AN INTRODUCTION TO NONCOMMUTATIVE GEOMETRY

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Introduction

These are lecture notes for a course given at the Summer School on Noncommutative Geometry and Applications, sponsored by the European Mathematical Society, at Monsaraz, Portugal and at Lisboa, from the 1st to the 10th of September, 1997.

Noncommutative geometry, which already occupies an extensive and wide-ranging area of mathematics, has come under increasing scrutiny from physicists interested in what it has to say about fundamental problems of Nature. This course sought to address a mixed audience of students and young researchers, both mathematicians and physicists, and to provide a gateway to some of its more recent developments.

Many approaches can be taken to introducing noncommutative geometry. I decided to focus on the geometry of Riemannian spin manifolds and their noncommutative cousins, which are geometries determined by a suitable generalization of the Dirac operator. These geometries underlie the NCG approach to phenomenological particle models and recent attempts to place gravity and matter fields on the same geometrical footing.

The first two lectures are devoted to commutative geometry; we set up the general framework and then compute a simple example, the two-sphere, in noncommutative terms. The general definition of a geometry is then laid out and exemplified with the noncommutative torus. Enough details are given so that one can see clearly that NCG is just ordinary geometry, extended by discarding the commutativity assumption on the coordinate algebra. Classification up to equivalence is dealt with briefly in lecture 7.

Other lectures explore some of the tools of the trade: the noncommutative integral, the rôle of quantization, and the spectral action functional. Physical models are not treated directly, since these were the subject of other lectures at the Summer School, but most of the mathematical issues needed for their understanding are dealt with here.

I wish to thank several people who contributed in no small way to assembling these lectures. José M. Gracia-Bondía gave decisive help at many points; he and Alejandro Rivero provided constructive criticism throughout. I thank Daniel Kastler, Bruno Iochum, Thomas Schücker and Daniel Testard for the opportunity to visit the Centre de Physique Théorique of the CNRS at Marseille, and the pleasure of learning and practising noncommutative geometry at the source. I am grateful for enlightening discussions with Alain Connes, Robert Coquereaux, Ricardo Estrada, Héctor Figueroa, Thomas Krajewski, Giovanni Landi, Fedele Lizzi, Carmelo Martín, William Ugalde and Mark Villarino. Thanks also to Jesús Clemente, Stephan de Bièvre and Markus Walze who provided indispensable references. Several improvements to the original draft notes were suggested by Eli Hawkins, Thomas Schücker and Georges Skandalis. Last but by no means least, I want to discharge a particular debt of gratitude to Paulo Almeida for his energy and foresight in organizing this Summer School in the right place at the right time.

Contents

1. Commutative Geometry from the Noncommutative Point of View

The traditional arena of geometry and topology is a *set of points* with some particular structure that, for want of a better name, we call a *space*. Thus, for instance, one studies curves and surfaces as subsets of an ambient Euclidean space. It was recognized early on, however, that even such a fundamental geometrical object as an elliptic curve is best studied not as a set of points (a torus) but rather by examining functions on this set, specifically the doubly periodic meromorphic functions. Weierstrass opened up a new approach to geometry by studying directly the collection of complex functions that satisfy an algebraic addition theorem, and derived the point set as a consequence. In probability theory, the set of outcomes of an experiment forms a measure space, and one may regard events as subsets of outcomes; but most of the information is obtained from "random variables", i.e., measurable functions on the space of outcomes.

In noncommutative geometry, under the influence of quantum physics, this general idea of replacing sets of points by classes of functions is taken further. In many cases the set is completely determined by an algebra of functions, so one forgets about the set and obtains all information from the functions alone. Also, in many geometrical situations the associated set is very pathological, and a direct examination yields no useful information. The set of orbits of a group action, such as the rotation of a circle by multiples of an irrational angle, is of this type. In such cases, when we examine the matter from the algebraic point of view, we often obtain a perfectly good operator algebra that holds the information we need; however, this algebra is generally not commutative. Thus, we proceed by first discovering how function algebras determine the structure of point sets, and then learning which relevant properties of function algebras do not depend on commutativity.

In a famous paper [52] that has become a cornerstone of noncommutative geometry, Gelfand and Naĭmark in 1943 characterized the involutive algebras of operators by just dropping commutativity from the most natural axiomatization for the algebra of continuous functions on a locally compact Hausdorff space. The starting point for noncommutative geometry that we shall adopt here is to study ordinary "commutative" spaces via their algebras of functions, omitting wherever possible any reference to the commutativity of these algebras.

The Gelfand–Naĭmark cofunctors

The Gelfand–Naĭmark theorem can be thought of as the construction of two contravariant functors (*cofunctors* for short) from the category of locally compact Hausdorff spaces to the category of C^*-algebras.

The first cofunctor C takes a compact space X to the C^*-algebra $C(X)$ of continuous complex-valued functions on X, and takes a continuous map $f\colon X \to Y$ to its transpose $Cf\colon h \mapsto h \circ f : C(Y) \to C(X)$. If X is only a locally compact space, the corresponding C^*-algebra is $C_0(X)$ whose elements are continuous functions vanishing at infinity, and we require that the continuous maps $f\colon X \to Y$ be proper (the preimage of a compact set is compact) in order that $h \mapsto h \circ f$ take $C_0(Y)$ into $C_0(X)$.

The other cofunctor M goes the other way: it takes a C^*-algebra A onto its space of *characters*, that is, nonzero homomorphisms $\mu\colon A \to \mathbb{C}$. If A is unital, $M(A)$ is closed in the weak* topology of the unit ball of the dual space A^* and hence is compact. If $\phi\colon A \to B$ is a unital $*$-homomorphism, the cofunctor M takes ϕ to its transpose $M\phi\colon \mu \mapsto \mu \circ \phi : M(B) \to M(A)$.

Write $X^+ := X \uplus \{\infty\}$ for the space X with a point at infinity adjoined (whether X is compact or not), and write $A^+ := \mathbb{C} \times A$ for the C^*-algebra A with an identity adjoined via the rule $(\lambda, a)(\mu, b) := (\lambda\mu, \lambda b + \mu a + ab)$, whether A is unital or not; then $C(X^+) \simeq C_0(X)^+$ as unital C^*-algebras. If $\mu_0\colon A^+ \to \mathbb{C} : (\lambda, a) \mapsto \lambda$, then $M(A) = M(A^+) \setminus \{\mu_0\}$ is locally compact when A is nonunital. Notice that $M(A)^+$ and $M(A^+)$ are homeomorphic.

That no information is lost in passing from spaces to C^*-algebras can be seen as follows. If $x \in X$, the *evaluation* $f \mapsto f(x)$ defines a character $\epsilon_x \in M(C(X))$, and the map $\epsilon_X\colon x \mapsto \epsilon_x : X \to M(C(X))$ is a homeomorphism. If $a \in A$, its *Gelfand transform* $\hat{a}\colon \mu \mapsto \mu(a) : M(A) \to \mathbb{C}$ is a continuous function on $M(A)$, and the map $\mathcal{G}\colon a \mapsto \hat{a} : A \to C(M(A))$ is a $*$-isomorphism of C^*-algebras, that preserves identities if A is unital. These maps are *functorial* (or "natural") in the sense that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \epsilon_X}\downarrow & & \downarrow{\scriptstyle \epsilon_Y} \\ M(C(X)) & \xrightarrow{MCf} & M(C(Y)) \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle \mathcal{G}_A}\downarrow & & \downarrow{\scriptstyle \mathcal{G}_B} \\ C(M(A)) & \xrightarrow{CM\phi} & C(M(B)) \end{array}$$

For instance, given a unital $*$-homomorphism $\phi\colon A \to B$, then for any $a \in A$ and $\nu \in M(B)$, we get

$$\begin{aligned} ((CM\phi \circ \mathcal{G}_A)a)\nu &= ((CM\phi)\hat{a})\nu = \hat{a}(M(\phi)\nu) = \hat{a}(\nu \circ \phi) \\ &= \nu(\phi(a)) = \widehat{\phi(a)}(\nu) = ((\mathcal{G}_B \circ \phi)a)\nu, \end{aligned}$$

by unpacking the various transpositions.

This "equivalence of categories" has several consequences. First of all, two commutative C^*-algebras are isomorphic if and only if their character spaces are homeomorphic. (If $\phi\colon A \to B$ and $\psi\colon B \to A$ are inverse $*$-isomorphisms, then $M\phi\colon M(B) \to M(A)$ and $M\psi\colon M(A) \to M(B)$ are inverse continuous proper maps.)

Secondly, the group of automorphisms $\mathrm{Aut}(A)$ of a commutative C^*-algebra A is isomorphic to the group of homeomorphisms of its character space. Note that, since A is commutative, there are no nontrivial inner automorphisms in $\mathrm{Aut}(A)$.

Thirdly, the topology of X may be dissected in terms of algebraic properties of $C_0(X)$. For instance, any *ideal* of $C_0(X)$ is of the form $C_0(U)$ where $U \subseteq X$ is an *open subset* (the closed set $X \setminus U$ being the zero set of this ideal).

If $Y \subseteq X$ is a *closed* subset of a compact space X, with inclusion map $j\colon Y \to X$, then $Cj\colon C(X) \to C(Y)$ is the restriction homomorphism (which is surjective, by Tietze's extension theorem). In general, $f\colon Y \to X$ is injective iff $Cf\colon C(X) \to C(Y)$ is surjective.

We may summarize several properties of the Gelfand–Naĭmark cofunctor with the following dictionary, adapted from [119, p. 24]:

TOPOLOGY	ALGEBRA
locally compact space	C^*-algebra
compact space	unital C^*-algebra
compactification	unitization
continuous proper map	$*$-homomorphism
homeomorphism	automorphism
open subset	ideal
closed subset	quotient algebra
second countable	separable
measure	positive functional

The C^*-algebra viewpoint also allows one to study the topology of non-Hausdorff spaces, such as arise in probing a continuum where points are unresolved: see the book by Landi on noncommutative spaces [79].

A commutative C^*-algebra has an abundant supply of characters, one for each point of the associated space. Looking ahead to noncommutative algebras, we can anticipate that characters will be fairly scarce, and we need not bother to search for points. There is, however, one rôle for points that survives in the noncommutative case: that of zero-dimensional elements of a homological skeleton or cell decomposition of a topological space. For that purpose, characters are not needed; we shall require functionals that are only *traces* on the algebra, but are not necessarily multiplicative.

The Γ functor

Continuous functions determine a space's topology, but to do geometry we need at least a differentiable structure. Thus we shall assume from now on that our "commutative space" is in fact a differential *manifold* M, of dimension n. For convenience, we shall usually assume that M is *compact*, even though this leaves aside important examples such as Minkowski space. (It turns out that noncommutative geometry has been developed so far almost entirely in the Euclidean signature, where compactness can be seen as a simplifying technical assumption. How to adapt the theory to deal with spaces with indefinite metric is very much an open problem at this stage.)

The C^*-algebra $A = C(M)$ of continuous functions must then be replaced by the algebra $\mathcal{A} = C^\infty(M)$ of *smooth* functions on the manifold M. This is not, of course, a C^*-algebra, and although it is a Fréchet algebra in its natural locally convex topology, we never use the theory of locally convex algebras: our tactic is to work with the dense subalgebra $\mathcal{A}$ of A in a purely algebraic fashion. We think of $\mathcal{A}$ as the subspace of "sufficiently regular" elements of A.

A character of $\mathcal{A}$ is a distribution μ on M that is positive, since $\mu(a^*a) = |\mu(a)|^2 \geq 0$, and as such is a measure [53] that extends to a character of $C(M)$; hence $\mathcal{A}$ also determines the point-space M.

To study a given compact manifold M, one uses the category of (complex) *vector bundles* $E \xrightarrow{\pi} M$; its morphisms are bundle maps $\tau: E \to E'$ satisfying $\pi' \circ \tau = \pi$ and so defining fibrewise maps $\tau_x: E_x \to E'_x$ $(x \in M)$ that are required to be linear.

Given any vector bundle $E \longrightarrow M$, write

$$\Gamma(E) := C^\infty(M, E)$$

for the space of *smooth sections* of M. If $\tau: E \to E'$ is a bundle map, the composition $\Gamma\tau: s \mapsto \tau \circ s : \Gamma(E) \to \Gamma(E')$ satisfies, for $a \in \mathcal{A}$, $x \in M$,

$$\Gamma\tau(sa)(x) = \tau_x(s(x)a(x)) = \tau_x(s(x))\, a(x) = (\Gamma\tau(s)a)(x)$$

so $\Gamma\tau(sa) = \Gamma\tau(s)a$; that is, $\Gamma\tau: \Gamma(E) \to \Gamma(E')$ is a morphism of (right) $\mathcal{A}$-modules.

Vector bundles over M admit operations such as duality, direct sum (i.e., Whitney sum) and tensor product; the Γ-functor carries these to analogous operations on $\mathcal{A}$-modules; for instance, if E, E' are vector bundles over M, then

$$\Gamma(E \otimes E') \simeq \Gamma(E) \otimes_{\mathcal{A}} \Gamma(E'),$$

where the right hand side is formed by finite sums $\sum_j s_j \otimes s'_j$ subject to the relations $sa \otimes s' - s \otimes as' = 0$, for $a \in \mathcal{A}$. One can show that any $\mathcal{A}$-linear map from $\Gamma(E)$ to $\Gamma(E')$ is of the form $\Gamma\tau$ for a unique bundle map $\tau: E \to E'$.

It remains to identify what the image of the Γ-functor is. First note that if $E = M \times \mathbb{C}^r$ is a trivial bundle, then $\Gamma(E) = \mathcal{A}^r$ is a *free* $\mathcal{A}$-module. Since M is compact, we can find nonnegative functions $\psi_1, \dots, \psi_q \in \mathcal{A}$ with $\psi_1^2 + \dots + \psi_q^2 = 1$ (a partition of unity) such that E is trivial over the set U_j where $\psi_j > 0$, for each j. If $f_{ij}: U_i \cap U_j \to GL(r, \mathbb{C})$ are the transition functions for E, satisfying $f_{ik} f_{kj} = f_{ij}$ on $U_i \cap U_j \cap U_k$, then the functions $p_{ij} = \psi_i f_{ij} \psi_j$ (defined to be zero outside $U_i \cap U_j$) satisfy $\sum_k p_{ik} p_{kj} = p_{ij}$, and so assemble into a $qr \times qr$ matrix $p \in M_{qr}(\mathcal{A})$ such that $p^2 = p$. A section in $\Gamma(E)$, given locally by smooth functions $s_j: U_j \to \mathbb{C}^r$ such that $s_i = f_{ij} s_j$ on $U_i \cap U_j$, can be regarded as a column vector $s = (\psi_1 s_1, \dots, \psi_q s_q)^t \in C^\infty(M)^{qr}$ satisfying $ps = s$. In this way, one identifies $\Gamma(E)$ with $p\mathcal{A}^{qr}$.

The *Serre–Swan theorem* [111] says that this is a two-way street: any (right) $\mathcal{A}$-module of the form $p\mathcal{A}^m$, for an idempotent $p \in M_m(\mathcal{A})$, is of the form $\Gamma(E) = C^\infty(M(\mathcal{A}), E)$. The fibre at the point $\mu \in M(\mathcal{A})$ is the vector space $p\mathcal{A}^m \otimes_{\mathcal{A}} (\mathcal{A}/\ker\mu)$ whose (finite) dimension is the trace of the matrix $\mu(p) \in M_m(\mathbb{C})$.

In general, a (right) $\mathcal{A}$-module of the form $p\mathcal{A}^m$ is called a **finite projective module** (more correctly, a *finitely generated* projective module). We summarize by saying that Γ is a (covariant) functor from the category of vector bundles over M to the category of finite projective modules over $C^\infty(M)$. The Serre–Swan theorem gives a recipe to construct an inverse functor going the other way, so that these categories are equivalent. (See the discussion by Brodzki [9] for more details in a modern style.)

What, then, is a *noncommutative vector bundle*? It is simply a finite projective right module $\mathcal{E}$ for a (not necessarily commutative) algebra $\mathcal{A}$, which will generally be a dense subalgebra of a C^*-algebra A.

Hermitian metrics and spin^c structures

Any complex vector bundle can be endowed (in many ways) with a *Hermitian metric*. The conventional practice is to define a positive definite sesquilinear form $(\cdot \mid \cdot)_x$ on each fibre E_x of the bundle, which must "vary smoothly with x". The noncommutative point of view is to eliminate x, whereupon what remains is a *pairing* $\mathcal{E} \times \mathcal{E} \to \mathcal{A}$ on a finite projective (right) $\mathcal{A}$-module with values in the algebra $\mathcal{A}$ that is $\mathcal{A}$-linear in the second variable, conjugate-symmetric and positive definite. In symbols:

$$\begin{aligned} (r \mid s+t) &= (r \mid s) + (r \mid t), \\ (r \mid sa) &= (r \mid s)\, a, \\ (r \mid s) &= (s \mid r)^*, \\ (s \mid s) &> 0 \qquad \text{for} \quad s \neq 0, \end{aligned} \tag{1.1}$$

with $r, s, t \in \mathcal{E}$, $a \in \mathcal{A}$. Notice the consequence $(rb \mid s) = b^* \, (r \mid s)$ if $b \in \mathcal{A}$.

