Structure Equations of Lie Pseudo-Groups

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Abstract

In 1904, Élie Cartan developed a new structure theory for Lie pseudo-groups based on his theory of exterior differential systems, [**?**]. About a century later, in 2005, Olver and Pohjanpelto proposed a new approach to derive the structure equations of Lie pseudo-groups, [**?**]. The two theories are compared and it is shown that for intransitive Lie pseudo-groups they do not agree. To make the two theories compatible, we show that Cartan's structure equations must be restricted to the orbits of the pseudo-group action. The repercussion of this modification on Cartan's concept of essential invariants is discussed. Also, the infinitesimal interpretation of Cartan's structure equations for transitive Lie pseudo-groups, given in 1965 by Singer and Sternberg, [**?**], is extended to intransitive Lie pseudo-groups.

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1 Introduction

Lie pseudo-groups are the infinite-dimensional counterparts of local Lie groups of transformations. When Sophus Lie began his work on continuous groups of transformations, no significant distinction was drawn between finite-dimensional and infinite-dimensional theory. But, since then the two subjects have evolved very differently. The definition of a Lie group as an abstract object in the early twentieth century was a breakthrough in our understanding of the finite-dimensional theory. The lack of an universally accepted abstract object playing the role of an infinite-dimensional Lie group has made the study of infinite-dimensional Lie pseudo-groups much more difficult. Infinite-dimensional Lie pseudo-groups only arise through their concrete action on a space. Lie pseudo-groups appear in many fundamental physical and geometrical contexts such as gauge theories, Hamiltonian mechanics, symplectic and Poisson geometry, conformal geometry of surfaces, conformal field theory, geometry of real hypersurfaces and as symmetry groups of partial differential equations.

A Lie pseudo-group $\mathcal{G}$ is defined in terms of an involutive system of differential equations whose solutions are the local diffeomorphisms constituting the pseudo-group. As for Lie groups, it is easier to study infinite-dimensional Lie pseudo-groups by looking at their infinitesimal properties. The infinitesimal structure of Lie pseudo-groups can be studied through their Maurer-Cartan structure equations or by computing the Lie brackets of their infinitesimal generators. In this paper our focus will be on the Maurer-Cartan structure equations of infinite-dimensional Lie pseudo-groups.

With his theory of exterior differential systems, Cartan associates to any Lie pseudo-group a set of invariant differential forms and shows that their differential are given by equations analogue to the Maurer-Cartan structure equations for Lie groups [**?**, ?]. For transitive Lie pseudo-groups a lot of efforts have been made to establish a proper rigorous foundation, [**?**, **?**, **?**, **?**, ?]. For transitive Lie pseudo-groups, Reid, *et al.*, also developed a method for determining Cartan's structure equations from the infinitesimal determining equations using only algebraic manipulations and differentiation, [**?**, **?**]. For intransitive Lie pseudo-groups the situation is much different, few general results are known. A difficulty encountered in the study of the local structure of intransitive Lie pseudo-groups is the possibility that the structure coefficients can depend on the manifold on which they act. But more importantly, as we will show, the fact that Cartan's structure equations do not recover the correct infinitesimal structure of intransitive Lie pseudo-groups is a problem.

Recently, Olver and Pohjanpelto have developed a new method to derive the structure equations of Lie pseudo-groups, [**?**, ?]. Their approach is completely algorithmic and requires only differentiation and algebraic manipulations. It bypasses Cartan's prolongation procedure and is completely general; it can be applied to Lie groups of transformations and infinite-dimensional Lie pseudo-groups, transitive or intransitive. It is based on a combination of the theories of Lie groupoids, [**?**, **?**], and variational bicomplexes, [**?**]. The first part of the paper is devoted to an overview of their theory. Then in Section 5 an infinitesimal interpretation of their structure equations is given. This result extends the correspondence between the Maurer-Cartan structure equations and the Lie algebra of infinitesimal generators given in [?] for transitive pseudo-group actions to intransitive Lie pseudo-groups.

In Section 6 we summarize Cartan's derivation of the structure equations for Lie pseudo-groups. Comparing Olver and Pohjanpelto's theory to Cartan's theory, we show in Section 7 that for intransitive Lie pseudo-groups the two theories do not completely agree. This is done by looking at some examples of intransitive Lie pseudo-group actions considered by Cartan in [?, ?]. To make the two theories compatible, we argue that Cartan's structure equations must be restricted to the orbits of the pseudo-group action. The repercussion of this observation on Cartan's definition of essential invariants is analyzed. It is shown that Cartan's definition of essential invariants in terms of the systatic system becomes vacuous. Yet essential invariants do exist, they correspond to the scalar invariants parametrizing the leaves of the group foliation and appearing in the structure equations.

2 Lie Pseudo-Groups

Definition 2.1. Let M be a smooth m-dimensional manifold and $\mathcal{G}$ be a collection of local diffeomorphisms of M. $\mathcal{G}$ is a *pseudo-group* if

1. $\mathcal{G}$ is closed under restriction: if $\phi : U \to M$ is in $\mathcal{G}$, then so is $\phi|_V$ for all open $V \subset U$,
2. we can piece together elements of $\mathcal{G}$: if $U \subset M$ is an open set with $U = \cup_i U_i$, and $\phi : U \to M$ is a diffeomorphism with $\phi|_{U_i} \in \mathcal{G}$, then $\phi \in \mathcal{G}$,
3. $\mathcal{G}$ is closed under composition: if $\phi : U \to M$, and $\psi : V \to M$ are two members of $\mathcal{G}$, then $\psi \circ \phi \in \mathcal{G}$ also, whenever the composition is defined,
4. $\mathcal{G}$ contains the identity diffeomorphism of M,
5. $\mathcal{G}$ is closed under inverse: if $\phi : U \to M$ is in $\mathcal{G}$, then $\phi^{-1} : \phi(U) \to M$ is also in $\mathcal{G}$.

We denote by $\mathcal{D}(M)$ be the pseudo-group of all local diffeomorphisms $\phi : U \to M$, $U \subset M$. For each $n \geq 0$, let $\mathcal{D}^{(n)}(M) \subset J^n(M, M)$ denote the bundle of their n-th order jets. Elements in $\mathcal{D}^{(n)}(M)$ are denoted by $j_n\phi$. Following Cartan, [?,?], we use lower case letters $z, x, y, u, \ldots$, for the source coordinates and corresponding upper case letters $Z, X, Y, U, \ldots$, for the target coordinates of diffeomorphisms $Z = \phi(z)$. With this notation

$$j_n\phi|_z = (z, Z^{(n)}),$$

where $Z^{(n)}$ denotes all the derivatives of $Z = \phi(z)$ with respect to z, up to order n. Throughout the paper we use the multi-index notation

$$Z^b_A = \frac{\partial^k Z^b}{(\partial z^1)^{a^1} \cdots (\partial z^m)^{a^m}}, \qquad A = (a^1, \ldots, a^m), \quad k = \#A = a^1 + \cdots + a^m,$$

$b = 1, \ldots, m$, to denote partial derivatives. Let $\mathcal{D}^{(\infty)}(M)$ be the inverse limit of the bundles $\pi^n_k : \mathcal{D}^{(n)}(M) \to \mathcal{D}^{(k)}(M)$ (where $n > k$ and π^n_k are the standard projection maps), which can be identified with the bundle of Taylor series of local diffeomorphisms. Each bundle $\mathcal{D}^{(n)}(M)$, $1 \leq n \leq \infty$, carries the structure of a groupoid. The source map $\boldsymbol{\sigma}^{(n)}(j_n\phi|_z) = z$ and target map $\boldsymbol{\tau}^{(n)}(j_n\phi|_z) = \phi(z) = Z$ induce the double fibration

$$\begin{array}{ccc} & \mathcal{D}^{(n)} & \\ {}^{\boldsymbol{\sigma}^{(n)}} \swarrow & & \searrow {}^{\boldsymbol{\tau}^{(n)}} \\ M & & M \end{array}$$

The groupoid multiplication follows from the composition of local diffeomorphisms. Given $g^{(n)} = j_n\phi|_z$, $h^{(n)} = j_n\psi|_Z$ with $Z = \boldsymbol{\tau}^{(n)}(j_n\phi|_z) = \boldsymbol{\sigma}^{(n)}(j_n\psi|_Z)$, we have $h^{(n)} \cdot g^{(n)} = j_n(\psi \circ \phi)|_z$. Local diffeomorphisms $\psi \in \mathcal{D}(M)$ can act on $\mathcal{D}^{(n)}(M)$ by either the left or right multiplication

$$L_\psi(j_n\phi|_z) = j_n(\psi \circ \phi)|_z, \qquad R_\psi(j_n\phi_z) = j_n(\phi \circ \psi^{-1})|_{\psi(z)}.$$

When the type of action is not specified, the right multiplication must be understood.

Definition 2.2. A sub-pseudo-group $\mathcal{G} \subset \mathcal{D}(M)$ is called a *Lie pseudo-group* if there exists $n^\star \geq 1$ such that the following assumptions are satisfied for all finite $n \geq n^\star$:

- $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ forms a smooth, embedded subbundle,
- $\pi^{n+1}_n : \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a bundle map,
- every smooth local solution $Z = \phi(z)$ to the determining system $\mathcal{G}^{(n)}$ belongs to $\mathcal{G}$,
- $\mathcal{G}^{(n)} = \mathrm{pr}^{(n-n^\star)}\mathcal{G}^{(n^\star)}$ is obtained by prolongation.

The minimal value of $n^\star$ is called the order of the Lie pseudo-group.

