The Equivariant Moving Frame Method and the Local Equivalence of $u_{xx} = r(x, u, v, u_x, v_x)$ Under Fiber-Preserving Transformations

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Abstract

We use the equivariant moving frame method to study the local equivalence problem of scalar control equations of the form $u_{xx} = r(x, u, v, u_x, v_x)$ under the pseudo-group of fiber-preserving transformations $X = \phi(x)$, $U = \beta(x, u)$, $V = \alpha(x, u, v)$. Three typical branches of the equivalence problem are considered: The degenerate case which contains the control systems with largest fiber-preserving symmetry group, the branch containing the Hilbert–Cartan equation and finally the generic case.

1 Introduction

A common problem in geometry consists of determining when two geometrical structures are locally equivalent up to some group of transformations. Using his theory of exterior differential systems, Cartan developed a powerful algorithm for answering such question, [13, 21]. Recently, Olver and his collaborators proposed a new theoretical foundation to Cartan's moving frame theory now known as the *equivariant moving frame method*, [9, 23, 24], and in [25], it was shown how to use this new method to solve local equivalence problems. The aim of this paper is to apply the results of [25] to study the local equivalence of non-autonomous one-dimensional control systems of the form

$$u_{xx} = r(x, u, v, u_x, v_x) \tag{1.1}$$

under the pseudo-group of fiber-preserving transformations

$$X = \phi(x), \qquad U = \beta(x, u), \qquad V = \alpha(x, u, v). \tag{1.2}$$

By assumption

$$\phi_x \neq 0, \qquad \beta_u \neq 0, \qquad \alpha_v \neq 0,$$

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so that (1.2) is a local diffeomorphisms of $\mathbb{R}^3$. In physical applications, the variable $u = u(x)$ describes the dynamics of a state while $v = v(x)$ is a scalar control parameter, and equation (1.1) can be interpreted as a one-dimensional Newtonian equation with a control in the force term.

Example 1.1. A simple control system prescribed by an equation of the form (1.1) comes from robotics where the position of a single-link rotational joint is controlled by a motor placed at the pivot, see Figure 1.

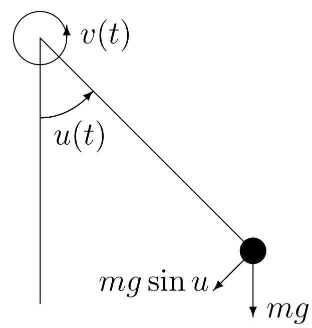

Figure 1: Controlled pendulum.

If we assume that the units of time and distance are chosen so that the mass m and the gravitational constant g are equal to one, and that the rod has unit length, then the differential equation governing the time evolution of the mass, when friction is not neglected, is given by

$$u_{tt} = f(t, v, v_t) - \sin u - \alpha\, u_t^2,$$

where the constant α is the *coefficient of friction* and $f(t, v, v_t)$ is a force term depending on the external torque v, its time derivative v_t and possibly on time.

The literature contains similar, but different, versions of the equivalence problem (1.1), (1.2). A more well-known problem in the geometry of underdetermined ordinary differential equations of the form

$$v_x = r(x, u, v, u_x, u_{xx})$$

is the problem of internal equivalence studied by Cartan, [5, 6], Nurowski, [19], Doubrov and Zelenko, [8] and Anderson and Kruglikov, [1]. Closer along the lines of the problem considered in this paper is the equivalence problem of control equations that do not depend on x and v_x, namely,

$$u_{xx} = r(u, v, u_x), \tag{1.3}$$

up to the group of feedback transformations

$$X = x, \qquad U = \beta(u), \qquad V = \alpha(u, v).$$

Provided some generic conditions on (1.3), this problem was solved in [17]. Finally, when the right-hand side of (1.1) does not depend on the control parameter v and its derivative v_x, i.e. when (1.1) is a standard second-order ordinary differential equation, the solution is well understood and appears in [10, 11, 15, 16]. To our knowledge, the equivalence problem (1.1), (1.2) satisfying the non-degeneracy condition

$$\frac{\partial r}{\partial v_x} \neq 0$$

has not been considered in the literature.

Mathematically, this problem is interesting in its own as it contains interesting examples of underdetermined ordinary differential equations. The most celebrated example is most likely the Hilbert–Cartan equation

$$u_{xx} = (v_x)^{1/2}. \tag{1.4}$$

An interesting feature of equation (1.4) is that its algebra of internal symmetries is isomorphic to the 14-dimensional non-compact real form of the exceptional Lie algebra $\mathfrak{g}_2$, [4], which is much larger than its 6-dimensional Lie algebra of external symmetries, [2]. The Hilbert–Cartan equation (1.4) is also an example of differential equation that does not admit parametric solutions of finite rank, [14].

Since we are interested in studying the equivalence problem (1.1), (1.2) using the new theory of equivariant moving frames, we begin this paper with a review of the structure theory of infinite-dimensional Lie pseudo-groups and the equivariant moving frame construction, [22, 23, 25]. One of the interesting features is that all computations can be done symbolically. Taking advantage of this feature, we symbolically analyze the solution to our proposed equivalence problem in Section 4. Three typical branches of the problem are studied. We first consider the degenerate branch which contains the control equations with larges fiber-preserving symmetry group. Then, we consider an intermediate branch which contains the Hilbert–Cartan equation (1.4) and finish with the generic case. These three cases highlight all the different features of the equivariant moving frame solution of a local equivalence problem. In Section 5, the coordinate expressions of some differential invariants found using our symbolic computations are obtained.

2 Lie Pseudo-Groups and Moving Frames

In this section we briefly review the structure theory of infinite-dimensional Lie pseudo-groups and the equivariant moving frame construction. We refer the reader to [22, 23, 24, 25] for more details.

2.1 Structure Equations of Lie Pseudo-Groups

Let M be an m-dimensional manifold and $\mathcal{D} = \mathcal{D}(M)$ the pseudo-group of all local diffeomorphisms of M. In the following, all manifolds, maps, vector fields and differential forms are assumed to be analytic. For all $0 \leq n \leq \infty$ we denote by $\mathcal{D}^{(n)}$ the bundle formed by the n^{th} order jets of local diffeomorphisms. The coordinates of the n-jet of a local diffeomorphism $Z = \varphi(z)$ are given by $\mathrm{j}_n\varphi = \phi^{(n)} = (z, Z^{(n)})$, where $z = (z^1, \ldots, z^m)$ are the *source coordinates* on M, $Z = (Z^1, \ldots, Z^m)$ the *target coordinates* also on M, and the corresponding *jet coordinates* Z^a_B representing the partial derivatives $\partial^k\varphi^a(z)/\partial z^{b^1} \cdots \partial z^{b^k}$, with $1 \leq a, b^1, \ldots, b^k \leq m$ and $1 \leq k = \#A \leq n$.

Definition 2.1. A pseudo-group $\mathcal{G} \subset \mathcal{D}$ is called *regular* of order $n_\star \geq 1$ if, for all finite $n \geq n_\star$, the set of n-jet of transformations $\mathcal{G}^{(n)}$ forms an embedded subbundle of $\mathcal{D}^{(n)}$ and the projection $\pi_n^{n+1}\colon \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a surjective submersion. An analytic pseudo-group $\mathcal{G} \subset \mathcal{D}$ is called a *Lie pseudo-group* if $\mathcal{G}$ is regular of order $n_\star \geq 1$ and, moreover, every local diffeomorphism φ of $\mathcal{D}$ satisfying $\mathrm{j}_{n_\star}\varphi \subset \mathcal{G}^{(n_\star)}$ belongs to the pseudo-group.

In local coordinates, for $n \geq n_\star$, the pseudo-group jet subbundle $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ is characterized by an involutive system of n^{th} order partial differential equations

$$F^{(n)}(z, Z^{(n)}) = 0 \tag{2.1}$$

called the n^{th} order *determining system* of $\mathcal{G}$.

Let $\mathcal{X}(M)$ be the space of locally defined vector fields on M. In local coordinates a vector field in $\mathcal{X}(M)$ will be denoted by

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a}. \tag{2.2}$$

For $0 \leq n \leq \infty$, let $\mathrm{J}^n TM$ denote the bundle of n^{th} order jets of sections of TM. Local coordinates on $\mathrm{J}^n TM$ are given by

$$(z, \zeta^{(n)}) = (z^a, \zeta^a_B), \qquad a = 1, \ldots, m, \qquad 0 \leq \#B \leq n,$$

where ζ^a_B represents the partial derivative $\partial^{\#B}\zeta^a/\partial z^B$. Let

$$L^{(n)}(z, \zeta^{(n)}) = \sum_{a=1}^{m} \sum_{\#B \leq n} h^B_a(z) \zeta^a_B = 0, \tag{2.3}$$

be the linear system of partial differential equations obtained by linearizing the determining system (2.1) at the identity n-jet $\mathbb{1}^{(n)} \in \mathcal{G}^{(n)}$. A vector field (2.2) is in the Lie algebra $\mathfrak{g}$ of infinitesimal generators of the Lie pseudo-group $\mathcal{G}$ if and only if[1] its n-jet is a solution of (2.3). For this reason, the system of equations (2.3) is called the (n^{th} order) *infinitesimal determining system* of $\mathfrak{g}$.

As with Lie groups, the structure equations of a Lie pseudo-group $\mathcal{G}$ are obtained by computing the structure equations of a $\mathcal{G}$-invariant coframe. For the diffeomorphism pseudo-group $\mathcal{D}$ an invariant coframe is given by the horizontal forms

$$\sigma^a = \sum_{b=1}^{m} Z^a_b dz^b, \qquad a = 1, \ldots, m = \dim M,$$

and the Maurer–Cartan forms

$$\mu^b_A, \qquad b = 1, \ldots, m, \quad \#A \geq 0, \tag{2.4}$$

on $\mathcal{D}^{(\infty)}$. The coordinate expressions of the Maurer–Cartan forms (2.4) can be found in [22] but they are not needed here. The structure equations of the diffeomorphism pseudo-group $\mathcal{D}(\mathbb{R})$ were first obtained by Cartan, [7]. For $M = \mathbb{R}^m$, $m \geq 2$, these equations were first explicitly obtained in [22].

[1] assuming that the pseudo-group is *tame*, [22].

Theorem 2.2. The structure equations of the diffeomorphism pseudo-group $\mathcal{D}(M)$ are

$$d\sigma^a = \sum_{b=1}^{m} \mu_b^a \wedge \sigma^b, \qquad d\mu_C^a = \sum_{b=1}^{m} \left[\sigma^b \wedge \mu_{C,b}^a + \sum_{\substack{C=(A,B),\\ \#B\geq 1}} \binom{C}{A} \mu_{A,b}^a \wedge \mu_B^b \right], \tag{2.5}$$

where the last sum ranges over all multi-indices A, B such that $\#B \geq 1$, and their concatenation equals the multi-index C. By definition

$$\binom{C}{A} = \frac{C!}{A!\,B!} \qquad \text{when} \qquad C = (A, B).$$

On $\mathcal{G}^{(\infty)}$ the Maurer–Cartan forms (2.4) are no longer linearly independent, the linear relations among them are easily obtained from the infinitesimal determining equations (2.3).

Proposition 2.3. Let $\mathcal{G} \subset \mathcal{D}$ be a Lie pseudo-group. Once restricted to $\mathcal{G}^{(\infty)}$ the Maurer–Cartan forms (2.4) satisfy the linear relations

$$L^{(\infty)}(Z, \mu^{(\infty)}) = 0, \tag{2.6}$$

obtained by making the formal replacements $z^a \mapsto Z^a$, $\zeta_B^a \mapsto \mu_B^a$ in the infinitesimal determining equations (2.3).

We refer the reader to [22] for the proof of Proposition 2.3 and the detailed justification of the formal replacement $\zeta_B^a \mapsto \mu_B^a$.

Theorem 2.4. The structure equations of a Lie pseudo-group $\mathcal{G} \subset \mathcal{D}$ are obtained by restricting the structure equations (2.5) to the kernel of (2.6).