With this structure, $\mathcal{E}$ is called a *pre-C^*-module* or "prehilbert module". More precisely, a pre-C^*-module over a dense subalgebra $\mathcal{A}$ of a C^*-algebra A is a right $\mathcal{A}$-module $\mathcal{E}$ (not necessarily finitely-generated or projective) with a sesquilinear pairing $\mathcal{E} \times \mathcal{E} \to \mathcal{A}$ satisfying (1.1). If desired, one can complete it in the norm

$$|||s||| := \sqrt{\|(s \mid s)\|}$$

where $\|\cdot\|$ is the C^*-norm of A; the resulting Banach space is then a *C^*-module*. In the case $\mathcal{E} = C^\infty(M, E)$, the completion is the Banach space of continuous sections $C(M, E)$. Indeed, in general this completion is *not* a Hilbert space. For instance, one can take $\mathcal{E} = \mathcal{A}$ itself, by defining $(a \mid b) := a^* b$; then $|||a|||$ equals the C^*-norm $\|a\|$, so the completion is the C^*-algebra A.

The free $\mathcal{A}$-module $\mathcal{A}^m$ is a pre-C^*-module in the obvious way: $(r \mid s) := \sum_{j=1}^m r_j^* s_j$. This column-vector scalar product also works for $p\mathcal{A}^m$ if $p = p^2 \in M_m(\mathcal{A})$, provided that $p = p^*$ also. If $q = q^2 \in M_m(\mathcal{A})$, one can always find a *projector* $p = p^2 = p^*$ in $M_m(\mathcal{A})$ that is similar and homotopic to q: see, for example, [119, p. 102]. (The choice of p selects a particular Hermitian structure on the right module $q\mathcal{A}^m$.) Thus we shall always assume from now on that the idempotent p is also selfadjoint.

One can similarly study *left* $\mathcal{A}$-modules. In fact, if $\mathcal{E}$ is any right $\mathcal{A}$-module, the *conjugate* space $\overline{\mathcal{E}}$ is a left $\mathcal{A}$-module: by writing $\overline{\mathcal{E}} = \{\, \bar{s} : s \in \mathcal{E} \,\}$, we can define

$$a\, \bar{s} := (sa^*)^-.$$

For $\mathcal{E} = p\mathcal{A}^m$, we get $\overline{\mathcal{E}} = \overline{\mathcal{A}}^m\, p$ where entries of $\overline{\mathcal{A}}^m$ are to be regarded as "row vectors".

Morita equivalence. Finite projective $\mathcal{A}$-modules with $\mathcal{A}$-valued scalar products play a rôle in noncommutative geometry as *mediating structures* that is partially hidden in commutative geometry: they allow the emergence of new algebras related, but not isomorphic, to $\mathcal{A}$. Consider the "ket-bra" operators on $\mathcal{E}$ of the form

$$|r\rangle\langle s| : t \mapsto r\,(s \mid t) : \mathcal{E} \to \mathcal{E},$$

for $r, s \in \mathcal{E}$. Since $r\,(s \mid ta) = r\,(s \mid t)\,a$ for $a \in \mathcal{A}$, these operators act "on the left" on $\mathcal{E}$ and commute with the right action of $\mathcal{A}$. Composing two ket-bras yields a ket-bra:

$$|r\rangle\langle s| \cdot |t\rangle\langle u| = |r(s \mid t)\rangle\langle u| = |r\rangle\langle u\,(t \mid s)|,$$

so all finite sums of ket-bras form an algebra $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$. When $\mathcal{E} = p\mathcal{A}^m$, we have $\mathcal{B} = p\,M_m(\mathcal{A})\,p$. Now $\mathcal{E}$ becomes a left $\mathcal{B}$-module, and we say that $\mathcal{E}$ is a "$\mathcal{B}$-$\mathcal{A}$-*bimodule*".

One can also regard $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ as $\mathcal{E} \otimes_{\mathcal{A}} \overline{\mathcal{E}}$, by $|r\rangle\langle s| \leftrightarrow r \otimes \bar{s}$. On the other hand, we can form $\overline{\mathcal{E}} \otimes_{\mathcal{B}} \mathcal{E}$, which is isomorphic to $\mathcal{A}$ as an $\mathcal{A}$-bimodule via $\bar{r} \otimes s \leftrightarrow (r \mid s)$. This is an instance of *Morita equivalence*. In general, we say that two algebras $\mathcal{A}$, $\mathcal{B}$ are **Morita-equivalent** if there is a $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{E}$ and an $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{F}$ such that

$$\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \simeq \mathcal{B}, \qquad \mathcal{F} \otimes_{\mathcal{B}} \mathcal{E} \simeq \mathcal{A} \tag{1.2}$$

as $\mathcal{B}$- and $\mathcal{A}$-bimodules respectively. With $\mathcal{E} = \mathcal{A}^m$ and $\mathcal{F} \simeq \overline{\mathcal{A}}^m$, we see that any full matrix algebra over $\mathcal{A}$ is Morita-equivalent to $\mathcal{A}$; nontrivial projectors over $\mathcal{A}$ offer a host of more "twisted" examples of algebras that are equivalent to $\mathcal{A}$ in this sense.

The importance of Morita equivalence of two algebras is that their representations match. More precisely, suppose that there is a Morita equivalence of two algebras $\mathcal{A}$ and $\mathcal{B}$, implemented by a pair of bimodules $\mathcal{E}$, $\mathcal{F}$ as in (1.2). Then the functors $\mathcal{H} \mapsto \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ and $\mathcal{H}' \mapsto \mathcal{F} \otimes_{\mathcal{B}} \mathcal{H}'$ implement opposing correspondences between representation spaces of $\mathcal{A}$ and $\mathcal{B}$.

Moral: if we study an algebra $\mathcal{A}$ only through its representations, we must simultaneously study the various algebras Morita-equivalent to $\mathcal{A}$. In particular, we package together the commutative algebra $C^\infty(M)$ and the noncommutative algebra $M_n(C^\infty(M))$ for the purpose of doing geometry.

In the category of C^*-algebras, one replaces finite projective modules by arbitrary C^*-modules and obtains a much richer theory; see, for instance, [78, 100]. The notion analogous to (1.2) is called "strong Morita equivalence". In particular, let us note that two C^*-algebras A and B are strongly Morita equivalent if and only if $A \otimes \mathcal{K} \simeq B \otimes \mathcal{K}$, where $\mathcal{K}$ is the elementary C^*-algebra of compact operators on a separable, infinite-dimensional Hilbert space [11].

Spin^c structures. Returning once more to ordinary manifolds, suppose that M is an n-dimensional orientable Riemannian manifold with a metric g on its tangent bundle TM. We build a Clifford algebra bundle $\mathbb{C}\ell(M) \to M$ whose fibres are full matrix algebras (over $\mathbb{C}$) as follows. If n is even, $n = 2m$, then $\mathbb{C}\ell_x(M) := C\ell(T_xM, g_x) \otimes_{\mathbb{R}} \mathbb{C} \simeq M_{2^m}(\mathbb{C})$ is the complexified Clifford algebra over the tangent space T_xM. If n is odd, $n = 2m+1$, the analogous fibre splits as $M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C})$, so we take only the *even* part of the Clifford algebra: $\mathbb{C}\ell_x(M) := C\ell^{\mathrm{even}}(T_xM) \otimes_{\mathbb{R}} \mathbb{C} \simeq M_{2^m}(\mathbb{C})$. The price we pay for this choice is that we lose the $\mathbb{Z}_2$-grading of the Clifford algebra bundle in the odd-dimensional case.

What we gain is that in all cases, the bundle $\mathbb{C}\ell(M) \to M$ is a locally trivial field of (finite-dimensional) elementary C^*-algebras. Such a field is classified, up to equivalence, by a third-degree Čech cohomology class $\delta(\mathbb{C}\ell(M)) \in H^3(M, \mathbb{Z})$ called the Dixmier–Douady class [38]. Locally, one finds trivial bundles with fibres S_x such that $\mathbb{C}\ell_x(M) \simeq \mathrm{End}(S_x)$; the class $\delta(\mathbb{C}\ell(M))$ is precisely the obstruction to patching them together (there is no obstruction to the existence of the algebra bundle $\mathbb{C}\ell(M)$). It was shown by Plymen [95] that $\delta(\mathbb{C}\ell(M)) = W_3(TM)$, the integral class that is the obstruction to the existence of a *spin^c structure* in the conventional sense of a lifting of the structure group of TM from $SO(n)$ to $\mathrm{Spin}^c(n)$: see [83, Appendix D] for more information on $W_3(TM)$.

Thus M admits spin^c structures if and only if $\delta(\mathbb{C}\ell(M)) = 0$. But in the Dixmier–Douady theory, $\delta(\mathbb{C}\ell(M))$ is the obstruction to constructing (within the C^*-category) a B-A-bimodule $\mathcal{S}$ that implements a Morita equivalence between $A = C_0(M)$ and $B = C_0(M, \mathbb{C}\ell(M))$. Let us paraphrase Plymen's redefinition of a spin^c structure, in the spirit of noncommutative geometry:

Definition. Let M be a Riemannian manifold, $A = C_0(M)$ and $B = C_0(M, \mathbb{C}\ell(M))$. We say that the tangent bundle TM *admits a spin^c structure* if and only if it is orientable and $\delta(\mathbb{C}\ell(M)) = 0$. In that case, a **spin$^{\mathbf{c}}$ structure** on TM is a pair $(\epsilon, \mathcal{S})$ where ϵ is an orientation on TM and $\mathcal{S}$ is a B-A-equivalence bimodule.

Following an earlier terminology introduced by Atiyah, Bott and Shapiro [2] in their seminal paper on Clifford modules, the pair $(\epsilon, \mathcal{S})$ is also called a **K-orientation** on M. Notice that K-orientability demands more than mere orientability in the cohomological sense.

What is this equivalence bimodule $\mathcal{S}$? By the Serre–Swan theorem, it is of the form $\Gamma(S)$ for some complex vector bundle $S \to M$ that also carries an irreducible left action of the Clifford algebra bundle $\mathbb{C}\ell(M)$. This is the *spinor bundle* whose existence displays the spin^c structure in the conventional picture. We call $\Gamma(S) = C^\infty(M, S)$ the *spinor module*; it is an irreducible Clifford module in the terminology of [2], and has rank 2^m over $C^\infty(M)$ if $n = 2m$ or $2m + 1$.

Another matter is how to fit into this picture *spin structures* on M (liftings of the structure group of TM from $SO(n)$ to $\mathrm{Spin}(n)$ rather than $\mathrm{Spin}^c(n)$). These are distinguished by the availability of a *conjugation operator* J on the spinors (which is antilinear); we shall take up this matter later.

To summarize: the language of bimodules and Morita equivalence gives us direct access to noncommutative (or commutative) vector bundles without ever invoking the concept of a "principal bundle". Although several proposals for defining a noncommutative principal bundle are available —see, for instance, [62]— for now we must pass them by.

The Dirac operator and the distance formula

As soon as a spinor module makes its appearance, one can introduce the **Dirac operator**. This is a selfadjoint first-order differential operator $D\!\!\!/$ defined on the space $\mathcal{H} := L^2(M, S)$ of square-integrable spinors, whose domain includes the smooth spinors $\mathcal{S} = C^\infty(M, S)$. If M is even-dimensional, there is a $\mathbb{Z}_2$-grading $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ arising from the grading of the Clifford algebra bundle $\Gamma(\mathbb{C}\ell(M))$, which in turn induces a grading of the Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$; let us call the grading operator χ, so that $\chi^2 = 1$ and $\mathcal{H}^\pm$ are its (± 1)-eigenspaces. The Dirac operator is fabricated by composing the natural *covariant derivative* on the modules $\mathcal{S}^\pm$ (or just on $\mathcal{S}$ in the odd-dimensional case) with the *Clifford multiplication by* 1*-forms* that reverses the grading.

We repeat that in more detail. The Riemannian metric $g = [g_{ij}]$ defines isomorphisms $T_xM \simeq T_x^*M$ and induces a metric $g^{-1} = [g^{ij}]$ on the cotangent bundle T^*M. Via this isomorphism, we can redefine the Clifford algebra as the bundle with fibres $\mathbb{C}\ell_x(M) := \mathrm{C}\ell(T_x^*M, g_x^{-1}) \otimes_{\mathbb{R}} \mathbb{C}$ (replacing $\mathrm{C}\ell$ by $\mathrm{C}\ell^{\mathrm{even}}$ when $\dim M$ is odd). Let $\mathcal{A}^1(M) := \Gamma(T^*M)$ be the $\mathcal{A}$-module of 1-forms on M. The spinor module $\mathcal{S}$ is then a $\mathcal{B}$-$\mathcal{A}$-bimodule on which the algebra $\mathcal{B} = \Gamma(\mathbb{C}\ell(M))$ acts irreducibly and obeys the anticommutation rule

$$\{\gamma(\alpha), \gamma(\beta)\} = -2g^{-1}(\alpha, \beta) = -2g^{ij}\alpha_i\beta_j \qquad \text{for} \quad \alpha, \beta \in \mathcal{A}^1(M).$$

The action γ of $\Gamma(\mathbb{C}\ell(M))$ on the Hilbert-space completion $\mathcal{H}$ of $\mathcal{S}$ is called the *spin representation*.

The metric g^{-1} on T^*M gives rise to a canonical *Levi-Civita connection* $\nabla^g\colon \mathcal{A}^1(M) \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{A}^1(M)$ that, as well as obeying the Leibniz rule

$$\nabla^g(\omega a) = \nabla^g(\omega)\, a + \omega \otimes da,$$

preserves the metric and is torsion-free. The *spin connection* is then a linear operator $\nabla^S\colon \Gamma(S) \to \Gamma(S) \otimes_{\mathcal{A}} \mathcal{A}^1(M)$ satisfying two Leibniz rules, one for the right action of $\mathcal{A}$ and the other, involving the Levi-Civita connection, for the left action of the Clifford algebra:

$$\begin{aligned} \nabla^S(\psi a) &= \nabla^S(\psi)\, a + \psi \otimes da, \\ \nabla^S(\gamma(\omega)\psi) &= \gamma(\nabla^g\omega)\, \psi + \gamma(\omega)\, \nabla^S\psi, \end{aligned} \tag{1.3}$$

for $a \in \mathcal{A}$, $\omega \in \mathcal{A}^1(M)$, $\psi \in \mathcal{S}$.

Once the spin connection is found, we define the Dirac operator as the composition $\gamma \circ \nabla^S$; more precisely, the local expression

$$\not{D} := \gamma(dx^j)\, \nabla^S_{\partial/\partial x^j} \tag{1.4}$$

is independent of the coordinates and defines $\not{D}$ on the domain $\mathcal{S} \subset \mathcal{H}$.

One can check that this operator is symmetric; it extends to an unbounded selfadjoint operator on $\mathcal{H}$, also called $\not{D}$. If M is compact, the latter $\not{D}$ is a Fredholm operator. Since the kernel $\ker \not{D}$ is finite-dimensional, on its orthogonal complement we may define $\not{D}^{-1}$, which is a compact operator.

The distance formula. The Dirac operator may be characterized more simply by its Leibniz rule. Since the algebra $\mathcal{A}$ is represented on the spinor space $\mathcal{H}$ by multiplication operators, we may form $\not{D}(a\psi)$, for $a \in \mathcal{A}$ and $\psi \in \mathcal{H}$. It is an easy consequence of (1.3) and (1.4) that

$$\not{D}(a\psi) = \gamma(da)\psi + a\, \not{D}\psi. \tag{1.5}$$

This is the rule that we need to keep in mind. We can equivalently write it as

$$[\not{D}, a] = \gamma(da).$$

In particular, since a is smooth and M is compact, the operator $\|[\not{D}, a]\|$ is *bounded*, and its norm is simply the sup-norm $\|da\|_\infty$ of the differential da. This also equals the *Lipschitz norm* of a, defined as

$$\|a\|_{\mathrm{Lip}} := \sup_{p \neq q} \frac{|a(p) - a(q)|}{d(p, q)},$$

where $d(p,q)$ is the geodesic distance between the points p and q of the Riemannian manifold M. This might seem to be an unwelcome return to the use of points in geometry; but in fact this simple observation (by Connes) led to one of the great coups of noncommutative geometry [21]. One can simply stand the previous formula on its head:

$$\begin{aligned} d(p,q) &= \sup\{\, |a(p) - a(q)| : a \in \mathcal{A},\ \|a\|_{\mathrm{Lip}} \leq 1 \,\}, \\ &= \sup\{\, |(\hat{p} - \hat{q})(a)| : a \in \mathcal{A},\ \|[D\!\!\!/, a]\| \leq 1 \,\}, \end{aligned} \tag{1.6}$$

and one discovers that *the metric* on the space of characters $M(\mathcal{A})$ *is entirely determined by the Dirac operator*.

This is, of course, just a tautology in commutative geometry; but it opens the way forward, since it shows that what one must carry over to the noncommutative case is precisely this operator, or a suitable analogue. One still must deal with the fact that for noncommutative algebras the characters will be scarce. The lesson that (1.6) teaches [24] is that the *length element* ds is in some sense inversely proportional to $D\!\!\!/$; we shall return to this matter later.

For a general overview of the many ways in which the noncommutative point of view enriches our insight at all levels: measurable, topological, differential and metric, consult the recent review [67].

