Applications of Moving Frames to Lie Pseudo-Groups

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Abstract

Recently, Olver and Pohjanpelto have successfully extended the theory of equivariant moving frames to infinite-dimensional Lie pseudo-groups. Based on its finite-dimensional counterpart, this new theory promises to be a source of interesting new results and applications. In this thesis, we look at two applications of this new theory.

By combining the powerful theories of Lie groupoids and variational bicomplexes, Olver and Pohjanpelto have developed a practical algorithm for determining the Maurer–Cartan structure equations of Lie pseudo-groups. The structure equations obtained with this new theory are compared with those derived by Cartan. It is shown that for transitive Lie pseudo-groups the two structure theories are isomorphic while for intransitive Lie pseudo-groups the two sets of structure equations do not agree. To make the two structure theory isomorphic we argue that Cartan's structure equations need to be slightly modified. The effect of this modification on Cartan's definition of essential invariants is analyzed.

In 1965, Singer and Sternberg gave an infinitesimal interpretation of Cartan's structure equations for transitive Lie pseudo-groups. This interpretation is extended to intransitive Lie pseudo-groups and the result is used to state a symmetry-based linearization theorem for systems of nonlinear partial differential equations which does not require the integration of the infinitesimal determining equations of the symmetry group.

The theory of equivariant moving frames is a powerful tool for determining a generating set of the differential invariant algebra of Lie pseudo-groups. After reviewing this theory, the method is illustrated with three applications. In the first two applications, generating sets of differential invariant algebra for the symmetry groups of the Infeld–Rowlands equation and the Davey–Stewartson equations are determined. Then we show that for two and three dimensional Riemannian manifolds the sectional curvatures generate the differential invariant algebra of the pseudo-group of locally invertible changes

of variables.

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Chapter 1

Introduction

When Sophus Lie began his work on continuous groups of transformations, no significant distinction was drawn between finite-dimensional and infinite-dimensional theory. But, since then the two subjects have evolved very differently. The definition of a Lie group as a manifold with smooth group structure in the early twentieth century was a major breakthrough in the finite-dimensional theory. The lack of a universally accepted abstract object playing the role of an infinite-dimensional Lie group has made the study of Lie pseudo-groups much more difficult. Presently, infinite-dimensional Lie pseudo-groups only manifest themselves through their action on a space. Such pseudo-groups appear in many branches of mathematics and physics: in fluid mechanics, [3, 81], Hamiltonian mechanics and symplectic and Poisson geometry, [81], conformal geometry of surfaces and conformal field theory, [32, 37], geometry of real hypersurfaces, [28], as gauge symmetries, [7], or symmetry groups of partial differential equations, [8, 9, 31, 36, 81, 92], and in geometric numerical integration [74].

Élie Cartan made remarkable contributions to the field of infinite-dimensional Lie pseudo-groups, [18, 20, 21, 22, 23]. Unfortunately, the complexity of his theory makes it very difficult to continue in his steps. A quote by André Weil, [112], summarizes well the situation:

> *Sur la théorie si importante sans doute, mais pour nous si obscure, des «groupes de Lie infinis», nous ne savons rien que ce qui ce trouve dans les mémoires de Cartan, première exploration à travers une jungles presque impénétrable; mais celle-ci menace de refermer sur les sentier déjà tracés, si l'on ne procède bientôt à un indispensable travail de défrichement.*

We had to wait until the 1960s to see new substantial results by Kuranishi, [63, 64], Guillemin, Singer, Sternberg and Quillen, [41, 42, 101, 102], on the subject. A lot of effort has been made to establish a proper rigorous foundation for transitive pseudo-groups, [52, 61, 63, 64, 95, 101], yet a lot of work remains to be done. This is without mentioning that for intransitive Lie pseudo-groups even less is known. Intransitive Lie pseudo-groups are difficult to understand partially due to the possible appearance of essential invariants in their Maurer–Cartan structure equations.

Time has shown that the theory of equivariant moving frames for Lie groups, developed by Olver and Fels, [38, 39], can be applied to many interesting problems. For example, the theory has been used to produce new algorithms for solving the symmetry and equivalence problems of polynomials that form the foundation of classical invariant theory, [6,58,81]. It found numerous applications in computer science. It has been applied to the problems of object recognition and symmetry detection, [11,16], and the applications to joint invariants and differential invariants, [12, 39, 79], led to the implementation of fully invariant finite difference numerical schemes, [53, 54, 55, 83]. The universal recurrence formulas lead to a complete characterization of the differential invariant algebra of group actions with results on minimal generating sets of invariants, [44,45,46,85,87]. It has successfully been used to solve the general problem from calculus of variation of directly constructing the invariant Euler–Lagrange equations from their invariant Lagrangians, [56], and to derive generalized Casimir invariants of Lie algebras, [13, 14]. Nowadays, many new applications are under investigation, [86].

In 2005, Olver and Pohjanpelto successfully extended the theory of equivariant moving frames to infinite-dimensional Lie pseudo-groups, [88, 90, 91]. Based on the numerous applications of the theory in the context of Lie groups, this new extension will for sure bring its wealth of new results. The first extensive application of this new theory was conducted in [26, 27] where the structure equations and differential invariant algebra of the symmetry group for the Kadomtsev–Petviashvili equation are computed in full detail. An immediate avenue of future research consists of determining to which extend results known for Lie group actions extent to infinite-dimensional Lie pseudo-groups. Also many new applications can be foreseen. For example, the theory should play an important role in the development of Vessiot's group foliation theory, [73, 78, 109]. This theory provides a powerful approach of determining explicit non-invariant solutions to partial differential equations. The application of the moving frame method to the theory of coverings of differential equations, [76, 77], to the symmetry classification of differential equations developed by Lisle and Reid [69] and to invariant variational problems admitting infinite-dimensional symmetry groups are just a few of many interesting sources of new research. In the long run the hoped is that this new theory of equivariant moving frames will shed a new light on the difficult theory of infinite-dimensional Lie pseudo-groups.

In this thesis we investigate two applications of the theory of equivariant moving frames to infinite-dimensional Lie pseudo-groups. By combining the two powerful theories of Lie algebroids, [71, 75], and variational bicomplexes, [1], Olver and Pohjanpelto developed a natural and completely algorithmic method of deriving the Maurer–Cartan structure equations of Lie pseudo-groups. After reviewing the notions of extended jet bundles, Lie groupoids and Lie pseudo-groups in Chapter 2, we proceed to explain Olver and Pohjanpelto's derivation of the Maurer–Cartan structure equations for Lie pseudo-groups in Section 3.1. In a natural way, an infinitesimal interpretation of the structure equations, in terms of the jets of infinitesimal generators is given. This interpretation extends to intransitive Lie pseudo-groups the infinitesimal interpretation of Cartan's structure equations for transitive Lie pseudo-groups given by Singer and Sternberg, [101]. It also shows in which sense Cartan's structure theory of Lie pseudo-groups is equivalent to Lie's structure theory based on the infinitesimal generators of Lie pseudo-group actions, [81, 82, 92].

With two sets of structure equations, namely those coming from the equivariant moving frame theory and those originating from Cartan's moving frame theory, a natural thing to do is to verify their compatibility. After reviewing Cartan's derivation of the structure equations in Section 3.3, we show in Section 3.4 that the two structure theories do not agree for intransitive Lie pseudo-groups. By working out some examples, considered by Cartan himself, [20, 22, 23], we come to the conclusion that the source of discrepancy in the two structure theories originates from Cartan's structure equations. As we explain, the two sets of structure equations are isomorphic modulo the restriction of Cartan's structure equations to the target fibers of the pseudo-group action. With the modified Cartan structure equations in hand we investigate Cartan's notion of essential invariants. Our conclusion that it is not possible to define the concept of essential invariants in terms of the systatic system of the Maurer–Cartan structure equations. To resolve this problem, an alternative definition is proposed. We believe that our definition still captures what Cartan had in mind when he defined his notion of essential invariants.

In the literature, one can find different, and equivalent, statements of the symmetry-based linearization theorem for nonlinear systems of partial differential equations, [10, 49, 50, 82]. Given a system of nonlinear partial differential equations, these theorems give necessary and sufficient conditions for the existence of a local smooth invertible change of variables mapping the nonlinear system of equations to a linear system of partial differential equations. To apply those theorems, the knowledge of the symmetry generators for the system of nonlinear partial differential equations is needed as the Lie algebra structure equations must be known. This means that the infinitesimal determining system for the symmetry group must be integrated. In Section 4 we state an equivalent symmetry-based linearization theorem in terms of the Maurer–Cartan structure equations. The advantage of this new version of the theorem is that the infinitesimal determining equations for the symmetry group do not need to be integrated.

The second application of the equivariant moving frame theory is a continuation of the work initiated in [25, 27], where the differential invariant algebra of the symmetry group action for the Kadomtsev-Petviashvili equation is completely characterized. Another illustration of the theory can be found in [106] where the algebra of differential invariants for the group of equivalence transformations of partial differential equations defined by vector fields is analyzed. For many Lie pseudo-groups, the number of functionally independent differential invariants is infinite-dimensional. Under suitable hypothesis, Lie [66, Theorem 42, p. 760] showed for finite-dimensional Lie groups, then extended by Tresse to infinite-dimensional pseudo-groups, [105], that the differential invariant algebra is finitely generated. This means that there exists a finite set of differential invariants and a well determined number of invariant derivatives such that every differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives. After reviewing the equivariant moving frame construction and stating the universal recurrence formula in Chapter 5, we apply the theory to the characterization of the differential invariant algebra for the Infeld–Rowlands equation and the Davey–Stewartson equations. Those two equations have been chosen based on the fact that they both admit an infinite-dimensional symmetry group with the distinction that the symmetry algebra of the Davey–Stewartson equations possesses the structure of a Kac–Moody Lie algebra while this is not the case for the Infeld–Rowlands equation. The long term project is to understand how the structure of the symmetry group affects the structure of the differential invariant algebra. Finally, we also analyze the differential invariant algebra of the pseudo-group of all locally invertible changes of variables for two and three dimensional Riemannian manifolds. The result of our computations is that for non-degenerate Riemannian manifolds the algebra of differential invariants is generated by the sectional curvatures. This result can be interpreted as an extension of the recent observations by Olver and Hubert, [46, 87], that for generic surfaces in three-dimensional Euclidean space the algebras of differential invariants for the Euclidean, equi-affine, conformal and projective groups are generated by only one invariant. As one will notice, the computation of the moving frames and recurrence formulas are computationally demanding. Some computations were thus implemented into MATHEMATICA routines.

Blanked Hypotheses and Notational Conventions

Throughout the thesis all the constructions and considerations are made in the analytic category. Thus Lie pseudo-groups, manifolds, submanifolds, differential functions, differential forms, vector fields, differential equations, etc., are assumed to be analytic. Some results can be extended to the smooth category by appealing to some more elaborate theorems but this will not be pursued in this work.

Frequently, the pull-back notation of differential forms will be omitted to avoid notational clutter. The context of the discussion should, we hope, make the identification of functions and forms with their pull-back not a source of confusion.

Furthermore, we use a global language, although most constructions are purely local. For example, the notation

$$\phi : M \to M,$$

will be used to denote a local diffeomorphism ϕ defined on an open subset U

of a manifold M.

Chapter 2

Preliminaries

2.1 Jet Bundles

The abstract theories of differential equations, differential invariants and equivariant moving frames find their roots in the theory of extended jet bundles. In this section, we review the definitions and constructions related to jet bundles and fix some notation. Our presentation follows [80, 111].

2.1.1 Extended Jet Bundles

Let M be an analytic manifold and $z_0 \in M$. Let $C^\omega(M, \mathbb{R})|_{z_0}$ denote the algebra of germs of analytic real valued functions on M at the point z_0. Let $I_{z_0} \subset C^\omega(M, \mathbb{R})|_{z_0}$ be the ideal of germs of functions which vanish at z_0, and let $I^n_{z_0}$ denote its n-th power, which consists of all finite linear combinations of n-fold products of elements of I_{z_0}.

Definition 2.1. The *n-th order cotangent jet bundle* of M at z_0 is

$$\mathcal{J}^n T^* M|_{z_0} = I_{z_0} / I^{n+1}_{z_0}.$$

The *n-th order tangent jet bundle* to M at z_0 is the dual space

$$\mathcal{J}^n TM|_{z_0} = (\mathcal{J}^n T^* M|_{z_0})^*.$$

Note that

$$\mathcal{J}^n T^* M|_{z_0} \cong \odot^n_{\mathbb{R}} T^* M|_{z_0}, \qquad \mathcal{J}^n TM|_{z_0} \cong \odot^n_{\mathbb{R}} TM|_{z_0},$$

where $\odot$ denotes the symmetric tensor product. The notion of n-th order tangent jet bundle can be tied with the usual notion of higher order derivative in Euclidean space by specifying a system of local coordinates around z_0. Let $z = (z^1, \ldots, z^m)$ be a local coordinate system around z_0. Given the m-tuple $A = (a^1, \ldots, a^m)$ of non-negative integers, we let

$$(z - z_0)^A = (z^1 - z_0^1)^{a^1} \cdots (z^m - z_0^m)^{a^m}, \qquad \partial_z^A = \frac{\partial^{\#A}}{(\partial z^1)^{a^1} \cdots (\partial z^m)^{a^m}},$$

where $\#A = a^1 + \cdots + a^m$ is the order of the m-tuple A. Also let

$$A! = a^1! a^2! \cdots a^m!$$

With this notation, the collection

$$\left\{ \frac{1}{A!} [(\mathbf{z} - \mathbf{z}_0)^A] : 1 \leq \#A \leq n \right\} \tag{2.1.1}$$

of germs of functions is seen to form a basis of $\mathcal{J}^n T^* M|_{z_0}$. Here $\mathbf{z}$ denotes the germ of the coordinate function z, $\mathbf{z}_0$ the germ of the constant function $f(z) = z_0$, and [ ] the equivalence class in $\mathcal{J}^n T^* M|_{z_0}$. Thus $\mathcal{J}^n T^* M|_{z_0}$ is finite-dimensional with

$$\dim(\mathcal{J}^n T^* M|_{z_0}) = \sum_{j=1}^{n} \binom{m+j-1}{j},$$

and $\mathcal{J}^n TM|_{z_0}$ is canonically isomorphic to $\mathcal{J}^n T^*M|_{z_0}$ with basis

$$\left\{\partial_z^A : 1 \le \#A \le n\right\}.$$

Definition 2.2. Two submanifolds S and $\widetilde{S}$ of M are said to have *n-th order contact* at $z \in S \cap \widetilde{S}$ if $\mathcal{J}^n TS|_z = \mathcal{J}^n T\widetilde{S}|_z$.

Definition 2.3. The *space of germs of p-dimensional submanifolds of M passing through z*, $C^\omega(M,p)|_z$, is the set of all smooth p-dimensional submanifolds of M passing through z modulo the equivalence relation that S and $\widetilde{S}$ define the same germ at z if and only if there is a neighborhood U of z with $S \cap U = \widetilde{S} \cap U$.

Definition 2.4. The *space of extended n-jets of p-dimensional submanifolds* of M at a point $z \in M$, $J^n(M,p)|_z$, is given by the space of germs of p-dimensional submanifolds of M passing through z modulo the equivalence relation of n-th order contact.

Definition 2.5. The *extended n-jet bundle of p-dimensional submanifolds* of M is

$$J^n(M,p) = \bigsqcup_{z \in M} J^n(M,p)|_z.$$

A p-dimensional submanifold S of M is locally described as the graph

$$(x^1, \ldots, x^p, f^1(x^1, \ldots, x^p), \ldots, f^q(x^1, \ldots, x^p)), \qquad p + q = m,$$

of q functions $f^\alpha : \mathbb{R}^p \to \mathbb{R}$, $\alpha = 1, \ldots, q$. This induces a splitting of the local coordinates $z = (x, u) = (x^1, \ldots, x^p,\ u^1, \ldots, u^q)$ on M into p independent and q dependent variables

$$M \cong X \times U.$$

Under this splitting the local coordinates of $J^n(M,p)$ are

$$j_n S|_z = z^{(n)} = (x, u^{(n)}) = (\ldots x^i \ldots u_J^\alpha \ldots),$$

where $u^{(n)}$ denotes all the derivatives of the dependent variables u with respect to the independent variables x up to order n. When $k > n$, we let $\widetilde{\pi}_n^k : J^k(M,p) \to J^n(M,p)$ denote the standard projection $\widetilde{\pi}_n^k(z^{(k)}) = z^{(n)}$. The *infinite extended jet bundle* $J^\infty(M,p)$ is defined to be the inverse limit, [98], of the inverse system

$$\widetilde{\pi}_n^{n+1} : J^{n+1}(M,p) \to J^n(M,p).$$

One of the advantages of working with $J^\infty(M,p)$ instead of $J^n(M,p)$, for some finite n, is that $J^\infty(M,p)$ carries a contact structure characterized by a certain distinguished subbundle of its cotangent bundle.

Definition 2.6. Let $\Omega^*(J^\infty(M,p))$ denote the exterior algebra of differential forms on $J^\infty(M,p)$. The *contact ideal* $\mathcal{C}(J^\infty(M,p))$ is the ideal in $\Omega^*(J^\infty(M,p))$ of forms such that for all local sections S of the bundle $\pi^0 : M \cong X \times U \to X$

$$(j^\infty S)^*\theta = 0.$$

Since $\theta \in \mathcal{C}(J^\infty(M,p))$ implies $d\theta \in \mathcal{C}(J^\infty(M,p))$, $\mathcal{C}(J^\infty(M,p))$ is a differential ideal, [15]. A local basis for $\mathcal{C}(J^\infty(M,p))$ is provided by the contact one-forms

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^{p} u_{J,i}^\alpha dx^i, \qquad \alpha = 1, \ldots, q,$$

where $J = (j^1, \ldots, j^p)$ is a p-tuple of nonnegative integers

$$u_J^\alpha = \frac{\partial^{\#J} u^\alpha}{(\partial x^1)^{j^1} \cdots (\partial x^p)^{j^p}}, \qquad \text{and} \qquad J,i = (j^1, \ldots, j^{i-1}, j^i + 1, j^{i+1}, \ldots, j^p).$$

One-forms in $\mathcal{C}(J^\infty(M,p)) \subset \Omega^*(J^\infty(M,p))$ are called *vertical*. The contact ideal provides a means to determine if a section σ of the n-th order jet bundle $\pi^n : J^n(M,p) \to X$ is the n-th prolongation of a local section of the initial bundle $\pi^0 : M \to X$. It will be the case if and only if

$$\sigma^*(\theta) = 0 \qquad \forall\, \theta \in \mathcal{C}(J^n(M,p)).$$

Definition 2.7. A vector field $\mathbf{v}$ is called a total vector field if it annihilates all contact one-forms. In the local coordinates $z = (x,u)$, a basis of total vector fields is given by the total differential operators

$$D_{x^j} = \frac{\partial}{\partial x^j} + \sum_{\alpha=1}^{q} \sum_{\#J \geq 0} u^\alpha_{J,j} \frac{\partial}{\partial u^\alpha_J}, \qquad j = 1, \ldots, p.$$

The vector fields

$$\left\{ D_{x^i}, \frac{\partial}{\partial u^\alpha_J} \right\}$$

form a frame on $J^\infty(M,p)$ with dual coframe

$$\{dx^i, \theta^\alpha_J\}$$

so that the differential function $F : J^\infty(M,p) \to \mathbb{R}$ has exterior differential

$$dF = \sum_{j=1}^{p} (D_{x^j} F) dx^j + \sum_{\alpha=1}^{q} \sum_{\#J \geq 0} \frac{\partial F}{\partial u^\alpha_J} \theta^\alpha_J. \tag{2.1.2}$$

Note that the second sum in (2.1.2) is finite since by definition of $\Omega^0(J^\infty(M,p))$ any differential function factors through $\Omega^0(J^n(M,p))$, for some n, [1].

2.1.2 Jets of Maps

To describe pseudo-groups adequately we need to work in the category of jet bundles of maps.

An analytic map $\phi : X \to U$ between analytic manifolds X and U of dimension p and q respectively, is completely determined by its graph

$$\mathrm{graph}(\phi) : X \to X \times U, \qquad x \mapsto (x, \phi(x)),$$

which is a submanifold of the product bundle

$$\mathrm{pr}_1 : X \times U \to X, \qquad (x, u) \mapsto x \tag{2.1.3}$$

transversal to each of the fibers $(\mathrm{pr}_1)^{-1}(x)$, $x \in X$.

Let Ω_X be a local volume form on X. In the local coordinate system $x = (x^1, \ldots, x^p)$, the volume form can be chosen to be the canonical volume form $\Omega_X = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^p$. Then a p-dimensional submanifold $S \subset X \times U$ is transversal to the fibers of the bundle (2.1.3) if and only if

$$\Omega_X|_S = \mathrm{pr}_1^*(\Omega_X)|_S \neq 0. \tag{2.1.4}$$

Every p-dimensional regular submanifold $S \subset X \times U$ satisfying the transversality condition $\Omega_X|_S \neq 0$ uniquely determines a section $s : X \to X \times U$ of the bundle (2.1.3), which in turn, uniquely defines the mapping

$$\mathrm{map}(s) : X \to U, \qquad x \mapsto \mathrm{pr}_2(s(x)),$$

where

$$\mathrm{pr}_2 : X \times U \to U, \qquad (x, u) \mapsto u,$$

is the projection of $X \times U$ onto U. The mappings "graph" and "map" satisfy

$$\mathrm{graph} \circ \mathrm{map} = \mathrm{id}, \qquad \mathrm{map} \circ \mathrm{graph} = \mathrm{id}.$$

Hence the space of maps $\phi : X \to U$ and the space of p-dimensional submanifolds of the product bundle (2.1.3) satisfying the transversality condition (2.1.4) are diffeomorphic.

Definition 2.8. Let X and U be a p and q dimensional manifolds respectively. The n-th order jet bundle $J^n(X, U)$ of maps from X to U is defined as the subbundle of $J^n(X \times U, p)$ containing all equivalence classes of p-dimensional submanifold $S \subset X \times U$ satisfying the transversality condition (2.1.4).

In a local coordinate system (x, u) of $X \times U$, the standard coordinate system of the n-th jet of a function $\phi : X \to U$ is

$$j_n\phi|_x = (x, u^{(n)}).$$

In the following, we will mostly be interested in jets of local diffeomorphisms $\phi : M \to M$. A local diffeomorphism will be denoted by $Z = \phi(z)$ and its n-th order jet by

$$j_n\phi|_z = (z, Z^{(n)}).$$

2.2 Lie Groupoids and Lie Algebroids

In this section, we review the definitions of Lie groupoids and Lie algebroids. We refer the reader to [71, 75] for a more detailed exposition.

2.2.1 Lie Groupoids

Definition 2.9. A *groupoid* G is a small[1] category with invertible morphisms.

The set of objects, also called the *base*, is denoted by G_0, while the set of morphisms is denoted by G_1. Every morphism g of G_1 has two objects assigned to it, its *source* $\boldsymbol{\sigma}(g)$ and its *target* $\boldsymbol{\tau}(g)$. The notation

$$g : x \to y$$

[1] A small category is a category in which both the objects and the morphisms are sets and not proper classes.

is used to indicate that $x = \boldsymbol{\sigma}(g)$ and $y = \boldsymbol{\tau}(g)$. The composition of morphisms induces a partial multiplication that is defined whenever source and target match:

$$m : G_1 \,{}^{\boldsymbol{\sigma}}\!\times^{\boldsymbol{\tau}}_{G_0} G_1 \to G_1, \qquad m(g,h) \to gh,$$

with

$$G_1 \,{}^{\boldsymbol{\sigma}}\!\times^{\boldsymbol{\tau}}_{G_0} G_1 = \{(g,h) \in G_1 \times G_1 : \boldsymbol{\sigma}(g) = \boldsymbol{\tau}(h)\}.$$

Via the mapping $i : G_0 \hookrightarrow G_1 : x \mapsto \mathbb{1}_x$, the base G_0 is embedded in G_1 as the identity morphisms.

Definition 2.10. A groupoid G is called a *Lie groupoid* if G_0 and G_1 are analytic manifolds, $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$, m, i and the inversion map $g \mapsto g^{-1}$ are analytic and $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ are surjective submersions.

Example 2.11. A Lie group G can be viewed as a Lie groupoid over a one point space with G as the manifold of morphisms.

Example 2.12. Let G be a Lie group acting on a manifold M. The source and target maps

$$\begin{aligned} \boldsymbol{\sigma} &: G \times M \to M, & (g,z) &\mapsto z, \\ \boldsymbol{\tau} &: G \times M \to M, & (g,z) &\mapsto g \cdot z, \end{aligned}$$

make the product manifold $G \times M$ into a Lie groupoid, called the *action groupoid* over the base space M. The multiplication on $G \times M$ is defined by

$$(\widetilde{g}, \widetilde{z}) \cdot (g, z) = (\widetilde{g}g, z),$$

where $\widetilde{z} = \boldsymbol{\sigma}(\widetilde{g}, \widetilde{z}) = \boldsymbol{\tau}(g,z) = g \cdot z$.

Example 2.13. Consider the n-th jet bundle $J^n(M,M)$ with source and projection maps

$$\boldsymbol{\sigma}^{(n)} : J^n(M,M) \to M, \qquad j_n\phi|_z \mapsto z,$$
$$\boldsymbol{\tau}^{(n)} : J^n(M,M) \to M, \qquad j_n\phi|_z \mapsto \phi(z).$$

Restricted to the open subset $\Pi^n \subset J^n(M,M)$ of invertible jets, the chain rule induces a partial multiplication on Π^n, which is respected by the natural projections $\pi_n^{n+r} : J^{n+r}(M,M) \to J^n(M,M)$. Π^n is called the *full jet groupoid* of order n and a *jet groupoid* $\mathcal{G}^n$ is a subbundle of Π^n, closed with respect to all groupoid operations.

Definition 2.14. An *orbit* of the groupoid G over the base G_0 is an equivalence class for the relation $x \sim_G y$ if and only if there is a morphism $g \in G_1$ with $\boldsymbol{\sigma}(g) = x$ and $\boldsymbol{\tau}(g) = y$.

Definition 2.15. The *isotropy group* of $x \in G_0$ consists of all morphisms $g \in G_1$ with $\boldsymbol{\sigma}(g) = x = \boldsymbol{\tau}(g)$.

2.2.2 Lie Algebroids

Lie algebroids arise naturally as the infinitesimal versions of Lie groupoids, in complete analogy to the way that Lie algebras arise as the infinitesimal versions of Lie groups. As for Lie groups, we consider the action of G on the tangent bundle of G_1. However this needs to be done with care since the action is not defined everywhere. For each $g \in G_1$ with $\boldsymbol{\sigma}(g) = x$ and $\boldsymbol{\tau}(g) = y$, the right multiplication

$$R_g : \boldsymbol{\sigma}^{-1}(x) \to \boldsymbol{\sigma}^{-1}(y), \qquad h \mapsto h \cdot g^{-1} \tag{2.2.1}$$

is a diffeomorphism of source fibers. Therefore the right multiplication of G on G_1 induces a right multiplication on the vector bundle

$$T^{\boldsymbol{\sigma}} G_1 = \mathrm{Ker}(d\boldsymbol{\sigma}) \subset TG_1$$

as follows: for $\mathbf{v}|_h \in T^{\boldsymbol{\sigma}}_h(G_1)$ and $g \in G_1$ with $\boldsymbol{\sigma}(h) = \boldsymbol{\tau}(g)$ we have

$$dR_g(\mathbf{v}|_h) \in T^{\boldsymbol{\sigma}}_{hg^{-1}} G_1.$$

Definition 2.16. A vector field $\mathbf{v} \in \mathcal{X}(G_1)$ is said to be *right-invariant* if it is tangent to the source fibers everywhere on G_1 and satisfies the right invariance condition

$$dR_g(\mathbf{v}|_h) = \mathbf{v}|_{hg^{-1}}$$

for all g and h with $\boldsymbol{\sigma}(h) = \boldsymbol{\tau}(g)$.

The set of right-invariant vector fields on G_1 is denoted by $\mathcal{X}^R(G_1)$.

Proposition 2.17. Let G be a Lie groupoid, then

1. $\mathcal{X}^R(G_1)$ is a Lie subalgebra of $\mathcal{X}(G_1)$,
2. any right-invariant vector field $\mathbf{v} \in \mathcal{X}^R(G_1)$ is projectable to a vector field on G_0 by the differential of the target map $d\boldsymbol{\tau} : \mathcal{X}(G_1) \to \mathcal{X}(G_0)$,
3. the restriction $d\boldsymbol{\tau} : \mathcal{X}^R(G_1) \to \mathcal{X}(G_0)$ induces a Lie algebra homomorphism,
4. each right-invariant vector field $\mathbf{v} \in \mathcal{X}^R(G_1)$ is uniquely determined by a section of the bundle

$$\mathfrak{g} = \bigsqcup_{x \in G_0} T^{\boldsymbol{\sigma}}_{\mathbb{1}_x} G_1.$$

Point 4 of Proposition 2.17 gives an isomorphism of vector spaces

$$\mathcal{X}^R(G_1) \cong \Gamma(\mathfrak{g}), \tag{2.2.2}$$

where $\Gamma(\mathfrak{g})$ denotes the space of sections of the vector bundle $\mathfrak{g}$. Thus there is a well defined Lie algebra structure on $\Gamma(\mathfrak{g})$. The differential of the target map $d\boldsymbol{\tau}$ restricts to an *anchor* homomorphism $\beta = d\boldsymbol{\tau}|_{\mathfrak{g}} : \mathfrak{g} \to TG_0$ of vector bundles over G_0 (taking into account the isomorphism (2.2.2)). The next proposition shows that the Lie bracket and the anchor are related by a Leibniz-type identity.

Proposition 2.18. For all X, $Y \in \Gamma(\mathfrak{g})$ and $f \in C^\infty(G_0)$

$$[X, fY] = f[X, Y] + \beta(X)(f)Y.$$

Definition 2.19. The *Lie algebroid* of a Lie groupoid G is a vector bundle $\Gamma(\mathfrak{g}) = \mathrm{Lie}(G)$, together with an anchor map $\beta : \mathrm{Lie}(G) \to TG_0$ and a Lie bracket $[\cdot, \cdot]$ on $\mathrm{Lie}(G)$.

Example 2.20. If G is a Lie group, viewed as a Lie groupoid over a one point space, the vector bundle $\mathrm{Lie}(G)$ is the Lie algebra of right-invariant vector fields on G, isomorphic to the tangent space of G at the unit element of the group.

Throughout the thesis, we shall make use of the isomorphism (2.2.2) and identify elements of a Lie algebroid $\mathrm{Lie}(G)$ with the invariant vector fields $\mathcal{X}^R(G_1)$.

2.3 Lie Pseudo-Groups

2.3.1 Pseudo-Groups

Two nonequivalent definitions of local diffeomorphism exist in the literature. In this thesis we use the following definition.

Definition 2.21. Let M be an analytic m-dimensional manifold and U an open subset of M. An analytic map $\phi : U \to M$ is said to be a local (analytic) diffeomorphism if $\phi^{-1} : \phi(U) \to M$ exists and is analytic.

Definition 2.22. Let M be an (analytic) m-dimensional manifold and $\mathcal{G}$ be a collection of local (analytic) diffeomorphisms of M. The collection $\mathcal{G}$ is a *pseudo-group* if

1. $\mathcal{G}$ is closed under restriction: if $U \subset M$ is an open set and $\phi : U \to M$ is in $\mathcal{G}$, then so is $\phi|_V$ for all open $V \subset U$,

2. we can piece together elements of $\mathcal{G}$: if $U \subset M$ is an open set with $U = \cup_i U_i$ and $\phi : U \to M$ is a local diffeomorphism with $\phi|_{U_i} \in \mathcal{G}$, then $\phi \in \mathcal{G}$,

3. $\mathcal{G}$ is closed under composition: if $\phi : U \to M$ and $\psi : V \to M$ are two members of $\mathcal{G}$ and $\phi(U) \subset V$ then $\psi \circ \phi \in \mathcal{G}$,

4. $\mathcal{G}$ contains the identity diffeomorphism of M,

5. $\mathcal{G}$ is closed under inverse: if $\phi : U \to M$ is in $\mathcal{G}$, then $\phi^{-1} : \phi(U) \to M$ is also in $\mathcal{G}$.

Every pseudo-group $\mathcal{G}$ carries the structure of a groupoid, [33, 35, 71, 75]. The groupoid multiplication follows from the composition of local diffeomorphisms. Following Cartan, [23] we use lower case letters, z, x, u, ... for the source coordinates $\boldsymbol{\sigma}(\phi)$ of local diffeomorphisms $\phi \in \mathcal{G}$ and the corresponding upper case letter Z, X, U, ... for the target coordinates $\boldsymbol{\tau}(\phi)$.

Definition 2.23. A pseudo-group is said to be *regular* if its orbits form a regular foliation, i.e., its leaves intersect small open sets in pathwise connected subsets.

Example 2.24. The largest pseudo-group of $\mathbb{R}^n$ is the set of all local diffeomorphisms $\mathcal{D}(\mathbb{R}^n)$:

$$(z^1, \ldots, z^n) \mapsto (Z^1, \ldots, Z^n) = (\phi^1(z^1, \ldots, z^n), \ldots, \phi^n(z^1, \ldots, z^n)).$$

Note that if a map $Z = \phi(z)$ is in $\mathcal{D}(\mathbb{R})$ then its Jacobian matrix

$$\nabla Z = \left(Z_a^b\right) = \frac{\partial(Z^1, \ldots, Z^n)}{\partial(z^1, \ldots, z^n)}$$

is invertible for all z in its domain of definition, but the converse is not true. The map

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{0\}, \qquad \phi(z) = e^z,$$

with $z = x + iy$ is a counter-example.

Example 2.25. The collection of local transformations of $\mathbb{R}^2$

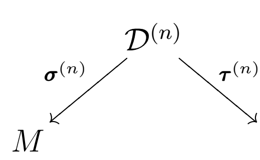
$$(X, U) = (f(x), f(u)),$$

where $f : \mathbb{R} \to \mathbb{R}$ is a local diffeomorphism forms a pseudo-group.

Let $\mathcal{D} = \mathcal{D}(M)$ be the pseudo-group of all local diffeomorphisms $Z = \phi(z)$ of M. For each $n \geq 0$, let $\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset J^{(n)}(M, M)$ denote the bundle of their n-th order jets. The local system of coordinates on $\mathcal{D}^{(n)}$ is given by

$$j_n\phi|_z = (z, Z^{(n)}), \tag{2.3.1}$$

where $Z^{(n)}$ parametrizes the fibers of the bundle $\pi_0^n : \mathcal{D}^{(n)} \to M$. The natural projections are written $\pi_k^n : \mathcal{D}^{(n)} \to \mathcal{D}^{(k)}$, $n \geq k$, and we let $\mathcal{D}^{(\infty)}$ be the inverse limit. The jet bundle $\mathcal{D}^{(n)}$ can be identified with the bundle of n-th order Taylor polynomials of local diffeomorphisms while $\mathcal{D}^{(\infty)}$ is identified with the bundle of infinite Taylor series. For every $n \geq 0$, the jet bundles $\mathcal{D}^{(n)}$ admit the structure of a groupoid. The source map $\boldsymbol{\sigma}^{(n)}(j_n\phi|_z) = z$ and target map $\boldsymbol{\tau}^{(n)}(j_n\phi|_z) = Z$ induce the double fibration



We use the notation $\mathcal{D}^{(n)}|_z$ to denote the jet fiber $(\boldsymbol{\sigma}^{(n)})^{-1}(z)$. The groupoid multiplication follows from the composition of n-th order jets. Local diffeomorphisms $\psi \in \mathcal{D}$ act on $\mathcal{D}^{(n)}$ by either left or right multiplication:

$$L_\psi(j_n\phi|_z) = j_n(\psi \circ \phi)|_z, \qquad R_\psi(j_n\phi|_z) = j_n(\phi \circ \psi^{-1})|_{\psi(z)}. \tag{2.3.2}$$

Throughout the thesis, if the action is not specified, the right multiplication must be understood.

Definition 2.26. A pseudo-group $\mathcal{G}$ acting on a manifold M is said to be *transitive* if for all $z \in M$ there exists an open neighborhood U of z such that for all $w \in U$ there exists $\phi \in \mathcal{G}$ such that $w = \phi(z)$. A pseudo-group is said to be *intransitive* if it is not transitive.

2.3.2 Lie Pseudo-Groups

Definition 2.27. A sub-pseudo-group $\mathcal{G} \subset \mathcal{D}$ is called a *Lie pseudo-group* if there exists $n^\star \geq 1$ such that the following assumptions are satisfied for all finite $n \geq n^\star$:

1. $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ forms a smooth, embedded subbundle,
2. $\pi_n^{n+1} : \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a fibration,
3. if $j_n\phi \subset \mathcal{G}^{(n)}$ then $\phi \in \mathcal{G}$,
4. $\mathcal{G}^{(n)} = \mathrm{pr}^{(n-n^\star)}\mathcal{G}^{(n^\star)}$ is obtained by prolongation.

The minimal value of $n^\star$ is called the *order* of the Lie pseudo-group.

The conditions (1-4) codify the formal integrability and local solvability requirements placed on the determining equations for the pseudo-group. We refer the reader to Appendix A for a brief discussion on formally integrable systems. For completeness we recall the definition of locally solvable systems of differential equations.

Definition 2.28. A system of k-th order differential equations $\Delta(x, u^{(k)}) = 0$ is *locally solvable at a point*

$$(x_0, u_0^{(k)}) \in \mathcal{S}_\Delta = \{(x, u^{(k)}) : \Delta(x, u^{(k)}) = 0\}$$

if there exists a smooth solution $u = f(x)$ of the system, defined for x in a neighborhood of x_0, which has the prescribed initial conditions $u_0^{(k)} = \mathrm{pr}^{(k)} f(x_0)$. The system is *locally solvable* if it is locally solvable at every point of $\mathcal{S}_\Delta$. A system of differential equations is *nondegenerate* if at every point $(x_0, u_0^{(k)}) \in \mathcal{S}_\Delta$ it is both locally solvable and of maximal rank.

In local coordinates, the *order n determining equations* defining the Lie pseudo-group subbundle $\mathcal{G}^{(n)}$ take the form of a system of nonlinear partial differential equations

$$F^{(n)}(z, Z^{(n)}) = 0, \tag{2.3.3}$$

whose local solutions $Z = \phi(z)$ are, for any $n \geq n^\star$, the pseudo-group transformations. The prolonged system $\mathrm{pr}^{(k)}\mathcal{G}^{(n)}$ is obtained by repeatedly applying the total differential operators

$$\mathbb{D}_{z^a} = \frac{\partial}{\partial z^a} + \sum_{b=1}^{m} \sum_{\#A \geq 0} Z^b_{A,a} \frac{\partial}{\partial Z^a_A}, \qquad a = 1, \ldots, m,$$

to (2.3.3):

$$0 = F^{(n+k)}(z, Z^{(n+k)}) = \mathbb{D}_z^A F^{(n)}(z, Z^{(n+k)}), \qquad 0 \leq \#A \leq k.$$

Not every nonlinear system of differential equations of the form (2.3.3) gives rise to a Lie pseudo-group. In view of definition 2.27, the system of nonlinear differential equations (2.3.3) must be a system of nonlinear *Lie equations*, [62]. This means that the independent and dependent variables are the coordinates of a manifold M, the identity map is a solution to the system, the composition of two solutions, whenever defined, is also a solution, and the inverse of a solution is still a solution.

Remark 2.29. Formal integrability of the determining system (2.3.3) can be difficult to verify in applications. Instead one will frequently require the determining equations to be in involution. By Cartan–Kuranishi's Theorem every system of differential equations can be completed to involution. For a brief overview of the theory of involutive systems of differential equations we refer the reader to Appendix A and the references therein.

Example 2.30. The collection of analytic maps $X = f(x)$ preserving the volume element $dx^1 \wedge \cdots \wedge dx^n$ is a Lie pseudo-group with defining equation

$$\begin{vmatrix} X^1_{x^1} & \cdots & X^1_{x^n} \\ \vdots & \ddots & \vdots \\ X^n_{x^1} & \cdots & X^n_{x^n} \end{vmatrix} = 1.$$

Example 2.31. Every finite-dimensional Lie group G is a Lie pseudo-group. The pseudo-group action can be taken to be the right multiplication

$$R_g : G \to G, \qquad h \mapsto hg^{-1}, \qquad g \in G, \tag{2.3.4}$$

for example. Let $\{\mu^1, \ldots, \mu^r\}$ be an invariant coframe under the action (2.3.4). A diffeomorphism $\phi : G \to G$ is a right translation if and only if

$$\phi^*(\mu^i) = \mu^i, \qquad i = 1, \ldots, r. \tag{2.3.5}$$

The first order system of equations (2.3.5) completely determines the elements of the Lie group G and are the corresponding determining equations.

Remark 2.32. Not every pseudo-group is a Lie pseudo-group. Example 2.25 is an instance of a pseudo-group which is not a Lie pseudo-group.

The geometric symbol (cf. Appendix A) of the defining system (2.3.3) immediately determines if the Lie pseudo-group is finite-dimensional or infinite-dimensional.

Proposition 2.33. A Lie pseudo-group is finite-dimensional if the geometric symbol of its determining system is of dimension zero and infinite-dimensional otherwise.

2.4 Infinitesimal Generators

Let J^nTM, $0 \leq n \leq \infty$, denote the n-th order jet bundle of the tangent bundle TM and $\mathrm{Lie}(\mathcal{D}^{(\infty)})$ the Lie algebroid of the diffeomorphism jet groupoid $\mathcal{D}^{(\infty)}$. We define the n-th order *lifting map*

$$\boldsymbol{\lambda}^{(n)} : J^nTM \to \mathrm{Lie}(\mathcal{D}^{(n)}), \qquad j_n\mathbf{v} \mapsto \mathbf{V}^{(n)} = \boldsymbol{\lambda}^{(n)}(j_n\mathbf{v}),$$

where $\boldsymbol{\lambda}^{(n)}(j_n\mathbf{v})$ denotes the infinitesimal generator of the prolonged left action on $\mathcal{D}^{(n)}$ of the local diffeomorphism $\exp(tj_n\mathbf{v}) \in \mathcal{D}^{(n)}$. That is, at each $j_n\phi|_z \in \mathcal{D}^{(n)}$ such that $j_n\mathbf{v}$ is defined at $\phi(z) = \boldsymbol{\tau}^{(n)}(j_n\phi|_z)$

$$\boldsymbol{\lambda}^{(n)}(j_n\mathbf{v}|_{\phi(z)})|_{j_n\phi|_z} = \left.\frac{d}{dt}\right|_{t=0} L_{\exp(tj_n\mathbf{v})}(j_n\phi|_z) = \left.\frac{d}{dt}\right|_{t=0} j_n(\exp(t\mathbf{v}) \circ \phi)|_z.$$

The lifting map has an inverse

$$(\boldsymbol{\lambda}^{(n)})^{-1} : \mathrm{Lie}(\mathcal{D}^{(n)}) \to J^nTM, \qquad \mathbf{V}^{(n)}|_{j_n\phi|_z} \mapsto j_n(d\boldsymbol{\tau}^{(n)}(\mathbf{V}^{(n)}))|_{\phi(z)},$$

where $\boldsymbol{\tau}^{(n)} : \mathcal{D}^{(n)} \to M$ is the usual target projection.

Proposition 2.34. The lifting map $\boldsymbol{\lambda}^{(n)} : J^nTM \to \mathrm{Lie}(\mathcal{D}^{(n)})$ is an isomorphism of vector bundles.

Proposition 2.35. In local coordinates, the lift of a vector field jet

$$j_\infty\mathbf{v} = j_\infty\left(\sum_{a=1}^{m} \zeta^a(z)\frac{\partial}{\partial z^a}\right)$$

in $J^\infty TM$ is

$$\mathbf{V}^{(\infty)} = \sum_{a=1}^{m} \sum_{\#A \geq 0} \mathbb{D}_z^A \zeta^a(Z) \frac{\partial}{\partial Z_A^a}. \tag{2.4.1}$$

Proof. At the identity jet $\mathbb{1}^\infty$

$$\mathbf{V}^{(\infty)}|_{\mathbb{1}^{(\infty)}} := \frac{d}{dt}\Big|_{t=0} j_n(\exp(t\mathbf{v})z) = \sum_{a=1}^{m} \sum_{\#A \geq 0} \mathbb{D}_z^A \zeta^a(z) \frac{\partial}{\partial Z_A^a}.$$

The right invariance of $\mathbf{V}^{(\infty)}$ implies that

$$\mathbf{V}^{(\infty)}|_{j_\infty \phi|_z} = \sum_{a=1}^{m} \sum_{\#A \geq 0} \mathbb{D}_z^A \zeta^a(Z) \frac{\partial}{\partial Z_A^a},$$

where $Z = \boldsymbol{\tau}^{(\infty)}(j_\infty \phi|_z)$. □

By expanding the right-hand-side of (2.4.1), the lifted vector field can be written as the linear combination of the Lie algebroid vector field basis elements

$$V_a^A \in \mathrm{Lie}(\mathcal{D}^{(\infty)}) \tag{2.4.2}$$

so that

$$\boldsymbol{\lambda}^{(\infty)}(j_\infty \mathbf{v}) = \sum_{a=1}^{m} \sum_{\#A \geq 0} \zeta_A^a(Z) V_a^A.$$

Example 2.36. Let $M = \mathbb{R}$, the lift of a vector field $j_\infty(\xi(x)\partial_x)$ is

$$\begin{aligned}\mathbf{V}^{(\infty)} =& \xi(X)\frac{\partial}{\partial X} + X_x \xi_X(X)\frac{\partial}{\partial X_x} + (X_x^2 \xi_{XX}(X) + X_{xx}\xi_X(X))\frac{\partial}{\partial X_{xx}} \\ &+ (X_x^3 \xi_{XXX}(X) + 3X_x X_{xx} \xi_{XX}(X) + X_{xxx}\xi_X(X))\frac{\partial}{\partial X_{xxx}} + \cdots ,\end{aligned}$$

and a basis of the Lie algebroid is

$$V^0 = \frac{\partial}{\partial X}, \qquad V^1 = X_x \frac{\partial}{\partial X_x} + X_{xx} \frac{\partial}{\partial X_{xx}} + X_{xxx} \frac{\partial}{\partial X_{xxx}} + \ldots,$$
$$V^2 = X_x^2 \frac{\partial}{\partial X_{xx}} + 3X_x X_{xx} \frac{\partial}{\partial X_{xxx}} + \cdots, \qquad V^3 = X_x^3 \frac{\partial}{\partial X_{xxx}} + \cdots,$$
$$\ldots.$$

Definition 2.37. Let $\mathcal{G}$ be a Lie pseudo-group acting on M. The *algebra of infinitesimal generators* of $\mathcal{G}$ is the Lie algebra $\mathfrak{g} \subset \mathcal{X}(M)$ of local vector fields $\mathbf{v}$ on M such that the local diffeomorphisms $\exp(t\mathbf{v})$ generated by $\mathbf{v} \in \mathfrak{g}$ belong to the Lie pseudo-group $\mathcal{G}$.

