1$, called the *determining order of the isotropy pseudo-group*, such that for all $n \geq n^\star$

- the isotropy pseudo-group $\mathcal{G}_{\mathcal{S}^\infty_n}$ is an embedded subbundle of $\mathcal{G}^{(\infty)}|_{\mathcal{S}^\infty_n}$ ($\mathcal{G}^{(\infty)}|_{\mathcal{S}^\infty_n}$ denotes the restriction of the pseudo-group jet bundle $\mathcal{G}^{(n)}$ to $\mathcal{S}^\infty_n$),
- the projection $\pi^{n+1}_n : \mathcal{G}_{\mathcal{S}^\infty_{n+1}} \to \mathcal{G}_{\mathcal{S}^\infty_n}$ is a fibration.

The Lie pseudo-group $\mathcal{G}$ acts on $\mathcal{G}_{\mathcal{S}^\infty}$ by conjugation:

$$K_{h^{(\infty)}}(g^{(\infty)}) = h^{(\infty)} \cdot g^{(\infty)} \cdot (h^{-1})^{(\infty)}, \quad \text{for all } g^{(\infty)} \in \mathcal{G}_{z^{(\infty)}} \text{ and } \boldsymbol{\sigma}(h^{(\infty)}) = z^{(\infty)}.$$

At the infinitesimal level let

$$\mathfrak{g}_{z^{(\infty)}} = \ker \mathbf{p}|_{z^{(\infty)}} \cap \mathrm{J}^\infty \mathfrak{g}|_z, \qquad z = \pi^\infty_0(z^{(\infty)}), \tag{3.50}$$

be the *isotropy Lie algebra* of $\mathcal{G}_{z^{(\infty)}}$. Then

$$\mathfrak{g}_{\mathcal{S}^\infty} = \bigcup_{z^{(\infty)} \in \mathcal{S}^\infty} \mathfrak{g}_{z^{(\infty)}}$$

is the *isotropy Lie algebroid* of $\mathcal{G}_{\mathcal{S}^\infty}$. In local coordinates, the regularity assumptions on the isotropy pseudo-group $\mathcal{G}_{\mathcal{S}^\infty}$ forces the system of equations (3.50) to be *(formally) integrable* in the sense that for all $n \geq n^\star$ and $k \geq 0$

$$(\pi^{n+k}_n)_*(\mathfrak{g}_{z^{(n+k)}}) = \mathfrak{g}_{z^{(n)}}, \qquad \text{with} \qquad \mathfrak{g}_{z^{(n)}} = (\pi^\infty_n)_*(\mathfrak{g}_{z^{(\infty)}}). \tag{3.51}$$

### 3.2.1 Partial Moving Frames

Though it is not possible to obtain a moving frame over $\mathcal{S}^\infty$, it is nevertheless possible to introduce the notion of a partial moving frame.

For $n \geq n^*$, the determining order of the isotropy pseudo-group $\mathcal{G}_{\mathcal{S}^\infty}$, let $\mathcal{E}_{\mathcal{S}^\infty_n}$ be the pull-back bundle of $\mathcal{G}_{\mathcal{S}^\infty_n} \to M$ via the projection $\pi^n_0 \colon \mathcal{S}^\infty_n \to M$. In the projective limit, $\mathcal{E}_{\mathcal{S}^\infty}$ is called the *prolongation bundle* of $\mathcal{S}^\infty$. A local diffeomorphism $\varphi \in \mathcal{G}$ acts on the set

$$\{(z^{(n)}, g^{(n)}) \in \mathcal{E}_{\mathcal{S}^\infty_n} \,|\, \pi^n_0(z^{(n)}) \in \mathrm{dom}\ \varphi\}$$

by

$$\varphi \cdot (z^{(n)}, g^{(n)}) = (\mathrm{j}_n\varphi|_z \cdot z^{(n)}, K_{\mathrm{j}_n\varphi|_z}(g^{(n)})).$$

**Definition 3.29.** Let $\mathcal{G}$ be a regular pseudo-group action on $\mathcal{S}^\infty$. An *n-th order partial moving frame* over $\mathcal{S}^\infty_n$ is a $\mathcal{G}$-equivariant bundle map

$$\rho^{(n)} \colon \mathcal{E}_{\mathcal{S}^\infty_n} \to \mathcal{E}^{(n)}|_{\mathcal{S}^\infty_n},$$

where $\mathcal{E}^{(n)}|_{\mathcal{S}^\infty_n}$ denotes the restriction of $\mathcal{E}^{(n)}$ to $\mathcal{S}^\infty_n$.

As in Definition 3.1 for moving frames, right $\mathcal{G}$-equivariance means that

$$\begin{aligned}
\varphi \cdot \rho^{(n)}(z^{(n)}, g^{(n)}) =& \varphi \cdot (z^{(n)}, \widetilde{\rho}^{(n)}(z^{(n)}, g^{(n)})) \\
=& (\mathrm{j}_n\varphi|_z \cdot z^{(n)}, \widetilde{\rho}^{(n)}(\mathrm{j}_n\varphi|_z \cdot z^{(n)}, K_{\mathrm{j}_n\varphi|_z}(g^{(n)}))) \\
=& (\mathrm{j}_n\varphi|_z \cdot z^{(n)}, \widetilde{\rho}^{(n)}(z^{(n)}, g^{(n)}) \cdot \mathrm{j}_n\varphi^{-1}|_{\varphi(z)}).
\end{aligned}$$

A right partial moving frame is constructed by following the algorithm in Section 3.1 leading to a moving frame. Namely, a partial moving frame is obtained by fixing a series of compatible *cross-sections* $\mathcal{K}^n \subset \mathcal{S}^\infty_n$, $n \geq n^\star$, to the pseudo-group action and solving the normalization equations (3.6). By assumption, the solution to the normalization equations will depend on the submanifold jets and the isotropy pseudo-group parameters. To obtain a unique solution, the isotropy pseudo-group parameters need to be specified. It is for this reason that the isotropy pseudo-group $\mathcal{G}_{\mathcal{S}^\infty}$ is given as an input in the definition of a partial moving frame (See Figure 2 for an illustration of partial moving frame). In view of the partial moving frame construction, the isotropy pseudo-group parameters $g^{(\infty)} \in \mathcal{G}_{z^{(\infty)}}$ will loosely be referred as the *unnormalizable parameters* of the pseudo-group action and the corresponding Maurer–Cartan forms as the *unnormalizable Maurer–Cartan forms.*

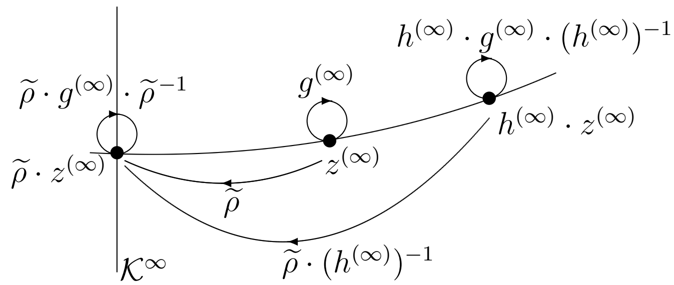


Figure 2: Right partial moving frame $\rho(z^{(\infty)}, g^{(\infty)}) = (z^{(\infty)}, \widetilde{\rho}(z^{(\infty)}, g^{(\infty)}))$, with $g^{(\infty)} \in \mathcal{G}_{z^{(\infty)}}$.

The definition of the invariantization map (3.17) and the recurrence formula (3.25) still hold for partial moving frames, with the obvious difference that these formulas hold on $\mathcal{E}_{\mathcal{S}^\infty}$. We note that by construction none of the normalized differential invariants $(H, I^{(\infty)}) = \iota(x, u^{(\infty)})$ can depend on the isotropy pseudo-group parameters $g^{(\infty)} \in \mathcal{G}_{\mathcal{S}^\infty}$ as otherwise those parameters could be normalized. On the other hand, the invariantization of the jet forms $dx^i$, $\theta^\alpha_J$ may depend on the isotropy pseudo-group parameters.

### 3.2.2 Algebra of Differential Invariants for Singular Submanifold Jets

In this section we revisit the algebraic constructions of Section 3.1.1 at a singular submanifold jet $(x, u^{(\infty)}) = z^{(\infty)} \in \mathcal{S}^\infty \subset \mathrm{J}^\infty$. The main conclusion is that, modulo the appropriate modifications, a Basis Theorem similar to Theorem 3.21 holds for singular submanifold jets (provided our regularity assumptions on $\mathcal{S}^\infty$ hold).

By hypothesis, $(x, u^{(\infty)})$ is a solution to the defining system of equations

$$E^{(n_0)}(x, u^{(n_0)}) \neq 0, \qquad F^{(\infty)}(x, u^{(\infty)}) = 0 \quad \text{for} \quad \mathcal{S}^\infty.$$

Under the identification,

$$dx^i \longleftrightarrow \widetilde{s}_i, \qquad du^\alpha_J \longleftrightarrow s_J S^\alpha,$$

we refer the reader to [35] for more details, the exterior differential of the functions $F^{(\infty)}(x, u^{(\infty)})$

$$dF(x, u^{(n)}) = \sum_{i=1}^{p} \frac{\partial F}{\partial x^i}(x, u^{(n)}) dx^i + \sum_{\alpha=1}^{q} \sum_{\#J \leq n} \frac{\partial F}{\partial u^\alpha_J}(x, u^{(n)}) du^\alpha_J, \qquad F \in F^{(\infty)}(x, u^{(\infty)}),$$

at a fixed $z^{(\infty)} = (x, u^{(\infty)}) \in \mathcal{S}^\infty$ gives a collection of polynomials in $\mathcal{S}$. Considering the span of the leading terms

$$\operatorname{span}\left\{\boldsymbol{\Sigma}(dF^{(\infty)}(x,u^{(\infty)}))\right\} = \operatorname{span}\left\{\sum_{\alpha=1}^{q}\sum_{\#J=n}\frac{\partial F}{\partial u_J^\alpha}(x,u^{(n)})s_J S_J^\alpha : F(x,u^{(n)}) \in F^{(\infty)}(x,u^{(\infty)})\right\} \tag{3.52}$$

we obtain a vector subspace of $\mathcal{S}$. Since $F^{(\infty)}(x,u^{(\infty)}) = 0$ is assumed to be formally integrable, the vector space (3.52) is in fact a submodule of $\mathcal{S}$. Let

$$\mathcal{S}_{z^{(\infty)}} = \mathcal{S}/\operatorname{span}\left\{\boldsymbol{\Sigma}(dF^{(\infty)}(x,u^{(\infty)}))\right\}$$

be the quotient module of $\mathcal{S}$ by (3.52). Geometrically, $\mathcal{S}_{z^{(\infty)}}$ can be identified with the independent submanifold jet coordinates of $z^{(\infty)} \in \mathcal{S}^\infty$. We can assume, possibly by restricting to an open subset in $\mathcal{S}^\infty$, that $\mathcal{S}_{z^{(\infty)}}$ does not depend on the particular $z^{(\infty)}$. We will then write $\mathcal{S}_{\mathcal{S}^\infty}$. We now defined the prolonged annihilator subbundle (3.36) and the prolonged symbol submodule (3.37) at a singular submanifold jet the same way as it is done for regular submanifold jets except that the module $\mathcal{S}$ is replaced by the quotient module $\mathcal{S}_{\mathcal{S}^\infty}$.

Under the identification of $(\mathrm{J}^\infty TM)^*$ with the symbol module $\mathcal{T}$ let

$$(\mathfrak{g}_{z^\infty})^\perp = \mathcal{T}_{z^{(\infty)}} \subset \mathcal{T}.$$

be the *isotropy algebra annihilator space* at the submanifold jet $z^{(\infty)}$.

**Proposition 3.30.** Let $z^{(\infty)} \in \mathcal{S}^\infty$, then

$$\mathbf{p}^*(\mathcal{S}_{\mathcal{S}^\infty}) + \mathcal{L}|_z = \mathcal{T}_{z^{(\infty)}}. \tag{3.53}$$

*Proof.* Let $z^{(\infty)} \in \mathcal{S}^\infty$, then

$$\mathfrak{g}_{z^{(\infty)}} = \ker\, \mathbf{p}|_{z^{(\infty)}} \cap \mathrm{J}^\infty\mathfrak{g}|_z = (\operatorname{rng}\,(\mathbf{p})^*)^\perp \cap (\mathcal{L}|_z)^\perp = (\mathbf{p}^*(\mathcal{S}_{\mathcal{S}^\infty}) + \mathcal{L}|_z)^\perp,$$

from which (3.53) follows. □

**Definition 3.31.** The isotropy algebra annihilator $\mathcal{T}_{z^{(\infty)}}$ is said to be *(formally) integrable* if there exists a finite $n^\star \geq 1$ such that for all $n \geq n^\star$ and $k \geq 0$

$$\mathcal{T}_{z^{(\infty)}}^{\leq n+k} \cap \mathcal{T}^{\leq n} = \mathcal{T}_{z^{(\infty)}}^{\leq n}. \tag{3.54}$$

The smallest integer $n^\star$ satisfying the definition is called the *order of integrability* of $\mathcal{T}_{z^{(\infty)}}$.

We note that Definition 3.31 is just a restatement of the integrability condition (3.51). Fixing a degree compatible term ordering, [39], let $\mathcal{N}|_{z^{(\infty)}} \subset \mathcal{T}$ denote the monomial module generated by the leading monomials of $\mathcal{T}_{z^{(\infty)}}$. Again, we can assume, possibly by restricting to an open subset in $\mathcal{S}^\infty$, that $\mathcal{N}|_{z^{(\infty)}} = \mathcal{N}$ does not depend on $z^{(\infty)} \in \mathcal{S}^\infty$. Under the correspondence

$$\mu_B^a \longleftrightarrow t_B T^a \tag{3.55}$$

the unormalizable Maurer–Cartan forms associated with a partial moving frame are in one-to-one correspondence with the monomials

$$t_B T^a \notin \mathcal{N}.$$

The important Lemma 3.18 leading to the Basis Theorem 3.21 also holds for singular submanifold jets.