In local coordinates, the order $n \geq n^\star$ determining equations defining the Lie pseudo-group subbundle $\mathcal{G}^{(n)}$ take the form of an involutive system, [**?**], of partial differential equations

$$F^{(n)}(z, Z^{(n)}) = 0, \tag{2.1}$$

whose local solutions $Z = \phi(z)$ are the pseudo-group transformations. The prolonged system defining $\mathrm{pr}^{(k)}\mathcal{G}^{(n)}$ is obtained by repeatedly applying the total differential operators

$$\mathbb{D}_{z^b} = \frac{\partial}{\partial z^b} + \sum_{a=1}^{m} \sum_{\#A \geq 0} Z^a_{A,b} \frac{\partial}{\partial Z^a_A}, \qquad b = 1, \ldots, m, \tag{2.2}$$

to (2.1),

$$F^{(n+k)}(z, Z^{(n+k)}) = \begin{cases} F^{(n)}(z, Z^{(n)}) = 0, \\ \mathbb{D}^z_A F^{(n)}(z, Z^{(n)}) = 0, \quad 1 \leq \#A \leq k. \end{cases}$$

In the definition of the total differential operators (2.2) we use the notation

$$Z^a_{A,b} = \frac{\partial^{\#A+1} Z^a}{(\partial z^1)^{a^1} \cdots (\partial z^{b-1})^{a^{b-1}} (\partial z^b)^{a^b+1} (\partial z^{b+1})^{a^{b+1}} \cdots (\partial z^m)^{a^m}}.$$

Definition 2.3. A Lie pseudo-group $\mathcal{G} \subset \mathcal{D}(M)$ is said to be *transitive* if for any point $z_0 \in M$ there is a neighborhood U of z_0 such that if $\overline{z}_0 \in U$ then there is a diffeomorphism $\phi \in \mathcal{G}$ such that $\phi(z_0) = \overline{z}_0$. If a Lie pseudo-group is not transitive, it is said to be *intransitive.*

Transitive Lie pseudo-groups do not possess any scalar invariants, and this is equivalent to the fact that the involutive defining system includes no zero order equations, [?,?]. Intransitive Lie pseudo-groups do possess scalar invariants. Those invariants can appear in the structure equations of the pseudo-groups, making their analysis more difficult.

3 Structure Equations of the Diffeomorphism Pseudo-Group

The cotangent bundle $T^*\mathcal{D}^{(\infty)}(M)$ naturally splits into horizontal and vertical components. In terms of local coordinates $g^{(\infty)} = (z, Z^{(\infty)})$, the horizontal subbundle of $T^*\mathcal{D}^{(\infty)}(M)$ is spanned by the 1-forms $dz^1 = d_M z^1, \ldots, dz^m = d_M z^m$, while the vertical subbundle is spanned by the contact forms

$$\Upsilon^a_A = d_G Z^a_A = dZ^a_A - \sum_{b=1}^m Z^a_{A,b} dz^b, \qquad a = 1, \ldots, m, \qquad \#A \geq 0.$$

This induces a splitting of the differential on $\mathcal{D}^{(\infty)}(M)$:

$$d = d_M + d_G.$$

The subscript on the vertical differential d_G refers to the groupoid structure of $\mathcal{D}^{(\infty)}(M)$. Given a differential function $F : \mathcal{D}^{(\infty)}(M) \to \mathbb{R}$, its horizontal and vertical differentials take the form

$$d_M F = \sum_{b=1}^m (\mathbb{D}_{z^b} F) dz^b, \qquad d_G F = \sum_{b=1}^m \sum_{\#A \geq 0} \frac{\partial F}{\partial Z^b_A} \Upsilon^b_A.$$

Definition 3.1. A differential form μ on $\mathcal{D}^{(n)}(M)$, $0 \leq n \leq \infty$, is *right-invariant* if it satisfies $(R_\psi)^* \mu = \mu$ (where defined) for every local diffeomorphism $\psi \in \mathcal{D}(M)$.

Since the splitting of the differential on $\mathcal{D}^{(\infty)}(M)$ into horizontal and contact components is also invariant under the action of $\mathcal{D}(M)$, if μ is a right-invariant differential form, so are $d_M \mu$ and $d_G \mu$. The right invariance of the target coordinate functions $Z^a : \mathcal{D}^{(0)}(M) \to \mathbb{R}$ implies that their differentials

$$dZ^a = d_M Z^a + d_G Z^a = \sigma^a + \mu^a, \qquad a = 1, \ldots, m,$$

split into right-invariant horizontal and contact forms. Thus, the one-forms

$$\sigma^a = d_M Z^a = \sum_{b=1}^m \frac{\partial Z^a}{\partial z^b} dz^b, \qquad a = 1, \ldots, m, \tag{3.1}$$

form an invariant horizontal coframe, while

$$\mu^a = d_G Z^a = \Upsilon^a = dZ^a - \sum_{b=1}^{m} \frac{\partial Z^a}{\partial z^b} dz^b, \qquad a = 1, \ldots, m, \tag{3.2}$$

are the zero-th order invariant contact forms. The total differential operators $\mathbb{D}_{Z^1}, \ldots, \mathbb{D}_{Z^m}$, dual to the horizontal forms $\sigma^1, \ldots, \sigma^m$, are defined by

$$d_M F = \sum_{a=1}^{m} (\mathbb{D}_{Z^a} F) \sigma^a,$$

where

$$\mathbb{D}_{Z^a} = \sum_{b=1}^{m} w_a^b \mathbb{D}_{z^b}, \qquad (w_a^b(z, Z^{(1)})) = \left(\frac{\partial Z^b}{\partial z^a} \right)^{-1}, \qquad a = 1, \ldots, m.$$

Invariance of the one-forms (3.1) implies that the Lie derivative of a right-invariant differential form with respect to $\mathbb{D}_{Z^a}$ is also right-invariant. The differential operators $\mathbb{D}_{Z^a}$ mutually commute and so the order of differentiation is immaterial. Therefore, higher-order invariant contact forms are defined by

$$\mu_A^a = \mathbb{D}_Z^A \mu^a, \qquad a = 1, \ldots, m, \qquad \#A \geq 0. \tag{3.3}$$

The right-invariant contact forms $\mu^{(\infty)}$, constructed in (3.3), are the Maurer-Cartan forms for the diffeomorphism pseudo-group $\mathcal{D}(M)$.

Given local coordinates $z = (z^1, \ldots, z^m)$ on M, we use $Z[\![h]\!]$ to denote the vector-valued Taylor series, depending on $h = (h^1, \ldots, h^m)$, of a diffeomorphism $Z = \phi(z + h)$ at the source point $z \in M$, with coordinates

$$Z^a[\![h]\!] = \sum_{\#A \geq 0} \frac{1}{A!} Z_A^a h^A, \qquad a = 1, \ldots, m,$$

where $h^A = (h^1)^{a^1} (h^2)^{a^2} \cdots (h^m)^{a^m}$, and $A! = a^1! a^2! \cdots a^m!$. Similarly, let $\mu[\![H]\!]$ be the vector-valued Maurer-Cartan form power series with components

$$\mu^a[\![H]\!] = \sum_{\#A \geq 0} \frac{1}{A!} \mu_A^a H^A, \qquad a = 1 \ldots, m. \tag{3.4}$$

The $m \times m$ Jacobian matrix power series obtained by differentiating $\mu[\![H]\!]$ with respect to $H = (H^1, \ldots, H^m)$ is denoted by

$$\nabla_H \mu[\![H]\!] = \left(\frac{\partial \mu^a[\![H]\!]}{\partial H^b} \right).$$

In [?] it is shown that the structure equations for the invariant coframe σ^a, μ_A^a, $a = 1, \ldots, m$, $\#A \geq 0$ are

$$\begin{aligned} d\mu[\![H]\!] &= \nabla_H \mu[\![H]\!] \wedge (\mu[\![H]\!] - dZ[\![0]\!]), \\ d\sigma &= \nabla_H \mu[\![0]\!] \wedge \sigma. \end{aligned} \tag{3.5}$$

where $\sigma = (\sigma^1, \ldots, \sigma^m)^T$. To obtain the structure equations of the diffeomorphism pseudo-group $\mathcal{D}^{(\infty)}(M)$ we restrict (3.5) to a target fiber $(\boldsymbol{\tau}^{(\infty)})^{-1}(Z)$. This amounts to setting $dZ[\![0]\!] = 0$ in the structure equations (3.5). Since $0 = dZ[\![0]\!] = \sigma + \mu[\![0]\!]$, on a target fiber,

$$\sigma = -\mu[\![0]\!],$$

and the structure equations for the horizontal forms σ are redundant with the structure equations of the zero-th order Maurer-Cartan forms $\mu[\![0]\!]$.

Proposition 3.2. The structure equations of the pseudo-group of local diffeomorphisms $\mathcal{D}(M)$ are

$$d\mu[\![H]\!] = \nabla_H \mu[\![H]\!] \wedge \mu[\![H]\!], \tag{3.6}$$

when restricted to a target fiber.

4 Structure Equations of a Lie Pseudo-Group

For a Lie pseudo-group $\mathcal{G} \subsetneq \mathcal{D}(M)$, the Maurer-Cartan forms (3.3) are no longer linearly independent. Remarkably, the explicit expressions for the Maurer-Cartan forms are not needed to find the linear relations between them. The linear dependencies follow from the infinitesimal determining system defining the infinitesimal generators of the pseudo-group action $\mathcal{G}^{(\infty)}$.

