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Applied Mathematics Letters 21 (2008) 248–253
Applied Mathematics Letters
www.elsevier.com/locate/aml

# Group analysis of nonlinear fin equations

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Received 3 October 2006; received in revised form 21 December 2006; accepted 6 February 2007

**Abstract**

Group classification of a class of nonlinear fin equations is carried out exhaustively. Additional equivalence transformations and conditional equivalence groups are also found. These enable us to simplify the classification results and their further applications. The derived Lie symmetries are used to construct exact solutions of *truly* nonlinear equations for the class under consideration. Nonclassical symmetries of the fin equations are discussed. Adduced results complete and essentially generalize recent works on the subject [M. Pakdemirli and A.Z. Sahin, Similarity analysis of a nonlinear fin equation, Appl. Math. Lett. 19 (2006) 378–384; A.H. Bokhari, A.H. Kara and F.D. Zaman, A note on a symmetry analysis and exact solutions of a nonlinear fin equation, Appl. Math. Lett. 19 (2006) 1356–1360].


*Keywords:* Fin equations; Equivalence transformations; Lie symmetries; Nonclassical symmetries

## 1. Introduction

Investigation of heat conductivity and diffusion processes leads to interesting mathematical models which can be often formulated in terms of partial differential equations. In the general case the coefficients of model equations include explicitly both dependent and independent model variables and this makes difficulties in studying such models.

In this letter the class of nonlinear fin equations of the general form

$$u_t = (D(u)u_x)_x + h(x)u, \tag{1}$$

where $D_u \neq 0$, is investigated within the symmetry framework. Here $u$ is treated as the dimensionless temperature, $t$ and $x$ the dimensionless time and space variables, $D$ the thermal conductivity, $h = -N^2 f(x)$, $N$ the fin parameter and $f$ the heat transfer coefficient.

The condition $D_u = 0$ corresponds to the linear case of (1) which was completely investigated from the Lie symmetry point of view a long time ago [6,12]. Moreover, the sets of linear and nonlinear equations of form (1)

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0893-9659/$ - see front matter 
doi:10.1016/j.aml.2007.02.023

Table 1
Results of group classification

| $N$ | $D(u)$ | $h(x)$ | Basis of $A^{\max}$ |
|---|---|---|---|
| 1 | $\forall$ | $\forall$ | $\partial_t$ |
| 2 | $\forall$ | 1 | $\partial_t, \partial_x$ |
| 3 | $\forall$ | $x^{-2}$ | $\partial_t, 2t\partial_t + x\partial_x$ |
| 4 | $u^n$ | $\varepsilon x^q$ | $\partial_t, -qnt\partial_t + nx\partial_x + (q+2)u\partial_u$ |
| 5 | $u^n$ | $\varepsilon e^x$ | $\partial_t, -nt\partial_t + n\partial_x + u\partial_u$ |
| 6 | $u^{-4/3}$ | $h^1(x)$ | $\partial_t, -4qt\partial_t + 4(x^2+p)\partial_x - 3(4x+q)u\partial_u$ |
| 7 | $\forall$ | 0 | $\partial_t, \partial_x, 2t\partial_t + x\partial_x$ |
| 8 | $(u+1)^{-1}$ | $\varepsilon$ | $\partial_t, \partial_x, e^{\varepsilon t}\partial_t + \varepsilon e^{\varepsilon t}(u+1)\partial_u$ |
| 9 | $e^u$ | 0 | $\partial_t, \partial_x, 2t\partial_t + x\partial_x, x\partial_x + 2\partial_u$ |
| 10 | $u^n, n \neq -\frac{4}{3}$ | $\varepsilon$ | $\partial_t, \partial_x, e^{-\varepsilon nt}(\partial_t + \varepsilon u\partial_u), nx\partial x + 2u\partial_u$ |
| 11 | $(u+\alpha)^n, n \neq -\frac{4}{3}$ | 0 | $\partial_t, \partial_x, 2t\partial_t + x\partial_x, nx\partial_x + 2(u+\alpha)\partial_u$ |
| 12 | $u^{-4/3}$ | $\varepsilon$ | $\partial_t, \partial_x, e^{\frac{4}{3}\varepsilon t}(\partial_t + \varepsilon u\partial_u), 2x\partial_x - 3u\partial_u, x^2\partial_x - 3xu\partial_u$ |
| 13 | $(u+\alpha)^{-4/3}$ | 0 | $\partial_t, \partial_x, 2t\partial_t + x\partial_x, 2x\partial_x - 3(u+\alpha)\partial_u, x^2\partial_x - 3x(u+\alpha)\partial_u$ |

can be separately investigated under restriction with point symmetries. That is why the linear case is excluded from consideration in the present letter.