**Proposition 3.32.** Let $n^\star$ be the order of integrability of $\mathcal{T}_{z^{(\infty)}}$ and $z^{(n)} = \pi_n^\infty(z^{(\infty)})$, then

$$\mathcal{U}^n|_{z^{(n)}} = \mathcal{J}^n|_{z^{(n)}} \tag{3.56}$$

for all $n > n^\star$.

*Proof.* By induction it suffices to prove (3.56) when $n = n^\star{+}1$. Let $Q \in \mathcal{J}^{n^\star+1}|_{z^{(n^\star+1)}}$ and $P = \mathbf{p}^*(Q)$. By (3.35) and (3.37)

$$\mathbf{H}(P) = \mathbf{H}(\mathbf{p}^*(Q)) = \boldsymbol{\beta}^*(\mathbf{H}(Q)) = \boldsymbol{\beta}^*(Q) \in \mathcal{I}^{n^\star+1}|_z.$$

The integrability of $\mathcal{T}_{z^{(\infty)}}$ implies that there exists $Y \in \mathcal{T}^{\leq n^\star}_{z^{(\infty)}}$ such that $P{+}Y \in \mathcal{L}^{n^\star+1}|_z$. Let $U \in \mathcal{S}^{\leq n^\star}_{\mathrm{g}^\infty}$ and $V \in \mathcal{L}^{\leq n^\star}|_z$ such that $Y = \mathbf{p}^*(U) + V$, then

$$\mathbf{p}^*(Q+U) = (P+Y) - V \in \mathcal{L}^{\leq n^\star+1}|_z.$$

Equation (3.36) implies that $Q + U \in \mathcal{Z}^{\leq n^\star+1}|_{z^{(n^\star+1)}}$. □

The proof of Proposition 3.32 is essentially the same as [35, Lemma 5.5], valid for Lie pseudo-groups acting freely. It is included to show where the integrability assumption on $\mathcal{T}_{z^{(\infty)}}$ comes into play. By virtue of Proposition 3.32 the constructions and results of [35] also hold when $\mathcal{T}_{z^{(\infty)}}$ is formally integrable. In particular, the Basis Theorem 3.21 still holds.

For certain types of Lie pseudo-group actions, the integrability assumption on the isotropy algebra annihilator can be replaced by involutivity. Locally, assume that $M \simeq X \times U \to X$ is a fiber bundle and that submanifolds of $M$ are local sections of the fiber bundle. Let $\mathcal{G}$ be a Lie pseudo-group action on $X \times U$, for the remainder of this section we assume that there exists a Lie pseudo-group action $\mathcal{H}$ on $X$ such that $\mathcal{G}$ is an isomorphic prolongation of $\mathcal{H}$ (recall Definition 2.8). Such Lie pseudo-group actions are of interest as they occur as symmetry groups of differential equations, for example the Kadomtsev–Petviashvili and the Khokhlov–Zabolotskaya equations [11, 38], but more importantly, Cartan's method of equivalence of coframes falls into this category of pseudo-group actions. We will discuss in more detail this aspect in Section 4.3.

Let

$$\mathcal{T}_\mathcal{H} = \text{span}\ \{t_J T^i : J = (j^1, \ldots, j^k), 1 \leq j^l \leq p;\ i = 1, \ldots, p\} \subset \mathcal{T}$$

be the symbol module of the Lie pseudo-group $\mathcal{H}$ acting on $X$. We defined the isotropy algebra annihilator in $\mathcal{T}_\mathcal{H}$ at a submanifold jet $z^{(\infty)} \in \mathrm{J}^\infty(M, p)$ by

$$\mathcal{T}_{\mathcal{H};z^{(\infty)}} = (\mathfrak{g}_{z^{(\infty)}})^\perp \cap \mathcal{T}_\mathcal{H}.$$

Since $\mathcal{G}$ is an isomorphic prolongation of $\mathcal{H}$ the involutivity of $\mathcal{T}_{\mathcal{H};z^{(\infty)}}$ automatically implies the involutivity of the annihilator $\mathcal{T}_{z^{(\infty)}} = \mathcal{T}_{\mathcal{G};z^{(\infty)}}$. Also, the assumption that $\mathcal{G}$ is an isomorphic prolongation of $\mathcal{H}$ implies that the linear polynomials

$$t_{p+\alpha}T^i \qquad \text{and} \qquad T^\alpha - L^\alpha(z, t_J T^i) \qquad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q,$$

where $L^\alpha(z, t_J T^i)$ are linear functions in $t_J T^i$ depending on $\mathcal{G}$, are in $\mathcal{L}|_z$. Setting

$$\mathcal{P}|_z = \text{span}\ \{t_A t_{p+\alpha} T^i, t_A(T^\alpha - L^\alpha(z, t_J T^i)) : A = (a^1, \ldots, a^k), 1 \le a^b \le m\}$$

we have $\mathcal{T}_{\mathcal{H};z^{(\infty)}} \simeq (\mathfrak{g}_{z^{(\infty)}})^\perp / \mathcal{P}|_z$. In other words, $\mathcal{T}_{\mathcal{H};z^{(\infty)}}$ is obtained from $(\mathfrak{g}_{z^{(\infty)}})^\perp$ by setting $t_A t_{p+\alpha} T^i = 0$ and replacing $t_A T^\alpha$ with $T_A L^\alpha(z, t_J T^i)$ in $\mathbf{p}^*(\mathcal{S}_{z^{(\infty)}})$. Now, at the symbol level, the $i$-th total derivative operator (3.2) becomes multiplication by $t_i$ in $\mathcal{T}_{\mathcal{H},z^{(\infty)}}$:

$$\mathbf{\Sigma}(D_{x^i} L) = t_i \mathbf{\Sigma}(L), \qquad L \in \mathcal{T}_{\mathcal{H},z^{(\infty)}}.$$

This means that the symbol of $D_{x^i}$ equals the symbol of the partial derivative operator $\partial_{x^i}$. Hence on $\mathcal{T}_{\mathcal{H};z^{(\infty)}}$ we can introduce the notion of involutivity by appealing to the standard theory of involutivity of partial differential equations as developed by Seiler, [39]. We now summarize this theory in the context of our algebraic constructions.

**Definition 3.33.** Let $z^{(\infty)}$ be a fix submanifold jet, $n \ge 1$, and

$$\mathcal{T}^n_{\mathcal{H};z^{(\infty)}} = \left\{ \eta_\upsilon(z^{(\infty)}; t, T) = \sum_{i=1}^{p} \sum_{\#J=n} h^J_{i;\upsilon}(z^{(\infty)}) t_J T^i, \quad \upsilon = 1, \ldots, \ell \right\}$$

the collection of degree $n$ polynomials in $\mathcal{T}_{\mathcal{H};z^{(\infty)}}$. The *symbol matrix*

$$\mathbf{T}_n = \mathbf{T}(\mathcal{T}^n_{z^{(\infty)}}) = \left( h^J_{i;\upsilon}(z^{(\infty)}) \right), \quad \upsilon = 1, \ldots, \ell, \quad i = 1, \ldots, p, \quad \#J = n \qquad (3.57)$$

is the $\ell \times p\binom{p+n-1}{n}$ matrix with the entries of the $\upsilon$-th row given by the coefficients $h^J_{i;\upsilon}(z^{(\infty)})$ of the polynomial $\eta_\upsilon(z^{(\infty)}; t, T)$.

To define the *class* of a symmetric multi-index $J = (j^1, \ldots, j^k)$ we rewrite the multi-index as $J = (\widetilde{j}^1, \ldots, \widetilde{j}^p)$, where $\widetilde{j}^i$ is the number of occurrences of the integer $i$ in $(j^1, \ldots, j^k)$.

**Definition 3.34.** The *class* of a multi-index $J = (\widetilde{j}^1, \ldots, \widetilde{j}^p)$ is

$$\text{cl}\ J = \min\ \{i : \widetilde{j}^i \ne 0\}.$$

The columns of the symbol matrix $\mathbf{T}_n$ are ordered in such a way that the column $(\eta^J_{i;1}, \ldots, \partial\eta^J_{i;\ell})^T$ is always to the left of the column $(\eta^K_{j;1}, \ldots, \eta^K_{j;\ell})^T$ if cl $J >$ cl $K$. For two multi-indices with the same class, the order of the columns does not matter. Once the columns of the symbol matrix are ordered it is put in row echelon form without performing any column permutations.

**Definition 3.35.** Let $\beta^{(j)}_n$, $j = 1, \ldots, p$, be the number of pivots with class $1 \le j \le p$ of the row echelon form symbol matrix $\mathbf{T}_n$. The numbers $\beta^{(j)}_n$ are called the *indices* of $\mathbf{T}_n$.

Definition 3.35 depends on the chosen coordinate system and one must always work with $\delta$-regular coordinate systems.

**Definition 3.36.** A coordinate system is said to be *$\delta$-regular* if the sum $\sum_{j=1}^{m} j\,\beta_n^{(j)}$ is maximal.

Any coordinate system can be transformed into a $\delta$-regular one with a linear transformation defined by a matrix coming from a Zariski open subset of $\mathbb{R}^{p\times p}$, [39].

**Definition 3.37.** The subspace $\mathcal{T}^n_{\mathcal{H};z^{(\infty)}}$ is said to be *involutive* if the symbol matrix $\mathbf{T}_{n+1}$ of $\mathcal{T}^{n+1}_{\mathcal{H};z^{(\infty)}}$ satisfies the algebraic equality

$$\operatorname{rank} \mathbf{T}_{n+1} = \sum_{j=1}^{p} j\beta_n^{(j)}. \tag{3.58}$$

**Definition 3.38.** The annihilator space $\mathcal{T}_{\mathcal{H};z^{(\infty)}}$ is said to be involutive if there exists $n \geq 1$ such that $\mathcal{T}^n_{\mathcal{H};z^{(\infty)}}$ is involutive.

By Cartan–Kuranishi Involutive Completion Theorem, [3, 39], we have the following result.

**Proposition 3.39.** There exists a finite $n \geq 1$, such that $\mathcal{T}^n_{\mathcal{H};z^{(\infty)}}$ is involutive. The smallest such $n = n^\star$ is called the *order of involutivity* of $\mathcal{T}_{\mathcal{H};z^{(\infty)}}$.

We remark that involution is preserved under prolongation, thus the involution of $\mathcal{T}^{n^\star}_{\mathcal{H};z^{(\infty)}}$ implies the involution of $\mathcal{T}^k_{\mathcal{H};z^{(\infty)}}$ for all $k \geq n^\star$. When $\mathcal{T}_{\mathcal{H};z^{(\infty)}}$ is involutive, the size of the isotropy algebra $\mathfrak{g}_{z^{(\infty)}}$ can be determined in terms of the *Cartan characters*, [39].

**Definition 3.40.** Let $\mathcal{T}^n_{\mathcal{H};z^{(\infty)}}$ be involutive with indices $\beta_n^{(j)}$, the *Cartan characters* of $\mathcal{T}^n_{\mathcal{H};z^{(\infty)}}$ are defined by the expressions

$$\alpha_n^{(j)} = p\binom{n+p-j-1}{n-1} - \beta_n^{(j)}, \qquad 1 \leq j \leq p. \tag{3.59}$$

**Theorem 3.41.** Let $\mathcal{T}^n_{z^{(\infty)}}$ be involutive with Cartan characters $\alpha_n^{(j)}$. Then the vector fields in $\mathfrak{g}_{z^{(\infty)}}$ depend on $f_j$ arbitrary functions of $j$ variables where the numbers $f_j$ are determined by the recursion relation

$$\begin{gathered} f_p = \alpha_n^{(p)}, \\ f_j = \alpha_n^{(j)} + \sum_{i=j+1}^{p} \frac{(j-1)!}{(p-1)!}\left(s_{i-j}^{(i-1)}(0)\alpha_n^{(i)} - s_{i-j}^{(i-1)}(n) f_i\right), \qquad 1 \leq j \leq p-1, \end{gathered} \tag{3.60}$$

provided all $f_j$ are non-negative integers. The numbers $s_i^{(j)}(k)$ are the modified Stirling numbers defined by the identity

$$(k+y+1)(k+y+2)\cdots(k+y+n) = \sum_{l=0}^{n} s_{n-l}^{(n)}(k) y^l$$

for all non-negative integers $n$, $l$, $k$ with $n \geq l$. Here $y$ is an arbitrary variable.

Since involutivity is stronger than formal integrability it follows that the Basis Theorem 3.21 holds when $\mathcal{T}_{\mathcal{H};z^{(\infty)}}$ is involutive. Finally, as in Section 3.1.1 the concepts of integrability and involutivity are also well defined when the algebraic constructions are invariantized since they coincide with their progenitor when restricted to the cross-section used to define the partial moving frame.

**Example 3.42.** Consider the diffeomorphism pseudo-group of $\mathbb{R}^2$

$$X = f(x,u), \qquad U = g(x,u),$$

acting on planar curves $(x,u(x))$. The Maurer–Cartan forms for this pseudo-group action are given in Example 2.2. The pseudo-group action on $\mathrm{J}^\infty(\mathbb{R}^2,1)$ is transitive and a cross-section to the pseudo-group action is

$$X = \widehat{U}_{X^k} = 0, \qquad k = 0,1,2,\ldots.$$

Let $\varpi = \iota(dx)$ be the invariantization of horizontal form $dx$ and $\vartheta_k = \iota(\theta_k)$ the invariantization of the contact forms. Solving the recurrence relations (3.26) for the Maurer–Cartan forms we obtain

$$\nu^x = -\varpi, \qquad \nu^u_{X^k} = \vartheta_k. \tag{3.61}$$

Under the correspondence (3.55), we have, by considering the left-hand side of (3.61), that

$$\widetilde{\mathcal{T}}_{\iota(z^{(\infty)})} = \iota(\mathcal{T}_{z^{(\infty)}}) = \text{span}\ \{T^1, t_1^k T^2 :\ k \geq 0\}. \tag{3.62}$$

The symbol module (3.62) is not involutive since it does not satisfy the involutivity test (3.58). Indeed, for all $n \geq 1$ the indices of $\mathbf{T}_n$ are $\beta_n^{(2)} = 1$, $\beta_n^{(1)} = 0$ while

$$\text{rank}\ \mathbf{T}_{n+1} = 1 \neq 2 = 2\beta_n^{(2)} + \beta_n^{(1)}.$$

On the other hand, the symbol module (3.62) is integrable of order $n^\star = 1$ as it satisfies the integrability condition (3.54).