A vector field $\mathbf{v} \in \mathcal{X}(M)$ belongs to the Lie algebra $\mathfrak{g}$ of infinitesimal generators of $\mathcal{G}$ if and only if $\mathbf{V}^{(n)} = \boldsymbol{\lambda}^{(n)}(j_n\mathbf{v}) \in \mathcal{X}(\mathcal{D}^{(n)})$ is tangent to $\mathcal{G}^{(n)}$ at the identity jet, i.e.,

$$(\mathbf{V}^{(n)}[F^{(n)}(z, Z^{(n)})])|_{\mathbb{1}^{(n)}} = 0, \tag{2.4.3}$$

where $F^{(n)}(z, Z^{(n)}) = 0$ is the determining system of $\mathcal{G}^{(n)}$. The system of equations (2.4.3) is called the *n-th order infinitesimal determining system* of $\mathcal{G}$. In local coordinates it takes the form of a linear system of partial differential equations in the unknown vector field coefficients $\zeta^1, \ldots, \zeta^m$:

$$L^{(n)}(z, \zeta^{(n)}) = 0. \tag{2.4.4}$$

If $\mathcal{G}$ is the symmetry pseudo-group of a system of differential equations, then the system of equations (2.4.4) is the usual infinitesimal symmetry determining system, [8, 9, 81, 82, 92].

As previously mentioned, Definition 2.27 implies the local solvability and formal integrability of the defining equations (2.3.3). At the infinitesimal level, Definition 2.27 also provides the formal integrability of the infinitesimal determining equations (2.4.4) but it is still an open problem to establish if the definition also implies local solvability, [89]. To deal with this issue, we assume our Lie pseudo-groups to be *tame*.

Definition 2.38. A Lie pseudo-group $\mathcal{G} \subset \mathcal{D}(M)$ is *tame* if for all $z \in M$ and all $n \geq n^\star$, each $\mathbf{V} \in T(\mathcal{G}^{(n)}|_z)|_{\mathbb{1}_z^{(n)}}$ is the lift of some $\mathbf{v} \in \mathfrak{g}$, that is, $\boldsymbol{\lambda}^{(n)}(j_n\mathbf{v})|_{\mathbb{1}_z^{(n)}} = \mathbf{V}$.

Proposition 2.39. A Lie pseudo-group $\mathcal{G}$ is tame at order n if and only if the n-th order infinitesimal determining equations for $\mathcal{G}$ are locally solvable.

Let

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z)\frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{2.4.5}$$

be a local infinitesimal generator of the diffemorphism pseudo-group $\mathcal{D}(M)$ acting on $J^0(M,p)$. The infinitesimal generator of the prolonged action of $\mathcal{D}(M)$ on $J^n(M,p)$ is called the *n-th prolongation* of (2.4.5). In local coordinates

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_{k=\#J=0}^{n} \phi_\alpha^J(x,u^{(k)})\frac{\partial}{\partial u_J^\alpha} \quad \in \quad \mathcal{X}(J^n(M,p)),$$

where the coefficients ϕ_α^J are given by the well-known recursive formula, [81, 82],

$$\phi_\alpha^{J,j}(u^{(n)},\zeta^{(n)}) = D_{x^j}\phi_\alpha^J - \sum_{i=1}^{p} D_{x^j}\xi^i \cdot u_{J,i}^\alpha. \tag{2.4.6}$$

The prolonged vector field coefficients are well-determined linear functions in the vector field jets, i.e., the partial derivatives ζ_A^a of the vector field coefficients, with coefficients depending polynomially on the jet coordinates u_J^α. Thus there is a well-defined map

$$\mathbf{p}_{z^{(n)}}^{(n)} : J^nTM|_z \to TJ^n(M,p)|_{z^{(n)}}, \qquad \mathbf{p}^{(n)}(j_n\mathbf{v}|_z) = \mathbf{v}^{(n)}|_z. \tag{2.4.7}$$

For a Lie pseudo-group $\mathcal{G}$, the prolonged infinitesimal generators of the pseudo-group action are obtained by restricting (2.4.7) to the fiber $J^n\mathfrak{g}|_z$

$$\mathfrak{g}^{(n)}|_{z^{(n)}} = \mathbf{p}^{(n)}(J^n\mathfrak{g}|_z).$$

2.5 Freeness

In this section we introduce the notion of *freeness* for Lie pseudo-group actions. This concept will be particularly important in Chapter 5 where the theory of equivariant moving frames is exposed. As we will see, to construct a complete equivariant moving frame we must require the pseudo-group action to be free.

Definition 2.40. The *n-th order isotropy jet subgroup of the point $z \in M$* is

$$\mathcal{G}_z^{(n)} = \left\{g^{(n)} \in \mathcal{G}^{(n)} : \boldsymbol{\tau}^{(n)}(g^{(n)}) = \boldsymbol{\sigma}^{(n)}(g^{(n)}) = z\right\} \subset \mathcal{G}^{(n)}|_z$$

For each $n < \infty$, the n-th isotropy subgroup of a point $z \in M$ is a finite-dimensional Lie group. In the limit, $\mathcal{G}_z^{(\infty)}$ has the structure of a pro-Lie group, [43, 102].

Definition 2.41. The *n-th order isotropy jet subgroup of $z^{(n)} \in J^n(M,p)$* is the closed Lie subgroup

$$\mathcal{G}_{z^{(n)}}^{(n)} = \left\{g^{(n)} \in \mathcal{G}_z^{(n)} : g^{(n)} \cdot z^{(n)} = z^{(n)}\right\} \subset \mathcal{G}_z^{(n)}.$$

Definition 2.42. A pseudo-group $\mathcal{G}$ acts *freely* at $z^{(n)} \in J^n(M,p)$ if $\mathcal{G}_{z^{(n)}}^{(n)} = \{\mathbb{1}_z^{(n)}\}$ and *locally freely* if $\mathcal{G}_{z^{(n)}}^{(n)}$ is a discrete subgroup of $\mathcal{G}_z^{(n)}$. A pseudo-group $\mathcal{G}$ is said to act (*locally*) *freely at order n* if it acts (locally) freely on an open subset $\mathcal{V}^n \subset J^n(M,p)$, called the set of regular n-jets.

Let

$$\mathcal{O}_{z^{(n)}}^{(n)} = \left\{g^{(n)} \cdot z^{(n)} : g^{(n)} \in \mathcal{G}^{(n)}|_z, z = \pi_0^n(z^{(n)})\right\} \subset J^n(M,p)$$

denote the prolonged pseudo-group orbit passing through the submanifold jet $z^{(n)} \in J^n(M,p)$. A theorem of Sussmann, [104] and the tameness condition implies that the pseudo-group orbits are immersed submanifolds.

Proposition 2.43. A pseudo-group $\mathcal{G}$ acts locally freely on the subset

$$\left\{ z^{(n)} \in J^n(M,p) : \dim\, \mathcal{O}^{(n)}_{z^{(n)}} = r_n = \dim\, \mathcal{G}^{(n)} \right\}$$

consisting of the jets whose orbit dimension equals the fiber dimension of the n-th order jet groupoid $\mathcal{G}^{(n)} \to M$.

Theorem 2.44. Let $\mathcal{G}$ be a regular pseudo-group acting on an m-dimensional manifold M. If $\mathcal{G}$ acts locally freely at $z^{(n)} \in J^n(M,p)$ for some $n > 0$, then it acts locally freely at any $z^{(k)} \in J^k(M,p)$ with $\pi^k_n(z^{(k)}) = z^{(n)}$, for $k \geq n$.

Proposition 2.45. A pseudo-group acts locally freely in a neighborhood of $z^{(n)}$ if and only if the prolongation map $\mathbf{p}^{(n)} : J^n\mathfrak{g}|_z \to \mathfrak{g}^{(n)}|_{z^{(n)}}$ is a monomorphism.

In applications, Proposition 2.45 is used to verify freeness. In local coordinates, the prolongation map $\mathbf{p}^{(n)}$ corresponds to the usual n-th order *Lie matrix*, [39, 84, 85]. The rank of the n-th order Lie matrix corresponds to the dimension of the orbit passing though $z^{(n)}$. Let $(\zeta^1, \ldots, \zeta^{r_n})$ be local coordinates of $J^n\mathfrak{g}|_z$, then the n-th order Lie matrix $L^{(n)}|_{z^{(n)}}$ at $z^{(n)}$ is the $q\binom{p+n}{n} \times r_n$ matrix in which the coefficients of the i-th column are the coefficients of the vector field $\mathbf{p}^{(n)}(e_i)$, where e_i is the vector of dimension r_n with a one in the i-th entry and zero elsewhere.

Example 2.46. Consider the Lie pseudo-group

$$X = f(x), \qquad Y = f'(x)y + g(x), \qquad U = u + \frac{f''(x)y + g'(x)}{f'(x)}, \qquad (2.5.1)$$

where $f \in \mathcal{D}(\mathbb{R})$ and $g \in C^\infty(\mathbb{R})$. This Lie pseudo-group was introduced in [88]. The determining system of this Lie pseudo-group is

$$X_y = X_u = 0, \qquad Y_u = 0, \qquad Y_x = (U - u)X_x, \qquad U_u = 1. \tag{2.5.2}$$

Linearizing (2.5.2) at the identity jet we obtain the infinitesimal determining equations for the infinitesimal generator $\mathbf{v} = \xi(x,y,u)\partial_x + \eta(x,y,u)\partial_y + \phi(x,y,u)\partial_u$:

$$\xi_x = \eta_y, \qquad \xi_y = \xi_u = \eta_u = \phi_u = 0, \qquad \eta_x = \phi. \tag{2.5.3}$$

From (2.5.3) it follows that $r_1 = \dim J^1\mathfrak{g} = 6$, while $\dim J^1(\mathbb{R}^3, 2) = 5$. Thus the action cannot be free at order 1. At order two, $r_2 = \dim J^2\mathfrak{g} = \dim J^2(\mathbb{R}^3, 2) = 8$. The second order Lie matrix

$$L^{(2)}|_{(x,y,u^{(2)})} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -u_y & 1 & 0 & 0 & 0 & -u_x \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -u_y \\ 0 & 0 & -2u_{xy} & -u_y & -u_x & 1 & 0 & -2u_{xx} \\ 0 & 0 & -u_{yy} & 0 & -u_y & 0 & 1 & -2u_{xy} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2u_{yy} \end{pmatrix}$$

is seen to be of full rank on the sets $\mathcal{V}^2_+ = J^2(\mathbb{R}^3, 2) \cap \{u_{yy} > 0\}$ and $\mathcal{V}^2_- = J^2(\mathbb{R}^3, 2) \cap \{u_{yy} < 0\}$. Thus we conclude that the pseudo-group action (2.5.1) is locally free on the sets $\mathcal{V}^n_+$ and $\mathcal{V}^n_-$, with $n \geq 2$.

Chapter 3

Structure Theory of Lie Pseudo-Groups

The obstruction preventing an immediate generalization of the finite-dimensional structure theory to infinite-dimensional Lie pseudo-groups is the lack of an abstract object to represent the pseudo-group itself. In our current state of knowledge, Lie pseudo-groups are inextricably bound to the manifold on which they act. The appropriate Maurer–Cartan forms thus must be suitably invariant differential forms living on the manifold or, better, some bundle connected with it. For intransitive Lie pseudo-groups a further difficulty is the possible dependence of the structure coefficients on essential invariants[1].

3.1 Maurer–Cartan Structure Equations

The identification of $\mathcal{D}^{(\infty)} \subset J^\infty(M, M)$ as a jet bundle over a Cartesian product bundle manifold induces a splitting of the cotangent bundle $T^*\mathcal{D}^{(\infty)}$ into horizontal and vertical (group) components. This induces a splitting of

[1]Most results of Sections 3.2–3.6 have been published in [107].

the differential on $\mathcal{D}^{(\infty)}$, which we denote by

$$d = d_M + d_G.$$

The group differential d_G corresponds to taking the differential of the pseudo-group parameters and plays a role equivalent to the group differential d_G in the finite-dimensional theory, [57].

In terms of the local coordinates $g^{(\infty)} = (z, Z^{(\infty)})$, the horizontal subbundle of $T^*\mathcal{D}^{(\infty)}$ is spanned by the one-forms

$$dz^a = d_M z^a, \qquad a = 1, \ldots, m,$$

while the vertical subbundle is spanned by the contact forms

$$\Upsilon^a_A = dZ^a_A - \sum_{b=1}^{m} Z^a_{A,b} dz^b, \qquad a = 1, \ldots, m, \qquad \#A \geq 0. \tag{3.1.1}$$

In the following we call the one-forms (3.1.1) *group forms* to distinguish them from the contact forms on the submanifold jet bundle $J^{(\infty)}(M, p)$. For a differential function $F : \mathcal{D}^{(\infty)} \to \mathbb{R}$, its horizontal and vertical differentials are given by

$$d_M F = \sum_{a=1}^{m} (\mathbb{D}_{z^a} F) dz^a, \qquad d_G F = \sum_{a=1}^{m} \sum_{\#A \geq 0} \frac{\partial F}{\partial Z^a_A} \Upsilon^a_A.$$

Definition 3.1. A differential form μ on $\mathcal{D}^{(\infty)}$ is *right-invariant* if and only if it satisfies $(R_\psi)^*\mu = \mu$ for every local diffeomorphism $\psi \in \mathcal{D}$ where the pull-back is defined.

Since the splitting of differential forms on $\mathcal{D}^{(\infty)}$ into horizontal and group components is invariant under the action of $\mathcal{D}$ on $\mathcal{D}^{(\infty)}$, if μ is a right-invariant differential form then so are $d_M\mu$ and $d_G\mu$. Since the target coordinate functions $Z^a : \mathcal{D}^{(0)} \to \mathbb{R}$ are right-invariant, their horizontal and group differentials are invariant one-forms:

$$\sigma^a = d_M Z^a = \sum_{b=1}^{m} Z^a_b dz^b, \qquad \mu^a = d_G Z^a = \Upsilon^a = dZ^a - \sum_{b=1}^{m} Z^a_b dz^b,$$

$a = 1, \ldots, m$. The one-forms $\sigma^1, \ldots, \sigma^m$ form an invariant horizontal coframe on $\mathcal{D}^{(\infty)}$. A complete invariant coframe on $\mathcal{D}^{(\infty)}$ is obtained as follows. Let $\mathbb{D}_{Z^1}, \ldots, \mathbb{D}_{Z^m}$ be total differential operators, dual to the horizontal forms $\sigma^1, \ldots, \sigma^m$, defined by the equality

$$d_M F = \sum_{a=1}^{m} (\mathbb{D}_{Z^a} F)\sigma^a,$$

for any differential function $F : \mathcal{D}^{(\infty)} \to \mathbb{R}$. More explicitly,

$$\mathbb{D}_{Z^a} = \sum_{b=1}^{m} w^b_a \mathbb{D}_{z^b}, \qquad \text{where} \qquad (w^b_a(z, Z^{(1)})) = \left(\frac{\partial Z^b}{\partial z^a}\right)^{-1}$$

denotes the inverse of the $m \times m$ Jacobian matrix $\nabla Z = (\partial Z^b / \partial z^a)$. The invariance of the one-forms $\sigma^1, \ldots, \sigma^m$ implies that the Lie derivatives of a right-invariant differential form with respect to $\mathbb{D}_{Z^1}, \ldots, \mathbb{D}_{Z^m}$ are also right-invariant. Hence, taking successive Lie derivatives of the invariant one-forms μ^a gives the higher-order invariant contact forms[2]

$$\mu^a_A = \mu^a_{Z^A} = \mathbb{D}^A_Z \mu^a, \qquad \text{where} \qquad \mathbb{D}^A_Z = (\mathbb{D}_{Z^1})^{a^1} \cdots (\mathbb{D}_{Z^m})^{a^m}, \tag{3.1.2}$$

$a = 1, \ldots, m$, $\#A \geq 0$.

Definition 3.2. The right-invariant one-forms $\mu^{(\infty)} = (\ldots \mu^a_A \ldots)$ are referred to as the *Maurer–Cartan forms* for the diffeomorphism pseudo-group $\mathcal{D}(M)$.

[2]The order in which we take Lie derivatives does not matter since the differential operators $\mathbb{D}_{Z^1}, \ldots, \mathbb{D}_{Z^m}$ commute.

The invariant differential forms $\mu^{(\infty)}$ play the same role as the standard Maurer–Cartan forms in the finite-dimensional Lie group theory, [57]. The differential forms σ , $\mu^{(\infty)}$ form a right-invariant coframe on $\mathcal{D}^{(\infty)}$.

By induction, one can verify that the Maurer–Cartan forms (3.1.2) form a dual basis to the Lie algebroid basis (2.4.2) of the diffeomorphism pseudo-group:

$$\langle V_a^A ; \mu_B^b \rangle = \delta_B^A \cdot \delta_a^b$$

where δ_B^A and δ_a^b are Kronecker deltas.

To write the structure equations of the diffeomorphism pseudo-group in a compact form we use the Taylor series notation introduced in [88]. Let $Z[\![h]\!]$ denote the vector valued Taylor series, depending on $h = (h^1, \ldots, h^m)$, of a diffeomorphism $Z = \phi(z + h)$ at the source point $z \in M$, with components

$$Z^a[\![h]\!] = \sum_{\#A \geq 0} \frac{1}{A!} Z_A^a h^A, \qquad a = 1, \ldots, m.$$

Similarly, we use the notation $\mu[\![H]\!]$ to denote the right-invariant contact form-valued power series with components

$$\mu^a[\![H]\!] = \sum_{\#A \geq 0} \frac{1}{A!} \mu_A^a H^A, \qquad a = 1, \ldots, m. \tag{3.1.3}$$

Theorem 3.3. The invariant coframe $\sigma, \mu^{(\infty)}$ satisfies the structure equations

$$\begin{aligned} d\mu[\![H]\!] &= \nabla_H \mu[\![H]\!] \wedge (\mu[\![H]\!] - dZ[\![0]\!]), \\ d\sigma &= \nabla_H \mu[\![0]\!] \wedge \sigma, \end{aligned} \tag{3.1.4}$$

where

$$\nabla_H \mu[\![H]\!] = \left(\frac{\partial \mu^a}{\partial H^b} [\![H]\!] \right)$$

denotes the $m \times m$ Jacobian matrix power series obtained by differentiating $\mu[\![H]\!]$ with respect to H.

Recall from Section 2.2.2 that the Lie algebroid $\mathrm{Lie}(\mathcal{D}^{(\infty)})$ of the diffeomorphism pseudo-group is the set of right-invariant vertical vector fields on $T\mathcal{D}^{(\infty)}$. Since the Maurer–Cartan forms (3.1.2) are dual to the Lie algebroid basis (2.4.2), the infinitesimal structure equations of the diffeomorphism pseudo-group $\mathcal{D}^{(\infty)}$ are obtained by restricting the structure equations (3.1.4) to the vertical subbundle of $T^*\mathcal{D}^{(\infty)}$. On each target fiber $(\boldsymbol{\tau}^{(\infty)})^{-1}(Z)$, the target coordinate functions $Z^1, \ldots, Z^m$ are constants and

$$dZ[\![0]\!] = 0. \tag{3.1.5}$$

Thus the structure equations of the Maurer–Cartan forms reduce to

$$d\mu[\![H]\!] = \nabla_H \mu[\![H]\!] \wedge \mu[\![H]\!]$$

on a target fiber. On the other hand, the equality (3.1.5) implies

$$0 = dZ[\![0]\!] = \mu[\![0]\!] + \sigma,$$

which means that

$$\mu[\![0]\!] = -\sigma, \tag{3.1.6}$$

when restricted to a target fiber. It follows that the structure equations for the one-forms $\sigma^1, \ldots, \sigma^m$ reduce to those of the zero order Maurer–Cartan forms

$$d\mu[\![0]\!] = \nabla_H \mu[\![0]\!] \wedge \mu[\![0]\!].$$

Theorem 3.4. The *Maurer–Cartan structure equations* for the diffeomorphism pseudo-group $\mathcal{D}^{(\infty)}$ are

$$d\mu[\![H]\!] = \nabla_H \mu[\![H]\!] \wedge \mu[\![H]\!]. \tag{3.1.7}$$

Remark 3.5. Another motivation behind the need to restrict the Maurer–Cartan forms to a target fiber can be readily understood in the context of finite-dimensional Lie group actions. In this situation, $\boldsymbol{\tau}^{(\infty)} : \mathcal{G}^{(\infty)} \to M$ will typically be a principal G bundle, and, consequently, the independent Maurer–Cartan forms on $\mathcal{G}^{(\infty)}$ and their structure equations, when restricted to a target fiber $(\boldsymbol{\tau}^{(\infty)})^{-1}(Z) \cong G$ coincide with the usual Maurer–Cartan forms and their usual structure equations. However, it is worth pointing out that the basis of $\mathfrak{g}^\star$ prescribed by the independent restricted invariant contact forms μ_A^b may vary from fiber to fiber as the target point Z ranges over M. Consequently, the structure coefficients in the pseudo-group structure equations (3.1.7) may very well be Z–dependent. It is a fact that, when $\mathcal{G}$ is of finite type and so represents the action of a finite-dimensional Lie group G on M, the resulting variable structure coefficients represent the *same* Lie algebra $\mathfrak{g}$ and so are all similar, modulo a Z–dependent change of basis, to the usual *constant structure coefficients* associated with a fixed basis of $\mathfrak{g}^\star$.

Example 3.6. The structure equations of the diffeomorphism pseudo-group $\mathcal{D}(\mathbb{R})$ are

$$\left(\sum_{k=0}^{\infty} d\mu_k \frac{H^k}{k!}\right) = \left(\sum_{k=1}^{\infty} \mu_k \frac{H^{k-1}}{(k-1)!}\right) \wedge \left(\sum_{k=0}^{\infty} \mu_k \frac{H^k}{k!}\right).$$

The individual components are

$$\begin{aligned} d\mu_n &= \sum_{k=0}^{n} \binom{n}{k} \mu_{k+1} \wedge \mu_{n-k} \\ &= \sum_{k=0}^{[(n+1)/2]} \frac{n-2k+1}{n+1} \binom{n+1}{k} \mu_{n+1-k} \wedge \mu_k, \qquad n \geq 0, \end{aligned} \tag{3.1.8}$$

thereby recovering the structure equations found by Cartan, [23, eq. (48)]. Since

$$\mu_0 = -\sigma = -X_x dx,$$

on a target fiber, the vector fields dual to the Maurer–Cartan forms μ_k, $k \geq 0$, are

$$
\begin{aligned}
&\widetilde{V^0} = -\frac{1}{X_x}\left(\frac{\partial}{\partial x} + X_{xx}\frac{\partial}{\partial X_x} + X_{xxx}\frac{\partial}{\partial X_{xx}} + \cdots\right) = -\frac{1}{X_x}\mathbb{D}_x + \frac{\partial}{\partial X},\\
&V^1 = X_x\frac{\partial}{\partial X_x} + X_{xx}\frac{\partial}{\partial X_{xx}} + X_{xxx}\frac{\partial}{\partial X_{xxx}} + \ldots,\\
&V^2 = X_x^2\frac{\partial}{\partial X_{xx}} + 3X_xX_{xx}\frac{\partial}{\partial X_{xxx}} + \cdots, \qquad V^3 = X_x^3\frac{\partial}{\partial X_{xxx}} + \cdots,\\
&\ldots.
\end{aligned}
\tag{3.1.9}
$$

It is straight forward to verify that commutation relations for the vector fields (3.1.9) are dual to the structure equations (3.1.8). Since the restriction to a target fiber does not affect the Maurer–Cartan forms $\mu_1, \mu_2, \ldots$, and the dual vector fields $V^1, V^2, \ldots$, given in (3.1.9), the only commutation relations that need to be verified are those involving $\widetilde{V^0}$. From the Maurer–Cartan structure equations (3.1.8) we need to verify the equalities

$$
[V^1, \widetilde{V^0}] = -\widetilde{V^0}, \qquad [V^{n+1}, \widetilde{V^0}] = -V^n, \qquad n \geq 1.
$$

By direct calculation we obtain

$$
\begin{aligned}
[V^1, \widetilde{V^0}] =& \left[\sum_{k=1}^{\infty} X_k\frac{\partial}{\partial X_k}, -\frac{1}{X_x}\mathbb{D}_x + \frac{\partial}{\partial X}\right]\\
=&\frac{1}{X_x}\mathbb{D}_x - \frac{\partial}{\partial X} - \frac{1}{X_x}\left(\mathbb{D}_x - \frac{\partial}{\partial x}\right) + \frac{\partial}{\partial X} + \frac{1}{X_x}\left(\mathbb{D}_x - \frac{\partial}{\partial x}\right) - \frac{\partial}{\partial X}\\
=& -\widetilde{V^0}.
\end{aligned}
$$

For the remaining commutation relations we use the fact that

$$
\left[V^{n+1}, \frac{1}{X_x}\mathbb{D}_x\right] = V^n, \qquad \left[V^{n+1}, \frac{\partial}{\partial X}\right] = 0, \qquad n \geq 1,
$$

which follows from the structure equations of the invariant coframe $\mu^{(\infty)}$, σ, c.f. [88, eq. 4.14]. Then for $n \geq 1$ we obtain

$$[V^{n+1}, \widetilde{V^0}] = -\left[V^{n+1}, \frac{1}{X_x}\mathbb{D}_x\right] = -V^n.$$

Remark 3.7. In equation (3.1.6) it is not too surprising that the restriction of the zero order Maurer–Cartan forms to a target fiber are horizontal forms as this reflects the fact that under the right action (2.3.2) a pseudo-group element ψ maps the source of a pseudo-group jet $j_n\phi|_z$ to $\psi(z)$. Instead of using the right action we could also develop the structure theory of Lie pseudo-groups using the left action (2.3.2). Since the source of a pseudo-group jet is invariant under the left action, the Maurer–Cartan structure equations will involve invariant differential forms which are linear combinations of the pseudo-group jet differential forms dZ^a_A. As an example we consider the pseudo-group $\mathcal{D}(\mathbb{R})$. A left invariant coframe is given by the differential forms

$$\psi = dx, \qquad \lambda_n = \mathbb{D}_x^n \left(\frac{dX - X_x dx}{X_x}\right), \qquad n \geq 0,$$

whose structure equations are

$$d\psi = 0, \qquad d\lambda_n = \psi \wedge \lambda_{n+1} - \sum_{k=0}^{n} \binom{n}{k} \lambda_{k+1} \wedge \lambda_{n-k}, \qquad n \geq 0. \tag{3.1.10}$$

On a source fiber $(\boldsymbol{\sigma}^{(\infty)})^{-1}(x)$ we have $dx = \psi = 0$ and the restricted left-invariant Maurer–Cartan forms λ_n are linear combinations of the dX_k. On a source fiber the structure equations (3.1.10) reduce to

$$d\lambda_n = -\sum_{k=0}^{n} \binom{n}{k} \lambda_{k+1} \wedge \lambda_{n-k}, \qquad n \geq 0. \tag{3.1.11}$$

Note that the structure equations (3.1.11) are isomorphic to the structure equations (3.1.8) for the right-invariant Maurer–Cartan forms.

Let $\mathcal{G} \subsetneq \mathcal{D}(M)$ be a Lie pseudo-group, then the restriction of the Maurer–Cartan forms (3.1.2) to $\mathcal{G}$ are no longer linearly independent. Remarkably, the linear dependencies can be determined without knowing the explicit expressions for the Maurer–Cartan forms. In [25, 88] it is shown that the linear dependencies follow from the infinitesimal determining equations (2.4.4).

Let $\mathcal{Z}^{(n)}$ denote the dual bundle to the vector field jet bundle J^nTM, and $\mathcal{Z}^{(\infty)}$ the direct limit.

Definition 3.8. The *lift* of a section ζ of $\mathcal{Z}^{(\infty)}$ is the right-invariant differential form $\boldsymbol{\lambda}(\zeta)$ on $\mathcal{D}^{(\infty)}$ which vanishes on all total vector fields and satisfties

$$\langle \boldsymbol{\lambda}(\zeta); \boldsymbol{\lambda}(j_\infty \mathbf{v})\rangle|_{g^{(\infty)}} = \langle \zeta; j_\infty \mathbf{v}\rangle|_Z$$

whenever $Z = \boldsymbol{\tau}^{(\infty)}(g^{(\infty)})$ and $g^{(\infty)} \in \mathcal{D}^{(\infty)}$.

In local coordinates we have the important equality

$$\mu_A^a = \boldsymbol{\lambda}(\zeta_A^a). \tag{3.1.12}$$

More generally, any linear function of the vector field jets $L(z, \zeta^{(n)})$ can be viewed as a section of $\mathcal{Z}^{(n)}$, whose lift

$$\boldsymbol{\lambda}\left[L(z, \zeta^{(n)})\right] = L(Z, \mu^{(n)})$$

is obtained by replacing the source variables z^a by their target counterparts Z^a and the vector field jet coordinates ζ_A^a by the Maurer–Cartan forms μ_A^a.

Theorem 3.9. The linear system

$$L^{(n)}(Z, \mu^{(n)}) = 0, \tag{3.1.13}$$

obtained by lifting the infinitesimal determining equations (2.4.4) serves to define the complete set of linear dependencies among the right-invariant Maurer–Cartan forms $\mu^{(n)}$.

The equations (3.1.13) are called the n-th order *lifted infinitesimal determining equations.*

Theorem 3.10. The Maurer–Cartan structure equations of a Lie pseudo-group are obtained by restricting the Maurer–Cartan structure equations of the diffeomorphism pseudo-group (3.1.7) to the kernel of the (formally integrable) lifted infinitesimal determining system (3.1.13).

We will use the notation

$$\left(d\mu[\![H]\!] = \nabla_H \mu[\![H]\!] \wedge \mu[\![H]\!]\right)\big|_{L^{(\infty)}(Z,\mu^{(\infty)})=0} \tag{3.1.14}$$

to denote the Maurer–Cartan structure equations of a Lie pseudo-group $\mathcal{G}$. It will also be convenient to write $\mu[\![H]\!]|_{\mathcal{G}}$ to denote the restriction of $\mu[\![H]\!]$ to the kernel of the lifted infinitesimal determining equations (3.1.13). We will use the notation "dim $\mu^{(n)}[\![H]\!]|_{\mathcal{G}}$" to denote the number of linearly independent Maurer–Cartan forms of order less or equal to n. Then

$$\dim\, \mu^{(n)}[\![H]\!]|_{\mathcal{G}} = \dim\, \mathcal{G}^{(n)} = r_n.$$

Example 3.11. In this example we compute the Maurer–Cartan structure equations of the symmetry group of the heat equation

$$u_t - u_{xx} = 0. \tag{3.1.15}$$

Let $\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u$ be an infinitesimal symmetry generator. Using Lie's standard method, [8,81,92], we derive the infinitesimal determining equations

$$\begin{gathered} \tau_x = 0, \qquad \tau_u = 0, \qquad \xi_u = 0, \qquad \tau_t - 2\xi_x = 0, \\ 2\phi_{xu} + \xi_t = 0, \qquad \phi_{uu} = 0, \qquad \phi_{xx} - \phi_t = 0. \end{gathered} \tag{3.1.16}$$

The lift of the infinitesimal determining equations (3.1.16) gives the linear relations

$$\mu^t_X = 0, \qquad \mu^t_U = 0, \qquad \mu^x_U = 0, \qquad \mu^t_T - 2\mu^x_X = 0,$$
$$2\mu^u_{XU} + \mu^x_T = 0, \qquad \mu^u_{UU} = 0, \qquad \mu^u_{XX} - \mu^u_T = 0,$$

among the low order Maurer–Cartan forms. Solving this system by Gaussian elimination we find that

$$\mu^x, \quad \mu^x_T, \quad \mu^t, \quad \mu^t_T, \quad \mu^t_{TT}, \quad \mu^u_U, \quad \mu^u_{X^k}, \quad k \geq 0, \tag{3.1.17}$$

is a basis of Maurer–Cartan forms. In this basis, the Maurer–Cartan valued Taylor series (3.1.3) is given by

$$\begin{pmatrix} \mu^t[\![H]\!] \\ \mu^x[\![H]\!] \\ \mu^u[\![H]\!] \end{pmatrix} = \begin{pmatrix} \mu^t + \mu^t_T H_t + \mu^t_{TT}\frac{H_t^2}{2} \\ \mu^x + \mu^t_T\frac{H_x}{2} + \mu^x_T H_t + \mu^t_{TT}\frac{H_t H_x}{2} \\ (\mu^u_U - \mu^x_T\frac{H_x}{2} - \mu^t_{TT}\frac{H_t}{4} - \mu^t_{TT}\frac{H_x^2}{8})H_u + \sum_{i,j\geq 0}\mu^u_{X^{i+2j}}\frac{H_t^j H_x^i}{j!i!} \end{pmatrix}.$$

Substituting this last expression into the Maurer–Cartan structure equations (3.1.7) we obtain (componentwise)

$$\begin{aligned}
d\mu^t &= \mu^t_T \wedge \mu^t, \\
d\mu^t_T &= \mu^t_{TT} \wedge \mu^t, \\
d\mu^t_{TT} &= \mu^t_{TT} \wedge \mu^t_T, \\
d\mu^x &= \mu^x_T \wedge \mu^t + \frac{1}{2}\mu^t_T \wedge \mu^x, \\
d\mu^x_T &= \frac{1}{2}\mu^x_T \wedge \mu^t_T + \frac{1}{2}\mu^t_{TT} \wedge \mu^x, \\
d\mu^u_U &= \frac{1}{4}\mu^t \wedge \mu^t_{TT} + \frac{1}{2}\mu^x \wedge \mu^x_T, \\
d\mu^u_{X^{i+2j}} &= \mu^u_{X^{i+2j+2}} \wedge \mu^t + \mu^u_{X^{i+2j+1}} \wedge \mu^x + \mu^u_U \wedge \mu^u_{X^{i+2j}} \\
&\qquad + \left(j + \frac{i}{2}\right)\mu^u_{X^{i+2j}} \wedge \mu^t_T + \left(j + \frac{i}{2}\right)\mu^u_{X^{i+2j-1}} \wedge \mu^x_T
\end{aligned} \tag{3.1.18}$$

$$+\left(\frac{ij}{2}+\frac{j}{4}+\frac{j(j-1)}{2}+\frac{i(i-1)}{8}\right)\mu^u_{X^{i+2j-2}}\wedge\mu^t_{TT},$$

with $i, j \geq 0$.

Alternatively, the infinitesimal determining equations (3.1.16) can be integrated. It is well-known, [81, 82], that the symmetry algebra of the heat equation (3.1.15) is spanned by the six infinitesimal generators

$$\begin{aligned}&\mathbf{v}_1=\frac{\partial}{\partial x},\qquad \mathbf{v}_2=\frac{\partial}{\partial t},\qquad \mathbf{v}_3=u\frac{\partial}{\partial u},\qquad \mathbf{v}_4=x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t},\\&\mathbf{v}_5=2t\frac{\partial}{\partial x}-xu\frac{\partial}{\partial u},\qquad \mathbf{v}_6=4tx\frac{\partial}{\partial x}+4t^2\frac{\partial}{\partial t}-(x^2+2t)u\frac{\partial}{\partial u},\end{aligned}\tag{3.1.19}$$

and the infinite-dimensional subalgebra

$$\mathbf{v}_\alpha=\alpha(x,t)\frac{\partial}{\partial u},\qquad \text{with}\qquad \alpha_t=\alpha_{xx}.\tag{3.1.20}$$

The commutation relations between these vector fields is given by the following table, the entry in row i and column j representing $[\mathbf{v}_i,\mathbf{v}_j]$:

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_\alpha$
$\mathbf{v}_1$	0	0	0	$\mathbf{v}_1$	$-\mathbf{v}_3$	$2\mathbf{v}_5$	$\mathbf{v}_{\alpha_x}$
$\mathbf{v}_2$	0	0	0	$2\mathbf{v}_2$	$2\mathbf{v}_1$	$4\mathbf{v}_4-2\mathbf{v}_3$	$\mathbf{v}_{\alpha_t}$
$\mathbf{v}_3$	0	0	0	0	0	0	$-\mathbf{v}_\alpha$
$\mathbf{v}_4$	$-\mathbf{v}_1$	$-2\mathbf{v}_2$	0	0	$\mathbf{v}_5$	$2\mathbf{v}_6$	$\mathbf{v}_{\alpha'}$
$\mathbf{v}_5$	$\mathbf{v}_3$	$-2\mathbf{v}_1$	0	$-\mathbf{v}_5$	0	0	$\mathbf{v}_{\alpha''}$
$\mathbf{v}_6$	$-2\mathbf{v}_5$	$2\mathbf{v}_3-4\mathbf{v}_4$	0	$-2\mathbf{v}_6$	0	0	$\mathbf{v}_{\alpha'''}$
$\mathbf{v}_\alpha$	$-\mathbf{v}_{\alpha_x}$	$-\mathbf{v}_{\alpha_t}$	$\mathbf{v}_\alpha$	$-\mathbf{v}_{\alpha'}$	$-\mathbf{v}_{\alpha''}$	$-\mathbf{v}_{\alpha'''}$	0

where

$$\alpha'=x\alpha_x+2t\alpha_t,\qquad \alpha''=2t\alpha_x+x\alpha,$$
$$\alpha'''=4tx\alpha_x+4t^2\alpha_t+(x^2+2t)\alpha.$$

Both the Maurer–Cartan structure equations (3.1.18) and the commutator relations of the infinitesimal generators (3.1.19) and (3.1.20) encode the infinitesimal structure of the symmetry group of the heat equation. In the following section we explain how the two approaches are related together.

3.2 Duality

To every r-dimensional Lie group G, we can define a set of r linearly independent invariant vector fields $\mathbf{v}_1, \ldots, \mathbf{v}_r$ on G generating a Lie algebra $\mathfrak{g}$ with commutator relations

$$[\mathbf{v}_i, \mathbf{v}_j] = \sum_{k=1}^{r} C^k_{ij} \mathbf{v}_k, \qquad i, j = 1, \ldots, r, \tag{3.2.1}$$

where the *structure constants* C^k_{ij} are skew-symmetric in their subscripts and satisfy the Jacobi identities

$$\sum_{k=1}^{r} (C^k_{ij} C^m_{kl} + C^k_{li} C^m_{kj} + C^k_{jl} C^m_{ki}) = 0, \qquad 1 \leq i, j, l, m \leq r. \tag{3.2.2}$$

The structure constants serve to uniquely characterize G up to a discrete subgroup. Dually, r linearly independent invariant one-forms $\mu^1, \ldots, \mu^r$ can be defined on G satisfying the Maurer–Cartan structure equations

$$d\mu^k = -\sum_{1 \leq i < j \leq r} C^k_{ij} \mu^i \wedge \mu^j, \tag{3.2.3}$$

where the constants C^k_{ij} are the same as in (3.2.1). The Jacobi identities (3.2.2) are equivalent to the identities $d^2\mu^k = 0$, $k = 1, \ldots, r$.

Cartan was skeptical that for infinite-dimensional Lie pseudo-groups a similar correspondence could be made between his structure equations and the infinitesimal theory advocated by S. Lie, [20, p. 1335]. But in the

1960s, Kuranishi, [63, 64], and Singer and Sternberg, [101], were able to give an infinitesimal interpretation of Cartan's structure equations for transitive Lie pseudo-groups of order one. This correspondence between the Maurer–Cartan structure equations and the infinitesimal generator commutators is in fact completely general. It holds for Lie pseudo-groups of arbitrary order, transitive or intransitive.

The infinite jet $j_\infty \mathbf{v}|_{z_0}$ of an infinitesimal generator (2.4.5) at $z_0 \in M$ can be identified with its Taylor expansion

$$\mathbf{v} = \sum_{a=1}^{m} \sum_{\#A \geq 0} \zeta_A^a(z_0) \frac{(z - z_0)^A}{A!} \frac{\partial}{\partial z^a}, \qquad z_0 \in M, \tag{3.2.4}$$

where the coefficients $\zeta_A^a(z_0)$ correspond to the jet coordinates of $j_\infty \mathbf{v}|_{z_0}$. From (2.1.1) and (3.2.4), it follows that

$$J^\infty TM|_{z_0} \cong \mathcal{J}^\infty T^* M|_{z_0} \otimes_{\mathbb{R}} TM|_{z_0}.$$

The monomial vector fields

$$\mathbf{v}_a^A|_{z_0} = \mathbf{v}_a^A = \frac{(z - z_0)^A}{A!} \frac{\partial}{\partial z^a}, \qquad a = 1, \ldots, m, \qquad \#A \geq 0, \tag{3.2.5}$$

provide a basis for the vector space $J^\infty TM|_{z_0}$. Any local analytic vector field defined in a sufficiently small neighborhood of z_0 can be written in terms of the basis element (3.2.5):

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{a=1}^{m} \sum_{\#A \geq 0} \zeta_A^a(z_0) \mathbf{v}_a^A|_{z_0}.$$

There is a well-defined Lie algebra structure on $J^\infty TM|_{z_0}$ obtained by interpreting (3.2.5) as vector fields on TM:

$$
\begin{aligned}
[\mathbf{v}_a^A, \mathbf{v}_b^B] =& \frac{(z-z_0)^{A+(B\setminus a)}}{A!(B\setminus a)!}\frac{\partial}{\partial z^b} - \frac{(z-z_0)^{(A\setminus b)+B}}{(A\setminus b)!B!}\frac{\partial}{\partial z^a} \\
=& \frac{(A+(B\setminus a))!}{A!(B\setminus a)!}\mathbf{v}_b^{A+(B\setminus a)} - \frac{((A\setminus b)+B)!}{(A\setminus b)!B!}\mathbf{v}_a^{(A\setminus b)+B} \\
=& \binom{A+(B\setminus a)}{A}\mathbf{v}_b^{A+(B\setminus a)} - \binom{(A\setminus b)+B}{B}\mathbf{v}_a^{(A\setminus b)+B},
\end{aligned}
\tag{3.2.6}
$$

$1 \le a, b \le m$, $\#A, \#B \ge 0$, where

$$B\setminus a = (b^1, \dots, b^{a-1}, b^a - 1, b^{a+1}, \dots, b^m),$$

and

$$A + B = (a^1 + b^1, \dots, a^m + b^m).$$

We use the convention that

$$\binom{A+(B\setminus a)}{A} = 0 \qquad \text{if} \qquad b^a - 1 < 0.$$

Theorem 3.12. The Lie algebra structure equations (3.2.6) are dual to the Maurer–Cartan structure equations (3.1.7).

Proof. The components of the Maurer–Cartan structure equations (3.1.7) for the diffeomorphism pseudo-group are

$$d\mu_C^a = \sum_{C=A+B}\sum_{b=1}^{m}\binom{C}{A}\mu_{A,b}^a \wedge \mu_B^b. \tag{3.2.7}$$

For fixed a, b and A, B, the two-form $\mu_A^a \wedge \mu_B^b$ appears twice on the right-hand side of (3.2.7). First in the structure equation

$$d\mu_{A+(B\setminus a)}^b = -\frac{(A+(B\setminus a))!}{A!(B\setminus a)!}\mu_A^a \wedge \mu_B^b + \cdots ,$$

then in

$$d\mu^a_{(A\setminus b)+B} = \frac{((A\setminus b)+B)!}{(A\setminus b)!B!}\mu^a_A \wedge \mu^b_B + \cdots .$$

□

Remark 3.13. An alternative proof of Theorem 3.12 consists of showing that the lifting map $\boldsymbol{\lambda} : J^\infty TM \to \mathrm{Lie}(\mathcal{D}^{(\infty)}(M))$ is a Lie algebroid isomorphism.

Remark 3.14. Since the higher order Maurer–Cartan forms μ^a_A are defined by (3.1.2), their structure equations (3.2.7) can also be derived by Lie differentiating the structure equations for the zero-th order invariant contact forms μ^a. By direct computation

$$d\mu^a = \sum_{b=1}^m \mu^a_b \wedge (\mu^b - dZ^b),$$

where $\mu^a_b = \mu^a_{Z^b}$. From the Leibniz rule, [34], we obtain

$$\begin{aligned} d\mu^a_C =& d\left(\mathbb{D}^Z_C \mu^a\right) = \mathbb{D}^Z_C\left(d\mu^a\right) = \mathbb{D}^Z_C\left(\sum_{b=1}^m \mu^a_b \wedge (\mu^b - dZ^b)\right) \\ =& \sum_{C=A+B}\sum_{b=1}^m \binom{C}{A} \mu^a_{A,b} \wedge \left(\mu^b_B - d(\mathbb{D}^Z_B Z^b)\right). \end{aligned}$$

Restricting the last equation to a target fiber we recover the Maurer–Cartan structure equations (3.2.7).

For a Lie pseudo-group $\mathcal{G} \subset \mathcal{D}(M)$, the infinitesimal interpretation of the Maurer–Cartan structure equations still holds.

Theorem 3.15. The Maurer–Cartan structure equations (3.1.14) of a Lie pseudo-group $\mathcal{G}$ at the target fiber $(\boldsymbol{\tau}^{(\infty)})^{-1}(z)$ are dual to the Lie algebra structure equations of its infinite jet of infinitesimal generators $J^\infty\mathfrak{g}$ at z.