Example 2.5. Let $p = u_x$ and $q = v_x$. In this example, we compute the structure equations of the Lie pseudo-group action

$$X = \phi(x), \quad U = \beta(x, u), \quad V = \alpha(x, u, v), \quad P = \frac{p\beta_u + \beta_x}{\phi_x}, \quad Q = \frac{q\alpha_v + p\alpha_u + \alpha_x}{\phi_x}, \tag{2.7}$$

obtained from (1.2) by considering its first-order prolongation, [20]. The linearization of (2.7) at the identity jet yields the infinitesimal generator

$$\begin{aligned} \mathbf{v} =& \xi(x)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u} + \gamma(x, u, v)\frac{\partial}{\partial v} + [p(\eta_u - \xi_x) + \eta_x]\frac{\partial}{\partial p} \\ &+ [q(\gamma_v - \xi_x) + p\gamma_u + \gamma_x]\frac{\partial}{\partial q}, \end{aligned} \tag{2.8}$$

where $\xi(x)$, $\eta(x, u)$ and $\gamma(x, u, v)$ are arbitrary analytic functions. Let $\tau(x, u, v, p, q)$, and $\zeta(x, u, v, p, q)$ be the p, q-components of the vector field (2.8), respectively. Then, the components of the vector field (2.8) are solution to the infinitesimal determining system

$$\begin{gathered} \xi_u = \xi_v = \xi_p = \xi_q = 0, \qquad \eta_v = \eta_p = \eta_q = 0, \qquad \gamma_p = \gamma_q = 0, \\ \tau = p(\eta_u - \xi_x) + \eta_x, \qquad \zeta = q(\gamma_v - \xi_x) + p\gamma_u + \gamma_x. \end{gathered} \tag{2.9}$$

By virtue of Proposition 2.3, the linear relations among the Maurer–Cartan forms μ^x_A, μ^u_A, μ^v_A, μ^p_A, μ^q_A are obtained by making the substitutions

$$
\begin{gathered}
x \mapsto X, \qquad u \mapsto U, \qquad v \mapsto V, \qquad p \mapsto P, \qquad q \mapsto Q, \\
\xi_A \mapsto \mu^x_A, \qquad \eta_A \mapsto \mu^u_A, \qquad \gamma_A \mapsto \mu^v_A, \qquad \tau_A \mapsto \mu^p_A, \qquad \zeta_A \mapsto \mu^q_A
\end{gathered}
$$

in (2.9). The result is

$$
\begin{gathered}
\mu^x_U = \mu^x_V = \mu^x_P = \mu^x_Q = 0, \qquad \mu^u_V = \mu^u_P = \mu^u_Q = 0, \qquad \mu^v_P = \mu^v_Q = 0, \\
\mu^p = P(\mu^u_U - \mu^x_X) + \mu^u_X, \qquad \mu^q = Q(\mu^v_V - \mu^x_X) + P\mu^v_U + \mu^v_X.
\end{gathered} \tag{2.10}
$$

It follows from (2.10) that a basis of Maurer–Cartan forms is given by

$$
\mu_i = \mu^x_{X^i}, \qquad \nu_{i,j} = \mu^u_{U^i X^j}, \qquad \alpha_{i,j,k} = \mu^v_{V^i U^j X^k}, \qquad i, j, k \geq 0. \tag{2.11}
$$

To simplify the notation, we introduce the functions

$$
\delta(x, u, p) = p\beta_u + \beta_x, \qquad \psi(x, u, p, q) = q\alpha_v + p\alpha_u + \alpha_x, \qquad \chi(x) = \frac{\phi_{xx}}{\phi_x}. \tag{2.12}
$$

Applying Theorem 2.4, we find that the structure equations of the invariant horizontal coframe

$$
\begin{gathered}
\sigma^x = \phi_x dx, \qquad \sigma^u = \beta_x dx + \beta_u du, \\
\sigma^v = \alpha_x dx + \alpha_u du + \alpha_v dv, \qquad \sigma^p = \frac{1}{\phi_x}\left[(\delta_x - \delta\chi)dx + \delta_u du + \beta_u dp\right], \\
\sigma^q = \frac{1}{\phi_x}\left[(\psi_x - \psi\chi)dx + \psi_u du + \psi_v dv + \alpha_u dp + \alpha_v dq\right],
\end{gathered} \tag{2.13}
$$

are

$$
\begin{aligned}
d\sigma^x &= \mu_X \wedge \sigma^x, \\
d\sigma^u &= \nu_X \wedge \sigma^x + \nu_U \wedge \sigma^u, \\
d\sigma^v &= \alpha_X \wedge \sigma^x + \alpha_U \wedge \sigma^u + \alpha_V \wedge \sigma^v, \\
d\sigma^p &= [P(\nu_{UX} - \mu_{XX}) + \nu_{XX}] \wedge \sigma^x + [P\nu_{UU} + \nu_{UX}] \wedge \sigma^u + [\nu_U - \mu_X] \wedge \sigma^p, \\
d\sigma^q &= [Q(\alpha_{VX} - \mu_{XX}) + P\alpha_{UX} + \alpha_{XX}] \wedge \sigma^x + [Q\alpha_{VU} + P\alpha_{UU} + \alpha_{UX}] \wedge \sigma^u \\
&\quad + [Q\alpha_{VV} + P\alpha_{VU} + \alpha_{VX}] \wedge \sigma^v + \alpha_U \wedge \sigma^p + [\alpha_V - \mu_X] \wedge \sigma^q,
\end{aligned} \tag{2.14a}
$$

while the structure equations for the Maurer–Cartan forms (2.11) are

$$
\begin{aligned}
& d\mu = \sigma^x \wedge \mu_X, \qquad d\nu = \sigma^x \wedge \nu_X + \sigma^u \wedge \nu_U, \qquad d\alpha = \sigma^x \wedge \alpha_X + \sigma^u \wedge \alpha_U + \sigma^v \wedge \alpha_V, \\
& d\mu_X = \sigma^x \wedge \mu_{XX}, \qquad d\nu_X = \sigma^x \wedge \nu_{XX} + \sigma^u \wedge \nu_{XU} + \nu_X \wedge (\mu_X - \nu_U), \\
& d\nu_U = \sigma^x \wedge \nu_{XU} + \sigma^u \wedge \nu_{UU}, \qquad d\alpha_V = \sigma^x \wedge \alpha_{XV} + \sigma^u \wedge \alpha_{UV} + \sigma^v \wedge \alpha_{VV}, \\
& d\alpha_U = \sigma^x \wedge \alpha_{XU} + \sigma^u \wedge \alpha_{UU} + \sigma^v \wedge \alpha_{VU} + \alpha_U \wedge (\nu_U - \alpha_V), \\
& d\alpha_X = \sigma^x \wedge \alpha_{XX} + \sigma^u \wedge \alpha_{UX} + \sigma^v \wedge \alpha_{VX} + \alpha_X \wedge (\mu_X - \alpha_V) + \alpha_U \wedge \nu_X, \\
& \qquad \vdots
\end{aligned} \tag{2.14b}
$$

2.2 Equivariant Moving Frames

Let $1 \leq p < m = \dim M$. For all $0 \leq n \leq \infty$, let $\mathrm{J}^n = \mathrm{J}^n(M,p)$ be the extended jet bundle of equivalence classes of p-dimensional submanifolds of M under n^{th} order contact, [20, 21]. We introduce adapted coordinates $z = (x,u) = (x^1,\ldots,x^p,u^1,\ldots,u^q)$ on M so that submanifolds of M are locally represented by graphs of functions $(x, f(x))$. The induced local coordinates on J^n are denoted by $z^{(n)} = (x, u^{(n)})$, where $u^{(n)}$ denotes the derivatives $u^\alpha_J = \partial^J u^\alpha / \partial x^J$ of the u's with respect to the x's of order $0 \leq \#J \leq n$. The algorithm leading to the construction of an equivariant moving frame contains two steps.

Step 1: Lift

Definition 2.6. The n^{th} order *lifted bundle* $\mathcal{E}^{(n)} \to \mathrm{J}^n$ is defined as the pull-back bundle of $\mathcal{G}^{(n)} \to M$ via the projection $\pi^n_0 : \mathrm{J}^n \to M$.

Local coordinates on $\mathcal{E}^{(n)}$ are given by $(z^{(n)}, g^{(n)})$, where the base coordinates $z^{(n)} = (x, u^{(n)}) \in \mathrm{J}^n$ are the submanifold jet coordinates, and the fiber coordinates $g^{(n)}$ parametrize the Lie pseudo-group jets. The lifted bundle has the structure of a groupoid, [18], with source map $\boldsymbol{\sigma}(z^{(\infty)}, g^{(\infty)}) = z^{(\infty)}$ and target map $\boldsymbol{\tau}(z^{(\infty)}, g^{(\infty)}) = Z^{(\infty)} = g^{(\infty)} \cdot z^{(\infty)}$, given by the *prolonged action*, [20]. In local coordinates, the prolonged action is found by differentiating the target coordinates $U^\alpha = g \cdot u^\alpha$ with respect to the *lifted total differential operators*

$$D_{X^i} = \sum_{j=1}^{p} W^j_i D_{x^j}, \qquad \text{where} \qquad (W^j_i) = (D_{x^i} X^j)^{-1}, \tag{2.15}$$

to obtain

$$U^\alpha_J = D^X_J(U^\alpha), \qquad \text{where } D^X_J = D_{X^{j^1}} \cdots D_{X^{j^k}}, \quad k = \#J \geq 0. \tag{2.16}$$

The action of $\mathcal{G}$ on J^n and $\mathcal{G}^{(n)}$ are combined together to induce the *lifted action* on $\mathcal{E}^{(n)}$:

$$h \cdot (z^{(n)}, g^{(n)}) = (h^{(n)} \cdot z^{(n)}, g^{(n)} \cdot (h^{-1})^{(n)}), \qquad h \in \mathcal{G}, \tag{2.17}$$

whenever the compositions in (2.17) are well-defined. By definition of the lifted action (2.17), the expressions (X^i, U^α_J) are invariant differential functions defined on the lifted bundle $\mathcal{E}^{(\infty)}$. These invariants are known as the *lifted differential invariants* of the Lie pseudo-group action.

Example 2.7. We consider the induced action of (1.2) on control equations of the form (1.1), where we set $p = u_x$, $q = v_x$, $r = u_{xx}$, and r is assumed to be a function of the independent variables x, u, v, p, q. By the chain rule, the action (1.2) induces an action on p, q, given by (2.7) and on r:

$$R = -\frac{(p\beta_u + \beta_x)\phi_{xx}}{\phi_x^3} + \frac{p^2\beta_{uu} + r\beta_u + 2p\beta_{ux} + \beta_{xx}}{\phi_x^2} = \frac{-\delta\chi + p\delta_u + \delta_x + r\beta_u}{\phi_x^2}. \tag{2.18}$$

To compute the prolonged action we apply the lifted total differential operators

$$\begin{gathered} D_Q = \frac{\phi_x}{\alpha_v} D_q, \quad D_P = \frac{1}{\beta_u}[\phi_x D_p - \alpha_u D_Q], \\ D_V = \frac{1}{\alpha_v}\left[D_v - \frac{\psi_v}{\phi_x} D_Q\right], \quad D_U = \frac{1}{\beta_u}\left[D_u - \alpha_u D_V - \frac{\delta_u}{\phi_x} D_P - \frac{\psi_u}{\phi_x} D_Q\right], \\ D_X = \frac{1}{\phi_x}\left[D_x - \beta_x D_U - \alpha_x D_V - \left(\frac{\delta_x - \delta\chi}{\phi_x}\right) D_P - \left(\frac{\psi_x - \psi\chi}{\phi_x}\right) D_Q\right], \end{gathered} \tag{2.19}$$

to the lifted invariant (2.18). For example, the first-order lifted differential invariants are

$$\begin{aligned} R_Q =& \frac{\beta_u r_q}{\alpha_v \phi_x}, \quad R_P = \frac{1}{\beta_u}\left[\frac{-\beta_u\chi + 2\delta_u + \beta_u r_p}{\phi_x} - \alpha_u R_Q\right], \quad R_V = \frac{1}{\alpha_v}\left[\frac{\beta_u r_v}{\phi_x^2} - \frac{\psi_v}{\phi_x} R_Q\right], \\ R_U =& \frac{1}{\beta_u}\left[\frac{-\delta_u\chi + p\delta_{uu} + \delta_{ux} + \beta_{uu} r + \beta_u r_u}{\phi_x^2} - \alpha_u R_V - \frac{\delta_u}{\phi_x} R_P - \frac{\psi_u}{\phi_x} R_Q\right], \\ R_X =& \frac{1}{\phi_x}\left[\frac{p\delta_{xu} + \delta_{xx} + \beta_{ux} r + \beta_u r_x - \delta_x\chi - \delta\chi_x}{\phi_x^2} - 2\gamma R - \beta_x R_U - \alpha_x R_V \right. \\ & \left. - \left(\frac{\delta_x - \delta\chi}{\phi_x}\right) R_P - \left(\frac{\psi_x - \psi\chi}{\phi_x}\right) R_Q\right]. \end{aligned} \tag{2.20}$$

Differentiating (2.20) with respect to (2.19) yields the second-order lifted differential invariants and so on.