The complement to (3.62) is spanned by

$$t_1^i t_2^j T^1, \qquad t_2^j T^2, \qquad i,j \geq 1,$$

and we conclude that the isotropy pseudo-group is parametrized by the pseudo-group parameters

$$f_{x^i u^j}, \qquad g_{u^j}, \qquad i,j \geq 1.$$

# 4 Equivalence of Submanifolds

Let $\mathcal{G}$ be a Lie pseudo-group action on a manifold $M$. Given two $p$-dimensional submanifolds $S$, $\overline{S}$ in $M$ the local equivalence problem for submanifolds consists of determining whether there exists or not a local diffeomorphism $\varphi \in \mathcal{G}$ mapping $S$ onto $\overline{S}$ locally. In accordance with Cartan's general philosophy, the solution to the equivalence problem is completely prescribed by the differential invariants of the pseudo-group action.

Within the equivariant moving frame framework the solution has a simple geometrical interpretation. Let $\rho$ be a moving frame with corresponding cross-section $\mathcal{K}^\infty$. To determine if two regular submanifolds $S$, $\overline{S}$ are locally equivalent up to a diffeormorphism $\varphi \in \mathcal{G}$ the liberty of "movement" of the pseudo-group action $\mathcal{G}$ is removed by projecting the submanifold jets $\mathrm{j}_\infty S$, $\mathrm{j}_\infty \overline{S}$ onto the cross-section $\mathcal{K}^\infty$ with the moving frame $\rho$. If the projections are locally the same then the submanifolds are locally equivalent and if the projections are different then the submanifolds are inequivalent. As the local coordinates on $\mathcal{K}^\infty$ are in one-to-one correspondence with the normalized differential invariants $(H, I^{(\infty)})$ of $\mathcal{G}$ via the invariantization map (3.18), the submanifolds $S$, $\overline{S}$ are locally equivalent it the restrictions of the differential invariants $(H, I^{(\infty)})$ to $S$ and $\overline{S}$ are the same.

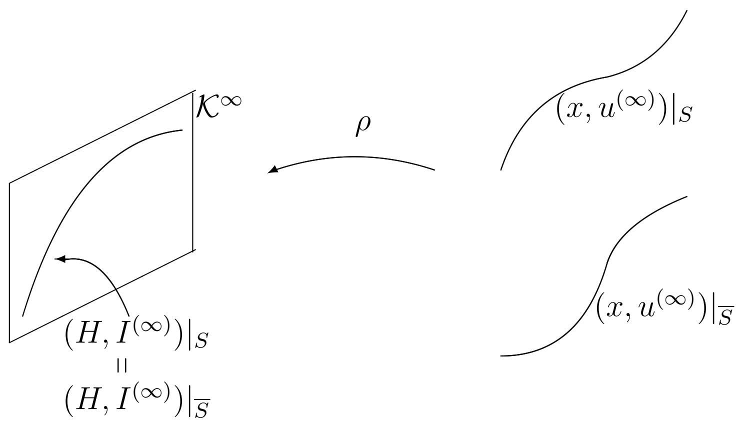


Figure 3: Signature of two equivalent submanifolds.

When the submanifolds $S$, $\overline{S}$ are singular, with the same type of singular jets, the above geometrical picture still holds, but there is one important distinction. While the equivalence map between two equivalent regular submanifolds is unique, the equivalence map between two singular submanifolds is not. An equivalence map $\varphi : S \to \overline{S}$ between two singular submanifolds $S$, $\overline{S}$ can always be precomposed at its source by a diffeomorphism in the isotropy pseudo-group of $\mathrm{j}_\infty S$ and composed at its range by a diffeomorphism in the isotropy pseudo-group of $\mathrm{j}_\infty \overline{S}$ to obtain a new equivalence map.

In applications, the identification of the regular and singular submanifolds naturally occurs as one tries to obtain a cross-section to the equivalence pseudo-group action so as to normalize the pseudo-group parameters. The submanifolds for which all pseudo-group parameters are normalizable are regular while the others are singular. There are fundamentally two ways of searching for a cross-section to a pseudo-group action. One can either use the coordinate expressions of the prolonged action (3.4) or the recurrence relations (3.24) for the lifted invariants. As the prolonged action is typically nonlinear, in practice, it is usually simpler to work with the recurrence relations. Modulo the lifted jet forms $\Omega^i$, $\Theta^\alpha_J$ the recurrence relations (3.24) determine how the lifted invariants depend on the pseudo-group jets through the Maurer–Cartan forms appearing on the right-hand side of the equalities, and can thus be used find a cross-section. Using the recurrence relations to find a cross-section is reminiscent to solving an equivalence problem in Cartan's framework using the *intrinsic method* popularized by Gardner, [14, 15, 27]. It is this approach that we will promote in our examples.

## 4.1 Equivalence of Regular Submanifolds

Let $\mathcal{G}$ be a pseudo-group action on $M$ and $\mathcal{V}^\infty$ the set of regular submanifold jets. In this section we assume that all submanifolds $S \subset M$ are regular, that is $\mathrm{j}_\infty S \subset \mathcal{V}^\infty$. In this setting a moving frame $\rho\colon \mathcal{V}^\infty \to \mathcal{E}^{(\infty)}$ exists in a neighborhood of every $z \in S$.

In the following we use the short-hand notation $\Omega|_S$ to denote the restriction of a differential form $\Omega$ to the submanifold $S$ (in other words $\Omega|_S = i^*(\Omega)$ where $i\colon S \hookrightarrow M$ is the inclusion map). But when the context is clear we will frequently make the abuse of notation $\Omega = \Omega|_S$ to lighten the notation.

By virtue of Proposition 3.18, a complete set of functionally independent differential invariants on $\mathcal{V}^{(\infty)}$ is given by the invariantization of the submanifold jet coordinates $\iota(x, u^{(\infty)}) = (H, I^{(\infty)})$. By the Basis Theorem 3.21, let $\mathcal{I} = \{I_\kappa : \kappa = 1, \ldots, \ell\}$ be a generating set for the algebra of differential invariants. Then all normalized invariants can be expressed in terms of the generating invariants and their invariant derivatives

$$\begin{aligned} H^i &= F^i(I_1, \ldots, I_\ell, \ldots, \mathcal{D}_K I_1, \ldots, \mathcal{D}_K I_\ell, \ldots), \\ I^\alpha_J &= F^\alpha_J(I_1, \ldots, I_\ell, \ldots, \mathcal{D}_K I_1, \ldots, \mathcal{D}_K I_\ell, \ldots). \end{aligned} \tag{4.1}$$

While the normalized invariants $(H, I^{(\infty)})$ form a complete set of functionally independent invariants on $\mathrm{J}^\infty$, when restricted to a submanifold $S$ they might no longer be functionally independent. This is definitely the case when there are more than $p$ of them. In principle, to solve an equivalence problem one needs to keep track of those functional relations. But by virtue of (4.1), instead of considering the functional relationships among the normalized invariants $(H, I^{(\infty)})|_S$, it is enough to consider the functional relations among the generating invariants and their invariant derivatives.

**Definition 4.1.** Let $\mathcal{G}$ be a Lie pseudo-group acting on $p$-dimensional submanifolds of $M$, and $\mathcal{I} = \{I_1, \ldots, I_\ell\}$ a generating set for the algebra of differential invariants associated to a moving frame $\rho\colon \mathcal{V}^\infty \to \mathcal{E}^{(\infty)}$. The *n-th order signature space* $\mathbb{K}^{(n)}$ is the Euclidean space of dimension $\ell(1 + p + p^2 + \cdots + p^n)$ coordinatized by $w^{(n)} = (\ldots, w_{\kappa;J}, \ldots)$, where $(\kappa; J) = (\kappa, j^1, \ldots, j^r)$ with $1 \le \kappa \le \ell$ and $(j^1, \ldots, j^r)$ ranging through all unordered multi-index with $1 \le j^i \le p$ and $0 \le r \le n$. The *n-th order signature map* associated with the moving frame $\rho$ is a map $\mathbf{I}^{(n)}_S\colon S \to \mathbb{K}^{(n)}$ whose compontents are

$$w_{\kappa;J} = (\mathcal{D}_J I_\kappa)|_S, \qquad \kappa = 1, \ldots, \ell, \qquad \#J \le n.$$

**Remark 4.2.** In Definition 4.1 the multi-index $J$ is not assumed to be symmetric as the invariant total differential operators $\mathcal{D}_i$ generally do not commute. Nevertheless, in applications we can reduce the dimension of the $n$-th order signature space $\mathbb{K}^{(n)}$ by ordering as many multi-indices $J$ as possible using the commutation relations (3.28) for the invariant total differential operators:

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^{p} T^k_{ij}(I_\kappa, \mathcal{D}_K I_\kappa)\mathcal{D}_k, \qquad i, j = 1, \ldots, p. \tag{4.2}$$

In (4.2), the commutator invariants $T^k_{ij}$ are expressed in terms of the generating invariants and their invariant derivatives.

**Definition 4.3.** A moving frame $\rho$ is said to be *fully regular* on $S$ if for each $n \ge 0$ the signature map $\mathbf{I}^{(n)}_S\colon S \to \mathbb{K}^{(n)}$ is regular.

**Definition 4.4.** Let $\rho$ be a fully regular moving frame. The image

$$\mathfrak{S}^{(n)}(\rho, S) = \{\mathbf{I}_S^{(n)}(z) : z \in S\} \subset \mathbb{K}^{(n)} \tag{4.3}$$

of the $n$-th order signature map $\mathbf{I}_S^{(n)}$ is called the *$n$-th order signature manifold.*

**Proposition 4.5.** Let $\rho$ be a fully regular moving frame, and let $\varrho_n$ denote the rank of the $n$-th order signature map $\mathbf{I}_S^{(n)}$. Then

$$0 \leq \varrho_0 < \varrho_1 < \cdots < \varrho_s = \varrho_{s+1} = \cdots = r \leq p,$$

and the stabilizing rank $r$ is referred as the *rank* of the moving frame, and the smallest $s$ for which $\varrho_s = \varrho_{s+1} = r$ is called the *order* of the moving frame.

**Theorem 4.6.** Let $\mathcal{G}$ be a Lie pseudo-group action on $M$, $\rho$ a fully regular moving frame, and $S, \overline{S} \subset M$ two regular $p$-dimensional submanifolds. There exists a local diffeomorphism $\varphi \in \mathcal{G}$ mapping $S$ onto $\overline{S}$ if and only if $\rho$ has the same order $\overline{s} = s$ on $S$ and $\overline{S}$, and the $(s+1)$-st order signature manifolds $\mathfrak{S}^{(s+1)}(\rho, S)$, $\mathfrak{S}^{(s+1)}(\rho, \overline{S})$ overlap. Moreover, if $z_0 \in S$ and $\overline{z}_0 \in \overline{S}$ are any points mapping to the same point

$$\mathbf{I}_S^{(s+1)}(z_0) = \mathbf{I}_{\overline{S}}^{(s+1)}(\overline{z}_0) \in \mathfrak{S}^{(s+1)}(\rho, S) \cap \mathfrak{S}^{(s+1)}(\rho, \overline{S})$$

on the overlap of the two signature manifolds, then there exists a local equivalence map $\varphi$ mapping $z_0$ to $\overline{z}_0 = \varphi(z_0)$.

*Proof.* Let $J_1, \ldots, J_r$ be a set of invariants parametrizing the the $s$-th order signature manifold $\mathfrak{S}^{(s)}(\rho, S)$. Then there exist *signature functions* $F_{\kappa;K}(z_1, \ldots, z_r)$, such that

$$\mathcal{D}_K I_\kappa = F_{K;\kappa}(J_1, \ldots, J_r), \qquad \kappa = 1, \ldots, \ell, \qquad \#K \leq s.$$

By hypothesis, the invariants $J_1, \ldots, J_r$ also parametrize the $(s+1)$-th signature manifold $\mathfrak{S}^{(n+1)}(\rho, S)$. Hence there exist signature functions $\widetilde{F}_{i;\upsilon}(z_1, \ldots, z_r)$ such that

$$\mathcal{D}_i J_\upsilon = \widetilde{F}_{i;\upsilon}(J_1, \ldots, J_r), \qquad \upsilon = 1, \ldots, r, \quad i = 1, \ldots, p.$$

By the chain rule

$$\mathcal{D}_i(\mathcal{D}_K I_\kappa) = \sum_{\upsilon=1}^{r} \frac{\partial F_{K;\kappa}}{\partial z^\upsilon}(J_1, \ldots, J_r) \cdot \widetilde{F}_{i;\upsilon}(J_1, \ldots, J_r),$$

and we conclude that once $\mathfrak{S}^{(s+1)}(\rho, S)$ is known, $\mathfrak{S}^{(s+k)}(\rho, S)$, $k \geq 2$, follows by differentiation.