The problem of group classification for the degenerate case $h = 0$ (i.e. the class of nonlinear one-dimensional diffusion equations) was first solved by Ovsiannikov [11,12]. Equations of the form (1) with $h$ being a constant are in the class of diffusion–reaction equations classified by Dorodnitsyn [3,5]. Group classification of the subclass where the thermal conductivity is a power function of the temperature was carried out in [17]. We keep the above cases in the presentation of results for reasons of classification usability. Note also that Lie symmetries of the class of quasi-linear parabolic equations in two independent variables, which has a wide equivalence group and covers all the aforementioned classes, were classified in [1,7].

Recently Lie symmetries and reductions of equations from class (1) were considered in a number of papers [2,13,14]. (See these works for references on the physical meaning and applications of equations from class (1).) In contrast to these works, the study in our letter is concentrated on rigorous and exhaustive group classification of the whole class (1) and construction of exact solutions for *truly* nonlinear 'variable-coefficient' equations from this class. Additional equivalence transformations and conditional equivalence groups are also found. These enable us to simplify the classification results and their further applications. To find exact solutions, we apply both classical Lie reduction and nonclassical symmetry approaches.

## 2. Group classification and related problems

Group classification of class (1) is performed in the framework of the classical approach [12]. All necessary objects (the equivalence group, the kernel and all inequivalent extensions of maximal Lie invariance algebras) are found. Moreover, we extend the classical approach with additional equivalence transformations and a conditional equivalence group for simplification of the classification results.

The *equivalence group* $G^{\sim}$ of class (1) is formed by the transformations

$$\tilde{t} = \delta_1 t + \delta_2, \qquad \tilde{x} = \delta_3 x + \delta_4, \qquad \tilde{u} = \delta_5 u, \qquad \tilde{D} = \delta_1^{-1}\delta_3^2 D, \qquad \tilde{h} = \delta_1^{-1} h,$$

where $\delta_i$, $i = 1, \ldots, 5$, are arbitrary constants, $\delta_1\delta_3\delta_5 \neq 0$. The connected component of the unity in $G^{\sim}$ is formed by continuous transformations having $\delta_1 > 0$, $\delta_3 > 0$ and $\delta_5 > 0$. The complement discrete component of $G^{\sim}$ is generated via three involutive transformations of alternating sign in the sets $\{t, D, h\}$, $\{x\}$ and $\{u\}$.

The *kernel* of the maximal Lie invariance algebras of equations from class (1) coincides with the one-dimensional algebra $\langle\partial_t\rangle$.

All possible $G^{\sim}$-inequivalent *cases of extension* of the maximal Lie invariance algebras are exhausted by ones adduced in Table 1, where

$$h^1(x) = \varepsilon \exp\left[\int \frac{q}{x^2+p}\mathrm{d}x\right]; \quad p \in \{-1, 0, 1\}, \varepsilon = \pm 1, \alpha \in \{0, 1\} \bmod G^{\sim}; n \neq 0, q \neq 0.$$

Case 6 was missed in [13] and in the subsequent papers on the subject. The parameter function $h^1$ equals the following functions, depending on values of $p$:

$$p=-1:\quad h^1=\varepsilon\left|\frac{x-1}{x+1}\right|^{q/2},\qquad p=0:\quad h^1=\varepsilon \mathrm{e}^{-q/x},\qquad p=1:\quad h^1=\varepsilon \mathrm{e}^{q\arctan x}.$$

Additionally we can assume that $q=-1 \bmod G^{\sim}$ if $p=0$.

Some cases from Table 1 are equivalent with respect to point transformations which obviously do not belong to $G^{\sim}$. These transformations are called *additional equivalence transformations* and lead to simplification of further application of the group classification results. The pairs of point-equivalent extension cases and the corresponding additional equivalence transformations are

$$6_{p=0}\to 5_{\tilde n=-4/3}:\quad \tilde t=t,\qquad \tilde x=x^{-1},\qquad \tilde u=x^3u;$$

$$6_{p=-1}\to 4_{\tilde n=-4/3,\tilde q=q/2}:\quad \tilde t=t,\qquad \tilde x=\frac{x-1}{x+1},\qquad \tilde u=2^{-3/2}(x+1)^3u;$$

$$11_{\alpha\neq0}\to 11_{\alpha=0},\ 13_{\alpha\neq0}\to 13_{\alpha=0}:\quad \tilde t=t,\qquad \tilde x=x,\qquad \tilde u=u+\alpha;$$

$$10\to 11_{\alpha=0},\ 12\to 13_{\alpha=0}\left(n=-\frac{4}{3}\right):\quad \tilde t=\frac{1}{\varepsilon n}\mathrm{e}^{\varepsilon nt},\qquad \tilde x=x,\qquad \tilde u=\mathrm{e}^{-\varepsilon t}u.$$