The ingredients for a reformulation of commutative geometry in algebraic terms are almost in place. We list them briefly: an algebra $\mathcal{A}$; a representation space $\mathcal{H}$ for $\mathcal{A}$; a selfadjoint operator $D\!\!\!/$ on $\mathcal{H}$; a conjugation operator J, still to be discussed; and, in even-dimensional cases, a $\mathbb{Z}_2$-grading operator χ on $\mathcal{H}$. This package of four or five terms is called a *real spectral triple* or a *real K-cycle* or, more simply, a **geometry**. Our task will be to study, to exemplify, and if possible, to parametrize these geometries.

2. Spectral Triples on the Riemann Sphere

We now undertake the construction of some spectral triples $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$ for a very familiar commutative manifold, the Riemann sphere $\mathbb{S}^2$. This is an even-dimensional Riemannian spin manifold, indeed it is the simplest nontrivial representative of that class. Nevertheless, the associated spectral triples are not completely transparent, and their construction is very instructive.

The sphere $\mathbb{S}^2$ can also be regarded as the complex projective line $\mathbb{C}P^1$, or as the compactified plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. As such, it is described by two charts, U_N and U_S, that omit respectively the north and south poles, with the respective local complex coordinates

$$z = e^{i\phi} \cot \tfrac{\theta}{2}, \qquad \zeta = e^{-i\phi} \tan \tfrac{\theta}{2},$$

related by $\zeta = 1/z$ on the overlap $U_N \cap U_S$. We write $q(z) := 1 + z\bar{z}$ for convenience. The Riemannian metric g and the area form Ω are given by

$$g = d\theta^2 + \sin^2\theta \, d\phi^2 = 4q(z)^{-2} \, dz \cdot d\bar{z} = 4q(\zeta)^{-2} \, d\zeta \cdot d\bar{\zeta},$$
$$\Omega = \sin\theta \, d\theta \wedge d\phi = -2i \, q(z)^{-2} \, dz \wedge d\bar{z} = -2i \, q(\zeta)^{-2} \, d\zeta \wedge d\bar{\zeta}.$$

Line bundles and the spinor bundle

Hermitian line bundles over $\mathbb{S}^2$ correspond to finite projective modules over $\mathcal{A} := C^\infty(\mathbb{S}^2)$, of *rank one*; these are of the form $\mathcal{E} = p\mathcal{A}^n$ where $p = p^2 = p^* \in M_n(\mathcal{A})$ is a projector of constant rank 1. (Equivalently, $\mathcal{E}$ is of rank one if $\operatorname{End}_{\mathcal{A}}(\mathcal{E}) \simeq \mathcal{A}$.) It turns out that it is enough to consider the case $p \in M_2(\mathcal{A})$. We follow the treatment of Mignaco *et al* [87].

Using Pauli matrices σ_1, σ_2, σ_3, we may write any projector in $M_2(\mathcal{A})$ as

$$p = \frac{1}{2} \begin{pmatrix} 1 + n_3 & n_1 - in_2 \\ n_1 + in_2 & 1 - n_3 \end{pmatrix} = \tfrac{1}{2}(1 + \vec{n} \cdot \vec{\sigma})$$

where $\vec{n}$ is then a smooth function from $\mathbb{S}^2$ to $\mathbb{S}^2$. Any homotopy between two such functions yields a homotopy between the corresponding projectors p and q; and one can then construct a unitary element $u \in M_4(\mathcal{A})$ such that $u(p \oplus 0)u^{-1} = q \oplus 0$. Thus inequivalent finite projective modules are classified by the homotopy group $\pi_2(\mathbb{S}^2) = \mathbb{Z}$, the corresponding integer m being the *degree* of the map $\vec{n}$. If $f(z) = (n_1 + in_2)/(1 - n_3)$ is the corresponding map on $\mathbb{C}_\infty$ after stereographic projection, then m is also the degree of f. As a representative degree-m map, one could choose $f(z) = z^m$ or $f(z) = 1/\bar{z}^m$.

Let us examine the projector corresponding to $f(z) = z$, of degree 1. We get

$$p_B = \frac{1}{1 + z\bar{z}} \begin{pmatrix} z\bar{z} & \bar{z} \\ z & 1 \end{pmatrix} = \frac{1}{1 + \zeta\bar{\zeta}} \begin{pmatrix} 1 & \zeta \\ \bar{\zeta} & \zeta\bar{\zeta} \end{pmatrix},$$

which is the well-known **Bott projector** that plays a key rôle in K-theory [119]. In general, if $m > 0$, suitable projectors for the modules $\mathcal{E}_{(m)}$, $\mathcal{E}_{(-m)}$ of degrees $\pm m$ are

$$p_m = \frac{1}{1 + (z\bar{z})^m} \begin{pmatrix} (z\bar{z})^m & \bar{z}^m \\ z^m & 1 \end{pmatrix}, \qquad p_{-m} = \frac{1}{1 + (z\bar{z})^m} \begin{pmatrix} (z\bar{z})^m & z^m \\ \bar{z}^m & 1 \end{pmatrix}.$$

One can identify $\mathcal{E}_{(1)}$ with the space of sections of the tautological line bundle $L \to \mathbb{C}P^1$ (the fibre at the point $[v] \in \mathbb{C}P^1$ being the subspace $\mathbb{C}v$ of $\mathbb{C}^2$), and $\mathcal{E}_{(-1)}$ with the space of sections of its dual, the so-called hyperplane bundle $H \to \mathbb{C}P^1$. In general, the integer m is the *Chern class* of the corresponding line bundle [54].

Let us choose basic local sections $\sigma_{mN}(z)$, $\sigma_{mS}(\zeta)$ for the module $\mathcal{E}_{(m)}$. We take, for $m > 0$,

$$\sigma_{mN}(z) := \frac{1}{\sqrt{1+(z\bar{z})^m}} \begin{pmatrix} z^m \\ 1 \end{pmatrix}, \qquad \sigma_{mS}(\zeta) := \frac{1}{\sqrt{1+(\zeta\bar{\zeta})^m}} \begin{pmatrix} 1 \\ \zeta^m \end{pmatrix},$$

normalized so that $(\sigma_{mN} \,|\, \sigma_{mN}) = (\sigma_{mS} \,|\, \sigma_{mS}) = 1$. A global section $\sigma = f_N \sigma_{mN} = f_S \sigma_{mS}$ is thus determined by a pair of functions $f_N(z,\bar{z})$ and $f_S(\zeta,\bar{\zeta})$ that are related on the overlap $U_N \cap U_S$ by the gauge transformation

$$f_N(z,\bar{z}) = (\bar{z}/z)^{m/2} f_S(z^{-1}, \bar{z}^{-1}). \tag{2.1}$$

Definition. The *spinor bundle* $S = S^+ \oplus S^-$ over $\mathbb{S}^2$ has rank two and is $\mathbb{Z}_2$-graded; the **spinor module** $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ over $\mathcal{A}$ is likewise graded by $\mathcal{S}^\pm := \Gamma(S^\pm)$. With the chosen conventions, we have $\mathcal{S}^+ \simeq \mathcal{E}_{(1)}$, $\mathcal{S}^- \simeq \mathcal{E}_{(-1)}$. Thus a *spinor* can be regarded as a pair of functions on each chart, $\psi_N^\pm(z,\bar{z})$ and $\psi_S^\pm(\zeta,\bar{\zeta})$, related by

$$\psi_N^+(z,\bar{z}) = \sqrt{\bar{z}/z}\, \psi_S^+(z^{-1}, \bar{z}^{-1}), \qquad \psi_N^-(z,\bar{z}) = \sqrt{z/\bar{z}}\, \psi_S^-(z^{-1}, \bar{z}^{-1}). \tag{2.2}$$

The spin connection. This is the connection ∇^S on the spinor module $\mathcal{S}$ determined by the Leibniz rule

$$\nabla^S(\gamma(\alpha)\psi) = \gamma(\nabla^g \alpha)\, \psi + \gamma(\alpha)\, \nabla^S \psi,$$

where ∇^g is the *Levi-Civita connection* on the cotangent bundle, determined by

$$\begin{aligned} \nabla^g_{q\partial_z}\left(\frac{dz}{q}\right) &= \bar{z}\,\frac{dz}{q}, & \nabla^g_{q\bar{\partial}_z}\left(\frac{d\bar{z}}{q}\right) &= z\,\frac{d\bar{z}}{q}, \\ \nabla^g_{q\partial_z}\left(\frac{d\bar{z}}{q}\right) &= -\bar{z}\,\frac{d\bar{z}}{q}, & \nabla^g_{q\bar{\partial}_z}\left(\frac{dz}{q}\right) &= -z\,\frac{dz}{q}, \end{aligned} \tag{2.3}$$

and $\gamma(\alpha)\psi$ is the Clifford action of the 1-form α on the spinor ψ, given by the spin representation. Concretely, we may use the gamma-matrices

$$\gamma^1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

It is convenient to introduce the complex combinations $\gamma^\pm := \frac{1}{2}(\gamma^1 \mp i\gamma^2)$. The grading operator for the spinor module $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ is then given by

$$\gamma^3 := i\gamma^1\gamma^2 = [\gamma^+, \gamma^-] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and we note that $\gamma^\pm\gamma^3 = \pm\gamma^\pm$. The Clifford action of 1-forms must satisfy

$$\{\gamma(dz),\gamma(dz)\} = -2g^{-1}(dz,dz) = 0, \qquad \{\gamma(dz),\gamma(d\bar z)\} = -2g^{-1}(dz,d\bar z) = -q(z)^2,$$

so we take simply $\gamma(dz) := q(z)\,\gamma^-$ and $\gamma(d\bar z) := q(z)\,\gamma^+$. (This choice eliminates the natural ambiguity of the matrix square root of $q(z)^2\,1$, and so is a gauge fixing.) Thus

$$\gamma(d\bar z)\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix} = q(z)\begin{pmatrix}0\\ \psi^+\end{pmatrix}, \qquad \gamma(dz)\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix} = -q(z)\begin{pmatrix}\psi^-\\ 0\end{pmatrix}. \tag{2.4}$$

From (2.3) and (2.4) we get

$$\nabla^S_{\partial_z} = \partial_z + \frac{\bar z}{2q}\,\gamma^3, \qquad \nabla^S_{\bar\partial_z} = \bar\partial_z - \frac{z}{2q}\,\gamma^3. \tag{2.5}$$

These operators commute with γ^3, and thus act on the rank-one modules $\mathcal{S}^+$, $\mathcal{S}^-$ by

$$\nabla^\pm_{\partial_z} = \partial_z \pm \frac{\bar z}{2q}, \qquad \nabla^\pm_{\bar\partial_z} = \bar\partial_z \mp \frac{z}{2q}. \tag{2.6}$$

The Dirac operator on the sphere

Definition. The **Dirac operator** $D\!\!\!/ := \gamma(dx^j)\nabla^S_{\partial_j}$ on $\mathbb{S}^2$ may be rewritten in complex coordinates as

$$D\!\!\!/ = \gamma(dz)\,\nabla^S_{\partial_z} + \gamma(d\bar z)\,\nabla^S_{\bar\partial_z} = \gamma(d\zeta)\,\nabla^S_{\partial_\zeta} + \gamma(d\bar\zeta)\,\nabla^S_{\bar\partial_\zeta}.$$

Recalling the form (2.5) of the spin connection, we get

$$\begin{aligned} D\!\!\!/ &= \gamma^-\,\nabla^S_{q\partial_z} + \gamma^+\,\nabla^S_{q\bar\partial_z} = \gamma^-\,(q\,\partial_z + \tfrac12\bar z\,\gamma^3) + \gamma^+\,(q\,\bar\partial_z - \tfrac12 z\,\gamma^3) \\ &= (q\,\partial_z - \tfrac12\bar z)\,\gamma^- + (q\,\bar\partial_z - \tfrac12 z)\,\gamma^+. \end{aligned}$$

The ð operator. At this point, it is handy to employ a first-order differential operator introduced by Newman and Penrose [92]:

$$ð_z := (1+z\bar z)\,\partial_z - \tfrac12\bar z \equiv q\,\partial_z - \tfrac12\bar z = q^{3/2}\cdot\partial_z\cdot q^{-1/2} \tag{2.7}$$

and its complex conjugate $\bar ð_z := q\,\bar\partial_z - \frac12 z$. Then

$$D\!\!\!/ = ð_z\,\gamma^- + \bar ð_z\,\gamma^+ = \begin{pmatrix}0 & -ð_z\\ \bar ð_z & 0\end{pmatrix}. \tag{2.8}$$

This operator is selfadjoint, since $ð_z$ is skewadjoint:

$$\langle \phi^+ \mid ð_z\psi^- \rangle = -\langle \bar ð_z\phi^+ \mid \psi^- \rangle,$$

on the Hilbert space $L^2(\mathbb{C}, -2i\, q^{-2}\, dz \wedge d\bar{z})$, in view of $\eth_z = q^{3/2} \cdot \partial_z \cdot q^{-1/2}$. The scalar product of spinors is then given by

$$\langle \psi_1 \mid \psi_2 \rangle = \langle \psi_1^+ \mid \psi_2^+ \rangle + \langle \psi_1^- \mid \psi_2^- \rangle := \int_{\mathbb{C}} (\overline{\psi_1^+}\psi_2^+ + \overline{\psi_1^-}\psi_2^-)\, \Omega.$$

$D\!\!\!/$ thus extends to a selfadjoint operator on this Hilbert space of spinors, which we call $\mathcal{H} := L^2(\mathbb{S}^2, S)$. Moreover, γ^3 extends to a grading operator (also called γ^3) on $\mathcal{H}$ for which $\mathcal{H}^\pm = L^2(\mathbb{S}^2, S^\pm)$, and it is immediate that $D\!\!\!/\gamma^3 = -\gamma^3 D\!\!\!/$.

Definition. The **conjugation operator** J on the Hilbert space $\mathcal{H}$ of spinors is defined as follows:

$$J\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} := \begin{pmatrix} -\bar{\psi}^- \\ \bar{\psi}^+ \end{pmatrix} \tag{2.9}$$

To see that J is well-defined, it suffices to recall that the gauge transformation rules for upper and lower spinors are conjugate (2.2). Clearly $J^2 = -1$ and J is antilinear, indeed *antiunitary* in the sense that $\langle J\psi_1 \mid J\psi_2 \rangle = \langle \psi_2 \mid \psi_1 \rangle$ for all $\psi_1, \psi_2 \in \mathcal{H}$. Moreover, J *anticommutes* with the grading: $J\gamma^3 = -\gamma^3 J$.

Finally, J *commutes* with the Dirac operator: $JD\!\!\!/ = D\!\!\!/J$. Here it is convenient to introduce the antilinear adjoint operator $J^\dagger$, defined by $\langle \psi_1 \,|\, J^\dagger \psi_2 \rangle := \langle \psi_2 \,|\, J\psi_1 \rangle$; of course, $J^\dagger = J^{-1} = -J$ since J is antiunitary. The desired identity $JD\!\!\!/J^\dagger = D\!\!\!/$ now follows from

$$JD\!\!\!/J^\dagger \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = JD\!\!\!/ \begin{pmatrix} \bar{\psi}^- \\ -\bar{\psi}^+ \end{pmatrix} = J\begin{pmatrix} \eth_z \bar{\psi}^+ \\ \bar{\eth}_z \bar{\psi}^- \end{pmatrix} = \begin{pmatrix} -\eth_z \psi^- \\ \bar{\eth}_z \psi^+ \end{pmatrix} = D\!\!\!/ \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}.$$

In the next chapter we shall see that the three signs that appear in the commutation relations for J, namely $J^2 = -1$, $J\gamma^3 = -\gamma^3 J$ and $JD\!\!\!/ = +D\!\!\!/J$, are *characteristic of dimension two*.

Definition. We call the data set $(C^\infty(\mathbb{S}^2), L^2(\mathbb{S}^2, S), D\!\!\!/, \gamma^3, J)$ the **fundamental spectral triple**, or *fundamental K-cycle*, for the commutative spin manifold $\mathbb{S}^2$.

The Lichnerowicz formula. This formula [7] relates the square of the Dirac operator on the spinor module to the spinor Laplacian; the difference between the two is one quarter of the scalar curvature K of the underlying spin manifold: $D\!\!\!/^2 = \Delta^S + \frac{1}{4}K$. For the sphere $\mathbb{S}^2$ with the metric already chosen, the scalar curvature (or Gaussian curvature) is $K = g^{ij} R^k_{ikj} = 2$, so that the Lichnerowicz formula in this case is just

$$D\!\!\!/^2 = \Delta^S + \tfrac{1}{2}. \tag{2.10}$$

The spinor is the generalized Laplacian [7] on the spinor module:

$$\Delta^S = -g^{ij}(\nabla^S_{\partial_i}\nabla^S_{\partial_j} - \Gamma^k_{ij}\nabla^S_{\partial_k}),$$

which in the isotropic basis $\{\partial_z, \overline{\partial}_z\}$ reduces to

$$\Delta^S = -q^2\, \partial_z \overline{\partial}_z + \tfrac{1}{4} z\bar{z} + \tfrac{1}{2} q(z\,\partial_z - \bar{z}\,\overline{\partial}_z)\,\gamma^3.$$

On the other hand, from (2.7) one gets directly

$$\not{D}^2 = \begin{pmatrix} -\eth_z\overline{\eth}_z & 0 \\ 0 & -\overline{\eth}_z\eth_z \end{pmatrix} = (-q^2\,\partial_z^2\overline{\partial}_z^2 + \tfrac14 z\bar z + \tfrac12) + \tfrac12 q(z\,\partial_z - \bar z\,\overline{\partial}_z)\,\gamma^3$$
$$= \Delta^S + \tfrac12.$$

Spinor harmonics and the spectrum of $\not{D}$

The eigenspinors of $\not{D}$ can now be found by turning up appropriate matrix elements of well-known representations of $SU(2)$; but a more pedestrian approach is quicker. These eigenspinors appear already in Newman and Penrose [92] under the name *spinor harmonics*, and were further studied by Goldberg *et al* [55].