Proof. At the target fiber $(\boldsymbol{\tau}^{(\infty)})^{-1}(z)$ the Maurer–Cartan forms satisfy the lifted infinitesimal determining equations

$$L^{(\infty)}(z, \mu^{(\infty)}) = 0 \tag{3.2.8}$$

while the vector field jet coordinates satisfy the equivalent infinitesimal determining equations

$$L^{(\infty)}(z, \zeta^{(\infty)}) = 0, \tag{3.2.9}$$

at z. The theorem follows from the observation that the Lie algebra structure equations for $J^\infty\mathfrak{g}$ at z are obtained by restricting the Lie algebra structure equations (3.2.6) to the kernel of the infinitesimal determining equations (3.2.9), while the Maurer–Cartan structure equations of the Lie pseudo-group $\mathcal{G}$ at the target fiber $(\boldsymbol{\tau}^{(\infty)})^{-1}(z)$ are, in turn, obtained by restricting the Maurer–Cartan structure equations (3.2.7) to the kernel of the lifted infinitesimal determining equations (3.2.8). □

Example 3.16. Consider the intransitive Lie pseudo-group

$$X = x, \qquad Y = ay + b, \qquad Z = a^x z + f(x), \tag{3.2.10}$$

where $a \in \mathbb{R}^+$, $b \in \mathbb{R}$ and $f \in C^\omega(\mathbb{R})$. The determining system for this Lie pseudo-group is

$$X = x, \qquad Y_x = 0, \qquad Y_{yy} = 0, \qquad Y_z = 0, \qquad Z_y = 0, \qquad Z_z = (Y_y)^x.$$

The corresponding infinitesimal determining equations, for an infinitesimal generator $\mathbf{v} = \xi(x,y,z)\partial_x + \eta(x,y,z)\partial_y + \phi(x,y,z)\partial_z$, are

$$\begin{gathered} \xi = 0, \qquad \eta_x = 0, \qquad \eta_{yy} = 0, \\ \eta_z = 0, \qquad \phi_y = 0, \qquad \phi_z = x\eta_y. \end{gathered} \tag{3.2.11}$$

The lift of (3.2.11) gives the linear relations

$$\begin{aligned} \mu^x &= 0, & \mu^y_X &= 0, & \mu^y_{YY} &= 0, \\ \mu^y_Z &= 0, & \mu^z_Y &= 0, & \mu^z_Z &= X\mu^y_Y. \end{aligned} \tag{3.2.12}$$

By Lie differentiating $\mu^z_Z = X\mu^y_Y$ with respect to $\mathbb{D}_X$ we obtain $\mu^z_{ZX} = \mu^y_Y$. Hence it follows that

$$\mu^y, \qquad \mu^y_Y, \qquad \mu^z_{X^k}, \qquad k \geq 0,$$

is a basis of Maurer–Cartan forms. Their structure equations are

$$\begin{aligned} &d\mu^y_Y = 0, \\ &d\mu^y = \mu^y_Y \wedge \mu^y, \\ &d\mu^z_{X^k} = X\mu^y_Y \wedge \mu^z_{X^k} + k\mu^y_Y \wedge \mu^z_{X^{k-1}}, \qquad k \geq 0. \end{aligned} \tag{3.2.13}$$

Setting $\mathbf{w}^k$ to be the vector dual to $\mu^z_{X^k}$, $k \geq 0$, $\mathbf{v}$ to be dual to μ^y and $\mathbf{v}^1$ to be dual to μ^y_Y, the Maurer–Cartan structure equations (3.2.13) yield the commutator relations

$$\begin{aligned} [\mathbf{v}^1, \mathbf{v}] &= -\mathbf{v}, \qquad [\mathbf{v}, \mathbf{w}^k] = 0, \\ [\mathbf{v}^1, \mathbf{w}^k] &= -x_0\mathbf{w}^k - (k+1)\mathbf{w}^{k+1}, \end{aligned} \tag{3.2.14}$$

at each fixed $x_0 \in \mathbb{R}$.

Taking Lie's approach, the space of infinitesimal generators for the pseudo-group action (3.2.10) is spanned by the vector fields

$$\mathbf{v} = \frac{\partial}{\partial y}, \qquad \mathbf{v}^1 = y\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}, \qquad \mathbf{v}_{f(x)} = f(x)\frac{\partial}{\partial z}.$$

In the analytic category, a basis of vector fields, in the neighborhood of the point (x_0, y_0, z_0), is given by

$$\begin{gathered}\mathbf{v}^1 = ((y-y_0)+y_0)\frac{\partial}{\partial y} + ((x-x_0)+x_0)((z-z_0)+z_0)\frac{\partial}{\partial z}, \\ \mathbf{v} = \frac{\partial}{\partial y}, \qquad \mathbf{w}^k = \frac{(x-x_0)^k}{k!}\frac{\partial}{\partial z}, \qquad k \geq 0.\end{gathered} \tag{3.2.15}$$

In terms of (3.2.15) we can write

$$\mathbf{v}_{f(x)} = \sum_{k=0}^{\infty} f^{(k)}(x_0)\mathbf{w}^k.$$

By direct computation we verify that the commutation relations for the vector fields (3.2.15) are given by (3.2.14).

3.3 Cartan Structure Equations

In this section we provide a brief overview of Cartan's method for constructing the structure equations of a Lie pseudo-group. For more detailed accounts, we refer the reader to Cartan's original works, [20, 23], and to the expository texts [40, 51, 103].

In an adapted set of local coordinates $z = (x, y)$ on M, it can be assumed that the pseudo-group action is locally given by

$$X^i = x^i, \qquad Y^\alpha = f^\alpha(x, y), \qquad i = 1, \ldots, s, \qquad \alpha = 1, \ldots, t, \tag{3.3.1}$$

where $s + t = m = \dim M$ and $\det(\partial Y^\alpha/\partial y^\beta) \neq 0$. If the action is transitive $s = 0$, otherwise $s > 0$. Cartan's starting point is the $n_\star$-th order involutive determining system

$$X = x, \qquad F^{(n_\star)}(x, y, Y^{(n_\star)}) = 0, \tag{3.3.2}$$

for the Lie pseudo-group action (3.3.1).

Remark 3.17. Note that $n_\star \geq n^\star$, where $n^\star$ is the order of the Lie pseudo-group, defined in Definition 2.27. The inequality comes from the fact that the $n^\star$-th order determining system (2.3.3) is locally solvable and formally integrable but not necessarily in involution. If this is the case the determining system needs to be completed to involution. We refer the reader to Appendix A for more details.

Motivated by his newly developed theory of exterior differential systems, [15,19], Cartan recasts the determining system (3.3.2) in terms of the Pfaffian system

$$\begin{aligned} &X^i - x^i = 0, \qquad i = 1, \ldots, s, \\ &\Upsilon^{s+\alpha}_A|_{F^{(n_\star)}(x,y,Y^{(n_\star)})=0} = (dY^\alpha_A - \sum_{b=1}^m Y^\alpha_{A,b} dz^b)|_{F^{(n_\star)}(x,y,Y^{(n_\star)})=0} = 0, \end{aligned} \tag{3.3.3}$$

$\alpha = 1, \ldots, t$ and $0 \leq \#A \leq n_\star - 1$. Let $Y_{[k]} = (Y^1_{[k]}, \ldots, Y^{t_k}_{[k]})$ be local parameterizations of the fibers of the bundles

$$\pi^k_{k-1} : \mathcal{G}^{(k)} \to \mathcal{G}^{(k-1)},$$

where $t_k = \dim \mathcal{G}^{(k)} - \dim \mathcal{G}^{(k-1)}$ is the dimension of the fibers and $k \geq 1$. The Pfaffian system (3.3.3) is then equivalent to

$$\begin{aligned} X^i - x^i &= 0, & i &= 1, \ldots, s, \\ dY^\alpha - \sum_{a=1}^m L^\alpha_a(z, Y, Y_{[1]}) dz^a &= 0, & \alpha &= 1, \ldots, t, \\ dY^i_{[1]} - \sum_{a=1}^m L^i_{[1],a}(z, Y, Y_{[1]}, Y_{[2]}) dz^a &= 0, & i &= 1, \ldots, t_1, \\ &\vdots \end{aligned}$$

$$dY^i_{[n_\star-1]} - \sum_{a=1}^{m} L^i_{[n_\star-1],a}(z, Y, Y_{[1]}, \ldots, Y_{[n_\star]})dz^a = 0, \qquad i = 1, \ldots, t_{n_\star-1}, \tag{3.3.4}$$

for some functions $L^\alpha_a, \ldots, L^i_{[n_\star-1],a}$, whose expressions are determined from the defining system (3.3.2).

From the differential forms appearing in (3.3.4), Cartan proceeds, in an inductive manner, to derive a system of invariant one-forms that serve to characterize the pseudo-group. Since the contact one-forms

$$dY^\alpha - \sum_{a=1}^{m} L^\alpha_a(z, Y, Y_{[1]})dz^a, \qquad \alpha = 1, \ldots, t, \tag{3.3.5}$$

and the differential forms $dY^1, \ldots, dY^t$ are right-invariant, the one-forms

$$\omega^{s+\alpha}_{[0]} = \sum_{a=1}^{m} L^\alpha_a(z, Y, Y_{[1]})dz^a, \qquad \alpha = 1, \ldots, t, \tag{3.3.6}$$

are likewise right-invariant. In a certain sense the passage from the invariant differential forms (3.3.5) to the invariant one-forms $\omega^{s+1}_{[0]}, \ldots, \omega^m_{[0]}$ is very similar to the restriction of the differential forms (3.3.5) to a target fiber $\boldsymbol{\tau}^{-1}(X, Y)$ (up to a negative sign), as it is done in Section 3.1, since on a target fiber $dY^\alpha = 0$, $\alpha = 1, \ldots, t$. But the obvious difference is that Cartan does not assume $Y^1, \ldots, Y^t$ to be constant.

Coming back to Cartan's derivation of the structure equations, Cartan establishes that the first m invariant one-forms characterizing the Lie pseudo-group are

$$\begin{aligned} &\omega^i_{[0]} = dx^i, && i = 1, \ldots, s, \\ &\omega^{s+\alpha}_{[0]} = \sum_{a=1}^{m} L^\alpha_a(z, Z, Z_{[1]})dz^a, && \alpha = 1, \ldots, t. \end{aligned} \tag{3.3.7}$$

The differential forms (3.3.7) constitute a basis of horizontal forms, therefore the differential forms $dz^1, \ldots, dz^m$ can be written as linear combinations of the $\omega^1_{[0]}, \ldots, \omega^m_{[0]}$. Hence the exterior derivative of the invariant one-forms $\omega^1_{[0]}, \ldots, \omega^m_{[0]}$ can be written as

$$d\omega^b_{[0]} = \sum_{a=1}^{m} d\left(L^b_a(z, Y, Y_{[1]})\right) \wedge dz^a = \sum_{a=1}^{m} \omega^a_{[0]} \wedge \pi^b_a, \qquad b = 1, \ldots, m,$$

where the one-forms π^b_a are certain linear combinations of $dY^1_{[1]}, \ldots, dY^{t_1}_{[1]}$, $dY^1, \ldots, dY^t$, and $\omega^1_{[0]}, \ldots, \omega^m_{[0]}$. The invariance of $\omega^1_{[0]}, \ldots, \omega^m_{[0]}$ implies

$$\sum_{a=1}^{m} \omega^a_{[0]} \wedge (R^*_\psi(\pi^b_a) - \pi^b_a) = 0, \qquad b = 1, \ldots, m, \qquad \forall\, \psi \in \mathcal{G}$$

such that the pull-back is defined. This means that

$$R^*_\psi(\pi^b_a) \equiv \pi^b_a \mod \omega^1_{[0]}, \ldots, \omega^m_{[0]}.$$

By hypothesis, $t_1 = \dim \mathcal{G}^{(1)} - \dim \mathcal{G}^{(0)}$ of the π^b_a are linearly independent modulo $\omega^1_{[0]}, \ldots, \omega^m_{[0]}$, $dY^1, \ldots, dY^t$. Hence those t_1 differential forms are of the form

$$\pi^i \equiv \sum_{j=1}^{t_1} c^i_j dY^j_{[1]} + \sum_{\alpha=1}^{q} e^i_\alpha dY^\alpha \mod \omega^1_{[0]}, \ldots, \omega^m_{[0]}, \qquad i = 1, \ldots, t_1,$$

with $\det\,(c^i_j) \neq 0$. The coefficients c^i_j and e^i_α may depend on the variables z, Y, and $Y_{[1]}$. By adding suitable multiples of the $\omega^a_{[0]}$ we can write

$$\pi^i \equiv \sum_{j=1}^{t_1} c^i_j \left(dY^j_{[1]} - \sum_{b=1}^{m} L^j_{[1],b}(z, Y, Y_{[1]}) dz^b \right) + \sum_{\alpha=1}^{t} e^i_\alpha \left(dY^\alpha - \omega^{p+\alpha}_{[0]} \right),$$

modulo $\omega^1_{[0]}, \ldots, \omega^m_{[0]}$, $i = 1, \ldots, t_1$. Defining

$$\omega^i_{[1]} := \sum_{j=1}^{t_1} c^i_j \left(dY^j_{[1]} - \sum_{b=1}^{m} L^j_{[1],b}(z, Y, Y_{[1]}) dz^b \right) + \sum_{\alpha=1}^{q} e^i_\alpha \left(dY^\alpha - \omega^{p+\alpha}_{[0]} \right), \tag{3.3.8}$$

$i = 1, \ldots, t_1$, Cartan shows that those one-forms are invariant, [23, pp. 597–600]. We refer to the one-forms (3.3.8) as the first order *Cartan forms* for the pseudo-group $\mathcal{G}$. Those invariant differential forms constitute a complete set of linearly independent first order Cartan forms. They are equivalent to the first order Maurer–Cartan forms (3.1.2) in the sense that

$$\text{span} \left\{ \omega^i_{[1]} \right\} = \text{span} \left\{ \mu^a_{Z^b} |_{L^{(n_\star)}(Z, \mu^{(n_\star)})=0} \right\}$$

over the ring of functions

$$F : (x, y, Y, Y_{[1]}) \to \mathbb{R}.$$

Next by computing the exterior derivatives of the first order Cartan forms (3.3.8) and repeating the above procedure, Cartan derives t_2 linearly independent second order Cartan forms, and so on, up to order $n_\star - 1$.

The $\widetilde{r}_{n_\star - 1} = m + t_1 + t_2 + \cdots + t_{n_\star - 1}$ invariant one-forms constructed are collectively denoted by $\omega^1, \omega^2, \ldots, \omega^{\widetilde{r}_{n_\star - 1}}$ without the subscripts[3]. Their exterior derivatives can be written as

$$d\omega^i = \sum_{1 \le j < k \le \widetilde{r}_{n_\star - 1}} C^i_{jk} \omega^j \wedge \omega^k + \sum_{j=1}^{\widetilde{r}_{n_\star - 1}} \sum_{\beta=1}^{t_{n_\star}} A^i_{j\beta} \omega^j \wedge \overline{\pi}^\beta, \tag{3.3.9}$$

$i = 1, \ldots, \widetilde{r}_{n_\star - 1}$, where

$$(\overline{\pi}^1, \ldots, \overline{\pi}^{t_{n_\star}}) \equiv (dY^1_{[n_\star]}, \ldots, dY^{t_{n_\star}}_{[n_\star]}) \qquad \text{mod } \omega^1, \ldots, \omega^{\widetilde{r}_{n_\star - 1}}$$

[3]Note that $\widetilde{r}_{n_\star - 1} = r_{n_\star - 1} + s$, where $r_{n_\star - 1}$ equals the fiber dimension of the $(n_\star - 1)$-th order jet groupoid $\mathcal{G}^{(n_\star - 1)} \to M$.

as modules of one-forms over the ring of functions

$$F : (x, y, Y, Y_{[1]}, \dots, Y_{[n_\star]}) \to \mathbb{R}.$$

The equations (3.3.9) are called the *Cartan structure equations*. If the pseudo-group is intransitive, the coefficients C^i_{jk}, and $A^i_{j\beta}$ may depend on the invariants $x^1, \dots, x^s$, [20, 23].

Remark 3.18. In the above derivation of the Cartan structure equations, no assumptions are made on the one-forms $dY^1, \dots, dY^t$. Thus we can set them equal to zero without loss of generality. This means that the target coordinates $Y^1, \dots, Y^t$ can be set equal to suitable constants, [103, pp. 382–383]. By doing so the one-forms (3.3.6) correspond to the negative of the restriction of the one-forms (3.3.5) to a target fiber, in analogy with the derivation of the Maurer–Cartan structure equations (3.1.14).

Example 3.19. Consider the infinite-dimensional Lie pseudo-group

$$X = x, \qquad Y = f(y), \qquad Z = z(f'(y))^x + \phi(x, y), \tag{3.3.10}$$

$f \in \mathcal{D}(\mathbb{R})$, $\phi \in C^\omega(\mathbb{R}^2)$, due to Cartan [22, 68]. The determining system for this pseudo-group is

$$X = x, \quad Y_x = 0, \quad Y_z = 0, \quad Z_z = (Y_y)^x. \tag{3.3.11}$$

The system of equations (3.3.11) is locally solvable and formally integrable but not involutive. To obtain an involutive determining system we must prolong (3.3.11) by including the second order determining equations. We now derive the Cartan forms and their structure equations. From (3.3.11) we see that the fibers of the bundle $\pi^1_0 : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$ are parameterized by

$$Y_{[1]} = (Y_y, Z_x, Z_y).$$

Hence we obtain the three invariant horizontal forms

$$\omega^1 = \omega^1_{[0]} = dx, \qquad \omega^2 = \omega^2_{[0]} = Y_y dy, \qquad \omega^3 = \omega^3_{[0]} = Z_x dx + Z_y dy + (Y_y)^x dz.$$

Taking their exterior derivative we obtain

$$\begin{aligned} d\omega^1 &= 0, \\ d\omega^2 &= -\omega^2 \wedge \pi^1, \\ d\omega^3 &= -\omega^1 \wedge \pi^2 - \omega^2 \wedge \pi^3 - x\omega^3 \wedge \pi^1, \end{aligned}$$

where

$$\begin{aligned} \pi^1 &= \frac{dY_y}{Y_y}, \\ \pi^2 &= dZ_x - \frac{xZ_x}{Y_y} dY_y - ((Y_y)^x \ln Y_y)\, dz, \\ \pi^3 &= \frac{1}{Y_y}\left(dZ_y - \frac{xZ_y}{Y_y} dY_y \right). \end{aligned} \tag{3.3.12}$$

The differential forms (3.3.12) are not invariant but can be made invariant by adding suitable linear combinations of the differential forms dx, dy and dz where the coefficients may depend on the pseudo-group jet coordinates

$$(x, y, z, Y_y, Z_x, Z_y, Y_{yy}, Z_{xx}, Z_{xy}, Z_{yy})$$

of $\mathcal{G}^{(2)}$. Following Cartan's algorithm we obtain

$$\begin{aligned} \omega^4 = \omega^1_{[1]} =& \pi^1 - \frac{Y_{yy}}{Y_y} dy = \frac{1}{Y_y}(dY_y - Y_{yy} dy), \\ \omega^5 = \omega^2_{[1]} =& \pi^2 - Z_{xx} dx - Z_{xy} dy + \frac{xZ_x Y_{yy}}{Y_y} dy \\ =& (dZ_x - Z_{xx}dx - Z_{xy}dy - (Y_y)^x \ln(Y_y) dz) - \frac{xZ_x}{Y_y}(dY_y - Y_{yy}dy), \\ \omega^6 = \omega^3_{[1]} =& \pi^3 - \frac{1}{Y_y}\left(Z_{xy} dx + \left(Z_{yy} - \frac{xZ_y Z_{yy}}{Y_y} \right) dy + \frac{x(Y_y)^x Y_{yy}}{Y_y} dz \right) \end{aligned}$$

$$=\frac{1}{Y_y}(dZ_y - Z_{xy}dx - Z_{yy}dy - x(Y_y)^{x-1}Y_{yy}dz) - \frac{xZ_y}{Y_y^2}(dY_y - Y_{yy}dy).$$

Taking the exterior derivative of the latter differential forms we obtain Cartan's structure equations

$$\begin{aligned}
d\omega^1 =&0,\\
d\omega^2 =&\omega^4 \wedge \omega^2,\\
d\omega^3 =&\omega^5 \wedge \omega^1 + \omega^6 \wedge \omega^2 + x\omega^4 \wedge \omega^3,\\
d\omega^4 =&\omega^2 \wedge \overline{\pi}^1,\\
d\omega^5 =&\omega^1 \wedge \overline{\pi}^2 + \omega^2 \wedge \overline{\pi}^3 + x\omega^4 \wedge \omega^5 + \omega^3 \wedge \omega^4,\\
d\omega^6 =&\omega^1 \wedge \overline{\pi}^3 + \omega^2 \wedge \overline{\pi}^4 + x\omega^4 \wedge \omega^6 + x\omega^3 \wedge \overline{\pi}^4,
\end{aligned}$$

where $\overline{\pi}^1$, $\overline{\pi}^2$, $\overline{\pi}^2$, $\overline{\pi}^4$ are equal to the restriction of μ^y_{YY}, μ^z_{XX}, μ^z_{XY}, μ^z_{YY} to $\mathcal{G}^{(2)}$ respectively. For example

$$\overline{\pi}^1 = \mu^y_{YY}|_{\mathcal{G}^{(2)}} = \frac{1}{Y_y^3}[Y_y dY_{yy} - Y_{yy}(dY_y - Y_{yy}dy)].$$

3.4 Comparison of the Two Structure Theories

The structure theory developed by Olver and Pohjanpelto, and explained in Section 3.1, has been applied to several transitive Lie pseudo-groups, [26,88]. As one expects, their structure equations are isomorphic to those obtained with Cartan's structure theory. Though the structure equations are equivalent, a fundamental distinction needs to be pointed out. While the Maurer–Cartan structure equations (3.1.14) involve contact invariant forms restricted to the target fibers of the pseudo-group action, the Cartan structure equations (3.3.9) mix horizontal and group forms. Unless there is a one-to-one correspondence between the differential forms appearing in Maurer–Cartan structure equations and the Cartan structure equations, there is no reason to believe that the two sets of structure equations are equivalent. As we now explain, the structure equations for intransitive Lie pseudo-groups do not agree.

Let $\mathcal{G} \subset \mathcal{D}(M)$ be an $n^\star$-th order Lie pseudo-group. We assume that locally it is given by (3.3.1), with determining system (3.3.2). In those adapted coordinates, we use the notation

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z)\frac{\partial}{\partial z^a} = \sum_{i=1}^{s} \xi^i(x,y)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{t} \phi^\alpha(x,y)\frac{\partial}{\partial y^\alpha}.$$

to denote a local vector field $\mathbf{v} \in \mathcal{X}(M)$. The corresponding Maurer–Cartan forms are denoted by

$$\widetilde{\mu}^i_A = \boldsymbol{\lambda}(\xi^i_A), \quad i = 1,\ldots,s, \quad \nu^\alpha_A = \boldsymbol{\lambda}(\phi^\alpha_A), \quad \alpha = 1,\ldots,t, \quad \#A \geq 0. \tag{3.4.1}$$

With this notation, the Maurer–Cartan-valued power series (3.1.3) is given by

$$\mu[\![H]\!] = \begin{pmatrix} \widetilde{\mu}[\![H]\!] \\ \nu[\![H]\!] \end{pmatrix}.$$

Furthermore, we split the formal parameters appearing in the expression of the vector-valued Maurer–Cartan power series (3.1.3) in the following way

$$(H,K) = (H^1,\ldots,H^s,K^1,\ldots,K^t).$$

The linearization of the determining system (3.3.2) at the identity jet gives the infinitesimal determining equations

$$\xi = 0, \qquad L^{(n^\star)}(x,y,\phi^{(n^\star)}) = 0. \tag{3.4.2}$$

Taking the lift of (3.4.2), we obtain the linear relations

$$\widetilde{\mu} = 0, \qquad L^{(n^\star)}(X,Y,\nu^{(n^\star)}) = 0. \tag{3.4.3}$$

It follows from (3.4.3) that the Maurer–Cartan structure equations (3.1.7) reduce to

$$(d\nu[\![H,K]\!] = \nabla_K \nu[\![H,K]\!] \wedge \nu[\![H,K]\!])\,|_{L^{(\infty)}(X,Y,\nu^{(\infty)})=0}, \tag{3.4.4}$$

where

$$\nabla_K \nu[\![H,K]\!] = \left(\frac{\partial \nu^\alpha}{\partial K^\beta}[\![H,K]\!]\right)$$

is the $t \times t$ Jacobian matrix power series obtained by differentiating $\nu[\![H,K]\!]$ with respect to $K = (K^1, \ldots, K^t)$.

Remark 3.20. The transitivity of the pseudo-group action on its orbits implies that the target fibers are isomorphic on each leaf of the foliation. Hence we can set $Y^1, \ldots, Y^t$ to suitable constants and assume that the structure coefficients in (3.4.4) only depend on the invariants $X^1, \ldots, X^s$.

For intransitive Lie pseudo-groups $s \geq 1$ and the Maurer–Cartan forms $\widetilde{\mu}^1, \ldots, \widetilde{\mu}^s$ do not enter the structure equations (3.4.4). From (3.1.6) we conclude that the horizontal forms $\sigma^i = \omega^i_{[0]}$, $i = 1, \ldots, s$, do not appear in the Maurer–Cartan structure equations (3.4.4). On the other hand, the horizontal forms $\omega^1_{[0]}, \ldots, \omega^s_{[0]}$ do appear in the Cartan structure equations. For instance, the first s structure equations in (3.3.9) are

$$d\omega^i_{[0]} = 0, \qquad i = 1, \ldots, s.$$

Thus for intransitive Lie pseudo-groups, the two sets of structure equations do not agree. To motivate the rest of the discussion we look at three examples.

Example 3.21. Consider the intransitive Lie group action

$$X = x \neq 0, \qquad Y = y + ax, \qquad a \in \mathbb{R}. \tag{3.4.5}$$

The infinitesimal generator of this one-parameter group of transformations is

$$\mathbf{v} = x\frac{\partial}{\partial y}.$$

Since the Lie algebra is one-dimensional it is automatically abelian. Cartan computed the structure equations of this group, [20, p. 1345], and obtained

$$d\omega^1 = 0, \qquad d\omega^2 = \frac{1}{x}\omega^1 \wedge \omega^2, \tag{3.4.6}$$

where

$$\omega^1 = dx \qquad \text{and} \qquad \omega^2 = dy - \frac{y}{x}dx.$$

Clearly, the structure equations (3.4.6) do not correspond to those of an abelian group. Furthermore, the group is one-dimensional and there should only be one independent (Cartan) Maurer–Cartan form associated to this group.

We now compute the Maurer–Cartan structure equations (3.1.14). The determining system for the group action (3.4.5) is

$$X = x, \qquad Y - y = xY_x, \qquad Y_y = 1.$$

An infinitesimal generator $\mathbf{v} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ of this group action must be a solution of the infinitesimal determining equations

$$\xi = 0, \qquad \eta = x\eta_x, \qquad \eta_y = 0.$$

The corresponding lifted infinitesimal determining equations are

$$\mu^x = 0, \qquad \mu^y = X\mu^y_X, \qquad \mu^y_Y = 0. \tag{3.4.7}$$

It follows from (3.4.7) that μ^y is a basis of Maurer–Cartan forms and that its structure equation is

$$d\mu^y = 0.$$

Hence we recover the expected structure equations. Namely, there is only one independent Maurer–Cartan form and it is closed.

Example 3.22. As a second example we consider the infinite-dimensional intransitive Lie pseudo-group

$$X = x, \qquad Y = y + f(x), \qquad f \in C^{\omega}(\mathbb{R}). \tag{3.4.8}$$

The infinitesimal generators of the pseudo-group action (3.4.8) are

$$\mathbf{v} = g(x)\frac{\partial}{\partial y}, \tag{3.4.9}$$

with $g \in C^{\omega}(\mathbb{R})$. The Lie algebra generated by the vector fields (3.4.9) is abelian. The structure equations for this Lie pseudo-group have been computed by Cartan, [20, p. 1346]. Those are

$$d\omega^1 = 0, \qquad d\omega^2 = \pi^1 \wedge \omega^1, \tag{3.4.10}$$

where $\omega^1 = dx$ and $\omega^2 = dy + Y_x dx$. The structure equations (3.4.10) do not correspond to those of the abelian algebra (3.4.9).

We now compute the Maurer–Cartan structure equations. The determining system of the Lie pseudo-group (3.4.8) is

$$X = x, \qquad Y_y = 1.$$

Hence the infinitesimal determining equations of an infinitesimal generator

$$\mathbf{v} = \xi(x,y)\frac{\partial}{\partial x} + \phi(x,y)\frac{\partial}{\partial y}$$

are

$$\xi = 0, \qquad \phi_y = 0. \tag{3.4.11}$$

The lift of (3.4.11) gives

$$\mu^x = 0, \qquad \mu^y_Y = 0,$$

and it follows that a basis of Maurer–Cartan forms for the pseudo-group (3.4.8) is given by

$$\mu^y_{X^k}, \qquad k \geq 0.$$

Their structure equations are

$$\sum_{k=0}^{\infty} d\mu^y_{X^k} \frac{H_x^k}{k!} = \frac{\partial}{\partial H_y}\left(\sum_{k=1}^{\infty} \frac{\mu^y_{X^k}}{k!} H_x^k\right) \wedge \sum_{k=0} \mu^y_{X^k} \frac{H_x^k}{k!} = 0, \tag{3.4.12}$$

which correspond to the structure equations of an infinite-dimensional abelian Lie pseudo-group.

Example 3.23. A further curious result of Cartan is his classification of infinite-dimensional first order Lie pseudo-goups in two variables, [23, 68]. Though the two Lie pseudo-groups

$$\mathcal{G}_1 : \begin{cases} X = x, \\ Y = y + f(x), \end{cases} \qquad \mathcal{G}_2 : \begin{cases} X = x + a, \\ Y = y + f(x), \end{cases} \tag{3.4.13}$$

have non-isomorphic Lie algebras, respectively spanned by

$$\left\{ f(x) \frac{\partial}{\partial y} \right\}, \qquad \left\{ \frac{\partial}{\partial x}, f(x)\frac{\partial}{\partial y} \right\}, \qquad f \in C^\omega(\mathbb{R}),$$

Cartan establishes that both Lie pseudo-groups have (3.4.10) for structure equations[4]. On the other hand, the Maurer–Cartan structure equations of the two Lie pseudo-groups (3.4.13) are non-isomorphic. The Maurer–Cartan

[4]Cartan distinguishes the two Lie pseudo-groups (3.4.13) by the fact that $\mathcal{G}_1$ has one invariant, namely x, while the pseudo-group $\mathcal{G}_2$ does not.

structure equations of $\mathcal{G}_1$ are

$$d\mu^y_{X^k} = 0, \qquad k \geq 0,$$

while the Maurer–Cartan structure equations of $\mathcal{G}_2$ are

$$d\mu^x = 0, \qquad d\mu^y_{X^k} = \mu^y_{X^{k+1}} \wedge \mu^x, \qquad k \geq 0.$$

Examples 3.21, 3.22 and 3.23 suggest that the Cartan structure equations for intransitive Lie pseudo-groups need to be modified in order to recover the adequate infinitesimal structure. Though it is true that the horizontal forms

$$\omega^i_{[0]} = dx^i, \qquad i = 1, \ldots, s,$$

are invariant under the identity transformation $X = x$, they are also invariant under the translation group $X = x + a$. By including the differential forms $\omega^1_{[0]}, \ldots, \omega^s_{[0]}$ into the structure equations of the Lie pseudo-group (3.3.1) the infinitesimal interpretation of the Maurer–Cartan equations given in Section 3.2 suggests that Cartan does not really compute the infinitesimal structure of an intransitive Lie pseudo-group action but rather computes the infinitesimal structure of the transformation

$$X^i = x^i + a^i, \qquad Y^\alpha = f^\alpha(x, y), \qquad i = 1, \ldots, s, \qquad \alpha = 1, \ldots, t, \quad (3.4.14)$$

$a \in \mathbb{R}^s$, $f \in C^\omega(\mathbb{R}^m, \mathbb{R}^t)$. This explains why Cartan obtains the same structure equations for the two non-isomorphic Lie pseudo-groups (3.4.13). Also, one can verify that the structure equations (3.4.10) of Example 3.22 correspond to the infinitesimal structure of the Lie pseudo-group

$$X = x + a, \qquad Y = y + f(x), \qquad a \in \mathbb{R}, \qquad f \in C^\omega(\mathbb{R}).$$

In general there is no guarantee that the set of transformations (3.4.14) is a Lie pseudo-group. Indeed, in Example 3.21 if we replace (3.4.5) by

$$X = x + b, \qquad Y = y + ax, \qquad (a,b) \in \mathbb{R}^2, \tag{3.4.15}$$

this is no longer a group action since it is not closed under composition. Also, let us mention that the Cartan forms $\omega^{s+1}, \ldots, \omega^{\widetilde{r}_{n_\star}-1}$ are generally not invariant under the transformations (3.4.14).

From the above discussion we conclude that we should set

$$\omega^i_{[0]} = dx^i = 0, \qquad i = 1, \ldots, s. \tag{3.4.16}$$

This is in complete agreement with Olver and Pohjanpelto's structure theory since from (3.4.3) we have

$$0 = \widetilde{\mu}^i = -\omega^i_{[0]} = -dx^i, \qquad i = 1, \ldots, s,$$

on a target fiber. Geometrically, the equations (3.4.16) say that Cartan's structure equations should be restricted to the orbits of the pseudo-group action. Alternatively, in light of Remark 3.18 and the fact that $X^i = x^i$, we can say that Cartan's structure equations should be restricted to the target fibers of the pseudo-group action. The latter point of view is consistent with the derivation of the Maurer–Cartan structure equations (3.1.14).

Finally, we note that for transitive Lie pseudo-groups the Cartan structure equations (3.3.9) and the Maurer–Cartan structure equations (3.1.14) are isomorphic since $s = 0$ in (3.3.1).

3.5 Systatic System

From the Cartan structure equations (3.3.9), Cartan defines the notion of essential invariants for intransitive Lie pseudo-groups. To state his definition of essential invariants we need to introduce the concept of systatic system.

The systatic system plays an important role in the structure theory of Lie pseudo-groups, [20, 23]. It is related to the isotropy algebra of Lie pseudo-groups, [103], and the latter has been used to classify infinite-dimensional primitive Lie pseudo-groups, [21, 41, 42].

Definition 3.24. Let $\mathcal{G}$ be a Lie pseudo-group with $n_\star$-th order involutive defining system and Cartan structure equations[5]

$$d\omega^i = \sum_{1\le j<k\le \widetilde{r}_{n_\star-1}} C^i_{jk}\omega^j\wedge\omega^k + \sum_{\beta=1}^{t_{n_\star}}\left(\sum_{b=1}^{m} A^i_{b\beta}\omega^b_{[0]}\right)\wedge\overline{\pi}^\beta,$$

$i=1,\ldots,\widetilde{r}_{n_\star-1}$. The module of one-forms generated by

$$\sum_{b=1}^{m} A^i_{b\beta}\omega^b_{[0]}, \qquad i=1,\ldots,\widetilde{r}_{n_\star-1}, \qquad \beta=1,\ldots,t_{n_\star}, \tag{3.5.1}$$

over the ring of invariants of $\mathcal{G}$ is called the *systatic system* of $\mathcal{G}$.

The concept of systatic system can also be defined for the Maurer–Cartan structure equations (3.1.14). On the lifted infinitesimal determining equations (3.1.13) we fix an order compatible term ordering of the partial derivatives that ranks derivatives of higher total order greater than those of lower total order, [99]. Gauss reduction of the lifted infinitesimal determining equations with respect to the ordering yields a solved form expressing certain dependents, the principal Maurer–Cartan forms, as functions of lower ranked non-principal (parametric) Maurer–Cartan forms, [67, 70, 96]. The parametric Maurer–Cartan forms form a basis of invariant contact forms on $\mathcal{G}^{(\infty)}$. We introduce the notation

$$\mu^i_{(n)}, \qquad i=1,\ldots,r_n=\dim\,\mathcal{G}^{(n)},$$

[5]A detailed analysis of Cartan's structure equations reveals that the second term in (3.3.9) is of the form $\sum_{\beta=1}^{t_{n_\star}}\sum_{b=1}^{m} A^i_{b\beta}\omega^b_{[0]}\wedge\pi^\beta$. In fact, more can be said. The coefficients $A^i_{b\beta}$ are zero if ω^i is a Cartan form of order $\le n_\star-2$ or an invariant horizontal form.

to denote a basis of Maurer–Cartan forms of order $\leq n$, and

$$\mu^i_{[n]}, \qquad i = 1, \ldots, t_n = r_n - r_{n-1},$$

to denote a basis of n-th order Maurer–Cartan forms ($r_{-1} = 0$). With this notation, the Maurer–Cartan structure equations (3.1.14), for the Maurer–Cartan forms of order $\leq n_\star - 1$, can be written as[6]

$$d\mu^i_{(n_\star-1)} = \sum_{1\leq j<k\leq r_{n_\star-1}} C^i_{jk}\mu^j_{(n_\star-1)}\wedge\mu^k_{(n_\star-1)} + \sum_{\beta=1}^{t_{n_\star}}\left(\sum_{j=1}^{t} A^i_{j\beta}\mu^j_{[0]}\right)\wedge\mu^\beta_{[n_\star]}, \quad (3.5.2)$$

where $i = 1, \ldots, r_{n_\star-1}$.

Definition 3.25. Let $\mathcal{G}$ be a Lie pseudo-group with Maurer–Cartan structure equations (3.5.2). The systatic system is defined as the module of one-forms generated by

$$\sum_{j=1}^{t} A^i_{j\beta}\mu^j_{[0]}, \qquad i = 1, \ldots, r_{n_\star-1}, \qquad \beta = 1, \ldots, t_{n_\star}, \quad (3.5.3)$$

over the ring of invariants of $\mathcal{G}$.

For transitive Lie pseudo-groups, the Definitions 3.24 and 3.25 are related together by the equalities

$$\mu^a = -d_M Z^a = -\omega^a_{[0]}, \qquad a = 1, \ldots, m, \quad (3.5.4)$$

on a target fiber $(\tau^{(\infty)})^{-1}(Z)$. For intransitive Lie pseudo-groups the definitions do not completely agree as the Cartan and Maurer–Cartan structure equations are different.

Example 3.26. Consider the Lie pseudo-group of conformal transformations

[6] As for the Cartan structure equations $A^i_{j\beta} = 0$ for all $\mu^i_{(n_\star-2)}$, $i = 1, \ldots, r_{n_\star-2}$.

of the plane

$$X = f(x,y), \qquad Y = g(x,y), \qquad f_x g_y - f_y g_x = 1. \tag{3.5.5}$$

The first order involutive infinitesimal determining system, for an infinitesimal generator

$$\mathbf{v} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y},$$

is

$$\xi_x + \eta_y = 0. \tag{3.5.6}$$

The lift of equation (3.5.6) gives the linear relation

$$\nu^x_X = -\nu^y_Y.$$

The structure equations for the zero order Maurer–Cartan forms ν^x and ν^y are

$$d\nu^x = \nu^x_X \wedge \nu^x + \nu^x_Y \wedge \nu^y, \qquad d\nu^y = \nu^y_X \wedge \nu^x - \nu^x_X \wedge \nu^y,$$

and we conclude that the systatic system is generated by

$$\{\nu^x, \nu^y\}. \tag{3.5.7}$$

Cartan computed the systatic system for the pseudo-group (3.5.5) in [23]. Our computations agree with his computations when we take into account the equalities (3.5.4).

3.6 Essential Invariants

In this section we recall Cartan's definition of essential invariants. Though numerous papers and books acknowledge the importance of essential invariants, very little seems to be known about those invariants. It is a challenge to understand the structure theory of Lie pseudo-groups that admit essential invariants.

Definition 3.27. A Pfaffian system generated by the one-forms $\omega^1, \ldots, \omega^r$ is said to be *complete* if

$$d\omega^i \equiv 0 \qquad \text{mod } \omega^1, \ldots, \omega^r, \qquad i = 1, \ldots, r.$$

In [23], Cartan shows that the systatic system (3.5.1) is complete. From this system he extracts as many linear combinations as possible that only depend on the invariants $x^1, \ldots, x^s$ and their differentials $dx^1, \ldots, dx^s$. Suppose there are $l \leq s$ linearly independent such combinations:

$$\Omega^j = \sum_{i=1}^{s} f_i^j(x^1, \ldots, x^s)dx^i, \qquad j = 1, \ldots, n. \tag{3.6.1}$$

The completeness of the systatic system implies that the Pfaffian system $(\Omega^1, \ldots, \Omega^l)$ is complete in the space of invariants $x^1, \ldots, x^s$. Cartan defines the first integrals[7] of $(\Omega^1, \ldots, \Omega^l)$ to be *essential invariants*, while the other invariants are said to be *inessential*. Reading Cartan's original work on this subject can be very confusing as the next example shows.

Example 3.28. Since the systatic systems of Lie group actions are always empty (the $A^i_{j\beta}$ are all zero in (3.5.1)) all invariants are inessential. Yet, in [20, p.1357] Cartan considers the group action

$$X = x, \qquad Y = y, \qquad Z = z + ax + by, \qquad a, b \in \mathbb{R}, \tag{3.6.2}$$

and writes that the ratio x/y is an essential invariant.

Example 3.29. A slight generalization of Example 3.28 is the pseudo-group

$$X = x, \qquad Y = y, \qquad Z = z + f(x)y + g(x), \tag{3.6.3}$$

[7] The existence of those first integrals is guaranteed by Frobenius' Theorem.

where f, $g \in C^\omega(\mathbb{R})$. The determining system of this Lie pseudo-group is

$$X = x, \qquad Y = y, \qquad Z_z = 1, \qquad Z_{yy} = 0.$$

Applying Cartan's algorithm we obtain the structure equations

$$\begin{gathered} d\omega^1 = 0, \qquad d\omega^2 = 0, \qquad d\omega^3 = \omega^4 \wedge \omega^1 + \omega^5 \wedge \omega^2, \\ d\omega^4 = \omega^1 \wedge \overline{\pi}^1 + \omega^2 \wedge \overline{\pi}^2, \qquad d\omega^5 = \omega^1 \wedge \overline{\pi}^2, \end{gathered} \tag{3.6.4}$$

where

$$\begin{gathered} \omega^1 = dx, \qquad \omega^2 = dy, \qquad \omega^3 = Z_x dx + Z_y dy + dz, \\ \omega^4 = dZ_x - Z_{xx} dx - Z_{xy} dy, \qquad \omega^5 = dZ_y - Z_{xy} dx, \\ \overline{\pi}^1 = dZ_{xx}, \qquad \overline{\pi}^2 = dZ_{xy}. \end{gathered}$$

From (3.6.4) we conclude that the systatic system is generated by the one-forms ω^1, ω^2 and that the invariants x and y are essential. But we can argue that the invariant y is not essential as the Lie pseudo-group (3.6.3) is isomorphic (the definition of isomorphism for Lie pseudo-groups is given below) to the pseudo-group

$$X = x, \qquad V = v + f(x), \qquad W = w + g(x),$$

which does not involve the invariant y.

Now recall that we made the observation that the differential forms $\omega^1_{[0]}$, $\ldots$, $\omega^s_{[0]}$ should be set equal to zero for the Cartan structure equations (3.3.9) to give the adequate infinitesimal structure of intransitive Lie pseudo-groups. If we include this observation in the above discussion of essential invariants we notice that the differential forms $\Omega^1, \ldots, \Omega^l$ are all identically zero. Hence we conclude that the definition of essential invariants in terms of the systatic system (3.5.3) of the Maurer–Cartan structure equations (3.5.2) is vacuous.

Of course we can recover Cartan's definition from the structure equations (3.1.4) for the invariant coframe σ, $\mu^{(\infty)}|_{\mathcal{G}^{(\infty)}}$. But Examples 3.28 and 3.29 suggest that this is not necessarily what one should do. It seems preferable to have a definition that only depends on the Maurer–Cartan structure equations. With this in mind we propose an alternative definition of essential invariants, which, we believe, still captures the essence of Cartan's original definition. In order to justify our definition, we need to review the notion of isomorphism for Lie pseudo-groups, [23, 67, 109]. To do so, some preliminary definitions need to be given.

Definition 3.30. A Lie pseudo-group $\mathcal{H} \subset \mathcal{D}(N)$ is *similar* to a Lie pseudo-group $\mathcal{G} \subset \mathcal{D}(M)$ if there is a local diffeomorphism $\phi : N \to M$ such that $\mathcal{H} = \phi^{-1} \circ \mathcal{G} \circ \phi$.

Lemma 3.31. If $\mathcal{G}$ and $\mathcal{H}$ are similar, their structure equations are isomorphic.

Definition 3.32. Let $\mathcal{G} \subset \mathcal{D}(M)$ and $\overline{\mathcal{G}} \subset \mathcal{D}(\overline{M})$ be two Lie pseudo-groups such that $\pi : \overline{M} \to M$ is a fiber bundle with base space M. If for all $\overline{\phi} \in \overline{\mathcal{G}}$ there exist $\phi \in \mathcal{G}$ such that $\pi \circ \overline{\phi} = \phi \circ \pi$, then $\overline{\mathcal{G}}$ is called a *Cartan prolongation* of $\mathcal{G}$.

In the literature, our definition of Cartan prolongation is often simply called prolongation. We introduce this new terminology to clearly distinguish the more general notion of Lie pseudo-group prolongation stated in Definition 3.32 from the usual definition of prolonged pseudo-group introduced in Definition 2.22.

Definition 3.33. A Cartan prolongation $\overline{\mathcal{G}} \subset \mathcal{D}(\overline{M})$ of $\mathcal{G} \subset \mathcal{D}(M)$ is said isomorphic if the only diffeomorphism of $\overline{\mathcal{G}}$ that projects to $\mathbb{1}_M$ is $\mathbb{1}_{\overline{M}}$.