The latter transformation was adduced e.g. in [5]. Case 6 with $p=1$ is reduced to Case 4 only over the complex field. Note also that Case 8 is reduced using the similar transformation

$$\tilde t=-\frac{1}{\varepsilon}\mathrm{e}^{-\varepsilon t},\qquad \tilde x=x,\qquad \tilde u=\mathrm{e}^{-\varepsilon t}(u+1)$$

into equation $\tilde u_{\tilde t}=(\tilde u^{-1}\tilde u_{\tilde x})_{\tilde x}-\varepsilon$ which is not a member of the class under consideration.

All other cases from Table 1 cannot be transformed into each other by point transformations. Therefore, up to point equivalence, possible cases of extension of maximal Lie invariance algebras are exhausted by Cases 1–5, $6_{p=1}$, 7–9, $11_{\alpha=0}$ and $13_{\alpha=0}$.

The singularity of the diffusion coefficient $D=u^{-4/3}$ with a number of different values of $h$ admitting extensions of the Lie invariance algebra can be explained in the framework of *conditional equivalence groups*. The equivalence group is extended under the condition $D=u^{-4/3}$. More precisely, the equivalence group $G_1^{\sim}$ of the subclass of class (1) with $D=u^{-4/3}$ is formed by the transformations

$$\tilde t=\delta_1 t+\delta_2,\qquad \tilde x=\frac{\delta_3x+\delta_4}{\delta_5x+\delta_6},\qquad \tilde u=\pm\delta_1(\delta_5x+\delta_6)^3u,\qquad \tilde h={\delta_1}^{-1}h,$$

where $\delta_i$, $i=1,\dots,6$, are arbitrary constants, $\delta_1>0$ and $\delta_3\delta_6-\delta_4\delta_5=\pm1$. $G_1^{\sim}$ is a nontrivial conditional equivalence group of class (1). We point out that the first two additional equivalence transformations belong to $G_1^{\sim}$.

Another example of a conditional equivalence group of class (1) arises under the condition $h=0$. The equivalence group $G^{\sim}$ of the whole class is then extended with translations with respect to $u$, i.e. the complete equivalence group $G_2^{\sim}$ of nonlinear diffusion equations ($h=0$) is formed by the transformations

$$\tilde t=\delta_1t+\delta_2,\qquad \tilde x=\delta_3x+\delta_4,\qquad \tilde u=\delta_5u+\delta_6,\qquad \tilde D=\delta_1^{-1}\delta_3^2D,$$

where $\delta_i$, $i=1,\dots,6$, are arbitrary constants, $\delta_1\delta_3\delta_5\neq0$. The third additional equivalence transformation belongs to $G_2^{\sim}$.

The subclass of class (1) with $D$ being a power function and $h$ being a constant admits an extension of the generalized equivalence group. The prefix "generalized" means that transformations of the variables $t$, $x$ and $u$ can depend on arbitrary elements [8–10]. The associated generalized equivalence group $G_3^{\sim}$ is generated by transformations from $G^{\sim}$ and the last of the above additional equivalence transformations, where $\varepsilon$ is replaced by $h$.

Knowledge on conditional equivalence groups enables us to describe the set of admissible (form-preserving) transformations in class (1) completely. See e.g. [17] and references therein.

Note also that the subclass of class (1) possessing nontrivial local conservation laws is exhausted by ones with $h$ a constant. Then the corresponding space CL of conservation laws is two dimensional. The conserved vectors and the characteristics of basis elements of CL are

$$\left(x\mathrm{e}^{-ht}u, \mathrm{e}^{-ht}\left(-xDu_x + \int D\right)\right), x\mathrm{e}^{-ht} \quad \text{and} \quad (\mathrm{e}^{-ht}u, -\mathrm{e}^{-ht}Du_x), \mathrm{e}^{-ht}.$$

## 3. Similarity solutions

Cases 7–13 of Table 1 are presented by 'constant coefficient' diffusion–reaction equations. Moreover, all of these cases either are usual nonlinear diffusion equations or can be reduced to them via additional equivalence transformations. Exact solutions of 'constant coefficient' diffusion–reaction equations have been already investigated intensively. See for example, [3–5,16]. That is why we choose Cases 4–6 as representatives among truly nonlinear variable-coefficient fin equations, which are most interesting for Lie reduction.