Now that we have lifted the submanifold jet coordinates $z^{(\infty)}$ to differential invariants $Z^{(\infty)}$ on $\mathcal{E}^{(\infty)}$, we do the same for the standard coframe on J^∞. The space of differential forms on $\mathcal{E}^{(\infty)}$ splits into

$$\mathbf{\Omega}^* = \bigoplus_{k,l} \mathbf{\Omega}^{k,l} = \bigoplus_{i,j,l} \mathbf{\Omega}^{i,j,l},$$

where l indicates the number of Maurer–Cartan forms (2.4), and $k = i + j$ the number of *jet forms*, with i indicating the number of horizontal forms dx^i, $1 \le i \le p$, and j the number of basic contact forms

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^p u_{J,i}^\alpha dx^i, \qquad \alpha = 1, \ldots, q, \qquad \#J \ge 0, \tag{2.21}$$

on the submanifold jet bundle J^∞. Let

$$\mathbf{\Omega}_J^* = \bigoplus_k \mathbf{\Omega}^{k,0} = \bigoplus_{i,j} \mathbf{\Omega}^{i,j,0} \tag{2.22}$$

denote the subspace of jet forms consisting of those differential forms containing no Maurer–Cartan forms. Let $\pi_J\colon \mathbf{\Omega}^* \to \mathbf{\Omega}_J^*$ be the projection that takes a differential form Ω on $\mathcal{E}^{(\infty)}$ to its jet component $\pi_J(\Omega)$ by annihilating all Maurer–Cartan forms in Ω. Similarly, let

$$\mathbf{\Omega}_{\mathcal{G}}^* = \bigoplus_l \mathbf{\Omega}^{0,l}$$

be the subspace of group forms consisting of those differential forms containing only Maurer–Cartan forms, and let $\pi_{\mathcal{G}}\colon \mathbf{\Omega}^* \to \mathbf{\Omega}_{\mathcal{G}}^*$ be the projection onto the group component.

Definition 2.8. The *lift* of a differential form Ω on J^∞ is the jet form

$$\boldsymbol{\lambda}(\Omega) := \pi_J[\boldsymbol{\tau}^*(\Omega)]. \tag{2.23}$$

Note that if Ω is a differential function, then (2.23) is equal to the prolonged action. In the following we denote by $\omega^i = \boldsymbol{\lambda}(dx^i)$ the lift of the horizontal coframe on J^∞ and by $\Theta^\alpha_J = \boldsymbol{\lambda}(\theta^\alpha_J)$ the lift of the contact forms. The lift map $\boldsymbol{\lambda}$ is extended to vector field jet coordinates as follows, [22, 23].

Definition 2.9. The lift of a vector jet coordinate ζ^b_A is defined to be the Maurer–Cartan form μ^b_A:

$$\boldsymbol{\lambda}(\zeta^b_A) = \mu^b_A, \qquad \text{for} \qquad b = 1, \ldots, m, \qquad \#A \geq 0.$$

More generally, the lift of any finite linear combination of vector field jet coordinates

$$\sum_{b=1}^{m} \sum_{\#A\geq 0} P^A_b(z^{(n)})\zeta^b_A$$

is defined to be the invariant group one-form

$$\boldsymbol{\lambda}\left[\sum_{b=1}^{m} \sum_{\#A\geq 0} P^A_b(z^{(n)})\zeta^b_A\right] = \sum_{b=1}^{m} \sum_{\#A\geq 0} P^A_b(Z^{(n)})\mu^b_A.$$

With this in hand, we can now write down the *universal recurrence* formula found in [23].

Theorem 2.10. Let Ω be a differential form on J^∞. Then

$$d[\boldsymbol{\lambda}(\Omega)] = \boldsymbol{\lambda}[d\Omega + \mathbf{v}^{(\infty)}(\Omega)], \tag{2.24}$$

where $\mathbf{v}^{(\infty)}$ is the prolongation of the vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \ \in \ \mathfrak{g} \tag{2.25}$$

given by

$$\mathbf{v}^{(\infty)} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{k=\#J\geq 0} \phi^{\alpha;J}(x,u^{(k)})\frac{\partial}{\partial u^\alpha_J} \ \in \ \mathfrak{g}^{(\infty)}, \tag{2.26}$$

with $\phi^{\alpha;J}$ defined recursively by the prolongation formula, [20],

$$\phi^{\alpha;J,j} = D_{x^j}\phi^{\alpha;J} - \sum_{i=1}^{p} (D_{x^j}\xi^i)u^\alpha_{J,i}. \tag{2.27}$$

In particular, the identity (2.24) applies to the lifted differential invariants X^i, U^α_J leading to

$$\begin{aligned} dX^i &= \omega^i + \mu^i, && i = 1, \ldots, p, \\ dU^\alpha_J &= \sum_{j=1}^{p} U^\alpha_{J,j}\omega^j + \Theta^\alpha_J + \widehat{\phi}^{\alpha;J}, && \alpha = 1, \ldots, q, \quad \#J \geq 0, \end{aligned} \tag{2.28}$$

where $\widehat{\phi}^{\alpha;J} = \boldsymbol{\lambda}(\phi^{\alpha;J})$ are *correction terms* obtained by lifting the prolonged vector field coefficients (2.27).

Example 2.11. We continue Example 2.7. Since the Lie pseudo-group (2.7), (2.18) acts projectably, [20], we have the equalities

$$\omega^x = \boldsymbol{\lambda}(dx) = \sigma^x, \qquad \omega^u = \boldsymbol{\lambda}(du) = \sigma^u, \qquad \omega^v = \boldsymbol{\lambda}(dv) = \sigma^v,$$
$$\omega^p = \boldsymbol{\lambda}(dp) = \sigma^p, \qquad \omega^q = \boldsymbol{\lambda}(dq) = \sigma^q,$$

where the coordinate expressions of the one-forms σ^i are given in (2.13). The lifted contact forms $\Theta_J = \boldsymbol{\lambda}(\theta_J)$ will not play an important role in the subsequent computations, thus we introduce the notation $\omega \equiv \Omega$ to denote equality up to a contact form, i.e. $\omega \equiv \Omega$ if and only if the difference $\omega - \Omega$ is a contact form.

To obtain the correction terms in the recurrence relations (2.28) one needs to compute the prolongation of the infinitesimal generator

$$\begin{aligned} \mathbf{v} =& \xi(x)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + \gamma(x,u,v)\frac{\partial}{\partial v} + [p(\eta_u - \xi_x) + \eta_x]\frac{\partial}{\partial p} \\ &+ [q(\gamma_v - \xi_x) + p\gamma_u + \gamma_x]\frac{\partial}{\partial q} + [p^2\eta_{uu} + p(2\eta_{ux} - \xi_{xx}) + r(\eta_u - 2\xi_x) + \eta_{xx}]\frac{\partial}{\partial r}, \end{aligned} \tag{2.29}$$

using the formula (2.27). Restricting our attention to the zero-order lifted differential invariants we obtain

$$\begin{aligned} &dX \equiv \omega^x + \mu, \qquad dU \equiv \omega^u + \nu, \qquad dV \equiv \omega^v + \alpha, \\ &dP \equiv \omega^p + P(\nu_U - \mu_X) + \nu_X, \\ &dQ \equiv \omega^q + Q(\alpha_V - \mu_X) + P\alpha_U + \alpha_X, \\ &dR \equiv R_X\omega^x + R_U\omega^u + R_V\omega^v + R_P\omega^p + R_Q\omega^q \\ &\qquad + P^2\nu_{UU} + P(2\nu_{UX} - \mu_{XX}) + R(\nu_U - 2\mu_X) + \nu_{XX}. \end{aligned} \tag{2.30}$$

Step 2: Normalization

Once the lifted differential invariants $(X, U^{(\infty)})$ have been computed, the next step in the construction of a moving frame is to normalize the pseudo-group parameters $g^{(\infty)}$ parametrizing the fibers of $\mathcal{G}^{(\infty)}$. This is done by choosing a cross-section to the prolonged action on J^∞, [23]. While the lift is well-defined at every submanifold jet, the normalization of the pseudo-group parameters depends on the geometry of the pseudo-group orbits.

Definition 2.12. A subset $\mathcal{S}^\infty \subset \mathrm{J}^\infty$ is said to be $\mathcal{G}$-invariant if for all $z^{(\infty)} \in \mathcal{S}^\infty$ and $g|_z \in \mathcal{G}|_z$, $z = \pi_0^\infty(z^{(\infty)})$, the submanifold jet $Z^{(\infty)} = g^{(\infty)}|_z \cdot z^{(\infty)}$ is also in $\mathcal{S}^\infty$. The set $\mathcal{S}^\infty$ is a $\mathcal{G}$-invariant subbundle of J^∞ if there exists a finite $n_0 \geq 1$ such that $\mathcal{S}^\infty_{n_0} = \pi^\infty_{n_0}(\mathcal{S}^\infty)$ is a $\mathcal{G}$-invariant subbundle of J^{n_0} locally described by

$$\mathcal{S}^\infty_{n_0} = \{(x, u^{(n_0)}) : E^{(n_0)}(x, u^{(n_0)}) \neq 0 \quad \text{and} \quad F^{(n_0)}(x, u^{(n_0)}) = 0\},$$

such that for all $n \geq n_0$ the subset $\mathcal{S}^\infty_n = \pi^\infty_n(\mathcal{S}^\infty)$ is obtained from $\mathcal{S}^\infty_{n_0}$ by "prolongation" in the sense that

$$\mathcal{S}^\infty_n = \{(x, u^{(n)}) : E^{(n_0)}(x, u^{(n_0)}) \neq 0 \quad \text{and} \quad (D^x_J F^{(n_0)})(x, u^{(n)}) = 0,\ 0 \leq \#J \leq n - n_0\}.$$

The integer n_0 is called the *determining order* of the $\mathcal{G}$-invariant subbundle $\mathcal{S}^\infty$.

By definition, the non-degeneracy conditions $E^{(n_0)}(x, u^{(n_0)}) \neq 0$ and the equations $F^{(n_0)}(x, u^{(n_0)}) = 0$ are $\mathcal{G}$-invariant. In a given equivalence problem, the different subbundles $\mathcal{S}^\infty$ satisfying the hypothesis of Definition 2.12 correspond to the different branches of the equivalence problem. The functions $E^{(n_0)}(x, u^{(n_0)})$ appear when non-degeneracy conditions are imposed on some differential invariants while the functions $F^{(n_0)}(x, u^{(n_0)})$ come from assuming that some differential invariants are identically zero.

Definition 2.13. The pseudo-group $\mathcal{G}$ is said to act regularly on a $\mathcal{G}$-invariant subbundle $\mathcal{S}^\infty \subset \mathrm{J}^\infty$ if the orbits in $\mathcal{S}^\infty$ form a regular foliation, i.e. its leaves intersect small open sets (in the relative topology) in $\mathcal{S}^\infty$ in pathwise connected subsets. The dimension of the leaves in $\mathcal{S}^\infty_n$ is denoted by d_n.

Definition 2.14. A pseudo-group $\mathcal{G}$ is said to act *freely* at $z^{(\infty)} \in \mathrm{J}^\infty$ if the only pseudo-group element that fixes the submanifold jet is the identity jet $\mathbb{1}^{(\infty)}$. Submanifold jets at which the action is free are called *regular*. Submanifold jets where the action is not free are called *singular*.

Definition 2.15. For $n \geq 1$, let

$$\mathcal{G}_{\mathcal{S}^\infty_n} = \bigcup_{z^{(\infty)} \in \mathcal{S}^\infty} \mathcal{G}_{\pi^\infty_n(z^{(\infty)})},$$

be the collection of isotropy groups of the submanifold jets $z^{(n)} \in \mathcal{S}^\infty_n$. The projective limit

$$\mathcal{G}_{\mathcal{S}^\infty} = \bigcup_{z^{(\infty)} \in \mathcal{S}^\infty} \mathcal{G}_{z^{(\infty)}} \tag{2.31}$$

is called *isotropy pseudo-group* of $\mathcal{S}^\infty$.

Two regularity assumptions are made on $\mathcal{G}_{\mathcal{S}^\infty}$. We assume there exists a finite $n^0 \geq n_0 \geq 1$, called the *determining order of the isotropy pseudo-group*, such that for all $n \geq n^0$

- the isotropy pseudo-group $\mathcal{G}_{\mathcal{S}^\infty_n}$ is an embedded subbundle of $\mathcal{G}^{(n)}|_{\mathcal{S}^\infty_n}$ ($\mathcal{G}^{(n)}|_{\mathcal{S}^\infty_n}$ denotes the restriction of the pseudo-group jet bundle $\mathcal{G}^{(n)}$ to $\mathcal{S}^\infty_n$),
- the projection $\pi^{n+1}_n : \mathcal{G}_{\mathcal{S}^\infty_{n+1}} \to \mathcal{G}_{\mathcal{S}^\infty_n}$ is a fibration.

The Lie pseudo-group $\mathcal{G}$ acts on $\mathcal{G}_{\mathcal{S}^\infty}$ by conjugation:

$$K_{h^{(\infty)}}(g^{(\infty)}) = h^{(\infty)} \cdot g^{(\infty)} \cdot (h^{-1})^{(\infty)}, \text{ for all } g^{(\infty)} \in \mathcal{G}_{z^{(\infty)}} \text{ and } \boldsymbol{\sigma}(h^{(\infty)}) = z^{(\infty)}.$$

For $n \geq n^0$, let $\mathcal{E}_{\mathcal{S}^\infty_n}$ be the pull-back bundle of $\mathcal{G}_{\mathcal{S}^\infty_n} \to M$ via the projection $\pi^n_0 \colon \mathcal{S}^\infty_n \to M$. In the projective limit, $\mathcal{E}_{\mathcal{S}^\infty}$ is called the $\mathcal{S}^\infty$ *prolonged bundle*.