Let $\mathcal{X}(M)$ denote the space of locally defined vector fields in TM, and J^nTM, $0 \leq n \leq \infty$, the tangent n-jet bundle of TM. A locally defined vector field $\mathbf{v} \in \mathcal{X}(M)$ induces a flow Φ_t on M. The left action of the flow Φ_t on $\mathcal{D}^{(n)}(M)$ induces an invariant infinitesimal generator $\mathbf{V}^{(n)}$ tangent to the source fibers $\mathcal{D}^{(n)}(M)|_z$. The vector field $\mathbf{V}^{(n)}$ is called the n-th order lift of the vector field $\mathbf{v}$, and the notation $\mathbf{V}^{(n)} = \boldsymbol{\lambda}^{(n)}(\mathbf{v})$ is used. The infinite order case is denoted by $\mathbf{V} = \boldsymbol{\lambda}(\mathbf{v})$. In local coordinates, the lift of a vector field

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} \tag{4.1}$$

on M is the right-invariant vector field

$$\mathbf{V} = \sum_{a=1}^{m} \sum_{\#A \geq 0} \mathbb{D}_z^A \zeta^a(Z) \frac{\partial}{\partial Z_A^a} \tag{4.2}$$

on $\mathcal{D}^{(\infty)}(M)$, where

$$\mathbb{D}_{z^b} = \sum_{a=1}^{m} \frac{\partial Z^a}{\partial z^b} \mathbb{D}_{Z^a}, \qquad b = 1, \ldots, m.$$

Let $\mathcal{Z}^{(n)}$ denote the dual bundle to the vector field jet bundle $J^{(n)}TM$, and $\mathcal{Z}^{(\infty)}$ the direct limit. The lift of a section ζ of $\mathcal{Z}^{(\infty)}$ is defined to be the right-invariant differential form $\boldsymbol{\lambda}(\zeta)$ on $\mathcal{D}^{(\infty)}(M)$ that vanishes on all total vector fields, and satisfies

$$\langle \boldsymbol{\lambda}(\zeta); \boldsymbol{\lambda}(\mathbf{v})\rangle|_{g^{(\infty)}} = \langle \zeta; j_\infty \mathbf{v}\rangle|_Z, \quad \text{whenever} \quad g^{(\infty)} \in \mathcal{D}^{(\infty)}(M), \quad Z = \boldsymbol{\tau}^{(\infty)}(g^{(\infty)}).$$

In [?, **?**] it is explained that in local coordinates, each vector field coordinate function ζ_A^a can be viewed as a section of $\mathcal{Z}^{(\infty)}$, and that

$$\boldsymbol{\lambda}(\zeta_A^a) = \mu_A^a. \tag{4.3}$$

More generally, the lift of a linear function of the vector field jets $L(z, \zeta^{(n)})$ is

$$\boldsymbol{\lambda}[L(z, \zeta^{(n)})] = L(Z, \mu^{(n)}).$$

Given a Lie pseudo-group $\mathcal{G}$, let $\mathfrak{g} \subset \mathcal{X}(M)$ denote the local Lie algebra of infinitesimal generators, i.e., the set of locally defined vector fields whose flows belong to the pseudo-group. Let $J^n\mathfrak{g} \subset J^nTM$ denote their n-jets. In local coordinates, we can view the subbundle $J^n\mathfrak{g} \subset J^nTM$ as defining a linear system of partial differential equations

$$L^{(n)}(z, \zeta^{(n)}) = 0 \tag{4.4}$$

for the vector field coefficients. The linear system of equations (4.4) is called the n-th order infinitesimal determining system of the Lie pseudo-group $\mathcal{G}$. In practice, they are constructed by linearizing the n-th order determining equations (2.1) at the n-th order identity jet $\mathbb{1}_z^{(n)}$:

$$L^{(n)}(z,\zeta^{(n)}) = \mathbf{V}^{(n)}[F^{(n)}(z,Z^{(n)})]\big|_{(z,Z^{(n)})=\mathbb{1}_z^{(n)}} = 0.$$

In [?] it is shown that

Proposition 4.1. The linear system

$$L^{(n)}(Z,\mu^{(n)}) = 0, \tag{4.5}$$

obtained by lifting the linear determining equations (4.4), gives the complete set of linear dependencies among the right-invariant Maurer-Cartan forms $\mu^{(n)}$.

The equations (4.5) are called the n-th order lifted determining equations for the Lie pseudo-group $\mathcal{G}$.

Theorem 4.2. The structure equations of a Lie pseudo-group $\mathcal{G}$, when restricted to a target fiber, are obtained by restricting the diffeomorphism structure equations (3.6) to the kernel of the linearized involutive system (4.5):

$$(d\mu[\![H]\!] = \nabla_H \mu[\![H]\!] \wedge \mu[\![H]\!])\big|_{L^{(\infty)}(Z,\mu^{(\infty)})=0}. \tag{4.6}$$

5 Correspondence Between the Maurer-Cartan Structure Equations and the Infinitesimal Generator Lie Brackets

Given an r-dimensional Lie group G, one can associate to it a set of r linearly independent invariant vector fields $\mathbf{v}_1,\ldots,\mathbf{v}_r$ on G generating a Lie algebra. The structure of the Lie algebra is given by the commutator relations

$$[\mathbf{v}_i,\mathbf{v}_j] = \sum_{k=1}^{r} C_{ij}^k \mathbf{v}_k, \qquad i,j=1,\ldots,r, \tag{5.1}$$

where the structure coefficients C_{ij}^k are skew-symmetric in their subscripts and satisfy the Jacobi identities

$$\sum_{k=1}^{r}(C_{ij}^k C_{kl}^m + C_{li}^k C_{kj}^m + C_{jl}^k C_{ki}^m) = 0, \qquad 1 \le i,j,l,m \le r. \tag{5.2}$$

Dually, one can associate to G a set of r linearly independent invariant one-forms $\mu^1,\ldots,\mu^r$ satisfying the Maurer-Cartan structure equations

$$d\mu^k = \sum_{1\le i<j\le r} C_{ij}^k \mu^j \wedge \mu^i,$$

where the coefficients C_{ij}^k are the same as in (5.1). The Jacobi identities (5.2) are equivalent to the identities $d^2\mu^k = 0$, $k=1,\ldots,r$.

Cartan was sceptical that for infinite-dimensional Lie pseudo-groups a similar correspondence could be made between his structure equations and the infinitesimal theory advocated by S. Lie, [?, p. 1335]. In the 1960s, Kuranishi, [?, ?], and Singer and Sternberg, [?], were able to give an infinitesimal interpretation of Cartan structure theory for transitive Lie pseudo-groups. We now explain how the Maurer-Cartan structure equations (4.6) of a Lie pseudo-group are related to the commutators of its infinitesimal generators.

Let $\mathcal{D}(M)$ be the pseudo-group of local diffeomorphisms of M. In local coordinates, an infinitesimal generator of the pseudo-group action is given by (4.1). In the category of analytic vector fields we can expand the vector coefficients of (4.1) in Taylor series:

$$\mathbf{v} = \sum_{a=1}^{m} \sum_{\#A\geq 0} \zeta_A^a(z_0) \frac{(z-z_0)^A}{A!} \frac{\partial}{\partial z^a} \simeq j_\infty \mathbf{v}|_{z_0}, \qquad z_0 \in M,$$

and the vector fields

$$\mathbf{v}_a^A|_{z_0} = \frac{(z-z_0)^A}{A!} \frac{\partial}{\partial z^a}, \qquad a = 1,\ldots,m, \qquad \#A \geq 0, \tag{5.3}$$

can be interpreted as a basis of $J^{(\infty)}TM|_{z_0}$. There is a well-defined Lie algebra structure on $J^{(\infty)}TM|_{z_0}$ given by

$$[\mathbf{v}_a^A, \mathbf{v}_b^B] = \frac{(A+B\setminus a)!}{A!(B\setminus a)!} \frac{(z-z_0)^{A+B\setminus a}}{(A+B\setminus a)!} \frac{\partial}{\partial z^b} - \frac{(A\setminus b+B)!}{(A\setminus b)!B!} \frac{(z-z_0)^{A\setminus b+B}}{(A\setminus b+B)!} \frac{\partial}{\partial z^a}, \tag{5.4}$$

$1 \leq a, b \leq m$, $\#A, \#B \geq 0$, where

$$B \setminus a = (b^1, \ldots, b^{a-1}, b^a - 1, b^{a+1}, \ldots, b^m),$$

with the convention that

$$\frac{(z-z_0)^{A+B\setminus a}}{A!(B\setminus a)!} \frac{\partial}{\partial z^b} = 0,$$

if $b^a - 1 < 0$, and similarly for the second term on the right-hand side of (5.4).

Proposition 5.1. The Lie brackets (5.4) are dual to the diffeomorphism pseudo-group structure equations (3.6).

Proof. The expression for the lift of a vector field $\mathbf{v}$ at z_0, given by equation (4.2), is a well-defined function of the vector field jet $j_\infty \mathbf{v}|_{z_0}$ in which the coordinate jets $\zeta_A^a(z_0)$ are replaced by $\zeta_A^a(Z_0)$, with source $\boldsymbol{\sigma}^{(0)}(Z_0) = z_0$. Thus the lift of a vector field defines a map

$$\boldsymbol{\lambda}|_{z_0} : J^\infty TM|_{z_0} \to \mathrm{Lie}(\mathcal{D}^{(\infty)}(M))|_{z_0}, \tag{5.5}$$

where $\mathrm{Lie}(\mathcal{D}^{(\infty)}(M))|_{z_0}$ denotes the set of right-invariant vector fields tangent to the source fiber $\mathcal{D}^{(\infty)}(M)|_{z_0}$. In local coordinates

$$\boldsymbol{\lambda}|_{z_0} \left(\frac{(z-z_0)^A}{A!} \frac{\partial}{\partial z^a} \right) = \mathbf{V}_a^A,$$

where $\mathbf{V}_a^A$ is the vector field dual to the Maurer-Cartan form μ_A^a. Since the map (5.5) is a Lie algebra isomorphism, [?], this finishes the proof. □

Remark 5.2. It is possible to give a combinatorial proof of Proposition 5.1 by directly computing the commutators of the vector fields (5.3), expanding (3.6) in powers of H, and verify that the commutators of the infinitesimal generators are dual to the Maurer-Cartan structure equations.

For Lie pseudo-groups, the duality between the Maurer-Cartan structure equations and the commutators of the infinitesimal generators still holds. This follows from the fact that the Maurer-Cartan forms satisfy the lifted determining equations (4.5) while the vector field jet coordinates satisfy the equivalent infinitesimal determining equations (4.4). Since we do not assume the Lie pseudo-group action to be transitive, the infinitesimal interpretation of the Maurer-Cartan structure equations (4.6) remains valid for intransitive Lie pseudo-group actions.