Their construction is based on two simple observations. The first is an elementary calculation with the $\eth$ operator:

$$\begin{aligned}\eth_z\big(q^{-l}z^r(-\bar z)^s\big) &= (l+\tfrac12-r)q^{-l}z^r(-\bar z)^{s+1} + rq^{-l}z^{r-1}(-\bar z)^s,\\ -\overline{\eth}_z\big(q^{-l}z^r(-\bar z)^s\big) &= (l+\tfrac12-s)q^{-l}z^{r+1}(-\bar z)^s + sq^{-l}z^r(-\bar z)^{s-1},\end{aligned} \tag{2.11}$$

where $q = 1 + z\bar z$. The first is easily checked, the second follows by complex conjugation. One sees at once that suitable combinations of the functions $q^{-l}z^r(-\bar z)^s$, with l and $(r-s)$ held fixed, will form eigenvectors for the operator $\not{D}$ on account of its presentation (2.8).

The other matter is that compatibility with gauge transformations of spinors (2.2) imposes restrictions on the exponents l, r, s. Indeed, if $\phi(z,\bar z) := \sum_{r,s\geq 0} a(r,s)q^{-l}z^r(-\bar z)^s$, then

$$(\bar z/z)^{1/2}\phi(z^{-1},\bar z^{-1}) = (-1)^{l+\frac12}\sum_{r,s\geq 0} a(r,s)q^{-l}z^{l-\frac12-r}(-\bar z)^{l+\frac12-s},$$

so that $\phi \in \mathcal{S}^+$ iff $l+\frac12$ is a positive integer, and $a(r,s)\neq 0$ only for $r = 0,1,\dots,l-\frac12$ and $s = 0,1,\dots,l+\frac12$. To have $\phi\in\mathcal{S}^-$, interchange the restrictions on r and s.

If we set $m := r - s \pm \frac12$, the corresponding restriction is $m = -l, -l+1, \dots, l-1, l$, a very familiar sight in the theory of angular momentum; but with the important difference that here l and m are *half-integers but not integers*, so the corresponding matrix elements do not drop to matrix elements of representations of $SO(3)$.

We can now display the spinor harmonics. They form two families, Y^+_{lm} and Y^-_{lm}, corresponding to upper and lower spinor components; they are indexed by

$$l \in \mathbb{N} + \tfrac12 = \{\tfrac12, \tfrac32, \tfrac52, \dots\}, \qquad m \in \{-l, -l+1, \dots, l-1, l\},$$

and the formulae are

$$\begin{aligned} Y^+_{lm}(z,\bar z) &:= C_{lm}q^{-l}\sum_{r-s=m-\frac12}\binom{l-\frac12}{r}\binom{l+\frac12}{s}z^r(-\bar z)^s,\\ Y^-_{lm}(z,\bar z) &:= C_{lm}q^{-l}\sum_{r-s=m+\frac12}\binom{l+\frac12}{r}\binom{l-\frac12}{s}z^r(-\bar z)^s,\end{aligned} \tag{2.12}$$

where the normalization constants C_{lm} are defined as

$$C_{lm} := (-1)^{l-m} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l+m)!\,(l-m)!}{(l+\frac{1}{2})!\,(l-\frac{1}{2})!}}.$$

Eigenspinors. The coefficients in (2.12) are chosen so as to satisfy

$$\eth_z Y^-_{lm} = (l+\tfrac{1}{2}) Y^+_{lm} \qquad \text{and} \qquad -\bar{\eth}_z Y^+_{lm} = (l+\tfrac{1}{2}) Y^-_{lm}.$$

in view of (2.11). If we then form normalized spinors by

$$Y'_{lm} := \frac{1}{\sqrt{2}} \begin{pmatrix} -Y^+_{lm} \\ Y^-_{lm} \end{pmatrix}, \qquad Y''_{lm} := \frac{1}{\sqrt{2}} \begin{pmatrix} Y^+_{lm} \\ Y^-_{lm} \end{pmatrix},$$

we get an orthonormal family of eigenspinors for the Dirac operator:

$$\not{D} Y'_{lm} = (l+\tfrac{1}{2}) Y'_{lm}, \qquad \not{D} Y''_{lm} = -(l+\tfrac{1}{2}) Y''_{lm},$$

where the eigenvalues are nonzero integers and each eigenvalue $\pm(l+\frac{1}{2})$ has multiplicity $(2l+1)$. In fact, these are all the eigenvalues of $\not{D}$; for that, we need the following completeness result, established in [55]:

$$\sum_{l,m} \overline{Y}^{\pm}_{lm}(z,\bar{z})\, Y^{\pm}_{lm}(z',\bar{z}') = \delta(\phi - \phi')\,\delta(\cos\theta - \cos\theta').$$

Consequently, the spinors $\{\, Y'_{lm}, Y''_{lm} : l \in \mathbb{N} + \frac{1}{2}, m \in \{-l,\dots,l\} \,\}$ form an orthonormal basis for the Hilbert space $\mathcal{H} = L^2(\mathbb{S}^2, S)$.

The spectrum. We have thus computed the *spectrum* of the Dirac operator:

$$\operatorname{sp}(\not{D}) = \{\, \pm(l+\tfrac{1}{2}) : l \in \mathbb{N} + \tfrac{1}{2} \,\} = \mathbb{Z} \setminus \{0\},$$

with the aforementioned multiplicities $(2l+1)$. Notice that, since the zero eigenvalue is missing, the Dirac operator $\not{D}$ is invertible and *it has index zero*.

The Lichnerowicz formula (2.10) gives at once the spectrum of the spinor Laplacian:

$$\operatorname{sp}(\Delta^S) = \{\, l^2 + l - \tfrac{1}{4} : l \in \mathbb{N} + \tfrac{1}{2} \,\}.$$

with respective multiplicities $2(2l+1)$.

Twisted spinor modules

To define other spectral triples over $\mathcal{A} = C^\infty(M)$, we may *twist* the spinor module $\mathcal{S}$ by tensoring it with some other finite projective $\mathcal{A}$-module $\mathcal{E}$, the Clifford action on $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}$ being given by

$$\gamma(\alpha)(\psi \otimes \sigma) := (\gamma(\alpha)\psi) \otimes \sigma \qquad \text{for} \quad \psi \in \mathcal{S},\ \sigma \in \mathcal{E}.$$

We call $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}$, with this action of the algebra $\Gamma(\mathbb{C}\ell(M))$, a *twisted spinor module*.

We now show, in the context of our example $\mathcal{A} = C^\infty(\mathbb{S}^2)$, how one can create new K-cycles by twisting the fundamental one. However, these K-cycles will not always respect the "real structure" J, as we shall see.

We examine first the case where $\mathcal{E} = \mathcal{E}_{(m)}$ is a module of sections of a complex line bundle of first Chern class m. Then the twisted spinor module is also $\mathbb{Z}_2$-graded; in fact, $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}_{(m)} \simeq \mathcal{E}_{(m+1)} \oplus \mathcal{E}_{(m-1)}$.

The twisted Dirac operator. The half-spinor modules $\mathcal{S}^\pm = \mathcal{E}_{(\pm 1)}$ have connections $\nabla^\pm$ given by (2.6). Now $\mathcal{E}_{(m)} \simeq \mathcal{E}_{(1)} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{E}_{(1)}$ (m times) if $m > 0$ and $\mathcal{E}_{(m)} \simeq \mathcal{E}_{(-1)} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{E}_{(-1)}$ ($|m|$ times) if $m < 0$, so we can define a connection $\nabla^{(m)}$ on $\mathcal{E}_{(m)}$ by

$$\nabla^{(m)}(s_1 \otimes \cdots \otimes s_{|m|}) := \sum_{j=1}^{|m|} s_1 \otimes \cdots \otimes \nabla^\pm(s_j) \otimes \cdots \otimes s_{|m|},$$

and from (2.6) it follows that

$$\nabla^{(m)}_{\partial_z} = \partial_z + \frac{m\bar{z}}{2q}, \qquad \nabla^{(m)}_{\bar{\partial}_z} = \bar{\partial}_z - \frac{mz}{2q}.$$

On the module $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}_{(m)}$, we define the *twisted spin connection* $\widetilde{\nabla}^S := \nabla^S \otimes 1 + 1 \otimes \nabla^{(m)}$. We obtain

$$\widetilde{\nabla}^S_{\partial_z} = \partial_z + \frac{\bar{z}}{2q}(m + \gamma^3), \qquad \widetilde{\nabla}^S_{\bar{\partial}_z} = \bar{\partial}_z - \frac{z}{2q}(m + \gamma^3).$$

The corresponding Dirac operator is

$$\begin{aligned} \not{D}_m &= \gamma^- \, \widetilde{\nabla}^S_{q\partial_z} + \gamma^+ \, \widetilde{\nabla}^S_{q\bar{\partial}_z} = \big(q(z)\,\partial_z + \tfrac{1}{2}(m-1)\bar{z}\big)\,\gamma^- + \big(q(z)\,\bar{\partial}_z - \tfrac{1}{2}(m+1)z\big)\,\gamma^+ \\ &= (\eth_z + \tfrac{1}{2}m\bar{z})\,\gamma^- + (\bar{\eth}_z - \tfrac{1}{2}mz)\,\gamma^+ \end{aligned}$$

or more pictorially,

$$\not{D}_m = \begin{pmatrix} 0 & \not{D}_m^- \\ \not{D}_m^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\eth_z - \frac{1}{2}m\bar{z} \\ \bar{\eth}_z - \frac{1}{2}mz & 0 \end{pmatrix}.$$

This extends to a selfadjoint operator on the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}_{(m)}$ where $\mathcal{H}^\pm_{(m)} = L^2(\mathbb{S}^2, L^{m\pm 1})$.

Computation of the index. Notice that

$$\not{D}_m^+ = q(z)^{(m+3)/2} \cdot \bar{\partial}_z \cdot q(z)^{-(m+1)/2},$$

so that a half-spinor ψ^+ lies in $\ker \not{D}_m^+$ if and only if $\psi_N^+(z,\bar{z}) = q(z)^{(m+1)/2}\,a(z)$ where a is an entire holomorphic function. The gauge transformation rule (2.1) shows that the function $\psi_S^+(z^{-1},\bar{z}^{-1}) = (z/\bar{z})^{(m+1)/2}\psi_N^+(z,\bar{z})$ is regular at $z = \infty$ only if either $a = 0$ or $m < 0$ and $a(z)$ is a polynomial of degree $< |m|$. Thus $\dim\ker \not{D}_m^+ = |m|$ if $m < 0$ and equals 0 for $m \geq 0$. A similar argument shows that $\dim\ker D\!\!\!/_m^- = m$ if $m > 0$ and is 0 for $m \leq 0$. We conclude that $D\!\!\!/_m$ is a Fredholm operator on $\mathcal{H}_{(m)}$, whose index is

$$\operatorname{ind} D\!\!\!/_m := \dim\ker D\!\!\!/_m^+ - \dim\ker D\!\!\!/_m^- = -m$$

which, up to a sign, is the first Chern class of the twisting bundle.

Incompatibility with the real structure. The twisting by $\mathcal{E}_{(m)}$ loses the property of commutation with the spinor conjugation J (2.9). In fact, it is easy to check that

$$J D\!\!\!/_m J^\dagger = \begin{pmatrix} 0 & -\eth_z + \frac{1}{2}m\bar{z} \\ \bar{\eth}_z + \frac{1}{2}mz & 0 \end{pmatrix} = D\!\!\!/_{-m}.$$

In conventional language, we could say that the twisted spinor bundle $S \otimes L^m$ is associated to a *spin^c structure* on $T\mathbb{S}^2$, and that this is a *spin structure* only if $m = 0$. Conjugation by J exchanges the spin^c structures, fixing only the spin structure; this exemplifies the general fact [25] that commutation (or anticommutation) of $D\!\!\!/$ with J picks out a spin structure when these are available. In view of this circumstance, we shall say that J defines a **real structure** on $(\mathcal{A}, \mathcal{H})$.

In summary, $(C^\infty(\mathbb{S}^2), \mathcal{H}_{(m)}, D\!\!\!/_m, \gamma^3)$ is a (complex) spectral triple, but is not a "real spectral triple" if $m \neq 0$.

A reducible spectral triple

The twisted spinor modules discussed above are irreducible for the action of the Clifford algebra $\mathcal{B} = \Gamma(\mathbb{C}\ell(\mathbb{S}^2))$. On the other hand, $\mathcal{B}$ acts reducibly on the algebra of differential forms $\mathcal{A}^\bullet(\mathbb{S}^2))$ by

$$\gamma(\alpha)\omega := \alpha \wedge \omega - \iota(\alpha^\sharp)\omega \qquad \text{for} \quad \alpha \in \mathcal{A}^1(\mathbb{S}^2),$$

where $\alpha^\sharp$ is the vector field determined by $\alpha^\sharp(f) := g^{-1}(\alpha, df)$, $f \in \mathcal{A}$. On the algebra of forms we can use the *Hodge star operator*, defined as the involutive $\mathcal{A}$-module isomorphism determined by $\star 1 = i\Omega$, $\star d\theta = i \sin\theta\, d\phi$ (the coefficient i is inserted to make $\star\star = 1$); in complex coordinates,

$$\star 1 = -2q^2\, dz \wedge d\bar{z}, \quad \star dz = dz, \quad \star d\bar{z} = -d\bar{z}.$$

The *codifferential* $\delta = -\star d \star$ is the adjoint of the differential d with respect to the scalar product of forms:

$$\langle \alpha \mid \beta \rangle = i(-1)^{k(k-1)/2} \int_{\mathbb{S}^2} \bar{\alpha} \wedge \star\beta \qquad \text{for} \quad \alpha, \beta \in \mathcal{A}^k(\mathbb{S}^2), \tag{2.13}$$

with which $\mathcal{A}^\bullet(\mathbb{S}^2)$ may be completed to a Hilbert space $L^{2,\bullet}(\mathbb{S}^2) := \bigoplus_{k=0}^n L^{2,k}(\mathbb{S}^2)$.

The Hodge–Dirac operator. One can construct a Dirac operator on this Hilbert space by twisting, along the following lines. One can identify $\mathcal{A}^\bullet(\mathbb{S}^2)$ with $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{S}'$ as $\mathcal{B}$-$\mathcal{A}$-bimodules, where $\mathcal{S}'$ denotes the spinor module with the opposite grading: $(\mathcal{S}')^\pm = \mathcal{S}^\mp$.

A detailed comparison of these bimodules and their Dirac operators is given in [114]. The spin connection on $\mathcal{S}'$ is given by (compare (2.5)):

$$\nabla^{S'}_{\partial_z} = \partial_z - \frac{\bar z}{2q}\,\gamma^3, \qquad \nabla^{S'}_{\bar\partial_z} = \bar\partial_z + \frac{z}{2q}\,\gamma^3,$$

and $\widetilde{\nabla} := \nabla^S \otimes 1 + 1 \otimes \nabla^{S'}$ gives the tensor product connection on $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{S}'$. The Dirac operator on this twisted module is then

$$\begin{aligned}\widetilde{D\!\!\!/} &:= \gamma(dz)\,\widetilde{\nabla}_{\partial_z} + \gamma(d\bar z)\,\widetilde{\nabla}_{\bar\partial_z} := (\gamma^- \otimes 1)\,\widetilde{\nabla}_{q\partial_z} + (\gamma^+ \otimes 1)\,\widetilde{\nabla}_{q\bar\partial_z} \\ &= D\!\!\!/ \otimes 1 + \gamma^- \otimes \nabla^{S'}_{q\partial_z} + \gamma^+ \otimes \nabla^{S'}_{q\bar\partial_z}.\end{aligned}$$

The Lichnerowicz formula for this operator is [114]:

$$\widetilde{D\!\!\!/}^2 = \widetilde{\Delta} + \tfrac12 + \tfrac12(\gamma^3 \otimes \gamma^3), \tag{2.14}$$

where the term $\frac12(\gamma^3 \otimes \gamma^3)$ is the "twisting curvature" [7].

Ugalde [114] has shown that, via an appropriate $\mathcal{A}$-module isomorphism $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{S}' \simeq \mathcal{A}^\bullet(\mathbb{S}^2)$, the corresponding operator on $\mathcal{A}^\bullet(\mathbb{S}^2)$ is precisely the operator $d + \delta$, that we call the "Hodge–Dirac operator". Its square is the *Hodge Laplacian* $\Delta^H := (d+\delta)^2 = d\delta + \delta d$. Under the aforementioned isomorphism, (2.14) transforms to $(d+\delta)^2 = \Delta^H + \frac12 - \frac12$.