Definition 3.34. Two Lie pseudo-groups $\mathcal{G}$ and $\mathcal{H}$ are said to be *isomorphic* if there exist isomorphic Cartan prolongations $\overline{\mathcal{G}}$ and $\overline{\mathcal{H}}$ such that $\overline{\mathcal{G}}$ is similar to $\overline{\mathcal{H}}$.

Of all possible isomorphic Cartan prolongations of $\mathcal{G}$, the infinite prolongation $\mathcal{G}^{(\infty)}$ is the most important. Indeed, let $\mathcal{G}$ and $\mathcal{H}$ be two Lie pseudo-groups such that $\mathcal{H}$ is an isomorphic Cartan prolongation of $\mathcal{G}$. Assuming that the invariants of $\mathcal{H}$ can be expressed by means of the local coordinates of the manifold that $\mathcal{G}$ acts on, Cartan shows in [23] that there exists n such that $\mathcal{G}^{(n)}$ is an isomorphic Cartan prolongation of $\mathcal{H}$. If $\mathcal{H}$ admits some invariants which are not acted upon by $\mathcal{G}$, $\mathcal{G}$ is extended by acting trivially on these, and then $\mathcal{G}^{(n)} \oplus \mathbb{1}$ is an isomorphic Cartan prolongation of $\mathcal{H}$ for some n. This implies that given two isomorphic Lie pseudo-groups $\mathcal{G}$, $\mathcal{H}$, up to the addition of scalar invariants, $\mathcal{G}^{(\infty)}$ is isomorphic to $\mathcal{H}^{(\infty)}$. In particular, their Maurer–Cartan structure equations are isomorphic.

Example 3.35. To illustrate the above definitions, consider the Lie pseudo-groups

$$\mathcal{H}: \qquad \widetilde{X} = \widetilde{x}, \qquad \widetilde{W} = \widetilde{w} + f(\widetilde{x}),$$

and

$$\mathcal{G}: \qquad X = x, \qquad Y = y + f(x)z + f'(x), \qquad Z = z,$$

where $f \in C^\omega(\mathbb{R})$. The Lie pseudo-group

$$\overline{\mathcal{H}}: \qquad \widetilde{X} = \widetilde{x}, \qquad \widetilde{Y} = \widetilde{y} + f(\widetilde{x})\widetilde{z} + f'(\widetilde{x}), \qquad \widetilde{Z} = \widetilde{z}, \qquad \widetilde{W} = \widetilde{w} + f(\widetilde{x}),$$

is an isomorphic Cartan prolongation of $\mathcal{H}$. Similarly

$$\overline{\mathcal{G}}: \qquad X = x, \qquad Y = y + f(x)z + f'(x), \qquad Z = z, \qquad W = w + f(x),$$

is an isomorphic Cartan prolongation of $\mathcal{G}$. Clearly, $\overline{\mathcal{H}}$ is similar to $\overline{\mathcal{G}}$, thus $\mathcal{H}$ and $\mathcal{G}$ are isomorphic Lie pseudo-groups. Alternatively we note that $\mathcal{H}^{(1)} \oplus \mathbb{1}_{\widetilde{z}}$ is similar to $\overline{\mathcal{G}}$ since

$$\mathcal{H}^{(1)} \oplus \mathbb{1}_{\widetilde{z}}: \qquad \widetilde{X} = \widetilde{x}, \qquad \widetilde{W} = \widetilde{w} + f(\widetilde{x}), \qquad \widetilde{Y} = \widetilde{y} - f'(\widetilde{x}), \qquad \widetilde{Z} = \widetilde{z},$$

and

$$\mathcal{H}^{(1)} \oplus \mathbb{1}_{\widetilde{z}} = \phi^{-1} \circ \overline{\mathcal{G}} \circ \phi,$$

with

$$\phi : (\widetilde{x}, \widetilde{w}, \widetilde{y}, \widetilde{z}) \mapsto (x, y, z, w) = (\widetilde{x}, \widetilde{w}\widetilde{z} - \widetilde{y}, \widetilde{z}, \widetilde{w}).$$

Proposition 3.36. Let $\mathcal{G}$ be an intransitive Lie pseudo-group locally represented by (3.3.1). If the Maurer–Cartan structure equations (3.4.4) depend on a (non-constant) invariant $I(X^1, \ldots, X^s)$ for one basis of Maurer–Cartan forms then the structure equations also depend on the invariant I for any other basis of Maurer–Cartan forms.

Definition 3.37. Let $\mathcal{G}$ be an intransitive Lie pseudo-group locally represented by (3.3.1). A (non-constant) function $I(X^1, \ldots, X^s)$ of the invariants $X^1, \ldots, X^s$ is called an *essential invariant* if for a basis of Maurer–Cartan forms (hence for all), the structure coefficients of the Maurer–Cartan structure equations (3.4.4) depend on $I(X^1, \ldots, X^s)$.

Example 3.38. In Example 3.22, the Cartan structure equations for the intransitive Lie pseudo-group (3.4.8) are given by (3.4.10). The systatic system is spanned by $\omega^1 = dx$, and in Cartan's sense x is considered to be an essential invariant. On the other hand, based on the Maurer–Cartan structure equations (3.4.12) and Definition 3.37, we say that the invariant x is not essential since it does not appear in the structure equations, and thus does not influence the infinitesimal structure of the pseudo-group.

Remark 3.39. In [20, 23], Cartan gives an algorithm to get rid of inessential invariants. With our definition of essential invariants no such claim is made. Example 3.38 is an illustration of this fact. Though the invariant x is not essential, in the sense of Definition 3.37, it is impossible to write the pseudo-group action (3.4.8), on a finite-dimensional manifold, without the use of the invariant x.

Example 3.40. The pseudo-group action (3.3.10) in Example 3.19 is an illustration of a Lie pseudo-group with essential invariant. The defining system of this Lie pseudo-group is

$$X = x, \qquad Y_x = 0, \qquad Y_z = 0, \qquad Z_z = (Y_y)^x,$$

and the infinitesimal determining equations, for an infinitesimal generator

$$\mathbf{v} = \xi(x,y,z)\frac{\partial}{\partial x} + \eta(x,y,z)\frac{\partial}{\partial y} + \phi(x,y,z)\frac{\partial}{\partial z},$$

are

$$\xi = 0, \qquad \eta_x = 0, \qquad \eta_z = 0, \qquad \phi_z = x\eta_y. \tag{3.6.5}$$

The lift of (3.6.5) gives the linear relations

$$\mu^x = 0, \qquad \mu^y_X = 0, \qquad \mu^y_Z = 0, \qquad \mu^z_Z = X\mu^y_Y,$$

and it follows that

$$\mu^y_{Y^k}, \qquad \mu^z_{X^kY^j}, \qquad k,j \geq 0,$$

is a basis of Maurer–Cartan forms. Focusing our attention on the differentials of μ^y and μ^z we obtain

$$\begin{aligned} d\mu^y =& \mu^y_X \wedge \mu^x + \mu^y_Y \wedge \mu^y + \mu^y_Z \wedge \mu^z = \mu^y_Y \wedge \mu^y, \\ d\mu^z =& \mu^z_X \wedge \mu^x + \mu^z_Y \wedge \mu^y + \mu^z_Z \wedge \mu^z = \mu^z_Y \wedge \mu^y + X\mu^y_Y \wedge \mu^z, \end{aligned}$$

and conclude that x is an essential invariant.

Example 3.16 is another instance of a Lie pseudo-group with essential invariant.

Chapter 4

Symmetry-Based Linearization Theorem

Given two m-dimensional manifolds M and $\overline{M}$, the linearization problem for k-th order systems of partial differential equations consists of determining if there exists a change of variables

$$\Lambda : J^{(k)}(M,p) \to J^{(k)}(\overline{M},p) \tag{4.0.1}$$

sending a k-th order system of $q = m - p$ independent nonlinear partial differential equations

$$0 = \Delta(x, u^{(k)}), \tag{4.0.2}$$

to a k-th order linear system of q independent partial differential equations

$$0 = f(y) - L[y]v, \tag{4.0.3}$$

where $L[y]$ is a linear differential operator. All symmetry-based linearization theorems found in the literature, [10, 49, 50, 82], use Lie's structural theory to establish the existence of (4.0.1). To apply those theorems, the first step consists of integrating the infinitesimal determining equations for the infinitesimal symmetry generators of the nonlinear system (4.0.2). In this section a different version of the linearization theorem, based on the Maurer–Cartan structure equations, is stated. Our version does not require the integration of the infinitesimal determining system.

4.1 Background Material

Some assumptions need to be made on the systems (4.0.2) and (4.0.3). Those seem to never be clearly stated in the literature. First the number of dependent variables is assumed to be equal to the number of independent differential equations appearing in the systems of equations, and the number of independent variables p is assumed to be greater than one. To avoid unnecessary technicalities, the system (4.0.2) is assumed to be locally solvable and to define a regular submanifold of $J^{(k)}(M,p)$. On the other hand, the linear system (4.0.3) is assumed to be normal.

Definition 4.1. A system of q differential equations $\Delta(x,u^{(k)}) = 0$ in q dependent variables is *normal at the point*

$$(x_0, u_0^{(k)}) \in \mathcal{S}_\Delta = \{(x,u^{(k)}) : \Delta(x,u^{(k)}) = 0\}$$

if there exists at least one noncharacteristic direction for Δ there. The system is *normal* if it is normal at each point of $\mathcal{S}_\Delta$.

The assumption that the linear system (4.0.3) is normal means that we can assume the system to be in Kovalevskaya form

$$v^\alpha_{ky^p} = \frac{\partial^k v^\alpha}{\partial (y^p)^k} = L^\alpha(y, \widetilde{v^{(k)}}), \qquad \alpha = 1,\ldots,q, \tag{4.1.1}$$

where $\widetilde{v^{(k)}}$ denotes all partial derivatives of v with respect to y up to order k except the derivatives $v^\alpha_{ky^p}$ which appear on the left-hand side of (4.1.1).

By a theorem due to Bäcklund, [2, 4], the general linearization problem previously stated reduces to determining the existence of an invertible map between first order jet bundles.

Theorem 4.2. If $q \geq 1$, a mapping Λ defines an invertible map from $J^{(k)}(M, p)$ to $J^{(k)}(\overline{M}, p)$, for any fixed $p \geq 1$, if and only if Λ is the prolongation of a first order contact transformation

$$\Phi : J^{(1)}(M, p) \to J^{(1)}(\overline{M}, p), \qquad (y, v^{(1)}) = \Phi(x, u^{(1)}).$$

If the number of dependent variables is greater than one, Bäcklund also proved that the map Λ is the prolongation of a point transformation.

Theorem 4.3. If $q \geq 2$ then a mapping Λ defines an invertible map from $J^{(k)}(M, p)$ to $J^{(k)}(\overline{M}, p)$, for any fixed $p \geq 1$, if and only if Λ is the prolongation of a point transformation

$$\Phi : M \to \overline{M}, \qquad (y, v) = \Phi(x, u). \tag{4.1.2}$$

In the following we restrict our considerations to point transformations. In light of Theorem 4.3 this restriction covers all systems of partial differential equations with at least two dependent variables. With a little extra work, the discussion can be extended to first order contact transformations.

The linear system of partial differential equations (4.0.3) admits the distinctive infinite-dimensional symmetry subgroup $\mathcal{G}_L$:

$$v \mapsto v + g(y), \qquad \text{with} \qquad L[y]g = 0, \tag{4.1.3}$$

corresponding to the superposition principle of solutions for the linear homogeneous system of partial differential equations

$$L[y]v = 0. \tag{4.1.4}$$

At the heart of any symmetry-based linearization theorem is the following proposition.

Proposition 4.4. Two systems of partial differential equations equivalent under a locally invertible change of variables have isomorphic symmetry groups.

Proposition 4.4 says that if $\Phi : (x, u) \mapsto (y, v)$ is a locally invertible change of variables mapping one system of differential equations to another and g is a symmetry of the first system, then $\overline{g} = \Phi \circ g \circ \Phi^{-1}$ is a symmetry of the second system. Thus for the nonlinear system of partial differential equations (4.0.2) to be linearizable it must admit an infinite-dimensional symmetry subgroup isomorphic to the symmetry group (4.1.3).

4.2 Maurer–Cartan Linearization Theorem

The infinitesimal structure of the symmetry group (4.1.3) is now completely characterized by coordinate free properties of its Maurer–Cartan structure equations. Let

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z)\frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i(y,v)\frac{\partial}{\partial y^i} + \sum_{\alpha=1}^{q} \phi_\alpha(y,v)\frac{\partial}{\partial v^\alpha}$$

denote a symmetry generator of the group action (4.1.3), and denote the lift of the vector field coefficient jets by

$$\mu[\![H]\!] = \begin{pmatrix} \widetilde{\mu}[\![H]\!] \\ \nu[\![H]\!] \end{pmatrix} = \boldsymbol{\lambda} \begin{pmatrix} \xi[\![h]\!] \\ \phi[\![h]\!] \end{pmatrix}. \tag{4.2.1}$$

The minimal infinitesimal determining system of $\mathcal{G}_L$ is

$$\begin{aligned} &\xi^i = 0, \qquad i = 1, \ldots, p, \qquad \phi^\alpha_{v^\beta} = 0, \qquad \alpha, \beta = 1, \ldots, q, \\ &\phi^\alpha_{ky^p} = \frac{\partial^k \phi^\alpha}{\partial (y^p)^k} = L^\alpha(y, \widetilde{\phi^{(k)}}), \qquad \alpha = 1, \ldots, q, \end{aligned} \tag{4.2.2}$$

Note that the order of the symmetry group $\mathcal{G}_L$ is equal to the order of the linear system (4.0.3). The lift of (4.2.2) implies the linear relations

$$\widetilde{\mu}^i = 0, \qquad i = 1, \ldots, p, \qquad \nu^\alpha_{V^\beta} = 0, \qquad \alpha, \beta = 1, \ldots, q, \tag{4.2.3a}$$

$$\nu^\alpha_{kY^p} = L^\alpha(Y, \widetilde{\nu^{(k)}}), \qquad \alpha = 1, \ldots, q, \tag{4.2.3b}$$

among the Maurer–Cartan forms of order $\leq k$. The system of equations (4.2.3) implies that the differential forms $\nu^\alpha_{Y^J}$, $0 \leq \#J \leq k$, are the only nonzero Maurer–Cartan forms of order $\leq k$. For $n < k$ all the Maurer–Cartan forms $\nu^\alpha_{Y^J}$, $0 \leq \#J \leq n$, are linearly independent. Hence the symmetry group $\mathcal{G}_L$ has

$$q\binom{p+n}{n}$$

linearly independent Maurer–Cartan forms of order at most $n < k$. At order k, the system of equations (4.2.3b) expresses the q Maurer–Cartan forms $\nu^1_{ky^p}, \ldots, \nu^q_{ky^p}$ in terms of the remaining Maurer–Cartan forms $\widetilde{\nu^{(k)}}$. Thus there are

$$q\binom{p+k}{k} - q$$

linearly independent Maurer–Cartan forms of order $\leq k$. Since the system (4.2.3b) is in Kovalewskaya form, each prolonged equation

$$\nu^\alpha_{kY^pY^J} = \mathbb{D}^J_Y(L^\alpha(Y, \widetilde{\nu^{(k)}})), \qquad \alpha = 1, \ldots, q, \quad \#J \geq 0,$$

introduces a new linear relation among higher order Maurer–Cartan forms. Hence for $n \geq k$, there are

$$q\left[\binom{p+n}{n} - \binom{p+(n-k)}{n-k}\right]$$

linearly independent Maurer–Cartan forms of order $\leq n$. In summary

$$\dim \mu^{(n)}[\![H]\!]\big|_{\mathcal{G}_L} = \begin{cases} q\binom{p+n}{n} & \text{if } 0 \leq n \leq k-1, \\ q\left[\binom{p+n}{n} - \binom{p+(n-k)}{n-k}\right] & \text{if } n \geq k. \end{cases} \tag{4.2.4}$$

Remark 4.5. Equation (4.2.4) is false if $p = 1$. Indeed, if $p = 1$, we have

$$\nu^{\alpha}_{kY} = L^{\alpha}(Y, \nu^{(k-1)}),$$

which implies, by prolongation, that all Maurer–Cartan forms of order greater or equal to k can be expressed in terms of Maurer–Cartan forms of order less than k. This means that the symmetry group is finite-dimensional and $\dim\, \mu^{(k+l)} = \dim\, \mu^{(k)}$ for all $l \geq 0$.

Finally, the commutativity property of the symmetry group $\mathcal{G}_L$ is reflected by the Maurer–Cartan structure equations

$$d\mu^{(n)}[\![H]\!]\big|_{\mathcal{G}_L} = 0, \qquad n \geq k.$$

Theorem 4.6. Let $\mathcal{G}$ be a k-th order tame Lie pseudo-group acting on an m-dimensional smooth manifold M. The Maurer–Cartan forms associated with $\mathcal{G}$ satisfy the

1. dimensional property

$$\dim \mu^{(n)}[\![H]\!]|_{\mathcal{G}} = \begin{cases} q\binom{p+n}{n} & \text{if } 0 \leq n \leq k-1, \\ q\left[\binom{p+n}{n} - \binom{p+n-k}{n-k}\right] & \text{if } n \geq k, \end{cases} \tag{4.2.5}$$

with $p = m - q > 1$,

2. and structural property

$$d\mu^{(k)}[\![H]\!]|_{\mathcal{G}} = 0,$$

if and only if there exists a local system of coordinates

$$y = (y^1, \ldots, y^p), \qquad\qquad v = (v^1, \ldots, v^q),$$

on M such that the pseudo-group action can be written as

$$Y = y, \qquad V = v + g(y), \qquad \text{where} \qquad L[y]g = 0$$

is a k-th order normal system of q independent linear partial differential equations.

Proof. The discussion preceding the theorem proves the necessity. To prove the sufficiency we exploit the infinitesimal interpretation of the Maurer–Cartan structure equations (3.1.14) discussed in Section 3.2.

The number of linearly independent zero order Maurer–Cartan forms equals the dimension of the group orbits on M, [68]. Since dim $\mu[\![0]\!] = q$, in a well-adapted coordinate system

$$y = (y^1, \ldots, y^p), \qquad v = (v^1, \ldots, v^q),$$

there exist, locally, q linearly independent infinitesimal generators of the group action of the form

$$\mathbf{v}_1 = g^1(y,v)\frac{\partial}{\partial v^1}, \qquad \ldots, \qquad \mathbf{v}_q = g^q(y,v)\frac{\partial}{\partial v^q}, \tag{4.2.6}$$

with $g^\alpha \neq 0$, $\alpha = 1, \ldots, q$. Expanding the vector fields (4.2.6) in Taylor series in the neighborhood of a point $(y_0, v_0) \in M$ and truncating at order k, we obtain

$$\mathbf{v}_\alpha^k = \sum_{\#J+\#K=0}^{k} g_{J,K}^\alpha(y_0,v_0)\frac{(y-y_0)^J}{J!}\frac{(v-v_0)^K}{K!}\frac{\partial}{\partial v^\alpha}, \qquad \alpha = 1, \ldots, q,$$

where

$$g^{\alpha}_{J,K}(y_0,v_0) = \frac{\partial^l g^{\alpha}(y_0,v_0)}{(\partial y^1)^{j^1}\cdots(\partial y^p)^{j^p}(\partial v^1)^{\kappa^1}\cdots(\partial v^q)^{\kappa^q}}, \qquad l = \#J + \#K.$$

Condition 2 of the theorem implies that the vector fields $\partial_{v^1},\ldots,\partial_{v^q}$ commute:

$$0 = \left[\frac{\partial}{\partial v^{\alpha}}, \frac{\partial}{\partial v^{\beta}}\right], \qquad \alpha,\beta = 1,\ldots,q.$$

It also implies the equalities

$$0 = \left[\frac{\partial}{\partial v^{\alpha}}, \mathbf{v}^k_{\beta}\right] = \sum_{\#J+\#K=0}^{k} g^{\beta}_{J,K}(y_0,v_0)\frac{(y-y_0)^J}{J!}\frac{(v-v_0)^{K\backslash\alpha}}{(K\backslash\alpha)!}\frac{\partial}{\partial v^{\beta}}, \tag{4.2.7}$$

$\alpha,\beta = 1,\ldots,q$. The equations (4.2.7) are satisfied provided $K = 0$, i.e., the vector field coefficients $g^1,\ldots,g^q$ do not depend on the coordinates $v = (v^1,\ldots,v^q)$. Thus the infinitesimal generators (4.2.6) reduce to

$$\mathbf{v}_1 = g^1(y)\frac{\partial}{\partial v^1}, \qquad \ldots, \qquad \mathbf{v}_q = g^q(y)\frac{\partial}{\partial v^q}. \tag{4.2.8}$$

Let $\nu^{\alpha} = \boldsymbol{\lambda}(g^{\alpha})$, $\alpha = 1,\ldots,q$. Note that $\nu^{\alpha}_{V^{\beta}} = 0$, $\alpha,\beta = 1,\ldots,q$, since the g^{α} do not depend on v. The dimensional constraint

$$\dim\, \mu^{(n)}[\![H]\!]|_{\mathcal{G}} = q\binom{p+n}{n}, \qquad n < k,$$

implies that the Maurer–Cartan forms $\nu^{\alpha}_{Y^J}$, $0 \le \#J \le k-1$, are linearly independent. The requirement

$$\dim\, \mu^{(k)}[\![H]\!]|_{\mathcal{G}} = q\binom{p+k}{k} - q$$

is satisfied if and only if there exist q independent relations

$$\nu^{\alpha^l}_{Y^{J^l}} = L^{\alpha^l}_{J^l}(Y, \widetilde{\nu^{(k)}}), \qquad \#J^l = k, \quad l = 1,\ldots,q, \tag{4.2.9}$$

where $\widetilde{\nu^{(k)}}$ denotes all the Maurer–Cartan forms of order $\leq k$ that do not appear on the left-hand side of (4.2.9). To satisfy the dimensional constraint (4.2.5) for all $n > k$, no two α^l can be equal. Indeed, if it were the case we would have two linear relations of the form

$$\nu^\alpha_{Y^J} = L^\alpha_J(Y, \widetilde{\nu^{(k)}}), \qquad \nu^\alpha_{Y^I} = L^\alpha_I(Y, \widetilde{\nu^{(k)}})$$

with $J \neq I$, and by choosing two multi-indices A, B such that

$$\mathbb{D}^A_Y(L^\alpha_J(Y, \widetilde{\nu^{(k)}})) = \nu^\alpha_{Y^{J+A}} = \nu^\alpha_{Y^{I+B}} = \mathbb{D}^B_Y(L^\alpha_I(Y, \widetilde{\nu^{(k)}}))$$

we would obtain the inequality

$$\dim\, \mu^{(n)}[\![H]\!]|_{\mathcal{G}} > q\binom{p+n}{n} - q\binom{p+n-k}{n-k}$$

for $n = \#J + \#A = \#I + \#B$. Thus system (4.2.9) is of the form

$$\nu^\alpha_{Y^{J^\alpha}} = L^\alpha(Y, \widetilde{\nu^{(k)}}), \qquad \alpha = 1, \ldots, q, \qquad \#J = k. \tag{4.2.10}$$

Equation (4.2.10) comes from the lift of

$$\phi^\alpha_{y^{J^\alpha}} = L^\alpha(y, \widetilde{\phi^{(k)}}), \qquad \alpha = 1, \ldots, q, \qquad \#J = k. \tag{4.2.11}$$

which is obviously nondegenerate. It is also locally solvable since the pseudo-group is assumed to be tame. Hence the linear system (4.2.11) is normal. □

Theorem 4.7. Let $\Delta(x, u^{(k)}) = 0$ be a nonlinear system of q functionally independent partial differential equations in $p \geq 2$ independent variables and q dependent variables. Suppose that the system admits a symmetry subgroup satisfying the properties of Theorem 4.6, then it can be mapped to a linear system of partial differential equations by a change of variables.

4.2.1 Lie Sub-Pseudo-Groups of a Lie Pseudo-Group

For a system of differential equations it is not difficult to determine the Maurer–Cartan structure equations of the whole symmetry group. The infinitesimal determining equations of the symmetry group are obtained using Lie's algorithm and the Maurer–Cartan structure equations follow from the theory discussed in Section 3.1. From the lifted infinitesimal determining equations and the Maurer–Cartan structure equations it is generally difficult to determine which sub-collections of Maurer–Cartan forms come from sub-pseudo-groups of the whole pseudo-group. Cartan developed an algorithm to deal with this problem in the context of his structure theory, [18]. But as it is frequently the case, his solution is difficult to understand. A similar solution must exist in the context of the structure theory discussed in Section 3.1. In this section we outline some of the first steps towards a complete solution. As we will see some important questions still need to be answered.

We start by restricting our attention to finite-dimensional Lie group actions. Let G be an r-dimensional Lie group of transformations with lifted infinitesimal determining system

$$L^{(n^\star)}(Z,\mu^{(n^\star)})=0. \tag{4.2.12}$$

Let

$$\mu^1_{(n^\star)},\ldots,\mu^r_{(n^\star)} \tag{4.2.13}$$

be a basis of Maurer–Cartan forms with structure equations

$$d\mu^i_{(n^\star)}=\sum_{1\leq j<k\leq r} C^i_{jk}\mu^j_{(n^\star)}\wedge\mu^k_{(n^\star)}, \qquad i=1,\ldots,r. \tag{4.2.14}$$

Assume that $H\subset G$ is a Lie subgroup of dimension $s<r$. Then there exist $r-s$ linear relations among the Maurer–Cartan forms (4.2.13). Without loss

of generality we assume that those relations are of the form

$$\begin{aligned} \mu^{s+1}_{(n^\star)} =& C^{s+1}_1(Z)\mu^1_{(n^\star)} + \cdots + C^{s+1}_s(Z)\mu^s_{(n^\star)}, \\ &\vdots \\ \mu^r_{(n^\star)} =& C^r_1(Z)\mu^1_{(n^\star)} + \cdots + C^r_s(Z)\mu^s_{(n^\star)}. \end{aligned} \tag{4.2.15}$$

The coefficients $C^\alpha_i(Z)$, $i = 1, \ldots, s$, $\alpha = s+1, \ldots, r$, are not arbitrary, they must satisfy two systems of equations. Substituting the linear relations (4.2.15) into the lifted infinitesimal determining system (4.2.12) gives a system of differential equations for the coefficients C^α_i. This system of equations should be considered as being equivalent to the system of differential equations [18, p. 752, eq. (4)] obtained by Cartan. Next, the requirement that $d^2\mu^i_{(n^\star)} = 0$, $i = 1, \ldots, s$, gives a second system of algebraic equations for the C^α_i which is analogous to the system of equations [18, p. 752, eq. (3)]. If it is not possible to find coefficients C^α_I satisfying the above requirements then we conclude that the Maurer–Cartan forms $\mu^1_{(n^\star)}, \ldots, \mu^s_{(n^\star)}$ are not those of a Lie sub-pseudo-group.

If a solution exists there are still some open questions that need to be answered:

- The linear relations (4.2.12) and (4.2.15) come from the lift of some infinitesimal determining system. Is this infinitesimal determining system integrable and locally solvable?

- The solution might not be unique. For different values of C^α_l do we obtain non-isomorphic Lie sub-pseudo-groups?

- Is there an algorithm to systematically obtain all Lie-sub-pseudo-groups of a given Lie pseudo-group?

Example 4.8. We consider the action of the projective group $PGL(2,\mathbb{R})$ on $\mathbb{RP}^1$ given by

$$X = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \alpha\delta - \beta\gamma = 1.$$

The lifted infinitesimal determining system for this transformation group is, [88],

$$\mu_{XXX} = 0. \tag{4.2.16}$$

Thus the Maurer–Cartan forms μ, μ_X, μ_{XX} form a basis of invariant differential forms and their Maurer–Cartan structure equations are

$$\begin{aligned} d\mu &= \mu_X \wedge \mu, \\ d\mu_X &= \mu_{XX} \wedge \mu, \\ d\mu_{XX} &= \mu_{XX} \wedge \mu_X. \end{aligned}$$

Suppose we search for a subgroup of transformations such that

$$\mu_{XX} = C(X)\mu + D(X)\mu_X. \tag{4.2.17}$$

Substituting (4.2.17) into (4.2.16) we obtain the equation

$$0 = \mathbb{D}_X(\mu_{XX}) = (C_X + CD)\mu + (C + D_X + D^2)\mu_X.$$

Since μ and μ_X are linearly independent Maurer–Cartan forms we conclude that the coefficients C and D must satisfy the system of equations

$$C_X + CD = 0, \qquad C + D_X + D^2 = 0. \tag{4.2.18}$$

The system (4.2.18) is equivalent to

$$C = -D_X - D^2, \qquad D_{XX} + 3DD_X + D^3 = 0.$$

An obvious solution to this system of differential equations is $C = D = 0$. For $D(X) \neq 0$ we obtain the solution

$$D(X) = \frac{2e^{2c_1/9}(X + c_2)}{3 + e^{2c_1/9}(X + c_2)^2}, \qquad C(X) = -\frac{2e^{2c_1/9}}{3 + e^{2c_1/9}(X + c_2)^2}, \tag{4.2.19}$$

where c_1 and c_2 are two constants of integration. Using MATHEMATICA we verified that the infinitesimal determining equation

$$\xi''(x) = -\frac{2e^{2c_1/9}}{3+e^{2c_1/9}(X+c_2)^2}\xi(x) + \frac{2e^{2c_1/9}(X+c_2)}{3+e^{2c_1/9}(X+c_2)^2}\xi'(x)$$

has a solution[1].

Under the substitution (4.2.17) the structure equations for μ and μ_X reduce to

$$d\mu = \mu_X \wedge \mu, \qquad d\mu_X = D(X)\mu_X \wedge \mu. \tag{4.2.20}$$

By transitivity of the group action we can fix X to any convenient constant. From (4.2.19) we choose to set $X = -c_2$ so that $D(X) = 0$. Then the Maurer–Cartan structure equations are isomorphic to the structure equations of the two-dimensional transformation group

$$X = \alpha x + \beta, \qquad \alpha \in \mathbb{R}^+, \qquad \beta \in \mathbb{R}. \tag{4.2.21}$$

Note that it is also possible to get rid of $D(X)$ in the structure equations (4.2.20) by redefining μ_X to be

$$\widetilde{\mu}_X = \mu_X - D(X)\mu.$$

Remark 4.9. The result of our computations in Example 4.8 is an illustration of the well-know fact that, up to a local isomorphism, every two-dimensional connected group acting on $\mathbb{R}$ locally effectively is locally equivalent to (4.2.21), [82].

A similar discussion holds for infinite-dimensional Lie pseudo-groups. But an aspect to be careful with is that the order of a Lie sub-pseudo-group can be greater than the order of the pseudo-group in which it is contained.

[1] We do not write down the solution since we do not need it and it would take too much space.

Example 4.10. The diffeomorphism pseudo-group $\mathcal{D}(\mathbb{R})$ is a Lie pseudo-group with empty determining system. On the other hand, the subgroup of translations

$$X = x + a, \qquad a \in \mathbb{R},$$

is a sub-pseudo-group of order one with defining system $X_x = 1$.

Let $\mathcal{H}$ be an n-th order Lie sub-pseudo-group contained in a Lie pseudo-group $\mathcal{G}$ of order $n^\star$. Let

$$L^{(k)}(Z, \mu^{(k)}) = 0, \tag{4.2.22}$$

be the lifted infinitesimal determining system of $\mathcal{G}^{(k)}$, where $k = n$ if $n \geq n^\star$ or $k = n^\star$ if $n < n^\star$. Let

$$\mu^1_{(k)}, \ldots, \mu^{r_k}_{(k)}$$

be a basis of Maurer–Cartan forms of order less or equal to k. As we did with Lie groups of transformations, we assume that $r_k - s$ Maurer–Cartan forms become linearly dependent when restricted to the sub-pseudo-group $\mathcal{H}$:

$$\begin{aligned} \mu^{s+1}_{(k)} &= C^{s+1}_1(Z)\mu^1_{(k)} + \cdots + C^{s+1}_s(Z)\mu^s_{(k)}, \\ &\vdots \\ \mu^{r_k}_{(k)} &= C^{r_k}_1(Z)\mu^1_{(k)} + \cdots + C^{r_k}_s(Z)\mu^s_{(k)}. \end{aligned} \tag{4.2.23}$$

Substituting (4.2.23) into (4.2.22) gives a system of differential equations for the $C^\alpha_i(Z)$, $\alpha = s+1, \ldots, r_k$, $i = 1, \ldots, s$. Let $\widetilde{L}^{(k)}(Z, \mu^{(k)}) = 0$, be the new system of lifted infinitesimal determining equations obtained by adding to (4.2.22) the relations (4.2.23). Then the Maurer–Cartan structure equations for the Lie sub-pseudo-group $\mathcal{H}$ are

$$(d\mu[\![H]\!] = \nabla_H \mu[\![H]\!] \wedge \mu[\![H]\!])_{\widetilde{L}^{(\infty)}(Z,\mu^{(\infty)})=0}.$$

The requirement that $d^2 = 0$ induces a second system of equations for the coefficients $C_i^\alpha(Z)$.

Example 4.11. We consider the diffeomorphism pseudo-group $\mathcal{D}(\mathbb{R})$, and we search for a Lie sub-pseudo-group generated by three Maurer–Cartan forms μ, μ_1, and μ_2. It is enough to set

$$\mu_3 = C(X)\mu + D(X)\mu_1 + E(X)\mu_2.$$

as the Maurer–Cartan forms μ_k, $k \geq 4$, are recovered by Lie differentiating μ_3 with respect to $\mathbb{D}_X$. Since the determining system of $\mathcal{D}(\mathbb{R})$ is empty, the only compatibility conditions on the coefficients C, D, and E come from the structure equations. Recall from Example 3.6 that the structure equations for μ, μ_1 and μ_2 are

$$\begin{aligned} d\mu &= \mu_1 \wedge \mu, \\ d\mu_1 &= \mu_2 \wedge \mu, \\ d\mu_2 &= \mu_3 \wedge \mu + \mu_2 \wedge \mu_1 = D\mu_1 \wedge \mu + E\mu_2 \wedge \mu + \mu_2 \wedge \mu_1. \end{aligned} \tag{4.2.24}$$

Since

$$0 = d^2\mu_2 = E\mu_2 \wedge \mu \wedge \mu_1,$$

$E = 0$. The identities $d^2\mu = d^2\mu_1 = 0$ do not impose any restrictions on C and D, so

$$\mu_3 = C(X)\mu + D(X)\mu_1,$$

and the structure equations (4.2.24) reduce to

$$\begin{aligned} d\mu &= \mu_1 \wedge \mu, \\ d\mu_1 &= \mu_2 \wedge \mu, \\ d\mu_2 &= D(X)\mu_1 \wedge \mu + \mu_2 \wedge \mu_1. \end{aligned} \tag{4.2.25}$$

If we set $C(X) = D(X) = 0$ the structure equations (4.2.25) reduce to those of $PGL(2,\mathbb{R})$. In fact for any admissible choice of $C(X)$ and $D(X)$ the

Maurer–Cartan structure equations (4.2.25) must be isomorphic to $PGL(2,\mathbb{R})$, [82].

Example 4.12. As a further example we consider the Lie pseudo-group (2.5.1) of Example 2.46. The first order lifted infinitesimal determining system for this Lie pseudo-group is

$$\mu^x_Y = \mu^x_U = \mu^y_U = \mu^u_U = 0, \qquad \mu^y_Y = \mu^x_X, \qquad \mu^u = \mu^y_X. \tag{4.2.26}$$

It follows that a basis of Maurer–Cartan forms is given by

$$\mu^x_{X^k}, \qquad \mu^y_{X^k}, \qquad k \geq 0. \tag{4.2.27}$$

The Maurer–Cartan structure equations are

$$\begin{aligned} d\mu_{X^n} &= \sum_{i=0}^{n} \binom{n}{i} \mu^x_{X^{i+1}} \wedge \mu^x_{X^{n-i}}, \\ d\mu^y_{X^n} &= \mu^y_{X^{n+1}} \wedge \mu^x + \mu^x_{X^{n+1}} \wedge \mu^y + \sum_{i=0}^{n-1} \left[\binom{n}{i} - \binom{n}{i+1} \right] \mu^y_{X^{i+1}} \wedge \mu^x_{X^{n-i}}. \end{aligned}$$

We search for a sub-pseudo-group generated by the Maurer–Cartan forms

$$\mu^x_{X^k}, \qquad k \geq 0.$$

Thus we set

$$\mu^y = \sum_{k=0}^{n} C^k(X,Y,U) \mu^x_{X^k},$$

for some $n \geq 1$. The linear relation $\mu^y_U = 0$ in (4.2.26) implies that the coefficients C^k do not depend on U. On the other hand the equation $\mu^y_Y = \mu^x_X$ implies

$$(C^1_Y - 1)\mu^x_{X^1} + C^0_Y \mu^x + C^2_Y \mu^x_{X^2} + C^3_Y \mu^x_{X^3} + \cdots + C^n_Y \mu^x_{X^n} = 0.$$

Hence the coefficients $C^0, C^2, C^3, C^4, \ldots, C^n$ do not depend on Y and $C^1 = Y + C^1(X)$. This means that

$$\mu^y = Y\mu^x_X + \sum_{k=0}^{n} C^k(X)\mu^x_{X^k}.$$

We note that the structure equations for the Maurer–Cartan forms $\mu^x_{X^k}$ are those of the one-dimensional diffeomorphism pseudo-group. Thus for admissible functions $C^0(X), \ldots, C^n(X)$ the sub-pseudo-group is isomorphic to the one-dimensional diffeomorphism pseudo-group. The sub-pseudo-group corresponding to the case $C^k(X) = 0$, $k = 0, 1, \ldots, n$ is

$$X = f(x), \qquad Y = f'(x)y, \qquad U = u + \frac{f''(x)}{f'(x)}.$$

4.2.2 Application of the Linearization Theorem

Though the theory of Section 4.2.1 is far from being complete it can nevertheless be successfully used to give an illustration of Theorem 4.7. As an application, we consider the potential Burgers' equation

$$u_t + \frac{1}{2}u_x^2 - u_{xx} = 0. \tag{4.2.28}$$

Let

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}$$

be an infinitesimal symmetry generator of (4.2.28). Then it is a solution of the infinitesimal determining system

$$\begin{gathered} \tau_x = 0, \qquad \tau_u = 0, \qquad \xi_u = 0, \qquad \tau_t - 2\xi_x = 0, \\ 2\phi_{tx} + \phi_x + \xi_t = 0, \qquad \phi_{uu} + \frac{1}{2}\phi_u = 0, \qquad \phi_{xx} - \phi_t = 0. \end{gathered} \tag{4.2.29}$$

At the Maurer–Cartan level we have, taking the lift of (4.2.29), the linear dependencies

$$\begin{gathered}\mu^t_X = 0, \qquad \mu^t_U = 0, \qquad \mu^x_U = 0, \qquad \mu^t_T - 2\mu^x_X = 0,\\ 2\mu^u_{UX} + \mu^u_X + \mu^x_T = 0, \qquad \mu^u_{UU} + \frac{1}{2}\mu^u_U = 0, \qquad \mu^u_{XX} - \mu^u_T = 0.\end{gathered} \tag{4.2.30}$$

It follows that the Maurer–Cartan forms.

$$\begin{gathered}\mu^x, \qquad \mu^x_T, \qquad \mu^t, \qquad \mu^t_T, \qquad \mu^t_{TT}, \qquad \mu^u_U,\\ \mu^u_{T^k}, \qquad \mu^u_{T^kX} \qquad k \geq 0,\end{gathered} \tag{4.2.31}$$

form a basis. We do not write down their structure equations since they take too much space. The facts that the differential forms μ^u_{XX} and μ^u_T are related by the heat equation $\mu^u_{XX} - \mu^u_T = 0$, and that the the set of Maurer–Cartan forms

$$\{\mu^u_{T^j}, \mu^u_{XT^j} : j \geq 0\} \tag{4.2.32}$$

is infinite-dimensional suggest that the Maurer–Cartan forms (4.2.32) are good candidates for Theorem 4.7. Let

$$\begin{aligned}\mu^x =& A(X,T,U)\mu^u,\\ \mu^t =& B(X,T,U)\mu^u,\\ \mu^u_U =& C(X,T,U)\mu^u + D(X,T,U)\mu^u_X\\ & + E(X,T,U)\mu^u_T + F(X,T,U)\mu^u_{XT} + G(X,T,U)\mu^u_{TT},\end{aligned}$$

Substituting those linear relations into (4.2.30) we find

$$C = -\frac{1}{2}, \qquad A = B = D = E = F = G = 0.$$

Thus we are left with the lifted infinitesimal determining equations

$$\mu^x = \mu^t = 0, \qquad \mu^u_U + \frac{1}{2}\mu^u = 0, \qquad \mu^u_{XX} - \mu^u_T = 0. \tag{4.2.33}$$

The Maurer–Cartan valued Taylor series satisfying (4.2.33) is

$$\mu^u[\![H_x, H_t, H_u]\!] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\mu^u_{XT^{i+j}}\frac{H_x^{2j+1}}{(2j+1)!} + \mu^u_{T^{i+j}}\frac{H_x^{2j}}{(2j)!}\right]\frac{H_t^i}{i!}e^{-H_u/2}. \tag{4.2.34}$$

The solution (4.2.34) satisfies the dimensional requirements of Theorem 4.7 and

$$d\mu^u[\![H_x, H_t, H_u]\!] = 0.$$

Hence we conclude that Burgers' equation can be mapped to a linear partial differential equation by an invertible change of variables. This is a well-known result, [10, 81]. By the Hopf-Cole map

$$(s, y, v) = (t, x, 2e^{u/2}),$$

the potential Burgers' equation is mapped to the heat equation (3.1.15).

Chapter 5

Equivariant Moving Frames

5.1 Lifted Jet Bundle

A local diffeomorphism $\phi \in \mathcal{D}(M)$ preserves the contact equivalence relation between p-dimensional submanifolds $S \subset M$, [1,81], thus it induces an action on the extended jet bundle $J^n = J^n(M,p)$, known as the n-th *prolonged action.* The chain rule implies that the n-jet of the transformed submanifold depends only on the n-jet of the diffeomorphism. Hence the action of the diffeomorphism jet groupoid $\mathcal{D}^{(n)}$ on J^n given by

$$j_n\phi|_z \cdot j_nS|_z = j_n(\phi(S))|_{\phi(z)} \tag{5.1.1}$$

is well-defined. It is convenient to combine the jet bundles $\mathcal{D}^{(n)}$ and J^n into a new bundle $\mathcal{E}^{(n)} \to J^n$, obtained by pulling back the bundle $\mathcal{D}^{(n)} \to M$ via the standard projection $\widetilde{\pi}^n_0 : J^n \to M$. This new bundle is called the *n-th order lifted jet bundle.*

For $k > n$ we let $\widehat{\pi}^k_n : \mathcal{E}^{(k)} \to \mathcal{E}^{(n)}$ denote the projection induced by $\widetilde{\pi}^k_n : J^k \to J^n$ and $\pi^k_n : \mathcal{D}^{(k)} \to \mathcal{D}^{(n)}$. Points in $\mathcal{E}^{(k)}$ are pair of jets

$$(j_nS|_z, j_n\phi|_z) \quad \in \quad J^n \times \mathcal{D}^{(n)}$$

such that $\pi_0^n(j_n\phi|_z) = \widetilde{\pi}_0^n(j_nS|_z) = z \in M$. The local coordinates on $\mathcal{E}^{(n)}$ are indicated by $\mathbf{Z}^{(n)} = (z^{(n)}, Z^{(n)})$, where $z^{(n)} = (x, u^{(n)})$ are the usual local coordinates on J^n and $Z^{(n)} = (X^{(n)}, U^{(n)})$ are identified with the fiber coordinates (2.3.1) of the diffeomorphism jet bundle.

The combined actions of local diffeomorphisms on submanifold jets (5.1.1) and on diffeomorphism jets (2.3.2) induces an action of $\mathcal{D}^{(n)}$ on the bundle $\mathcal{E}^{(n)}$, called the *lifted action*:

$$j_n\psi|_z \cdot (j_nS|_z, j_n\phi|_z) = (j_n\psi(S)|_{\psi(z)}, j_n(\phi\circ\psi^{-1})|_{\psi(z)}). \tag{5.1.2}$$

The n-th order lifted jet bundle $\mathcal{E}^{(n)}$ has a groupoid structure induced by that on $\mathcal{D}^{(n)}$:

$$\begin{array}{ccccc} & & \mathcal{E}^{(n)} & & \\ & \overset{\widetilde{\boldsymbol{\sigma}}^{(n)}}{\swarrow} & & \overset{\widetilde{\boldsymbol{\tau}}^{(n)}}{\searrow} & \\ J^n & & & & J^n \end{array}$$

where the source map $\widetilde{\boldsymbol{\sigma}}^{(n)}(z^{(n)}, Z^{(n)}) = z^{(n)}$ is the projection onto the first factor and the target map is the prolonged action of $\mathcal{D}^{(n)}$ on J^n:

$$(X, \widehat{U}^{(n)}) = \widehat{Z}^{(n)} = \widetilde{\boldsymbol{\tau}}^{(n)}(\mathbf{Z}^{(n)}) = \mathbf{Z}^{(n)} \cdot z^{(n)}. \tag{5.1.3}$$

Hats are added over the target submanifold jet coordinates to avoid confusion with the diffeomorphism jet coordinates. The entries of the target map are invariant under the lifted action (5.1.2). In analogy with the finite-dimensional moving frame theory, [38, 39], the entries of the target map are referred to as the *lifted invariants*. In local coordinates, the target map coordinate functions encode the implicit differentiation formulas

$$\widehat{U}_J^\alpha = F_J^\alpha(z^{(n)}, Z^{(n)}) = F_J^\alpha(x, u^{(n)}, X^{(n)}, U^{(n)}) \tag{5.1.4}$$

for the jets of transformed submanifolds. We now explain how to systematically derive the expressions $F_J^\alpha(x, u^{(n)}, X^{(n)}, U^{(n)})$.