Example 5.3. Consider the intransitive Lie pseudo-group

$$X = x, \qquad Y = ay + b, \qquad Z = a^x z + f(x), \qquad a > 0, \qquad f \in C^\infty(\mathbb{R}). \tag{5.6}$$

The minimal involutive determining system is

$$X = x, \qquad Y_x = 0, \qquad Y_{yy} = 0, \qquad Y_z = 0, \qquad Z_y = 0, \qquad Z_z = (Y_y)^x.$$

The associated infinitesimal determining equations, for an infinitesimal generator

$$\mathbf{v} = \xi(x,y,z)\partial_x + \eta(x,y,z)\partial_y + \phi(x,y,z)\partial_z,$$

are

$$\xi = 0, \qquad \eta_x = 0, \qquad \eta_{yy} = 0, \qquad \eta_z = 0, \qquad \phi_y = 0, \qquad \phi_z = x\eta_y. \tag{5.7}$$

Taking the lift of (5.7) we obtain the linear relations

$$\mu^x = 0, \qquad \mu^y_X = 0, \qquad \mu^y_{YY} = 0, \qquad \mu^y_Z = 0, \qquad \mu^z_Y = 0, \qquad \mu^z_Z = X\mu^y_Y. \tag{5.8}$$

We note that the last equation of (5.8) implies the non trivial relation $\mu^z_{ZX} = \mu^y_Y$. From (5.8) it follows that

$$\mu^y, \qquad \mu^y_Y, \qquad \mu^z_{X^k}, \qquad k \geq 0,$$

is a basis of Maurer-Cartan forms for the pseudo-group (5.6). Setting

$$\mu[\![H]\!] = \left(0, \mu^y + \mu^y_Y H_y, X\mu^y_Y H_z + \mu^y_Y H_x H_z + \sum_{k=0}^{\infty} \mu^z_{X^k} \frac{H_x^k}{k!}\right)^T$$

one obtains the Maurer-Cartan structure equations ($dX = 0$ on a target fiber)

$$\begin{pmatrix} 0 \\ d\mu^y + d\mu^y_Y H_y \\ X d\mu^y_Y H_z + d\mu^y_Y H_x H_z + \sum_{k=0}^\infty d\mu^z_{X^k} \frac{H_x^k}{k!} \end{pmatrix} =$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu^y_Y & 0 \\ \mu^y_Y H_z + \sum_{k=1}^\infty \mu^z_{X^k} \frac{H_x^{k-1}}{(k-1)!} & 0 & X\mu^y_Y + \mu^y_Y H_x \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \mu^y + \mu^y_Y H_y \\ X\mu^y_Y H_z + \mu^y_Y H_x H_z + \sum_{k=0}^\infty \mu^z_{X^k} \frac{H_x^k}{k!} \end{pmatrix}.$$

More explicitly

$$d\mu^y_Y = 0, \qquad d\mu^y = \mu^y_Y \wedge \mu^y, \qquad d\mu^z_{X^k} = X\mu^y_Y \wedge \mu^z_{X^k} + k\mu^y_Y \wedge \mu^z_{X^{k-1}}, \qquad k \geq 0. \tag{5.9}$$

The structure equations (5.9) induce the structure of an abstract Lie algebra. Setting W_k to be the vector dual to $\mu^z_{X^k}$, $k \geq 0$, V_1 to be dual to μ^y_Y and V_2 to be dual to μ^y, (5.9) yields the commutator relations

$$[V_1, V_2] = -V_2, \qquad [V_2, W_k] = 0, \qquad [V_1, W_k] = -x_0 W_k - (k+1)W_{k+1}, \tag{5.10}$$

at each fixed x_0.

Taking Lie's approach to the problem, the space of infinitesimal generators for the group action (5.6) is spanned by

$$V_1 = y\partial_y + xz\partial_z, \qquad V_2 = \partial_y, \qquad V_{f(x)} = f(x)\partial_z.$$

In the analytic category, a basis of vector fields, in the neighborhood of (x_0, y_0, z_0), is given by

$$\begin{aligned} V_1 &= ((y-y_0)+y_0)\partial_y + ((x-x_0)+x_0)((z-z_0)+z_0)\frac{\partial}{\partial z}, \\ V_2 &= \frac{\partial}{\partial y}, \qquad W_k = \frac{(x-x_0)^k}{k!}\frac{\partial}{\partial z}, \qquad k \geq 0. \end{aligned} \tag{5.11}$$

In terms of (5.11) we can write

$$V_{f(x)} = \sum_{k=0}^{\infty} f^{(k)}(x_0) W_k.$$

By direct computation one can verify that the commutation relations for the vector fields (5.11) are given by (5.10).

6 Cartan's Structure Equations

In this section we summarize Cartan's derivation of the structure equations for a Lie pseudo-group $\mathcal{G} \subset \mathcal{D}(M)$. We refer the reader to Cartan's original work [?, ?] and to [**?**, **?**, **?**] for a detailed exposition of the material. This section follows mainly the discussion given in [?].

In an adapted set of local coordinates $z = (x, y)$ on M, we can assume that the pseudo-group action is locally given by

$$X^i = x^i, \qquad Y^\alpha = f^\alpha(x,y), \quad i = 1,\ldots,p, \quad \alpha = 1,\ldots,q, \tag{6.1}$$

$p + q = m = \dim M$, with $p = 0$ if the action is transitive, and $p > 0$ if the action is intransitive.

The starting point of Cartan's structure theory is the $n^\star$-th involutive defining system

$$X = x, \qquad F^{(n^\star)}(x, y, Y^{(n^\star)}) = 0, \tag{6.2}$$

for the Lie pseudo-group action (6.1). Motivated by his theory of exterior differential systems, Cartan recasts the determining system in terms of the Pfaffian system

$$\begin{aligned} &X^i = x^i, \qquad i = 1,\ldots,p, \\ &\Upsilon^{p+\alpha}_A|_{F^{(n^\star)}(x,y,Y^{(n^\star)})=0} = (dY^\alpha_A - \sum_{b=1}^m Y^\alpha_{A,b}dz^b)|_{F^{(n^\star)}(x,y,Y^{(n^\star)})=0} = 0, \end{aligned} \tag{6.3}$$

$\alpha = 1,\ldots,q$, $0 \leq A \leq n^\star - 1$. Let $Y_{[k]} = (Y^1_{[k]},\ldots,Y^{s_k}_{[k]})$ be local parameterizations of the fibers of the bundles

$$\begin{array}{c} \mathcal{G}^{(k)} \\ \big\downarrow \pi^k_{k-1} \\ \mathcal{G}^{(k-1)} \end{array}$$

where $s_k = \dim \mathcal{G}^{(k)} - \dim \mathcal{G}^{(k-1)}$, $k \geq 1$. Then the system (6.3) is equivalent to

$$\begin{aligned}
&X^i - x^i = 0, && i = 1, \ldots, p, \\
&dY^\alpha - \sum_{a=1}^m L_a^\alpha(z, Y, Y_{[1]}) dz^a = 0, && \alpha = 1, \ldots, q, \\
&dY_{[1]}^i - \sum_{a=1}^m L_{[1],a}^i(z, Y, Y_{[1]}, Y_{[2]}) dz^a = 0, && i = 1, \ldots, s_1, \\
&\vdots \\
&dY_{[n^\star-1]}^i - \sum_{a=1}^m L_{[n^\star-1],a}^i(z, Y, Y_{[1]}, \ldots, Y_{[n^\star]}) dz^a = 0, && i = 1, \ldots, s_{n^\star-1},
\end{aligned} \tag{6.4}$$

for some functions $L_a^\alpha, \ldots, L_{[n^\star-1],a}^i$, whose expressions follow from the determining system (6.2). From the differential forms appearing in (6.4), Cartan proceeds to derive a set of invariant one-forms in an inductive fashion.

Since the one-forms

$$\omega_{[0]}^{p+\alpha} = dY^\alpha - \sum_{a=1}^m L_a^\alpha(z, Y, Y_{[1]}) dz^a, \qquad \alpha = 1, \ldots, q, \tag{6.5}$$

are invariant and dY^α, $\alpha = 1, \ldots, q$, are also (right) invariant one-forms, Cartan replaces (6.5) by

$$\omega_{[0]}^{p+\alpha} = \sum_{a=1}^m L_a^\alpha(z, Y, Y_{[1]}) dz^a, \qquad \alpha = 1, \ldots, q.$$

Thus a set of order zero invariant one-forms is given by

$$\begin{aligned}
\omega_{[0]}^i &= dx^i, && i = 1, \ldots, p, \\
\omega_{[0]}^{p+\alpha} &= \sum_{a=1}^m L_a^\alpha(z, Z, Z_{[1]}) dz^a, && \alpha = 1, \ldots, q.
\end{aligned} \tag{6.6}$$

The differential forms (6.6) form a basis of horizontal forms, therefore $dz^1, \ldots, dz^m$ can be written as linear combinations of the $\omega_{[0]}^b$, $b = 1, \ldots, m$. Hence the differential of the invariant one-forms $\omega_{[0]}^b$ can be written as

$$d\omega_{[0]}^b = \sum_{a=1}^m d\left(L_a^b(z, Y, Y_{[1]})\right) \wedge dz^a = \sum_{a=1}^m \omega_{[0]}^a \wedge \pi_a^b, \qquad b = 1, \ldots, m,$$

where the π_a^b are certain linear combinations of $dY_{[1]}^1, \ldots, dY_{[1]}^{s_1}$, $dY^1, \ldots, dY^q$, and $\omega_{[0]}^1, \ldots, \omega_{[0]}^m$. The invariance of $\omega_{[0]}^a$, $a = 1, \ldots, m$, implies

$$\sum_{a=1}^m \omega_{[0]}^a \wedge (R_\psi^*(\pi_a^b) - \pi_a^b) = 0, \qquad b = 1, \ldots, m, \qquad \psi \in \mathcal{G},$$

which means that

$$R_\psi^*(\pi_a^b) \equiv \pi_a^b \pmod{\omega_{[0]}^1, \ldots, \omega_{[0]}^m}.$$