Spectrum of $d + \delta$. The eigenforms for the Hodge Laplacian on spheres have been determined by Folland [49]. For $n = 2$, the eigenvalues of Δ^H are $\{\, l(l+1) : l \in \mathbb{N} \}$ with multiplicities $4(2l+1)$ for $l = 1, 2, 3, \dots$; for $l = 0$, there is a 2-dimensional kernel of harmonic forms, generated by 1 and $i\Omega$. The other eigenforms are interchanged by d and δ, and so may be combined to get a complete set of eigenvectors for $d + \delta$; this yields

$$\operatorname{sp}(d+\delta) = \{\, \pm\sqrt{l(l+1)} : l \in \mathbb{N} \}, \tag{2.15}$$

with respective multiplicities $2(2l+1)$.

Grading and real structure. We have two $\mathbb{Z}_2$-grading operators at our disposal on the Hilbert space of forms $\mathcal{H} = L^{2,\bullet}(\mathbb{S}^2)$: the even/odd form-degree grading ε and the Hodge star operator $\star$. In differential geometric language, these correspond to selecting the *de Rham complex* or the *signature complex* as the object of interest [54]. The Dirac operator $(d + \delta)$ is odd for either grading.

Thus there are two (complex) K-cycles $(\mathcal{A}, \mathcal{H}, d+\delta, \varepsilon)$ and $(\mathcal{A}, \mathcal{H}, d+\delta, \star)$ from the $\mathcal{A}$-module of forms. To distinguish them, we look for a real structure J: an antilinear isometry, satisfying $J^2 = -1$ and $JaJ^\dagger = \bar a$ if $a \in C^\infty(\mathbb{S}^2)$, that commutes with $d + \delta$ and anticommutes with the grading operator. In particular, J must preserve the two-dimensional space $\ker(d + \delta)$. Since the harmonic forms have even degree, any such J cannot anticommute with the grading ε. However, it turns out that the eigenspaces for $d+\delta$ can be split into selfdual and antiselfdual subspaces of dimension $2l+1$ each. One can then find a conjugation J that does anticommute with the Hodge star operator: $J\star J^\dagger = -\star$. This yields a real spectral triple:

$$(C^\infty(\mathbb{S}^2), L^{2,\bullet}(\mathbb{S}^2), d+\delta, \star, J).$$

This is the "Dirac–Kähler" geometry that has been studied by Mignaco *et al* [87]. It also appears prominently in [51].

3. Real Spectral Triples: the Axiomatic Foundation

Having exemplified how differential geometry may be made algebraic in the commutative case of Riemannian spin manifolds, we now extract the essential features of this formulation, with a view to relaxing the constraint of commutativity on the underlying algebra. We shall follow quite closely the treatment of Connes in [25, 26], wherein an axiomatic scheme for noncommutative geometries is set forth. Indeed, one could say that these lectures are essentially an extended meditation on those axioms.

The data set

The fundamental object of study is a *K-cycle* (so called because it is a building block for a K-homology theory) or a *spectral triple*. This consists of three pieces of data $(\mathcal{A}, \mathcal{H}, D)$, sometimes accompanied by other data Γ and J, satisfying several compatibility conditions which we formulate as axioms. If Γ is present, we say that the K-cycle is *even*, otherwise it is *odd*. If J is present, we say the K-cycle is *real*; if not, we can call it "complex". We shall, however, concentrate on the real case.

Definition. An **even, real, spectral triple** or ***K*-cycle** consists of a set of five objects $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$, of the following types:

(1) $\mathcal{A}$ is a *pre-C^*-algebra*;

(2) $\mathcal{H}$ is a *Hilbert space* carrying a faithful representation π of $\mathcal{A}$ by bounded operators;

(3) D is a *selfadjoint operator* on $\mathcal{H}$, with compact resolvent;

(4) Γ is a *selfadjoint unitary operator* on $\mathcal{H}$, so that $\Gamma^2 = 1$;

(5) J is an *antilinear isometry* of $\mathcal{H}$ onto itself.

Before introducing the further relations and properties that these objects must satisfy, we make some comments on the data themselves.

Pre-C^*-algebras.

(1) A pre-C^*-algebra $\mathcal{A}$ is a dense involutive subalgebra of a C^*-algebra A that is *stable under the holomorphic functional calculus*. This means that for any $a \in \mathcal{A}$ and any function f holomorphic in a neighbourhood of the spectrum $\mathrm{sp}(a)$ in A, the element $f(a) \in A$ actually belongs to $\mathcal{A}$. (We may suppose that $\mathcal{A}$ has a unit, otherwise we adjoin one and work with the unitized algebras $\mathcal{A}^+$ and A^+.) Here $f(a)$ is defined by the Dunford integral

$$f(a) := \frac{1}{2\pi i} \oint_\gamma f(\zeta)(\zeta - a)^{-1}\, d\zeta,$$

where γ is any circuit winding once around $\mathrm{sp}(a)$.

This happens whenever $\mathcal{A} = \bigcap_{k=1}^\infty A_k$, where the A_k form a decreasingly nested sequence of Banach algebras with continuous inclusions $A_k \hookrightarrow A$. For example, if $\mathcal{A}$ is the set of smooth elements of A under the action of a one-parameter group of automorphisms with generator L, one can take $A_k := \mathrm{Dom}(L^k)$. In the case $\mathcal{A} = C^\infty(M)$ with M a spin manifold, one can use $A_k := \mathrm{Dom}(D\!\!\!/^{2k})$.

The major consequence of stability under the holomorphic functional calculus is that the K-theories of $\mathcal{A}$ and of A are the same. That is to say, the inclusion $j: \mathcal{A} \to A$ induces isomorphisms $j_*: K_0(\mathcal{A}) \to K_0(A)$ and $j_*: K_1(\mathcal{A}) \to K_1(A)$. Thus, one may use the technology of C^*-algebraic K-theory [119] with the dense subalgebra $\mathcal{A}$. For more information on this point, see [22, III.C], and the appendices of [17, 19].

(2) When $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$, we shall usually write $a\xi := \pi(a)\xi$. On a few occasions, though, we shall need to refer explicitly to the representation π.

(3) That D has "compact resolvent" means that $(D - \lambda)^{-1}$ is compact, whenever $\lambda \notin \mathbb{R}$. Equivalently, D has a finite-dimensional kernel, and the inverse D^{-1} (defined on the orthogonal complement of this kernel) is compact. In particular, D has a discrete spectrum of eigenvalues of finite multiplicity. This generalizes the case of a Dirac operator on a *compact* spin manifold; thus the K-cycles discussed here represent "noncommutative compact manifolds". Noncompact manifolds can be treated in a parallel manner by supposing that the algebra has no unit, whereupon we require only that for each $a \in \mathcal{A}$, the operator $a(D - \lambda)^{-1}$ has compact resolvent [24].

On the basis of the distance formula (1.6), we shall interpret the inverse of D as a *length element*. Since D need not be positive, one may prefer the inverse of its modulus $|D| = (D^2)^{1/2}$; we shall write $ds := |D|^{-1}$. (Actually, the point of view advocated in [25] is that one should treat ds as an abstract symbol adjoined to the algebra $\mathcal{A}$ and consider D^{-1} as its representative on $\mathcal{H}$; but we shall ignore this distinction here.)

(4) The *grading operator* Γ, available for even K-cycles, splits the Hilbert space as $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^\pm$ is the (± 1)-eigenspace of Γ. In this case, we suppose that the representation of $\mathcal{A}$ on $\mathcal{H}$ is *even* and that the operator D is *odd*, that is, $a\Gamma = \Gamma a$ for $a \in \mathcal{A}$ and $D\Gamma = -\Gamma D$. We display this symbolically as

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix},$$

where $D^+: \mathcal{H}^+ \to \mathcal{H}^-$ and $D^-: \mathcal{H}^- \to \mathcal{H}^+$ are adjoints.

(5) The *real structure* J must satisfy $J^2 = \pm 1$ and commutation relations $JD = \pm DJ$, $J\Gamma = \pm \Gamma J$; for the signs, see the reality axiom below. Its adjoint is $J^\dagger = J^{-1} = \pm J$. The recipe

$$\pi^0(b) := J\pi(b^*)J^\dagger \tag{3.1}$$

defines an antirepresentation of $\mathcal{A}$, that is, it reverses the product. It is convenient to think of π^0 as a true representation of the *opposite algebra* $\mathcal{A}^0$, consisting of elements $\{ a^0 : a \in \mathcal{A} \}$ with product $a^0 b^0 = (ba)^0$. Thus we shall usually abbreviate (3.1) to $b^0 = Jb^*J^\dagger$. The important property that we require is that *the representations π and π^0 commute*; that is,

$$[a, b^0] = [a, Jb^*J^\dagger] = 0 \qquad \text{for all} \quad a, b \in \mathcal{A}. \tag{3.2}$$

When $\mathcal{A}$ is commutative, we demand also that $J\pi(b^*)J^\dagger = \pi(b)$, whereupon (3.2) is automatic. This requires that $\mathcal{A}$ act as scalar multiplication operators on the spinor space, as exemplified in §2.

The stage is set. We now deal with the further conditions needed to ensure that these data underlie a *geometry*.

Infinitesimals and dimension

Axiom 1 (Dimension). There is an integer n, the *dimension* of the K-cycle, such that the *length element* $ds := |D|^{-1}$ is an *infinitesimal* of order $1/n$.

By "infinitesimal" we mean simply a *compact operator* on $\mathcal{H}$. Since the days of Leibniz, an infinitesimal is conceptually a nonzero quantity smaller than any positive ϵ. Since we work on the arena of an infinite-dimensional Hilbert space, we may forgive the violation of the requirement $T < \epsilon$ over a finite-dimensional subspace (that may depend on ϵ). T must then be an operator with discrete spectrum, with any nonzero λ in $\operatorname{sp}(T)$ having finite multiplicity; in other words, the operator T must be compact.

The *singular values* of T, i.e., the eigenvalues of the positive compact operator $|T| := (T^*T)^{1/2}$, are arranged in decreasing order: $\mu_0 \geq \mu_1 \geq \mu_2 \geq \cdots$. We then say that T is an *infinitesimal of order* α if

$$\mu_k(T) = O(k^{-\alpha}) \qquad \text{as } k \to \infty.$$

Notice that infinitesimals of *first order* have singular values that form a *logarithmically divergent series*:

$$\mu_k(T) = O\Big(\frac{1}{k}\Big) \implies \sigma_N(T) := \sum_{k<N} \mu_k(T) = O(\log N). \tag{3.3}$$

The dimension axiom can then be reformulated as: "there is an integer n for which the singular values of D^{-n} form a logarithmically divergent series".

The coefficient of logarithmic divergence will be denote by $⨍ |D|^{-n}$, where $⨍$ denotes the *noncommutative integral*; we shall have more to say about it later.

Example. Let us compute the dimension of the sphere $\mathbb{S}^2$ from its fundamental K-cycle. From the spectrum of $\not{D}$ we get the eigenvalues of the positive operator $\not{D}^{-2}$:

$$\operatorname{sp}(\not{D}^{-2}) = \{\, (l + \tfrac{1}{2})^2 : l \in \mathbb{N} + \tfrac{1}{2} \,\} = \{\, k^{-2} : k = 1, 2, 3, \ldots \,\}$$

where the eigenvalue k^{-2} has multiplicity $4k = 2(2l+1)$. For $N = 2M(M+1)$, we get

$$\sigma_N(\not{D}^{-2}) = \sum_{k<M} \frac{4k}{k^2} \sim 4 \log M \sim 2 \log N \qquad \text{as } N \to \infty,$$

so that $\not{D}^{-1}$ is an infinitesimal of order $\frac{1}{2}$ and therefore the dimension is 2 (surprise!). Also, the coefficient of logarithmic divergence is

$$⨍ ds^2 := ⨍ \not{D}^{-2} = 2.$$

As we shall see later on, this coefficient equals $1/2\pi$ times the area in the case of any 2-dimensional surface, so the area of the sphere is hereby computed to be 4π.

Exercise. The Dirac operator for the circle $\mathbb{S}^1$ is just $-i\,d/d\theta$. Use the Fourier series expansion of functions in $C^\infty(\mathbb{S}^1)$ to check that $|d/d\theta|^{-1}$ is an infinitesimal of order 1; the circle thus has dimension 1. $\diamondsuit$

The order-one condition

Axiom 2 (Order one). For all $a, b \in \mathcal{A}$, the following commutation relation holds:

$$[[D, a], Jb^* J^\dagger] = 0. \tag{3.4}$$

This could be rewritten as $[[D, a], b^0] = 0$ or, more precisely, $[[D, \pi(a)], \pi^0(b)] = 0$. Using (3.2) and the Jacobi identity, we see that this condition is symmetric in the representation π and π^0, since

$$[a, [D, b^0]] = [[a, D], b^0] + [D, [a, b^0]] = -[[D, a], b^0] = 0.$$

In the commutative case, the condition (3.4) expresses the fact that the Dirac operator is a first-order differential operator:

$$[[\not{D}, a], Jb^* J^\dagger] = [[\not{D}, a], b] = [\gamma(da), b] = 0.$$

(Contrast this with a second-order operator like a Laplacian, that satisfies $[[\Delta, a], b] = -2g^{-1}(da, db)$, generally nonzero [7].)

We can interpret (3.4) as saying that the operators $\pi^0(b)$ commute with the subalgebra of operators on $\mathcal{H}$ generated by all operators $\pi(a)$ and $[D, \pi(a)]$. This gives rise to a linear representation of the tensor product of several copies of $\mathcal{A}$:

$$\pi_D(a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) := \pi(a)\,[D, \pi(a_1)]\,[D, \pi(a_2)] \dots [D, \pi(a_n)],$$

or, more simply, $a_0\,[D, a_1]\,[D, a_2] \dots [D, a_n]$. In view of the order one condition, we could even replace the first entry $a \in \mathcal{A}$ by $a \otimes b^0 \in \mathcal{A} \otimes \mathcal{A}^0$, writing

$$\pi_D((a \otimes b^0) \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) := \pi(a)\pi^0(b)\,[D, \pi(a_1)]\,[D, \pi(a_2)] \dots [D, \pi(a_n)], \tag{3.5}$$

Now $C_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0) := (\mathcal{A} \otimes \mathcal{A}^0) \otimes \mathcal{A}^{\otimes n}$ is a bimodule over the algebra $\mathcal{A}$, and this recipe represents it by operators on $\mathcal{H}$. Its elements are called *Hochschild n-chains with coefficients in the $\mathcal{A}$-bimodule $\mathcal{A} \otimes \mathcal{A}^0$*.

Smoothness of the algebra

Axiom 3 (Regularity). For any $a \in \mathcal{A}$, $[D, a]$ is a bounded operator on $\mathcal{H}$, and both a and $[D, a]$ belong to the domain of smoothness $\bigcap_{k=1}^\infty \operatorname{Dom}(\delta^k)$ of the derivation δ on $\mathcal{L}(\mathcal{H})$ given by $\delta(T) := [|D|, T]$.

In the commutative case, where $[\not{D}, a] = \gamma(da)$, this condition amounts to saying that a has derivatives of all orders, i.e., that $\mathcal{A} \subseteq C^\infty(M)$. This can be proved with pseudodifferential calculus, since the principal symbol of the modulus of the Dirac operator is $\sigma_{|D|}(x,\xi) = |\xi|\,1$. From there one obtains that all multiplication operators in $\bigcap_{k=1}^{\infty} \mathrm{Dom}(\delta^k)$ are multiplications by smooth functions.