The assumption that the signature manifolds $\mathfrak{S}^{(s+1)}(\rho, S)$, $\mathfrak{S}^{(s+1)}(\rho, \overline{S})$ overlap implies that the generating invariants $\mathcal{I} = \{I_1, \ldots, I_\ell\}$ and their invariant derivatives are equal when restricted to $S$ and $\overline{S}$: $\mathcal{D}_K I_\kappa|_S = \mathcal{D}_K I_\kappa|_{\overline{S}}$, with $\#K \geq 0$. From (4.1) it then follows that the normalized invariants $H^i|_S = H^i|_{\overline{S}}$ and $I_K^\alpha|_S = I_K^\alpha|_{\overline{S}}$ are equal on the overlap. Equivalently, this means that the projections of $\mathrm{j}_\infty S$ and $\mathrm{j}_\infty \overline{S}$ onto the cross-section $\mathcal{K}^\infty$ by the moving frame $\rho$ are the same. Choose $z_0 \in S$ and $\overline{z}_0 \in \overline{S}$ such that $\rho(\mathrm{j}_\infty S|_{z_0}) = \rho(\mathrm{j}_\infty \overline{S}|_{\overline{z}_0})$. Let $\Phi(z)$ and $\overline{\Phi}(\overline{z})$ be the local diffeomorphisms in $\mathcal{G}$ such that their jets at $z_0$ and $\overline{z}_0$ satisfy

$$\mathrm{j}_\infty \Phi|_{z_0} = \widetilde{\rho}(\mathrm{j}_\infty S|_{z_0}), \qquad \mathrm{j}_\infty \overline{\Phi}|_{\overline{z}_0} = \widetilde{\rho}(\mathrm{j}_\infty \overline{S}|_{\overline{z}_0}).$$

Since $\boldsymbol{\sigma}((\mathrm{j}_\infty \overline{\Phi}|_{\overline{z}_0})^{-1}) = \boldsymbol{\tau}(\mathrm{j}_\infty \Phi|_{z_0})$, the map $\varphi = \overline{\Phi}^{-1} \circ \Phi \in \mathcal{G}$ is locally well defined and by construction $\mathrm{j}_\infty \varphi \circ \mathrm{j}_\infty S|_{z_0} = \mathrm{j}_\infty \overline{S}|_{\overline{z}_0}$. Now, since we work in the analytic category, the equality also holds for all $z$, $\overline{z}$ in some neighborhoods of $z_0$ and $\overline{z}_0$ respectively: $\mathrm{j}_\infty \varphi \circ \mathrm{j}_\infty S|_z = \mathrm{j}_\infty \overline{S}|_z$. This implies $\varphi(S) = \overline{S}$ locally. $\square$

## 4.2 Equivalence of Singular Submanifolds

We now turn to the equivalence of singular submanifolds. Again, let $\mathcal{G}$ be a Lie pseudo-group acting on $p$-dimensional submanifolds of $M$ and let $\mathcal{S}^\infty$ be a fix set of singular submanifold jets satisfying the regularity assumptions stated below Definition 3.27. Since the algebra of differential invariants is also finitely generated for singular submanifolds, the notion of signature manifold given in Definition 4.4 is also well defined.

**Theorem 4.7.** Let $\rho$ be a (fully regular) partial moving frame defined on the bundle of singular submanifold jets $\mathcal{S}^\infty$ with isotropy pseudo-group $\mathcal{G}_{\mathcal{S}^\infty}$. Let $S$, $\overline{S}$ be to singular submanifolds with $\mathrm{j}_\infty S$, $\mathrm{j}_\infty\overline{S} \subset \mathcal{S}^\infty$. There exists a local diffeomorphism $\varphi \in \mathcal{G}$ sending $S$ onto $\overline{S}$ if and only if $\rho$ has the same order $s = \overline{s}$ on $S$ and $\overline{S}$, and the $(s{+}1)$-st order signature manifolds $\mathfrak{S}^{(s+1)}(\rho, S)$, $\mathfrak{S}^{(s+1)}(\rho, \overline{S})$ overlap. Moreover, if $z_0 \in S$ and $\overline{z}_0 \in \overline{S}$ are any points mapping to the same point

$$\mathbf{I}_S^{(s+1)}(z_0) = \mathbf{I}_{\overline{S}}^{(s+1)}(\overline{z}_0) \in \mathfrak{S}^{(s+1)}(\rho, S) \cap \mathfrak{S}^{(s+1)}(\rho, \overline{S})$$

on the overlap of the two signature manifolds, then there is a family of local equivalence maps mapping $z_0$ to $\overline{z}_0$. Any two equivalence maps $\varphi$, $\psi$ are related by

$$\psi = \alpha \circ \varphi \circ \gamma, \qquad \text{with} \qquad \mathrm{j}_\infty\alpha|_{\overline{z}_0} \in \mathcal{G}_{\mathrm{j}_\infty\overline{S}|_{\overline{z}_0}} \text{ and } \mathrm{j}_\infty\gamma|_{z_0} \in \mathcal{G}_{\mathrm{j}_\infty S|_{z_0}}.$$

*Proof.* The proof is the same as the proof of Theorem 4.6. The only difference is that the diffeomorphism $\varphi \in \mathcal{G}$ mapping $S$ onto $\overline{S}$ in the neighborhoods of $z_0 \in S$ and $\overline{z}_0 \in \overline{S}$ is not uniquely defined. The diffeomorphism $\varphi \in \mathcal{G}$ can be precomposed by any $\gamma \in \mathcal{G}$ such that $\mathrm{j}_\infty\gamma|_{z_0} \in \mathcal{G}_{\mathrm{j}_\infty S|_{z_0}}$ and compose by $\alpha \in \mathcal{G}$ with $\mathrm{j}_\infty\alpha|_{\overline{z}_0} \in \mathcal{G}_{\mathrm{j}_\infty\overline{S}|_{\overline{z}_0}}$ to obtain a new equivalence map $\psi = \alpha \circ \varphi \circ \gamma$. □

**Remark 4.8.** We note that in Theorem 4.7 the isotropy groups $\mathcal{G}_{\mathrm{j}_\infty\overline{S}|_{\overline{z}_0}}$, $\mathcal{G}_{\mathrm{j}_\infty S|_{z_0}}$ are isomorphic as they are conjugate to each other.

## 4.3 Equivalence of Coframes

In Cartan's framework, a local equivalence problem is solved by recasting it as an equivalence problem between coframes. We now specialize the preceding results to this type of problem. Our exposition follows the treatments [19, 27]. Let $\mathcal{H}$ be a Lie pseudo-group action on a $p$-dimensional manifold $X$ and let

$$\boldsymbol{\gamma} = \{\gamma^i = \sum_{j=1}^p u^i_j(x)dx^j, i = 1, \ldots, p\}, \qquad \overline{\boldsymbol{\gamma}} = \{\overline{\gamma}^i = \sum_{j=1}^p \overline{u}^i_j(\overline{x})d\overline{x}^j, i = 1, \ldots, p\} \quad (4.4)$$

be two coframes on $X$ adapted to a given equivalence problem. Clearly, the functions $u^i_j(x)$ and $\overline{u}^i_j(\overline{x})$ depend on the geometry of the equivalence problem. We refer the reader to [27] for an extensive discussion as to how to formulate equivalence problems in terms of differential forms. The equivalence problem for the coframes (4.4) consists of determining whether there exists or not a local diffeomorphism $\varphi \in \mathcal{H}$ such that

$$d\varphi^*(\overline{\gamma}^i) = \sum_{j=1}^p h^i_j(x)\gamma^j, \qquad \text{for} \qquad i = 1, \ldots, p. \quad (4.5)$$

The matrix $(h^i_j(x)) \in GL(p)$ in (4.5) is an element of a certain Lie group $H$ called the *structure group* of the equivalence problem. The primary goal of Cartan's equivalence method is to try to reduce the $H$-structure to an $\{e\}$-structure through a series of "invariant operations" after which it is possible to determine if the two coframes (4.4) are equivalent.

Considering coframes on $X$ as sections of the coframe bundle $\mathcal{F}(X)$, the equivalence problem for coframes can be interpreted as an equivalence problem of sections in $\mathcal{F}(X)$. To do so, let $M \subset \mathcal{F}(X)$ be the subbundle of all coframes (4.4) adapted to an equivalence problem. Then the action of $\mathcal{H}$ on $X$ naturally induces a Lie pseudo-group action $\mathcal{G}$ on the subbundle $M$ via the the equivalence criterion (4.5). In terms of the pseudo-group action $\mathcal{G}$ on $M$, two coframes $\boldsymbol{\gamma}, \overline{\boldsymbol{\gamma}}$ are locally equivalent if and only if the corresponding sections $S$, $\overline{S}$ in $M$ are equivalent up to a transformation $\varphi \in \mathcal{G}$. Introducing the jet bundle of sections of $M$ one can apply the equivariant moving frame apparatus to the pseudo-group action $\mathcal{G}$ to find the differential invariants of the equivalence problem.

**Remark 4.9.** In a system of local coordinates $M \simeq X \times U$ we note that the pseudo-group action $\mathcal{G}$ on $M$ is an isomorphic prolongation of the pseudo-group action $\mathcal{H}$ on $X$ (recall Definition 2.8). This observation will be used in the next section.

**Example 4.10.** Consider the local equivalence of first order Lagrangians (2.16) under the pseudo-group of fiber preserving transformations

$$\mathcal{H}: \qquad X = \phi(x), \qquad U = \beta(x,u), \qquad P = \frac{p\beta_u + \beta_x}{\phi_x} = \frac{\psi(x,u,p)}{\phi_x}. \tag{4.6}$$

In Cartan's formalism, the coframes adapted to the equivalence problem are

$$\boldsymbol{\gamma} = \{\gamma^1 = du - p dx,\ \gamma^2 = l dx,\ \gamma^3 = dp\}, \tag{4.7}$$

where $l = l(x,u,p)$ is any nonzero Lagrangian. The subbundle $M \subset \mathcal{F}(X)$ adapted to the equivalence problem is thus parametrized by $(x,u,p,l)$ where $(x,u,p)$ play the role of the independent variables and $l$ the role of the dependent variable. In terms of the differential forms (4.7) the equivalence problem is encoded by requiring that

$$d\varphi^* \begin{pmatrix} dU - PdX \\ LdX \\ dP \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ b & c & e \end{pmatrix} \begin{pmatrix} du - pdx \\ ldx \\ dp \end{pmatrix} \qquad \text{for a diffeomorphism } \varphi \text{ in } \mathcal{H}.$$

In terms of the pseudo-group action (4.6), the structure group parameters are given by

$$a = \beta_u, \qquad b = \frac{\psi_u}{\phi_x}, \qquad c = \frac{\psi_x + p\psi_u - \gamma\psi}{l\phi_x}, \qquad e = \frac{\beta_u}{\phi_x}.$$

The induced action on the fibers parametrized by the Lagrangian variable $l$ is obtained by requiring that the equality $LdX = ldx$ holds. This gives

$$L = \frac{l}{\phi_x}.$$

At the end of Cartan's equivalence algorithm there are three possible outcomes, each having their counterparts in the equivariant moving frame theory.

**a) Complete Normalization:** The $H$-structure is reducible to an $\{e\}$-structure on $X$. This occurs when the sections of $M \subset \mathcal{F}(X)$ are regular and lie in the domain of definition of an equivariant moving frame.

**b) Prolongation:** The equivalence problem is *prolonged* to a larger space on which the equivalence problem reduces to an $\{e\}$-structure problem. This situation occurs when the sections of $M$ are singular and their isotropy pseudo-groups are finite-dimensional.

**c) Involution:** The structure equations of the invariant coframe are in involution. This happens when the sections of $M$ are singular and their isotropy pseudo-groups are infinite-dimensional.

We now consider each cases separately and illustrate them with examples. Our focus is on recovering results obtained with Cartan's equivalence method using the equivariant moving frame apparatus.

### 4.3.1 Complete Normalization

Let $\mathcal{G}$ be the Lie pseudo-group action on $M \subset \mathcal{F}(X)$ induced by a Lie pseudo-group $\mathcal{H}$ acting on $X$. Given a moving frame $\rho\colon \mathcal{V}^\infty \to \mathcal{E}^{(\infty)}$ and a regular section $S \subset M$ there is a canonical invariant coframe on $X$. This coframe is obtain by restricting the invariantization of the horizontal forms $dx^i$ to the section $S$:

$$\boldsymbol{\varpi} = \boldsymbol{\varpi}|_S = \{\varpi^i = \iota(dx^i)\}|_S = \{\omega^i = \rho^*(d_H X^i)\}|_S = \boldsymbol{\omega}|_S = \boldsymbol{\omega}. \tag{4.8}$$

The equality between $\boldsymbol{\varpi}|_S$ and $\boldsymbol{\omega}|_S$ in (4.8) follows from the fact that contact forms vanish on $S$ (In fact, since the pseudo-group actions we are presently considering are projectable, we have equality between $\boldsymbol{\varpi}$ and $\boldsymbol{\omega}$ before their restriction to $S$). The *coframe derivatives*, which are the dual operators to $\boldsymbol{\omega}$, are obtained by restricting the invariant total differential oparators $\mathcal{D}_i$ to the section $S$:

$$\mathcal{D}_i = \mathcal{D}_i|_S = \frac{\partial}{\partial \omega^i}, \qquad i = 1, \ldots, p.$$

Provided the cross-section defining the equivariant moving frame is compatible with the normalizations leading to an $\{e\}$-structure in Cartan's algorithm, the coframe (4.8) will be identical to the one obtained by Cartan's equivalence method. Now there are three ways to derive the structure equations of the invariant coframe (4.8):

$$d\omega^i = \sum_{1 \le j < k \le p} T^i_{jk}\, \omega^j \wedge \omega^k, \qquad i = 1, \ldots, p: \tag{4.9}$$

One can of course compute the exterior differential of the invariant one-forms $\omega^i$ using their coordinate expressions but a better strategy is do the computations symbolically by either using the recurrence relations (3.25) or the structure equations (2.8) of the equivalence pseudo-group $\mathcal{H}$ (or[4] $\mathcal{G}$). We privilege the latter approach as it offers a unified approach to recover the three possible outcomes of Cartan's equivalence method.