The bundle structure $\widetilde{\boldsymbol{\sigma}}^{(\infty)} : \mathcal{E}^{(\infty)} \to J^\infty$ splits the cotangent bundle $T^*\mathcal{E}^{(\infty)}$ into jet and group forms, spanned, respectively by the *jet forms*, consisting of the horizontal and contact one-forms

$$dx^i, \qquad \theta^\alpha_J, \qquad i = 1, \ldots, p, \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0,$$

for the submanifold jet bundle J^∞, and the group forms (3.1.1) coming from the diffeomorphism jet bundle $\mathcal{D}^{(\infty)}$. This induces a splitting of the differential on $\mathcal{E}^{(\infty)}$ into jet and group components:

$$d = d_J + d_G,$$

where the jet component furthermore splits into horizontal and vertical components:

$$d_J = d_H + d_V.$$

The identity $d^2 = 0$ implies the equalities

$$d_J^2 = d_G^2 = d_H^2 = d_V^2 = 0,$$
$$d_J d_G = -d_G d_J, \quad d_H d_V = -d_V d_H, \quad d_H d_G = -d_G d_H, \quad d_V d_G = -d_G d_V.$$

The horizontal differential of a function $F : \mathcal{E}^{(\infty)} \to \mathbb{R}$ has the local coordinate formula

$$d_H F = \sum_{j=1}^{p} (D_{x^j} F) dx^j,$$

where

$$D_{x^j} = \mathbb{D}_{x^j} + \sum_{\alpha=1}^{q} \left[u^\alpha_j \mathbb{D}_{u^\alpha} + \sum_{\#J \geq 1} u^\alpha_{J,j} \frac{\partial}{\partial u^\alpha_J} \right], \qquad i = 1, \ldots, p,$$

are the *lifted total derivative operators* on $\mathcal{E}^{(\infty)}$. The local coordinate formula for the vertical differential is

$$d_V F = \sum_{\alpha=1}^{q} \left[(\mathbb{D}_{u^\alpha} F)\theta^\alpha + \sum_{\#J \geq 1} \frac{\partial F}{\partial u_J^\alpha} \theta_J^\alpha \right],$$

while the group differential is

$$d_G F = \sum_{b=1}^{m} \sum_{\#A \geq 0} \frac{\partial F}{\partial Z_A^b} \Upsilon_A^b.$$

The target independent variables X^i on $\mathcal{E}^{(\infty)}$ are used to construct the *lifted horizontal coframe*

$$d_H X^i = \sum_{j=1}^{p} (D_{x^j} X^i) dx^j = \sum_{j=1}^{p} \left(X_{x^j}^i + \sum_{\alpha=1}^{q} u_j^\alpha X_{u^\alpha}^i \right) dx^j, \tag{5.1.5}$$

$i = 1, \ldots, p$. In local coordinate computations, to ensure that the one-forms (5.1.5) are linearly independent, we restrict our considerations to the open dense set where the Jacobian determinant $\det(D_{x^j} X^i)$ is not zero. This excludes the submanifolds which do not intersect the vertical fiber transversally when acted on by the diffeomorphism jet. Those manifolds can be taken care of by considering a different system of local coordinates. The formula

$$d_H F = \sum_{i=1}^{p} (D_{X^i} F) d_H X^i,$$

serves to define the *lifted total differential operators*

$$D_{X^i} = \sum_{j=1}^{p} W_i^j D_{x^j}, \qquad \text{where} \qquad (W_i^j) = (D_{x^i} X^j)^{-1} \tag{5.1.6}$$

is the inverse of the total Jacobian matrix. By successively differentiating[1]

[1] The differential operators (5.1.6) commute, so the order of differentiation is irrelevant.

the target dependent variables U^α with respect to the target independent variables X^i we obtain explicit expressions for (5.1.4):

$$\widehat{U}^\alpha_J = D^J_X U^\alpha = (D_{X^1})^{j^1} \cdots (D_{X^p})^{j^p} U^\alpha. \tag{5.1.7}$$

5.2 Equivariant Moving Frames

Let $\mathcal{H}^{(n)} \subset \mathcal{E}^{(n)}$ denote the subgroupoid obtained by pulling back $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ via the projection $\widetilde{\pi}^n_0 : J^n \to M$. The groupoid structure on $\mathcal{H}^{(n)}$ is induced by that of $\mathcal{E}^{(n)}$. A local coordinate system on $\mathcal{H}^{(n)}$ is given by

$$(x, u^{(n)}, g^{(n)})$$

where $g^{(n)} = (g^1, \ldots, g^{r_n})$ are the fiber coordinates of the Lie pseudo-group $\mathcal{G}^{(n)}$ over the base point $z = (x, u) \in M$. The explicit expressions of the target map coordinates $\boldsymbol{\tau}^{(n)} : \mathcal{G}^{(n)} \to M$ are obtained by restricting the target map coordinates of the diffeomorphism pseudo-group (5.1.3) to the solution space of the defining equations (2.3.3):

$$\widetilde{\boldsymbol{\tau}}^{(n)}(x, u^{(n)}, g^{(n)}) = (\ldots, X^i, \ldots, D^J_X U^\alpha, \ldots)|_{F^{(n)}(z, Z^{(n)})=0}.$$

Definition 5.1. A (right) *moving frame* $\rho^{(n)}$ of *order* n is a $\mathcal{G}^{(n)}$ equivariant local section of the bundle $\mathcal{H}^{(n)} \to J^n$, i. e., $\rho^{(n)} : J^n \to \mathcal{H}^{(n)}$ satisfies

$$\rho^{(n)}(g^{(n)} \cdot z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot (g^{(n)})^{-1}, \qquad \text{for all } g^{(n)} \in \mathcal{G}^{(n)},$$

with $z = \widetilde{\pi}^n_0(z^{(n)})$ and groupoid inverse $(g^{(n)})^{-1} \in \mathcal{G}^{(n)}|_{\boldsymbol{\tau}^{(n)}(z^{(n)}, g^{(n)})}$, such that both $z^{(n)}$ and $g^{(n)} \cdot z^{(n)}$ lie in the domain of definition of $\rho^{(n)}$.

Theorem 5.2. Let $\mathcal{G}$ be a regular Lie pseudo-group acting on an m-dimensional manifold M. If $\mathcal{G}$ acts locally freely at $z^{(n)} \in J^n$ for some $n > 0$, then it acts locally freely at any $z^{(k)} \in J^k$ with $\pi^k_n(z^{(k)}) = z^{(n)}$, for $k \geq n$.

Proposition 5.3. Suppose $\mathcal{G}$ acts locally freely on $\mathcal{V}^n \subset J^n$ with its orbits forming a regular foliation. Let $\mathcal{K}^n \subset \mathcal{V}^n$ be a local cross-section to the pseudo-group orbits. Let $\mathcal{U}^n \subset \mathcal{V}^n$ be a neighborhood intersecting the cross-section $\mathcal{K}^n$ such that $\mathcal{G}^{(n)}$ acts freely on $\mathcal{U}^n$. For $z^{(n)} \in \mathcal{U}^n$ define $\rho^{(n)}(z^{(n)}) \in \mathcal{H}^{(n)}$ to be the unique pair of jets such that $\widetilde{\boldsymbol{\sigma}}(\rho^{(n)}(z^{(n)})) = z^{(n)}$ and $\widetilde{\boldsymbol{\tau}}(\rho^{(n)}(z^{(n)})) \in \mathcal{K}^n$. Then $\rho^{(n)} : J^n \to \mathcal{H}^{(n)}$ is a moving frame for $\mathcal{G}^{(n)}$.

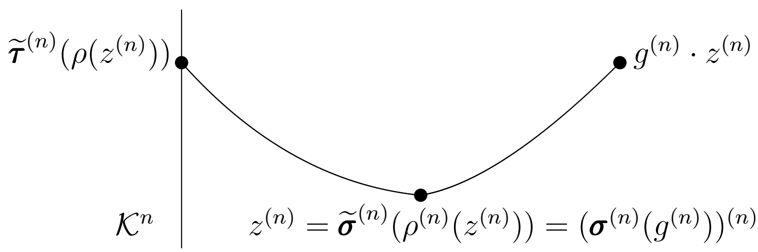


Figure 5.1: Moving frame.

In applications, a moving frame of order n is obtained by *normalizing* the group jet coordinates $g^{(n)} = (g^1, \ldots, g^{r_n})$. The normalization procedure is given by the following steps:

1. Using Formula (5.1.7) and taking their restriction to the solution space of the defining system (2.3.3), write out explicitly the n-th order target map coordinates

$$(X, \widehat{U}^{(n)}) = P^{(n)}(x, u^{(n)}, g^{(n)}). \tag{5.2.1}$$

2. Choose r_n components from (5.2.1) so that the *normalization equations*

$$P_k(x, u^{(n)}, g^{(n)}) = c_k, \qquad k = 1, \ldots, r_n, \tag{5.2.2}$$

for some well-chosen constants c_k, form a cross-section to the pseudo-group orbits in $\mathcal{V}^n$.

3. Solve the normalization equations (5.2.2) for the pseudo-group parameters in terms of the submanifold jets $(x, u^{(n)}) \in \mathcal{V}^n$:

$$g^{(n)} = g^{(n)}(x, u^{(n)}). \tag{5.2.3}$$

4. The n-th order moving frame $\rho^{(n)} : J^n \to \mathcal{H}^{(n)}$ is locally given by

$$\rho^{(n)}(x, u^{(n)}) = (x, u^{(n)}, g^{(n)}(x, u^{(n)})).$$

Definition 5.4. A moving frame $\rho^{(k)} : J^k \to \mathcal{H}^{(k)}$ of order $k > n$ is *compatible* with a moving frame $\rho^{(n)} : J^n \to \mathcal{H}^{(n)}$ of order n provided $\widehat{\pi}^k_n \circ \rho^{(k)} = \rho^{(n)} \circ \widetilde{\pi}^k_n$, when defined. A *complete moving frame* is a collection of compatible moving frames of all orders $k \geq n^\star$.

In applications we work with compatible moving frames so that they can be constructed incrementally.

5.3 Invariantization

Once a moving frame has been determined for a Lie pseudo-group $\mathcal{G}$, it is possible to define an *invariantization map* which assigns to any differential form $\omega \in T^*J^\infty$ a right-invariant differential form on T^*J^∞.

Let

$$\pi_J : \Omega^*\mathcal{E}^{(\infty)} \to (\boldsymbol{\sigma}^{(\infty)})^*(\Omega^* J^\infty)$$

be the natural projection which annihilates the contact ideal generated by the group forms (3.1.1) of the diffeomorphism pseudo-group $\mathcal{D}(M)$.

Definition 5.5. Let $\rho^{(\infty)} : J^\infty \to \mathcal{H}^{(\infty)}$ be a complete moving frame. The *invariantization* of a differential form ω on J^∞ is the jet form

$$\iota(\omega) = (\rho^{(\infty)})^*[\pi_J((\widetilde{\boldsymbol{\tau}}^{(\infty)})^*\omega)]. \tag{5.3.1}$$

Note from the definition that if ω is already invariant, then $\iota(\omega) = \omega$. Thus the invariantization map is a projection map in the sense that

$$\iota^2 = \iota. \tag{5.3.2}$$

The invariantization map can be given a geometrical interpretation. The invariantization of a differential form is the unique invariant differential form that has the same value when restricted to the cross-section defining the moving frame. In the particular case of a zero-form, namely a differential function $f : J^\infty \to \mathbb{R}$, the invariantization map reduces to

$$\iota(f(z^{(n)})) = (\rho^{(\infty)})^*[(\widetilde{\boldsymbol{\tau}}^{(\infty)})^* f(z^{(n)})] = f(\widetilde{\boldsymbol{\tau}}^{(\infty)}(\rho^{(\infty)}(z^{(n)}))).$$

Of particular importance is the invariantization of the jet coordinate functions on J^∞:

$$H^i = \iota(x^i), \quad i = 1, \ldots, p, \quad I^\alpha_J = \iota(u^\alpha_J), \quad \alpha = 1, \ldots, q, \quad \#J \geq 0. \tag{5.3.3}$$

The invariants (5.3.3) are referred to as the *normalized invariants*. Also the invariantization of the basic jet forms are denoted by

$$\begin{aligned} \varpi^i &= \iota(dx^i), \qquad i = 1, \ldots, p, \\ \vartheta^\alpha_J &= \iota(\theta^\alpha_J), \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0. \end{aligned} \tag{5.3.4}$$

Given a differential invariant

$$I = f(x, u^{(n)}),$$

it can easily be written in terms of the normalized invariants (5.3.3) using property (5.3.2) of the invariantization map:

$$I = \iota(f(x, u^{(n)})) = f(H, I^{(n)}). \tag{5.3.5}$$

The equality (5.3.5) is referred to as the *replacement principle*. A differential invariant is expressed in terms of the normalized invariants simply by replacing x^i by H^i and u^α_J by I^α_J.

In terms of the invariant jet forms (5.3.4) we split the pull-back of the Maurer–Cartan forms (3.1.2) by a moving frame $\rho^{(\infty)}$ into horizontal and vertical components:

$$(\rho^{(\infty)})^*\mu^a_A = \pi_{\mathcal{H}}((\rho^{(\infty)})^*\mu^a_A) + \pi_{\mathcal{V}}((\rho^{(\infty)})^*\mu^a_A) = \nu^a_A + \gamma^a_A.$$

5.4 Recurrence Formulas

Definition 5.6. A differential operator $\mathcal{D}$ is said to be an *invariant differential operator* if when applied to any (non-trivial) differential invariant the output is again a differential invariant.

A convenient basis of invariant differential operators

$$\mathcal{D}_1, \ldots, \mathcal{D}_p \tag{5.4.1}$$

is the one given by the dual vector fields to the invariant horizontal forms $\varpi^1, \ldots, \varpi^p$:

$$dF = \sum_{i=1}^{p} (\mathcal{D}_i F)\varpi^i \qquad \text{mod (invariant contact ideal)},$$

valid for any differential function $F : J^\infty \to \mathbb{R}$. We note that the invariant differential operators (5.4.1) can be obtained from the lifted differential operators (5.1.6) by substituting the pseudo-group parameters with their normalization (5.2.3). But while the lifted differential operators (5.1.6) do commute, the invariant differential operators (5.4.1) generally do not. Their commutator relations are of the form

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^{p} Y_{ij}^k \mathcal{D}_k, \qquad i, j = 1, \ldots, p, \tag{5.4.2}$$

where the coefficients Y_{ij}^k are certain differential invariants, called *commutator invariants*, [90].

The invariantization map and the exterior differential operator on jet forms do not commute in general. For example, if the independent variable x^i is normalized to a constant value c then

$$0 \neq \varpi^i = \iota(dx^i) \neq d\iota(x^i) = dc = 0.$$

The key result of this section is the *universal recurrence formula* for invariantized jet forms.

Theorem 5.7. Let $\rho^{(\infty)} : J^\infty \to \mathcal{H}^{(\infty)}$ be a complete moving frame and ω a differential form on J^∞. Then

$$d\iota(\omega) = \iota(d\omega + \mathbf{v}^{(\infty)}(\omega)). \tag{5.4.3}$$

In its full generality, equation (5.4.3) plays a fundamental role in the theory of invariant variational bicomplexes [56, 57]. In this thesis, we will only be interested in the horizontal component of (5.4.3).

Remark 5.8. An order convention needs to be respected when computing the *correction term* $\iota[\mathbf{v}^{(\infty)}(\omega)]$. The vector field jet coordinates in $\mathbf{v}^{(\infty)}(\omega)$ must always be at the left of the jet forms

$$\mathbf{v}^{(\infty)}(\omega) = \sum_{b=1}^{m} \sum_{\#A \leq n} \zeta_A^b \omega_A^b, \qquad \omega, \omega_A^b \in \Omega^k(T^* J^\infty), \tag{5.4.4}$$

so that

$$\iota[\mathbf{v}^{(\infty)}(\omega)] = \sum_{b=1}^{m} \sum_{\#A\leq n} (\rho^{(\infty)})^* \mu_A^b \wedge \iota(\omega_A^b).$$

As we did in Section 3.1, the vector field jet coordinates must be interpreted as differential forms and the product $\zeta_A^b \omega_A^b$ must be understood as the wedge product in order for the equality

$$(\rho^{(\infty)})^* \mu_A^b \wedge \iota(\omega_A^b) = \iota(\zeta_A^b \omega_A^b) = (-1)^k \iota(\omega_A^b \zeta_A^b) = (-1)^k \iota(\omega_A^b) \wedge (\rho^{(\infty)})^* \mu_A^b$$

to be true.

Lemma 5.9. The recurrence relations for the normalized differential invariants (5.3.3) are

$$\begin{aligned} dH^i =& \iota(dx^i + \xi^i) = \varpi^i + \iota(\xi^i), \qquad i = 1, \dots, p, \\ dI_J^\alpha =& \iota(du_J^\alpha + \phi_\alpha^J) = \iota\left(\sum_{i=1}^{p} u_{J,i}^\alpha dx^i + \theta_J^\alpha + \phi_\alpha^J\right) \\ =& \sum_{i=1}^{p} I_{J,i}^\alpha \varpi^i + \vartheta_J^\alpha + \iota(\phi_\alpha^J), \end{aligned} \tag{5.4.5}$$

$\alpha = 1, \dots, q, \quad \#J \geq 0.$

Taking the horizontal projection $\pi_{\mathcal{H}}$ of (5.4.5) we obtain the key relations needed to study the algebra of differential invariants.

Theorem 5.10. The horizontal components of the recurrence formulas (5.4.5) are

$$\begin{aligned} \sum_{j=1}^{p} (\mathcal{D}_j H^i) \varpi^j =& \varpi^i + \nu^i, \qquad & i = 1, \dots, p, \\ \sum_{j=1}^{p} (\mathcal{D}_j I_J^\alpha) \varpi^j =& \sum_{j=1}^{m} I_{J,j}^\alpha \varpi^j + \psi_\alpha^J, \qquad & \alpha = 1, \dots, q, \quad \#J \geq 0. \end{aligned} \tag{5.4.6}$$

where $\nu^i = \pi_{\mathcal{H}}(\iota(\xi^i))$ and $\psi_\alpha^J = \pi_{\mathcal{H}}(\iota(\phi_\alpha^J))$.

The differential forms ν^i, ψ_α^J are certain linear expressions of the invariant horizontal forms ϖ^j, and are called *correction terms*. Once the correction terms are determined, the equations (5.4.6) give identities relating the normalized invariants H^i, I_J^α and their invariant derivatives $\mathcal{D}_j H^i$, $\mathcal{D}_j I_J^\alpha$. A remarkable fact about equations (5.4.6) is that the correction terms can be obtained without knowing the explicit expressions for the normalized invariants, the invariant horizontal forms and the invariant differential operators. Once a complete moving frame has been determined, the correction terms are found using only algebraic manipulations. Each phantom differential invariant is, by definition, normalized to a constant value, and hence its invariant derivatives are zero. Therefore, the recurrence relations for the phantom invariants form a system of linear equations for the horizontal component of the pulled-back Maurer–Cartan forms ν_A^a. If the pseudo-group acts locally freely on J^n, then it is proven in [90] that those equations can be uniquely solved for the one-forms ν_A^a of order $\leq n$ as invariant linear combinations of the invariant horizontal forms ϖ^i. Substituting the resulting solution into the remaining recurrence relations of (5.4.6) yields a complete system of recurrence relations for the non-phantom differential invariants.

The universal recurrence relation (5.4.3) is also used to obtain the commutator invariants Y_{ij}^k appearing in (5.4.2). If we let ω in (5.4.3) be one of the horizontal forms dx^i then the horizontal component of the recurrence relation yields

$$d_{\mathcal{H}} \varpi^k = - \sum_{1 \leq i < j \leq p} Y_{ij}^k \varpi^i \wedge \varpi^j,$$

from which we can read the commutator invariants Y_{ij}^k.

5.5 Infeld–Rowlands Equation

As a first application of the theory exposed in Sections 5.1-5.3, we characterize the differential invariant algebra of the Infeld–Rowlands equation

$$\Delta_{IR} = u_t + 2u_x u_{xx} + u_{xxxx} + u_{xy} = 0. \tag{5.5.1}$$

This equation appears in the study of soliton stability of the Landau–Ginzburg equation, [47]. Let

$$\mathbf{v} = \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \tau(x,y,t,u)\frac{\partial}{\partial t} + \phi(x,y,t,u)\frac{\partial}{\partial u}, \tag{5.5.2}$$

be an infinitesimal symmetry generator of (5.5.1). From Lie's algorithm, [81, 82], we obtain the infinitesimal determining system

$$\begin{gathered} \xi_t = \xi_u = 0, \qquad \xi_{xx} = \xi_{xy} = 0, \qquad \eta_x = \eta_t = \eta_u = 0, \\ \eta_y = 3\xi_x, \qquad \tau_x = \tau_y = \tau_u = 0, \qquad \tau_t = 4\xi_x, \\ \phi_u = -\xi_x, \qquad \phi_x = \frac{1}{2}\xi_y, \qquad \phi_t = -\frac{1}{2}\xi_{yy}. \end{gathered} \tag{5.5.3}$$

The solution to the system of equations (5.5.3) was found in [36]:

$$\begin{aligned} \xi(x,y,t,u) =& \lambda x + f(y), \\ \eta(x,y,t,u) =& \alpha + 3\lambda y, \\ \tau(x,y,t,u) =& \epsilon + 4\lambda t, \\ \phi(x,y,t,u) =& -\lambda u + \frac{x}{2}f'(y) - \frac{t}{2}f''(y) + g(y), \end{aligned} \tag{5.5.4}$$

where λ, α, ϵ are constants and $f(y)$, $g(y)$ are analytic functions. Exponentiating the infinitesimal generator (5.5.4), [81, 82], we obtain the pseudo-group action

$$\begin{aligned} X =& \lambda x + F(Y), \\ Y =& \lambda^3 y + \alpha, \\ T =& \lambda^4 t + \epsilon, \\ U =& \frac{u}{\lambda} + \frac{X}{2}F'(Y) - \frac{T}{2}F''(Y) + G(Y). \end{aligned} \tag{5.5.5}$$

The lifted horizontal coframe associated to the pseudo-group action (5.5.5) is

$$d_H X = \lambda dx + \lambda^3 F'(Y) dy, \qquad d_H Y = \lambda^3 dy, \qquad d_H T = \lambda^4 dt. \tag{5.5.6}$$

Dual to the lifted horizontal coframe (5.5.6) are the lifted total differential operators

$$D_X = \frac{1}{\lambda} D_x, \qquad D_Y = \frac{1}{\lambda^3}(-\lambda^2 F'(Y) D_x + D_y), \qquad D_T = \frac{1}{\lambda^4} D_t. \tag{5.5.7}$$

The prolonged pseudo-group action is obtained by repeated applications of the differential operators (5.5.7) to U:

$$\begin{aligned} \widehat{U}_X &= \frac{u_x}{\lambda^2} + \frac{1}{2}F'(Y), \\ \widehat{U}_Y &= -\frac{u_x}{\lambda^2}F'(Y) + \frac{u_y}{\lambda^4} + \frac{X}{2}F''(Y) - \frac{T}{2}F'''(Y) + G'(Y), \\ \widehat{U}_T &= \frac{u_t}{\lambda^5} - \frac{1}{2}F''(Y), \\ \widehat{U}_{XX} &= \frac{u_{xx}}{\lambda^3}, \\ \widehat{U}_{XY} &= -\frac{u_{xy}}{\lambda^3}F'(Y) + \frac{u_{xy}}{\lambda^5} + \frac{1}{2}F''(Y), \\ \widehat{U}_{YY} &= -\frac{u_{xx}}{\lambda^3}F'(Y)^2 - 2\frac{u_{xy}}{\lambda^5}F'(Y) + \frac{u_{yy}}{\lambda^7} + \frac{X}{2}F'''(Y) - \frac{T}{2}F''''(Y) + G''(Y), \\ \widehat{U}_{XT} &= \frac{u_{xt}}{\lambda^6}, \\ \widehat{U}_{TT} &= \frac{u_{tt}}{\lambda^9}, \end{aligned}$$

$$\widehat{U}_{YT} = -\frac{u_{xt}}{\lambda^6}F'(Y) + \frac{u_{yt}}{\lambda^8} - \frac{1}{2}F'''(Y),$$
$$\vdots$$
$$\widehat{U}_{XXXX} = \frac{u_{xxxx}}{\lambda^5},$$
$$\vdots$$

A possible cross-section to the prolonged pseudo-group action is given by

$$\widehat{U}_{XX} = 1, \quad X = Y = T = U = \widehat{U}_X = \widehat{U}_{TY^k} = \widehat{U}_{Y^{k+1}} = 0, \quad k \geq 0. \tag{5.5.8}$$

Solving the normalization equations for the pseudo-group parameters we obtain

$$\begin{gathered} \lambda = u_{xx}^{1/3}, \qquad \alpha = -yu_{xx}, \qquad \epsilon = -tu_{xx}^{4/3}, \\ F(Y) = -xu_{xx}^{1/3}, \qquad G(Y) = -\frac{u}{u_{xx}^{1/3}} \qquad F'(Y) = -2\frac{u_x}{u_{xx}^{2/3}}, \\ G'(Y) = -\frac{2u_x^2 + u_y}{u_{xx}^{4/3}}, \qquad F''(Y) = 2\frac{u_t}{u_{xx}^{5/3}}, \qquad \ldots \end{gathered} \tag{5.5.9}$$

which is well-defined provided $u_{xx} \neq 0$. Substituting the normalized pseudo-group parameters (5.5.9) into the transformed submanifold jet coordinates yields the normalized differential invariants

$$\begin{gathered} I_{1,1,0} = \iota(u_{xy}) = 2\frac{u_x}{u_{xx}^{2/3}} + \frac{u_{xy}}{u_{xx}^{5/3}} + \frac{u_t}{u_{xx}^{5/3}}, \qquad I_{1,0,1} = \iota(u_{xt}) = \frac{u_{xt}}{u_{xx}^2}, \\ I_{0,0,2} = \iota(u_{tt}) = \frac{u_{tt}}{u_{xx}^3}, \qquad \ldots, \qquad I_{4,0,0} = \frac{u_{xxxx}}{u_{xx}^{5/3}}, \qquad \ldots \end{gathered} \tag{5.5.10}$$

With the equivariant moving frame machinery at our disposition, the Infeld–Rowlands equation (5.5.1) can easily be rewritten in terms of the normalized differential invariants (5.5.10) using the invariantization map

$$0 = \iota(\Delta_{IR}) = I_{4,0,0} + I_{1,1,0} = \frac{\Delta_{IR}}{u_{xx}^{5/3}}. \tag{5.5.11}$$

Remark 5.11. Any invariant system of differential equations

$$\Delta(x, u^{(n)}) = 0 \tag{5.5.12}$$

can be rewritten in terms of the normalized differential invariants by invariantization:

$$\iota(\Delta(x, u^{(n)})) = \Delta(\ldots, H^i, \ldots, I^\alpha_J, \ldots) = 0. \tag{5.5.13}$$

But as one can notice from (5.5.11), the left hand side of the original system of differential equations (5.5.12) may only be relatively invariant, while the left hand side of (5.5.13) is fully invariant.

Substituting the normalized pseudo-group parameters (5.5.9) into the lifted differential operators (5.5.7), we obtain the invariant differential operators

$$\mathcal{D}_1 = \frac{1}{u_{xx}^{1/3}} D_x, \qquad \mathcal{D}_2 = \frac{1}{u_{xx}}(2u_x D_x + D_y), \qquad \mathcal{D}_3 = \frac{1}{u_{xx}^{4/3}} D_t. \tag{5.5.14}$$

We end this example by showing that the algebra of differential invariants is generated by the invariants $I_{1,1,0}$, $I_{1,0,1}$ and the invariant differential operators (5.5.14). The prolonged vector field coefficients (2.4.6), restricted to the kernel of the infinitesimal determining system (5.5.3), are

$$\begin{aligned}
\phi^x &= \frac{1}{2}\xi_y - 2\xi_x u_x, & \phi^y &= \phi_y - 4\xi_x u_y - \xi_y u_x, \\
\phi^t &= -\frac{1}{2}\xi_{yy} - 5\xi_x u_t, & \phi^{xx} &= -3\xi_x u_{xx}, \\
\phi^{xy} &= \frac{1}{2}\xi_{yy} - 5\xi_x u_{xy} - \xi_y u_{xx}, & \phi^{xt} &= -6\xi_x u_{xt}, \\
\phi^{yy} &= \phi_{yy} - 7\xi_x u_{yy} - 2\xi_y u_{xy}, & \phi^{tt} &= -9\xi_x u_{tt}, \\
\phi^{yt} &= -\frac{1}{2}\xi_{yyy} - 8\xi_x u_{yt} - \xi_y u_{xt}, & & \ldots
\end{aligned}$$

Introducing the notation

$$\begin{aligned} \nu_k = \pi_{\mathcal{H}} \circ \iota(\xi_{y^k}), \qquad \mu_k = \pi_{\mathcal{H}} \circ \iota(\phi_{y^k}), \qquad k \leq 0, \\ \gamma = \pi_{\mathcal{H}} \circ \iota(\xi_x), \qquad \alpha = \pi_{\mathcal{H}} \circ \iota(\eta), \qquad \beta = \pi_{\mathcal{H}} \circ \iota(\tau), \end{aligned} \tag{5.5.15}$$

to denote the horizontal components of the invariantization of the vector field jet coordinates, the first few recurrence relations (5.4.5), restricted to the cross-section (5.5.8), are

$$\begin{aligned}
0 &= \varpi^1 + \nu, \\
0 &= \varpi^2 + \alpha, \\
0 &= \varpi^3 + \beta, \\
0 &= \mu, \\
0 &= \varpi^1 + I_{1,1,0}\varpi^2 + I_{1,0,1}\varpi^3 + \frac{1}{2}\nu_1, \\
0 &= I_{1,1,0}\varpi^1 + \mu_1, \\
0 &= I_{1,0,1}\varpi^1 + I_{0,0,2}\varpi^3 - \frac{1}{2}\nu_2, \\
0 &= I_{3,0,0}\varpi^1 + I_{2,1,0}\varpi^2 + I_{2,0,1}\varpi^3 - 3\gamma, \\
\sum_{i=1}^{3} \mathcal{D}_i I_{1,1,0} &= I_{2,1,0}\varpi^1 + I_{1,2,0}\varpi^2 + I_{1,1,1}\varpi^3 + \frac{1}{2}\nu_2 - 5I_{1,1,0}\gamma - \nu_1, \\
\sum_{i=1}^{3} \mathcal{D}_i I_{1,0,1} &= I_{2,0,1}\varpi^1 + I_{1,1,1}\varpi^2 + I_{1,0,2}\varpi^3 - 6I_{1,0,1}\gamma, \\
0 &= I_{1,2,0}\varpi^1 + \mu_2 - 2I_{1,1,0}\nu_1, \\
0 &= I_{1,1,1}\varpi^1 + I_{0,1,2}\varpi^3 - \frac{1}{2}\nu_3 - I_{1,0,1}\nu_1, \\
\sum_{i=1}^{3} \mathcal{D}_i I_{0,0,2} &= I_{1,0,2}\varpi^1 + I_{0,1,2}\varpi^2 + I_{0,0,3}\varpi^3 - 9I_{0,0,2}\gamma, \\
&\vdots
\end{aligned}$$

The recurrence relations for the phantom invariants are used to solve for the one-forms (5.5.15):

$$
\begin{gathered}
\nu = -\varpi^1, \qquad \alpha = -\varpi^2, \qquad \beta = -\varpi^3, \qquad \mu = 0, \\
\nu_1 = -2(\varpi^1 + I_{1,1,0}\varpi^2 + I_{1,0,1}\varpi^3), \qquad \mu_1 = -I_{1,1,0}\varpi^1, \\
\nu_2 = 2(I_{1,0,1}\varpi^1 + I_{0,0,2}\varpi^3), \\
\gamma = \frac{1}{3}(I_{3,0,0}\varpi^1 + I_{2,1,0}\varpi^2 + I_{2,0,1}\varpi^3), \\
\mu_2 = -I_{1,2,0}\varpi^1 - 4I_{1,1,0}(\varpi^1 + I_{1,1,0}\varpi^2 + I_{1,0,1}\varpi^3), \\
\nu_3 = 2I_{1,1,1}\varpi^2 + 2I_{0,1,2}\varpi^3 + 4I_{1,0,1}(\varpi^1 + I_{1,1,0}\varpi^2 + I_{1,0,1}\varpi^3), \\
\vdots
\end{gathered}
\tag{5.5.16}
$$

Substituting the expressions (5.5.16) into the recurrence relations for the non-phantom invariants of order two we obtain the recurrence relations

$$
\begin{aligned}
\mathcal{D}_1 I_{1,1,0} =& I_{2,1,0} + I_{1,0,1} - \frac{5}{3} I_{1,1,0} I_{3,0,0} + 2, \\
\mathcal{D}_2 I_{1,1,0} =& I_{1,2,0} - \frac{5}{3} I_{1,1,0} I_{2,1,0} + 2I_{1,1,0}, \\
\mathcal{D}_3 I_{1,1,0} =& I_{1,1,1} + I_{0,0,2} - \frac{5}{3} I_{1,1,0} I_{2,0,1} + 2I_{1,0,1}, \\
\mathcal{D}_1 I_{1,0,1} =& I_{2,0,1} - 2I_{1,0,1} I_{3,0,0}, \\
\mathcal{D}_2 I_{1,0,1} =& I_{1,1,1} - 2I_{1,0,1} I_{2,1,0}, \\
\mathcal{D}_3 I_{1,0,1} =& I_{1,0,2} - 2I_{1,0,1} I_{2,0,1}, \\
\mathcal{D}_1 I_{0,0,2} =& I_{1,0,2} - 3I_{0,0,2} I_{3,0,0}, \\
\mathcal{D}_2 I_{0,0,2} =& I_{0,1,2} - 3I_{0,0,2} I_{2,1,0}, \\
\mathcal{D}_3 I_{0,0,2} =& I_{0,0,3} - 3I_{0,0,2} I_{2,0,1}.
\end{aligned}
$$

From the above equations we can express the differential invariants $I_{2,1,0}$, $I_{1,2,0}$, $I_{1,1,1}$, $I_{2,0,1}$, $I_{1,0,2}$, $I_{0,1,2}$, $I_{0,0,3}$ and $I_{0,0,2}$ in terms of $I_{1,1,0}$, $I_{1,0,1}$ $I_{3,0,0}$ and their invariant derivatives:

$$\begin{aligned}
I_{2,1,0} =& \mathcal{D}_1 I_{1,1,0} - I_{1,0,1} + \frac{5}{3} I_{1,1,0} I_{3,0,0} - 2,\\
I_{1,2,0} =& \mathcal{D}_2 I_{1,1,0} + \frac{5}{3} I_{1,1,0} \left(\mathcal{D}_1 I_{1,1,0} - I_{1,0,1} + \frac{5}{3} I_{1,1,0} I_{3,0,0} - 2 \right) - 2 I_{1,1,0},\\
I_{2,0,1} =& \mathcal{D}_1 I_{1,0,1} + 2 I_{1,0,1} I_{3,0,0},\\
I_{1,1,1} =& \mathcal{D}_3 I_{1,1,0} - I_{0,0,2} + \frac{5}{3} I_{1,1,0} (\mathcal{D}_1 I_{1,0,1} + 2 I_{1,0,1} I_{3,0,0}) - 2 I_{1,0,1},\\
I_{1,0,2} =& \mathcal{D}_3 I_{1,0,1} + 2 I_{1,0,1} (\mathcal{D}_1 I_{1,0,1} + 2 I_{1,0,1} I_{3,0,0}),\\
I_{0,1,2} =& \mathcal{D}_2 I_{0,0,2} + 3 I_{0,0,2} \left(\mathcal{D}_1 I_{1,1,0} - I_{1,0,1} + \frac{5}{3} I_{1,1,0} I_{3,0,0} - 2 \right),\\
I_{0,0,3} =& \mathcal{D}_3 I_{0,0,2} + 3 I_{0,0,2} (\mathcal{D}_1 I_{1,0,1} + 2 I_{1,0,1} I_{3,0,0}),
\end{aligned}$$

where

$$\begin{aligned}
I_{0,0,2} =& \mathcal{D}_3 I_{1,1,0} - \mathcal{D}_2 I_{1,0,1} + \frac{5}{3} I_{1,1,0} (\mathcal{D}_1 I_{1,0,1} + 2 I_{1,0,1} I_{3,0,0}) - 2 I_{1,0,1}\\
& - 2 I_{1,0,1} \left(\mathcal{D}_1 I_{1,1,0} - I_{1,0,1} - \frac{5}{3} I_{1,1,0} I_{3,0,0} - 2 \right).
\end{aligned}$$

The commutation relations between the invariant differential operators (5.5.14) is used to express $I_{3,0,0}$ in terms of $I_{1,1,0}$ and $I_{1,0,1}$ and their invariant derivatives. From the fundamental recurrence relation (5.4.3) we have

$$\begin{aligned}
d_{\mathcal{H}} \varpi^1 =& - \left(\frac{1}{3} I_{2,1,0} + 2 \right) \varpi^1 \wedge \varpi^2 - \frac{1}{3} I_{2,0,1} \varpi^1 \wedge \varpi^3 + 2 I_{1,0,1} \varpi^2 \wedge \varpi^3,\\
d_{\mathcal{H}} \varpi^2 =& I_{3,0,0} \varpi^1 \wedge \varpi^2 - I_{2,0,1} \varpi^2 \wedge \varpi^3,\\
d_{\mathcal{H}} \varpi^3 =& \frac{4}{3} I_{3,0,0} \varpi^1 \wedge \varpi^3 + \frac{4}{3} I_{2,1,0} \varpi^2 \wedge \varpi^3,
\end{aligned}$$

and conclude that the commutation relations for the invariant differential operators are

$$[\mathcal{D}_1, \mathcal{D}_2] = \left(\frac{I_{2,1,0}}{3} + 2\right)\mathcal{D}_1 - I_{3,0,0}\mathcal{D}_2, \tag{5.5.17a}$$

$$[\mathcal{D}_1, \mathcal{D}_3] = \frac{I_{2,0,1}}{3}\mathcal{D}_1 - \frac{4}{3}I_{3,0,0}\mathcal{D}_3, \tag{5.5.17b}$$

$$[\mathcal{D}_2, \mathcal{D}_3] = -2I_{1,0,1}\mathcal{D}_1 + I_{2,0,1}\mathcal{D}_2 - \frac{4}{3}I_{2,1,0}\mathcal{D}_3. \tag{5.5.17c}$$

Applying the the commutation relation (5.5.17b) to the differential invariant $I_{1,0,1}$ we obtain

$$[\mathcal{D}_1, \mathcal{D}_3]I_{1,0,1} = \frac{1}{3}(\mathcal{D}_1 I_{1,0,1} + 2I_{1,0,1}I_{3,0,0})\mathcal{D}_1 I_{1,0,1} - \frac{4}{3}I_{3,0,0}\mathcal{D}_3 I_{1,0,1}.$$

Provided that

$$I_{1,0,1}\mathcal{D}_1 I_{1,0,1} - 2\mathcal{D}_3 I_{1,0,1} \neq 0,$$

which is generically the case, we can solve for $I_{3,0,0}$:

$$I_{3,0,0} = \frac{3}{2I_{1,0,1}\mathcal{D}_1 I_{1,0,1} - 4\mathcal{D}_3 I_{1,0,1}}\left([\mathcal{D}_1, \mathcal{D}_3]I_{1,0,1} - \frac{(\mathcal{D}_1 I_{1,0,1})^2}{3}\right).$$

Since the correction terms for the recurrence relations of the non-phantom invariants of order $n \geq 3$ involve differential invariants of order at most n, the above discussion proves the following proposition.