We are now in a position to define the notion of an equivariant moving frame on $\mathcal{E}_{\mathcal{S}^\infty_n}$.

Definition 2.16. Let $\mathcal{G}$ be a regular pseudo-group action on $\mathcal{S}^\infty$. An n^{th} *order moving frame* on $\mathcal{S}^\infty_n$ is a $\mathcal{G}$-equivariant bundle map

$$\rho^{(n)} \colon \mathcal{E}_{\mathcal{S}^\infty_n} \to \mathcal{E}^{(n)}|_{\mathcal{S}^\infty_n}.$$

It is always possible to ask for right or left $\mathcal{G}$-equivariance. In the following we work with right moving frames which means that

$$\begin{aligned} h\cdot\rho^{(n)}(z^{(n)},g^{(n)}) =& h\cdot(z^{(n)},\widetilde{\rho}^{(n)}(z^{(n)},g^{(n)})) = (h^{(n)}\cdot z^{(n)},\widetilde{\rho}^{(n)}(h^{(n)}\cdot z^{(n)},K_{h^{(n)}}(g^{(n)}))) \\ =& (h^{(n)}\cdot z^{(n)},\widetilde{\rho}^{(n)}(z^{(n)},g^{(n)})\cdot(h^{-1})^{(n)}), \qquad h\in\mathcal{G}|_z,\ g^{(n)}\in\mathcal{G}_{z^{(n)}}. \end{aligned}$$

Theorem 2.17. Let $\mathcal{S}^\infty$ be a $\mathcal{G}$-invariant subbundle of J^∞ and n^0 the determining order of the isotropy group $\mathcal{G}_{\mathcal{S}^\infty}$. For $n\geq n^0$ let $\mathcal{K}^n\subset\mathcal{S}^\infty_n$ be a (local) cross-section to the pseudo-group orbits. Given $z^{(n)}\in\mathcal{S}^\infty_n$ and $g^{(n)}\in\mathcal{G}_{z^{(n)}}$, define $\widetilde{\rho}^{(n)}(z^{(n)},g^{(n)})\in\mathcal{E}^{(n)}|_{z^{(n)}}$ to be the unique pseudo-group jet with the property that $\widetilde{\rho}^{(n)}(z^{(n)},g^{(n)})\cdot z^{(n)}\in\mathcal{K}^n$. Then $\rho^{(n)}:\mathcal{E}_{\mathcal{S}^\infty_n}\to\mathcal{E}^{(n)}|_{\mathcal{S}^\infty_n}$ is a right moving frame defined on $\mathcal{E}_{\mathcal{S}^\infty_n}$. The local cross-section coordinates of the induced map $I^{(n)}=\boldsymbol{\tau}\circ\rho^{(n)}:\mathcal{E}_{\mathcal{S}^\infty_n}\to\mathcal{K}^n$ provide a complete system of functionally independent n^{th} order differential invariants on the domain of definition of the moving frame.

In applications, a (coordinate) cross-section is obtained by fixing $d_n=\operatorname{codim}\mathcal{K}^n$ individual jet coordinates of $z^{(n)}=(x,u^{(n)})$ equal to some constants (recall that d_n is also equal to the dimension of the pseudo-group orbits in $\mathcal{S}^\infty_n$.) Writing the coordinate formulas for the prolonged action

$$(X,U^{(n)})=F^{(n)}(x,u^{(n)},h^{(n)}) \tag{2.32}$$

in terms of the submanifold jet coordinates $(x,u^{(n)})$ and the Lie pseudo-group parameters $h^{(n)}$, the d_n components of (2.32) corresponding to the cross-section variables serve to define the *normalization equations*

$$F_1(z^{(n)},h^{(n)})=c_1,\quad \ldots\quad F_{r_n}(z^{(n)},h^{(n)})=c_{d_n}. \tag{2.33}$$

Writing $h^{(n)}=(\widetilde{h}^{(n)},g^{(n)})$, so that $g^{(n)}$ are the pseudo-group jets parametrizing the isotropy group $\mathcal{G}_{z^{(n)}}$, we can solve for the pseudo-group parameters $\widetilde{h}^{(n)}$ in terms of $z^{(n)}$ and $g^{(n)}$:

$$\widetilde{h}^{(n)}=\widetilde{h}^{(n)}(z^{(n)},g^{(n)}).$$

This leads to the moving frame

$$\rho^{(n)}(z^{(n)},g^{(n)})=(z^{(n)},\widetilde{h}^{(n)}(z^{(n)},g^{(n)}),g^{(n)})=(z^{(n)},\widetilde{\rho}^{(n)}(z^{(n)},g^{(n)})). \tag{2.34}$$

Substituting the moving frame (2.34) into the lifted invariants (2.32) yields the *normalized differential invariants*

$$I^{(n)}=F^{(n)}(z^{(n)},\widetilde{\rho}^{(n)}(z^{(n)},g^{(n)}))=(H^i(z^{(n)}),I^\alpha_K(z^{(n)})), \tag{2.35}$$

with $1\leq i\leq p$, $1\leq\alpha\leq q$, $0\leq\#K\leq n$. Note that the normalized invariants (2.35) cannot depend on the pseudo-group parameters $g^{(n)}$ since they parametrize the isotropy group $\mathcal{G}|_{z^{(n)}}$. The normalized invariants used to obtain the normalization equations (2.33) are called *phantom invariants.*

Definition 2.18. A moving frame $\rho^{(k)}:\mathcal{E}_{\mathcal{S}^\infty_k}\to\mathcal{E}^{(k)}|_{\mathcal{S}^\infty_k}$ of order $k>n$ is *compatible* with a moving frame $\rho^{(n)}:\mathcal{E}_{\mathcal{S}^\infty_n}\to\mathcal{E}^{(n)}|_{\mathcal{S}^\infty_n}$ of order n provided $\pi^k_n\circ\rho^{(k)}=\rho^{(n)}\circ\pi^k_n$ where defined.

A *complete moving frame* is a collection of mutually compatible moving frames of all orders $k \geq n$. In the projective limit we write $\rho : \mathcal{E}_{\mathcal{S}^\infty} \to \mathcal{E}^{(\infty)}|_{\mathcal{S}^\infty}$.

Remark 2.19. When the isotropy pseudo-group $\mathcal{G}_{\mathcal{S}^\infty}$ only contains the identity jet $\mathbb{1}^{(\infty)}$, the preceding considerations reduce to the usual moving frame construction appearing in [23]. When this is not the case we obtain what is called a *partial moving frame* in [25].

Definition 2.20. Let $z^{(\infty)} \in \mathcal{S}^\infty$ and ρ a moving frame defined in a neighborhood of $z^{(\infty)}$ in $\mathcal{S}^\infty$. The *invariantization* of a jet form $\Omega \in T^*\mathrm{J}^\infty$ is the invariant jet form

$$\iota(\Omega) = \rho^* \circ \pi_J[\boldsymbol{\tau}^*(\Omega)] \tag{2.36}$$

defined on $\mathcal{E}_{\mathcal{S}^\infty}$.

We note that if the action is not free at $z^{(\infty)}$ then the coefficients of an invariantized jet form (2.36) may depend on the isotropy pseudo-group jet parameters.

3 Local Equivalence of Submanifolds

In this section we briefly review the solution to the local equivalence problem of submanifolds.

Proposition 3.1. Let $z^{(\infty)} = (x, u^{(\infty)}) \in \mathcal{S}^\infty$ and ρ a moving frame defined in a neighborhood of $(x, u^{(\infty)})$. The invariantization of the submanifold jet coordinates $(H, I^{(\infty)}) = \iota(x, u^{(\infty)})$ forms a complete set of differential invariants.

Definition 3.2. A total differential operator $\mathcal{D}$ is said to be an *invariant differential operator* if for all differential invariant I, $\mathcal{D}I$ is also a differential invariant.

Once a moving frame ρ is constructed, a basis of invariant differential operators is easily obtained from the lifted total differential operators (2.15). One simply has to pull-back the coefficients W_i^j by ρ:

$$\mathcal{D}_i = \sum_{j=1}^{p} \rho^*(W_i^j) D_{x^j}, \qquad i = 1, \ldots, p.$$

Definition 3.3. A set of differential invariants $\{I_\kappa\}$ is said to generate the algebra of differential invariants if all differential invariants can be written in terms of the invariants I_κ and their invariant derivatives $\mathcal{D}_J I_\kappa$.

The key result which makes it possible to solve local equivalence problems is that while there might be infinitely many functionally independent normalized differential invariants $(H, I^{(\infty)})$, there always exists a finite generating set, [24, 25].

Proposition 3.4. The algebra of differential invariants is finitely generated.

Definition 3.5. Let $\mathcal{I} = \{I_1, \ldots, I_\ell\}$ be a generating set of differential invariants. The n^{th} *order signature space* $\mathbb{K}^{(n)}$ is the Euclidean space of dimension $\ell(1+p+p^2+\cdots+p^n)$ coordinatized by $w^{(n)} = (\ldots, w_{\kappa;J}, \ldots)$, where $(\kappa; J) = (\kappa, j^1, \ldots, j^r)$ with $1 \leq \kappa \leq \ell$, and $(j^1, \ldots, j^r)$ ranging through all unordered multi-index with $1 \leq j^i \leq p$ and $0 \leq$

$r \leq n$. Given a submanifold S, the n^{th} *order signature map* $\mathbf{I}_S^{(n)} \colon S \to \mathbb{K}^{(n)}$ is the map whose compontents are

$$w_{\kappa;J} = (\mathcal{D}_J I_\kappa)|_S, \qquad \kappa = 1, \ldots, \ell, \qquad \#J \leq n.$$

Definition 3.6. A moving frame ρ is said to be *fully regular* on S if for each $n \geq 0$ the signature map $\mathbf{I}_S^{(n)} \colon S \to \mathbb{K}^{(n)}$ is regular.

Definition 3.7. Let ρ be a fully regular moving frame. The image

$$\mathfrak{S}^{(n)}(\rho, S) = \{\mathbf{I}_S^{(n)}(z) : z \in S\} \subset \mathbb{K}^{(n)} \tag{3.1}$$

of the n^{th} order signature map $\mathbf{I}_S^{(n)}$ is called the n^{th} *order signature manifold.*

Proposition 3.8. Let ρ be a fully regular moving frame and denote by ϱ_n the rank of the n^{th} order signature map $\mathbf{I}_S^{(n)}$. Then

$$0 \leq \varrho_0 < \varrho_1 < \cdots < \varrho_s = \varrho_{s+1} = \cdots = r \leq p = \dim S,$$

the stabilizing rank r is called the *rank* of the moving frame and the smallest s for which $\varrho_s = \varrho_{s+1} = r$ is called the *order* of the moving frame.

Theorem 3.9. Let $\mathcal{G}$ be a Lie pseudo-group action on M, ρ a fully regular moving frame on $\mathcal{S}^\infty \subset \mathrm{J}^\infty$ and $S, \overline{S} \subset M$ two p-dimensional submanifolds such that for all $z \in S$ and $\overline{z} \in \overline{S}$, $\mathrm{j}_\infty S|_z, \mathrm{j}_\infty \overline{S}|_{\overline{z}} \in \mathcal{S}^\infty$. Then there exists a local diffeomorphism $\varphi \in \mathcal{G}$ mapping S onto $\overline{S}$ if and only if ρ has the same order $\overline{s} = s$ on S and $\overline{S}$ and the $(s{+}1)^{\text{st}}$ order signature manifolds $\mathfrak{S}^{(s+1)}(\rho, S)$, $\mathfrak{S}^{(s+1)}(\rho, \overline{S})$ overlap. Moreover, if $z_0 \in S$ and $\overline{z}_0 \in \overline{S}$ are any points mapping to the same point

$$\mathbf{I}_S^{(s+1)}(z_0) = \mathbf{I}_{\overline{S}}^{(s+1)}(\overline{z}_0) \in \mathfrak{S}^{(s+1)}(\rho, S) \cap \mathfrak{S}^{(s+1)}(\rho, \overline{S})$$

on the overlap of the two signature manifolds, the local equivalence map φ sending z_0 to $\overline{z}_0 = \varphi(z_0)$ is uniquely defined up to precomposition at the source by a local diffeomorphism $\psi \in \mathcal{G}_{z_0^{(\infty)}}$ and composition at the target by a local diffeomorphism $\overline{\psi} \in \mathcal{G}_{\overline{z}_0^{(\infty)}}$.

Finally, given an equivariant moving frame ρ, we note that the structure equations obtained with Cartan's equivalence method of coframes are readily obtained by pulling-back the structure equations (2.5) of the equivalence pseudo-group by ρ, [25].

4 Symbolic Computations

In this section we use the recurrence relations (2.28) to investigate the local equivalence problem for control equations of the form (1.1) under the group of transformations (1.2) symbolically.