Hochschild cycles and orientation

Axiom 4 (Orientability). There is a *Hochschild cycle* $c \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0)$ whose representative on $\mathcal{H}$ is

$$\pi(c) = \begin{cases} \Gamma, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Here c is a Hochschild n-chain as defined above, and $\pi(c)$ is given by (3.5). We say c is a *cycle* if its boundary is zero, where the Hochschild boundary operator is

$$\begin{aligned} b(m_0 \otimes a_1 \otimes \dots \otimes a_n) := m_0 a_1 \otimes a_2 \otimes \dots \otimes a_n - m_0 \otimes a_1 a_2 \otimes \dots \otimes a_n + \cdots \\ + (-1)^{n-1} m_0 \otimes a_1 \otimes \dots \otimes a_{n-1} a_n \\ + (-1)^n a_n m_0 \otimes a_1 \otimes \dots \otimes a_{n-1}. \end{aligned}$$

(Here $m_0 \in \mathcal{A} \otimes \mathcal{A}^0$.) This satisfies $b^2 = 0$ and thus makes $C_\bullet(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0)$ a chain complex, whose homology is the Hochschild homology $H_\bullet(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0)$. (For the full story, see [84] or [120].) Notice that if $x = (a \otimes b^0) \otimes a_1 \otimes \dots \otimes a_n$, then, by telescoping:

$$\pi_D(bx) = (-1)^{n-1} \big(ab^0\,[D, a_1] \dots [D, a_{n-1}] a_n - a_n ab^0\,[D, a_1] \dots [D, a_{n-1}] \big)$$

This Hochschild cycle c is the algebraic equivalent of a *volume form* on our noncommutative manifold. To see that, let us look briefly at the commutative case, where we may replace $\mathcal{A} \otimes \mathcal{A}^0$ simply by $\mathcal{A}$. A differential form in $\mathcal{A}^k(M)$ is a sum of terms $a_0\, da_1 \wedge \dots \wedge da_k$, but in the noncommutative case the antisymmetry of the wedge product is lost, so we replace such a form with

$$c' := \sum_\sigma (-1)^\sigma a_0 \otimes a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(n)} \tag{3.6}$$

(sum over n-permutations) in $\mathcal{A}^{\otimes(n+1)} = C_n(\mathcal{A}, \mathcal{A})$. Then $bc' = 0$ by cancellation since $\mathcal{A}$ is commutative; for instance:

$$b(a \otimes a' \otimes a'' - a \otimes a'' \otimes a') = (aa' - a'a) \otimes a'' - a \otimes (a'a'' - a''a') + (a''a - aa'') \otimes a'.$$

In the commutative case $\mathcal{A} = C^\infty(M)$, chains are represented by Clifford products: $\pi_{\not{D}}(a_0 \otimes a_1 \otimes \dots \otimes a_n) = a_0\, \gamma(da_1) \dots \gamma(da_n)$. The Riemannian volume form on M can be written as $\Omega = i^{\lfloor (n+1)/2 \rfloor} \theta^1 \wedge \dots \wedge \theta^n$ where $\{\theta^1, \dots, \theta^n\}$ is an oriented orthonormal basis of 1-forms, and the corresponding cycle c is represented by $\pi_{\not{D}}(c) = i^{\lfloor (n+1)/2 \rfloor} \gamma(\theta^1) \dots \gamma(\theta^n)$. It is now an easy exercise in Clifford algebra [83] to check that $\pi_{\not{D}}(c) = 1$ if n is odd and $\pi_{\not{D}}(c) = \chi$ if n is even, where χ is the grading operator of the spin representation.

Finiteness of the K-cycle

Axiom 5 (Finiteness). The space of smooth vectors $\mathcal{H}_\infty := \bigcap_{k=1}^\infty \mathrm{Dom}(D^k)$, is a *finite projective left $\mathcal{A}$-module* with a Hermitian structure $(\cdot \mid \cdot)$ given by

$$⨍ (\xi \mid \eta)\, ds^n := \langle \xi \mid \eta \rangle.$$

The representation $\pi\colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and the regularity axiom already make $\mathcal{H}_\infty$ a left $\mathcal{A}$-module. It is clear how to adapt the definition (1.1) of Hermitian structures for right $\mathcal{A}$-modules to the case of left $\mathcal{A}$-modules; for instance, one has $(\xi \mid a\eta) = a\,(\xi \mid \eta)$, so that the previous equation entails

$$⨍ a\,(\xi \mid \eta)\, ds^n := \langle \xi \mid a\eta \rangle. \tag{3.7}$$

To see how (3.7) defines a Hermitian structure implicitly, notice that whenever $a \in \mathcal{A}$ then $a\, ds^n = a|D|^{-n}$ is an infinitesimal of first order, so that the left hand side is defined provided $(\xi \mid \eta) \in \mathcal{A}$.

As a finite projective left $\mathcal{A}$-module, $\mathcal{H}_\infty \simeq \mathcal{A}^m p$ with $p = p^2 = p^*$ in some $M_m(\mathcal{A})$, so we can write $\xi \in \mathcal{H}_\infty$ as a row vector $(\xi_1, \dots, \xi_m)$ satisfying $\sum_j \xi_j p_{jk} = \xi_k$. Furthermore,

$$(\xi \mid \eta) = \sum_{j=1}^m \eta_j \xi_j^* \in \mathcal{A}.$$

In the commutative case, Connes's trace theorem (see below) shows that $(\xi \mid \eta)$ is just the hermitian product of spinors given by the metric on the spinor bundle.

A point to notice is that

$$⨍ a\,(\xi \mid \eta)\, ds^n = \langle \xi \mid a\eta \rangle = \langle a^*\xi \mid \eta \rangle = ⨍ (a^*\xi \mid \eta)\, ds^n = ⨍ (\xi \mid \eta)\, a\, ds^n,$$

so this axiom implies that $⨍(\cdot)|D|^{-n}$ defines a *finite trace* on the algebra $\mathcal{A}$. As shown by Cipriani *et al* [15, Prop. 1.6], the tracial property follows from the regularity axiom (one only requires that both a and $[D, a]$ lie in $\mathrm{Dom}(\delta)$); this refutes an earlier complaint [117] that one needed an extra assumption of "tameness" on the K-cycle.

The existence of a finite trace on $\mathcal{A}$ implies that the von Neumann algebra $\mathcal{A}''$ generated by $\mathcal{A}$ also has a finite normal trace, so it cannot have components of types I_∞, II_∞ or III [68, §8.5].

The finiteness and regularity axioms entail [25] that

$$\mathcal{A} = \left\{ T \in \mathcal{A}'' : T \in \bigcap_{k=1}^\infty \mathrm{Dom}(\delta^k) \right\}. \tag{3.8}$$

As such, $\mathcal{A}$ becomes automatically a pre-C^*-algebra, so this assumption of ours is in fact redundant.

Poincaré duality and K-theory

Axiom 6 (Poincaré duality). The Fredholm index of the operator D yields a *nondegenerate* intersection form on the K-theory ring of the algebra $\mathcal{A} \otimes \mathcal{A}^0$.

On a compact oriented n-dimensional manifold M, Poincaré duality is usually formulated [36] as an isomorphism of cohomology (in degree k) with homology (in degree $n-k$), or equivalently as a nondegenerate bilinear pairing on the cohomology ring $H^\bullet(M)$. (For noncompact manifolds, the pairing is between the ordinary and the compactly supported cohomologies.) If $\alpha \in \mathcal{A}^k(M)$ and $\eta \in \mathcal{A}^{n-k}(M)$ are closed forms, integration over M pairs them by

$$(\alpha, \eta) \mapsto \int_M \alpha \wedge \eta;$$

since the right hand side depends only on the cohomology classes of α and β (it vanishes if either α or β is exact), it gives a bilinear map $H^k(M) \times H^{n-k}(M) \to \mathbb{C}$. Now each $\mathcal{A}^k(M)$ carries a scalar product $(\cdot \mid \cdot)$ induced by the metric and orientation on M, given by

$$\alpha \wedge \star\beta =: \epsilon_k \,(\alpha \mid \beta)\, \Omega \qquad \text{for} \quad \alpha, \beta \in \mathcal{A}^k(M),$$

where $\epsilon_k = \pm 1$ or $\pm i$ and Ω is the volume form on M. [Compare (2.13)]. This pairing is nondegenerate since

$$\int_M \alpha \wedge (\epsilon_k^{-1} \star\alpha) = \int_M (\alpha \mid \alpha)\, \Omega > 0 \qquad \text{for} \quad \alpha \neq 0.$$

In view of the existence of isomorphisms between $K^\bullet(M) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $H^\bullet(M; \mathbb{Q})$ given by the Chern character, one could hope to reformulate this as a canonical pairing on the K-theory ring. This can be done if M is a spin^c manifold; the rôle of the orientation $[\Omega]$ in cohomology is replaced by the K-orientation, so that the corresponding pairing of K-rings is mediated by the Dirac operator: see [22, IV.1.γ] or [19] for a discussion of how this pairing arises. We leave aside the translation from K-theory to cohomology (by no means a short story) and explain briefly how the intersection form may be computed in the K-context.

***K*-theory of algebras.** There are two abelian groups, $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$, associated to a pre-C^*-algebra $\mathcal{A}$, as follows [119]. The group $K_0(\mathcal{A})$ gives a rough classification of finite projective modules over $\mathcal{A}$. If $M_\infty(\mathbb{C})$ is the algebra of compact operators of finite rank, then $M_\infty(\mathcal{A}) = \mathcal{A} \otimes M_\infty(\mathbb{C})$ is a pre-C^*-algebra dense in $A \otimes \mathcal{K}$. Two projectors in $M_\infty(\mathcal{A})$ have a direct sum

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

Two such projectors p and q are equivalent if $p = uqu^{-1}$ for some unitary u in some $M_\infty(\mathcal{A})$ (this makes sense if $\mathcal{A}$ is unital; otherwise, we work with $\mathcal{A}^+$). Adding the equivalence classes by $[p] + [q] := [p \oplus q]$, we get a semigroup, and the group $K_0(\mathcal{A})$ is the corresponding group of formal differences $[p] - [q]$.

The other group $K_1(\mathcal{A})$ is generated by classes of *unitary* matrices over $\mathcal{A}$. We nest the unitary groups of various sizes by identifying $u \in U_m(\mathcal{A})$ with $u \oplus 1 \in U_{m+k}(\mathcal{A})$, and call u, v equivalent if $v^{-1}u$ lies in the identity component of $U_\infty(\mathcal{A}) := \bigcup_{m\geq 1} U_m(\mathcal{A})$. The equivalence classes form the discrete group of components of $U_\infty(\mathcal{A})$: this is $K_1(\mathcal{A})$. It turns out that $[uv] = [u \oplus v] = [v \oplus u] = [vu]$, so that $K_1(\mathcal{A})$ is abelian. (This is the standard definition of K_1 for a C^*-algebra A; to define it thus for a pre-C^*-algebra $\mathcal{A}$ is a slight abuse of notation on our part, that amounts to conferring on $\mathcal{A}$ a "topological" K-theory. In "algebraic" K-theory, the definition of the K_1 group is not the same and may give different groups. In fact, one can equivalently define $K_1(\mathcal{A})$ with invertible rather than unitary matrices, as the quotient of $GL_\infty(\mathcal{A})$ by its neutral component; for $K_1^{\mathrm{alg}}(\mathcal{A})$, we forget the topology of $GL_\infty(\mathcal{A})$ and factor by the smaller subgroup generated by its commutators. See [9, 104] for the algebraic theory.)

Both groups are homotopy invariant: if $\{\, p_t : 0 \leq t \leq 1 \,\}$ is a homotopy of projectors in $M_\infty(\mathcal{A})$ and if $\{\, u_t : 0 \leq t \leq 1 \,\}$ is a homotopy in $U_\infty(\mathcal{A})$, then $[p_0] = [p_1]$ in $K_0(\mathcal{A})$ and $[u_0] = [u_1]$ in $K_1(\mathcal{A})$. In the commutative case, we have $K_j(C^\infty(M)) = K^j(M)$ for $j = 0, 1$.

When $\mathcal{A}$ is represented on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, any odd selfadjoint Fredholm operator D on $\mathcal{H}$ defines an index map $\phi_D \colon K_0(\mathcal{A}) \to \mathbb{Z}$, as follows. Denote by $a \mapsto a^+ \oplus a^-$ the representation of $M_m(\mathcal{A})$ on $\mathcal{H}_m^+ \oplus \mathcal{H}_m^- = \mathcal{H}_m = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (m times); write $D_m := D \oplus \cdots \oplus D$, acting on $\mathcal{H}_m$. Then $p^- D_m p^+$ is a Fredholm operator from H_m^+ to $\mathcal{H}_m^-$, whose index depends only on the class $[p]$ in $K_0(\mathcal{A})$. We define

$$\phi_D\big([p]\big) := \operatorname{ind}(\pi^-(p) D_m \pi^+(p)).$$

On the other hand, when $\mathcal{A}$ is represented on an ungraded Hilbert space $\mathcal{H}$, a selfadjoint Fredholm operator D on $\mathcal{H}$ defines an index map $\phi_D \colon K_1(\mathcal{A}) \to \mathbb{Z}$. Let $\mathcal{H}^>$ be the range of the spectral projector $P_> = P_{(0,\infty)}$ determined by the positive part of the spectrum of D. Then if $u \in U_m(\mathcal{A})$, $P_> \, u \, P_>$ is a Fredholm operator on $\mathcal{H}_m^>$, whose index depends only on the class $[u]$ in $K_1(\mathcal{A})$. We define

$$\phi_D\big([u]\big) := \operatorname{ind}(P_> \, u \, P_>).$$

Finally, it is possible to work with K_0 alone since there is a natural isomorphism ("suspension") from $K_1(A)$ to $K_0(A \otimes C_0(\mathbb{R}))$ for any C^*-algebra A [119, 7.2.5].

The intersection form. Coming back now to the spectral triple under discussion, we define a pairing on $K_\bullet(\mathcal{A}) = K_0(\mathcal{A}) \oplus K_1(\mathcal{A})$ as follows. The commuting representations π, π^0 determine a representation of the algebra $\mathcal{A} \otimes \mathcal{A}^0$ on $\mathcal{H}$ by

$$a \otimes b^0 \longmapsto aJb^*J^\dagger = Jb^*J^\dagger a.$$

If $[p], [q] \in K_0(\mathcal{A})$, then $[p \otimes q^0] \in K_0(\mathcal{A} \otimes \mathcal{A}^0)$, and the *intersection form* due to D is

$$\big\langle [p], [q] \big\rangle := \phi_D\big([p \otimes q^0]\big).$$

Combined with the suspension isomorphism, we get three other maps $K_i(\mathcal{A}) \times K_j(\mathcal{A}^0) \to K_{i+j}(\mathcal{A} \otimes \mathcal{A}^0) \to \mathbb{Z}$, where the second arrow is ϕ_D.

Poincaré duality is the assertion that this pairing on $K_\bullet(\mathcal{A})$ is nondegenerate. For the case of the finite-dimensional algebra $\mathcal{A} = \mathbb{C} \oplus \mathcal{H} \oplus M_3(\mathbb{C})$ that acts on the space of fermions of the Standard Model [106], the intersection form has been computed in [24] (see also [86, §6.2]). For more general finite-dimensional algebras, see [77, 93]. In that context, Poincaré duality is a very efficient discriminator that rules out several plausible alternatives to the Standard Model.

The real structure

Axiom 7 (Reality). There is an antilinear isometry $J\colon \mathcal{H} \to \mathcal{H}$ such that the representation $\pi^0(b) := J\pi(b^*)J^\dagger$ commutes with $\pi(\mathcal{A})$, satisfying

$$J^2 = \pm 1, \qquad JD = \pm DJ, \qquad J\Gamma = \pm \Gamma J, \tag{3.9}$$

where the signs are given by the following tables:

n even:

n mod 8	0	2	4	6
$J^2 = \pm 1$	$+$	$-$	$-$	$+$
$JD = \pm DJ$	$+$	$+$	$+$	$+$
$J\Gamma = \pm \Gamma J$	$+$	$-$	$+$	$-$

n odd:

n mod 8	1	3	5	7
$J^2 = \pm 1$	$+$	$-$	$-$	$+$
$JD = \pm DJ$	$-$	$+$	$-$	$+$

These tables, with their periodicity in steps of 8, arise from the structure of real Clifford algebra representations that underlie real K-theory. See [2, 83] for the algebraic foundation of this real Bott periodicity. We claim that, in the commutative case of Riemannian spin manifolds, one can find conjugation operators J on spinors that satisfy the foregoing sign rules. Thus, for instance, $J^2 = -1$ for Dirac spinors over 4-dimensional spaces with Euclidean signature. (We make no attempt to extend the theory to Minkowskian spaces at this stage.) Notice also that the signs for dimension two are those we have used in the example of the Riemann sphere.

The Tomita involution. It is time to explain where the antilinear operator J comes from in the noncommutative case. The bicommutant $\mathcal{A}''$ of the involutive algebra $\mathcal{A}$, or more precisely of $\pi(\mathcal{A})$, is a weakly closed algebra of operators on $\mathcal{H}$, i.e., a von Neumann algebra (generally much larger than the norm closure A). Let us assume that $\mathcal{H}$ contains a *cyclic and separating vector* ξ_0 for $\mathcal{A}''$, that is, a vector such that (i) $\mathcal{A}''\xi_0$ is a dense subspace of $\mathcal{H}$ (cyclicity) and (ii) $a\xi_0 = 0$ in $\mathcal{H}$ only if $a = 0$ in $\mathcal{A}''$ (separation). A basic result of operator algebras, Tomita's theorem [68, 112], says that the antilinear mapping

$$a\xi_0 \longmapsto a^*\xi_0 \tag{3.10}$$

extends to a closed antilinear operator S on $\mathcal{H}$, whose polar decomposition $S = J\Delta^{1/2}$ determines an antiunitary operator $J\colon \mathcal{H} \to \mathcal{H}$ with $J^2 = 1$ such that $a \mapsto Ja^*J^\dagger$ is an isomorphism from $\mathcal{A}''$ onto its commutant $\mathcal{A}'$. (Since these commuting operator algebras are isomorphic, the space $\mathcal{H}$ can be neither too small nor too large; this is what the cyclic and separating vector ensures.)

When the state of $\mathcal{A}''$ given by $a \mapsto \langle \xi_0 \mid a\xi_0 \rangle$ is a trace, the operator $\Delta = S^* S$ is just 1, and so the mapping (3.10) is J itself. From (3.7), the trace $⨍(\cdot)\,|D|^{-n}$ on $\mathcal{A}$ gives rise to a tracial vector state on $\mathcal{A}''$. Thus the Tomita theorem already provides us with an antiunitary operator J satisfying $[a, Jb^* J^\dagger] = 0$; we shall see in the next chapter how to modify it to obtain $J^2 = -1$ when that is required.