As shown in the previous section, equation

$$u_t = (u^{-4/3}u_x)_x + h^1(x)u \tag{2}$$

(Case 6 of Table 1) admits the two-dimensional (noncommutative) Lie invariance algebra $\mathfrak{g}$ generated by the operators

$$X_1 = \partial_t, \qquad X_2 = -4qt\partial_t + 4(x^2 + p)\partial_x - 3(4x + q)u\partial_u.$$

A complete list of inequivalent nonzero subalgebras of $\mathfrak{g}$ is exhausted by the algebras $\langle X_1\rangle$, $\langle X_2\rangle$ and $\langle X_1, X_2\rangle$.

Lie reduction of Eq. (2) to an algebraic equation can be made with the two-dimensional subalgebra $\langle X_1, X_2\rangle$ which coincides with the whole algebra $\mathfrak{g}$. The associated ansatz and the reduced algebraic equation have the form

6.0. $\langle X_1, X_2\rangle$: $u = C(x^2 + p)^{-3/2}(h^1(x))^{-3/4}$, $C^{4/3} = \frac{3}{16}(q^2 + 16p)$.

Substituting the solution $C = \pm\frac{3^{3/4}}{8}(q^2 + 16p)^{3/4}$ of the reduced algebraic equation into the ansatz, we construct the exact solution

$$u = \pm\frac{3^{3/4}}{8}(q^2 + 16p)^{3/4}(x^2 + p)^{-3/2}(h^1(x))^{-3/4}$$

of Eq. (2).

The ansätze and reduced equations corresponding to the one-dimensional subalgebras from the optimal system are the following:

6.1. $\langle X_1\rangle$: $u = (\varphi(\omega))^{-3}$, $\omega = x$; $3\varphi_{\omega\omega} = h^1(\omega)\varphi^{-3}$;
6.2. $\langle X_2\rangle$: $u = ((x^2 + p)^{1/2}(h^1(x))^{1/4}\varphi(\omega))^{-3}$, $\omega = th^1(x)$;

$$3q^2\omega^2\varphi_{\omega\omega} + \frac{9}{2}q^2\omega\varphi_\omega - 3\varphi^{-4}\varphi_\omega + \frac{3}{16}(q^2 + 16p)\varphi - \varepsilon\varphi^{-3} = 0.$$

The reduced equations obtained obviously have partial exact solutions which lead to the above exact solution of Eq. (2) and can be constructed via reduction to algebraic equations. The problem is finding some different solutions. We are only able to reduce the order of equation 6.1. Namely, in the variables

$$y = (\omega^2 + p)^{-1/2}(h^1(\omega))^{-1/4}\varphi, \qquad \psi = (\omega^2 + p)^{-1/2}(h^1(\omega))^{-1/4}((\omega^2 + p)\varphi_\omega - \omega\varphi)$$

constructed with the induced symmetry operator $4(\omega^2 + p)\partial_\omega + (4\omega + q)\varphi\partial_\varphi$ equation 6.1 takes the form $(4\psi - qy)\psi_y + q\psi + 4py = \frac{4}{3}\varepsilon y^{-3}$. A better way of constructing exact solutions for the equations of Case 6 with $p \leqslant 0$ is to map them to Cases 4 and 5 with additional equivalence transformations and then study the latter cases.

Let us review results on Lie reduction for Cases 4 and 6. For each of these cases we denote the basis symmetry operators adduced in Table 1 by $X_1$ and $X_2$. The structure and list of inequivalent subalgebras of the Lie invariance algebras are the same as those in Case 6. The associated ansätze and reduced equations have the form ($\varepsilon' = \operatorname{sign} t$):

4.0. $\langle X_1, X_2\rangle$: $u = Cx^{\frac{q+2}{n}}$, $(q + 2)(nq + n + q + 2)C^{n+1} + \varepsilon n^2 C = 0$;

4.1. $\langle X_1\rangle$: $u = (\varphi(\omega))^{\frac{1}{n+1}}$, $\omega = x$, $\varphi_{\omega\omega} + \varepsilon(n+1)\omega^q\varphi^{\frac{1}{n+1}} = 0$ if $n \neq -1$;

$u = \exp(\varphi(\omega))$, $\qquad \omega = x$, $\quad \varphi_{\omega\omega} + \varepsilon\omega^q e^{\varphi} = 0 \quad$ if $n = -1$;

4.2. $\langle X_2\rangle$: $u = |t|^{-\frac{q+2}{nq}}\varphi(\omega)$, $\omega = |t|^{\frac{1}{q}}x$, $(\varphi^n\varphi_\omega)_\omega + \varepsilon\omega^q\varphi + \varepsilon'\frac{q+2}{nq}\varphi - \varepsilon'\frac{1}{q}\omega\varphi_\omega = 0$;