The first step towards the solution consists of constructing a moving frame. This is done by determining a cross-section to the equivalence pseudo-group action. Since there is a correspondence between the normalization of the pseudo-group parameters and the normalization of the Maurer–Cartan forms (2.11), the recurrence relations (2.28) can be used to find a cross-section. Considering the group differential

$$d_{\mathcal{G}} = \pi_{\mathcal{G}} \circ d$$

component of the recurrence relations (2.28) it is possible to symbolically determine how the lifted invariants depend on the pseudo-group parameters. For example, for the zero order lifted invariants we have

$$d_{\mathcal{G}} X = \mu, \qquad d_{\mathcal{G}} U = \nu, \qquad d_{\mathcal{G}} V = \alpha, \qquad d_{\mathcal{G}} P = P(\nu_U - \mu_X) + \nu_X,$$
$$d_{\mathcal{G}} Q = Q(\alpha_V - \mu_X) + P\alpha_U + \alpha_X,$$
$$d_{\mathcal{G}} R = P^2 \nu_{UU} + P(2\nu_{UX} - \mu_{XX}) + R(\nu_U - 2\mu_X) + \nu_{XX}.$$

Since the group differential of X equals μ we conclude that it is possible to translate X to zero. Similarly, since the group differential of U, V, P, Q, R depend on the linearly independent Maurer–Cartan forms ν, α, ν_X, α_X, ν_{XX}, respectively, it is also possible to translate U, V, P, Q and R to zero. Said differently, we can set $X = U = V = P = Q = R = 0$, and use the recurrence relations to solve for the Maurer–Cartan forms μ, ν, α, ν_X, α_X, ν_{XX}. Computing the group differential of the higher order lifted differential invariants, the objective is to normalize as many Maurer–Cartan forms as possible. Since the expressions for the group differential grow rapidly as the order of the lifted invariants increases those computations were done with the assistance of MATHEMATICA. In the following we only include the main intermediate steps. Also, to simplify the expressions, once a Maurer–Cartan form is normalized this normalization is taken into account in following expressions. For example, setting

$$X = U = V = P = Q = R = 0,$$

leads to the normalizations

$$\mu = \nu = \alpha = \nu_X = \alpha_X = \nu_{XX} = 0 \qquad \text{(mod submanifold jet forms)}. \tag{4.1}$$

Thus, when writing the group differential of higher order invariants we systematically set the Maurer–Cartan forms (4.1) equal to zero. With this in mind, the group differential of the first order lifted invariants reduce to

$$\begin{aligned} d_{\mathcal{G}} R_Q &= R_Q\left(\nu_U - \alpha_V - \mu_X\right), \\ d_{\mathcal{G}} R_P &= -R_P \mu_X - R_Q \alpha_U + 2\nu_{UX} - \mu_{XX}, \\ d_{\mathcal{G}} R_V &= R_V\left(\nu_U - \alpha_V - 2\mu_X\right) - R_Q \alpha_{VX}, \\ d_{\mathcal{G}} R_U &= -R_P \nu_{UX} - R_Q \alpha_{UX} - R_V \alpha_U - 2R_U \mu_X + \nu_{UXX}, \\ d_{\mathcal{G}} R_X &= R_X(\nu_U - 3\mu_X) - R_Q \alpha_{XX} + \nu_{XXX}. \end{aligned} \tag{4.2}$$

The group differential of R_Q reveals that the equivalence pseudo-group acts by scaling on R_Q. Since at the identity transformation R_Q is equal to r_q and that we assume $r_q \neq 0$ it follows that $R_Q \neq 0$. Hence, from (4.2) we see that there is enough liberty in the prolonged action to set

$$R_P = R_V = R_U = R_X = 0 \quad \text{and} \quad R_Q = 1,$$

which leads to the normalization of the Maurer–Cartan forms μ_{XX}, α_{VX}, ν_{UXX}, ν_{XXX} and α_V, respectively. From the prolongation formula (2.27) for the coefficients of a vector field, we see that there is enough liberty in the prolonged action to set

$$R_{PX^k} = R_{V^iU^jX^k} = 0, \qquad i,j,k \geq 0. \tag{4.3}$$

The normalization equations (4.3) are used to normalize the Maurer–Cartan forms $\mu_{X^{k+2}}$, $\nu_{U^jX^{k+2}}$ and $\alpha_{V^iU^jX^{k+1}}$, with $i \geq 1$. For the remainder of the discussion the normalizations (4.3) are performed in the background, and all the expressions that follow take into account these normalizations. Thus, the group differential of the remaining unnormalized second order differential invariants are

$$\begin{aligned}
d_{\mathcal{G}}R_{QQ} =& R_{QQ}\,(2\mu_X - \nu_U)\,, \quad d_{\mathcal{G}}R_{QP} = R_{QP}\,(\mu_X - \nu_U) - R_{QQ}\alpha_U \\
d_{\mathcal{G}}R_{QV} =& R_{QV}\,(\mu_X - \nu_U) - \alpha_{VV}, \quad d_{\mathcal{G}}R_{QX} = -R_{QQ}\alpha_{XX} - R_{QX}\mu_X + \alpha_U - \nu_{UX}, \\
d_{\mathcal{G}}R_{PP} =& - R_{PP}\nu_U - 2R_{QP}\alpha_U + 2\nu_{UU}, \quad d_{\mathcal{G}}R_{PV} = -R_{PV}\nu_U - R_{QV}\alpha_U - \alpha_{VU}, \\
d_{\mathcal{G}}R_{QU} =& - R_{QV}\alpha_U - R_{QQ}\alpha_{UX} - R_{QU}\nu_U - R_{QP}\nu_{UX} - \alpha_{VU} + \nu_{UU}, \\
d_{\mathcal{G}}R_{PU} =& - R_{PP}\nu_{UX} - R_{PU}(\nu_U + \mu_X) - R_{PV}\alpha_U - R_{QP}\alpha_{UX} - R_{QU}\alpha_U \\
& - \alpha_{UU} + 2\nu_{UUX}.
\end{aligned} \tag{4.4}$$

At this stage we can set

$$R_{QV} = R_{QX} = R_{PP} = R_{PV} = R_{PU} = 0, \tag{4.5}$$

and normalize the Maurer–Cartan forms α_{VV}, ν_{UX}, ν_{UU}, α_{VU}, ν_{UUX}. More generally, we can set

$$R_{PPU^i} = R_{PU^{i+1}} = R_{QV^{i+1}} = R_{PV^{i+1}U^j} = 0, \qquad i,j \geq 0,$$

and normalize of the Maurer–Cartan forms $\nu_{U^{i+2}}$, $\nu_{U^{i+2}X}$, $\alpha_{V^{i+2}}$ and $\alpha_{V^{i+1}U^{j+1}}$. Once these normalizations are done, the group differential for R_{QQ}, R_{QP} and R_{QU} reduce to

$$\begin{aligned}
d_{\mathcal{G}}R_{QQ} =& R_{QQ}(2\mu_X - \nu_U), \\
d_{\mathcal{G}}R_{QP} =& R_{QP}\,(\mu_X - \nu_U) - R_{QQ}\alpha_U, \\
d_{\mathcal{G}}R_{QU} =& R_{QQ}(R_{QP}\alpha_{XX} - \alpha_{UX}) - R_{QU}\nu_U.
\end{aligned}$$

At this stage, the equivalence problem splits into four branches:

Second order degenerate branch: $R_{QQ} = R_{QP} = R_{QU} = 0$.

Second order intermediate branch 1: $R_{QQ} = R_{QP} = 0$, $R_{QU} \neq 0$.

Second order intermediate branch 2: $R_{QQ} = 0$, $R_{QP} \neq 0$.

Second order generic branch: $R_{QQ} \neq 0$.

In Section 4.1 we consider the degenerate branch with the aim of determining the control equations with largest fiber-preserving symmetry group. In Section 4.2 we study the generic branch while the intermediate branches are omitted.

4.1 Second Order Degenerate Branch

In this section we assume that the second order invariants

$$R_{QQ} = R_{QP} = R_{QU} = 0 \tag{4.6}$$

are identically equal to zero. Under assumption (4.6), the group differential of the invariants R_{QQ}, R_{QP}, R_{QU} is identically zero and these invariants cannot be used to normalize any Maurer–Cartan form. We must thus proceed to the next order. But before doing so we must take into account the consequences of (4.6) on the third order lifted invariants. Taking the horizontal differential of (4.6) we obtain

$$\begin{aligned}
0 = d_{\mathcal{H}} R_{QQ} =& R_{QQQ}\omega^q + R_{QQP}\omega^p + R_{QQV}\omega^v + R_{QQU}\omega^u + R_{QQX}\omega^x,\\
0 = d_{\mathcal{H}} R_{QP} =& R_{QQP}\omega^q + R_{QPP}\omega^p + R_{QPV}\omega^v + R_{QPU}\omega^u + R_{QPX}\omega^x,\\
0 = d_{\mathcal{H}} R_{QU} =& R_{QQU}\omega^q + R_{QPU}\omega^p + R_{QVU}\omega^v + R_{QUU}\omega^u + R_{QUX}\omega^x\\
&- \frac{1}{2}\Big[R_{QPP}\omega^q + R_{PPP}\omega^p + R_{PPV}\omega^v + R_{PPX}\omega^x\\
&\qquad + 2R_{QPV}\omega^q + 2R_{PPV}\omega^p + 2R_{PVX}\omega^x\Big],
\end{aligned}$$

from which we conclude that

$$\begin{gathered}
R_{QQQ} = R_{QQP} = R_{QQV} = R_{QQU} = R_{QQX} = R_{QPP} = 0\\
R_{QPV} = R_{QPU} = R_{QPX} = R_{QUU} = 0,\\
R_{PPP} = -2R_{PPV}, \qquad R_{PPV} = 2R_{QVU}, \qquad R_{PPX} = 2R_{QUX} - 2R_{PVX}.
\end{gathered}$$

It follows that the only remaining independent unnormalized third order invariants are

$$R_{QVU}, \qquad R_{PVX}, \qquad R_{PUX}, \qquad R_{QVX}, \qquad R_{QUX}, \qquad R_{QXX}.$$

Computing their group differential we obtain

$$\begin{aligned}
d_{\mathcal{G}} R_{QVU} &= R_{QVU}(\mu_X - 2\nu_U), \qquad d_{\mathcal{G}} R_{QVX} = -R_{QVX}\nu_U,\\
d_{\mathcal{G}} R_{QUX} &= - R_{QVX}\alpha_U - R_{QUX}(\nu_U + \mu_X) + \frac{1}{2}\alpha_{UU},\\
d_{\mathcal{G}} R_{QXX} &= - 2R_{QXX}\mu_X, \qquad d_{\mathcal{G}} R_{PVX} = -R_{QVX}\alpha_U - R_{PVX}(\nu_U + \mu_X),\\
d_{\mathcal{G}} R_{PUX} &= (R_{PVX} - 3R_{QUX})\alpha_U - R_{PUX}(\nu_U + 2\mu_X) + \alpha_{UUX}.
\end{aligned}$$

Setting $R_{QUX} = R_{PUX} = 0$ we can normalize the Maurer–Cartan forms α_{UU}, α_{UUX}. More generally, we can set

$$R_{QU^{i+1}X} = R_{PU^{i+1}X^{j+1}} = 0, \qquad i, j \geq 0,$$

and normalize the Maurer–Cartan forms $\alpha_{U^{i+2}}$ and $\alpha_{U^{i+2}X^{j+1}}$.