By hypothesis, $s_1 = \dim \mathcal{G}^{(1)} - \dim \mathcal{G}$ of the π_a^b are linearly independent modulo $\omega_{[0]}^1, \ldots, \omega_{[0]}^m$, $dY^1, \ldots, dY^q$. Hence those s_1 forms are of the form

$$\pi^i \equiv \sum_{j=1}^{s_1} c_j^i dY_{[1]}^j + \sum_{\alpha=1}^q e_\alpha^i dY^\alpha \pmod{\omega_{[0]}^1, \ldots, \omega_{[0]}^m}, \qquad i = 1, \ldots, s_1,$$

with det $(c^i_j) \neq 0$. By adding suitable multiples of the $\omega^a_{[0]}$ we can write

$$\pi^i \equiv \sum_{j=1}^{s_1} c^i_j \left(dY^j_{[1]} - \sum_{b=1}^{m} L^j_{[1],b}(z, Y, Y_{[1]}) dz^b \right) + \sum_{\alpha=1}^{q} e^i_\alpha \left(dY^\alpha - \omega^{p+\alpha}_{[0]} \right),$$

modulo $\omega^1_{[0]}, \dots, \omega^m_{[0]}$, $i = 1, \dots, s_1$. Defining

$$\omega^i_{[1]} = \sum_{j=1}^{s_1} c^i_j \left(dY^j_{[1]} - \sum_{b=1}^{m} L^j_{[1],b}(z, Y, Y_{[1]}) dz^b \right) + \sum_{\alpha=1}^{q} e^i_\alpha \left(dY^\alpha - \omega^{p+\alpha}_{[0]} \right), \tag{6.7}$$

$i = 1, \dots, s_1$, Cartan shows that those one-forms are invariant, [?, pp. 597–600]. We refer to the one-forms (6.7) as the first order Maurer-Cartan forms of the pseudo-group $\mathcal{G}$. Those invariant differential forms are equivalent to some of the first order Maurer-Cartan forms constructed in (3.3). Taking the differential of the first order Maurer-Cartan forms (6.7) and repeating the above discussion, Cartan derives s_2 linearly independent second order Maurer-Cartan forms, and so on, up to order $n^\star - 1$.

The $r_{n^\star-1} = m + s_1 + s_2 + \cdots + s_{n^\star-1}$ invariant one-forms constructed are collected together and denoted by $\omega^1, \omega^2, \dots, \omega^{r_{n^\star-1}}$, where we drop their subscripts. Their exterior derivatives are of the form

$$d\omega^i = \sum_{1 \leq j < k \leq r_{n^\star-1}} C^i_{jk} \omega^j \wedge \omega^k + \sum_{j=1}^{r_{n^\star-1}} \sum_{\beta=1}^{s_{n^\star}} A^i_{j\beta} \omega^j \wedge \pi^\beta, \qquad i = 1, \dots, r_{n^\star-1}, \tag{6.8}$$

where

$$(\pi^1, \dots, \pi^{s_{n^\star}}) \equiv (dY^1_{[n^\star]}, \dots, dY^{s_{n^\star}}_{[n^\star]}) \qquad (\text{mod } \omega^1, \dots, \omega^{r_{n^\star-1}})$$

as modules of one-forms over the ring of functions $F : (x, y, Y, Y_{[1]}, \dots, Y_{[n^\star]}) \to \mathbb{R}$. If the pseudo-group is intransitive, the coefficients C^i_{jk}, and $A^i_{j\beta}$ can depend on the invariants x^i, $i = 1, \dots, p$.

Example 6.1. Consider the infinite-dimensional Lie pseudo-group

$$X = x, \qquad Y = f(y), \qquad Z = z(f'(y))^x + \phi(x, y), \tag{6.9}$$

$f \in \mathcal{D}(\mathbb{R})$, $\phi \in C^\infty(\mathbb{R}^2)$, due to Cartan [**?**]. The defining system for this pseudo-group is given by

$$X = x, \quad Y_x = 0, \quad Y_z = 0, \quad Z_z = (Y_y)^x. \tag{6.10}$$

The fibers of the bundle $\mathcal{G}^{(1)} \to \mathcal{G}$ are parameterized by

$$Y_{[1]} = (Y_y, Z_x, Z_y).$$

Since the determining system is of order one, Cartan's algorithm gives three invariant one-forms:

$$\omega^1 = dx, \qquad \omega^2 = Y_y dy, \qquad \omega^3 = Z_x dx + Z_y dy + (Y_y)^x dz.$$

Taking their differentials, we obtain Cartan's structure equations

$$\begin{aligned} d\omega^1 &= 0, \\ d\omega^2 &= -\omega^2 \wedge \pi^1, \\ d\omega^3 &= -\omega^1 \wedge \pi^2 - \omega^2 \wedge \pi^3 - x\omega^3 \wedge \pi^1, \end{aligned}$$

where

$$\begin{aligned}
\pi^1 &= \frac{dY_y}{Y_y},\\
\pi^2 &= dZ_x - \frac{xZ_x}{Y_y}dY_y - ((Y_y)^x \ln Y_y)\, dz, \qquad (6.11)\\
\pi^3 &= \frac{1}{Y_y}\left(dZ_y - \frac{xZ_y}{Y_y}dY_y\right).
\end{aligned}$$

7 Comparison of the Two Theories

The new structure theory by Olver, *et al.*, has been applied to several transitive Lie pseudo-groups, [?, ?]. As one expects, their structure equations are equivalent to those obtained with Cartan's theory. Though the structure equations are equivalent, a fundamental distinction needs to be pointed out. While the structure equations by Olver, *et al.*, depend only on group forms restricted to the target fibers, Cartan's structure equations mix group and horizontal forms. Indeed, the first m invariant one-foms (6.6) constructed by Cartan are horizontal, while the others are group forms of order greater or equal to one. For the two theories to be equivalent, the horizontal forms appearing in Cartan's structure equations must be equivalent to the zero order Maurer-Cartan forms (3.2), when restricted to a target fiber. In Section 7.1.1, in accordance with all examples investigated so far in the literature, we show that for transitive Lie pseudo-groups the two sets of structure equations are equivalent. On the other hand, in Section 7.2.2 we demonstrate that the two structure theories do not give the same structure equations for intransitive Lie pseudo-groups.

7.1 Transitive Lie Pseudo-Groups

Let $\mathcal{G} \subset \mathcal{D}(M)$ be a transitive Lie pseudo-group. The transitivity of the action implies that its involutive infinitesimal determining system does not contain any zero order equation. Hence the lift of the infinitesimal determining system does not introduce any linear relations among the order zero Maurer-Cartan forms $\mu^1, \ldots, \mu^m$, defined in (3.2). We conclude that those differential forms are non-zero and linearly independent. Since on a target fiber $(\boldsymbol{\tau}^{(\infty)})^{-1}(Z)$

$$0 = dZ^a = \sigma^a + \mu^a, \qquad a = 1, \ldots, m,$$

the Maurer-Cartan forms μ^a are in one to one correspondence with the horizontal forms σ^a. Making the substitutions

$$\mu^a = -\sigma^a = -d_M Z^a = -\omega^a_{[0]}, \qquad a = 1, \ldots, m,$$

into the structure equations (4.6), we conclude that for transitive Lie pseudo-groups, Cartan's structure equations are equivalent to Olver, *et al.*, equations.

7.2 Intransitive Lie Pseudo-Groups

Let $\mathcal{G} \subset \mathcal{D}(M)$ be an intransitive Lie pseudo-group. We assume it is locally given by (6.1), with $p > 0$. In those adapted coordinates, we use the notation

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z)\frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i(x,y)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x,y)\frac{\partial}{\partial y^\alpha}.$$

to denote a local vector field $\mathbf{v} \in \mathcal{X}(M)$, and the corresponding Maurer-Cartan forms are denoted by

$$\varpi^i_A = \boldsymbol{\lambda}(\xi^i_A), \qquad i = 1, \ldots, p, \qquad \nu^\alpha_A = \boldsymbol{\lambda}(\phi^\alpha_A), \qquad \alpha = 1, \ldots, q, \qquad \# A \geq 0. \tag{7.1}$$

Furthermore we split the formal parameter $H = (H^1, \ldots, H^m)$ appearing in the expression of the vector-valued Maurer-Cartan power series as

$$H = (H^1, \ldots, H^p, K^1, \ldots, K^q), \qquad p + q = m.$$

With the above notations, the vector-valued Maurer-Cartan form power series (3.4) is given by

$$\mu[\![H]\!] = (\varpi^1[\![H,K]\!], \ldots, \varpi^p[\![H,K]\!], \nu^1[\![H,K]\!], \ldots, \nu^q[\![H,K]\!])^T = \begin{pmatrix} \varpi[\![H,K]\!] \\ \nu[\![H,K]\!] \end{pmatrix}.$$

The determining system for the infinite prolongation of the pseudo-group action (6.1) is of the form

$$X = x, \qquad F^{(\infty)}(x, y, Y^{(\infty)}) = 0. \tag{7.2}$$

Linearizing (7.2) at the identity jet we obtain the infinitesimal determining equations

$$\xi = 0, \qquad L^{(\infty)}(x, y, \phi^{(\infty)}) = 0. \tag{7.3}$$

Taking the lift of (7.3), we obtain the linear relations

$$\varpi = 0, \qquad L^{(\infty)}(X, Y, \nu^{(\infty)}) = 0. \tag{7.4}$$

We have thus shown

Proposition 7.1. The Maurer-Cartan structure equations of an intransitive Lie pseudo-group $\mathcal{G}$ locally given by (6.1), are

$$(d\nu[\![H,K]\!] = \nabla_K \nu[\![H,K]\!] \wedge \nu[\![H,K]\!])\,|_{L^{(\infty)}(X,Y,\nu^{(\infty)})=0}, \tag{7.5}$$

when restricted to a target fiber, where

$$\nabla_K \nu[\![H,K]\!] = \left(\frac{\partial \nu^\alpha}{\partial K^\beta}[\![H,K]\!] \right)$$

is the $q \times q$ Jacobian matrix power series obtained by differentiating $\nu[\![H,K]\!]$ with respect to $K = (K^1, \ldots, K^q)$.