5.0. $\langle X_1, X_2\rangle$: $u = Ce^{\frac{x}{n}}$, $(n+1)C^{n+1} + \varepsilon n^2 C = 0$;

5.1. $\langle X_1\rangle$: $u = (\varphi(\omega))^{\frac{1}{n+1}}$, $\omega = x$, $\varphi_{\omega\omega} + \varepsilon(n+1)e^{\omega}\varphi^{\frac{1}{n+1}} = 0$ if $n \neq -1$;

$u = \exp(\varphi(\omega))$, $\qquad \omega = x$, $\quad \varphi_{\omega\omega} + \varepsilon e^{\varphi+\omega} = 0 \quad$ if $n = -1$;

5.2. $\langle X_2\rangle$: $u = |t|^{-\frac{1}{n}}\varphi(\omega)$, $\omega = x + \ln|t|$, $(\varphi^n\varphi_\omega)_\omega + \varepsilon e^{\omega}\varphi + \varepsilon' n^{-1}\varphi - \varepsilon'\varphi_\omega = 0$.

Reduction to algebraic equations gives the following solutions of the initial equations:

4. $u = (-\frac{q+2}{\varepsilon n^2}(nq + n + q + 2))^{-\frac{1}{n}} x^{\frac{q+2}{n}}$;

5. $u = (-\frac{n+1}{\varepsilon n^2})^{-\frac{1}{n}} e^{\frac{x}{n}}$.

There are Emden–Fowler and Lane–Emden equations and their different modifications among the reduced ordinary differential equations. Solutions of these equations are known for a number of parameter values (see e.g. [15]). As a result, classes of exact solutions can be constructed for fin equations corresponding to Cases 4 and 5 of Table 1 for a wide set of the parameters $n$ and $q$.

## 4. On nonclassical symmetries

We also study conditional (nonclassical) symmetries of equations which are members of the class (1). It is well known that the operators with vanishing coefficient of $\partial_t$ give the so-called 'no-go' case in the study of conditional symmetries of an arbitrary $(1+1)$-dimensional evolution equation since the problem of finding them is reduced to that of solving a single equation which is equivalent to the initial one (see e.g. [18]). Since the determining equation has more independent variables and, therefore, more degrees of freedom, it is more convenient often to guess a simple solution or a simple ansatz for the determining equation, which can give a parametric set of complicated solutions of the initial equation. For example, the fin equation

$$u_t = (u^{-1}u_x)_x + xu \tag{3}$$

is conditionally invariant with respect to the operator $\partial_x + tu\partial_u$. The associated ansatz $u = e^{tx}\varphi(\omega)$, $\omega = t$, reduces Eq. (3) to the equation $\varphi_\omega = 0$, i.e. $u = Ce^{tx}$ is its non-Lie exact solution which can be additionally extended with symmetry transformations.

The known cases of the standard nonlinear diffusion equations (i.e. the equations of form (1) with $h = 0$) possessing nontrivial conditional symmetry operators with nonvanishing coefficients of $\partial_t$ are exhausted by the cases equivalent to $D = e^u$ and $D = u^{-1/2}$. Moreover, for $D = e^u$ solutions associated with such conditional symmetry operators are still Lie invariant. If $h \neq 0$, there exist other values of $D$ such that the equations of form (1) possess conditional symmetry operators which have nonvanishing coefficients of $\partial_t$, are inequivalent to Lie invariance operators and even lead to truly non-Lie exact solutions. Let us consider again Eq. (3). It also admits the conditional symmetry operator $\partial_t + xu\partial_u$. The associated ansatz $u = e^{tx}\varphi(\omega)$, $\omega = x$, reduces Eq. (3) to the equation $(\varphi^{-1}\varphi_\omega)_\omega = 0$. The general solution $\varphi = C_1e^{C_2x}$ of the reduced equation gives a solution of (3), which is simplified to the above constructed one with symmetry transformations.

Exhaustive description of nonclassical symmetry operators of equations from class (1) will be a subject of a forthcoming paper.

## Acknowledgements

The research of R.P. was supported by the Austrian Science Fund (FWF), Lise Meitner project M923-N13. The research of O.V. was partially supported by a grant of the President of Ukraine for young scientists GF/F11/0061. A.G.J. would like to thank Eastern University, Sri Lanka, for its continuing support of his research work. R.P., O.V. and A.G.J. are grateful for the hospitality and financial support of the University of Cyprus. The authors also wish to thank the referees for their constructive suggestions for the improvement of this letter.

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