Sub-branches to the equivalence problem appear depending on the values of the invariants R_{PVX}, R_{QXX}, R_{QVX}, R_{QVU}. In the following, we consider the degenerate case where all these invariants are identically equal to zero:

$$R_{PVX} = R_{QXX} = R_{QVX} = R_{QVU} = 0. \tag{4.7}$$

In this case all differential invariants of order ≤ 3 are constant. This observation, combined with our choice of cross-section made thus far, implies that all higher order invariants are also constant and we conclude that there is no further normalizations possible. Hence, the Maurer–Cartan forms

$$\alpha_{XX},\ \alpha_{XXX},\ \ldots,\qquad \alpha_U,\ \alpha_{UX},\ \alpha_{UXX},\ \ldots,\qquad \mu_X,\ \nu_U \tag{4.8}$$

cannot be normalized. In (4.8) we identify two sequences of un-normalized Maurer–Cartan forms, namely $\alpha_{X^{i+2}}$ and α_{UX^i}. This suggests that the general equivalence map between two control equations in this branch of the equivalence problem depends on two functions of one variable. This is verified to be the case using Cartan's involutivity test, [3, 21, 25]. Indeed, substituting the normalizations

$$\begin{gathered}
\mu \equiv -\omega^x,\quad \nu \equiv -\omega^u,\quad \alpha \equiv -\omega^v,\quad \nu_X \equiv -\omega^p,\quad \alpha_X \equiv -\omega^q,\quad \alpha_V \equiv \nu_U - \mu_X,\\
\nu_{XX} \equiv -\omega^q,\qquad \mu_{XX} \equiv \alpha_U,\qquad \alpha_{VX} \equiv 0,\qquad \alpha_{VV} \equiv 0,\\
\nu_{UX} \equiv \alpha_U,\qquad \nu_{UU} \equiv 0,\qquad \alpha_{VU} \equiv 0,\qquad \alpha_{UU} \equiv 0,\qquad \ldots
\end{gathered} \tag{4.9}$$

into the structure equations (2.14) we obtain the involutive structure equations

$$\begin{aligned}
&d\omega^x \equiv \mu_X \wedge \omega^x,\qquad d\omega^u \equiv \omega^x \wedge \omega^p + \nu_U \wedge \omega^u,\\
&d\omega^v \equiv \omega^x \wedge \omega^q + \alpha_U \wedge \omega^u + (\nu_U - \mu_X) \wedge \omega^v,\\
&d\omega^p \equiv \omega^x \wedge \omega^q + \alpha_U \wedge \omega^u + (\nu_U - \mu_X) \wedge \omega^p,\\
&d\omega^q \equiv \alpha_{XX} \wedge \omega^x + \alpha_{UX} \wedge \omega^u + \alpha_U \wedge \omega^p + (\nu_U - 2\mu_X) \wedge \omega^q,\\
&d\mu_X \equiv \omega^x \wedge \alpha_U,\qquad d\nu_U \equiv \omega^x \wedge \alpha_U,\qquad d\alpha_U \equiv \omega^x \wedge \alpha_{UX} + \alpha_U \wedge \mu_X,\\
&\qquad\vdots
\end{aligned} \tag{4.10}$$

with Cartan characters $s_1 = 2$, $s_2 = 0$.

4.2 Second Order Generic Branch

We now assume that $R_{QQ} \neq 0$. From the group differential expressions (4.4) we see that the normalization equations

$$R_{QQ} = 1,\qquad R_{QP} = R_{QU} = 0, \tag{4.11}$$

can be added to (4.5). With the three equations (4.11) we can normalize the Maurer–Cartan forms ν_U, α_U, α_{UX}. More generally, we can set

$$R_{QPU^{i+1}} = 0,\quad i \geq 0,\qquad R_{QU^iX^j} = 0,\quad i+j \geq 2,$$

to normalize $\alpha_{U^{i+1}}$, $i \geq 0$ and $\alpha_{U^iX^{j+1}}$, $i+j \geq 2$, respectively. Considering the third order differential invariants, the group differential of the unnormalized invariants are

$$\begin{aligned}
&d_{\mathcal{G}} R_{QQQ} = 0,\qquad d_{\mathcal{G}} R_{QQP} = -R_{QQP}\mu_X, && d_{\mathcal{G}} R_{QQV} = -R_{QQV}\mu_X,\\
&d_{\mathcal{G}} R_{QQU} = R_{QQP}\alpha_{XX} - 2R_{QQU}\mu_X, && d_{\mathcal{G}} R_{QPP} = -2R_{QPP}\mu_X,\\
&d_{\mathcal{G}} R_{QQX} = -[1 + R_{QQQ}]\alpha_{XX} - R_{QQX}\mu_X, && d_{\mathcal{G}} R_{QPV} = -2R_{QPV}\mu_X,\\
&d_{\mathcal{G}} R_{QPX} = -2R_{QPX}\mu_X - R_{QQP}\alpha_{XX}, && d_{\mathcal{G}} R_{PPP} = -3R_{PPP}\mu_X,\\
&d_{\mathcal{G}} R_{QVU} = R_{QPV}\alpha_{XX} - 3R_{QVU}\mu_X, && d_{\mathcal{G}} R_{PPV} = -3R_{PPV}\mu_X,\\
&d_{\mathcal{G}} R_{QVX} = -R_{QQV}\alpha_{XX} - 2R_{QVX}\mu_X, && d_{\mathcal{G}} R_{PPX} = -3R_{PPX}\mu_X,
\end{aligned}$$

$$d_{\mathcal{G}} R_{PUX} = \Big[R_{PPX} - \frac{1}{2} R_{PPP} - R_{PPV}\Big] \alpha_{XX} - 4 R_{PUX} \mu_X,$$
$$d_{\mathcal{G}} R_{PVX} = -3 R_{PVX} \mu_X - R_{QPV} \alpha_{XX}. \tag{4.12}$$

At this stage we are left with two Maurer–Cartan forms to normalize, namely α_{XX} and μ_X. The different scenarios leading to the normalization of those two Maurer–Cartan forms lead to different branches of the equivalence problem. In the next two subsections we consider the following two cases:

Hilbert–Cartan sub-branch: $R_{QQQ} \neq -1$ and the invariants

$$R_{QQP} = R_{QQV} = R_{QQU} = R_{QPP} = R_{QPV} = R_{QPX} = R_{QVU} = R_{QVX} = 0$$
$$R_{PPP} = R_{PPV} = R_{PPX} = R_{PVX} = R_{PUX} = R_{QQQX} = R_{QQXX} = 0$$

are identically equal to zero.

Third order generic branch: $R_{QQP} \neq 0$.

The motivation for calling the first case the "Hilbert–Cartan sub-branch" comes from the fact that the Hilbert–Cartan equation (1.4) is contained in this branch of the equivalence problem. Also, we note that the generic branch $R_{QQP} \neq 0$ is just one of many cases leading to the complete normalization of the pseudo-group parameters. Indeed, from the recurrence relations (4.12), we see that when $R_{QQP} \neq 0$ the remaining Maurer–Cartan forms ν_X, α_{XX} can be normalized by setting $R_{QQP} = 1$ and $R_{QQU} = 0$ or $R_{QPX} = 0$. But there are other obvious cases leading to the normalization of ν_X, α_{XX}. For example, we could replace the non-degeneracy condition $R_{QQP} \neq 0$ by $R_{QPV} \neq 0$ and make the normalizations $R_{QPV} = 1$ and $R_{QVU} = 0$ or $R_{PVX} = 0$, It is also possible to assume $R_{QQV} \neq 0$, and make the normalizations $R_{QQV} = 1$, $R_{QVX} = 0$ or to assume $R_{PPX} - R_{PPP}/2 - R_{PPV} \neq 0$ and make the normalizations $R_{PPX} - R_{PPP}/2 - R_{PPV} = 1$, $R_{PUX} = 0$. With the appropriate modifications, the analysis of each for each of these cases is very similar to the computations appearing in Section 4.2.2 where the case $R_{QQP} \neq 0$ is considered.

4.2.1 Hilbert–Cartan Sub-Branch

If we assume that $R_{QQQ} + 1 \neq 0$, then it follows from (4.12) that the group differential of R_{QQX} depends on α_{XX}. We can thus set

$$R_{QQX} = 0 \tag{4.13}$$

and normalize the Maurer–Cartan form α_{XX}. Once α_{XX} is normalized, the group differential of the remaining third order unnormalized invariants, except for R_{QQQ}, is of the form

$$d_{\mathcal{G}} R_J = C_J R_J \mu_X, \qquad \text{where } C_J \text{ is some nonzero constant.}$$

If we impose that the invariants

$$\begin{gathered} R_{QQP} = R_{QQV} = R_{QQU} = R_{QPP} = R_{QPV} = R_{QPX} = R_{QVU} = R_{QVX} = 0 \\ R_{PPP} = R_{PPV} = R_{PPX} = R_{PVX} = R_{PUX} = 0 \end{gathered} \tag{4.14}$$

are identically zero, the Maurer–Cartan form μ_X cannot be normalized. Combining (4.14) together with the normalization equations chosen thus far, it can be verified with the recurrence relations (2.28) that all fourth order invariants are zero except possibly for R_{QQQQ}, R_{QQQX}, R_{QQXX}. Computing their group differential we obtain

$$d_{\mathcal{G}} R_{QQQQ} = 0, \qquad d_{\mathcal{G}} R_{QQQX} = -R_{QQQX}\mu_X, \qquad d_{\mathcal{G}} R_{QQXX} = -2R_{QQXX}\mu_X.$$

Assuming that

$$R_{QQQX} = R_{QQXX} = 0 \tag{4.15}$$

are identically zero, the Maurer–Cartan form μ_X cannot be normalized using the fourth order invariants. In fact, with the help of the recurrence relations (2.28) we can conclude that, under the hypotheses (4.14), (4.15) and our choice of cross-section, all invariants of order ≥ 4 are equal to zero except for R_{Q^k}, $k \geq 3$. But since $R_{Q^k} = I_{q^k}$, $k \geq 3$, are genuine differential invariants, i.e. they do not depend on the pseudo-group parameters (after normalization), we conclude that the Maurer–Cartan form μ_X cannot be normalized.

Since the normalized invariants I_{q^k}, $k \geq 3$, can be expressed in terms of I_{qqq} and its invariants derivatives $\mathcal{D}_q^k I_{qqq}$ we obtain the following result.

Theorem 4.1. If $r_q \neq 0$, $r_{qq} \neq 0$, $I_{qqq} \neq -1$ and the invariants (4.14), (4.15) are identically zero, then I_{qqq} together with the invariant differential operator $\mathcal{D}_q$ generate the algebra of differential invariants $\{I_{q^k} : k \geq 3\}$.

Theorem 4.2. Two control equations Δ, $\overline{\Delta}$ of the form (1.1) that satisfy $r_q \neq 0$, $r_{qq} \neq 0$, $I_{qqq} \neq -1$ and (4.14), (4.15) are locally equivalent if and only if their signature manifolds

$$\mathfrak{S}(\rho, \Delta) = \{\mathcal{D}_q^k(I_{qqq})|_\Delta : k \geq 0\} \qquad \mathfrak{S}(\rho, \overline{\Delta}) = \{\mathcal{D}_q^k(I_{qqq})|_{\overline{\Delta}} : k \geq 0\}$$

have the same order and overlap.

To obtain the structure equations of the invariant coframe $\boldsymbol{\omega} = \{\omega^x, \omega^u, \omega^v, \omega^p, \omega^q, \mu_X\}$ we use the recurrence relations of the phantom invariants to solve for the Maurer–Cartan forms. Modulo contact forms, the result is

$$\begin{gathered}
\mu \equiv -\omega^x, \quad \nu \equiv -\omega^u, \quad \alpha \equiv -\omega^v, \quad \nu_U \equiv (-2 + I_{qqq})\omega^q + 2\mu_X, \quad \nu_X \equiv -\omega^p, \\
\alpha_V \equiv (-1 + I_{qqq})\omega^q + \mu_X, \quad \alpha_U \equiv 0, \quad \alpha_X \equiv -\omega^q, \quad \mu_{XX} \equiv 0, \\
\nu_{UU} \equiv 0, \quad \nu_{UX} \equiv 0, \quad \nu_{XX} \equiv -\omega^q, \quad \alpha_{VV} \equiv 0, \quad \alpha_{VU} \equiv 0, \quad \alpha_{VX} \equiv 0, \\
\alpha_{UU} \equiv 0, \quad \alpha_{UX} \equiv 0, \quad \alpha_{XX} \equiv 0, \quad \ldots .
\end{gathered} \tag{4.16}$$

Substituting the expressions (4.16) into the structure equations (2.14) we obtain

$$\begin{aligned}
&d\omega^x \equiv \mu_X \wedge \omega^x, \qquad d\omega^q \equiv 0, \qquad d\mu_X \equiv 0, \\
&d\omega^u \equiv \omega^x \wedge \omega^p + [I_{qqq} - 2]\omega^q \wedge \omega^u + 2\mu_X \wedge \omega^u, \\
&d\omega^v \equiv \omega^x \wedge \omega^q + [I_{qqq} - 1]\omega^q \wedge \omega^v + \mu_X \wedge \omega^v, \\
&d\omega^p \equiv \omega^x \wedge \omega^q + [I_{qqq} - 2]\omega^q \wedge \omega^p + \mu_X \wedge \omega^p.
\end{aligned} \tag{4.17}$$

Theorem 4.3. Under the conditions of Theorem 4.1, a control equation of the form (1.1) has a 6-dimensional fiber-preserving symmetry group with Maurer–Cartan structure equations isomorphic to (4.17) if and only if I_{qqq} is constant.