In particular the structure equations of the Lie pseudo-group (6.1) do not involve the Maurer-Cartan forms ϖ^i_A, $i = 1, \ldots, p$, $\# A \geq 0$. Since $\varpi^i = -dx^i$ on a target fiber, this implies that the structure equations (7.5) do not depend on dx^i, $i = 1, \ldots, p$. On the other hand, those differential forms do appear in Cartan's structure equations. Recalling equation (6.6), the differential forms dx^i correspond to the first p differential forms $\omega^i_{[0]}$, $i = 1, \ldots, p$, and their exterior differentials are

$$d\omega^i_{[0]} = 0, \qquad i = 1, \ldots, p.$$

Example 7.2. Consider the intransitive Lie group action

$$X = x \neq 0, \qquad Y = y + ax, \qquad a \in \mathbb{R}. \tag{7.6}$$

The infinitesimal generator of this one-parameter group of transformations is

$$\mathbf{v} = x\frac{\partial}{\partial y},$$

which clearly spans an abelian Lie algebra. Cartan computed the structure equations of this group, [?, p. 1345] , and obtained

$$d\omega^1 = 0, \qquad d\omega^2 = \frac{1}{x}\omega^1 \wedge \omega^2, \tag{7.7}$$

where $\omega^1 = dx$ and $\omega^2 = dy - \frac{y}{x}dx$. Clearly, the structure equations (7.7) do not correspond to those of an abelian group. Even worse, the structure equations (7.7) cannot be those of a finite Lie group since the structure constants depend on x and it is well known that all Lie groups have constant structure coefficients. Furthermore, the group is one-dimensional and there should only be one independent Maurer-Cartan form associated to this group.

Let us now compute the structure equation of the pseudo-group (7.6) using (4.6). The minimal involutive defining system of the group action is

$$X = x, \qquad Y - y = xY_x, \qquad Y_y = 1.$$

An infinitesimal generator $\mathbf{v} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ must be a solution of the infinitesimal determining equations

$$\xi = 0, \qquad \eta = x\eta_x, \qquad \eta_y = 0,$$

and the corresponding lifted determining equations are

$$\mu^x = 0, \qquad \mu^y = X\mu^y_X, \qquad \mu^y_Y = 0. \tag{7.8}$$

It follows from (7.8) that μ^y is a basis of Maurer-Cartan forms. The structure equation for μ^y is

$$d\mu^y = \mu^y_Y \wedge \mu^y = 0,$$

which corresponds to the structure equation of the abelian group action (7.6).

Example 7.3. Consider the intransitive Lie pseudo-group

$$X = x, \qquad Y = y + f(x), \qquad f \in C^\infty(\mathbb{R}). \tag{7.9}$$

The infinitesimal generator of the pseudo-group action (7.9) is

$$\mathbf{v} = g(x)\frac{\partial}{\partial y}, \tag{7.10}$$

with $g \in C^\infty(\mathbb{R})$. The Lie algebra generated by the vector field (7.10) is abelian. The structure equations for this Lie pseudo-group have been computed by Cartan, [?, p. 1346]. Those are given by

$$d\omega^1 = 0, \qquad d\omega^2 = \pi^1 \wedge \omega^1, \tag{7.11}$$

where $\omega^1 = dx$ and $\omega^2 = dy + Y_x dx$. The structure equations (7.11) do not correspond to those of the abelian algebra (7.10). On the other hand, one can verify, using the infinitesimal interpretation of the Maurer-Cartan structure equations given in Section 5, that the structure equations (7.11) correspond to the pseudo-group action

$$X = x + a, \qquad Y = y + f(x), \qquad a \in \mathbb{R}, \qquad f \in C^\infty(\mathbb{R}), \tag{7.12}$$

with infinitesimal generators

$$\partial_x, \qquad f(x)\partial_y.$$

The two Lie pseudo-groups (7.9) and (7.12) clearly have nonisomorphic infinitesimal structures, yet Cartan's structure theory gives the same structure equations for those two Lie pseudo-groups. This again highlights a problem with Cartan's structure theory.

Now let us use Olver and Pohjanpelto's theory to derive the structure equations of the Lie pseudo-group (7.9). The involutive defining system is given by the equations

$$X = x, \qquad Y_y = 1.$$

The infinitesimal determining equations of an infinitesimal generator $\mathbf{v} = \xi(x,y)\partial_x + \phi(x,y)\partial_y$ are

$$\xi = 0, \qquad \phi_y = 0. \tag{7.13}$$

The lift of (7.13) gives

$$\mu^x = 0, \qquad \mu^y_Y = 0,$$

and it follows that a basis of Maurer-Cartan forms for the pseudo-group (7.9) is given by

$$\mu^y_{X^k}, \qquad k \geq 0.$$

The structure equations for those differential forms are

$$\sum_{k=0}^{\infty} d\mu^y_{X^k} \frac{H_x^k}{k!} = \frac{\partial}{\partial H_u}\left(\sum_{k=1}^{\infty} \frac{\mu^y_{X^k}}{k!} H_x^k\right) \wedge \sum_{k=0} \mu^y_{X^k} \frac{H_x^k}{k!} = 0, \tag{7.14}$$

which do correspond to the structure equations of an abelian pseudo-group.

Returning to the pseudo-group (7.12), one can verify that Olver and Pohjanpelto's theory gives the structure equations

$$d\mu^x = 0, \qquad d\mu^y_{X^k} = \mu^y_{X^{k+1}} \wedge \mu^x, \qquad k \geq 0,$$

which is in accordance with the fact that two Lie pseudo-groups with nonisomorphic infinitesimal structures should have different structure equations.

The two previous examples show that Cartan's structure theory does not recover adequately the infinitesimal properties of intransitive Lie pseudo-groups. Under the assumption that the pseudo-group action is locally given by (6.1), the source of the problem is the inclusion of the differential forms $\omega^i_{[0]} = dx^i$, $i = 1, \ldots, p$, in the structure equations. The differential forms dx^i, $i = 1, \ldots, p$, are indeed invariant under the identity transformation $x \mapsto x$, but they are also invariant under the translation group $x \mapsto x + a$. By including the differential forms $dx^1, \ldots, dx^p$, in the structure equations (6.8), Cartan does not compute the structure of the Lie pseudo-group (6.1), but rather computes the infinitesimal structure of the transformation

$$X^i = x^i + a^i, \qquad Y^\alpha = f^\alpha(x,y), \qquad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q. \tag{7.15}$$

There is no guarantee that (7.15) is a Lie pseudo-group action. Indeed if we go back to Example 7.2 and replace the action (7.6) by

$$X = x + b, \qquad Y = y + ax, \qquad (a,b) \in \mathbb{R}^2,$$

this is no longer a group action since it is not closed under composition. The Lie brackets of the infinitesimal generators

$$\partial_x, \qquad x\partial_y$$

are seen to be dual to the structure equations (7.7) and as previously remarked, those do not form a Lie algebra.

A way to fix the problem is to set

$$\omega^i_{[0]} = dx^i = 0, \qquad i = 1, \ldots, p, \tag{7.16}$$

in Cartan's structure equations. One can verify in the two previous examples that under the equalities (7.16), Cartan's structure equations do give the infinitesimal structure one expects. The equalities (7.16) amounts to saying that Cartan's structure equations must be restricted to an orbit of the pseudo-group action where x is constant. Once the substitutions (7.16) are done, Cartan's structure theory becomes compatible with Olver and Pohjanpelto's approach.

In Olver and Pohjanpelto's formalism, the equalities (7.16) are direct consequences of the theory. From equation (7.4), we have that the order zero Maurer-Cartan forms ϖ^i, $i = 1, \ldots, p$, are identically zero. Since on a target fiber $(\tau^\infty)^{-1}(Z)$, $Z = (X, Y)$ is constant we have the equalities

$$0 = dX^i = d_M X^i + d_G X^i = \sigma^i + \varpi^i = \sigma^i = dx^i, \qquad i = 1, \ldots, p.$$

As for transitive Lie pseudo-groups, the equalities

$$0 = dY^\alpha = d_M Y^\alpha + \nu^\alpha = \omega^{p+\alpha}_{[0]} + \nu^\alpha, \qquad \alpha = 1, \ldots, q,$$

on a target fiber, relate the non-zero Maurer-Cartan forms ν^α, $\alpha = 1, \ldots, q$, to the horizontal forms $\omega^{p+\alpha}_{[0]}$, $\alpha = 1, \ldots, q$, defined by Cartan.

The restriction of Cartan's structure equations to an orbit of the pseudo-group action has a direct repercussion on Cartan's theory of essential invariants. As we explain in Section 9, Cartan's definition of essential invariants in terms of the systatic system becomes vacuous.

To simplify the following discussion, we assume the infinitesimal determining systems of Lie pseudo-groups to be of first order for the rest of the paper. There is no loss of generality in doing so since in [?, **?**] the authors give an algorithm that transforms an $n^\star$-th order infinitesimal involutive determining system into an equivalent first order involutive system.

8 Systatic System

Cartan's systatic system plays an important role in the structure theory of Lie pseudo-groups, [?, ?]. It is related to the isotropy algebra of pseudo-groups, [?], and the latter plays an important role in the classification of infinite-dimensional primitive Lie pseudo-groups, [**?**,**?**,**?**].