**Proposition 4.11.** The structure equations (4.9) are obtained by taking the pull-back of the structure equations of the equivalence pseudo-group $\mathcal{H}$ (or $\mathcal{G}$) by the moving frame $\rho$ and restricting the result to a section $S \subset M \subset \mathcal{F}(X)$.

[4]Since $\mathcal{G}$ is an isomorphic prolongation of $\mathcal{H}$, the structural properties of $\mathcal{G}$ and $\mathcal{H}$ is essentially the same.

*Proof.* Let $(z^1,\ldots,z^p,z^{p+1},\ldots,z^m) = (x^1,\ldots,x^p,u^1,\ldots,u^q)$ be adapted coordinates on $M \simeq X \times U$. The equalities $dZ^a = \sigma^a + \mu^a$, $a = 1,\ldots,m$, combined with the recurrence relations (3.24):

$$dX^i = \Omega^i + \mu^i, \quad i = 1,\ldots,p, \qquad dU^\alpha = \sum_{j=1}^{p} \widehat{U}^\alpha_j \Omega^j + \Theta^\alpha + \mu^{p+\alpha}, \quad \alpha = 1,\ldots,q,$$

lead to the identities $\sigma^i = \Omega^i$ and $\sigma^{p+\alpha} = \sum_{j=1}^p \widehat{U}^\alpha_j \Omega^j + \Theta^\alpha$. Hence once pulled-back by a moving frame $\rho$ and restricted to a section $S$ we obtain the equalities

$$[\rho^*\sigma^i]|_S = \omega^i|_S, \qquad [\rho^*\sigma^{p+\alpha}]|_S = \sum_{i=1}^{p} I^\alpha_j \omega^j|_S. \tag{4.10}$$

On the other hand, the Maurer–Cartan forms reduce to

$$\nu^b_A = \nu^b_A|_S = \rho^*(\mu^b_A)|_S = \sum_{j=1}^{p} F^b_{A;j}(H, I^{(\infty)})\omega^j|_S, \quad b = 1,\ldots,m, \quad \#A \geq 0. \tag{4.11}$$

Substituting the expressions (4.10), (4.11) into the structure equations of the pseudo-group $\mathcal{H}$ (or $\mathcal{G}$) leads to the structure equations (4.9). □

**Remark 4.12.** Only the structure equations for $\sigma^1,\ldots,\sigma^p$ are needed to obtain the structure equations (4.9). The pull-back of the remaining structure equations lead to syzygies among the normalized invariants. We note that those syzygies can also be recovered from the recurrence relations (3.25), and in [35] the authors give a computational algorithm for locating a finite system of generating differential syzygies among the generating differential invariants.

**Example 4.13.** The local equivalence under the fiber preserving transformations (2.17) of first order variational problems (2.16) satisfying the non-degeneracy conditions

$$l \neq 0, \qquad l_p \neq 0, \qquad l_{pp} \neq 0, \tag{4.12}$$

is an example of equivalence problem where all submanifold jets are regular. The first two non-degeneracy conditions in (4.12) are clear. The Lagrangian is assumed to be nonzero and to depend on the derivative coordinate $p$. The third hypothesis says that the Lagrangian should not be an affine function in the derivative coordinate

$$l(x,u,p) \neq a(x,u)p + b(x,u)$$

as such Lagrangian is equivalent, modulo the addition of a suitable divergence, to a degenerate Lagrangian that does not depend on the derivative coordinate, [21]. The construction of a moving frame, the computation of differential invariants and their recurrence relations was the content of Examples 3.7 and 3.22. The last step in the solution to the equivalence problem is to analyze the signature manifold. The detailed analysis, using Cartan's approach, can be found in [21] and in [27, pp. 321–327]. The rank of the moving frame (3.12) is either 0, 1, 2 or 3. The simplest case is when the rank is zero which means that all the differential invariants are constant. It is important to note that the constant values taken by the invariants are not completely arbitrary. For example, in the problem we are concerned with it follows from (3.39) that $I_{pp} \neq 0$ since

$l_{pp} \neq 0$. Also the normalized invariants must satisfy the recurrence relations (3.46), (3.47). For example, the recurrence relations (3.46) imply

$$I_{ppx} = I_{ppu} = 0, \qquad I_{ppp} = I_{pp}(2I_{pp} - 1),$$

while the fourth and last recurrence relations in (3.47) give

$$0 = (I_{pp} + 1)I_{pux}.$$

Thus unless $I_{pp} \neq -1$ we must have $I_{pux} = 0$. In any case, by a standard result in the equivalence theory of coframes, [27], the structure equations of the invariant coframe $\boldsymbol{\omega} = \{\omega^x, \omega^u, \omega^p\}$ associated with a rank zero moving frame will be those of a three-dimensional Lie group. The structure equations of the invariant coframe $\boldsymbol{\omega}$ are obtained by taking the pull-back of the structure equations of $\sigma^x = \Omega^x$, $\sigma^u = \Omega^u$, $\sigma^p = \Omega^p$ in (2.23) by the moving frame (3.12) and restricting the result to a submanifold $(x, u, p, l(x, u, p))$. Symbolically this is done by substituting the horizontal component of the expressions (3.44) for pulled–back Mauer–Cartan forms in (2.23), and setting $P$ equal to zero as $\rho^*(P) = 0$ by virtue of our choice of cross-section (3.11). The result are the structure equations

$$d\omega^x = \omega^p \wedge \omega^x, \quad d\omega^u = I_{pp}\, \omega^p \wedge \omega^u + \omega^x \wedge \omega^p, \quad d\omega^p = \frac{I_{pux}}{I_{pp}} \omega^u \wedge \omega^x + \frac{I_{ppx}}{I_{pp}} \omega^p \wedge \omega^x. \tag{4.13}$$

The reader interested in comparing the structure equations (4.13) with those that one would obtain with Cartan's equivalence method is invited to look at the structure equations [27, eq. (10.58)]. The passage between the two sets of equations is given by the equalities

$$\theta^1 = \omega^u, \qquad \theta^2 = \omega^x, \qquad \theta^3 = \omega^p, \qquad I_1 = I_{pp}, \qquad I_2 = \frac{I_{pux}}{I_{pp}}, \qquad I_3 = -\frac{I_{ppx}}{I_{pp}}.$$

As previously mentioned, the syzygies [27, eq. (10.61)], obtained by requiring that $d^2\theta^1 = d^2\theta^2 = d^2\theta^3 = 0$, can also be found using the recurrence relations (3.46), (3.47). First, from (3.46) we immediately see that

$$-\frac{I_{ppx}}{I_{pp}} = -\frac{1}{I_{pp}} \mathcal{D}_x(I_{pp}),$$

which is the first syzygy of [27, eq. (10.61)]. Then from the eighth identity in (3.47) we have

$$I_{ppux} = \mathcal{D}_u(I_{ppx}) + I_{pux}\left(1 + \frac{I_{ppp}}{I_{pp}}\right),$$

which when substituted into the last identity of (3.47) yields the second syzygy of [27, eq. (10.61)]:

$$\mathcal{D}_p\left(\frac{I_{pux}}{I_{pp}}\right) - \mathcal{D}_u\left(\frac{I_{ppx}}{I_{pp}}\right) + (1 + I_{pp})\frac{I_{pux}}{I_{pp}} = 0.$$

We end this example by pointing out that the structure equations (4.13) can be used to obtain a simple expression for the invariant $I_{pux}$ in terms of $I_{pp}$. Indeed, from the structure equations (4.13) it follows that the commutation relations between the invariant differential operators are

$$[\mathcal{D}_x, \mathcal{D}_p] = \mathcal{D}_x - \mathcal{D}_u + \frac{I_{ppx}}{I_{pp}}\mathcal{D}_p, \quad [\mathcal{D}_u, \mathcal{D}_p] = I_{pp}\mathcal{D}_u, \quad [\mathcal{D}_x, \mathcal{D}_u] = \frac{I_{pux}}{I_{pp}}\mathcal{D}_p.$$

Hence if $\mathcal{D}_p I_{pp} \neq 0$ we conclude from the third commutation relation that

$$I_{pux} = \frac{I_{pp}}{\mathcal{D}_p I_{pp}} [\mathcal{D}_x, \mathcal{D}_u] I_{pp}.$$

**Proposition 4.14.** If $\mathcal{D}_p I_{pp} \neq 0$ the algebra of differential invariants of the pseudo-group action (2.17) is generated by the single invariant $I_{pp} = \dfrac{l l_{pp}}{l_p^2}$.

### 4.3.2 Prolongation

Let $\mathcal{G}$ be a Lie pseudo-group action on $M \subset \mathcal{F}(X)$ and $\mathcal{S}^\infty \subset \mathrm{J}^\infty$ a subbundle of singular submanifold jets. For the moment we assume that the fibers of isotropy pseudo-group $\mathcal{G}_{\mathcal{S}^\infty}$ are finite-dimensional. Let $g = (g_1, \ldots, g_r)$, $r = \dim \mathcal{G}_{z^{(\infty)}} < \infty$, be the pseudo-group jets parametrizing the fibers of $\mathcal{G}_{\mathcal{S}^\infty}$.

**Theorem 4.15.** Let $\rho$ be a (fully regular) partial moving frame defined on the bundle of singular submanifold jets $\mathcal{S}^\infty$ with $r$-dimensional isotropy pseudo-group $\mathcal{G}_{\mathcal{S}^\infty}$, i.e. $\dim \mathcal{G}_{z^{(\infty)}} = r$ for all $z^{(\infty)} \in \mathcal{S}^\infty$. Let $S$, $\overline{S}$ be two singular sections of $M \subset \mathcal{F}(X)$ with $\mathrm{j}_\infty S$, $\mathrm{j}_\infty \overline{S} \subset \mathcal{S}^\infty$. There exists a local diffeomorphism $\varphi \in \mathcal{G}$ sending $S$ onto $\overline{S}$ if and only if $\rho$ has the same order $s = \overline{s}$ on $S$ and $\overline{S}$, and the $(s{+}1)$-st order signature manifolds $\mathfrak{S}^{(s+1)}(\rho, S)$, $\mathfrak{S}^{(s+1)}(\rho, \overline{S})$ overlap. Moreover, if $z_0 \in S$ and $\overline{z}_0 \in \overline{S}$ are any points mapping to the same point

$$\mathbf{I}_S^{(s+1)}(z_0) = \mathbf{I}_{\overline{S}}^{(s+1)}(\overline{z}_0) \in \mathfrak{S}^{(s+1)}(\rho, S) \cap \mathfrak{S}^{(s+1)}(\rho, \overline{S})$$

on the overlap of the two signature manifolds, then there is an $r$-dimensional family of local equivalence maps sending $z_0$ to $\overline{z}_0$ locally parametrized the fibers of the isotropy pseudo-group $\mathcal{G}_{\mathcal{S}^\infty}$.

Let $\boldsymbol{\mu} = \{\mu^1, \ldots, \mu^r\}$ be the Maurer–Cartan forms associated with the pseudo-group parameters $g = (g_1, \ldots, g_r)$. Then the invariant horizontal forms (4.8) together with $\boldsymbol{\nu} = \rho^*(\boldsymbol{\mu})|_S$ form a finite-dimensional invariant coframe on the finite-dimensional space $\mathcal{E}_{\mathcal{S}^\infty}|_S$. Since the prolonged coframe $\{\boldsymbol{\omega}, \boldsymbol{\nu}\}$ contains the moving frame pull-back of the Maurer–Cartan forms $\boldsymbol{\mu}$, their structure equations are contained in the pull-back of the structure equations of $\mathcal{H}$ (or $\mathcal{G}$) by the partial moving frame (after restriction to a singular section $S \subset \mathcal{F}(X)$). Indeed, when $\rho$ is a partial moving frame, the pull-backs (4.11) are linear combinations of the "Maurer–Cartan" forms $\nu^1, \ldots, \nu^r$ and the invariant horizontal forms $\omega^1, \ldots, \omega^p$:

$$\nu_A^b = \nu_A^b|_S = \rho^*(\mu_A^b)|_S = \sum_{i=1}^p F_{A;i}^b(H, I^{(\infty)})\omega^i + \sum_{l=1}^r G_{A;l}^b(H, I^{(\infty)})\nu^l,$$

$b = 1, \ldots, m$, $\#A \geq 0$.

**Example 4.16.** In this example we consider the local equivalence of second order ordinary differential equations

$$u_{xx} = F(x, u, u_x)$$

under the pseudo-group of fiber preserving transformations

$$X=\phi(x), \qquad U=\beta(x,u), \qquad P=\frac{p\beta_u+\beta_x}{\phi_x}=\frac{\psi}{\phi_x}, \tag{4.14a}$$

$$Q=\frac{p\psi_u+\psi_x+q\beta_u-\gamma\psi}{\phi_x^2}, \qquad \text{with} \qquad \gamma=\frac{\phi_{xx}}{\phi_x}, \tag{4.14b}$$

and where we use the notation $p=u_x$, $q=u_{xx}$. The solution based on Cartan's equivalence method can be found in [27, p. 397]. We now reconsider this problem using the method of equivariant moving frames.