Proposition 5.12. The algebra of differential invariants for the Infeld–Rowlands equation is generated by

$$I_{1,1,0} \qquad \text{and} \qquad I_{1,0,1}.$$

5.6 Davey–Stewartson Equations

As a second illustration of the equivariant moving frame theory we consider the Davey–Stewartson equations. The Davey–Stewartson equations describe the propagation of two-dimensional water waves under the force of gravity in water of finite depth, [30]. We restrict our attention to the integrable version, [24]. In that regime the Davey–Stewartson equations are given by

$$\Delta_{DS} : \begin{cases} i\psi_t + \psi_{xx} + \epsilon\psi_{yy} - \delta|\psi|^2 - \psi w = 0, \\ w_{xx} - \epsilon w_{yy} - \alpha(|\psi|^2)_{yy} = 0, \end{cases} \tag{5.6.1}$$

where $\psi(x, y, t)$ and $w(x, y, t)$ are complex and real functions, respectively, and $\epsilon = \pm 1$, $\delta = \pm 1$, $\alpha \in \mathbb{R}^{\times}$. Setting $\psi = u + iv$, the system of equations (5.6.1) is equivalent to

$$\begin{aligned} \Delta^1_{DS} &= u_t + v_{xx} + \epsilon v_{yy} - \delta v(u^2 + v^2) - vw = 0, \\ \Delta^2_{DS} &= -v_t + u_{xx} + \epsilon u_{yy} - \delta u(u^2 + v^2) - uw = 0, \\ \Delta^3_{DS} &= w_{xx} - \epsilon w_{yy} - 2\alpha[uu_{yy} + (u_y)^2 + vv_{yy} + (v_y)^2] = 0. \end{aligned} \tag{5.6.2}$$

Let

$$\begin{aligned} \mathbf{v} =& \xi(x, y, t, u, v, w)\frac{\partial}{\partial x} + \eta(x, y, t, u, v, w)\frac{\partial}{\partial y} + \tau(x, y, t, u, v, w)\frac{\partial}{\partial t} \\ &+ \phi(x, y, t, u, v, w)\frac{\partial}{\partial u} + \gamma(x, y, t, u, v, w)\frac{\partial}{\partial v} + \beta(x, y, t, u, v, w)\frac{\partial}{\partial w}, \end{aligned}$$

be an infinitesimal symmetry generator for the system of partial differential equations (5.6.2). By Lie's algorithm, the coefficients of the symmetry generator satisfy the infinitesimal determining system

$$\begin{gathered} \tau_x = \tau_y = \tau_u = \tau_v = \tau_w = 0, \\ \xi_y = \xi_u = \xi_v = \xi_w = 0, \qquad 2\xi_x = \tau_t, \\ \eta_x = \eta_u = \eta_v = \eta_w = 0, \qquad 2\eta_y = \tau_t, \\ \phi_w = 0, \qquad -2\phi_u = \tau_t, \qquad 2v\phi_v = 2\phi + u\tau_t, \\ -2\phi_x = v\xi_t, \qquad -2\phi_y = \epsilon v\eta_t, \qquad 2\gamma = -v\tau_t - 2u\phi_v, \qquad \beta = \phi_{vt}. \end{gathered} \tag{5.6.3}$$

The infinitesimal determining equations (5.6.3) can be integrated, [24]. The solution is

$$\begin{aligned}
\xi &= g(t) + \frac{f'(t)}{2}x, \\
\eta &= h(t) + \frac{f'(t)}{2}y, \\
\tau &= f(t), \\
\phi &= -\frac{f'(t)}{2}u + \left(m(t) - \frac{f''(t)}{8}(x^2+\epsilon y^2) - \frac{g'(t)}{2}x - \frac{\epsilon h'(t)}{2}y\right)v, \\
\gamma &= -\frac{f'(t)}{2}v - \left(m(t) - \frac{f''(t)}{8}(x^2+\epsilon y^2) - \frac{g'(t)}{2}x - \frac{\epsilon h'(t)}{2}y\right)u, \\
\beta &= -\frac{f'''(t)}{8}(x^2+\epsilon y^2) - \frac{g''(t)}{2}x - \frac{\epsilon h''(t)}{2}y + m'(t),
\end{aligned} \tag{5.6.4}$$

where $f(t)$, $g(t)$, $h(t)$ and $m(t)$ are analytic functions. The corresponding Lie pseudo-group action is

$$\begin{aligned}
T =& F(t), \\
X =& a^{1/2}(T)x + B(t), \\
Y =& a^{1/2}(T)y + C(t), \\
\Psi =& \frac{\psi}{a^{1/2}(T)} \exp\left[i\left(\frac{a'(T)}{8a(T)}(X^2+\epsilon Y^2) + \frac{b(T)}{2a(T)}X + \frac{\epsilon c(T)}{2a(T)}Y + E(T)\right)\right], \\
W =& \frac{w}{a(T)} - \left(a(T)a''(T) - \frac{a'(T)^2}{2}\right)\frac{(X^2+\epsilon Y^2)}{8a^2(T)} \\
&- \left(\frac{2b'(T)a(T) - b(T)a'(T)}{4a^2(T)}\right)X - \left(\frac{2c'(T)a(T) - c(T)a'(T)}{4a^2(T)}\right)Y \\
&- e'(T) - \frac{b^2(T)}{4a^2(T)} - \epsilon\frac{c^2(T)}{4a^2(T)},
\end{aligned} \tag{5.6.5}$$

where

$$a(T) = a(T(t)) = F'(t), \qquad e(T) = e(T(t)) = E(t),$$

$$b(T) = b(T(t)) = B'(t) - \frac{B(t)F''(t)}{2F'(t)}, \quad c(T) = c(T(t)) = C'(t) - \frac{C(t)F''(t)}{2F'(t)},$$

and $E(t)$, $B(t)$, $C(t)$ are arbitrary analytic functions and $F(t)$ is a local analytic diffeomorphism. The corresponding lifted horizontal coframe is spanned by the three horizontal forms

$$\begin{aligned}
d_H T &= a(T)dt, \\
d_H X &= a^{1/2}(T)dx + \left(\frac{a'(T)X}{2} + b(T)\right) dt, \\
d_H Y &= a^{1/2}(T)dy + \left(\frac{a'(T)Y}{2} + c(T)\right) dt.
\end{aligned}$$

Thus the dual lifted total differential operators are

$$\begin{aligned}
D_X =& \frac{1}{a^{1/2}(T)} D_x, \\
D_Y =& \frac{1}{a^{1/2}(T)} D_y, \\
D_T =& - \frac{1}{a^{3/2}(T)} \left(\frac{a'(T)X}{2} + b(T)\right) D_x + \frac{1}{a(T)} D_t \\
& - \frac{1}{a^{3/2}(T)} \left(\frac{a'(T)Y}{2} + c(T)\right) D_y.
\end{aligned} \tag{5.6.6}$$

To simplify the expressions of the transformed submanifold jet coordinates we rewrite (5.6.5) as

$$\begin{aligned}
\Psi =& \frac{\psi}{a^{1/2}(T)} \exp[i\zeta(X, Y, T)], \\
W =& \frac{w}{a(T)} - \xi(T)(X^2 + \epsilon Y^2) - \alpha(T)X - \epsilon\gamma(T)Y + \beta(T).
\end{aligned}$$

With this notation we obtain

$$\widehat{\Psi}_X = \frac{\psi_x}{a(T)} \exp[i\zeta(X, Y, T)] + \Psi i \left(\frac{a'(T)X}{4a(T)} + \frac{b(T)}{2a(T)}\right),$$

$$
\begin{aligned}
\widehat{\Psi}_Y =& \frac{\psi_y}{a(T)} \exp[i\zeta(X,Y,T)] + \epsilon \Psi i \left(\frac{a'(T)Y}{4a(T)} + \frac{c(T)}{2a(T)} \right), \\
\widehat{\Psi}_T =& \left[-\frac{\psi a'(T)}{2a^{3/2}(T)} - \frac{1}{a^2(T)} \left(\frac{a'(T)X}{2} + b(T) \right) \psi_x \right. \\
& \left. - \frac{1}{a^2(T)} \left(\frac{a'(T)Y}{2} + C(T) \right) \psi_y + \frac{\psi_t}{a^{3/2}(T)} \right] \exp[i\zeta(X,Y,T)] \\
& + i\Psi \left[\left(\frac{a''(T)}{8a(T)} - \frac{(a'(T))^2}{8a^2(T)} \right) (X^2 + \epsilon Y^2) + \left(\frac{b'(T)}{2a(T)} - \frac{b(T)a'(T)}{2a^2(T)} \right) X \right. \\
& \left. + \epsilon \left(\frac{c'(T)}{2a(T)} - \frac{c(T)a'(T)}{2a^2(T)} \right) Y + e'(T) \right], \\
\widehat{W}_X =& \frac{w_x}{a^{3/2}(T)} + 2\xi(T)X + \alpha(T), \\
\widehat{W}_Y =& \frac{w_y}{a^{3/2}(T)} + 2\epsilon\xi(T)Y + \epsilon\gamma(T), \\
\widehat{W}_T =& - \frac{a'(T)w}{a^2(T)} - \frac{1}{a^{5/2}(T)} \left(\frac{a'(T)X}{2} + b(T) \right) w_x \\
& - \frac{1}{a^{5/2}(T)} \left(\frac{a'(T)Y}{2} + c(T) \right) w_y + \frac{w_t}{a^2(T)} + \xi'(T)(X^2 + \epsilon Y^2) \\
& + \alpha'(T)X + \epsilon\gamma'(T)Y + \beta'(T), \\
\widehat{W}_{XX} =& \frac{w_{xx}}{a^2(T)} + 2\xi(T), \\
\widehat{W}_{XT} =& - \frac{3}{2} \frac{a'(T)}{a^{5/2}(T)} w_x - \frac{1}{a^3(T)} \left(\frac{a'(T)X}{2} + b(T) \right) w_{xx}, \\
\widehat{W}_{YT} =& - \frac{3}{2} \frac{a'(T)}{a^{5/2}(T)} w_y - \frac{1}{a^3(T)} \left(\frac{a'(T)X}{2} + b(T) \right) w_{xy} \\
& - \frac{1}{a^3(T)} \left(\frac{a'(T)Y}{2} + c(T) \right) w_{yy} + \frac{1}{a^{5/2}(T)} w_{ty} + 2\epsilon\xi'(T)Y + \epsilon\gamma'(T), \\
& \vdots
\end{aligned}
$$

We choose the cross-section

$$
\begin{gathered}
V = 1, \qquad X = Y = T = U = W = \widehat{U}_X = \widehat{U}_Y = \widehat{V}_T = 0, \\
\widehat{W}_{XT^k} = \widehat{W}_{XXT^k} = \widehat{W}_{YT^k} = \widehat{W}_{T^k} = 0, \qquad k \geq 0.
\end{gathered} \tag{5.6.7}
$$

Solving the normalization equations we obtain

$$
\begin{aligned}
F(t) =& 0, \\
B(t) =& -(u^2+v^2)^{1/2}x, \\
C(t) =& -(u^2+v^2)^{1/2}y, \\
a(T) =& u^2+v^2, \\
b(T) =& 2\frac{vu_x-uv_x}{(u^2+v^2)^{1/2}}, \\
c(T) =& 2\epsilon\frac{vu_y-uv_y}{(u^2+v^2)^{1/2}}, \\
e(T) =& \arctan\left(\frac{u}{v}\right), \\
a'(T) =& \frac{2}{(u^2+v^2)^2}[2(uv_x-vu_x)(uu_x+vv_x)+2\epsilon(vu_y-uv_y)(uu_y+vv_y) \\
& +(u^2+v^2)(uu_t+vv_t)], \\
b'(T) =& \frac{2w_x}{(u^2+v^2)^{1/2}}+\frac{2(vu_x-uv_x)}{(u^2+v^2)^{7/2}}[2(vu_x-uv_x)(uu_x+vv_x) \\
& +2\epsilon(vu_y-uv_y)(uu_y+vv_y)-(u^2+v^2)(uu_t+vv_t)], \\
c'(T) =& \frac{2w_y}{(u^2+v^2)^{1/2}}+\frac{2\epsilon(vu_y-uv_y)}{(u^2+v^2)^{7/2}}[2(vu_x-uv_x)(uu_x+vv_x) \\
& +2\epsilon(vu_y-uv_y)(uu_y+vv_y)-(u^2+v^2)(uu_t+vv_t)], \\
e'(T) =& \frac{w}{u^2+v^2}-\frac{(vu_x-uv_x)^2}{(u^2+v^2)^3}-\epsilon\frac{(vu_y-uv_y)^2}{(u^2+v^2)^3}, \\
a''(T) =& \frac{4w_{xx}}{u^2+v^2}+\frac{1}{(u^2+v^2)^3}[2(vu_x-uv_x)(uu_x+vv_x) \\
& +2\epsilon(vu_y-uv_y)(uu_y+vv_y)-(u^2+v^2)(uu_t+vv_t)], \\
b''(T) =& \frac{2}{(u^2+v^2)^{5/2}}[(u^2+v^2)w_{tx}+2\epsilon(uv_y-vu_y)w_{xy}], \\
c''(T) =& \frac{2\epsilon}{(u^2+v^2)^{5/2}}[(u^2+v^2)w_{ty}+2(uv_x-vu_x)w_{xy}+(vu_y-uv_y)w_{xx} \\
& +2\epsilon(uv_y-vu_y)w_{yy}],
\end{aligned}
$$

$$
\begin{aligned}
e''(T) =& \frac{w_t}{(u^2+v^2)^2} + \frac{2w}{(u^2+v^2)^4}[2(vu_x - uv_x)(uu_x + vv_x) \\
&+ 2\epsilon(vu_y - uv_y)(uu_y + vv_y) - (u^2+v^2)(uu_t + vv_t)] \\
&- \frac{6}{(u^2+v^2)^3}[(vu_x - uv_x)w_x + \epsilon(vu_y - uv_y)w_y], \\
&\vdots
\end{aligned}
$$

which is well-defined provided $u^2 + v^2 \neq 0$. Substituting the normalized pseudo-group parameters into the non-phantom differential invariants gives

$$
\begin{aligned}
I^1_{0,0,1} = \iota(u_t) =& \frac{vu_t - uv_t}{(u^2+v^2)^2} - \frac{w}{u^2+v^2} - \frac{(vu_x - uv_x)^2}{(u^2+v^2)^3} - \epsilon\frac{(vu_y - uv_y)^2}{(u^2+v^2)^3}, \\
I^2_{1,0,0} = \iota(v_x) =& \frac{vv_x + uu_x}{(u^2+v^2)^{3/2}}, \qquad I^2_{0,1,0} = \iota(v_y) = \frac{vv_y + uu_y}{(u^2+v^2)^{3/2}}, \\
I^1_{2,0,0} = \iota(u_{xx}) =& \frac{vu_{xx} - uv_{xx}}{(u^2+v^2)^2} - \frac{(vv_x + uu_x)(vu_x - uv_x)}{(u^2+v^2)^3} \\
&+ \epsilon\frac{(vv_y - uv_y)(uu_y + vv_y)}{(u^2+v^2)^3} - \frac{uu_t + vv_t}{2(u^2+v^2)^2}, \\
I^1_{0,2,0} = \iota(u_{yy}) =& \frac{vu_{yy} - uv_{yy}}{(u^2+v^2)^2} - \frac{(vv_y - uv_y)(uu_y + vv_y)}{(u^2+v^2)^3} \\
&+ \epsilon\frac{(vv_x + uu_x)(vu_x - uv_x)}{(u^2+v^2)^3} - \epsilon\frac{uu_t + vv_t}{2(u^2+v^2)^2}, \\
I^2_{2,0,0} = \iota(v_{xx}) =& \frac{vv_{xx} + uu_{xx}}{(u^2+v^2)^2} + \frac{(vu_x - uv_x)^2}{(u^2+v^2)^3}, \\
I^2_{0,2,0} = \iota(v_{yy}) =& \frac{vv_{yy} + uu_{yy}}{(u^2+v^2)^2} + \frac{(vu_y - uv_y)^2}{(u^2+v^2)^3}, \\
I^2_{1,0,1} = \iota(v_{xt}) =& 2\frac{uu_y + vv_y}{(u^2+v^2)^{9/2}}[2(uv_x - vu_x)(uu_x + vv_x) \\
&+ 2\epsilon(uv_y - vu_y)(uu_y + vv_y) + (u^2+v^2)(uu_t + vv_t)] \\
&+ 2\frac{(uv_x - vu_x)(uu_{xy} - vv_{xy})}{(u^2+v^2)^{7/2}} \\
&+ 2\epsilon\frac{(uv_y - vu_y)(uu_{yy} + vv_{yy})}{(u^2+v^2)^{7/2}} + \frac{uu_{ty} + vv_{ty}}{(u^2+v^2)^{5/2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(uv_y - vu_y)}{(u^2+v^2)^{9/2}}[2(vu_x - uv_x)^2 + 2\epsilon(vu_y - uv_y)^2 \\
&+ (u^2+v^2)(uv_t - vu_t)], \\
I^3_{1,1,0} = \iota(w_{xy}) =&\frac{w_{xy}}{(u^2+v^2)^2}, \\
I^3_{0,2,0} = \iota(w_{yy}) =&\frac{w_{yy}}{(u^2+v^2)^2} - \epsilon\frac{w_{xx}}{(u^2+v^2)^2}, \\
&\vdots
\end{aligned}
$$

The Davey–Stewartson equations are related to the normalized invariants in the following way:

$$
\begin{aligned}
\iota(\Delta^1_{DS}) = I^1_{0,0,1} + I^2_{2,0,0} + \epsilon I^2_{2,0,0} - \delta = \frac{v\Delta^1_{DS} + u\Delta^2_{DS}}{(u^2+v^2)^2} = 0, \\
\iota(\Delta^2_{DS}) = I^1_{2,0,0} + \epsilon I^1_{0,2,0} = \frac{v\Delta^2_{DS} - u\Delta^1_{DS}}{(u^2+v^2)^2} = 0, \\
\iota(\Delta^3_{DS}) = -\epsilon I^3_{0,2,0} - 2\alpha[I^2_{0,2,0} + (I^2_{0,1,0})^2] = \frac{\Delta^3_{DS}}{(u^2+v^2)^2} = 0.
\end{aligned}
\tag{5.6.8}
$$

The expressions for the invariant total differential operators are

$$
\begin{aligned}
\mathcal{D}_1 =&\frac{1}{(u^2+v^2)^{1/2}}D_x, \qquad \mathcal{D}_2 = \frac{1}{(u^2+v^2)^{1/2}}D_y, \\
\mathcal{D}_3 =&\frac{1}{(u^2+v^2)^2}[2(uv_x - vu_x)D_x + 2\epsilon(uv_y - vu_y)D_y + (u^2+v^2)D_t].
\end{aligned}
\tag{5.6.9}
$$

We now analyze the algebra of differential invariants. In the following equations we use the notation

$$
\begin{aligned}
\mu_k = \pi_{\mathcal{H}} \circ \iota(\xi_{t^k}), \qquad \nu_k = \pi_{\mathcal{H}} \circ \iota(\eta_{t^k}), \\
\alpha_k = \pi_{\mathcal{H}} \circ \iota(\tau_{t^k}), \qquad \zeta_k = \pi_{\mathcal{H}} \circ \iota(\phi_{vt^k}),
\end{aligned}
\tag{5.6.10}
$$

$k \geq 0$. The first few recurrence relations (5.4.5), restricted to the cross-section (5.6.7), are

$$
\begin{aligned}
0 &= \varpi^1 + \mu,\\
0 &= \varpi^2 + \nu,\\
0 &= \varpi^3 + \alpha,\\
0 &= I^1_{0,0,1}\varpi^3 + \zeta,\\
0 &= I^2_{1,0,0}\varpi^1 + I^2_{0,1,0}\varpi^2 - \frac{1}{2}\alpha_1,\\
0 &= \zeta_1,\\
0 &= I^1_{2,0,0}\varpi^1 + I^1_{1,1,0}\varpi^2 + I^1_{1,0,1}\varpi^3 - \frac{1}{2}\mu_1 + I^2_{1,0,0}\zeta,\\
0 &= I^1_{1,1,0}\varpi^1 + I^1_{0,2,0}\varpi^2 + I^1_{0,1,1}\varpi^3 - \frac{\epsilon}{2}\nu_1 + I^2_{0,1,0}\zeta,\\
\sum_{i=1}^{3}\mathcal{D}_i I^1_{0,0,1}\varpi^i &= I^1_{1,0,1}\varpi^1 + I^1_{0,1,1}\varpi^2 + I^1_{0,0,2}\varpi^3 + \zeta_1 - \frac{3}{2}I^1_{0,0,1}\alpha_1,\\
\sum_{i=1}^{3}\mathcal{D}_i I^2_{1,0,0}\varpi^i &= I^2_{2,0,0}\varpi^1 + I^2_{1,1,0}\varpi^2 + I^2_{1,0,1}\varpi^3 - I^2_{1,0,0}\alpha_1,\\
\sum_{i=1}^{3}\mathcal{D}_i I^2_{0,1,0}\varpi^i &= I^2_{1,1,0}\varpi^1 + I^2_{0,2,0}\varpi^2 + I^2_{0,1,1}\varpi^3 - I^2_{0,1,0}\alpha_1,\\
0 &= I^2_{1,0,1}\varpi^1 + I^2_{0,1,1}\varpi^2 + I^2_{0,0,2}\varpi^3 - \frac{1}{2}\alpha_2 - I^1_{0,0,1}\zeta - I^2_{1,0,0}\mu_1 - I^2_{0,1,0}\nu_1,\\
0 &= I^3_{1,1,0}\varpi^2 - \frac{1}{2}\mu_2,\\
0 &= I^3_{1,1,0}\varpi^1 + I^3_{0,2,0}\varpi^2 - \frac{\epsilon}{2}\nu_2,\\
0 &= \zeta_2,\\
\sum_{i=1}^{3}\mathcal{D}_i I^1_{2,0,0}\varpi^i &= I^1_{3,0,0}\varpi^1 + I^1_{2,1,0}\varpi^2 + I^1_{2,0,1}\varpi^3 + I^2_{2,0,0}\zeta - I^2_{1,0,0}\mu_1\\
&\quad - \frac{3}{2}I^1_{2,0,0}\alpha_1 - \frac{1}{4}\alpha_2,\\
\sum_{i=1}^{3}\mathcal{D}_i I^1_{1,1,0}\varpi^i &= I^1_{2,1,0}\varpi^1 + I^1_{1,2,0}\varpi^2 + I^1_{1,1,1}\varpi^3 + I^2_{1,1,0}\zeta - \frac{1}{2}I^2_{0,1,0}\mu_1
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\epsilon}{2} I^2_{1,0,0}\nu_1 - \frac{3}{2} I^1_{1,1,0}\alpha_1,\\
\sum_{i=1}^{3} \mathcal{D}_i I^1_{0,2,0}\varpi^i &= I^1_{1,2,0}\varpi^1 + I^1_{0,3,0}\varpi^2 + I^1_{0,2,1}\varpi^3 + I^2_{0,2,0}\zeta - \epsilon I^2_{0,1,0}\nu_1\\
&- \frac{3}{2} I^1_{0,2,0}\alpha_1 - \frac{\epsilon}{4}\alpha_2,\\
\sum_{i=1}^{3} \mathcal{D}_i I^1_{1,0,1}\varpi^i &= I^1_{2,0,1}\varpi^1 + I^1_{1,1,1}\varpi^2 + I^1_{1,0,2}\varpi^3 + I^2_{1,0,0}\zeta_1 + I^2_{1,0,1}\zeta - I^1_{2,0,0}\mu_1\\
&- \frac{1}{2}\mu_2 - I^1_{1,1,0}\nu_1 - 2I^1_{1,0,1}\alpha_1\\
\sum_{i=1}^{3} \mathcal{D}_i I^1_{0,0,2}\varpi^i &= I^1_{1,0,2}\varpi^1 + I^1_{0,1,2}\varpi^2 + I^1_{0,0,3}\varpi^3 + \zeta_2 + I^2_{0,0,2}\zeta - 2I^1_{1,0,1}\mu_1\\
&- 2I^1_{0,1,1}\nu_1 - \frac{5}{2} I^1_{0,0,2}\alpha_1 - 2I^1_{0,0,1}\alpha_2,\\
\sum_{i=1}^{3} \mathcal{D}_i I^1_{0,1,1}\varpi^i &= I^1_{1,1,1}\varpi^1 + I^1_{0,2,1}\varpi^2 + I^1_{0,1,2}\varpi^3 + I^2_{0,1,0}\zeta_1 + I^2_{0,1,1}\zeta - I^1_{1,1,0}\mu_1\\
&- I^1_{0,2,0}\nu_1 - \frac{\epsilon}{2}\nu_2 - 2I^1_{0,1,1}\alpha_1,\\
\sum_{i=1}^{3} \mathcal{D}_i I^2_{2,0,0}\varpi^i &= I^2_{3,0,0}\varpi^1 + I^2_{2,1,0}\varpi^2 + I^2_{2,0,1}\varpi^3 - I^1_{2,0,0}\zeta - \frac{3}{2} I^2_{2,0,0}\alpha_1,\\
\sum_{i=1}^{3} \mathcal{D}_i I^2_{1,1,0}\varpi^i &= I^2_{2,1,0}\varpi^1 + I^2_{1,2,0}\varpi^2 + I^2_{1,1,1}\varpi^3 - I^1_{1,1,0}\zeta - \frac{3}{2} I^2_{1,1,0}\alpha_1,\\
\sum_{i=1}^{3} \mathcal{D}_i I^2_{0,2,0}\varpi^i &= I^2_{1,2,0}\varpi^1 + I^2_{0,3,0}\varpi^2 + I^2_{0,2,1}\varpi^3 - I^1_{0,2,0}\zeta - \frac{3}{2} I^2_{0,2,0}\alpha_1,\\
\sum_{i=1}^{3} \mathcal{D}_i I^2_{1,0,1}\varpi^i &= I^2_{2,0,1}\varpi^1 + I^2_{1,1,1}\varpi^2 + I^2_{1,0,2}\varpi^3 - I^1_{1,0,1}\zeta + \frac{1}{2} I^1_{0,0,1}\mu_1 - I^2_{2,0,0}\mu_1\\
&- I^2_{1,1,0}\nu_1 - 2I^2_{1,0,1}\alpha_1 - I^2_{1,0,0}\alpha_2,\\
\sum_{i=1}^{3} \mathcal{D}_i I^2_{0,0,2}\varpi^i &= I^2_{1,0,2}\varpi^1 + I^2_{0,1,2}\varpi^2 + I^2_{0,0,3}\varpi^3 - 2I^1_{0,0,1}\zeta_1 - I^1_{0,0,2}\zeta - 2I^2_{1,0,1}\mu_1\\
&- I^2_{1,0,0}\mu_2 - 2I^2_{0,1,1}\nu_1 - I^2_{0,1,0}\nu_2 - \frac{5}{2} I^2_{0,0,2}\alpha_1 - \frac{1}{2}\alpha_3,
\end{aligned}
$$

$$\sum_{i=1}^{3} \mathcal{D}_i I^2_{0,1,1}\varpi^i = I^2_{1,1,1}\varpi^1 + I^2_{0,2,1}\varpi^2 + I^2_{0,1,2}\varpi^3 - I^1_{0,1,1}\zeta - I^2_{1,1,0}\mu_1 + \frac{\epsilon}{2}I^1_{0,0,1}\nu_1$$
$$- I^2_{0,2,0}\nu_1 - 2I^2_{0,1,1}\alpha_1 - I^2_{0,1,0}\alpha_2,$$
$$0 = I^3_{3,0,0}\varpi^1 + I^3_{2,1,0}\varpi^2 - \frac{1}{4}\alpha_3,$$
$$\sum_{i=1}^{3} \mathcal{D}_i I^3_{1,1,0}\varpi^i = I^3_{2,1,0}\varpi^1 + I^3_{1,2,0}\varpi^2 + I^3_{1,1,1}\varpi^3 - I^3_{1,1,0}\alpha_1,$$
$$\sum_{i=1}^{3} \mathcal{D}_i I^3_{0,2,0}\varpi^i = I^3_{1,2,0}\varpi^1 + I^3_{0,3,0}\varpi^2 + I^3_{0,2,1}\varpi^3 - I^3_{0,2,0}\alpha_1 - \frac{\epsilon}{4}\alpha_3,$$
$$0 = I^3_{1,1,1}\varpi^2 - \frac{1}{2}\mu_3 - I^3_{1,1,0}\nu_1,$$
$$0 = I^3_{0,1,2}\varpi^2 + \zeta_3,$$
$$0 = I^3_{1,1,1}\varpi^1 + I^3_{0,2,1}\varpi^2 - I^3_{1,1,0}\mu_1 - I^3_{0,2,0}\nu_1 - \frac{\epsilon}{2}\nu_3,$$
$$\vdots$$

We use the recurrence relations for the phantom invariants to solve for the differential forms (5.6.10) and obtain

$$\mu = -\varpi^1, \qquad \nu = -\varpi^2, \qquad \alpha = -\varpi^3, \qquad \zeta = - I^1_{0,0,1}\varpi^3,$$
$$\alpha_1 = 2(I^2_{1,0,0}\varpi^1 + I^2_{0,1,0}\varpi^2), \qquad \zeta_1 = 0, \qquad \zeta_2 = 0,$$
$$\mu_1 = 2(I^1_{2,0,0}\varpi^1 + I^1_{1,1,0}\varpi^2 + I^1_{1,0,1}\varpi^3 - I^1_{0,0,1}I^2_{1,0,0}\varpi^3),$$
$$\nu_1 = 2\epsilon(I^1_{1,1,0}\varpi^1 + I^1_{0,2,0}\varpi^2 + I^1_{0,1,1}\varpi^3 - I^1_{0,0,1}I^2_{0,1,0}\varpi^3), \qquad \mu_2 = 2I^3_{1,1,0}\varpi^2,$$
$$\nu_2 = 2\epsilon(I^3_{1,1,0}\varpi^1 + I^3_{0,2,0}\varpi^2), \quad \alpha_3 = 4(I^3_{3,0,0}\varpi^1 + I^3_{2,1,0}\varpi^2), \quad \zeta_3 = -I^3_{0,1,2}\varpi^2,$$
$$\alpha_2 = 2[I^2_{1,0,1}\varpi^1 + I^2_{0,1,1}\varpi^2 + (I^1_{0,0,1})^2\varpi^3 + I^2_{0,0,2}\varpi^3 - 2\epsilon I^2_{0,1,0}(I^1_{1,1,0}\varpi^1 + I^1_{0,2,0}\varpi^2$$
$$+(I^1_{0,1,1} - I^1_{0,0,1}I^2_{0,1,0})\varpi^3) - 2I^2_{1,0,0}(I^1_{2,0,0}\varpi^1 + I^1_{1,1,0}\varpi^2 + (I^1_{1,0,1} - I^1_{0,0,1}I^2_{1,0,0})\varpi^3)],$$
$$\mu_3 = 2[I^3_{1,1,1}\varpi^2 - 2\epsilon I^3_{1,1,0}(I^1_{1,1,0}\varpi^1 + I^1_{0,2,0}\varpi^2 + I^1_{0,1,1}\varpi^3 - I^2_{0,1,0}I^1_{0,0,1}\varpi^3)],$$
$$\nu_3 = 2\epsilon[I^3_{1,1,1}\varpi^1 + I^3_{0,2,1}\varpi^2 + I^3_{0,1,2}\varpi^3 - 2I^3_{1,1,0}(I^1_{2,0,0}\varpi^1 + I^1_{1,1,0}\varpi^2 + I^1_{1,0,1}\varpi^3$$
$$-I^2_{1,0,0}I^1_{0,0,1}\varpi^3) - 2\epsilon I^3_{0,2,0}(I^1_{1,1,0}\varpi^1 + I^1_{0,2,0}\varpi^2 + I^1_{0,1,1}\varpi^3 - I^2_{0,1,0}I^1_{0,0,1}\varpi^3)],$$

for the first few differential forms. Substituting those expressions into the recurrence relations for the non-phantom invariants yields the following recurrence relations

$$
\begin{aligned}
\mathcal{D}_1 I^1_{0,0,1} =& I^1_{1,0,1} - 3I^1_{0,0,1}I^2_{1,0,0},\\
\mathcal{D}_2 I^1_{0,0,1} =& I^1_{0,1,1} - 3I^1_{0,0,1}I^2_{0,1,0},\\
\mathcal{D}_3 I^1_{0,0,1} =& I^1_{0,0,2},\\
\mathcal{D}_1 I^2_{1,0,0} =& I^2_{2,0,0} - 2I^2_{1,0,0}I^2_{1,0,0},\\
\mathcal{D}_2 I^2_{1,0,0} =& I^2_{1,1,0} - 2I^2_{1,0,0}I^2_{0,1,0},\\
\mathcal{D}_3 I^2_{1,0,0} =& I^2_{1,0,1},\\
\mathcal{D}_2 I^2_{0,1,0} =& I^2_{0,2,0} - 2(I^2_{0,1,0})^2,\\
\mathcal{D}_3 I^2_{0,1,0} =& I^2_{0,1,1},\\
\mathcal{D}_1 I^1_{2,0,0} =& I^1_{3,0,0} + \epsilon I^1_{1,1,0}I^2_{0,1,0} - 4I^1_{2,0,0}I^2_{1,0,0} + \frac{1}{2}I^2_{1,0,1},\\
\mathcal{D}_2 I^1_{2,0,0} =& I^1_{2,1,0} + \epsilon I^1_{0,2,0}I^2_{0,1,0} - 3I^1_{2,0,0}I^2_{0,1,0} - \frac{1}{2}I^2_{0,1,1} - I^1_{1,1,0}I^2_{1,0,0},\\
\mathcal{D}_3 I^1_{2,0,0} =& I^1_{2,0,1} + \epsilon I^2_{0,1,0}(I^1_{0,1,1} - I^1_{0,0,1}I^2_{0,1,0}) - \frac{1}{2}((I^1_{0,0,1})^2 + I^2_{0,0,2} + 2I^1_{1,0,1}I^2_{1,0,0}\\
&- 2I^1_{0,0,1}(I^2_{1,0,0})^2 + 2I^1_{0,0,1}I^2_{2,0,0}),\\
\mathcal{D}_2 I^1_{1,1,0} =& I^1_{1,2,0} - 4I^1_{1,1,0}I^2_{0,1,0} - I^1_{0,2,0}I^2_{1,0,0},\\
\mathcal{D}_3 I^1_{1,1,0} =& I^1_{1,1,1} - I^1_{1,0,1}I^2_{0,1,0} - I^1_{0,1,1}I^2_{1,0,0} + 2I^1_{0,0,1}I^2_{0,1,0}I^2_{1,0,0} - I^1_{0,0,1}I^2_{1,1,0},\\
\mathcal{D}_2 I^1_{0,2,0} =& I^1_{0,3,0} - 4I^1_{0,2,0}I^2_{0,1,0} + \epsilon(I^1_{1,1,0}I^2_{1,0,0} - \frac{1}{2}I^2_{0,1,1}),\\
\mathcal{D}_3 I^1_{0,2,0} =& I^1_{0,2,1} - \frac{\epsilon}{2}((I^1_{0,0,1})^2 + I^2_{0,0,2}) + I^1_{0,1,1}I^2_{0,1,1} + I^1_{0,0,1}(I^2_{0,1,0})^2 - I^1_{0,0,1}I^2_{0,2,0}\\
&+ \epsilon(I^1_{1,0,1}I^2_{1,0,0} - I^1_{0,0,1}(I^2_{1,0,0})^2),\\
\mathcal{D}_1 I^1_{0,0,2} =& I^1_{1,0,2} - 4I^1_{1,0,1}I^1_{2,0,0} - 4\epsilon I^1_{1,1,0}(I^1_{0,1,1} - 2I^1_{0,0,1}I^2_{0,1,0}) - 5I^1_{0,0,2}I^2_{1,0,0}\\
&+ 8I^1_{0,0,1}I^1_{2,0,0}I^2_{1,0,0} - 4I^1_{0,0,1}I^2_{1,0,0},\\
\mathcal{D}_2 I^1_{0,0,2} =& I^1_{0,1,2} - 4I^1_{1,0,1}I^1_{1,1,0} - 5I^1_{0,0,2}I^2_{0,1,0} - 4\epsilon I^1_{0,2,0}(I^1_{0,1,1} - 2I^1_{0,0,1}I^2_{0,1,0})\\
&- 4I^1_{0,0,1}I^2_{0,1,1} + 8I^1_{0,0,1}I^1_{1,1,0}I^2_{1,0,0},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_3 I^1_{0,0,2} =& I^1_{0,0,3} - 4(I^1_{0,0,1})^3 - 4\epsilon(I^1_{0,1,1})^2 - 4(I^1_{1,0,1})^2 - 5I^1_{0,0,1}I^2_{0,0,2} \\
& + 12\epsilon I^1_{0,0,1}I^1_{0,1,1}I^2_{0,1,0} - 8\epsilon(I^1_{0,0,1})^2(I^2_{0,1,0})^2 + 12I^1_{0,0,1}I^1_{1,0,1}I^2_{1,0,0} \\
& - 8(I^1_{0,0,1})^2(I^2_{1,0,0})^2, \\
\mathcal{D}_1 I^2_{2,0,0} =& I^2_{3,0,0} - 3I^2_{1,0,0}I^2_{2,0,0}, \\
\mathcal{D}_2 I^2_{2,0,0} =& I^2_{2,1,0} - 3I^2_{0,1,0}I^2_{2,0,0}, \\
\mathcal{D}_3 I^2_{2,0,0} =& I^2_{2,0,1} + I^1_{0,0,1}I^1_{2,0,0}, \\
\mathcal{D}_2 I^2_{1,1,0} =& I^2_{1,2,0} - 3I^2_{0,1,0}I^2_{1,1,0}, \\
\mathcal{D}_3 I^2_{1,1,0} =& I^2_{1,1,1} + I^1_{0,0,1}I^1_{1,1,0}, \\
\mathcal{D}_2 I^2_{0,2,0} =& I^2_{0,3,0} - 3I^2_{0,1,0}I^2_{0,2,0}, \\
\mathcal{D}_3 I^2_{0,2,0} =& I^2_{0,2,1} + I^1_{0,0,1}I^1_{0,2,0}, \\
\mathcal{D}_3 I^2_{1,0,1} =& I^2_{1,0,2} + 2I^1_{0,0,1}I^1_{1,0,1} - 3(I^1_{0,0,1})^2I^2_{1,0,0} - 2I^2_{0,0,2}I^2_{1,0,0} + 4\epsilon I^1_{0,1,1}I^2_{0,1,0}I^2_{1,0,0} \\
& - 4\epsilon I^1_{0,0,1}(I^2_{0,1,0})^2I^2_{1,0,0} + 4I^1_{1,0,0}(I^2_{1,0,0})^3 - 2\epsilon I^1_{0,1,1}I^2_{1,1,0} \\
& + 2\epsilon I^1_{0,0,1}I^2_{0,1,0}I^2_{1,1,0} - 2I^1_{1,0,1}I^2_{2,0,0} + 2I^1_{0,0,1}I^2_{1,0,0}I^2_{2,0,0}, \\
\mathcal{D}_2 I^2_{0,0,2} =& I^2_{0,1,2} - 5I^2_{0,0,2}I^2_{0,1,0} - 4\epsilon I^1_{0,2,0}I^2_{0,1,1} - 4I^1_{1,1,0}I^2_{1,0,0} - 2\epsilon I^2_{0,1,0}I^3_{0,2,0} \\
& - 2I^2_{1,0,0}I^3_{1,1,0} - 2I^3_{2,1,0}, \\
\mathcal{D}_3 I^2_{0,0,2} =& I^2_{0,0,3} + I^1_{0,0,1}I^1_{0,0,2} - 4\epsilon(I^1_{0,1,1} - I^1_{0,0,1}I^2_{0,1,0})I^2_{0,1,1} \\
& - 4(I^1_{1,0,1} - I^1_{0,0,1}I^2_{1,0,0})I^2_{1,0,1}, \\
\mathcal{D}_1 I^3_{0,2,0} =& I^3_{1,2,0} - 2I^3_{0,2,0}I^2_{1,0,0} - \epsilon I^3_{3,0,0}, \\
\mathcal{D}_2 I^3_{0,2,0} =& I^3_{0,3,0} - 2I^3_{0,2,0}I^2_{0,1,0} - \epsilon I^3_{2,1,0}, \\
\mathcal{D}_3 I^3_{0,2,0} =& I^3_{0,2,1}, \\
\mathcal{D}_1 I^3_{1,1,0} =& I^3_{2,1,0} - 2I^3_{1,1,0}I^2_{1,0,0}, \\
\mathcal{D}_2 I^3_{1,1,0} =& I^3_{1,2,0} - 2I^3_{1,1,0}I^2_{0,1,0}, \\
\mathcal{D}_3 I^3_{1,1,0} =& I^3_{1,1,1}, \\
& \vdots
\end{aligned}
$$

From the above recurrence relations we conclude that all differential invariants of order less or equal to three are certain combinations of the invariants

$$\begin{gathered} I^1_{0,0,1}, \qquad I^2_{1,0,0}, \qquad I^2_{0,1,0}, \qquad I^1_{2,0,0}, \\ I^1_{1,1,0}, \qquad I^1_{0,2,0}, \qquad I^2_{0,0,2}, \qquad I^3_{1,1,0}, \qquad I^3_{0,2,0}, \end{gathered} \tag{5.6.11}$$

and their invariant differentiation. Using the commutation relations

$$[\mathcal{D}_1, \mathcal{D}_2] = I^2_{0,1,0}\mathcal{D}_1 - I^2_{1,0,0}\mathcal{D}_2, \tag{5.6.12a}$$

$$[\mathcal{D}_1, \mathcal{D}_3] = -2I^1_{2,0,0}\mathcal{D}_1 - 2\epsilon I^1_{1,1,0}\mathcal{D}_2 - 2I^2_{1,0,0}\mathcal{D}_3, \tag{5.6.12b}$$

$$[\mathcal{D}_2, \mathcal{D}_3] = -2I^1_{1,1,0}\mathcal{D}_1 - 2\epsilon I^1_{0,2,0}\mathcal{D}_2 - 2I^2_{0,1,0}\mathcal{D}_3, \tag{5.6.12c}$$

we can reduce the number of differential invariants appearing in (5.6.11). Under the hypothesis that

$$\det \begin{pmatrix} \mathcal{D}_1 I^1_{0,0,1} & \mathcal{D}_2 I^1_{0,0,1} & \mathcal{D}_3 I^1_{0,0,1} \\ \mathcal{D}_1 I^3_{1,1,0} & \mathcal{D}_2 I^3_{1,1,0} & \mathcal{D}_3 I^3_{1,1,0} \\ \mathcal{D}_1 I^3_{0,2,0} & \mathcal{D}_2 I^3_{0,2,0} & \mathcal{D}_3 I^3_{0,2,0} \end{pmatrix} \neq 0,$$

which is generically satisfied, we apply the commutation relations (5.6.12b) and (5.6.12c) to the invariants $I^1_{0,0,1}$, $I^3_{1,1,0}$ and $I^3_{0,2,0}$ to solve for the invariants $I^1_{2,0,0}$, $I^1_{1,1,0}$, $I^2_{1,0,0}$, $I^1_{0,2,0}$ and $I^2_{0,1,0}$. Finally, subtracting the two equations

$$\begin{aligned} \mathcal{D}_3 I^1_{2,0,0} =& I^1_{2,0,1} + \epsilon I^2_{0,1,0}(I^1_{0,1,1} - I^1_{0,0,1}I^2_{0,1,0}) - \frac{1}{2}((I^1_{0,0,1})^2 + I^2_{0,0,2} + 2I^1_{1,0,1}I^2_{1,0,0} \\ &- 2I^1_{0,0,1}(I^2_{1,0,0})^2 + 2I^1_{0,0,1}I^1_{2,0,0}), \\ \mathcal{D}_1 I^1_{1,0,1} =& I^1_{2,0,1} - 2\epsilon(I^1_{1,1,0})^2 - 2(I^1_{2,0,0})^2 - 4I^1_{1,0,1}I^2_{1,0,0}, \end{aligned}$$

we find that

$$\begin{aligned} I^2_{0,0,2} =& -(I^1_{0,0,1})^2 - I^1_{1,0,1}I^2_{1,0,0} + 2I^1_{0,0,1}(I^2_{1,0,0})^2 - 2I^1_{0,0,1}I^2_{2,0,0} + 2(-\mathcal{D}_3 I^1_{2,0,0} \\ &+ \epsilon I^2_{0,1,0}(I^1_{0,1,1} - I^1_{0,0,1}I^2_{0,1,0}) + \mathcal{D}_1 I^1_{1,0,1} + 2\epsilon(I^1_{1,1,0})^2 + 2(I^1_{2,0,0})^2 \\ &+ 4I^1_{1,0,1}I^2_{1,0,0}). \end{aligned} \tag{5.6.13}$$

Since the differential invariants $I^1_{1,0,1}$, $I^2_{1,0,0}$, $I^2_{2,0,0}$, $I^2_{0,1,0}$, $I^1_{0,1,1}$, $I^1_{1,1,0}$ can be expressed solely in terms of the invariants $I^1_{0,0,1}$, $I^3_{1,1,0}$, $I^3_{0,2,0}$ and their invariant differentiation, we conclude from (5.6.13) that the same is true for $I^2_{0,0,2}$.

Proposition 5.13. The algebra of differential invariants for the symmetry group of the Davey–Stewartson equations (5.6.2) is generated by the differential invariants

$$I^1_{0,0,1}, \qquad I^3_{1,1,0}, \qquad I^3_{0,2,0}. \tag{5.6.14}$$

Proof. The above computations show that all normalized differential invariants of order less or equal to three are generated by the invariants (5.6.14). Let $k \geq 3$, from the formulas for the prolonged vector field coefficients (2.4.6) and the infinitesimal determining equations (5.6.3) we notice that the recurrence relations are of the form

$$\sum_{i=1}^{3} \mathcal{D}_i I^\alpha_J \varpi^i = \sum_{i=1}^{3} I^\alpha_{J,i} \varpi^i + P^\alpha_J(I^{(k)}, \mu^{(k+1)}, \nu^{(k+1)}, \alpha^{(k+1)}, \zeta^{(k+1)}),$$

with $\alpha = 1,2,3$ and $\#J = k$. Focusing on the correction term P^α_J , we define the operator $\mathbf{H}_k$ which projects the correction term onto the highest order differential forms μ_{k+1}, ν_{k+1}, α_{k+1}, ζ_{k+1}. For example

$$\mathbf{H}_3(I^2_{1,0,0}\mu_4 + \nu_3 + \alpha) = I^2_{1,0,0}\mu_4.$$

Then $\mathbf{H}_k(P^\alpha_J) = 0$ except for

$$\mathbf{H}_k(P^2_{0,0,k}) = -\frac{1}{2}\alpha_{k+1}, \qquad \mathbf{H}_k(P^3_{2,0,k-2}) = -\frac{1}{4}\alpha_{k+1},$$
$$\mathbf{H}_k(P^3_{0,2,k-2}) = -\frac{\epsilon}{4}\alpha_{k+1}, \qquad \mathbf{H}_k(P^3_{1,0,k-1}) = -\frac{1}{2}\mu_{k+1},$$
$$\mathbf{H}_k(P^3_{0,0,k}) = \zeta_{k+1}, \qquad \mathbf{H}_k(P^3_{0,1,k-1}) = -\frac{\epsilon}{2}\nu_{k+1}.$$

From the recurrence relation for the phantom invariant $I^3_{2,0,k-2} = 0$ we obtain

$$\alpha_{k+1} = 4(I^3_{3,0,k-2}\varpi^1 + I^3_{2,1,k-2}\varpi^2 + \widetilde{P}^3_{2,0,k-2}),$$

where $P^3_{2,0,k-2} = -\frac{1}{4}\alpha_{k+1} + \widetilde{P}^3_{2,0,k-2}$ and $\mathbf{H}_k(\widetilde{P}^3_{2,0,k-2}) = 0$. Since

$$\begin{aligned} I^3_{3,0,k-2}\varpi^3 &= \mathcal{D}_3 I^3_{3,0,k-3}\varpi^3 - \pi_3(P^3_{3,0,k-3}), \\ I^3_{2,1,k-2}\varpi^1 &= \mathcal{D}_1 I^3_{1,1,k-2}\varpi^1 - \pi_1(P^3_{1,1,k-2}), \end{aligned}$$

where π_i denotes the projection onto the invariant differential form ϖ^i, $i = 1, 2, 3$, and $\mathbf{H}_k(P^3_{3,0,k-3}) = \mathbf{H}_k(P^3_{1,1,k-2}) = 0$, we conclude that α_{k+1} is a linear combination of $\varpi^1, \varpi^2, \varpi^3$ with coefficients expressible in terms of differential invariants of order $\leq k$ and their invariant differentiation. On the other hand, the differential forms μ_{k+1}, ν_{k+1} and ζ_{k+1} only appear in the recurrence relations of the phantom invariants $I^3_{1,0,k-1} = I^3_{0,1,k-1} = I^3_{0,0,k} = 0$ at order k. Hence we conclude that all non-phantom differential invariants of order $k+1$ are expressible in terms of differential invariants of order $\leq k$ and their invariant differentiation. □

5.7 Riemannian Manifolds

The general problem of equivalence for Riemannian manifolds was originally studied by Riemann, [97], who was motivated by a problem on heat flow in a solid, and also by Christoffel, [29]. The necessary and sufficient conditions for the equivalence of Riemannian metrics is given by the following theorem, [82, 108].

Theorem 5.14. A complete set of structure invariants for a Riemannian manifold are provided by the invariant components of the higher order curvature tensors $\nabla^k \mathbf{R}$, $k = 0, 1, 2, \ldots$. In the regular case, two Riemannian manifolds are locally isometric if and only if all their curvature tensors parametrize overlapping manifolds.

In this section we characterize the algebra of differential invariants $\nabla^k \mathbf{R}$, $k = 0, 1, 2, \ldots$, for Riemannian manifolds of dimension two and three. The computation of a minimal generating set for the algebra of differential invariants is a computationally challenging problem. In Section 5.7.1 we solve, in somewhat extensive detail, the two-dimensional case, and in Section 5.7.2, we give a less detailed solution of the three-dimensional problem. Partial results have been obtained for higher dimensional Riemannian manifolds but those are not included in the present work.