We sum up our discussion with the basic definition.

Definition. A **noncommutative geometry** is a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$ or $(\mathcal{A}, \mathcal{H}, D, J)$, according as its dimension is even or odd, that satisfies the seven axioms set out above.

Riemannian spin manifolds provide the commutative examples. It is not hard to manufacture noncommutative examples with finite-dimensional matrix algebras [77, 93]; these are zero-dimensional geometries in the sense of Axiom 1. In the next chapter we study a more elaborate noncommutative example which, like the Riemann sphere, has dimension two.

4. Geometries on the Noncommutative Torus

We turn now to an algebra that is not commutative, in order to see how the algebraic formulation of geometries, as laid out in the previous chapter, allows us to go beyond Riemannian spin manifolds. Of course, the mere act of moving from commutative to noncommutative algebras is a very familiar one: it is the mathematical point of departure for quantum mechanics. As was already made clear by von Neumann in his 1931 study of the canonical commutation relations [91], the Schrödinger representation arises by replacing convolution of functions of two real variables by a noncommutative variant nowadays called "twisted convolution" [72]. This was reinterpreted by Moyal [90] as a variant of the ordinary product of functions on a two-dimensional phase space, related to Weyl's method of quantization.

We begin, then, with Weyl quantization. We wish to quantize a compact phase space, since our formalism so far relies heavily on compactness, e.g., by demanding that the length element $ds = D^{-1}$ be a compact operator. The Riemann sphere would seem to be a good candidate, since several studies exist of its quantization both in the Moyal framework [116] and from the noncommutative geometry point of view [60, 85]. However, all of these involve an approximating sequence of algebras rather a single algebra and so do not define a solitary K-cycle.

We turn instead to the torus $\mathbb{T}^2$ (with the flat metric). Via the Gelfand cofunctor, this is determined by an algebra of *doubly periodic* functions on $\mathbb{R}^2$ (or on $\mathbb{C}$). If ω_1, ω_2 are the periods, with ratio $\tau := \omega_2/\omega_1$ in the upper half plane $\mathbb{C}_+$, so that $\Im\tau > 0$, one identifies $\mathbb{T}^2$ with the quotient space $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$. These "complex tori" are homeomorphic but are not all equivalent as complex manifolds. In fact, if one chooses to study these tori through the algebras of *meromorphic* functions with the required double periodicity, the resulting *elliptic curves* E_τ are classified by the orbit of τ under the action $\tau \mapsto (a\tau + b)/(c\tau + d)$ of the modular group $PSL(2,\mathbb{Z})$ on $\mathbb{C}_+$.

Algebras of Weyl operators

Our starting point is the canonical commutation relations of quantum mechanics on a one-dimensional configuration space $\mathbb{R}$. In Weyl form [34, 42], these are represented by a family of unitary operators on $L^2(\mathbb{R})$:

$$W_\theta(a,b)\psi : t \longmapsto e^{-\pi i\theta ab}\, e^{2\pi i\theta bt}\, \psi(t-a) \qquad \text{for} \quad a,b \in \mathbb{R}. \tag{4.1}$$

Here θ is a nonzero real parameter; the reader is invited to think of θ as $1/\hbar$, the reciprocal of the Planck constant. The linear space generated by $\{\, W_\theta(a,b) : a,b \in \mathbb{R} \,\}$ is an involutive algebra, wherein

$$[W_\theta(a,b), W_\theta(c,d)] = -2i \sin\bigl(\pi\theta(ad-bc)\bigr)\, W_\theta(a+c, b+d),$$

so that $W_\theta(a,b)$ and $W_\theta(c,d)$ commute if and only if

$$\theta(ad - bc) \in \mathbb{Z}. \tag{4.2}$$

Moreover, the unitary operators $U_\theta := W_\theta(1,0)$ and $V_\theta := W_\theta(0,1)$ obey the commutation relation

$$V_\theta U_\theta = e^{2\pi i\theta}\, U_\theta V_\theta.$$

The full set of operators $\{\, W_\theta(a,b) : a,b \in \mathbb{R} \,\}$ acts irreducibly on $L^2(\mathbb{R})$, but by restricting to *integral* parameters we get reducible actions. We examine first the von Neumann algebra

$$\mathcal{N}_\theta := \{\, W_\theta(m,n) : m,n \in \mathbb{Z} \,\}'',$$

whose commutant can be shown [99] to be

$$\mathcal{N}_\theta' := \{\, W_\theta(r/\theta, s/\theta) : r,s \in \mathbb{Z} \,\}''.$$

The change of scale given by $(R_\theta\psi)(t) := \theta^{-1/2}\psi(t/\theta)$ transforms these operators to

$$R_\theta\, W_\theta(r/\theta, s/\theta)\, R_\theta^{-1} = W_{1/\theta}(r,s),$$

from which we conclude that

$$\mathcal{N}_\theta' \simeq \mathcal{N}_{1/\theta}.$$

The *centre* $\mathcal{Z}(\mathcal{N}_\theta) = \mathcal{N}_\theta \cap \mathcal{N}_\theta'$ depends sensitively on the value of θ. Suppose θ is rational, $\theta = p/q$ where p, q are integers with $\gcd(p,q) = 1$. Then

$$\mathcal{Z}(\mathcal{N}_{p/q}) = \{\, W_{p/q}(qm, qn) : m,n \in \mathbb{Z} \,\}''.$$

This commutative von Neumann algebra is generated by the translation $\psi(t) \mapsto \psi(t-q)$ and the multiplication $\psi(t) \mapsto e^{2\pi i p t}\,\psi(t)$, and can be identified to the multiplication operators on periodic functions (of period q); thus $\mathcal{Z}(\mathcal{N}_{p/q}) \simeq L^\infty(\mathbb{S}^1)$.

On the other hand, if θ is *irrational*, then $\mathcal{N}_\theta$ is a *factor*, i.e., it has trivial centre $\mathcal{Z}(\mathcal{N}_\theta) = \mathbb{C}1$.

If we try to apply the Tomita theory at this stage, we get an unpleasant surprise [42]: $\mathcal{N}_\theta$ has a cyclic vector iff $|\theta| \le 1$, whereas it has a separating vector iff $|\theta| \ge 1$. This tells us that (if $\theta \ne \pm 1$) the space $L^2(\mathbb{R})$ is not the right Hilbert space, and we should represent the algebra generated by the $W_\theta(m,n)$ somewhere else.

To find the good Hilbert space, we must first observe that there is a faithful normal trace on $\mathcal{N}_\theta$ determined by

$$\tau_0(W_\theta(m,n)) := \begin{cases} 1, & \text{if } (m,n) = (0,0), \\ 0, & \text{otherwise.} \end{cases}$$

The GNS Hilbert space associated to this trace is what we need. Before constructing it, notice that, for θ irrational, $\mathcal{N}_\theta$ is a factor with a finite normal trace; and it will soon become clear that its relative dimension function [22, V.1.γ] has range $[0,1]$, so that $\mathcal{N}_\theta$ is a factor of type II_1 in the Murray–von Neumann classification.

The algebra of the noncommutative torus

We now leave the "measure-theoretic" level of von Neumann algebras and focus on the pre-C^*-algebra $\mathcal{A}_\theta$ generated by the operators $W_\theta(m,n)$. Since $W_\theta(m,n) = e^{\pi i m n \theta}\, U_\theta^m V_\theta^n$ and since we shall need to use a GNS representation, it is better to start afresh with a more abstract approach. We redefine $\mathcal{A}_\theta$ as follows.

Definition. For a fixed irrational real number θ, let A_θ be the unital C^*-algebra generated by two elements u, v subject only to the relations $uu^* = u^*u = 1$, $vv^* = v^*v = 1$, and

$$vu = \lambda\, uv \qquad \text{where} \quad \lambda := e^{2\pi i\theta}. \tag{4.3}$$

Let $\mathcal{S}(\mathbb{Z}^2)$ denote the double sequences $\underline{a} = \{a_{rs}\}$ that are *rapidly decreasing* in the sense that

$$\sup_{r,s\in\mathbb{Z}} (1 + r^2 + s^2)^k\, |a_{rs}|^2 < \infty \qquad \text{for all} \quad k \in \mathbb{N}.$$

The **irrational rotation algebra** $\mathcal{A}_\theta$ is defined as

$$\mathcal{A}_\theta := \bigl\{\, a = \textstyle\sum_{r,s} a_{rs}\, u^r v^s : \underline{a} \in \mathcal{S}(\mathbb{Z}^2) \bigr\}. \tag{4.4}$$

It is a *pre-C^*-algebra* that is dense in A_θ.

The product and involution in $\mathcal{A}_\theta$ are computable from (4.3):

$$ab = \textstyle\sum_{r,s} a_{r-n,m}\, \lambda^{mn}\, b_{n,s-m}\, u^r v^s, \qquad a^* = \textstyle\sum_{r,s} \lambda^{rs}\, \bar a_{-r,-s}\, u^r v^s. \tag{4.5}$$

The Weyl operators U_θ, V_θ are unitary and obey (4.3); thus they generate a faithful concrete representation of this pre-C^*-algebra. On the other hand, the relation (4.3) alone generates the algebra $\mathbb{C}_\lambda[u,v]$ known to quantum-group theorists as the Manin q-plane [71] for $q = \lambda \in \mathbb{T}$. Here we also require unitarity of the generators, so that $\mathcal{A}_\theta$ is (a completion of) a quotient algebra of this q-plane.

The irrational rotation algebra gets its name from another representation on $L^2(\mathbb{T})$: the multiplication operator U and the rotation operator V given by $(U\psi)(z) := z\psi(z)$ and $(V\psi)(z) := \psi(\lambda z)$ satisfy (4.3). In the C^*-algebraic framework, U generates the C^*-algebra $C(\mathbb{T})$ and conjugation by V gives an automorphism α of $C(\mathbb{T})$. Under such circumstances, the C^*-algebra generated by $C(\mathbb{T})$ and the unitary operator V is called the *crossed product* of $C(\mathbb{T})$ by α (more pedantically, by the automorphism group $\{\, \alpha^n : n \in \mathbb{Z} \,\}$). In symbols,

$$A_\theta \simeq C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}.$$

The corresponding action by the rotation angle $2\pi\theta$ on the circle is ergodic and minimal (all orbits are dense); it is known [96] that the C^*-algebra A_θ is therefore *simple*.

One advantage of using the abstract presentation by (4.3) and (4.4) to define the algebras $\mathcal{A}_\theta$ is that certain *isomorphisms* become evident. First of all, $\mathcal{A}_\theta \simeq \mathcal{A}_{\theta+n}$ for any $n \in \mathbb{Z}$, since λ is the same for both. (Please note, however, that their representations by Weyl operators, while equivalent, are not identical: indeed, $V_{\theta+n} = e^{2\pi i n t} V_\theta$. Hence the von Neumann algebras $\mathcal{N}_\theta$ and $\mathcal{N}_{\theta+n}$ do not coincide.)

Next, $\mathcal{A}_\theta \simeq \mathcal{A}_{-\theta}$ via the isomorphism determined by $u \mapsto v$, $v \mapsto u$. There are no more isomorphisms among the $\mathcal{A}_\theta$, however: by computing the K_0-groups of these algebras, Rieffel [98] has shown that the abelian group $\mathbb{Z} + \mathbb{Z}\theta$ is an isomorphism invariant of $\mathcal{A}_\theta$.

Some *automorphisms* of $\mathcal{A}_\theta$ are also easy to find. The map $u \mapsto u^{-1}$, $v \mapsto v^{-1}$ is one such. More generally, if $a, b, c, d \in \mathbb{Z}$, then

$$\sigma(u) := u^a v^b, \quad \sigma(v) := u^c v^d \tag{4.6}$$

yields $\sigma(v)\sigma(u) = \lambda^{ad-bc}\sigma(u)\sigma(v)$, so this map extends to an automorphism of $\mathcal{A}_\theta$ if and only if $ad - bc = 1$.

For *rational* θ, we can define $\mathcal{A}_\theta$ in the same way. When $\theta = 0$, we identify u, v with multiplications by $z_1 = e^{2\pi i\phi_1}$, $z_2 = e^{2\pi i\phi_2}$ on $\mathbb{T}^2$, so that $\mathcal{A}_0 = C^\infty(\mathbb{T}^2)$. The presentation $a = \sum_{r,s} a_{rs}\, u^r v^s$ is the expansion of a smooth function on the torus in a *Fourier series*:

$$a(\phi_1, \phi_2) = \sum_{r,s\in\mathbb{Z}} a_{rs}\, e^{2\pi i r\phi_1} e^{2\pi i s\phi_2},$$

and (4.6) gives the hyperbolic automorphisms of the torus.

For other rational values $\theta = p/q$, it turns out that $\mathcal{A}_{p/q}$ is *Morita-equivalent* to $C^\infty(\mathbb{T}^2)$; for an explicit construction of the equivalence bimodules, we remit to [101]. The algebras $\mathcal{A}_{p/q}$, for $0 \leq p/q < \frac{1}{2}$, are mutually non-isomorphic.

The normalized trace. The linear functional $\tau\colon \mathcal{A}_\theta \to \mathbb{C}$ given by

$$\tau_0(a) := a_{00}$$

is positive definite since $\tau_0(a^*a) = \sum_{r,s} |a_{rs}|^2 > 0$ for $a \neq 0$; it satisfies $\tau_0(1) = 1$ and is a trace, since $\tau_0(ab) = \tau_0(ba)$ from (4.5). Also, τ_0 extends to a faithful continuous trace on the C^*-algebra A_θ; and, in fact, this normalized trace on A_θ is unique.

The Weyl operators (4.1) allow us to quantize $\mathcal{A}_0$ in such a way that $a(\phi_1, \phi_2)$ is the symbol of the operator $a \in \mathcal{A}_\theta$. Then it can be proved [34, Thm. 3] that $\tau_0(a) = \int_{\mathbb{T}^2} a(\phi_1, \phi_2)\, d\phi_1\, d\phi_2$, so that τ_0 is just the integral of the classical symbol.

The *GNS representation space* $\mathcal{H}_0 = L^2(\mathcal{A}_\theta, \tau_0)$ may be described as the completion of the vector space $\mathcal{A}_\theta$ in the Hilbert norm

$$\|a\|_2 := \sqrt{\tau_0(a^*a)}.$$

Since τ_0 is faithful, the obvious map $\mathcal{A}_\theta \to \mathcal{H}_0$ is injective; to keep the bookkeeping straight, we shall denote by $\underline{a}$ the image in $\mathcal{H}_0$ of $a \in \mathcal{A}_\theta$.

The GNS representation of $\mathcal{A}_\theta$ is just $\pi(a)\colon \underline{b} \mapsto \underline{ab}$. Notice that the vector $\underline{1}$ is obviously cyclic and separating, so the Tomita involution is given by

$$J_0(\underline{a}) := \underline{a^*}.$$

The commuting representation π^0 is then given by

$$\pi^0(a)\,\underline{b} := J_0 \pi(a^*) J_0^\dagger\, \underline{b} = J_0\, \underline{a^* b^*} = \underline{ba}.$$

To build a two-dimensional geometry, we need to have a $\mathbb{Z}_2$-graded Hilbert space on which there is an antilinear involution J that anticommutes with the grading and satisfies $J^2 = -1$. There is a simple device that solves all of these requirements: we simply *double* the GNS Hilbert space by taking $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_0$ and define

$$J := \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}.$$

It remains to introduce the operator D. Before doing so, we make a brief excursion into two-dimensional topology.

The skeleton of the noncommutative torus

An ordinary 2-torus may be built up from the following ingredients: (i) a single point; (ii) two lines, adjoined at their ends to the point to form a pair of circles; and (iii) a plane sheet, attached along its borders to the two circles. If we take the torus apart again, we get its "cell decomposition" into a 0-cell, two 1-cells and a 2-cell.

In more technical language, these cells form the *skeleton* of the torus, and are represented by independent homology classes: one in $H_0(\mathbb{T}^2)$, two in $H_1(\mathbb{T}^2)$ and one in $H_2(\mathbb{T}^2)$. The Euler characteristic of the torus is then computed as $1 - 2 + 1 = 0$.

Guided by the Gelfand cofunctor, homology of spaces is replaced by cohomology of algebras; thus the skeleton of the noncommutative torus will consist of a 0-cocycle, two 1-cocycles and a 2-cocycle on the algebra $\mathcal{A}_\theta$. The appropriate theory for that is *cyclic* cohomology [9, 19, 22]. It is a topological theory insofar as it depends only on the algebra $\mathcal{A}_\theta$ and not on the geometries determined by its K-cycles.

Definition. A *cyclic n-cochain* over an algebra $\mathcal{A}$ is an $(n+1)$-linear form $\phi\colon \mathcal{A}^{n+1} \to \mathbb{C}$ that satisfies the cyclicity condition

$$\phi(a_0, \dots, a_{n-1}, a_n) = (-1)^n \phi(a_n, a_0, \dots, a_{n-1}).$$

It is a **cyclic n-cocycle** if $b\phi = 0$, where the coboundary operator b is defined by

$$\begin{aligned} b\phi(a_0, \dots, a_{n+1}) := \phi(a_0 a_1, a_2, \dots, a_{n+1}) - \cdots + (-1)^j \phi(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}) \\ + \cdots + (-1)^{n+1} \phi(a_{n+1} a_0, a_1, \dots, a_n). \end{aligned}$$

One checks that $b^2 = 0$, so that the cyclic cochains form a complex whose n-th cyclic cohomology group is denoted $HC^n(\mathcal{A})$.