The infinitesimal generators of the pseudo-group action (4.14) are

$$\mathbf{v}=\xi(x)\frac{\partial}{\partial x}+\eta(x,u)\frac{\partial}{\partial u}+[p(\eta_u-\xi_x)+\eta_x]\frac{\partial}{\partial p}+[p^2\eta_{uu}+p(2\eta_{ux}-\xi_{xx})+q(\eta_u-2\xi_x)+\eta_{xx}]\frac{\partial}{\partial q},$$

where $\xi(x)$ and $\eta(x,u)$ are two arbitrary differentiable functions. Computing their prolongation and substituting the result into (3.26) leads, modulo the lifted contact forms $\Theta_J$, to the recurrence relations

$$\begin{aligned}
dX =&\Omega^x+\mu^x, \qquad dU=\Omega^u+\mu^u, \qquad dP=\Omega^p+P(\mu^u_X-\mu^x_X)+\mu^u_X,\\
dQ \equiv&Q_X\Omega^x+Q_U\Omega^u+Q_P\Omega^p+P^2\mu^u_{UU}+P(2\mu^u_{UX}-\mu^x_{XX})-Q(\mu^u_U-2\mu^x_X)+\mu^u_{XX},\\
dQ_P \equiv&Q_{PX}\Omega^x+Q_{PU}\Omega^u+Q_{PP}\Omega^p+2P\mu^u_{UU}+2\mu^u_{UX}-\mu^x_{XX}-Q_P\mu^x_X,\\
dQ_U \equiv&Q_{UX}\Omega^x+Q_{UU}\Omega^u+Q_{UP}\Omega^p+P^2\mu^u_{UUU}+2P\mu^u_{UUX}-2Q_U\mu^x_X+Q\mu^u_{UU}\\
&+\mu^u_{UXX}-Q_P(P\mu^u_{UU}+\mu^u_{UX}),\\
dQ_X \equiv&Q_{XX}\Omega^x+Q_{XU}\Omega^u+Q_{XP}\Omega^p+P^2\mu^u_{UUX}+P(2\mu^u_{UXX}-\mu^x_{XXX})\\
&+Q_X(\mu^u_U-3\mu^x_X)+Q(\mu^u_{UX}-2\mu^x_{XX})+\mu^u_{XXX}-Q_U\mu^u_X\\
&-Q_P[p(\mu^u_{UX}-\mu^x_{XX})+\mu^u_{XX}],
\end{aligned} \tag{4.15}$$

and so on. With the recurrence relations (4.15) in hand, the objective is to normalize as many Maurer–Cartan forms as possible without imposing any non-degeneracy conditions on the invariants. For example, the equation

$$dX=\Omega^x+\mu^x$$

implies that we can always translate the invariant $X$ to take the value zero. On the other hand, if we wanted to use the recurrence relation

$$dQ_{PPP}\equiv Q_{PPPX}\Omega^x+Q_{PPPU}\Omega^u+Q_{PPPP}\Omega^p+(\mu^x_X-2\mu^u_U)Q_{PPP}$$

to normalize the Maurer–Cartan form $\mu^u_U$ we would need to impose the non-degeneracy $Q_{PPP}\neq 0$, which for the moment we do not want to impose.

From the recurrence relations (4.15) (and their prolongations) we come to the conclusion that all the Maurer–Cartan forms, except for $\mu^x_X$, $\mu^u_U$, $\mu^u_{UX}$, can be normalized without making non-degeneracy hypothesis on the invariants. An admissible cross-section is given by

$$X=U=P=0, \qquad Q_{U^iX^j}=Q_{PU^i}=Q_{PPU^i}=Q_{PX^j}=0, \qquad i,j\geq 0. \tag{4.16}$$

Using the recurrence relations of the phantom invariants (4.16) to solve for the (moving frame pulled-back) Maurer–Cartan forms we obtain

$$\begin{gathered}\nu^x = -\omega^x, \qquad \nu^u = -\omega^u, \qquad \nu^p = \nu^u_X = -\omega^p, \qquad \nu^u_{XX} = 0,\\ \nu^u_{UU} = -\frac{1}{2}(I_{ppx}\omega^x + I_{ppp}\omega^p), \qquad \nu^x_{XX} = 2\nu^u_{UX},\\ \nu^u_{XXX} = \nu^u_{UXX} = \nu^u_{UUX} = 0, \quad \nu^u_{UUU} = -\frac{1}{2}(I_{ppux}\omega^x + I_{pppu}\omega^p - I_{ppp}\nu^u_{UX}),\\ \nu^u_{U^iX^j} = 0, \quad i+j \geq 4, \quad \nu^x_{X^k} = 0, \quad k \geq 4, \quad (\text{mod } \omega^x, \omega^u, \omega^p, \nu^x_X, \nu^u_U, \nu^u_{XU}).\end{gathered} \tag{4.17}$$

To obtain the structure equations of the prolonged coframe $\{\omega^x, \omega^u, \omega^p, \nu^x_X, \nu^u_U, \nu^u_{XU}\}$ we simply substitute the expressions (4.17) into the structure equations of the pseudo-group action (4.14a). The structure equations of the pseudo-group (4.14a) are given by (2.23) where the differential of $\sigma^l$ is omitted. The result is

$$\begin{gathered}d\omega^x = \nu^x_X \wedge \omega^x, \qquad d\omega^u = \nu^u_U \wedge \omega^u + \omega^x \wedge \omega^p,\\ d\omega^p = \nu^u_{XU} \wedge \omega^u + (\nu^u_U - \nu^x_X) \wedge \omega^p, \qquad d\nu^x_X = -2\nu^x_X \wedge \omega^x,\\ d\nu^u_U = -\frac{I_{ppp}}{2}\omega^u \wedge \omega^p - \frac{I_{ppx}}{2}\omega^u \wedge \omega^x - \nu^u_{XU} \wedge \omega^x,\\ d\nu^u_{XU} = -\frac{I_{pux}}{2}\omega^u \wedge \omega^x - \frac{I_{ppx}}{2}\omega^p \wedge \omega^x + \nu^u_{XU} \wedge \nu^x_X.\end{gathered} \tag{4.18}$$

The structure equations (4.18) are equivalent to the structure equations [27, eq. (12.61)] obtained with Cartan's equivalence method. The correspondence between the differential forms and the invariants is given by

$$\begin{gathered}\theta^1 = \omega^u, \qquad \theta^2 = \omega^p, \qquad \theta^3 = \omega^x, \qquad \pi^1 = \nu^u_U, \qquad \pi^2 = \nu^u_{XU}, \qquad \pi^6 = \nu^x_{XU},\\ J_1 = \frac{I_{ppp}}{2}, \qquad J_2 = -\frac{I_{ppx}}{2}, \qquad J_3 = -\frac{I_{pux}}{2}.\end{gathered}$$

Further reduction of the Maurer–Cartan forms $\nu^u_U$, $\nu^u_{XU}$, $\nu^x_{XU}$ depends on the values of $I_{pux}$, $I_{ppp}$, $I_{ppx}$. If the three invariants are equal to zero then all higher order invariants are automatically zero. This can be seen from the recurrence relations (3.24) as follows. Combining the assumption that $I_{pux} = I_{ppp} = I_{ppx} = 0$ with the cross-section (4.16) we conclude that all invariants of order less or equal to three are zero. Since the correction terms $\widehat{\phi}^{\alpha;J}$ in (3.24) only involve invariants of order less or equal to the order of the multi-index $J$, the correction terms in the recurrence relations for the invariants $I_{pux}$, $I_{ppp}$, $I_{ppx}$ are identically zero. This implies

$$\begin{gathered}I_{pppp} = \mathcal{D}_p(I_{ppp}) = 0, \qquad I_{pppu} = \mathcal{D}_u(I_{ppp}) = 0, \qquad I_{pppx} = \mathcal{D}_x(I_{ppp}) = 0,\\ I_{ppux} = \mathcal{D}_u(I_{ppx}) = 0, \qquad I_{ppxx} = \mathcal{D}_x(I_{ppx}) = 0, \qquad I_{puux} = \mathcal{D}_u(I_{pux}) = 0,\\ I_{puxx} = \mathcal{D}_x(I_{pux}) = 0,\end{gathered}$$

and when combined with the cross-section (4.16) this means that all fourth order invariants are zero. Iterating the argument order by order we come to the conclusion that all invariants are zero, that it is not possible to normalize the Maurer–Cartan forms $\nu^u_U$, $\nu^u_{XU}$, $\nu^x_{XU}$ and that

$$\mathcal{S}^\infty = \{(x, u, p, l^{(\infty)}) : I_{J,pux} = I_{J,ppp} = I_{J,ppx} = 0,\ \#J \geq 0\} \subset J^\infty \tag{4.19}$$

is the bundle of singular submanifold jets. Since the Maurer–Cartan forms $\nu^u_U$, $\nu^u_{XU}$, $\nu^x_{XU}$ cannot be normalized on (4.19), the isotropy pseudo-group of any submanifold jet in (4.19) is parametrized by the pseudo-group parameters $\phi_x$, $\beta_u$, $\beta_{xu}$. For this branch of the equivalence problem, the structure equations (4.18) are the structure equations of the six-parameter fiber-preserving symmetry group of all second order differential equations satisfying $I_{pux} = I_{ppp} = I_{ppx} = 0$, [27]. The coordinate expressions for the invariants $I_{pux}$, $I_{ppp}$, $I_{ppx}$ and the invariant coframe $\{\omega^x, \omega^u, \omega^p, \nu^x_X, \nu^u_U, \nu^u_{XU}\}$ are obtained by implementing the (partial) moving frame algorithm of Section 3.

If certain of the invariants $I_{pux}$, $I_{ppp}$, $I_{ppx}$ are non-zero then some of the Maurer–Cartan forms $\nu^u_U$, $\nu^u_{XU}$, $\nu^x_{XU}$ can be normalized leading to the different branches of the equivalence problem. For example, in the generic case $I_{ppp}I_{ppx} \neq 0$ we can normalize all three Maurer–Cartan forms $\nu^x_X$, $\nu^u_U$, $\nu^u_{XU}$ by setting $I_{ppp} = I_{ppx} = 1$ and $I_{pux} = 0$. Using the recurrence relations for $I_{ppp}$, $I_{ppx}$ and $I_{pux}$, and solving for the Maurer–Cartan forms $\nu^x_X$, $\nu^u_U$, $\nu^u_{XU}$ we obtain

$$\begin{aligned}
\nu^x =& \frac{2I_{ppxx} - I_{pppx}}{3}\omega^x + \frac{2I_{ppxu} - I_{pppu}}{3}\omega^u + \frac{2I_{pppx} - I_{pppp}}{3}\omega^p,\\
\nu^u_U =& \frac{I_{pppx} + I_{ppxx}}{3}\omega^x + \frac{I_{pppu} + I_{ppux}}{3}\omega^u + \frac{I_{pppp} + I_{pppx}}{3}\omega^p,\\
\nu^u_{XU} =& I_{puxx}\omega^x + I_{puux}\omega^u + I_{ppux}\omega^p.
\end{aligned}$$

Substituting the latter expressions into the structure equations of $\omega^x$, $\omega^u$, $\omega^p$ in (4.18) leads to

$$\begin{aligned}
d\omega^x =& \frac{2I_{ppux} - I_{pppu}}{3}\omega^u \wedge \omega^x + \frac{2I_{pppx} - I_{pppp}}{3}\omega^p \wedge \omega^x,\\
d\omega^u =& \frac{I_{pppx} - I_{ppxx}}{3}\omega^x \wedge \omega^u + \frac{I_{pppp} - I_{pppx}}{3}\omega^p \wedge \omega^u + \omega^x \wedge \omega^p,\\
d\omega^p =& I_{puxx}\omega^x \wedge \omega^u + \frac{4I_{ppux} - 2I_{pppu}}{3}\omega^p \wedge \omega^u + \frac{2I_{pppx} - I_{ppxx}}{3}\omega^x \wedge \omega^p.
\end{aligned} \tag{4.20}$$

The structure equations (4.20) are the same as [27, eq. (12.66)] with

$$I_1 = \frac{I_{pppx} - I_{ppxx}}{3}, \qquad I_2 = \frac{2I_{pppx} - I_{pppp}}{3}, \qquad I_3 = 2\frac{I_{pppx} - I_{ppxx}}{3},$$
$$I_4 = \frac{2I_{ppux} - I_{pppu}}{3}, \qquad I_5 = -I_{puxx}.$$

### 4.3.3 Involution

Finally, we consider the case when the fibers of $\mathcal{G}_{\mathcal{S}^\infty}$ are infinite-dimensional. By virtue of Theorem 3.41 the size of the fibers is determined by the Cartan characters, and Theorem 4.7 can be specialized as follows.

**Theorem 4.17.** Let $\rho$ be a (fully regular) partial moving frame defined on the bundle of singular submanifold jets $\mathcal{S}^\infty$ with infinite-dimensional isotropy pseudo-group $\mathcal{G}_{\mathcal{S}^\infty}$. Let $S$, $\overline{S}$ be two singular sections of $M \subset \mathcal{F}(X)$ with $\mathrm{j}_\infty S$, $\mathrm{j}_\infty \overline{S} \subset \mathcal{S}^\infty$. There exists a local diffeomorphism $\varphi \in \mathcal{G}$ sending $S$ onto $\overline{S}$ if and only if $\rho$ has the same order $s = \overline{s}$ on $S$ and $\overline{S}$, and the $(s+1)$-st order signature manifolds $\mathfrak{S}^{(s+1)}(\rho, S)$, $\mathfrak{S}^{(s+1)}(\rho, \overline{S})$ overlap. Moreover, if $z_0 \in S$ and $\overline{z}_0 \in \overline{S}$ are any points mapping to the same point

$$\mathbf{I}^{(s+1)}_S(z_0) = \mathbf{I}^{(s+1)}_{\overline{S}}(\overline{z}_0) \in \mathfrak{S}^{(s+1)}(\rho, S) \cap \mathfrak{S}^{(s+1)}(\rho, \overline{S})$$

on the overlap of the two signature manifolds, then the set of local equivalence maps sending $z_0$ to $\overline{z}_0$ depend on $f_k$ arbitrary functions of $k$ variables, where the $f_k$'s are given by formula (3.60).

As in the preceding section, let $g^{(n)} = (g_1, \ldots, g_{r_n})$ be the pseudo-group parameters parametrizing $\mathcal{G}_{\mathbb{S}_n^\infty}$, $\boldsymbol{\mu}^{(n)}$ the corresponding Maurer–Cartan forms and $\boldsymbol{\nu}^{(n)} = \rho^*(\boldsymbol{\mu}^{(n)})$ their (partial) moving frame pull-back. Assuming compatibility of the pseudo-group jet normalizations in the equivariant moving method and the normalizations of the structure group parameters in Cartan's equivalence method, the structure equations obtained with Cartan's method are recovered with the equivariant moving frame method by computing the structure equations of the prolonged coframe $\{\boldsymbol{\omega}, \boldsymbol{\nu}^{(n^\star-1)}\}$ where $n^\star$ is the order of involutivity of the isotropy algebra annihilator bundle. A particular feature of the structure equations of the prolonged coframe $\{\boldsymbol{\omega}, \boldsymbol{\nu}^{(n^\star-1)}\}$ is that they depend on the $n^\star$-th order unnormalizable Maurer–Cartan forms. Those play an important role in Cartan's involutivity test, [3, 18, 27].