4.2.2 Third Order Generic Branch

We now assume that the normalizations made in Section 4.2 still hold except for the normalization (4.13). Also, we no longer require that $R_{QQQ} \neq -1$. Instead, we assume $R_{QQP} \neq 0$. In local coordinates, the non-degeneracy conditions $R_Q \neq 0$, $R_{QQ} \neq 0$, $R_{QQP} \neq 0$ imply

$$r_q \neq 0, \qquad r_{qq} \neq 0 \qquad \text{and} \qquad r_{qq}r_{qqp} - r_{qp}r_{qqq} \neq 0. \tag{4.18}$$

Provided (4.18) is satisfied, the group differential of R_{QQP} and R_{QQU} in (4.12) reveals that there is enough liberty in the prolonged action to set

$$R_{QQP} = 1, \qquad R_{QQU} = 0.$$

This leads to the normalization of the last two un-normalized Maurer–Cartan forms μ_X, α_{XX}. Introducing the notation

$$z^1 = x, \qquad z^2 = u, \qquad z^3 = v, \qquad z^4 = p, \qquad z^5 = q,$$

the recurrence relations for the third order normalized invariants $I_J = \iota(r_J)$ are

$$\begin{aligned}
&I_{qqq,i} = \mathcal{D}_i I_{qqq} + M^i_{qqq},\\
&I_{qqv,i} = \mathcal{D}_i I_{qqv} + I_{qqv}I_{qqp,i} + M^i_{qqv},\\
&I_{qqx,i} = \mathcal{D}_i I_{qqx} + I_{qqx}I_{qqp,i} - (1 + I_{qqq})I_{qqu,i} + M^i_{qqx},\\
&I_{qpp,i} = \mathcal{D}_i I_{qpp} + 2I_{qpp}I_{qqp,i} + M^i_{qpp},\\
&I_{qpv,i} = \mathcal{D}_i I_{qpv} + 2I_{qpv}I_{qqp,i} + M^i_{qpv},\\
&I_{qpx,i} = \mathcal{D}_i I_{qpx} + 2I_{qpx}I_{qqp,i} - I_{qqu,i} + M^i_{qpx},\\
&I_{qvu,i} = \mathcal{D}_i I_{qvu} + I_{pvv,i} + I_{qpv}I_{qqu,i} + 3I_{qvu}I_{qqp,i} + M^i_{qvu},\\
&I_{qvx,i} = \mathcal{D}_i I_{qvx} + 2I_{qvx}I_{qqp,i} - I_{qqv}I_{qqu,i} + M^i_{qvx},\\
&I_{ppp,i} = \mathcal{D}_i I_{ppp} + 3I_{ppp}I_{qqp,i} + M^i_{ppp},\\
&I_{ppv,i} = \mathcal{D}_i I_{ppv} + 3I_{ppv}I_{qqp,i} + M^i_{ppv},\\
&I_{ppx,i} = \mathcal{D}_i I_{ppx} + 3I_{ppx}I_{qqp,i} - I_{qpu,i} + M^i_{ppx},\\
&I_{pvx,i} = \mathcal{D}_i I_{pvx} + 3I_{pvx}I_{qqp,i} - I_{qpv}I_{qqu,i} + M^i_{pvx},\\
&I_{pux,i} = \mathcal{D}_i I_{pux} + [I_{ppx} - \frac{I_{ppp}}{2} - I_{ppv}]I_{qqu,i} + 4I_{pux}I_{qqp,i} - I_{quu,i} + \frac{I_{ppu,i}}{2} + I_{pvu,i} + M^i_{pux},
\end{aligned} \tag{4.19}$$

where $1 \leq i \leq 5$ and M^l_{ijk} are correction terms involving only third order normalized invariants. It follows from (4.19) that all fourth order normalized invariants are expressible in terms of the third order invariants, their invariant derivatives and the fourth order invariants I_{qqpu}, I_{qquu}. Furthermore, since the recurrence relations for the normalized invariants of order ≥ 4 are of the form

$$I_{J,i} = \mathcal{D}_i I_J + M^i_J, \qquad \#J \geq 4, \qquad 1 \leq i \leq 5,$$

where the correction terms M^i_J depend on invariants of order $\leq \#J$, we obtain the following result.

Theorem 4.4. Provided that $r_q \neq 0$, $r_{qq} \neq 0$ and $r_{qq}r_{qqp} - r_{qp}r_{qqq} \neq 0$, the algebra of differential invariants is generated by the third order invariants

$$I_{qqq},\ I_{qqv},\ I_{qpp},\ I_{qpv},\ I_{ppp},\ I_{ppv},\ I_{qqx},\ I_{qpx},\ I_{qvu},\ I_{qvx},\ I_{ppx},\ I_{pvx},\ I_{pux}, \tag{4.20a}$$

and the fourth order invariants

$$I_{qqpu},\ I_{qquu}. \tag{4.20b}$$

In Theorem 4.4 the generating set is not assumed to be minimal. In fact there are many syzygies among them. For example, each time a fourth order normalized invariant I_{ijkl} appears twice on the left-hand side of (4.4), equating the respective right-hand sides gives a syzygy. For example, the recurrence relations

$$I_{qqqv} = \mathcal{D}_v I_{qqq} + M^v_{qqq}, \qquad I_{qqqv} = \mathcal{D}_q I_{qqv} + I_{qqv}I_{qqqp} + M^q_{qqv},$$

give the syzygy

$$\mathcal{D}_v I_{qqq} + M^v_{qqq} = \mathcal{D}_q I_{qqv} + I_{qqv}I_{qqqp} + M^q_{qqv}.$$

Also, there are the commutator syzygies to consider, [24].

Finally, pulling-back the structure equations (2.14) via the moving frame we obtain the structure equations of the invariant horizontal coframe $\{\omega^x, \omega^u, \omega^v, \omega^p, \omega^q\}$.

Theorem 4.5. Provided that $r_q \neq 0$, $r_{qq} \neq 0$ and $r_{qq}r_{qqp} - r_{qp}r_{qqq} \neq 0$, the structure equations of the invariant horizontal coframe $\{\omega^x, \omega^u, \omega^v, \omega^p, \omega^q\}$ are

$$\begin{aligned}
d\omega^x \equiv\ & [2 - 3I_{qqq} + I_{qqqp}]\omega^q \wedge \omega^x + [I_{qqpp} - I_{qpp}I_{qqq} - 2]\omega^p \wedge \omega^x \\
& + [I_{qqpv} - I_{qpv}I_{qqq} - 2I_{qqv}]\omega^v \wedge \omega^x + I_{qqpu}\omega^u \wedge \omega^x, \\
d\omega^u \equiv\ & \omega^x \wedge \omega^p + [2 - 5I_{qqq} + 2I_{qqqp}]\omega^q \wedge \omega^u + [2I_{qqpp} - 2I_{qpp}I_{qqq} - 3]\omega^p \wedge \omega^u \\
& + [2I_{qqpv} - 2I_{qpv}I_{qqq} - 3I_{qqv}]\omega^v \wedge \omega^u + [2I_{qqpx} - 2I_{qpx}I_{qqq} - 3I_{qqx}]\omega^x \wedge \omega^u, \\
d\omega^v \equiv\ & \omega^q \wedge \omega^u + \omega^x \wedge \omega^q + [I_{qqpx} - I_{qpx}I_{qqq} - I_{qqx}]\omega^x \wedge \omega^v + [I_{qpv} - I_{qqpu}]\omega^v \wedge \omega^u \\
& + [1 - 2I_{qqq} + I_{qqqp}]\omega^q \wedge \omega^v + [I_{qqqp} - 1 - I_{qpp}I_{qqq}]\omega^p \wedge \omega^v + I_{qpp}\omega^p \wedge \omega^u \\
& + I_{qpx}\omega^x \wedge \omega^u, \\
d\omega^p \equiv\ & \omega^x \wedge \omega^q + [I_{qpp}(I_{qqq} - 1)/2 + I_{qpv}(I_{qqq} - 2) + I_{qqqu} - I_{qqv}]\omega^q \wedge \omega^u \\
& + [I_{ppp}(I_{qqq} - 1)/2 + I_{ppv}(I_{qqq} - 2) - I_{qpp}I_{qqv}]\omega^p \wedge \omega^u \\
& + [I_{ppv}(I_{qqq} - 1)/2 - I_{qpv}I_{qqv} + I_{qqvu} - I_{qqq}I_{qvu}]\omega^v \wedge \omega^u \\
& + [I_{ppx}(I_{qqq} - 1)/2 + I_{pvx}(I_{qqq} - 2) + I_{qqux} - I_{qpx}I_{qqv}]\omega^x \wedge \omega^u \\
& + [I_{qqqp} - 2I_{qqq}]\omega^q \wedge \omega^p + [I_{qqpv} - I_{qpv}I_{qqq} - I_{qqv}]\omega^v \wedge \omega^p \\
& + [I_{qqpx} - I_{qpx}I_{qqq} - I_{qqx}]\omega^x \wedge \omega^p, \\
d\omega^q \equiv\ & [1 + I_{qpp}(1 - I_{qqq})/2 + I_{qpv}(2 - I_{qqq}) - I_{qqqu} + I_{qqv}]\omega^q \wedge \omega^x \\
& + [I_{qpp} - I_{qqpu} + I_{ppp}(1 - I_{qqq})/2 + I_{ppv}(2 - I_{qqq}) + I_{qpp}I_{qqv}]\omega^p \wedge \omega^x \\
& + [I_{qpv}(1 + I_{qqv}) + I_{ppv}(1 - I_{qqq})/2 - I_{qqvu} + I_{qqq}I_{qvu} + I_{qvx}]\omega^v \wedge \omega^x \\
& + [I_{ppx}/2 + I_{pvx} - I_{qquu}]\omega^u \wedge \omega^x + [I_{qpp}/2 + I_{qpv}]\omega^u \wedge \omega^q + I_{qqv}\omega^v \wedge \omega^q \\
& + [I_{ppp}/2 + I_{ppv}]\omega^u \wedge \omega^p + [I_{qvu} - I_{ppv}/2]\omega^v \wedge \omega^u + I_{qpv}\omega^v \wedge \omega^p.
\end{aligned}$$

5 Coordinate Expressions

In this section we obtain the local coordinate expressions of some of the differential invariants found in the previous section.

5.1 Second Order Degenerate Branch

We begin by considering the degenerate branch discussed in Section 4.1. We found that this branch of the equivalence problem occurs when the lifted invariants

$$R_{QQ} = R_{QP} = R_{QU} = R_{PVX} = R_{QXX} = R_{QVX} = R_{QVU} = 0 \tag{5.1}$$

are identically zero and the cross-section is

$$\begin{gathered} X = U = V = P = Q = 0, \quad R_Q = 1, \\ R_{V^iU^jX^k} = R_{PU^iX^j} = R_{PPU^i} = R_{PV^{i+1}U^j} = R_{QV^{i+1}} = R_{QU^iX} = 0, \qquad i,j,k \geq 0. \end{gathered} \tag{5.2}$$

In this setting we concluded that the Maurer–Cartan forms (4.8) cannot be normalized. In terms of the pseudo-group jet parameters this translates into the observation that the solution to the normalization equations (5.2) will involve the submanifold jets coordinates $(x, u^{(\infty)})$ and the pseudo-group parameters

$$\phi_x, \qquad \beta_u, \qquad \alpha_{x^{k+2}}, \qquad \alpha_{ux^k}, \qquad k \geq 0,$$

or equivalently, in terms of the functions (2.12),

$$\phi_x, \qquad \beta_u, \qquad \alpha_u, \qquad \psi_{x^{k+1}}, \qquad \psi_{ux^k}, \qquad k \geq 0.$$

Solving the normalization equations (5.2) we obtain

$$\begin{gathered} \phi = 0, \quad \beta = 0, \quad \alpha = 0, \quad \delta = 0, \quad \psi = 0, \quad \alpha_v = \frac{\beta_u r_q}{\phi_x}, \quad \psi_v = \frac{\beta_u r_v}{\phi_x}, \quad \alpha_{vv} = \frac{\beta_u r_{qv}}{\phi_x}, \\ \beta_{uu} = -\frac{\beta_u r_{pp}}{2}, \quad \alpha_{uv} = \frac{\beta_u r_{pv}}{\phi_x}, \quad \psi_{vv} = \frac{\beta_u r_{vv}}{\phi_x}, \quad \alpha_{vvv} = \frac{\beta_u r_{qvv}}{\phi_x}, \quad \alpha_{uvv} = \frac{\beta_u r_{pvv}}{\phi_x}, \\ \beta_{uuu} = \frac{\beta_u}{2}\left[\frac{r_{pp}^2}{2} - r_{ppu}\right], \quad \psi_{uv} = \frac{\beta_u}{\phi_x}\left[r_{vu} - \frac{r_v r_{pp}}{2}\right], \quad \alpha_{uuv} = \frac{\beta_u}{\phi_x}\left[r_{pvu} - \frac{r_{pv} r_{pp}}{2}\right], \end{gathered} \tag{5.3}$$

and so on. The system of differential equations defining the control equations (1.1) contained in the degenerate branch of the equivalence problem are obtained by substituting (5.3) into the lifted invariants (5.1). The result is

$$\begin{gathered} r_{qq} = 0, \qquad r_{pq} = 0, \qquad r_{ppp} = 0, \qquad 2r_{pv} + r_{pp} r_q = 2 r_{qu}, \qquad r_{qv}\mathcal{A} = r_q \mathcal{A}_v, \\ r_{pv}\mathcal{A} + \frac{r_v r_q r_{pp}}{2} = r_q \mathcal{B}_v, \qquad \frac{r r_q^2 r_{pp}}{2} + 2\mathcal{A}^2 + r_q(\mathcal{A}_x + q\mathcal{A}_v + p\mathcal{A}_u + r_p\mathcal{A}) = r_v\mathcal{A} + r_q^2\mathcal{B}, \end{gathered} \tag{5.4}$$

where

$$\mathcal{A} = r_v - p r_{qu} - q r_{qv} - r_{qx}, \qquad \mathcal{B} = r_u - p r_{pu} - q r_{pv} - r_{px}$$

and subscripts on $\mathcal{A}$, $\mathcal{B}$ denote total differentiation: $\mathcal{A}_v = D_v(\mathcal{A})$.