We now explain where Cartan's systatic system sits in Olver and Pohjanpelto's structure theory. Given a system of linear homogenous partial differential equations, we fix an order compatible term ordering of the partial derivatives which ranks derivatives of higher total order greater than those of lower total order, [?]. Gauss reduction of such system of differential equations with respect to the ordering yields a solved form expressing certain dependents, the principal derivatives, as functions of lower ranked non-principal (parametric) derivatives, [?, ?, **?**]. We denote by $\mathcal{P}_k$ the set of parametric derivatives of k-th order, and by $\overline{\mathcal{P}}_k$ the set of principal derivatives of k-th order. The number of principal and parametric derivatives of order $k \geq 1$ is given by

$$\begin{aligned} \#\mathcal{P}_k &= s_k = \dim \mathcal{G}^{(k)} - \mathcal{G}^{(k-1)}, \\ \#\overline{\mathcal{P}}_k &= m\binom{m+k-1}{k} - s_k. \end{aligned}$$

While $\#\mathcal{P}_0$ equals the dimension of the pseudo-group orbits on M, and $\#\overline{\mathcal{P}}_0 = m - \#\mathcal{P}_0$ their codimension. Using the notation conventions of Section 7.2, the pseudo-group orbits of the action (6.1) are of dimension $\#\mathcal{P}_0 = q$ and codimension $\#\overline{\mathcal{P}}_0 = p$. The first order infinitesimal involutive defining system $L^{(1)}(x, y, \xi^{(1)}, \phi^{(1)}) = 0$, can be written in the form

$$\begin{aligned} &\xi^i = 0, \qquad i = 1, \ldots, p, \\ &\frac{\partial \phi^\alpha}{\partial z^a} = \sum_{(\beta,b)\in\mathcal{P}_1} b^\alpha_{a,\beta b}(z) \frac{\partial \phi^\beta}{\partial z^b} + \sum_{\beta=1}^{q} b^\alpha_{a,\beta}(z)\phi^\beta, \qquad (\alpha, a) \in \overline{\mathcal{P}}_1. \end{aligned} \tag{8.1}$$

The lift of (8.1) implies the linear relations

$$\begin{aligned} \varpi^i =& 0, \\ \nu^\alpha_{Z^a} =& \sum_{(\beta,b)\in\mathcal{P}_1} b^\alpha_{a,\beta b}(Z)\nu^\beta_{Z^b} + \sum_{\beta=1}^{q} b^\alpha_{a,\beta}(Z)\nu^\beta \\ =& \sum_{\tau=1}^{s_1} b^\alpha_{a,\tau}(Z)\widetilde{\nu}^\tau + \sum_{\beta=1}^{q} b^\alpha_{a,\beta}(Z)\nu^\beta \qquad (\alpha, a) \in \overline{\mathcal{P}}_1. \end{aligned} \tag{8.2}$$

among the zero and first order Maurer-Cartan forms (7.1). In equation (8.2) we have relabeled the parametric Maurer-Cartan forms $\nu^\beta_{Z^b}$, $(\beta, b) \in \mathcal{P}_1$ by $\widetilde{\nu}^\tau$, $\tau = 1, \ldots, s_1$.

The zero order terms of the structure equations (7.5) give the differentials for the Maurer-Cartan forms $\nu^1, \ldots, \nu^q$. More explicitly, those are obtained by restricting

$$d\nu^\alpha = \sum_{\beta=1}^{q} \nu^\alpha_{Y^\beta} \wedge \nu^\beta, \qquad \alpha = 1, \ldots, q,$$

to (8.2). Hence the structure equations for $\nu^1, \ldots, \nu^q$ are of the form

$$d\nu^\alpha = \sum_{1\le\beta<\gamma\le q} C^\alpha_{\beta\gamma}\nu^\beta \wedge \nu^\gamma + \sum_{\gamma=1}^{q}\sum_{\tau=1}^{s_1} A^\alpha_{\gamma\tau}\nu^\gamma \wedge \widetilde{\nu}^\tau, \qquad \alpha = 1, \ldots, q. \tag{8.3}$$

Since all the differential forms involved in (8.3) are right invariant, it follows that the structure coefficients $C^\alpha_{\beta\gamma}$, and $A^\alpha_{\gamma\tau\beta}$ depend only on the target coordinates $Z^1, \ldots, Z^m$ of the Lie pseudo-group $\mathcal{G}$. This is also clear from (8.2). By transitivity of the pseudo-group action on its orbits, we can set $Y^\alpha = Y^\alpha_0$, $\alpha = 1, \ldots, q$, to some appropriate constants without losing any information, and assume that the structure coefficients in (8.3) depend only on the scalar invariants X^i, $i = 1, \ldots, p$.

To obtain structure equations analogous to (6.8), we restrict (8.3) to $\mathcal{G}^{(1)}$. We denote the restriction of $\widetilde{\nu}^\tau$ to $\mathcal{G}^{(1)}$ by π^τ. Because the Maurer-Cartan forms $\widetilde{\nu}^\tau$, $1, \ldots, s_1$ live in $T^*\mathcal{G}^{(2)}$, their restriction π^τ are no longer invariant one-forms, except when the Lie pseudo-group is finite-dimensional (Recall we are assuming the determining system to be of order one.). So the restriction of the structure equations (8.3) to $\mathcal{G}^{(1)}$ are

$$d\nu^\alpha = \sum_{1\le\beta<\gamma\le q} C^\alpha_{\beta\gamma}(X, Y_0)\nu^\beta \wedge \nu^\gamma + \sum_{\gamma=1}^{q}\sum_{\tau=1}^{s_1} A^\alpha_{\gamma\tau}(X, Y_0)\nu^\gamma \wedge \pi^\tau, \tag{8.4}$$

$\alpha = 1, \ldots, q$. In analogy with Cartan, we define the systatic system to be the Pfaffian system generated by the one-forms

$$\sum_{\gamma=1}^{q} A^\alpha_{\gamma\tau}\nu^\gamma, \qquad \alpha = 1, \ldots, m, \quad \tau = 1, \ldots, s. \tag{8.5}$$

It is related to Cartan's definition, [?, ?], by the equalities

$$\nu^\alpha = -d_M Y^\alpha = -\omega^{p+\alpha}_{[0]}, \qquad \alpha = 1, \ldots, q,$$

on a target fiber $(\tau^{(\infty)})^{-1}(Z)$.

Definition 8.1. The Pfaffian system generated by

$$\sum_{\gamma=1}^{q} A^\alpha_{\gamma\tau}\nu^\gamma, \qquad \alpha = 1, \ldots, q, \quad \tau = 1, \ldots, s_1, \tag{8.6}$$

is called the *systatic system* of $\mathcal{G}$.

Example 8.2. Consider the Lie pseudo-group of conformal transformations of the plane

$$X = f(x, y), \qquad Y = g(x, y), \qquad f_x g_y - f_y g_x = 1. \tag{8.7}$$

The first order involutive infinitesimal determining system, for an infinitesimal generator

$$\mathbf{v} = \xi(x, y)\partial_x + \eta(x, y)\partial_y,$$

is

$$\xi_x + \eta_y = 0. \tag{8.8}$$

The lift of (8.8) gives

$$\nu^x_X = -\nu^y_Y,$$

and the structure equations for ν^x and ν^y are

$$d\nu^x = \nu^x_X \wedge \nu^x + \nu^x_Y \wedge \nu^y,$$
$$d\nu^y = \nu^y_X \wedge \nu^x - \nu^x_X \wedge \nu^y.$$

Thus the systatic system is generated by

$$\{\nu^x, \nu^y\}. \tag{8.9}$$

On the other hand, Cartan computed the structure equations of the pseudo-group action (8.7) in [?]. With

$$\omega^1 = \sigma^x = X_x dx + X_y dy, \qquad \omega^2 = \sigma^y = Y_x dx + Y_y dy, \qquad X_x Y_y - X_y Y_x = 1,$$

he obtained

$$d\omega^1 = \pi^1 \wedge \omega^1 + \pi^2 \wedge \omega^2,$$
$$d\omega^2 = \pi^3 \wedge \omega^1 - \pi^1 \wedge \omega^2,$$

and the systatic system is generated by

$$\{\omega^1, \omega^2\}. \tag{8.10}$$

Using the fact that $\nu^x = -\sigma^x = -\omega^1$, $\nu^y = -\sigma^y = -\omega^2$ on a target fiber, the systatic systems (8.9) and (8.10) are equivalent.

9 Essential Invariants

For intransitive Lie pseudo-groups, Cartan draws a distinction between essential and inessential invariants.

Definition 9.1. A Pfaffian system generated by $\omega^1, \ldots, \omega^r$ is said to be *complete* if

$$d\omega^i \equiv 0 \qquad (\text{mod } \omega^1, \ldots, \omega^r), \qquad i = 1, \ldots, r.$$

In [?], Cartan shows that the systatic system is complete, and then extracts from it as many as possible linear combinations that only depend on the invariants $x^1, \ldots, x^p$ and their differentials $dx^1, \ldots, dx^p$. Suppose there are s linearly independent such combinations,

$$\Omega^j = \sum_{i=1}^{p} f_i^j(x^1, \ldots, x^p) dx^i, \qquad j = 1, \ldots, s. \tag{9.1}$$

The completeness of the systatic system implies that the Pfaffian system $(\Omega^1, \ldots, \Omega^s)$ is complete in the space of invariants $x^1, \ldots, x^p$, and so Cartan defines the first integrals[1] of $(\Omega^1, \ldots, \Omega^s)$ to be essential invariants, while the other invariants are said to be inessential. The problem with this definition, once the substitutions (7.16) are done, is that all the differential forms $\Omega^1, \ldots, \Omega^s$ are zero. Indeed, none of the differential forms $dx^1, \ldots, dx^p$ appear in the systatic system (8.6), hence there are no differential forms of the form (9.1).

Even though, Cartan's definition of essential invariants becomes vacuous, it is possible to give an alternative definition, which reflects Cartan's idea of what essential invariants should be. In order to justify our definition, we need to review the concept of isomorphism for Lie pseudo-groups, [?, **?**]. To do so, some preliminary definitions need to be given.

Definition 9.2. A Lie pseudo-group $\mathcal{H} \subset \mathcal{D}(N)$ is *similar* to a Lie pseudo-group $\mathcal{G} \subset \mathcal{D}(M)$ if there is a local diffeomorphism $\phi : N \to M$ such that $\mathcal{H} = \phi^{-1} \circ \mathcal{G} \circ \phi$.

Lemma 9.3. If $\mathcal{G}$ and $\mathcal{H}$ are similar, their structure equations are isomorphic.

Since any Lie pseudo-group is trivially similar to itself we have

Corollary 9.4. Let $\mathcal{G}$ be an intransitive Lie pseudo-group locally represented by (6.1). If the structure equations

$$\left(d\mu[\![H]\!] = \nabla_H \mu[\![H]\!] \wedge \mu[\![H]\!]\right)|_{L^{(\infty)}(Z,\mu^{(\infty)})=0},$$

depend on X^i for one basis of Maurer-Cartan forms then they also depend on X^i for any other basis of Maurer-Cartan forms.