Let N be a Riemannian manifold of dimension p with Riemannian metric locally given by

$$g = \sum_{i,j=1}^{p} g_{ij}(x) dx^i dx^j, \qquad g_{ij}(x) = g_{ji}(x). \tag{5.7.1}$$

Let $\mathcal{G}$ be the Lie pseudo-group all locally invertible changes of variables $\psi : N \to N$. By the usual transformation rule for tensors, the components of the Riemannian metric (5.7.1) are mapped to

$$G_{kl} = \sum_{i,j=1}^{p} g_{ij} \frac{\partial x^i}{\partial X^k} \frac{\partial x^j}{\partial X^l}, \qquad 1 \leq k, l \leq p, \tag{5.7.2}$$

under a local diffeomorphism $X = \psi(x)$. To analyze the algebra of differential invariants of the pseudo-group $\mathcal{G}$, we need to compute the infinitesimal generator of the transformation (5.7.2). Let

$$\overline{\mathbf{v}} = \sum_{i=1}^{p} \xi^i(x) \frac{\partial}{\partial x^i}, \tag{5.7.3}$$

be an infinitesimal generator whose flow is a local diffeomorphism on N. The vector field (5.7.3) induces an infinitesimal transformation of the metric

$$\widetilde{\mathbf{v}} = \sum_{1\le i\le j\le p} \phi_{ij}(x,g)\frac{\partial}{\partial g_{ij}}. \tag{5.7.4}$$

The infinitesimal generator (5.7.4) is obtained by linearizing (5.7.2) at the identity transformation $\mathbb{1}_N$. Let

$$\begin{aligned} X^i =& x^i + \epsilon\xi^i(x) + O(\epsilon^2), & i = 1,\dots,p,\\ G_{ij} =& g_{ij} + \epsilon\phi_{ij}(x,g) + O(\epsilon^2), & i,j = 1,\dots,p, \end{aligned}$$

be a one-parameter transformation group. Then

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \nabla_X x = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} (\nabla_x X)^{-1} = -\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \nabla_x X = -\nabla_x \xi, \tag{5.7.5}$$

where $\nabla_x X = (X^i_{x^j})$, $\nabla_X x = (x^i_{X^j})$, and $\nabla_x \xi = (\xi^i_{x^j})$ denote $p\times p$ Jacobian matrices. Thus the expression for ϕ_{kl} is

$$\begin{aligned} \phi_{kl} =& \frac{d}{d\epsilon}\bigg|_{\epsilon=0} G_{kl} = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \left(\sum_{i,j=1}^{p} g_{ij}\frac{\partial x^i}{\partial X^k}\frac{\partial x^j}{\partial X^l}\right)\\ =& \sum_{i,j=1}^{p} g_{ij}\left(\frac{\partial x^j}{\partial X^l}\cdot\frac{d}{d\epsilon}\left(\frac{\partial x^i}{\partial X^k}\right) + \frac{\partial x^i}{\partial X^k}\cdot\frac{d}{d\epsilon}\left(\frac{\partial x^j}{\partial X^l}\right)\right)\bigg|_{\epsilon=0}\\ =& -\sum_{i,j=1}^{p} g_{ij}\left(\delta^j_l\cdot\frac{\partial \xi^i}{\partial x^k} + \delta^i_k\cdot\frac{\partial \xi^j}{\partial x^l}\right)\\ =& -\sum_{i=1}^{p}\left(g_{il}\frac{\partial \xi^i}{\partial x^k} + g_{ik}\frac{\partial \xi^i}{\partial x^l}\right). \end{aligned}$$

In summary, an infinitesimal generator of the pseudo-group $\mathcal{G}$ acting on the manifold N and the space of Riemannian metrics is given by

$$\mathbf{v} = \overline{\mathbf{v}} + \widetilde{\mathbf{v}} = \sum_{i=1}^{p} \xi^i(x) \frac{\partial}{\partial x^i} - \sum_{1 \le k \le l \le p} \left[\sum_{i=1}^{p} \left(g_{il} \frac{\partial \xi^i}{\partial x^k} + g_{ik} \frac{\partial \xi^i}{\partial x^l} \right) \right] \frac{\partial}{\partial g_{kl}}. \quad (5.7.6)$$

We introduce the notation

$$g_{ij;J} = D_J^x g_{ij}$$

to denote total derivatives of the metric coefficients. With this notation, the components of the prolonged infinitesimal generator

$$\mathbf{v}^{(\infty)} = \sum_{i=1}^{p} \xi^i(x) \frac{\partial}{\partial x^i} + \sum_{1 \le i \le j \le p} \left[\sum_{k=\#J \ge 0} \phi_{ij}^J(x, g^{(k)}) \right] \frac{\partial}{\partial g_{ij;J}} \quad (5.7.7)$$

are

$$\phi_{kl}^J = -D_J^x \left(\sum_{i=1}^{p} g_{il} \frac{\partial \xi^i}{\partial x^k} + g_{ik} \frac{\partial \xi^i}{\partial x^l} + \xi^i g_{kl;i} \right) + \sum_{i=1}^{p} \xi^i g_{kl;J,i}.$$

Applying Leibniz' formula, [34], we obtain

$$\phi_{kl}^J = \sum_{i=1}^{p} \left(\xi^i g_{kl;J,i} - \sum_{J=K+L} \binom{J}{K} [g_{il;K} \xi^i_{L,k} + g_{ik;K} \xi^i_{L,l} + g_{kl;K,i} \xi^i_L] \right). \quad (5.7.8)$$

In the following we set

$$H^i = \iota(x^i), \qquad I_J^{ij} = \iota(g_{ij;J}), \qquad i, j = 1, \ldots, p, \qquad \#J \ge 0,$$

and

$$\nu_J^i = \pi_{\mathcal{H}} \left((\rho^{(\infty)})^* \left(\boldsymbol{\lambda} \left(\xi_J^i \right) \right) \right), \qquad i = 1, \ldots, p, \qquad \#J \ge 0.$$

5.7.1 Two-Dimensional Riemannian Manifolds

For a two-dimensional Riemannian manifold, the recurrence formulas (5.4.6), for the normalized invariants of order less or equal to two, are

$$
\begin{aligned}
\sum_{i=1}^{2}(\mathcal{D}_i H^1)\varpi^i =& \varpi^1 + \nu^1,\\
\sum_{i=1}^{2}(\mathcal{D}_i H^2)\varpi^i =& \varpi^2 + \nu^2,\\
\sum_{i=1}^{2}(\mathcal{D}_i I^{11}_{0,0})\varpi^i =& I^{11}_{1,0}\varpi^1 + I^{11}_{0,1}\varpi^2 - 2(I^{12}_{0,0}\nu^2_{1,0} + I^{11}_{0,0}\nu^1_{1,0}),\\
\sum_{i=1}^{2}(\mathcal{D}_i I^{12}_{0,0})\varpi^i =& I^{12}_{1,0}\varpi^1 + I^{12}_{0,1}\varpi^2 - I^{11}_{0,0}\nu^1_{0,1} - I^{12}_{0,0}(\nu^1_{1,0} + \nu^2_{0,1}) - I^{22}_{0,0}\nu^2_{1,0},\\
\sum_{i=1}^{2}(\mathcal{D}_i I^{22}_{0,0})\varpi^i =& I^{22}_{1,0}\varpi^1 + I^{22}_{0,1}\varpi^2 - 2(I^{12}_{0,0}\nu^1_{0,1} + I^{22}_{0,0}\nu^2_{0,1}),\\
\sum_{i=1}^{2}(\mathcal{D}_i I^{11}_{1,0})\varpi^i =& I^{11}_{2,0}\varpi^1 + I^{11}_{1,1}\varpi^2 - (2I^{12}_{1,0} + I^{11}_{0,1})\nu^2_{1,0} - 3I^{11}_{1,0}\nu^1_{1,0} - 2I^{22}_{0,0}\nu^2_{2,0}\\
&- 2I^{11}_{0,0}\nu^1_{2,0},\\
\sum_{i=1}^{2}(\mathcal{D}_i I^{11}_{0,1})\varpi^i =& I^{11}_{1,1}\varpi^1 + I^{11}_{0,2}\varpi^2 - I^{11}_{1,0}\nu^1_{0,1} - 2I^{11}_{0,1}\nu^1_{1,0} - I^{11}_{0,1}\nu^2_{0,1} - 2I^{11}_{0,0}\nu^1_{1,1}\\
&- 2I^{12}_{0,1}\nu^2_{1,0} - 2I^{12}_{0,0}\nu^2_{1,1},\\
\sum_{i=1}^{2}(\mathcal{D}_i I^{12}_{1,0})\varpi^i =& I^{12}_{2,0}\varpi^1 + I^{12}_{1,1}\varpi^2 - 2I^{12}_{1,0}\nu^1_{1,0} - (I^{22}_{1,0} + I^{12}_{0,1})\nu^2_{1,0} - I^{12}_{1,0}\nu^2_{0,1}\\
&- I^{11}_{1,0}\nu^1_{0,1} - I^{11}_{0,0}\nu^1_{1,1} - I^{12}_{0,0}\nu^1_{2,0} - I^{12}_{0,0}\nu^2_{1,1} - I^{22}_{0,0}\nu^2_{2,0},\\
\sum_{i=1}^{2}(\mathcal{D}_i I^{12}_{0,1})\varpi^i =& I^{12}_{1,1}\varpi^1 + I^{12}_{0,2}\varpi^2 - (I^{11}_{0,1} + I^{12}_{1,0})\nu^1_{0,1} - 2I^{12}_{0,1}\nu^2_{0,1} - I^{12}_{0,1}\nu^1_{1,0}\\
&- I^{22}_{0,1}\nu^2_{1,0} - I^{11}_{0,0}\nu^1_{0,2} - I^{22}_{0,0}\nu^2_{1,1} - I^{12}_{0,0}\nu^1_{1,1} - I^{12}_{0,0}\nu^2_{0,2},
\end{aligned}
$$

$$\begin{aligned}
\sum_{i=1}^{2}(\mathcal{D}_i I^{22}_{1,0})\varpi^i =& I^{22}_{2,0}\varpi^1 + I^{22}_{1,1}\varpi^2 - 2I^{12}_{1,0}\nu^1_{0,1} - 2I^{12}_{0,0}\nu^1_{1,1} - 2I^{22}_{1,0}\nu^2_{0,1} - 2I^{22}_{0,0}\nu^2_{1,1}\\
&- I^{22}_{1,0}\nu^1_{1,0} - I^{22}_{0,1}\nu^2_{1,0},\\
\sum_{i=1}^{2}(\mathcal{D}_i I^{22}_{0,1})\varpi^i =& I^{22}_{1,1}\varpi^1 + I^{22}_{0,2}\varpi^2 - (2I^{12}_{0,1} + I^{22}_{1,0})\nu^1_{0,1} - 2I^{12}_{0,0}\nu^1_{0,2} - 3I^{22}_{0,1}\nu^2_{0,1}\\
&- 2I^{22}_{0,0}\nu^2_{0,2},\\
\sum_{i=1}^{2}(\mathcal{D}_i I^{11}_{2,0})\varpi^i =& I^{11}_{3,0}\varpi^1 + I^{11}_{2,1}\varpi^2 - 4I^{11}_{2,0}\nu^1_{1,0} - 5I^{11}_{1,0}\nu^1_{2,0} - 2I^{11}_{0,0}\nu^1_{3,0}\\
&- 2(I^{11}_{1,1} + I^{12}_{2,0})\nu^2_{1,0} - I^{11}_{0,1}\nu^2_{2,0} - 4I^{12}_{1,0}\nu^2_{2,0} - 2I^{12}_{0,0}\nu^2_{3,0},\\
\sum_{i=1}^{2}(\mathcal{D}_i I^{11}_{1,1})\varpi^i =& I^{11}_{2,1}\varpi^1 + I^{11}_{1,2}\varpi^2 - I^{11}_{2,0}\nu^1_{0,1} - 3I^{11}_{1,1}\nu^1_{1,0} - 3I^{11}_{1,0}\nu^1_{1,1} - 2I^{11}_{0,1}\nu^1_{2,0}\\
&- 2I^{11}_{0,0}\nu^1_{2,1} - I^{11}_{1,1}\nu^2_{0,1} - I^{11}_{0,2}\nu^2_{1,0} - 2I^{12}_{11}\nu^2_{1,0} - I^{11}_{0,1}\nu^2_{1,1}\\
&- 2I^{12}_{1,0}\nu^2_{1,1} - 2I^{12}_{0,1}\nu^2_{2,0} - 2I^{12}_{0,0}\nu^2_{2,1},\\
\sum_{i=1}^{2}(\mathcal{D}_i I^{11}_{0,2})\varpi^i =& I^{11}_{1,2}\varpi^1 + I^{11}_{0,3}\varpi^2 - 2I^{11}_{1,1}\nu^1_{0,1} - I^{11}_{1,0}\nu^1_{0,2} - 2I^{11}_{0,2}\nu^1_{1,0} - 4I^{11}_{0,1}\nu^1_{1,1}\\
&- 2I^{11}_{0,0}\nu^1_{1,2} - 2I^{11}_{0,2}\nu^2_{0,1} - I^{11}_{0,1}\nu^2_{0,2} - 2I^{12}_{0,2}\nu^2_{1,0} - 4I^{12}_{0,1}\nu^2_{1,1}\\
&- 2I^{12}_{0,0}\nu^2_{1,2},\\
\sum_{i=1}^{2}(\mathcal{D}_i I^{12}_{2,0})\varpi^i =& I^{12}_{3,0}\varpi^1 + I^{12}_{2,1}\varpi^2 - I^{11}_{2,0}\nu^1_{0,1} - 3I^{12}_{2,0}\nu^1_{1,0} - 2I^{11}_{1,0}\nu^1_{1,1} - 3I^{12}_{1,0}\nu^1_{2,0}\\
&- I^{11}_{0,0}\nu^1_{2,1} - I^{12}_{0,0}\nu^1_{3,0} - I^{12}_{2,0}\nu^2_{0,1} - 2I^{12}_{1,1}\nu^2_{1,0} - I^{22}_{2,0}\nu^2_{1,0} - 2I^{12}_{1,0}\nu^2_{1,1}\\
&- I^{12}_{0,1}\nu^2_{2,0} - 2I^{22}_{1,0}\nu^2_{2,0} - I^{12}_{0,0}\nu^2_{2,1} - I^{22}_{0,0}\nu^2_{3,0},\\
\sum_{i=1}^{2}(\mathcal{D}_i I^{12}_{1,1})\varpi^i =& I^{12}_{2,1}\varpi^1 + I^{12}_{1,2}\varpi^2 - I^{11}_{1,1,}\nu^1_{0,1} - I^{12}_{2,0}\nu^1_{0,1} - I^{11}_{1,0}\nu^1_{0,2} - 2I^{12}_{1,1}\nu^1_{1,0}\\
&- I^{11}_{0,1}\nu^1_{1,1} - 2I^{12}_{1,0}\nu^1_{1,1} - I^{11}_{0,0}\nu^1_{1,2} - I^{12}_{0,1}\nu^1_{2,0} - I^{12}_{0,0}\nu^1_{2,1} - 2I^{12}_{1,1}\nu^2_{0,1}\\
&- I^{12}_{1,0}\nu^2_{0,2} - I^{12}_{0,2}\nu^2_{1,0} - I^{22}_{1,1}\nu^2_{1,0} - 2I^{12}_{0,1}\nu^2_{1,1} - I^{22}_{1,0}\nu^2_{1,1} - I^{12}_{0,0}\nu^2_{1,2}\\
&- I^{22}_{0,1}\nu^2_{2,0} - I^{22}_{0,0}\nu^2_{2,1},
\end{aligned}$$

$$\sum_{i=1}^{2}(\mathcal{D}_i I^{12}_{0,2})\varpi^i = I^{12}_{1,2}\varpi^1 + I^{12}_{0,3}\varpi^2 - I^{11}_{0,2}\nu^1_{0,1} - 2I^{12}_{1,1}\nu^1_{0,1} - 2I^{11}_{0,1}\nu^1_{0,2} - I^{12}_{1,0}\nu^1_{0,2} - I^{11}_{0,0}\nu^1_{0,3} - I^{12}_{0,2}\nu^1_{1,0} - 2I^{12}_{0,1}\nu^1_{1,1} - I^{12}_{0,0}\nu^1_{12} - 3I^{12}_{0,2}\nu^2_{0,1} - 3I^{12}_{0,1}\nu^2_{0,2} - I^{12}_{0,0}\nu^2_{0,3} - I^{22}_{0,2}\nu^2_{1,0} - 2I^{22}_{0,1}\nu^2_{1,1} - I^{22}_{0,0}\nu^2_{1,2},$$

$$\sum_{i=1}^{2}(\mathcal{D}_i I^{22}_{2,0})\varpi^i = I^{22}_{3,0}\varpi^1 + I^{22}_{2,1}\varpi^2 - 2I^{12}_{2,0}\nu^1_{0,1} - 2I^{22}_{2,0}\nu^1_{1,0} - 4I^{12}_{1,0}\nu^1_{1,1} - I^{22}_{1,0}\nu^1_{2,0} - 2I^{12}_{0,0}\nu^1_{2,1} - 2I^{22}_{2,0}\nu^2_{0,1} - 2I^{22}_{1,1}\nu^2_{1,0} - 4I^{2,2}_{1,0}\nu^2_{1,1} - I^{22}_{0,1}\nu^2_{2,0} - 2I^{22}_{0,0}\nu^2_{2,1},$$

$$\sum_{i=1}^{2}(\mathcal{D}_i I^{22}_{1,1})\varpi^i = I^{22}_{2,1}\varpi^1 + I^{22}_{1,2}\varpi^2 - 2I^{12}_{1,1}\nu^1_{0,1} - I^{22}_{2,0}\nu^1_{0,1} - 2I^{12}_{1,0}\nu^1_{0,2} - I^{22}_{1,1}\nu^1_{1,0} - 2I^{12}_{0,1}\nu^1_{1,1} - I^{22}_{1,0}\nu^1_{1,1} - 2I^{12}_{0,0}\nu^1_{1,2} - 3I^{22}_{1,1}\nu^2_{0,1} - 2I^{22}_{1,0}\nu^2_{0,2} - I^{22}_{0,2}\nu^2_{1,0} - 3I^{22}_{0,1}\nu^2_{1,1} - 2I^{22}_{0,0}\nu^2_{1,2},$$

$$\sum_{i=1}^{2}(\mathcal{D}_i I^{22}_{0,2})\varpi^i = I^{22}_{1,2}\varpi^1 + I^{22}_{0,3}\varpi^2 - 2(I^{12}_{0,2} + I^{22}_{1,1})\nu^1_{0,1} - 4I^{12}_{0,1}\nu^1_{0,2} - I^{22}_{1,0}\nu^1_{0,2} - 2I^{12}_{0,0}\nu^1_{0,3} - 4I^{22}_{0,2}\nu^2_{0,1} - 5I^{22}_{0,1}\nu^2_{0,2} - 2I^{22}_{0,0}\nu^2_{0,3}.$$

Though dim $J^2\mathfrak{g} =$ dim $\mathfrak{g}^{(2)} = 18$, the pseudo-group action is not free at order two. This is verified by computing the rank of the second order Lie matrix for the infinitesimal generator (5.7.7). The result of the computation is that the rank of the Lie matrix $L^{(2)}$ is equal to 17. This means that there is one differential invariant at order two. Computing the rank of the third order Lie matrix shows that the action becomes free at order three. Consequently we need to append to the above recurrence relations the recurrence relations for the third order normalized invariants in order to solve for the invariant differential forms $\nu^i_{j,k}$ of order ≤ 4. To avoid the proliferation of formulas, we do not write the recurrence formulas for the third order normalized invariants.

Many different cross-sections can be chosen, giving similar results. Here we write down the solution for a Euclidean type cross-section[2]:

$$
\begin{gathered}
H^1 = H^2 = 0, \\
I^{11}_{0,0} = I^{22}_{0,0} = 1, \qquad I^{11}_{i,j} = 0, \qquad I^{22}_{i,j} = 0, \qquad i+j \geq 1, \\
I^{12}_{0,0} = 0, \qquad I^{12}_{2,1} = 1, \qquad I^{12}_{i,0} = I^{12}_{0,i} = 0, \qquad i \geq 0.
\end{gathered} \tag{5.7.9}
$$

Since the differential invariants $I^{11}_{i,j}$ and $I^{22}_{i,j}$ are all set equal to some constants, we set $I^{12}_{i,j} = I_{i,j}$ in the formulas below. The recurrence relations for the phantom invariants are

$$
\begin{aligned}
&0 = \varpi^1 + \nu^1, && 0 = \varpi^2 + \nu^2, \\
&0 = -2\nu^1_{1,0}, && 0 = -\nu^1_{0,1} - \nu^2_{1,0}, \\
&0 = -2\nu^2_{0,1}, && 0 = -2\nu^1_{2,0}, \\
&0 = I_{1,1}\varpi^2 - \nu^1_{1,1} - \nu^2_{2,0}, && 0 = -2\nu^2_{1,1}, \\
&0 = -2\nu^1_{1,1}, && 0 = I_{1,1}\varpi^1 - \nu^1_{0,2} - \nu^2_{1,1}, \\
&0 = -2\nu^2_{0,2}, && 0 = -2\nu^1_{3,0}, \\
&0 = \varpi^2 - \nu^1_{2,1} - 2I_{1,1}\nu^2_{1,0} - \nu^2_{3,0}, && 0 = -2\nu^2_{2,1}, \\
&0 = -2\nu^1_{2,1} - 2I_{1,1}\nu^2_{1,0}, && 0 = -2I_{1,1}\nu^1_{0,1} - 2\nu^2_{1,2}, \\
&0 = -2\nu^1_{1,2}, && 0 = I_{1,2}\varpi^1 - 2I_{1,1}\nu^1_{0,1} - \nu^1_{0,3} - \nu^2_{1,2}, \\
&0 = -2\nu^2_{0,3}, && 0 = -2\nu^1_{4,0}, \\
&0 = I_{3,1}\varpi^2 - \nu^1_{3,1} - 3\nu^2_{1,0} - 3I_{1,1}\nu^2_{2,0} && 0 = -2\nu^2_{3,1}, \\
&\quad - \nu^2_{4,0}, && \\
&0 = -2\nu^1_{3,1} - 2\nu^2_{1,0} - 4I_{1,1}\nu^2_{2,0}, && 0 = I_{3,1}\varpi^1 + I_{2,2}\varpi^2 - 3\nu^1_{1,0} - 3I_{1,1}\nu^1_{2,0} \\
& && \quad - \nu^1_{2,2} - 2\nu^2_{0,1} - 2I_{1,2}\nu^2_{1,0} - 4I_{1,1}\nu^2_{1,1} \\
& && \quad - \nu^2_{3,1}, \\
&0 = -2\nu^1_{0,1} - 4I_{1,1}\nu^1_{1,1} - 2\nu^2_{2,2}, && 0 = -2\nu^1_{2,2} - 2I_{1,2}\nu^2_{1,0} - 4I_{1,1}\nu^2_{1,1},
\end{aligned}
$$

[2]The terminology stems from the fact that G_{11}, G_{22} and G_{12} are normalized to be equal to the coefficients of the standard Euclidean metric. Namely $G_{11} = G_{22} = 1$ and $G_{12} = 0$.

$$
\begin{aligned}
&0 = -2I_{1,2}\nu^1_{0,1} - 4I_{1,1}\nu^1_{0,2} - 2\nu^2_{1,3}, && 0 = -2\nu^1_{1,3},\\
&0 = I_{1,3}\varpi^1 - 3I_{1,2}\nu^1_{0,1} - 3I_{1,1}\nu^1_{0,2} && 0 = -2\nu^2_{0,4},\\
&\quad - \nu^1_{0,4} - \nu^2_{1,3},\\
&\quad\vdots
\end{aligned}
$$

Solving for the unknown one-forms $\nu^1_{i,j}$, $\nu^2_{i,j}$ we obtain

$$
\begin{aligned}
&\nu^1_{1,0} = \nu^1_{2,0} = \nu^1_{1,1} = \nu^1_{3,0} = \nu^1_{1,2} = \nu^1_{4,0} = \nu^1_{1,3} = 0,\\
&\nu^2_{0,1} = \nu^2_{1,1} = \nu^2_{0,2} = \nu^2_{2,1} = \nu^2_{0,3} = \nu^2_{3,1} = \nu^2_{0,4} = 0,\\
&\nu^1 = -\varpi^1, \qquad \nu^2 = -\varpi^2,\\
&\nu^1_{0,2} = I_{1,1}\varpi^1, \qquad \nu^2_{2,0} = I_{1,1}\varpi^2,\\
&-\nu^1_{0,1} = \nu^2_{1,0} = \frac{I_{3,1}\varpi^1 + I_{2,2}\varpi^2}{I_{1,2}},\\
&-\nu^1_{2,1} = \nu^2_{1,2} = \frac{I_{1,1}(I_{3,1}\varpi^1 + I_{2,2}\varpi^2)}{I_{1,2}},\\
&\nu^1_{0,3} = \frac{((I_{1,2})^2 + I_{1,1}I_{3,1})\varpi^1 + I_{1,1}I_{2,2}\varpi^2}{I_{1,2}},\\
&\nu^2_{3,0} = \frac{-I_{1,1}I_{3,1}\varpi^1 + (I_{1,2} + I_{1,1}I_{2,2})\varpi^2}{I_{1,2}},\\
&\nu^1_{3,1} = -\frac{I_{3,1}\varpi^1 + (2(I_{1,1})^2 I_{1,2} + I_{2,2})\varpi^2}{I_{1,2}},\\
&\nu^1_{2,2} = -I_{3,1}\varpi^1 - I_{2,2}\varpi^2,\\
&\nu^1_{0,4} = (-(I_{1,1})^2 + I_{1,3} + 2I_{3,1})\varpi^1 + 2I_{2,2}\varpi^2,\\
&\nu^2_{4,0} = \frac{-2I_{3,1}\varpi^1 - ((I_{1,1})^2 I_{1,2} + 2I_{2,2} - I_{1,2}I_{3,1})\varpi^2}{I_{1,2}},\\
&\nu^2_{2,2} = \frac{I_{3,1}\varpi^1 + I_{2,2}\varpi^2}{I_{1,2}},\\
&\nu^2_{1,3} = (-2(I_{1,1})^2 + I_{3,1})\varpi^1 + I_{2,2}\varpi^2.
\end{aligned}
\tag{5.7.10}
$$

The solution (5.7.10) is well-defined provided $I_{1,2} \neq 0$. Replacing (5.7.10) in the recurrence relations for the non-phantom invariants $I_{1,1}$ and $I_{1,2}$:

$$\begin{aligned}(\mathcal{D}_1 I_{1,1})\varpi^1 + (\mathcal{D}_2 I_{1,1})\varpi^2 =& \varpi^1 + I_{1,2}\varpi^2 - 2I_{1,1}\nu^1_{1,0} - \nu^1_{1,2} - 2I_{1,1}\nu^2_{0,1} - \nu^2_{2,1},\\ (\mathcal{D}_1 I_{1,2})\varpi^1 + (\mathcal{D}_2 I_{1,2})\varpi^2 =& I_{2,2}\varpi^1 + I_{1,3}\varpi^2 - 2\nu^1_{0,1} - 2I_{1,2}\nu^1_{1,0} - 4I_{1,1}\nu^1_{1,1}\\ & - \nu^1_{1,3} - 3I_{1,2}\nu^2_{0,1} - 3I_{1,1}\nu^2_{0,2} - \nu^2_{2,2},\end{aligned}$$

we obtain the relations

$$\begin{aligned} \mathcal{D}_1 I_{1,1} &= 1, & \mathcal{D}_2 I_{1,1} &= I_{1,2},\\ \mathcal{D}_1 I_{1,2} &= I_{2,2} + \frac{I_{3,1}}{I_{1,2}}, & \mathcal{D}_2 I_{1,2} &= I_{1,3} + \frac{I_{2,2}}{I_{1,2}}.\end{aligned} \tag{5.7.11}$$

The identities (5.7.11) are used to solve for the differential invariants $I_{1,2}$, $I_{1,3}$, $I_{3,1}$ in terms of the invariants $I_{1,1}$ and $I_{2,2}$:

$$\begin{aligned} I_{1,2} &= \mathcal{D}_2 I_{1,1},\\ I_{3,1} &= \mathcal{D}_2 I_{1,1}(\mathcal{D}_1\mathcal{D}_2 I_{1,1} - I_{2,2}),\\ I_{1,3} &= \mathcal{D}_2^2 I_{1,1} - \frac{I_{2,2}}{\mathcal{D}_2 I_{1,1}}.\end{aligned} \tag{5.7.12}$$

Lemma 5.15. The differential invariants $I_{1,1}$ and $I_{2,2}$ generate the algebra of differential invariants.

Proof. The equalities (5.7.12) show that all non-phantom differential invariants of order at most four follow from invariant differentiation of the invariants $I_{1,1}$ and $I_{2,2}$. We now show that all differential invariants of order greater than four can be obtained by invariant differentiation of lower order invariants. From (5.7.8) and our choice of cross-section (5.7.9)

$$\iota(\phi^J_{kl}) = -\nu^l_{J,k} - \nu^k_{J,l} + \psi^J_{kl}(I^{(\#J)}, \nu^{(\#J)}), \tag{5.7.13}$$

where ψ^J_{kl} depends linearly on the one-forms $\nu^{(\#J)}$ of order $\leq \#J$ and the differential invariants $I^{(\#J)}$ of order $\leq \#J$. To obtain "well-adapted algebraic recurrence formulas", [90], we make the substitutions

$$g_{12;(j^1,j^2)} \mapsto \widetilde{g}_{12;(j^i,j^2)} = g_{12;(j^1,j^2)} - \frac{1}{2}g_{11;(j^1-1,j^2+1)} - \frac{1}{2}g_{22;(j^1+1,j^2-1)},$$

$j^1+j^2 \geq 4$, and $(j^1,j^2) \notin \{(i,0),(0,i) : i \geq 0\}$. Then the recurrence relations for the non-phantom invariants are of the form

$$\mathcal{D}_1\widetilde{I}_{j^1,j^2}\varpi^1 + \mathcal{D}_2\widetilde{I}_{j^1,j^2}\varpi^2 = \widetilde{I}_{j^1+1,j^2}\varpi^1 + \widetilde{I}_{j^1,j^2+1}\varpi^2 + \widetilde{\psi}^{j_1,j_2}(I^{(\#J)},\nu^{(\#J)}), \tag{5.7.14}$$

$j^1+j^2 \geq 4$ and $(j^1,j^2) \notin \{(i,0),(0,i) : i \geq 0\}$. Since the expressions for the horizontal forms $\nu^{(\#J)}$ depend on invariants of order $\leq \#J$, the recurrence relations (5.7.14) are used to express the invariants of order $\#J+1$ in terms of lower order invariants and their invariant differentiation. □

The fundamental recurrence formula (5.4.3) applied to the differential forms dx^1 and dx^2 gives

$$\begin{aligned} d_{\mathcal{H}}\varpi^1 &= \nu^1_{1,0}\wedge\varpi^1 + \nu^1_{0,1}\wedge\varpi^2 = -\frac{I_{3,1}}{I_{1,2}}\varpi^1\wedge\varpi^2, \\ d_{\mathcal{H}}\varpi^2 &= \nu^2_{1,0}\wedge\varpi^1 + \nu^2_{0,1}\wedge\varpi^2 = -\frac{I_{2,2}}{I_{1,2}}\varpi^1\wedge\varpi^2. \end{aligned}$$

Hence the commutation relation for the invariant differential operators $\mathcal{D}_1$ and $\mathcal{D}_2$ is

$$[\mathcal{D}_1,\mathcal{D}_2] = \frac{I_{1,3}}{I_{1,2}}\mathcal{D}_1 + \frac{I_{2,2}}{I_{1,2}}\mathcal{D}_2. \tag{5.7.15}$$

Applying the commutation relation to the invariant $I_{1,1}$, we obtain

$$I_{2,2} = \left(\mathcal{D}_2 I_{1,1}\cdot\mathcal{D}_1\mathcal{D}_2 I_{1,1} - \mathcal{D}_2^2 I_{1,1}\right)\frac{\mathcal{D}_2 I_{1,1}}{(\mathcal{D}_2 I_{1,1})^2-1} \tag{5.7.16}$$

provided $I_{1,2} = \mathcal{D}_2 I_{1,1} \neq \pm 1$.

Proposition 5.16. The algebra of differential invariants for the pseudo-group of local changes of variables for a generic two-dimensional Riemannian manifold is generated by the differential invariant $I_{1,1}$.

Coordinate Expression of $I_{1,1}$

In theory, the coordinate expression of the differential invariant $I_{1,1}$ can be found by applying the normalization procedure discussed in Section 5.2. Since the pseudo-group action becomes free at order three, this involves solving a large system of nonlinear algebraic equations, which is computationally challenging. This difficult task is bypassed by taking advantage of the replacement principle (5.3.5) and known differential invariants given in Theorem 5.14.

We start by recalling some well-known formulas from Riemannian geometry, [17,93]. Let N be a p-dimensional Riemannian manifold with Riemannian metric (5.7.1), then the components of the Riemannian curvature tensor are given by

$$R^s_{ijk} = \sum_{l=1}^{p} \left(\Gamma^l_{ik}\Gamma^s_{jl} - \Gamma^l_{jk}\Gamma^s_{il}\right) + D_j\Gamma^s_{ik} - D_i\Gamma^s_{jk}, \tag{5.7.17}$$

$i, j, k, s = 1, \ldots, p$, where

$$\Gamma^m_{ij} = \sum_{k=1}^{p} \frac{1}{2} g^{km}(D_i g_{jk} + D_j g_{ki} - D_k g_{ij}) \tag{5.7.18}$$

are the Christoffel symbols. The sectional curvatures are defined by the expressions

$$\kappa_{ik} = R_{ikik} = \sum_{j=1}^{p} g_{jk} R^j_{iki}, \qquad i, k = 1, \ldots, p. \tag{5.7.19}$$

The components $\nabla_m R_{ijkl}$ of the covariant derivative of the Riemannian curvature tensor are

$$\nabla_m R_{ijkl} = D_m R_{ijkl} - \sum_{n=1}^{p} \left(R_{njkl}\Gamma^n_{im} + R_{inkl}\Gamma^n_{jm} + R_{ijnl}\Gamma^n_{km} + R_{ijkn}\Gamma^n_{lm} \right), \tag{5.7.20}$$

with $i, j, k, l, m = 1, \ldots, p$.

For surfaces, the sectional curvature is equal to the Gaussian curvature, [17, 82, 93],

$$\kappa = \kappa_{12} = R_{1212}.$$

Since all normalized invariants of order one are set equal to zero we immediately obtain from the definition of the Christoffel symbols (5.7.18) that

$$\iota(\Gamma^m_{ij}) = 0, \qquad m, i, j = 1, \ldots, p.$$

By the replacement principle, we find that the differential invariant $I_{1,1}$ is equal to the Gaussian curvature:

$$\kappa = \iota\left(\sum_{i=1}^{p} g_{i2} R^i_{121}\right) = \iota(R^2_{121}) = \iota(D_2\Gamma^2_{11}) - \iota(D_1\Gamma^2_{21}) = I_{1,1}. \tag{5.7.21}$$

Coordinate Expressions of the Invariant Differential Operators

To determine the coordinate expressions of the invariant differential operators $\mathcal{D}_1$, $\mathcal{D}_2$ we once more appeal the replacement principle. It is known, [60], that

$$\text{grad } \kappa \qquad \text{and} \qquad \text{sgrad } \kappa,$$

are two linearly independent invariant differential operators of $\mathcal{G}$. By definition, [93], the gradient of the Gaussian curvature is the unique vector field satisfying

$$g(\mathbf{v}, \text{grad } \kappa) = d\kappa(\mathbf{v}), \qquad \forall\, \mathbf{v} \in TN.$$

In local coordinates this reads

$$\text{grad } \kappa = \left(\frac{g_{22} D_1 \kappa - g_{12} D_2 \kappa}{g_{11} g_{22} - g_{12}^2} \right) D_1 + \left(\frac{-g_{12} D_1 \kappa + g_{11} D_2 \kappa}{g_{11} g_{22} - g_{12}^2} \right) D_2.$$

By definition

$$\text{sgrad } \kappa = J \text{ grad } \kappa, \qquad \text{with} \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

hence in local coodinates

$$\text{sgrad } \kappa = \left(\frac{g_{12} D_1 \kappa - g_{11} D_2 \kappa}{g_{11} g_{22} - g_{12}^2} \right) D_1 + \left(\frac{g_{22} D_1 \kappa - g_{12} D_2 \kappa}{g_{11} g_{22} - g_{12}^2} \right) D_2.$$

The dual invariant one-forms to grad κ and sgrad κ are

$$\begin{aligned} \varpi_{\text{grad } \kappa} = & \frac{g_{11} g_{22} - g_{12}^2}{(g_{11} D_2 \kappa - g_{12} D_1 \kappa)^2 + (g_{22} D_1 \kappa - g_{12} D_2 \kappa)^2} \\ & \times [(g_{22} D_1 \kappa - g_{12} D_2 \kappa) dx^1 + (g_{11} D_2 \kappa - g_{12} D_1 \kappa) dx^2], \end{aligned}$$

and

$$\begin{aligned} \varpi_{\text{sgrad } \kappa} = & \frac{g_{11} g_{22} - g_{12}^2}{(g_{11} D_2 \kappa - g_{12} D_1 \kappa)^2 + (g_{22} D_1 \kappa - g_{12} D_2 \kappa)^2} \\ & \times [-(g_{11} D_2 \kappa - g_{12} D_1 \kappa) dx^1 + (g_{22} D_1 \kappa - g_{12} D_2 \kappa) dx^2], \end{aligned}$$

respectively. Since the pseudo-group action is projectable and

$$\iota(D_1 \kappa) = 1, \qquad \iota(D_2 \kappa) = I_{1,2},$$

we have

$$\varpi_{\text{grad } \kappa} = \iota(\varpi_{\text{grad } \kappa}) = \frac{1}{(I_{1,2})^2 + 1} [\varpi^1 + I_{1,2} \varpi^2]$$

$$\varpi_{\mathrm{sgrad}\ \kappa} = \iota(\varpi_{\mathrm{sgrad}\ \kappa}) = \frac{1}{(I_{1,2})^2+1}[-I_{1,2}\varpi^1 + \varpi^2].$$

It follows that

$$\begin{aligned} \mathrm{grad}\ \kappa &= \mathcal{D}_1 + I_{1,2}\mathcal{D}_2, \\ \mathrm{sgrad}\ \kappa &= -I_{1,2}\mathcal{D}_1 + \mathcal{D}_2. \end{aligned} \tag{5.7.22}$$

Inverting (5.7.22) we conclude that

$$\begin{aligned} \mathcal{D}_1 =& \frac{1}{1+(I_{1,2})^2}(\mathrm{grad}\ \kappa - I_{1,2}\ \mathrm{sgrad}\ \kappa), \\ \mathcal{D}_2 =& \frac{}{1+(I_{1,2})^2}(I_{1,2}\ \mathrm{grad}\ \kappa + \mathrm{sgrad}\ \kappa). \end{aligned} \tag{5.7.23}$$

By the replacement principle and Theorem 5.14 we obtain

$$\nabla_2\kappa = \iota(\nabla_2\kappa) = I_{1,2}. \tag{5.7.24}$$

In conclusion,

$$\begin{aligned} \mathcal{D}_1 &= \frac{1}{1+(\nabla_2\kappa)^2}(\mathrm{grad}\ \kappa - \nabla_2\kappa\cdot\mathrm{sgrad}\ \kappa), \\ \mathcal{D}_2 &= \frac{1}{1+(\nabla_2\kappa)^2}(\nabla_2\kappa\cdot\mathrm{grad}\ \kappa + \mathrm{sgrad}\ \kappa). \end{aligned} \tag{5.7.25}$$

5.7.2 Three-Dimensional Riemannian Manifolds

For Riemannian manifolds of dimension greater than two, the pseudo-group action becomes free at order 2. For a three-dimensional Riemannian manifold, we consider a Euclidean type cross-section given by

$$\begin{gathered} H^1 = H^2 = H^3 = 0, \qquad I^{11}_{0,0,0} = I^{22}_{0,0,0} = I^{33}_{0,0,0} = 1, \\ I^{11}_{i,j,k} = I^{22}_{i,j,k} = I^{33}_{i,j,k} = 0, \qquad \text{if} \qquad i+j+k \geq 1, \\ I^{12}_{i,j,k} = 0, \qquad \text{if} \qquad i+j+k \leq 2 \qquad \text{and} \qquad (i,j,k) \neq (1,1,0), \\ I^{13}_{i,j,k} = 0, \qquad \text{if} \qquad i+j+k \leq 2 \qquad \text{and} \qquad (i,j,k) \neq (1,0,1), \end{gathered} \tag{5.7.26}$$

$$
\begin{aligned}
I^{23}_{i,j,k} &= 0, \qquad \text{if} \qquad i+j+k \le 2 \qquad \text{and} \qquad (i,j,k) \ne (0,1,1),\\
I^{12}_{0,i,0} &= I^{12}_{i-j,0,j} = 0, \qquad \text{if} \qquad i \ge 3, \qquad \text{and} \qquad 0 \le j \le i,\\
I^{13}_{i,0,0} &= I^{13}_{0,i-j,j} = 0, \qquad \text{if} \qquad i \ge 3, \qquad \text{and} \qquad 0 \le j \le i,\\
I^{23}_{0,0,i} &= I^{23}_{i-j,j,0} = 0, \qquad \text{if} \qquad i \ge 3, \qquad \text{and} \qquad 0 \le j \le i.
\end{aligned}
$$

By a similar argument to the proof of Lemma 5.15 we conclude that

$$
\begin{gathered}
I^{12}_{1,1,0},\ I^{13}_{1,0,1},\ I^{23}_{0,1,1},\ I^{12}_{2,1,0},\ I^{12}_{1,2,0},\ I^{12}_{0,2,1},\ I^{12}_{0,1,2},\ I^{12}_{1,1,1},\ I^{13}_{2,1,0},\ I^{13}_{1,2,0},\\
I^{13}_{2,0,1},\ I^{13}_{1,1,1},\ I^{13}_{1,0,2},\ I^{23}_{2,0,1}, I^{23}_{1,0,2},\ I^{23}_{1,1,1},\ I^{23}_{0,2,1},\ I^{23}_{0,1,2},
\end{gathered}
\tag{5.7.27}
$$

is a generating set for the algebra of the differential invariants. From the recurrence relations

$$
\begin{aligned}
\mathcal{D}_1 I^{12}_{1,1,0}\varpi^1 + \mathcal{D}_2 I^{12}_{1,1,0}\varpi^2 + \mathcal{D}_3 I^{12}_{1,1,0}\varpi^3 &= I^{12}_{2,1,0}\varpi^1 + I^{12}_{1,2,0}\varpi^2 + I^{12}_{1,1,1}\varpi^3,\\
\mathcal{D}_1 I^{13}_{1,0,1}\varpi^1 + \mathcal{D}_2 I^{13}_{1,0,1}\varpi^2 + \mathcal{D}_3 I^{13}_{1,0,1}\varpi^3 &= I^{13}_{2,0,1}\varpi^1 + I^{13}_{1,1,1}\varpi^2 + I^{13}_{1,0,2}\varpi^3,\\
\mathcal{D}_1 I^{23}_{0,1,1}\varpi^1 + \mathcal{D}_2 I^{23}_{0,1,1}\varpi^2 + \mathcal{D}_3 I^{23}_{0,1,1}\varpi^3 &= I^{23}_{1,1,1}\varpi^1 + I^{23}_{0,2,1}\varpi^2 + I^{23}_{0,1,2}\varpi^3,
\end{aligned}
\tag{5.7.28}
$$

we reduce the generating set (5.7.27) to

$$
I^{12}_{1,1,0},\ I^{13}_{1,0,1},\ I^{23}_{0,1,1},\ I^{12}_{0,2,1},\ I^{12}_{0,1,2},\ I^{13}_{2,1,0},\ I^{13}_{1,2,0},\ I^{23}_{2,0,1},\ I^{23}_{1,0,2}.
$$

The commutation relations for the invariant differential operators are

$$
\begin{aligned}
[\mathcal{D}_1,\mathcal{D}_2] = &-\frac{I^{13}_{1,1,1}+I^{23}_{2,0,1}}{2(I^{13}_{1,0,1}-I^{23}_{0,1,1})}\mathcal{D}_1 + \frac{I^{12}_{0,1,2}-I^{23}_{1,1,1}}{2(I^{13}_{1,0,1}-I^{23}_{0,1,1})}\mathcal{D}_2\\
&+\left[\frac{I^{13}_{2,1,0}}{2(I^{12}_{1,1,0}-I^{13}_{1,0,0})}-\frac{I^{12}_{0,2,1}}{2(I^{12}_{1,0,1}-I^{23}_{0,1,1})}\right]\mathcal{D}_3,
\end{aligned}
\tag{5.7.29a}
$$

$$
\begin{aligned}
[\mathcal{D}_2,\mathcal{D}_3] = &\left[\frac{I^{23}_{1,0,2}}{2(I^{13}_{1,0,1}-I^{23}_{0,1,1})}-\frac{I^{12}_{0,2,1}}{2(I^{12}_{1,1,0}-I^{23}_{0,1,1})}\right]\mathcal{D}_1\\
&-\frac{I^{12}_{1,1,1}+I^{13}_{1,2,0}}{2(I^{12}_{1,1,0}-I^{13}_{1,0,1})}\mathcal{D}_2 + \frac{I^{23}_{2,0,1}-I^{13}_{1,1,1}}{2(I^{12}_{1,1,0}-I^{13}_{1,0,1})}\mathcal{D}_3,
\end{aligned}
\tag{5.7.29b}
$$

$$[\mathcal{D}_1, \mathcal{D}_3] = \frac{I^{13}_{1,2,0} - I^{12}_{1,1,1}}{2(I^{12}_{1,1,0} - I^{23}_{0,1,1})}\mathcal{D}_1 - \left[\frac{I^{23}_{1,0,2}}{2(I^{13}_{1,0,1} - I^{23}_{0,1,1})} + \frac{I^{13}_{2,1,0}}{2(I^{12}_{1,1,0} - I^{13}_{1,0,1})}\right]\mathcal{D}_2 - \frac{I^{23}_{1,1,1} + I^{12}_{0,1,2}}{2(I^{12}_{1,1,0} - I^{23}_{0,1,1})}\mathcal{D}_3. \tag{5.7.29c}$$

They are well defined provided

$$I^{12}_{1,1,0} - I^{13}_{1,0,1} \neq 0, \qquad I^{12}_{1,1,0} - I^{23}_{0,1,1} \neq 0, \qquad I^{13}_{1,0,1} - I^{23}_{0,1,1} \neq 0.$$

Applying (5.7.29a) to $I^{12}_{1,1,0}$ and $I^{13}_{1,0,1}$, (5.7.29b) to $I^{13}_{1,0,1}$ and $I^{23}_{0,1,1}$ and (5.7.29c) to $I^{12}_{1,1,0}$ and $I^{23}_{0,1,1}$, we can solve for $I^{12}_{0,1,2}$, $I^{12}_{0,2,1}$, $I^{13}_{1,2,0}$, $I^{13}_{2,1,0}$, $I^{23}_{1,0,2}$ and $I^{23}_{2,0,1}$ under the assumption that

$$\mathcal{D}_i I^{12}_{1,1,0} \neq 0, \qquad \mathcal{D}_i I^{13}_{1,0,1} \neq 0, \qquad \mathcal{D}_i I^{23}_{0,1,1} \neq 0, \qquad i = 1, 2, 3.$$

Proposition 5.17. The differential invariants

$$I^{12}_{1,1,0}, \qquad I^{13}_{1,0,1}, \qquad I^{23}_{0,1,1}, \tag{5.7.30}$$

form a generating set for the differential invariant algebra.