Theorem 5.1. All control equations of the form (1.1) with

$$r = a(x,u,v)q + \left[\frac{a_u - c_v}{a}\right]p^2 + c(x,u,v)p + d(x,u,v), \qquad a(x,u,v) \neq 0, \tag{5.5}$$

such that the functions $a(x,u,v)$, $c(x,u,v)$, $d(x,u,v)$ satisfy

$$a(a_u - c_v)_v = a_v(a_u - c_v), \quad a(d_v - a_x)_v = a_v(d_v - a_x), \quad a(d_v - c_x)_v = d_v a_u - c_v a_x,$$
$$ad(a_u - c_v) + 2(d_v - a_x)^2 + a((d_v - a_x)_x + c(d_v - a_x)) = d_v(d_v - a_x) + a^2(d_u - c_x)$$

are equivalent. These control systems admit an infinite-dimensional fiber-preserving symmetry group with structure equations (4.10).

Proof. The control equations (5.5) are the general solution of the system of differential equations (5.4). □

Corollary 5.2. All control systems satisfying the hypothesis of Theorem 5.1 are equivalent to

$$u_{xx} = v_x. \tag{5.6}$$

5.2 Second Order Generic Branch

We now consider the generic branch discussed in Section 4.2. First, we consider the sub-branch obtained in Section 4.2.1.

5.2.1 Hilbert–Cartan Sub-Branch

For this branch of the equivalence problem, recall that the control equation (1.1) must satisfy the non-degeneracy conditions

$$R_Q \neq 0, \qquad R_{QQ} \neq 0, \qquad R_{QQQ} \neq -1, \tag{5.7}$$

while the third order differential invariants (4.14) and the fourth order invariants (4.15) are identically equal to zero and that our chosen cross-section is

$$\begin{gathered} X = U = V = P = Q = 0, \qquad R_Q = R_{QQ} = 1, \\ R_{V^iU^jX^k} = R_{PX^i} = R_{PPU^i} = R_{QV^{i+1}} = R_{QPU^i} = R_{QQX^{i+1}} = 0, \qquad i,j,k \geq 0, \\ R_{PV^iU^j} = R_{QU^iX^j} = 0, \qquad i+j \geq 1. \end{gathered} \tag{5.8}$$

As in the previous section, to find the control equations that are part of this branch of the equivalence problem we must solve the normalization equations (5.8) for the pseudo-group parameters and substitute the result in the invariants (4.14), (4.15). The result is a complicated system of partial differential equations. Solving the system of equations appears to be very complicated. Fortunately, it is possible to obtain particular solutions, and this without knowing the actual expressions of the equations.

Proposition 5.3. A control equation that satisfies the non-degeneracy conditions $r_q \neq 0$, $r_{qq} \neq 0$, $r_q r_{qqq} \neq r_{qq}^2$ and only depends on the variable q, i.e. $r = r(q)$, is part of the Hilbert–Cartan sub-branch.

Proof. The restrictions $r_q \neq 0$, $r_{qq} \neq 0$, $r_q r_{qqq} \neq -r_{qq}^2$ come from the non-degeneracy conditions (5.7) imposed on the invariants. When

$$r_{x^iu^jv^kp^l} = 0, \qquad i,j,k,l \geq 0,$$

the solution to the normalization equations (5.8) for the pseudo-group jets simplifies to

$$\phi_{x^i} = \beta = \beta_{x^i u^{j+1}} = \alpha = \alpha_{x^i u^j v^{k+1}} = 0, \qquad i, j, k \geq 0, \tag{5.9a}$$

and

$$\beta_u = \frac{\phi_x^2 r_{qq}}{r_q^2}, \qquad \alpha_v = \frac{\phi_x r_{qq}}{r_q}. \tag{5.9b}$$

Substituting (5.9) into the lifted invariants R_J we find that the invariants (4.14), (4.15) are identically zero. □

From (5.9b) we obtain the differential invariants

$$I_{qqq} = \iota(r_{qqq}) = \frac{r_q r_{qqq}}{r_{qq}^2}, \qquad I_{qqqq} = \iota(r_{qqqq}) = \frac{r_q^2 r_{qqqq}}{r_{qq}^3}, \qquad \ldots . \tag{5.10}$$

Finally, substituting (5.9b) into the lifted total differential operator D_Q given in (2.19) yields the invariant differential operator

$$\mathcal{D}_q = \frac{r_q}{r_{qq}} D_q$$

mentioned in Theorem 4.1.

The Hilbert–Cartan equation (1.4) satisfies the hypotheses of Proposition 5.3, and has the property that $I_{qqq} = 3$. It thus satisfies the hypotheses of Theorem 4.3 from which we conclude that it admits a 6-dimensional fiber-preserving symmetry group with Maurer–Cartan structure equations isomorphic to

$$\begin{gathered} d\omega^x = \mu_X \wedge \omega^x, \quad d\omega^q = 0, \quad d\mu_X = 0, \quad d\omega^u = -\omega^p \wedge \omega^x + \omega^q \wedge \omega^u + 2\mu_X \wedge \omega^u, \\ d\omega^v = -\omega^q \wedge \omega^x + 2\omega^q \wedge \omega^v + \mu_X \wedge \omega^v, \quad d\omega^p = -\omega^q \wedge \omega^x + \omega^q \wedge \omega^p + \mu_X \wedge \omega^p. \end{gathered}$$

This fact is well-known, [2, 12], it is nevertheless comforting to recover this result using the equivariant moving frame method.

More generally, we can find all control equations of the form $r = r(q)$ that admit a six-dimensional fiber-preserving symmetry group by solving the ordinary differential equation

$$\frac{r_q r_{qqq}}{r_{qq}^2} = I, \qquad \text{where } I \neq -1 \text{ is a constant.} \tag{5.11}$$

Integrating once, we obtain

$$r_{qq}(r_q)^{-I} = C \neq 0. \tag{5.12}$$

The constant of integration C cannot be equal to zero since by assumption $r_q \neq 0$ and $r_{qq} \neq 0$. If $I = 1$ we obtain

$$r(q) = A + \frac{e^{Cq+B}}{C}, \qquad \text{where } A,\, B \text{ are constants.}$$

When $I \neq 1$, integrating (5.12) we obtain

$$r_q = [(1-I)(Cq+B)]^{\frac{1}{1-I}}. \tag{5.13}$$

If $I = 2$, the solution of (5.13) is

$$r(q) = A - \frac{\ln|Cq+B|}{C},$$

otherwise

$$r(q) = A + \frac{[(1-I)(Cq+B)]^{\frac{2-I}{1-I}}}{C(2-I)}. \tag{5.14}$$

The Hilbert–Cartan equation is recovered from (5.14) by setting $I = 3$, $C = -2$ and $A = B = 0$.

5.2.2 Third Order Generic Branch

We end by writing down the coordinate expressions of the simplest invariants of the generic branch of the equivalence problem. Let

$$\frac{D(r_J, r_K)}{D(z^i, z^l)} = r_{J,i}\, r_{K,l} - r_{J,l}\, r_{K,i}$$

denote the total Jacobian determinant of r_J and r_K with respect to z^i, z^l. Introducing the notation

$$\Delta = \frac{D(r_{qq}, r_q)}{D(q,p)} \neq 0, \qquad \Gamma = -\frac{r_{qq}}{r_q}\frac{D(r, r_p)}{D(q,v)} + \frac{r_q}{2}\frac{D(r_q, r_p)}{D(p,q)} + r_{qq} r_{qu},$$

$$\Xi = \frac{D(r_q, r_{qq})}{D(q,u)} + 2\frac{r_{qq}}{r_q}\frac{D(r_q, r_v)}{D(p,q)} + \frac{D(r_p, r_{qq})}{D(v,q)} + \frac{r_{qq}}{2}\frac{D(r_q, r_p)}{D(p,q)}(1 - I_{qqq}),$$

$$\Lambda = \frac{1}{r_{qq}}\left[r_{qx} - r_v + \frac{q}{r_q}\frac{D(r, r_q)}{D(q,v)} + p r_{qu} + \frac{r_q}{r_{qq}}\frac{D(r, r_q)}{D(q,p)} - \frac{r_q}{\Delta}\Xi + q\frac{r_v \Delta}{r_{qq}^2} + \frac{r r_q r_{qp}\Delta}{r_{qq}^3}\right],$$

the coordinate expressions of the first six differential invariants (4.20a) are

$$I_{qqq} = \frac{r_q r_{qqq}}{r_{qq}^2}, \qquad I_{qqv} = \frac{r_{qq}}{r_q^3 \Delta}\left[r_q \frac{D(r, r_{qq})}{D(q,v)} - 2 r_{qq}\frac{D(r, r_q)}{D(q,v)}\right],$$

$$I_{qpp} = \frac{r_{qq}^2}{r_q \Delta^2}\left[r_{qq}\frac{D(r_q, r_{qp})}{D(q,p)} - r_{qp}\frac{D(r_{qq}, r_q)}{D(p,q)}\right],$$

$$I_{qpv} = \frac{r_{qq}^3}{r_q^3 \Delta^2}\left[r_{qq}\frac{D(r_q, r_v)}{D(p,q)} + r_q \frac{D(r_q, r_{qv})}{D(q,p)} - r_v \frac{D(r_{qq}, r_q)}{D(p,q)}\right],$$

$$I_{ppp} = \frac{r_{qq}^6}{r_q^2 \Delta^3}\left[\frac{D(r_q, r_{pp})}{D(q,p)} + \frac{2 r_{qp}}{r_{qq}}\frac{D(r_q, r_{qp})}{D(p,q)} + \frac{r_{qp}^2}{r_{qq}^2}\frac{D(r_{qq}, r_q)}{D(p,q)}\right],$$

$$I_{ppv} = \frac{r_{qq}^5}{r_q^3 \Delta^3}\left[r_{qq}\frac{D(r_q, r_{pv})}{D(q,p)} + r_{pq}\frac{D(r_q, r_{qv})}{D(p,q)} + \frac{r_v}{r_q}\left(r_{qp}\frac{D(r_{qq}, r_q)}{D(p,q)} - r_{qq}\frac{D(r_q, r_{qp})}{D(q,p)}\right)\right].$$

The remaining differential invariants (4.4) can easily be obtained with a symbolic software. Unfortunately, the expressions obtained take too much space to write them down. Finally, the coordinate expressions of the invariant horizontal coframe in Theorem 4.5 are

$$\omega^x = \frac{r_q^2 \Delta}{r_{qq}^3} dx, \qquad \omega^v = \frac{r_q \Delta}{r_{qq}^3}[r_{qp}(du - p dx) + r_{qq}(dv - q dx)],$$

$$\omega^u = \frac{r_q^2 \Delta^2}{r_{qq}^2}(du - p dx), \qquad \omega^p = \frac{1}{r_{qq}^2}[\Xi(du - p dx) + \Delta(dp - r dx)],$$

$$\omega^q = \frac{r_{qq}}{r_q}\left[\frac{\Gamma}{r_{qq}^2}(du - p dx) + \frac{r_v}{r_q}(dv - q dx) + \frac{r_{qp}}{r_{pp}}(dp - r dx) + (dq + \Lambda dx)\right].$$

6 Concluding Remarks

The solution to a complicated equivalence problem involves many challenges. One of the main difficulties is to determine all the different branches of the problem and to characterize the algebra of differential invariants for each cases. Now, with the universal recurrence relations (2.28) these questions can readily be answered symbolically. In this paper we considered three typical branches of the equivalence problem (1.1), (1.2) that illustrate the possible outcomes of the moving frame method:

Generic branch: all pseudo-group parameters are normalized and the result is a standard invariant coframe $\{\omega^x, \omega^u, \omega^v, \omega^p, \omega^q\}$.

Hilbert–Cartan branch: finitely many pseudo-group parameters cannot be normalized and the result is a prolonged invariant coframe $\{\omega^x, \omega^u, \omega^v, \omega^p, \omega^q, \mu_X\}$.

Degenerate branch: Infinitely many pseudo-group parameters cannot be normalized and the result is an infinite-dimensional invariant coframe $\{\omega^x, \omega^u, \omega^v, \omega^p, \omega^q, \mu_X, \nu_U,, \alpha_{X^{k+2}},, \alpha_{UX^k}\}$ with involutive structure equations.

The author believes that doing the similar computations with Cartan's equivalence method of coframes would have required a lot more work. By possibly using symbolic softwares, the hope is that the equivariant moving frame method will lead to the solution of new equivalence problems or at least give valuable information on some key branches of complicated equivalence problems.

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