Definition 9.5. Let $\mathcal{G} \subset \mathcal{D}(M)$ and $\overline{\mathcal{G}} \subset \mathcal{D}(\overline{M})$ be two Lie pseudo-groups such that $\pi : \overline{M} \to M$ is a fiber bundle with base space M. If for all $\overline{\phi} \in \overline{\mathcal{G}}$ there exist $\phi \in \mathcal{G}$ such that $\pi \circ \overline{\phi} = \phi \circ \pi$, then $\overline{\mathcal{G}}$ is called a *generalized prolongation* of $\mathcal{G}$.

In the literature, our definition of generalized prolongation is often simply called prolongation. We introduce this new terminology to distinguish between the more general notion of prolongation of a Lie pseudo-group $\mathcal{G}$ given in Definition 9.5 and the usual definition of prolonged pseudo-group $\mathcal{G}^{(n)}$ introduced in Definition 2.2.

Definition 9.6. A generalized prolongation $\overline{\mathcal{G}} \subset \mathcal{D}(\overline{M})$ of $\mathcal{G} \subset \mathcal{D}(M)$ is called isomorphic if the only diffeomorphism of $\overline{\mathcal{G}}$ that projects to $\mathbb{1}_M$ is $\mathbb{1}_{\overline{M}}$.

[1]The existence those first integrals is guaranteed by Frobenius' Theorem.

Definition 9.7. Two Lie pseudo-groups $\mathcal{G}$, $\mathcal{H}$ are said to be *isomorphic* if there exist isomorphic generalized prolongations $\overline{\mathcal{G}}$, $\overline{\mathcal{H}}$ such that $\overline{\mathcal{G}}$ is similar to $\overline{\mathcal{H}}$.

Of all possible isomorphic prolongations of $\mathcal{G}$, the infinite prolongation $\mathcal{G}^{(\infty)}$ is the most important. Indeed, let $\mathcal{G}$ and $\mathcal{H}$ be two Lie pseudo-groups such that $\mathcal{H}$ is an isomorphic generalized prolongation of $\mathcal{G}$. Assuming that the invariants of $\mathcal{H}$ can be expressed by means of the local coordinates of the manifold that $\mathcal{G}$ acts on, Cartan shows in [?] that there exists n such that $\mathcal{G}^{(n)}$ is an isomorphic generalized prolongation of $\mathcal{H}$. If $\mathcal{H}$ admits some invariants which are not acted upon by $\mathcal{G}$, $\mathcal{G}$ is extended by acting trivially on these, and then $\mathcal{G}^{(n)} \oplus \mathbb{1}$ is an isomorphic generalized prolongation of $\mathcal{H}$ for some n. This discussion implies that given two isomorphic Lie pseudo-groups $\mathcal{G}$, $\mathcal{H}$, up to the addition of scalar invariants, $\mathcal{G}^{(\infty)}$ is isomorphic to $\mathcal{H}^{(\infty)}$. In particular, their structure equations (4.6) are isomorphic.

Example 9.8. To illustrate the above definitions, consider the Lie pseudo-groups

$$\mathcal{H}: \qquad \widetilde{X} = \widetilde{x}, \qquad \widetilde{W} = \widetilde{w} + f(\widetilde{x}),$$

and

$$\mathcal{G}: \qquad X = x, \qquad Y = y + f(x)z + f'(x), \qquad Z = z,$$

where $f \in C^\infty(\mathbb{R})$. The Lie pseudo-group

$$\overline{\mathcal{H}}: \qquad \widetilde{X} = \widetilde{x}, \qquad \widetilde{Y} = \widetilde{y} + f(\widetilde{x})\widetilde{z} + f'(\widetilde{x}), \qquad \widetilde{Z} = \widetilde{z}, \qquad \widetilde{W} = \widetilde{w} + f(\widetilde{x}),$$

is an isomorphic generalized prolongation of $\mathcal{H}$. Similarly

$$\overline{\mathcal{G}}: \qquad X = x, \qquad Y = y + f(x)z + f'(x), \qquad Z = z, \qquad W = w + f(x),$$

is an isomorphic generalized prolongation of $\mathcal{G}$. Clearly, $\overline{\mathcal{H}}$ is similar to $\overline{\mathcal{G}}$, thus $\mathcal{H}$ is isomorphic to $\mathcal{G}$. Alternatively we note that $\mathcal{H}^{(1)} \oplus \mathbb{1}_{\widetilde{z}}$ is similar to $\overline{\mathcal{G}}$ since

$$\mathcal{H}^{(1)} \oplus \mathbb{1}_{\widetilde{z}}: \qquad \widetilde{X} = \widetilde{x}, \qquad \widetilde{W} = \widetilde{w} + f(\widetilde{x}), \qquad \widetilde{Y} = \widetilde{y} - f'(\widetilde{x}), \qquad \widetilde{Z} = \widetilde{z},$$

and

$$\mathcal{H}^{(1)} \oplus \mathbb{1}_{\widetilde{z}} = \phi^{-1} \circ \overline{\mathcal{G}} \circ \phi,$$

with

$$\phi : (\widetilde{x}, \widetilde{w}, \widetilde{y}, \widetilde{z}) \mapsto (x, y, z, w) = (\widetilde{x}, \widetilde{w}\widetilde{z} - \widetilde{y}, \widetilde{z}, \widetilde{w}).$$

The above discussion motivates the following definition of essential invariant.

Definition 9.9. Let $\mathcal{G}$ be an intransitive Lie pseudo-group locally represented by (6.1). An invariant X^i, $i \in \{1, \ldots, p\}$, is said to be *essential* if for a basis of Maurer-Cartan forms (hence for all), the structure coefficients of

$$(d\mu[\![H]\!] = \nabla_H \mu[\![H]\!] \wedge \mu[\![H]\!])\,|_{L^{(\infty)}(Z,\mu^{(\infty)})=0}$$

depend on X^i.

Example 9.10. In Example 7.3, Cartan's structure equations for the intransitive Lie pseudo-group (7.9) are given by (7.11). The systatic system is spanned by $\omega^1 = dx$, and in Cartan's theory x is an essential invariant. On the other hand, the observation that the transitive Lie pseudo-group (7.12) also has (7.11) for structure equations implies that the x appearing in the structure equations (7.11) is not even an invariant.

Based on the structure equations (7.14) and Definition 9.9, we say that the invariant x is not essential since it does not appear in the structure equations, and thus does not influence the structure of the pseudo-group.

Remark 9.11. In Cartan's structure theory, there is an algorithm to get rid of nonessential invariants as defined by Cartan, [?, ?]. On the other hand, no claim is made that such an algorithm exists for invariants not satisfying Definition 9.9. Example 9.10 is an illustration of this fact. In Example 9.10, the variable x is not an essential invariant, in the sense of Definition 9.9, yet it is clear that the pseudo-group action (7.9) cannot be written without the use of the invariant x.

Example 9.12. Consider the pseudo-group action (6.9) of Example 6.1. We show that x is an essential invariant. The involutive defining system of (6.9) is

$$X = x, \qquad Y_x = 0, \qquad Y_z = 0, \qquad Z_z = (Y_y)^x,$$

and the infinitesimal determining system, for an infinitesimal generator

$$\mathbf{v} = \xi(x,y,z)\partial_x + \eta(x,y,z)\partial_y + \phi(x,y,z)\partial_z,$$

is

$$\xi = 0, \qquad \eta_x = 0, \qquad \eta_z = 0, \qquad \phi_z = x\eta_y. \tag{9.2}$$

The lift of (9.2) gives the linear relations

$$\mu^x = 0, \qquad \mu^y_X = 0, \qquad \mu^y_Z = 0, \qquad \mu^z_Z = X\mu^y_Y,$$

and it follows that

$$\mu^y_{Y^k}, \qquad \mu^z_{X^kY^j}, \qquad k,j \geq 0,$$

is a basis of Maurer-Cartan forms. Focusing our attention to the differentials of μ^y and μ^z we have

$$\begin{aligned} d\mu^y &= \mu^y_X \wedge \mu^x + \mu^y_Y \wedge \mu^y + \mu^y_Z \wedge \mu^z = \mu^y_Y \wedge \mu^y, \\ d\mu^z &= \mu^z_X \wedge \mu^x + \mu^z_Y \wedge \mu^y + \mu^z_Z \wedge \mu^z = \mu^z_Y \wedge \mu^y + X\mu^y_Y \wedge \mu^z. \end{aligned}$$

Thus x is an essential invariant.

10 Conclusion

In this paper we have shown that Olver and Pohjanpelto's structure equations capture the infinitesimal properties of Lie pseudo-groups. In the language of jets, the information of a Lie pseudo-group is contained in the target fibers of the bundle $\tau^{(\infty)} : \mathcal{G}^{(\infty)} \to M$. The infinitesimal structural properties are studied by introducing right invariant group forms and computing their differentials on the target fibers. If the pseudo-group is transitive, every target fibers are isomorphic and the structure coefficients are constant. If the action is intransitive, the structure of the pseudo-group on each target fiber can vary if the structure equations depend on essential invariants.

Cartan's structure theory mixes horizontal and group forms and we have shown that the appearance of the horizontal forms into the structure equations is a source of problem for intransitive Lie pseudo-groups. In fact, we noticed that Cartan's structure equations of intransitive Lie pseudo-groups always correspond to the infinitesimal structure of some transitive transformations. A way to correct Cartan's structure theory is to restrict his considerations to the orbits the pseudo-group actions. Since transitive Lie pseudo-groups have only one orbit, we conclude that his theory is good for such Lie pseudo-groups. For intransitive Lie pseudo-groups, this is not the case. The restriction to an orbit modifies

Cartan's structure equations and has an important repercussion on his notion of essential invariants.

Our observation explains, in part, some difficulties encountered by past researchers when trying to extend results for transitive Lie pseudo-groups to intransitive Lie pseudo-groups. For example, as we have shown in Section 5, Singer and Sternberg infinitesimal interpretation of Cartan's structure equations can be extended to intransitive Lie pseudo-groups when working with the right structure equations. Also, Lisle and Reid's method of deriving Cartan's structure equations for transitive Lie pseudo-groups from the infinitesimal defining system, [?, ?, ?], extends to intransitive Lie pseudo-groups. Indeed, once restricted to a pseudo-group orbit, the pseudo-group action is transitive and their algorithm still holds.

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