Coordinate Expressions of $I^{12}_{1,1,0}$, $I^{13}_{1,0,1}$, and $I^{23}_{0,1,1}$

As for the two-dimensional case, we use the replacement principle to obtain the coordinate expressions of $I^{12}_{1,1,0}$, $I^{13}_{1,0,1}$, and $I^{23}_{0,1,1}$. The invariantization of the sectional curvatures gives

$$\begin{aligned}\kappa_{ij} = \iota(R_{ijij}) = \sum_{m,s=1}^{3} \iota(g_{js})[\iota(\Gamma^m_{ii})\iota(\Gamma^s_{jm}) - \iota(\Gamma^m_{ji})\iota(\Gamma^s_{im}) + \iota(D_j\Gamma^s_{ii}) - \iota(D_i\Gamma^s_{ji})] \\ = \iota(D_j\Gamma^j_{ii}) - \iota(D_i\Gamma^j_{ji}).\end{aligned} \tag{5.7.31}$$

From (5.7.18) we have

$$D_j\Gamma^j_{ii} = \frac{1}{2}D_j g^{kj}(2D_i g_{ik} - D_k g_{ii}) + \frac{1}{2}g^{kj}(2D_iD_j g_{ik} - D_kD_j g_{ii}),$$

and invariantization of the preceding equality yields

$$\iota(D_j\Gamma^j_{ii}) = \iota(D_iD_j g_{ij}). \tag{5.7.32}$$

On the other hand

$$D_i\Gamma^j_{ji} = \frac{1}{2}D_i g^{kj}(D_i g_{jk} + D_j g_{ki} - D_k g_{ij}) + \frac{1}{2}g^{kj}(D_i^2 g_{jk} + D_jD_i g_{ki} - D_kD_i g_{ij}),$$

hence

$$\iota(\Gamma^j_{ji,i}) = 0. \tag{5.7.33}$$

Combining (5.7.31), (5.7.32) and (5.7.33) we obtain

$$\kappa_{ij} = \iota(D_iD_j g_{ij}),$$

and conclude that the three differential invariants (5.7.30) are equal to the sectional curvatures:

$$\kappa_{12} = I^{12}_{1,1,0}, \qquad \kappa_{13} = I^{13}_{1,0,1}, \qquad \kappa_{23} = I^{23}_{0,1,1}.$$

Coordinate Expressions of the Invariant Differential Operators

In a similar fashion to the two-dimensional case we have the three invariant differential operators

$$\begin{gathered}\text{grad } \kappa_{12} = g^{-1}\nabla^{\mathbb{R}^3}\kappa_{12}, \qquad \text{grad } \kappa_{13} = g^{-1}\nabla^{\mathbb{R}^3}\kappa_{13},\\ \text{grad } \kappa_{23} = g^{-1}\nabla^{\mathbb{R}^3}\kappa_{23},\end{gathered} \tag{5.7.34}$$

where $\nabla^{\mathbb{R}^3}$ is the standard gradient in $\mathbb{R}^3$, and g^{-1} is the inverse matrix of $g = (g_{ij})$. In matrix notation

$$\begin{pmatrix} \text{grad } \kappa_{12} \\ \text{grad } \kappa_{13} \\ \text{grad } \kappa_{23} \end{pmatrix} = g^{-1} \begin{pmatrix} D_1\kappa_{12} & D_2\kappa_{12} & D_3\kappa_{12} \\ D_1\kappa_{13} & D_2\kappa_{13} & D_3\kappa_{13} \\ D_1\kappa_{23} & D_2\kappa_{23} & D_3\kappa_{23} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}.$$

From Theorem 5.14 we have

$$\begin{aligned}
\iota(D_1\kappa_{12}) &= I^{12}_{2,1,0} = \iota(\nabla_1\kappa_{12}) = \nabla_1\kappa_{12}, \\
\iota(D_2\kappa_{12}) &= I^{12}_{1,2,0} = \iota(\nabla_2\kappa_{12}) = \nabla_2\kappa_{12}, \\
\iota(D_3\kappa_{12}) &= I^{12}_{1,1,1} = \iota(\nabla_3\kappa_{12}) = \nabla_3\kappa_{12}, \\
\iota(D_1\kappa_{13}) &= I^{13}_{2,0,1} = \iota(\nabla_1\kappa_{13}) = \nabla_1\kappa_{13}, \\
\iota(D_2\kappa_{13}) &= I^{13}_{1,1,1} = \iota(\nabla_2\kappa_{13}) = \nabla_2\kappa_{13}, \\
\iota(D_3\kappa_{13}) &= I^{13}_{1,0,2} = \iota(\nabla_3\kappa_{13}) = \nabla_3\kappa_{13}, \\
\iota(D_1\kappa_{23}) &= I^{23}_{1,1,1} = \iota(\nabla_1\kappa_{23}) = \nabla_1\kappa_{23}, \\
\iota(D_2\kappa_{23}) &= I^{23}_{0,2,1} = \iota(\nabla_2\kappa_{23}) = \nabla_2\kappa_{23}, \\
\iota(D_3\kappa_{23}) &= I^{23}_{0,1,2} = \iota(\nabla_3\kappa_{23}) = \nabla_3\kappa_{23}.
\end{aligned}$$

By computations identical to the two dimensional case we conclude that

$$\begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_2 \\ \mathcal{D}_3 \end{pmatrix} = \begin{pmatrix} \nabla_1\kappa_{12} & \nabla_2\kappa_{12} & \nabla_3\kappa_{12} \\ \nabla_1\kappa_{13} & \nabla_2\kappa_{13} & \nabla_3\kappa_{13} \\ \nabla_1\kappa_{23} & \nabla_2\kappa_{23} & \nabla_3\kappa_{23} \end{pmatrix}^{-1} \begin{pmatrix} \text{grad } \kappa_{12} \\ \text{grad } \kappa_{13} \\ \text{grad } \kappa_{23} \end{pmatrix},$$

provided the matrix is invertible.

Chapter 6

Conclusion

6.1 Conclusions

Two important formulas in the theory of equivariant moving frames are the Maurer–Cartan structure equations (3.1.14) and the universal recurrence relation (5.4.3). Understanding and applying those two formulas has been the focus of this thesis.

The Maurer–Cartan structure equations (3.1.14) characterize the infinitesimal properties of Lie pseudo-groups. Those equations are dual to the Lie commutator structure equations of the jets of infinitesimal generators in the same sense as the finite-dimensional version (3.2.1), (3.2.3). The duality also holds for the Cartan structure equations (3.3.9) modulo their restriction to the target fibers of the pseudo-group action. This follows from the fact that the Maurer–Cartan structure equations and the Cartan structure equations are isomorphic on target fibers. From the systatic system of the Maurer–Cartan structure equations it is not possible to recover Cartan's definition of essential invariants. To resolve this problem, an alternative definition, invariant under Lie pseudo-group isomorphisms, is proposed in Definition 3.37. Finally, in Section 4.2, the structure theory of Lie pseudo-groups is used to state a symmetry-based linearization theorem for partial differential equations that does not require the integration of the infinitesimal determining equations.

The universal recurrence relation (5.4.3) unveils the structure of the differential invariant algebra of Lie pseudo-groups. Using this equation we characterized the algebra of differential invariants for the Infeld–Roland equation and the Davey–Stewartson equations. As a third application we have shown that for two and three dimensional Riemannian manifolds the differential invariant algebra for the equivalence pseudo-group of Rimannian manifolds is generated by the sectional curvatures. We believe that this last observation should also be true in higher dimension.

6.2 Future Research

The theory of equivariant moving frames for infinite-dimensional Lie pseudo-groups exposed in this thesis is quite new. There are many open problems and applications to consider.

1. In comparison to the structure theory of Lie algebras, the structure theory of infinite-dimensional Lie pseudo-groups is still in its early stage of development. The role of essential invariants and their influence on the structure theory of intransitive Lie pseudo-group is something that needs to be clarified. The classification of infinite-dimensional Lie pseudo-groups is another obvious avenue of research. In this direction we mention Cartan's classification of primitive complex Lie pseudo-groups, [21, 41], and [100] for primitive real Lie pseudo-groups. To those results we can add the classification of infinite-dimensional Kac–Moody Lie algebras, [48]. Also, the extension (if possible) of Lie algebra structural results to infinite-dimensional Lie pseudo-groups is of great interest. For example we believe that Levi's decomposition of Lie algebras into solvable and semisimple subalgebras should extend to infinite-dimensional Lie pseudo-groups, the semisimple component

being replaced by an infinite-dimensional Kac–Moody Lie algebra.

2. Morozov has recently shown that the covering of some nonlinear differential equations can be derived from the Maurer–Cartan forms of their symmetry pseudo-groups, [76, 77]. An interesting problem would be to revisit Morozov papers using Olver and Pohjanpelto structure theory and see if their method sheds additional light on the theory of coverings of differential equations.

3. One advantage of Cartan's structure theory in terms of differential forms over Lie's structure theory formulated in terms of vector fields is that the infinitesimal determining equations do not have to be integrated to obtain the Maurer–Cartan structure equations. Hence the structure theory exposed in Section 3.1 is completely algorithmic and can be implemented in a computer, something that still needs to be done.

4. The key for studying the differential invariant algebra of a Lie pseudo-group is the universal recurrence relation (5.4.3). Since the infinitesimal generator of the pseudo-group action is an important component of the universal recurrence formula, the infinitesimal structure of the pseudo-group should influence the structure of the algebra of differential invariants. For example, the symmetry algebra of the Infeld–Rolands equation is known to have no Kac–Moody structure, [36], while the symmetry algebra of the Davey–Stewartson equations (5.6.1) can be embedded into a Kac–Moody type loop algebra, [24]. At the moment it is not clear how this distinction between the two Lie algebras affects (if it does) the two algebras of differential invariants.

5. In Chapter 5 we determined generating sets for the differential invariant algebra of three pseudo-groups. Unless the generating set contains only one invariant, there is, as of now, no systematic way of determining if a generating set is minimal, i.e., the cardinality of the generating

set cannot be made smaller. An important open problem consists of finding a systematic way of determining the cardinality of the minimal generating set from the structure of the pseudo-group action. Also, as the applications considered in this thesis show, the computation of the normalized differential invariants using the equivariant moving frame method usually requires a lot of calculations. Adapting Kogan's recursive construction of moving frames, [58,59], to Lie pseudo-groups should simplify the computations by splitting complicated pseudo-group actions into simpler sub-pseudo-group actions. Implementing the moving frame method for Lie pseudo-groups into a MATHEMATICA or MAPLE package would also be very helpful.

6. The application of the equivariant moving frame theory to the group foliation method of Vessiot, [73, 109], is another interesting research direction. The group foliation method provides a powerful approach to the construction of explicit non-invariant solutions to partial differential equations. It relies on the classification of the differential invariants and their syzygies. The universal recurrence relation (5.4.3) should help put the group foliation method on solid theoretical grounds. We refer the reader to [94] for a formulation in terms of differential forms.

Appendix: Formal Theory of Differential Equations

In this appendix, we survey some of the basic ideas of the formal theory of differential equations. The most important concept to be defined is that of an involutive system of partial differential equations. Different, but equivalent, definitions of involutivity exist in the literature, [65]. The involutive form of a system of partial differential equations can be defined in the language of exterior differential systems, [15] (or dually in terms of vector fields, [110]), in terms of the Spencer cohomology machinery, [72, 95], or directly in terms of the system of differential equations, [5, 99]. In this section, we use the last point of view to define the notion of involution . Our exposition follows mainly [99]. We refer the reader to this reference for more details and for the proofs.

A.1 Prolongation and Projection

Geometric approaches to differential equations are based on jet bundles, [98]. The independent and dependent variables are modeled by a fibered manifold $\pi : M \to X$. Since our considerations are all local there is no loss of generality in assuming that the fibered manifold is trivial $M = X \times U$. Then X corresponds to the space of independent variables and U to the space of dependent variables. Every local section $s : X \to M$ defines a $p = \dim X$

dimensional regular submanifold of M transversal to the fibers $\pi^{-1}(x)$, $x \in X$. Inversely, locally every p-dimensional regular submanifold transversal to the fibers $\pi^{-1}(x)$, $x \in X$ determines a local section. We define the space of n-th order jets of local sections J^n of a fibered manifold $\pi : M \to X$ to be the submanifold of $J^n(M,p)$ consisting of those equivalence classes of p-dimensional submanifolds transversal to the fibers.

Definition A.1. A *system of differential equations* of order n is a fibered manifold $\mathcal{R}_n \subseteq J^n$. A *solution* is a section $s : X \to M$ such that the image of the prolonged section $j_n s : X \to J^n$ is a subset of $\mathcal{R}_n$.

Locally, $\mathcal{R}_n$ is described by some equations

$$\Delta_k(x, u^{(n)}) = 0, \qquad k = 1, \ldots, \nu, \tag{A.1.1}$$

and a section is a solution if $\Delta_k(j_n s) = 0$, $k = 1, \ldots, \nu$.

There are two natural operations with systems of differential equations, the operations of projection and prolongation. The projection to order $r < n$ of the system of differential equations $\mathcal{R}_n$ is defined as $\pi^n_r(\mathcal{R}_n) \subseteq J^r$, where $\pi^n_r : J^n \to J^r$ is the usual jet bundle projection. In local coordinates, the projection requires the elimination, by purely algebraic operations, of the jet variables of order greater than r in as many local equations $\Delta_k = 0$ as possible. If we cannot construct any equation depending only on derivatives of order $\leq r$, then $\pi^n_r(\mathcal{R}_n) = J^r$. To define the operation of prolongation we recall that J^{n+r} is always strictly contained in $J^r(J^n)$. The jet bundle J^{n+r} is embedded in $J^r(J^n)$ using the r-th prolongation of the n-th order identity jet $j_r\mathbb{1}^{(n)} : J^n \hookrightarrow J^{n+r}$. We denote by $\overline{j_r\mathbb{1}^{(n)}} : \mathcal{R}_n \hookrightarrow J^{n+r}$ the restriction of $j_r\mathbb{1}^{(n)}$ to $\mathcal{R}_n$. Then the r-th prolongation of an n-th order system of differential equations $\mathcal{R}_n$ is defined as

$$\mathcal{R}_{n+r} = D^r\mathcal{R}_n = (j_r\mathbb{1}^{(n)})^{-1}\left(\overline{j_r\mathbb{1}^{(n)}}(J^r\mathcal{R}_n) \cap j_r\mathbb{1}^{(n)}(J^{n+r})\right) \subseteq J^{n+r}.$$

In local coordinates, the prolonged system of differential equations $\mathcal{R}_{n+r}$ is obtained by adding to (A.1.1) all total derivatives of (A.1.1) up to order r:

$$D_x^J \Delta_k(x, u^{(n+r)}) = 0, \qquad 0 \leq \#J \leq r, \qquad k = 1, \ldots, \nu.$$

We introduce the notation $\mathcal{R}_k^{(1)} = \pi_k^{k+1}(\mathcal{R}_{k+1})$ to denote the first prolongation $D\mathcal{R}_k$ of $\mathcal{R}_k$ followed by its projection $\pi_k^{k+1}(D\mathcal{R}_k)$ for any $k \geq 1$. More generally

$$\mathcal{R}_k^{(s)} = \pi_k^{k+s}(\mathcal{R}_{k+s}), \qquad s \geq 0.$$

The operations of prolongation and projection of a system of differential equations do not commute. In general we only have the containment $\mathcal{R}_{n+r}^{(1)} \subseteq \mathcal{R}_{n+r}$, for any $r \geq 0$. If it is a proper submanifold, this implies the existence of an integrability condition.

Example A.2. Consider the linear system of partial differential equations

$$\mathcal{R}_1 : \begin{cases} u_z + y u_x = 0, \\ u_y = 0. \end{cases}$$

The first prolongation of this system is

$$\mathcal{R}_2 : \begin{cases} u_{xz} + y u_{xx} = 0, & u_{yy} = 0, \\ u_{yz} + u_x + y u_{xy} = 0, & u_{yz} = 0, \\ u_{zz} + y u_{xz} = 0, & u_z + y u_x = 0, \\ u_{xy} = 0, & u_y = 0. \end{cases}$$

Substituting the equations $u_{yz} = 0$ and $u_{xy} = 0$ in $u_{yz} + u_x + y y_{xy} = 0$ yields $u_x = 0$. Thus

$$\mathcal{R}_1^{(1)} : \qquad u_x = u_y = u_z = 0$$

is strictly contained in $\mathcal{R}_1$. The new equation $u_x = 0$ is an example of an integrability condition.

A.2 Formal Integrability

The integrability conditions appearing when taking the prolongation of a system of differential equations $\mathcal{R}_n$ followed by its projection can be interpreted as obstructions to the construction of power series solutions. Assume that in a neighborhood of $x_0 \in X$ the system of differential equations is given by (A.1.1). Consider the formal power series solution[1]

$$u^\alpha(x) = \sum_{\#J\geq 0} \frac{a_J^\alpha}{J!}(x-x_0)^J, \qquad \alpha = 1,\ldots,q, \tag{A.2.1}$$

with real coefficients a_J^α. Substituting the formal power series solution into the system of differential equations and evaluating at x_0 yields a system of algebraic equations for the coefficients a_J^α with $\#J \leq n$:

$$\Delta_k(x_0, a^{(n)}) = 0, \qquad k = 1,\ldots,\nu. \tag{A.2.2}$$

The system of equations (A.2.2) is generally non-linear and the solution space can be very complicated. The first prolongation of $\mathcal{R}_n$ is described in a neighborhood of x_0 by the original equations $\Delta_k(x, u^{(n)}) = 0$ plus the equations $D_i\Delta_k(x, u^{(n+1)}) = 0$, $i = 1,\ldots,p$, $k = 1,\ldots,\nu$. Substituting our ansatz into this system and evaluating at x_0 will give the new equations

$$D_i\Delta_k(x_0, a^{(n+1)}) = 0, \qquad i = 1,\ldots,p, \qquad k = 1,\ldots,\nu. \tag{A.2.3}$$

The system of equations (A.2.3) is an inhomogeneous linear system in the coefficients of order $n+1$. This system can be used to solve for as many a_J^α of order $n+1$ as possible. Taking higher and higher prolongations of the differential system yields infinitely many equations which can be solved for some of the coefficients a_J^α. Those coefficients are called *principal* and the remaining ones are called *parametric*.

[1] We are only dealing with formal series, as we do not discuss their convergence.

In concrete computations we can only perform a finite number of prolongations. Say we stop at order $s > n$. The prolonged system will give the correct coefficients a^α_J, with $0 \leq \#J \leq s$, of the truncated power series if and only if there are no integrability conditions of order less than or equal to s hidden in the system $\mathcal{R}_s$, as otherwise there are some relations between the coefficients a^α_J, $0 \leq \#J \leq s$, not taken into account.

Definition A.3. A system of differential equations $\mathcal{R}_n$ is said to be *formally integrable* if the equality

$$\mathcal{R}^{(1)}_{n+r} = \mathcal{R}_{n+r} \tag{A.2.4}$$

holds for all integers $r \geq 0$.

Thus for formally integrable systems, we are certain that we do not overlook any hidden conditions on the lower order coefficients, if we build only a truncated series.

Remark A.4. Integrability conditions do not represent additional restrictions to the solution space. They are simply equations hidden in the structure of the system.

A.3 Involution

The difficulty with the definition of formal integrability of a system of differential equations is that the equality (A.2.4) needs to be verified for infinitely many orders. The fact that there are no integrability condition up to a certain order does not imply that there is no hidden integrability condition at a higher order. This difficulty is taken care of by introducing the stronger notion of involution which can be verified in a finite number of algorithmic operations on the system of differential equations. The following property of the n-th jet bundle of sections J^n is the key for the introduction of algebraic techniques into the geometric theory of differential equations.

Proposition A.5. The n-th order jet bundle J^n is an affine bundle over the $(n-1)$-th jet bundle J^{n-1} modeled on the vector bundle $\odot^n_{\mathbb{R}}(TX) \otimes_{\mathbb{R}} VM$, where $VM = \text{Ker } d\pi$ is the vertical bundle of TM.

Definition A.6. The *geometric symbol* of a system of differential equations $\mathcal{R}_n$ at $\rho \in J^n$ is the vector bundle $\mathcal{S}_n|_\rho = V^{(n)}_\rho \mathcal{R}_n \subset V^{(n)}_\rho J^n$, where $V^{(n)}_\rho J^n = \text{Ker } d\pi^n_{n-1}|_\rho$ is the vertical bundle with respect to the projection $\pi^n_{n-1} : J^n \to J^{n-1}$ at ρ.

Thus the geometric symbol is the vertical part of the tangent space of $\mathcal{R}_n$ with respect to the fibration π^n_{n-1}. Let

$$v = \sum_{\alpha=1}^{q} \sum_{0 \le \#J \le n} v^\alpha_J \, dx^J \otimes \frac{\partial}{\partial u^\alpha} \in \odot^n_{\mathbb{R}} T^*X \otimes_{\mathbb{R}} VM,$$

then the symbol $\mathcal{S}_n|_\rho$ consist of the points v for which

$$\sum_{\alpha=1}^{q} \sum_{\#J=n} \frac{\partial \Delta_k}{\partial u^\alpha_J}(\rho) \cdot v^\alpha_J = 0, \qquad 1 \le k \le \nu. \tag{A.3.1}$$

The rank of the matrix $(\partial\Delta_k/\partial u^\alpha_J)(\rho)$ can vary with ρ. We assume that this is not the case so that $\mathcal{S}_n$ forms a vector bundle over $\mathcal{R}_n$, and we drop the reference to the point ρ.

Let $\{\partial_{x^1}, \dots, \partial_{x^p}\}$ be the standard basis on TX and consider the subspaces

$$\mathcal{S}_{n,k} = \{\sigma \in \mathcal{S}_n : \sigma(\partial_{x^k}, v_1, \dots, v_{n-1}) = 0, \ \forall \ v_1, \dots, v_{n-1} \in TX\},$$

$k = 1, \dots, p$, and set $\mathcal{S}_{n,0} = \mathcal{S}_n$.

Definition A.7. The *Cartan characters* of the symbol $\mathcal{S}_n$ are the integers

$$\alpha^{(k)}_n = \dim \mathcal{S}_{n,k-1} - \dim \mathcal{S}_{n,k}, \qquad 1 \le k \le p.$$

The Cartan characters satisfy the descending sequence $\alpha_n^{(1)} \geq \cdots \geq \alpha_n^{(p)} \geq 0$.

Proposition A.8. The inequality

$$\dim \mathcal{S}_{n+1} \leq \dim \mathcal{S}_{n,0} + \dim \mathcal{S}_{n,1} + \cdots + \dim \mathcal{S}_{n,p-1} = \sum_{k=1}^{p} k\alpha_n^{(k)} \qquad \text{(A.3.2)}$$

always holds.

Definition A.9. The symbol $\mathcal{S}_n$ is *involutive* if there exist local coordinates on the base manifold X such that we have equality in (A.3.2).

A.4 Computational Criterion for Involution

In applications, it is easier to work with the differential equations instead of their solutions, so we now explain how involutivity of the symbol $\mathcal{S}_n$ can be verified in terms of the linear system (A.3.1) describing it locally. The first step is to order the columns of the *symbol matrix* $\mathbf{S}_n = \mathbf{S}(\mathcal{R}_n) = (\partial\Delta_k/\partial u_J^\alpha)$, $\#J = n$, relative to their class.

Definition A.10. The *class* of a multi-index $J = (j^1, \ldots, j^p)$ is

$$\text{cl } J = \min \{k : j^k \neq 0\}.$$

We order the columns of $\mathbf{S}_n$ by requiring that the column corresponding to the unknown v_J^α is always to the left of the column corresponding to v_I^β if cl $J >$ cl I. For two multi-indices with the same class, the order of the columns does not matter. This can be achieved by using the reverse lexicographic order: $v_J^\alpha \prec v_I^\beta$ if either the first non-vanishing entry of $J - I$ is positive or $J = I$ and $\alpha < \beta$. Next we put the matrix in row echelon form, without performing any column permutations. Let $\beta_n^{(k)}$ be the number of pivots that lie in a column corresponding to an unknown v_J^α with class k.

The $\beta_n^{(k)}$ are called the *indices* of $\mathcal{S}_n$. The definition of the $\beta_n^{(k)}$ depends on the chosen coordinate system.

Definition A.11. A coordinate system is said to be *δ-regular* if the sum $\sum_{k=1}^{p} k\beta_n^{(k)}$ is maximal.

Any coordinate system can be transformed into a δ-regular one with a linear transformation defined by a matrix coming from a Zariski open subset of $\mathbb{R}^{p\times p}$, [99].

Proposition A.12. The symbol $\mathcal{S}_n$ is involutive if there exist local coordinates on X such that the matrix $\mathbf{S}_{n+1}$ of the prolonged symbol $\mathcal{S}_{n+1}$ satisfies

$$\text{rank } \mathbf{S}_{n+1} = \sum_{k=1}^{p} k\beta_n^{(k)}.$$

Remark A.13. The rank of the symbol matrix $\mathbf{S}_n$, tells us how many coefficients of order n in the formal power series solution (A.2.1) are determined by the system of algebraic equations (A.2.2). More generally, the rank of $\mathbf{S}_k$ with $k \geq n$, determines the number of principal coefficients of order k in the formal power series solution.

A.5 Involutive Differential Equations

Definition A.14. The system of differential equations $\mathcal{R}_n$ is said to be *involutive* if it is formally integrable and if its symbol $\mathcal{S}_n$ is involutive.

Proposition A.15. Assume the symbol $\mathcal{S}_n$ of the system of differential equations $\mathcal{R}_n$ to be involutive, then $\mathcal{R}_n$ is involutive if and only if $\mathcal{R}_n^{(1)} = \mathcal{R}_n$.

Proposition A.15 implies that for a system of differential equations $\mathcal{R}_n$ with involutive symbol it is no longer necessary to check an infinite number of prolongations for the existence of integrability conditions. If there are no integrability conditions in the next prolongation, none will appear at higher order. By a theorem due to Cartan and Kuranishi, every system of differential equations has an equivalent involutive representation.

Theorem A.16. (Cartan–Kuranishi) For every (sufficiently regular) system of differential equations $\mathcal{R}_n$ there exists two integers $r, s \geq 0$ such that the system of differential equations $\mathcal{R}^{(s)}_{n+r} = \mathcal{R}_{n+r}$ is involutive.

Cartan–Kuranishi's theorem implies that every system of differential equations $\mathcal{R}$ can be completed to involution. This can be achieved by the following completion algorithm:

- **Input** $\mathcal{R}$
- **Repeat**
  (a) **While** $\mathbf{S}(\mathcal{R})$ is not involutive **repeat** $\mathcal{R} := D\mathcal{R}$
  (b) **If** $\mathcal{R} \neq \mathcal{R}^{(1)}$, **then** $\mathcal{R} := \mathcal{R}^{(1)}$ **and go to loop** (a)
- **Until** $\mathbf{S}(\mathcal{R})$ is involutive and $\mathcal{R} = \mathcal{R}^{(1)}$
- **Output** $\mathcal{R}$

Example A.17. Consider the system of second order partial differential equations

$$\mathcal{R}_2 : \qquad u_{xx} = 0, \qquad u_{yy} = 0. \tag{A.5.1}$$

If we order $y \prec x$, the first equation in (A.5.1) is of class 2 and the second equation is of class 1. The symbol matrix for the system (A.5.1) is the two by two identity matrix $\mathbf{S}(\mathcal{R}_2) = I_{2\times 2}$. Thus the symbol matrix is already in row echelon form. The indices of $\mathbf{S}(\mathcal{R}_2)$ are $\beta_2^{(2)} = \beta_2^{(1)} = 1$. The first prolongation of (A.5.1) is

$$\mathcal{R}_3 : \qquad u_{xx} = u_{yy} = u_{xxx} = u_{xxy} = u_{xyy} = u_{yyy} = 0. \tag{A.5.2}$$

Thus $\mathbf{S}(\mathcal{R}_3) = I_{4\times 4}$ and the rank $\mathbf{S}(\mathcal{R}_3) = 4 \neq \beta_2^{(1)} + 2\beta_2^{(2)} = 3$. We conclude that the system of equations (A.5.1) is not involutive. On the other hand, it is verified that (A.5.2) is involutive since rank $\mathbf{S}(\mathcal{R}_4) = 5 = \beta_3^{(1)} + 2\beta_3^{(2)}$.

Note that the original system of equations is formally integrable since $\mathcal{R}_k^{(1)} = \mathcal{R}_k$ for all $k \geq 2$.

Bibliography

[1] Anderson, I. M., *The variational bicomplex*, Utah State Technical Report, http://math.usu.edu/~fg_mp,1989.

[2] Anderson, R. L., Ibragimov, N. H., *Lie–Bäcklund Transformations in Applications*, SIAM Studies in Applied Mathematics, Philadelphia, 1979.

[3] Andreev, V. K., Kaptsov, O. V., Pukhnachov, V. V., Rodionov, A. A., *Applications of Group-Theoretical Methods in Hydrodynamics*, Mathematics and Its Applications, Kluwer Academic Publisher, Boston, 1998.

[4] Bäcklund, A. V., Ueber Flächentransformationen, *Math. Ann.* **9** (1876) 297–320.

[5] Baker, R., Doran, C., Jet bundles and the formal theory of partial differential equations, in: *Applications of Geometric Algebra in Computer Science and Engineering*, Birkhäuser Boston, Boston (2002) 133–143.

[6] Berchenko, I. A., Olver, P. J., Symmetries of polynomials, *J. Symb. Comp.* **29** (2000) 485–514.

[7] Bleecker, D., *Gauge Theory and Variational Principles*, Addison–Wesley Publ. Co., Reading, Mass., 1981.

[8] Blumen, G. W., Anco, S. C., *Symmetry and Integration Methods for Differential Equations*, Applied Mathematical Sciences, vol. 154, Springer, New York, 2002.

[9] Blumen, G. W., Kumei, S., *Symmetries and Differential Equations*, Applied Mathematical Sciences, vol. 81, Springer, New York, 1989.

[10] Blumen, G. W., Kumei, S., Symmetry-based algorithms to relate partial differential equations: I. Local symmetries, *Euro. J. of App. Math.* **1** (1990) 189–216.

[11] Boutin, M., Numerically invariant signature curves, *Int. J. Computer Vision* **40** (2000) 235–248.

[12] Boutin, M., On orbit dimensions under a simultaneous Lie group action on n copies of a manifold, *J. Lie Theory* **12** (2002) 191–203.

[13] Boyko, V., Patera, J., Popovych, R., Computation of invariants of Lie algebras by means of moving frames, *J. Phys. A* **39** (2006) 5749–5762.

[14] Boyko, V., Patera, J., Popovych, R., Invariants of solvable Lie algebras with triangular nilradicals and diagonal nilindependent elements, *Linear Algebra Appl.* **428** (2008) 834–854.

[15] Bryant, R. L., Chern, S. S., Gardner, R. B., Goldschmidt, H. L., Griffiths, P. A., *Exterior Differential Systems*, Springer-Verlag, New York, 1991.

[16] Calabi, E., Olver, P. J., Shakiban, C., Tannenbaum, A., Haker, S., Differential and numerically invariant signature curves applied to object recognition, *Int. J. Computer Vision* **26** (1998) 107–135.

[17] Carmo, Manfredo do, *Riemannian Geometry, Mathematics*, Theory & Applications, Birkhäuser, Boston, 1992.

[18] Cartan, É., Les sous-groupes des groupes continus de transformations, *Annales scientifiques de l' É.N.S.*, 3e série, tome 25 (1908) 57–194.

[19] Cartan, É, *Les Systèmes Différentiels Extérieurs et leurs Applications Géométriques*, Éxposés de Géométrie, Hermann & Cie, Paris, 1945.

[20] Cartan, É., La structure des groupes infinis, in: *Oeuvres Complètes, Part. II*, vol. 2, Gauthier–Villars, Paris (1953) 1335–1384.

[21] Cartan, É., Les groupes de transformations continus, infinis, simples, in: *Oeuvres Complètes, Part. II*, vol. 2, Gauthier–Villars, Paris (1953) 857–925.

[22] Cartan, É., Sur la structure des groupes infinis, in: *Oeuvres Complètes, Part. II*, vol. 2, Gauthier–Villars, Paris (1953) 567–569.

[23] Cartan, É., Sur la structure des groupes infinis de transformations, in: *Oeuvres Complètes, Part. II*, vol. 2, Gauthier–Villars, Paris (1953) 571–714.

[24] Champagne, B., Winternitz, P., On the infinite-dimensional symmetry group of the Davey–Stewartson equations, *J. Math. Phys.* **29** (1988) 1–8.

[25] Cheh, J., *Symmetry Pseudogroups of Differential Equations*, Ph.D. thesis, University of Minnesota, 2005.

[26] Cheh, J., Olver, P. J., Pohjanpelto, J., Maurer–Cartan equations for Lie symmetry pseudo-groups of differential equations, *J. Math. Phys.* **46** (2005) 023504.

[27] Cheh, J., Olver, P. J., Pohjanpelto, J., Algorithms for differential invariants of symmetry groups of differential equations, *Found. Comput. Math.* **8** (2008) 501–532.

[28] Chern S. S., Moser J. K. , Real hypersufaces in complex manifolds, *Acta. Math.* **133** (1974) 219–262.

[29] Christoffel, E. B., Ueber die transformation der homogenen differentialausdrücke zweiten grades, *J. für die Reine und Angew. Math.* **70** (1969) 46–70.

[30] Davey, A., Stewartson, K., On three dimensional packets of surface waves, *Proc. R. Soc.* **338** (1974) 101–110.

[31] David, D., Kamran, N., Levi, D., Winternitz, P., subalgebras of loop algebras and symmetries of the Kadomtsev–Petviashivili equation, *Phys. Rev. Lett.* **55** (1985) 2111–2113.

[32] Di Francesco, P., Mathieu, P., Sénéchal, D., *Conformal Field Theory*, Springer-Verlag, New York, 1997.

[33] Ehresmann, C., Introduction à la théorie des structure infinitésimales et des pseudo-groupes de Lie, in: *Géometrie Différentielle*, Colloq. Inter. du Centre Nat. de la Rech. Sci., Strasbourg (1953) 97–110.

[34] Evans, L. C., *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Rhode Island, 1998.

[35] Evtushik, L. E., Pseudo-groups of transformations of geometrical-differential structures and their invariants, *J. Math. Sci.* **94** (1999) 1643–1684.

[36] Faucher, M., Winternitz, P., Symmetry analysis of the Infeld–Rowlands equation, *Phys. rev. E* **48** 4 (1993) 3066–3071.

[37] Fefferman, C., Graham, C. R., Conformal Invariants, in: *Élie Cartan et les Mathématiques d'aujourd'hui*, Astérisque, hors séries, Soc. Math. France, Paris (1985) 95–116.

[38] Fels, M., Olver, P. J., Moving coframes. I. A practical algorithm, *Acta Appl. Math.* **51** (1998) 161–213.

[39] Fels, M., Olver, P. J., Moving coframes. II. Regularization and theoretical foundations, *Acta Appl. Math.* **55** (1999) 127–208.

[40] Gardner, R. B., *The Method of Equivalence and its Applications*, CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial Mathematics, Philadelphia, 1989.

[41] Guillemin, V., Quillen, D., Sternberg, S., The classification of the complex primitive infinite pseudogroups, *Proceedings of the National Academy of Sciences of the United States of America* **55** 4 (1966) 687–690.

[42] Guillemin, V., Quillen, D., Sternberg, S., The classification of the irreducible complex algebras of infinte type, *J. Anal. Math.* **18** 1 (1967) 107–112.

[43] Hofmann, K. H., Morris, S. A., *The Lie Theory of Connected Pro-Lie Groups. A Structure Theory for Pro-Lie Algebras, Pro-Lie Groups, and Connected Locally Compact Groups*, Tracts in Mathematics 2, European Mathematical Society, 2007.

[44] Hubert, E., Generation properties of Maurer–Cartan invariants, preprint, INRIA, 2008.

[45] Hubert, E., Differential invariants of a Lie group action: Syzygies on a generating set, *J. Symb. Comp.* **44** (2009) 382–416.

[46] Hubert, E., Olver, P. J., Differential invariants of conformal and projective surfaces, *SIGMA* **3** (2007) 097.

[47] Infeld, E., Rowlands, G., Theory of soliton transition from lower to higher dimension, *Phys. Rev. A* **43** 8 (1991) 4537–4539.

[48] Kac, V. G., *Infinite Dimensional Lie Algebras*, Third Edition, Cambridge University Press, Cambridge, 1990.

[49] Kalnins, E. G., Miller, W., Related evolution equations and Lie symmetries, *SIAM J. Math. Anal.* **16** (1985) 221–232.

[50] Kalnins, E. G., Miller, W., Equivalence classes of related evolution equations and Lie symmetries, *J. Phys. A* **20** (1987) 5434–5446.

[51] Kamran, N., *Contributions to the Study of the Equivalence Problem of Élie Cartan and its Applications to Partial and Ordinary Differential Equations*, Acad. Roy. Belg. Cl. Sci. Mém. Collect. 8o (2) 45, no. 7, Palais des Acadmémies, Bruxelles, 1989.

[52] Kamran, N., Robart, T., A manifold structure for analytic isotropy Lie pseudogroups of infinite type, *J. Lie Theory* **11** (2001) 57–80.

[53] Kim, P., *Invariantization of numerical schemes using moving frames*, *BIT* **47** (2007) 525–546.

[54] Kim, P., Invariantization of the Crank–Nicolson method for Burger's equation, *Physica D* **237** (2008) 213–226.

[55] Kim, P., Olver, P. J., Geometric integration via multi-space, *Regular and Chaotic Dynamics* **9** (2004) 213–226.

[56] Kogan, I., Olver, P. J., The invariant variational bicomplex, *Contemp. Math.* **285** (2001) 131–141.

[57] Kogan, I., Olver, P. J., Invariant Euler–Lagrange equations and the invariant variational bicomplex, *Acta Appl. Math.* **76** (2003) 137–193.

[58] Kogan, I. A., *Inductive Approach to Cartan's Moving Frame Method with Applications to Classical Invariant Theory*, Ph.D. thesis, University of Minnesota, 2000.

[59] Kogan, I. A., Inductive construction of moving frames, *Contemp. Math.* **285** (2001) 157–170.

[60] Kruglikov, B., Lychagin, V., Compatibility, multi-brackets and integrability of systems of PDEs, arXiv:math/0610930v2.

[61] Kumpera, A., Invariants différentiels d'un pseudo-groupe de Lie, *J. Diff. Geom.* **10** (1975) 289–416.

[62] Kumpera, A., Spencer, D., *Lie Equations*, Princeton University Press, Princeton, N. J., 1972.

[63] Kuranishi, M., On the local theory of continuous infinite pseudo-groups I, *Nagoya Math. J.* **15** (1959) 225–260.

[64] Kuranishi, M., On the local theory of continuous infinite pseudo-groups II, *Nagoya Math. J.* **19** (1961) 55–91.

[65] Lambe, L. A., Seiler, W. M., Differential equations, Spencer cohomology, and computing resolutions, *Georgian Math. J.* **9** 4 (2002) 723–774.

[66] Lie, S., Scheffers, G., *Vorlesungen über Continuierliche Gruppen mit Geometrishen und Anderen Anwendungen*, B. G. Teubner, Leipzig, 1893.

[67] Lisle, I. G., Reid, G. J., Geometry and structure of Lie pseudogroups from infinitesimal defining systems, *J. Symbolic Computation* **26** (1998) 355–379.

[68] Lisle, I. G., Reid, G. J., Cartan structure of infinite Lie pseudogroups, in: *Geometric Approaches to Differential Equations*, Austral. Math. Soc. Lect. Ser., vol. 15, Cambridge Univ. Press, Cambridge (2000) 116–145.

[69] Lisle, I. G., Reid, G. J., Symmetry classification using non-commutative invariant differential operators, *Found. Comput. Math.* **6** (2006) 353–386.

[70] Lisle, I. G., Reid, G. J., Boulton, A., Algorithmic determination of structure of infinite Lie pseudogroups of symmetries of PDEs, in: *Proceedings of the 1995 International Symposium on Symbolic and Algebraic Computation*, Proc. ISSAC '95, ACM Press, New York (1995) 1–6.

[71] Mackenzie, K., *Lie Groupoids and Lie Algebroids in Differential Geometry*, London Mathematical Society Lecture Note Series, vol. 124, Cambridge University Press, 1987.

[72] Malgrange, B., *Système Différentiels Involutifs*, Panoramas et Synthèses, vol. 19, Société Mathématiques de France, Paris, 2005.

[73] Martina, L., Sheftel, M. B., Winternitz, P., Group foliation and non-invariant solutions of the heavenly equation, *J. Phys. A* **34** (2001) 9243–9263.

[74] McLachlan, R. I., Quispel, G. R. W., What kinds of dynamics are there? Lie pseudogroups, dynamical systems and geometric integration, *Nonlinearity*, **14** (2001) 1689–1705.

[75] Moerdijk, I., Mrcun, J., *Introduction to Foliations and Lie Groupoids*, Graduate studies in advanced mathematics, Cambridge University Press, Cambridge,2003.

[76] Morozov, O. I., Coverings of differential equations and Cartan's structure theory of Lie pseudo-groups, *Acta Appl. Math.* **99** (2007) 309–319.

[77] Morozov, O. I., Cartan's structure theory of symmetry pseudo-groups, covering and multi-valued solutions for the Khokhlov–Zabolotskaya equation, *Acta Appl. Math.* **101** (2008) 231–241.

[78] Nutku, Y., Sheftel, M. B., Differential invariants and group foliation for the complex Monge–Ampère equation, *J. Phys. A* **34** (2001) 137–156.

[79] Olver, P., Joint invariant signatures, *Found. Comput. Math.* **1** (2001) 3–67.

[80] Olver, P. J., Symmetry groups and group invariant solutions of partial differential equations, *J. Diff. Geom.* **14** (1979) 497-542.

[81] Olver, P. J., *Applications of Lie Groups to Differential Equation*, second, Graduate Texts in Mathematics, vol. 107, Springer, New York, 1993.

[82] Olver, P. J., *Equivalence, Invariants, and Symmetry*, Cambridge University Press, Cambridge, 1995.

[83] Olver, P. J., Geometric foundations of numerical algorithms and symmetry, *Appl. Alg. Engin. Comp. Commun.* **11** (2001) 417–436.

[84] Olver, P. J., Moving frames, *J. Symb. Comp.* **36** (2003) 501–512.

[85] Olver, P. J., Generating differential invariants, *J. Math. Anal. Appl.* **333** (2007) 450–471.

[86] Olver, P. J., Lectures on moving frames, preprint, University of Minnesota, 2008.

[87] Olver, P. J., Differential invariants of surfaces, *Diff. Geom. Appl.* **27** (2009) 230–239.

[88] Olver, P. J., Pohjanpelto, J., Maurer–Cartan equations and structure of Lie pseudo-groups, *Selecta Math.* **11** (2005) 99–126.

[89] Olver, P. J., Pohjanpelto, J., Regularity of pseudogroup orbits, in: *Symmetry and Perturbation Theory*, World Scientific (2005) 244–254.

[90] Olver, P. J., Pohjanpelto, J., Differential invariant algebras of Lie pseudo-groups, preprint, University of Minnesota, 2007.

[91] Olver, P. J., Pohjanpelto, J., Moving frames for Lie pseudo-groups, *Canadian J. Math.* **60** (2008) 1336–1386.

[92] Ovsiannikov, L. V., *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[93] Peterson, P., *Riemannian Geometry*, Graduate Texts in Mathematics, vol. 171, Springer, New York, 2006.

[94] Pohjanpelto, J., Reduction of exterior differential systems with infinite dimensional symmetry groups, *BIT* **48** 2 (2008) 337–355.

[95] Pommaret, J.-F., *Systems of Partial Differential Equations and Lie Pseudogroups*, Gordon and Breach, New York, 1978.

[96] Reid, G. J., Algorithms for reducing a system of PDEs to standard form, determining the dimension of its solution space and calculating its Taylor series solution, *Euro. J. Appl. Math.* **2** (1991) 319–340.

[97] Riemann, B., Commentatio mathematica, qua respondere tentatur quaestioni ab Ill$^{\text{ma}}$ Academia Parisiensi propositae, in: *Gesammelte Mathematische Werke und Wissenshaftlicher Nachlass*, B. G. Teubner, Leipzig (1892) 391–423.

[98] Saunders, D. J., *The Geometry of Jet Bundles*, London Mathematical Society Lecture Note Series, vol. 142, Cambridge University Press, Cambridge, 1989.

[99] Seiler, W. M., *Involution – The Formal Theory of Differential Equations and its Applications in Computer Algebra and Numerical Analysis*, Habilitation thesis, Dept. of Mathematics, Universität Mannheim, 2002.

[100] Shnider, S, Classification of the real infinite simple and real infinite primitive Lie algebras, *Proc. Natl Acad. Sci. U.S.A.* **64** 2 (1969) 466–467.

[101] Singer, I., Sternberg, S., The infinite groups of Lie and Cartan I. The transitive groups , *J. d'Analyse Math.* **15** (1965) 1–115.

[102] Sternberg, S., Infinite Lie groups and the formal aspects of dynamical systems, *J. Math. and Mech.* **10** 3 (1961) 451–474.

[103] Stormark, O., *Lie's Structural Approach to PDE Systems*, Encyclopedia of Mathematics and its Applications, vol. 80, Cambridge University Press, Cambridge, 2000.

[104] Sussmann, H. J., Orbits of families of vector fields and integrability of distributions, *Trans. Amer. Math. Soc.* **180** (1973) 171–188.

[105] Tresse, A., Sur les invariants différentiels des groupes continus de transformations, *Acta Math.* **18** (1894) 1–88.

[106] Valiquette, F., Comment on 'Invariants of differential equations defined by vector fields', *J. Phys. A: Math. Theor.* **41** (2008) 478001.

[107] Valiquette, F., Structure equations of Lie pseudo-groups, *J. of Lie Theory* **18** 4 (2008) 869–895.

[108] Veblen, O., *Invariants of Quadratic Differential Forms*, Cambridge Tracts in Mathematics and Mathematical Physics, Cambridge University Press, Cambridge, 1927.

[109] Vessiot, E., Sur l'intégration des systèmes différentiels qui admettent des groupes continus de transformations, *Acta Math.* **28** (1904) 307–349.

[110] Vessiot, E., Sur une théorie nouvelle des problème généraux d'intégration, *Bull. Soc. Math. France* **52** (1924) 336–395.

[111] Warner, F. W., *Foundations of Differentiable Manifolds and Lie Groups*, Graduate Text in Mathematics, vol. 94, Springer, New York, 1983.

[112] Weil, A., *Oeuvres Scientifiques*, vol. 1, Springer-Verlag, New York, 1979.