**Remark 4.18.** The equivalence problems discussed in Sections 4.3.1 and 4.3.2 can be seen as particular instances of the general framework exposed in this section. These equivalence problems are characterized by the property that all their Cartan characters vanish which implies that for $n \geq n^\star$ all the $n$-th order pseudo-group parameters are normalizable. If the pseudo-group action is free on $\mathrm{J}^{n^\star}$ then all pseudo-group parameters of order $\leq n^\star$ are normalizable and we are in the framework of Section 4.3.1 where a moving frame exits. If the action on $\mathrm{J}^{n^\star}$ is not free then finitely many pseudo-group parameters cannot be normalized and those parametrize the finite-dimensional isotropy pseudo-group of the singular submanifold jets.

**Example 4.19.** In this example we extend the fiber preserving equivalence pseudo-group of Example 4.16 to contact transformations:

$$X = \phi(x,u,p), \quad U = \beta(x,u,p), \quad P = \psi(x,u,p), \quad Q = \frac{\psi_x + p\psi_u + q\psi_p}{\phi_x + p\phi_u + q\phi_p}, \tag{4.21}$$

where the functions $\phi$, $\beta$ and $\psi$ satisfy the contact conditions

$$\beta_p = \psi\phi_p, \quad \beta_x - \psi\phi_x = -p(\beta_u - \psi\phi_u), \quad \text{and} \quad \det\left(\frac{\partial(\phi,\beta,\psi)}{\partial(x,u,p)}\right) \neq 0. \tag{4.22}$$

The corresponding infinitesimal generators are

$$\mathbf{v} = \xi(x,u,p)\frac{\partial}{\partial x} + \eta(x,u,p)\frac{\partial}{\partial u} + \tau(x,u,p)\frac{\partial}{\partial p} + [\tau_x + p\tau_u + q(\tau_p - \xi_x - p\xi_u - q\xi_p)]\frac{\partial}{\partial q},$$

where $\xi(x,u,p)$, $\eta(x,u,p)$ and $\tau(x,u,p)$ are solution to the system of partial differential equations

$$\eta_p = p\xi_p, \qquad \eta_x = p(\xi_x - \eta_u) + \tau + p^2\xi_u, \tag{4.23}$$

obtained by linearizing the determining system (4.22) at the identity jet. The lift of (4.23) implies the linear relations

$$\mu^u_P - P\mu^x_P = 0, \qquad \mu^p - [\mu^u_X + P(\mu^u_U - \mu^x_X) - P^2\mu^x_X] = 0, \tag{4.24}$$

among the Maurer–Cartan forms. Linear relations among higher order Maurer–Cartan forms are obtained by taking the prolongation of (4.24). The computation of the recurrence relations (3.24) for the pseudo-group action (4.21) gives

$$\begin{aligned} dX =&\Omega^x + \mu^x, \qquad dU = \Omega^u + \mu^u, \qquad dX^p = \Omega^p + \mu^p, \\ dQ_J \equiv& Q_{J,X}\Omega^x + Q_{J,U}\Omega^u + Q_{J,P}\Omega^p + \mu^p_{J,X} + \mathbb{D}_J[P\mu^p_U + Q\mu^p_P - Q(\mu^x_X + P\mu^x_U + Q\mu^x_P) \\ &- \mu^x Q_X - \mu^u Q_U - \mu^p Q_P] + \mu^x Q_{J,X} + \mu^u Q_{J,U} + \mu^P Q_{J,P}, \end{aligned} \tag{4.25}$$

modulo the lifted contact forms $\Theta_J$. Since the Maurer–Cartan forms $\mu^P_{J,X}$ are linearly independent it follows from (4.25) that the equivalence pseudo-group is transitive on $\mathrm{J}^\infty$. Since $\dim \mathcal{G}^{(n)} > \dim \mathrm{J}^n$ for all $n$, every submanifold jet is singular, $\mathcal{S}^\infty = \mathrm{J}^\infty$. Choosing the cross-section

$$X = U = P = Q_J = 0, \qquad \#J \geq 0, \tag{4.26}$$

we obtain

$$\nu^x = -\omega^x, \qquad \nu^u = -\omega^u, \qquad \nu^p = -\omega^p, \qquad \nu^p_{J,X} = 0. \tag{4.27}$$

Thus once pulled-back by the partial moving frame (and restricted to a section $S$), the Maurer–Cartan forms of the equivalence pseudo-group must satisfy the relations (4.27) together with the system of equations (4.24) and their prolongations. Under the correspondence (3.55), we can use the right-hand side of those relations to compute the Cartan characters of the isotropy algebra annihilator bundle at the cross-section (4.26). For example, collecting all the relations among the Maurer–Cartan forms in $\nu^x$, $\nu^u$, $\nu^p$ of order $\leq 2$ we obtain the equations

$$\begin{gathered} \nu^x = -\omega^x, \qquad \nu^u = -\omega^u, \qquad \nu^p = -\omega^p, \\ \nu^u_P = 0, \qquad \nu^u_X - \nu^p = 0, \qquad \nu^p_P + \nu^x_X - \nu^u_U = 0, \qquad \nu^p_X = 0, \\ \nu^u_{PX} = 0, \quad \nu^u_{PU} = 0, \quad \nu^u_{PP} = 0, \quad \nu^u_{XX} - \nu^p_X = 0, \quad \nu^u_{UX} - \nu^p_U = 0, \\ \nu^p_{XX} = 0, \quad \nu^p_{UX} = 0, \quad \nu^p_{PX} = 0, \quad \nu^p_{PX} + \nu^x_{XX} - \nu^u_{UX} = 0, \\ \nu^p_{PU} + \nu^x_{UX} - \nu^u_{UU} = 0, \qquad \nu^p_{PP} + 2(\nu^x_{PX} - \nu^u_{PU}) - 2\nu^x_X = 0, \end{gathered} \tag{4.28}$$

to which we associate the collection of polynomials

$$\begin{gathered} T^x, \qquad T^u, \qquad T^p, \\ t_pT^u, \qquad t_xT^u - T^p, \qquad t_pT^p + t_xT^x - t_uT^u, \qquad t_xT^p, \\ t_pt_xT^u, \qquad t_pt_uT^u, \qquad t_p^2T^u, \qquad t_x^2T^u - t_xT^p, \qquad t_ut_xT^u - t_uT^p, \\ t_x^2T^p, \qquad t_ut_xT^p, \qquad t_pt_xT^p, \qquad t_pt_xT^p + t_x^2T^x - t_ut_xT^u, \\ t_pt_uT^p + t_ut_xT^x - t_u^2T^u, \qquad t_p^2T^p + 2(t_pt_xT^x - t_pt_uT^u) - 2t_xT^x. \end{gathered} \tag{4.29}$$

Considering the polynomials of degree one on the second line of (4.29) and computing the symbol matrix (3.57) we obtain

$$\mathbf{T}_1 = \begin{array}{c} \begin{matrix} t_xT^p & t_xT^u & t_xT^x & t_pT^p & t_pT^u & t_pT^x & t_uT^p & t_uT^u & t_uT^x \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}. \tag{4.30}$$

The symbol of the matrix (4.30) are $\beta_1^{(3)} = 3$, $\beta_1^{(2)} = 1$, $\beta_1^{(1)} = 0$. Now, the rank of the symbol matrix $\mathbf{T}_2$ is 11 as the symbol of the polynomials of degree two on lines 3–5 in (4.29) are linearly independent. Since $3\beta_1^{(3)} + 2\beta_1^{(2)} + \beta_1^{(1)} = 11$ the system (4.29) is in involution and the order of involution is $n^\star = 1$. The Cartan characters (3.59) are $\alpha_1^{(3)} = 0$, $\alpha_1^{(2)} = 2$, $\alpha_1^{(1)} = 3$, and we conclude that the general equivalence map depends on two arbitrary analytic functions, each depending on two variables. This is in accordance with the results obtained with Cartan's equivalence method, [27, Example 11.10].

Finally, the structure equations of the invariant coframe $\boldsymbol{\omega} = \{\omega^x, \omega^u, \omega^p\}$ are obtained by substituting $\nu^i = -\omega^i$ in the structure equations (2.8b). Note that since $\rho^*(P) = 0$ the equation (4.24) implies $\nu^p_U = \nu^u_{UX}$. This leads to the recurrence relations

$$\begin{aligned} d\omega^x =&\nu^x_X \wedge \omega^x + \nu^x_U \wedge \omega^u + \nu^x_P \wedge \omega^p, \\ d\omega^u =&\nu^u_U \wedge \omega^u + \omega^x \wedge \omega^p, \\ d\omega^p =&\nu^u_{XU} \wedge \omega^u + (\nu^u_U - \nu^x_X) \wedge \omega^p. \end{aligned} \tag{4.31}$$

The coordinate expressions of the invariant coframe $\boldsymbol{\omega}$ are found by the usual pseudo-group parameter normalization procedure. The requirements that $X = U = P = Q = 0$ implies the normalizations

$$\phi = 0, \quad \beta = 0, \quad \psi = 0, \quad \beta_p = 0, \quad \beta_x = -p\beta_u, \quad \psi_x = -p\psi_u - q\psi_p, \tag{4.32}$$

which leads to

$$\begin{aligned} \omega^x =& d_H X|_{(4.32)} = (\phi_x + p\phi_u + q\phi_p)dx + \phi_u(du - pdx) + \phi_p(dp - qdx), \\ \omega^u =& d_H U|_{(4.32)} = \beta_u(du - pdx), \quad \omega^p = d_H P|_{(4.32)} = \psi_u(du - pdx) + \psi_p(dp - qdx). \end{aligned}$$

This completes the solution to the equivalence problem under contact transformations. We refer the reader to [27, pp. 348–356] for the solution in terms of Cartan's moving frame method. For comparison purposes, the correspondence between (4.31) and [27, eq. (11.5)] is given by

$$\begin{gathered} \theta^1 = \omega^u, \qquad \theta^2 = \omega^p, \qquad \theta^3 = \omega^x, \qquad \pi^1 = \nu^u_U, \qquad \pi^2 = \nu^u_{XU}, \\ \pi^3 = \nu^u_U - \nu^x_X, \qquad \pi^4 = \nu^x_U, \qquad \pi^5 = \nu^x_P. \end{gathered}$$

**Example 4.20.** In our last example we consider the simultaneous local equivalence of a two-form and a vector field on $\mathbb{R}^3$, [16]. This example is of interest as it is one of the rare known equivalence problems whose solution, in terms of Cartan's method, leads to structure equations with an essential invariant.

Let

$$\Omega = a(x,y,z)dx \wedge dy + b(x,y,z)dx \wedge dz + c(x,y,z)dy \wedge dz, \qquad a(x,y,z) \neq 0,$$

be a non-vanishing two-form and

$$\mathbf{v} = e(x,y,z)\frac{\partial}{\partial x} + f(x,y,z)\frac{\partial}{\partial y} + g(x,y,z)\frac{\partial}{\partial z}, \qquad g(x,y,z) \neq 0,$$

a non-zero vector field on $\mathbb{R}^3$. Assume $\overline{\Omega}$ is another non-vanishing two-form and $\overline{\mathbf{v}}$ another non-zero vector field. The equivalence problem consists of determining if there exists or not a local diffeomorphism of $\mathbb{R}^3$

$$\varphi: \quad X = \phi(x,y,z), \quad Y = \beta(x,y,z), \quad Z = \alpha(x,y,z) \quad \in \quad \mathcal{D}(\mathbb{R}^3), \tag{4.33}$$

such that

$$d\varphi^*(\overline{\Omega}) = \Omega \qquad \text{and} \qquad d\varphi^{-1}(\overline{\mathbf{v}}) = \mathbf{v}. \tag{4.34}$$

The equivalence problem splits in two branches: $\mathbf{v} \lrcorner \Omega = 0$ or $\mathbf{v} \lrcorner \Omega \neq 0$. In the following we consider the case $\mathbf{v} \lrcorner \Omega = 0$. This imposes the restrictions

$$e(x,y,z) = \frac{g(x,y,z)c(x,y,z)}{a(x,y,z)} \qquad \text{and} \qquad f(x,y,z) = -\frac{g(x,y,z)b(x,y,z)}{a(x,y,z)}$$

on the vector field coefficients. In local coordinates, the equivalence criterions (4.34) lead to the transformation rules

$$\begin{gathered} A(\phi_x\beta_y - \beta_x\phi_y) + B(\phi_x\alpha_y - \alpha_x\phi_y) + C(\beta_x\alpha_y - \alpha_x\beta_y) = a, \\ A(\phi_x\beta_z - \beta_x\phi_z) + B(\phi_x\alpha_z - \alpha_x\phi_z) + C(\beta_x\alpha_z - \alpha_x\beta_z) = b, \\ A(\phi_y\beta_z - \beta_y\phi_z) + B(\phi_y\alpha_z - \alpha_y\phi_z) + C(\beta_y\alpha_z - \alpha_y\beta_z) = c, \\ G = \frac{g}{a}(c\alpha_x - b\alpha_y + a\alpha_z), \end{gathered} \tag{4.35}$$

for the two-form and vector field components. The infinitesimal generators corresponding to the transformations (4.33), (4.35) are given by the vector fields

$$\begin{aligned} \mathbf{w} =& \xi(x,y,z)\frac{\partial}{\partial x} + \eta(x,y,z)\frac{\partial}{\partial y} + \tau(x,y,z)\frac{\partial}{\partial z} - [a(\xi_x+\eta_y) + b\tau_y - c\tau_x]\frac{\partial}{\partial a} \\ &- [a\eta_z + b(\xi_x+\tau_z) + c\eta_x]\frac{\partial}{\partial b} - [-a\xi_z + b\xi_y + c(\eta_y+\tau_z)]\frac{\partial}{\partial c} \\ &+ \frac{g}{a}[a\tau_z - b\tau_y + c\tau_x]\frac{\partial}{\partial g}, \end{aligned}$$

where $\xi(x,y,z)$, $\eta(x,y,z)$ and $\tau(x,y,z)$ are arbitrary differentiable functions. The rank of the Lie matrix, [28], of the first prolongation $\mathbf{w}^{(1)}$ reveals that the orbits of the first order prolonged action are of codimension one in $\mathrm{J}^1$. Hence the equivalence pseudo-group (4.33), (4.35) admits a first order differential invariant.

Respecting the geometric features of the equivalence problem, namely that the lifted invariants $A \neq 0$ and $G \neq 0$, we see from the recurrence relations (3.24) that in general there is enough liberty in the pseudo-group action to set

$$\begin{gathered} X = Y = Z = 0, \quad A = G = 1, \quad B_J = C_J = 0, \quad \#J \geq 0, \\ G_K = 0, \quad \#K > 1, \qquad A_{X^iY^j} = 0, \qquad i+j \geq 1, \end{gathered} \tag{4.36}$$

without imposing non-degeneracy conditions on some lifted invariants. We stress the fact that the solution to the normalization equations (4.36) does not lead to the normalization of all the pseudo-group parameters. Nevertheless, solving (4.36) for as many pseudo-group parameters as possible it is possible to obtain the local coordinate expression of the anticipated first order differential invariant:

$$I = \iota(a_z) = \frac{g}{a}(a_z - b_y + c_x). \tag{4.37}$$

Also, we observe that the invariantization of the standard horizontal coframe $\{dx, dy, dz\}$ by the equivariant moving frame method leads to an invariant horizontal coframe $\boldsymbol{\omega} = \{\omega^x, \omega^y, \omega^z\} = \iota\{dx, dy, dz\}$ adapted to the geometry of the equivalence problem. First, the invariant one-forms $\omega^x$, $\omega^y$ are such that $\Omega = \omega^x \wedge \omega^y$ while $\omega^z$ is such that $\mathbf{v} \lrcorner \omega^z = 1$. In the adapted invariant coframe, the differential invariant (4.37) has a clear geometrical interpretation, it measures the obstruction of $\Omega$ to be closed since

$$d\Omega = I\, \omega^x \wedge \omega^y \wedge \omega^z.$$

Further normalization of the pseudo-group parameters depends on the invariant $I = \iota(a_z) = \rho^*(A_Z)$. If $I$ is constant it follows from the recurrence relations (3.24) that the invariants $\rho^*(A_{Z,J})$ are also constant. In this case no further normalization is possible and (4.36) determines the cross-section to a partial moving frame. The partial moving frame is involutive as we now verify. From the recurrence relations for the phantom invariants we obtain

$$\nu^x = -\omega^x, \qquad \nu^y = -\omega^y, \qquad \nu^z = -\omega^z,$$

for the zero order Maurer–Cartan forms and

$$\nu^y_Y = -\nu^x_X \pmod{\boldsymbol{\omega}}, \qquad \nu^y_Z = \nu^x_Z = \nu^z_Z = 0, \tag{4.38}$$

with their prolongation, for the higher order Maurer–Cartan forms. The computation of indices of the first order symbol matrix $\mathbf{T}_1$ associated to the system (4.38) yields $\beta_1^{(3)} = 3$, $\beta_1^{(2)} = 1$, $\beta_1^{(1)} = 0$. Computing the rank of the symbol matrix $\mathbf{T}_2$ corresponding to the first prolongation of (4.38) gives: rank $\mathbf{T}_2 = 11 = 3\beta_1^{(3)} + 2\beta_1^{(2)} + \beta_1^{(1)}$. Thus the involutivity test (3.58) is satisfied. As in the previous example, the Cartan characters are $\alpha_1^{(3)} = 0$, $\alpha_1^{(2)} = 2$, $\alpha_1^{(1)} = 3$, and we conclude that the general equivalence map depends on two arbitrary analytic functions, each depending on two variables.

Finally, the structure equations of the invariant differential forms $\omega^x$, $\omega^y$, $\omega^z$ are

$$\begin{aligned} d\omega^x =& \nu^x_X \wedge \omega^x + \nu^x_Y \wedge \omega^y, \\ d\omega^y =& \nu^y_X \wedge \omega^x - \nu^x_X \wedge \omega^y - I\omega^y \wedge \omega^z, \\ d\omega^z =& \nu^z_X \wedge \omega^x + \nu^z_Y \wedge \omega^y, \end{aligned} \tag{4.39}$$

where the invariant $I$ appears. The structure equations (4.39) are equivalent to [27, eq. (11.29)]. The correspondence is given by

$$\theta^1 = \omega^1, \qquad \theta^2 = \omega^y, \qquad \theta^3 = \omega^z, \qquad \alpha^1 = \nu^x_X, \qquad \alpha^2 = \nu^x_Y, \qquad \alpha^3 = \nu^x_Y,$$
$$\beta^1 = \nu^z_X, \qquad \beta^2 = \nu^z_Y.$$

When $I = \iota(a_z)$ is not constant further normalizations are possible leading to different branches of the equivalence problems. The complete analysis, based on Cartan's equivalence method, can be found in [16].

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## References


[1] Akivis, M.A., and Rosenfeld, B.A., *Élie Cartan (1869–1951)*, Translations Math. Monographs, Vol. 123, American Math. Soc., Providence, R.I., 1993.

[2] Anderson, I.M., *The Variational Bicomplex*, Utah State Technical Report, 1989, `http://math.usu.edu/~fg_mp`.

[3] Bryant, R.L, Chern, S.S., Gardner, R.B., Goldshmidt, H.L., and Griffiths, P.A., *Exterior Differential Systems*, Mathematical Sciences Research Institute Publications, Vol. 18, Springer, New York, 1991.

[4] Cartan, É., *La Méthode du Repère Mobile, la Théorie des Groupes Continus, et les Espaces Généralisés*, Exposés de Géométrie No. 5, Hermann, Paris, 1935.

[5] Cartan, É., *Les Systèmes Différentiels Extérieurs et leurs Applications Géométriques*, Exposés de Géométrie, no. 14, Hermann, Paris, 1945.

[6] Cartan, É., Sur la structure des groupes infinis de transformations, in: *Oeuvres Complètes*, Part. II, Vol. 2, Gauthier–Villars, Paris, 1953, pp. 571–714.

[7] Cartan, É., La structure des groupes infinis, in: *Oeuvres Complètes*, Part. II, Vol. 2, Gauthier–Villars, Paris, 1953, pp. 1335–1384.

[8] Cheh, J., Olver, P.J., and Pohjanpelto, J., Maurer–Cartan equations for Lie symmetry pseudo-groups of differential equations, *J. Math. Phys.* **46** (2005) 023504.

[9] Cheh, J., Olver, P.J., and Pohjanpelto, J., Algorithms for differential invariants of symmetry groups of differential equations, *Found. Comput. Math.* **8** (2008) 501–532.

[10] Cox, D., Little, J., and O'Shea, D., *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Second Edition, Springer, New York, 1996.

[11] David, D., Levi, D., and Winternitz, P., Equations invariant under the symmetry group of the Kadomtsev–Petviashvili equation, *Phys. Lett. A*, **129**: 3 (1988) 161–164.

[12] Fels, M., and Olver, P.J., Moving coframes. I. A practical algorithm, *Acta Appl. Math.* **51** (1998) 161–213.

[13] Fels, M., and Olver, P.J., Moving coframes. II. Regularization and theoretical foundations, *Acta Appl. Math.* **55** (1999) 127–208.

[14] Gardner, R.B., Differential geometric methods interfacing control theory, in: *Differential Geometric Control Theory*, R.W. Brockett et. al., eds., Birkhäuser, Boston, 1983, pp. 117–180.

[15] Gardner, R.B., *The Method of Equivalence and Its Applications*, CBMS–NSF Regional Conference Series in Applied Mathematics, Vol. 58, Capital City Press, Montpelier, 1989.

[16] Gardner, R.B., and Shadwick, W.F., An equivalence problem for a two-form and a vector field on $\mathbb{R}^3$, *Can. Math. Soc. Conf. Proc.* **12** (1992) 41–50.

[17] Hubert, E., Differential invariants of a Lie group action: Syzygies on a generating set, *J. of Symb. Comp.* **44** (2009) 382–416.

[18] Ivey, T.A., and Landsberg, J.M., *Cartan for Beginners: Differential Geometry Via Moving Frames and Exterior Differential Systems*, Graduate Studies in Mathematics, Vol. 61, American Mathematical Society, 2003.

[19] Kamran, N., Contributions to the study of the equivalence problem of Elie Cartan and its applications to partial and ordinary differential equations, *Mém. Cl. Sci. Acad. Roy. Belg.* **45** (1989) Fac. 7.

[20] Kamran, K., Lamb, K.G., and Shadwick, W.F., The local equivalence problem for $d^2y/dx^2 = F(x, y, dy/dx)$ and the Painlevé transcendents, *J. Diff. Geo.* **22** (1985) 139–150.

[21] Kamran, N., and Olver, P.J., Equivalence problems for first order Lagrangian on the line, *J. Diff. Eq.* **80** (1989) 32–78.

[22] Kamran, N., and Shadwick W.F., The solution of the equivalence problem for $y'' = F(x, y, y')$ under the pseudo-group $\overline{x} = \phi(x), \overline{y} = \psi(x, y)$, in: *Field Theory, Quantum Gravity and Strings*, ed. de Vega, H., and Sánchez, N., Springer Lecture Notes in Physics, Vol. 246, Springer–Verlag, New York, pp. 320–334, 1986.

[23] Kogan, I., and Olver, P.J., Invariant Euler–Lagrange equations and the invariant variational bicomplex, *Acta Appl. Math.* **76** (2003) 137–193.

[24] Lie, S., and Scheffers, G., *Vorlesungen über Continuierliche Guppen mit Geometrischen und Anderen Anwendungen*, B.G. Teubner, Leipzig, 1893.

[25] Mackenzie, K., *Lie Groupoids and Lie Algebroids in Differential Geometry*, London Math. Soc. Lect. Note Series, Vol. 124, Cambridge Univ. Press, Great Britain, 1987.

[26] Olver, P.J., *Applications of Lie Groups to Differential Equations*, Second Edition, Springer–Verlag, New York, 1993.

[27] Olver, P.J., *Equivalence, Invariants, and Symmetry*, Cambridge University Press, Cambridge, 1995.

[28] Olver, P.J., Moving frames and singularities of prolonged group actions, *Selecta Math.* **6** (2000) 41–77.

[29] Olver, P.J., Geometric foundations of numerical algorithms and symmetry, *Appl. Alg. Engin. Comp. Commun.* **11** (2001) 417–436.

[30] Olver, P.J., Moving frames, *J. Symb. Comp.* **36** (2003) 501–512.

[31] Olver, P.J., Invariant submanifold flows, *J. Phys. A* **41** (2008) 344017.

[32] Olver, P.J., Lectures on moving frames, in: *Symmetries and Integrability of Difference Equations*, Levi, D., Olver, P., Thomova, Z., and Winternitz, P., eds., Cambridge University Press, Cambridge, to appear.

[33] Olver, P.J., and Pohjanpelto, J., Maurer–Cartan forms and the structure of Lie pseudo-groups, *Selecta Math.* **11** (2005) 99–126.

[34] Olver, P.J., and Pohjanpelto, J., Moving frames for Lie pseudo-groups, *Canadian J. Math.* **60** (2008) 1336–1386.

[35] Olver, P.J., and Pohjanpelto, J., Differential invariant algebras of Lie pseudo-groups, *Adv. Math.* **222** (2009) 1746–1792.

[36] Olver, P.J., and Pohjanpelto, J., Persistence of freeness for pseudo-group actions, preprint, University of Minnesota, 2009.

[37] Olver, P.J., Pohjanpelto, J., and Valiquette, F., On the structure of Lie pseudo-groups, *SIGMA* **5** (2009), 077, 14 pages.

[38] Schwarz, F., Symmetries of the Khokhlov–Zabolotskaya equation, *J. Phys. A: Math. Gen* **20** (1987) 1613–1614.

[39] Seiler, W.M., *Involution: The Formal Theory of Differential Equations and Its Applications in Computer Algebra and Numerical Analysis*, Algorithms and Computation in Mathematics, Vol. 24, Springer, New York, 2009.

[40] Stormark, O., *Lie's Structural Approach to PDE Systems*, Encyclopedia of Mathematics and Its Applications, Vol. 80, Cambridge University Press, Cambridge, 2000.

[41] Shemyakova, E., and Mansfield, E.L., Moving frames for Laplace invariants, in: *Proceedings ISSAC 2008*, D. Jeffrey, ed., ACM, New York, 2008, pp. 295–302.

[42] Thompson, T., and Valiquette, F., On the cohomology of the invariant EulerLagrange complex, McGill University, 2010.

[43] Tresse, A., Sur les invariants différentiels des groupes continus de transformations, *Acta Math.* **18** (1894) 1–88.

[44] Vessiot, E., Sur la théorie des groupes continues, *Ann. École Norm. Sup.* **20** (1903) 411–451.

[45] Valiquette, F., Structure equations of Lie pseudo-groups, *J. of Lie Theory* **18**, No. 4 (2008) 869–895.

[46] Valiquette, F. *Applications of Moving Frames to Lie Pseudo-Groups*, Ph.D. Thesis, University of Minnesota, 2009.

[47] Warner, F.W., *Foundations of Differentiable Manifolds and Lie Groups*, Graduate Texts in Mathematics, Vol. 94, Springer, New York, 1983.