The normalized trace τ_0 is a cyclic 0-cocycle on $\mathcal{A}_\theta$, since

$$b\tau_0(a, b) := \tau_0(ab) - \tau_0(ba) = 0.$$

In fact, a cyclic 0-cocycle is clearly the same thing as a trace. The uniqueness of the normalized trace shows that $HC^0(\mathcal{A}_\theta) = \mathbb{C}[\tau_0]$.

The two basic derivations. The partial derivatives $\partial/\partial\phi_1$, $\partial/\partial\phi_2$ on the algebra of Fourier series $\mathcal{A}_0 = C^\infty(\mathbb{T}^2)$ can be rewritten as

$$\begin{aligned} \delta_1(a_{rs}\, u^r v^s) &:= 2\pi i r\, a_{rs}\, u^r v^s, \\ \delta_2(a_{rs}\, u^r v^s) &:= 2\pi i s\, a_{rs}\, u^r v^s. \end{aligned} \tag{4.7}$$

These formulae also make sense on $\mathcal{A}_\theta$, and define *derivations* δ_1, δ_2, i.e.,

$$\delta_j(ab) = (\delta_j a)\, b + a\, (\delta_j b), \qquad j = 1, 2.$$

They are symmetric, i.e., $(\delta_j a)^* = \delta_j(a^*)$. Each δ_j extends to an unbounded operator on A_θ whose smooth domain is exactly $\mathcal{A}_\theta$. Notice that $\tau(\delta_1 a) = \tau(\delta_2 a) = 0$ for all a.

The two cyclic 1-cocycles we need are then given by:

$$\psi_1(a, b) := \tau_0(a\, \delta_1 b), \qquad \psi_2(a, b) := \tau_0(a\, \delta_2 b).$$

These are cocycles because δ_1, δ_2 are derivations:

$$b\psi_j(a, b, c) = \tau_0(ab\, \delta_j c - a\, \delta_j(bc) + a\, (\delta_j b)c) = 0.$$

It turns out [19] that $HC^1(\mathcal{A}_\theta) = \mathbb{C}[\psi_1] \oplus \mathbb{C}[\psi_2]$.

Next, there is a 2-cocycle obtained by promoting the trace τ_0 to a cyclic trilinear form:

$$S\tau_0(a, b, c) := \tau_0(abc).$$

In fact, one can always promote a cyclic m-cocycle on an algebra to a cyclic $(m+2)$-cocycle by the *periodicity operator* S of Connes [22, III,1.β]. For instance, for $m = 1$ we get

$$S\psi_j(a, b, c, d) := \psi_j(abc, d) - \psi_j(ab, cd) + \psi_j(a, bcd).$$

However, there is another cyclic 2-cocycle that is not in the range of S:

$$\phi(a, b, c) := \frac{1}{2\pi i} \tau_0(a\, \delta_1 b\, \delta_2 c - a\, \delta_2 b\, \delta_1 c). \tag{4.8}$$

Its cyclicity $\phi(a, b, c) = \phi(c, a, b)$ and the condition $b\phi = 0$ are easily verified. It turns out that $HC^2(\mathcal{A}_\theta) = \mathbb{C}[S\tau_0] \oplus \mathbb{C}[\phi]$.

For $m \geq 3$, the cohomology groups are stable under repeated application of S, i.e., $HC^m(\mathcal{A}_\theta) = S(HC^{m-2}(\mathcal{A}_\theta)) \simeq \mathbb{C} \oplus \mathbb{C}$. The inductive limit of these groups yields a $\mathbb{Z}_2$-graded ring $HP^0(\mathcal{A}_\theta) \oplus HP^1(\mathcal{A}_\theta)$ called *periodic cyclic cohomology*, with HP^0 generated by $[\tau_0]$ and $[\phi]$, while HP^1 is generated by $[\psi_1]$ and $[\psi_2]$. (This ring is the range of the Chern character in noncommutative topology: for that, we refer to [9, 89] or to [22, III].) In this way, the four cyclic cocycles defined above yield a complete description, in algebraic terms, of the homological structure of the noncommutative torus.

The cocycle (4.8) plays an important rôle in the theory of the integer quantum Hall effect [22, IV.6.γ]: for a comprehensive review, see Bellissard *et al* [5]. In essence, the Brillouin zone $\mathbb{T}^2$ of a periodic two-dimensional crystal may be replaced in the nonperiodic case by a *noncommutative Brillouin zone* that is none other than the algebra $\mathcal{A}_\theta$ (where θ is a magnetic flux in units of h/e). Provided that a certain parameter μ (the Fermi level) lies in a gap of the spectrum of the Hamiltonian, with corresponding spectral projector $E_\mu \in \mathcal{A}_\theta$, the Hall conductivity is given by the Kubo formula: $\sigma_H = (e^2/h)\, \phi(E_\mu, E_\mu, E_\mu)$. What happens is that $\phi(E_\mu, E_\mu, E_\mu) = \big\langle [\phi], [E_\mu] \big\rangle$ where the latter is an integer-valued pairing of $HP^0(\mathcal{A}_\theta)$ with $K_0(\mathcal{A}_\theta)$, so σ_H is predicted to be an integral multiple of e^2/h.

A family of geometries on the torus

We search now for suitable operators D so that $(\mathcal{A}_\theta, \mathcal{H}, D, J, \Gamma)$ is a two-dimensional geometry. Here Γ is the grading operator on $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ where $\mathcal{H}^+$, $\mathcal{H}^-$ are two copies of $L^2(\mathcal{A}_\theta, \tau_0)$. The known operators on $\mathcal{H}$ are

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}.$$

Also, in order that D be selfadjoint and anticommute with Γ, it must be of the form

$$D = \begin{pmatrix} 0 & \underline{\partial}^\dagger \\ \underline{\partial} & 0 \end{pmatrix},$$

for a suitable closed operator $\underline{\partial}$ on $L^2(\mathcal{A}_\theta, \tau_0)$.

The order-one axiom demands that $[\mathcal{D}, \pi(a)]$ commute with each $\pi^0(b) = J\pi(b^*)J^\dagger$, so that $[\mathcal{D}, \pi(a)] \in (\pi^0(\mathcal{A}_\theta))' = \pi(\mathcal{A}_\theta)''$, by Tomita's theorem. The regularity axiom and the finiteness property (3.8) now show that

$$[\mathcal{D}, \pi(a)] \in \pi(\mathcal{A}_\theta)'' \cap \bigcap_{k=1}^{\infty} \mathrm{Dom}(\delta^k) = \pi(\mathcal{A}_\theta).$$

We conclude that

$$[\mathcal{D}, \pi(a)] = \begin{pmatrix} 0 & [\underline{\partial}^\dagger, a] \\ [\underline{\partial}, a] & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial^* a \\ \partial a & 0 \end{pmatrix}$$

where ∂, ∂^* are linear maps of $\mathcal{A}_\theta$ into itself. Let us assume also that $\underline{\partial}(\underline{1}) = \underline{\partial}^\dagger(\underline{1}) = 0$. It follows that $\underline{\partial}(\underline{b}) = [\underline{\partial}, b](\underline{1}) = \underline{\partial b}$, and that

$$\tau_0(a^* \, \partial^* b) = \langle \underline{a} \mid \underline{\partial}^\dagger \, \underline{b} \rangle = \langle \underline{\partial} \, \underline{a} \mid \underline{b} \rangle = \tau_0((\partial a)^* \, b). \tag{4.9}$$

Since $[\mathcal{D}, \pi(ab)] = [\mathcal{D}, \pi(a)] \, \pi(b) + \pi(a) \, [\mathcal{D}, \pi(b)]$, the maps ∂, ∂^* are *derivations* of $\mathcal{A}_\theta$. Then $\partial(1) = 0$ and (4.9) shows that $\tau \circ \partial^* = 0$, so

$$\tau_0(b \, (\partial a)^*) = \tau_0((\partial^* b) \, a^*) = -\tau_0(b \, \partial^* a^*),$$

and therefore $(\partial a)^* = -\partial^* a^*$.

The reality condition $JDJ^\dagger = D$ is equivalent to the condition that $J_0 \, \underline{\partial} \, J_0 = -\underline{\partial}^\dagger$ on $\mathcal{H}_0 = L^2(\mathcal{A}_\theta, \tau_0)$. Since

$$[J_0 \, \underline{\partial} \, J_0, a^*] = (\partial a)^* = -\partial^* a^* = -[\underline{\partial}^\dagger, a^*],$$

the operator $J_0 \, \underline{\partial} \, J_0 + \underline{\partial}^\dagger$ commutes with $\mathcal{A}_\theta''$ and kills the cyclic vector $\underline{1}$, so it vanishes identically.

The derivation ∂_τ. For concreteness, we take ∂ to be a linear combination of the basic derivations δ_1, δ_2 of (4.7). Apart from a scale factor, the most general such derivation is

$$\partial = \partial_\tau := \delta_1 + \tau \delta_2 \qquad \text{with} \quad \tau \in \mathbb{C}. \tag{4.10}$$

(We shall soon see that real values of τ must be excluded.) Also, since we could replace ∂_τ by $\tau^{-1}\partial_\tau = \delta_2 + \tau^{-1}\delta_1$, we may assume that $\Im\tau > 0$. It follows from (4.9) that $\partial_\tau^* = -\delta_1 - \bar\tau\delta_2$.

To verify that this putative geometry is two-dimensional, we must check that $ds = D_\tau^{-1}$ is an infinitesimal of order $\frac{1}{2}$. Notice that $D_\tau^2 = \underline{\partial}_\tau^\dagger\underline{\partial}_\tau \oplus \underline{\partial}_\tau\underline{\partial}_\tau^\dagger$ and that the vectors $\underline{u^m v^n}$ form an orthonormal basis of eigenvectors for both $\underline{\partial}_\tau^\dagger\underline{\partial}_\tau$ and $\underline{\partial}_\tau\underline{\partial}_\tau^\dagger$. In fact,

$$\begin{aligned}\partial_\tau^*\partial_\tau(u^m v^n) = \partial_\tau\partial_\tau^*(u^m v^n) &= -(\delta_1 + \tau\delta_2)(\delta_1 + \bar\tau\delta_2)(u^m v^n) \\ &= 4\pi^2|m + n\tau|^2\, u^m v^n.\end{aligned}$$

Thus D_τ^{-2} has a discrete spectrum of eigenvalues $(4\pi^2)^{-1}|m + n\tau|^{-2}$, each with multiplicity 2, and hence is a compact operator. Now the Eisenstein series

$$G_{2k}(\tau) := \sum_{m,n}{}' \frac{1}{(m + n\tau)^{2k}},$$

with primed summation ranging over integer pairs $(m, n) \neq (0, 0)$, converges absolutely for $k > 1$ and only conditionally for $k = 1$. We shall see below that $\sum'_{m,n} |m + n\tau|^{-2}$ in fact diverges logarithmically, thereby establishing the two-dimensionality of the geometry.

The orientation cycle. In terms of the generators $u = e^{2\pi i\phi_1}$, $v = e^{2\pi i\phi_2}$ of $\mathcal{A}_0$, the usual volume form on the torus $\mathbb{T}^2$ is $d\phi_1 \wedge d\phi_2 = (2\pi i)^{-2}u^{-1}v^{-1}\, du \wedge dv$, with the corresponding Hochschild cycle

$$(2\pi i)^{-2}\big(u^{-1}v^{-1} \otimes u \otimes v - u^{-1}v^{-1} \otimes v \otimes u\big). \tag{4.11}$$

In $\mathcal{A}_\theta$, this formula must be modified since u^{-1} and v^{-1} do not commute. Consider $c \in C_2(\mathcal{A}_\theta, \mathcal{A}_\theta \otimes \mathcal{A}_\theta^0)$ of the form

$$c := m \otimes u \otimes v - n \otimes v \otimes u.$$

Then

$$bc = (mu \otimes v - m \otimes uv + vm \otimes u) - (nv \otimes u - n \otimes vu + un \otimes v),$$

so that c is a 2-cycle if and only if $mu = un$, $vm = nv$ and $m = \lambda n$ in $\mathcal{A}_\theta \otimes \mathcal{A}_\theta^0$. For instance, since $\mathcal{A}_\theta \simeq \mathcal{A}_\theta \otimes \pi^0(1) \subset \mathcal{A}_\theta \otimes \mathcal{A}_\theta^0$, we can take $m = \alpha v^{-1}u^{-1}$, $n = \alpha u^{-1}v^{-1}$ with a suitable constant $\alpha \in \mathbb{C}$.

The representative $\pi(c)$ on $\mathcal{H}$ satisfies

$$\begin{aligned}\alpha^{-1}\,\pi(c) &= \pi(v^{-1}u^{-1})\,[D_\tau, \pi(u)]\,[D_\tau, \pi(v)] - \pi(u^{-1}v^{-1})\,[D_\tau, \pi(v)]\,[D_\tau, \pi(u)] \\ &= \begin{pmatrix} v^{-1}u^{-1}\,\partial_\tau^* u\,\partial_\tau v - u^{-1}v^{-1}\,\partial_\tau^* v\,\partial_\tau u & 0 \\ 0 & v^{-1}u^{-1}\,\partial_\tau u\,\partial_\tau^* v - u^{-1}v^{-1}\,\partial_\tau v\,\partial_\tau^* u \end{pmatrix}.\end{aligned}$$

Since

$$\partial_\tau u = 2\pi i u, \quad \partial_\tau^* u = -2\pi i u, \quad \partial_\tau v = 2\pi i\tau v, \quad \partial_\tau^* v = -2\pi i\bar\tau v,$$

this reduces to

$$\pi(c) = 4\pi^2\alpha \begin{pmatrix} \tau - \bar\tau & 0 \\ 0 & \bar\tau - \tau \end{pmatrix} = 4\pi^2\alpha(\tau - \bar\tau)\,\Gamma.$$

Thus the *orientation cycle* is given by

$$c := \frac{1}{4\pi^2(\tau-\bar\tau)}(v^{-1}u^{-1}\otimes u\otimes v - u^{-1}v^{-1}\otimes v\otimes u). \tag{4.12}$$

This makes sense only if $\tau-\bar\tau\neq 0$, i.e., $\tau\notin\mathbb{R}$, which explains why we chose $\Im\tau>0$. Thus $(\Im\tau)^{-1}$ is a *scale factor* in the metric determined by D_τ. There is, however, a difference with the commutative volume form (4.11): since $v^{-1}u^{-1}=\lambda\, u^{-1}v^{-1}$, there is also a *phase factor* $\lambda = e^{2\pi i\theta}$ in the orientation cycle.

The area of the noncommutative torus. To determine the total area, we compute the coefficient of logarithmic divergence of the series given by $\operatorname{sp}(D_\tau^{-2})$. Partially summing over lattice points with $m^2+n^2\leq R^2$, we get, with $\tau = s+it$:

$$\begin{aligned}
⨍ D^{-2} &= \frac{2}{4\pi^2}\lim_{R\to\infty}\frac{1}{2\log R}\sum_{m^2+n^2\leq R^2}{}' \frac{1}{|m+n\tau|^2} \\
&= \frac{1}{4\pi^2}\lim_{R\to\infty}\frac{1}{\log R}\int_1^R \frac{r\,dr}{r^2}\int_{-\pi}^{\pi}\frac{d\theta}{(\cos\theta+s\sin\theta)^2+t^2\sin^2\theta} \\
&= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\frac{d\theta}{(\cos\theta+s\sin\theta)^2+t^2\sin^2\theta} \\
&= \frac{1}{4\pi^2}\left(\frac{2\pi}{t}\right) = \frac{i}{\pi(\tau-\bar\tau)},
\end{aligned}$$

after an unpleasant contour integration. The area is then $2\pi ⨍ D^{-2} = 1/\Im\tau$. This area is inversely proportional to the area of the period parallelogram of the elliptic curve E_τ with periods $\{1,\tau\}$.

A second method of computing the area relies on the existence of a Chern character homomorphism from K-homology (a classification ring for K-cycles) to cyclic cohomology: see [22, IV.1] for a brief discussion of this. The general theory suggests that the area will be given by pairing the orientation class $[c]\in H_2(\mathcal{A}_\theta,\mathcal{A}_\theta\bar\otimes\mathcal{A}_\theta^0)$ with a class $[\phi]\in HC^2(\mathcal{A}_\theta)$, where ϕ is the cocycle (4.8) that represents the highest level of the skeleton of the torus. (This is the image under the Gelfand cofunctor of the familiar process of integrating the volume form over the fundamental cycle of the manifold.) At the level of cocycles and cycles, the pairing is defined by $\langle\phi, a_0\otimes a_1\otimes a_2\rangle := \phi(a_0,a_1,a_2)$. Thus

$$\begin{aligned}
\langle\phi,c\rangle &= \frac{1}{4\pi^2(\tau-\bar\tau)}\big(\phi(v^{-1}u^{-1},u,v)-\phi(u^{-1}v^{-1},v,u)\big) \\
&= \frac{(2\pi i)^{-1}}{4\pi^2(\tau-\bar\tau)}\tau_0\big(v^{-1}u^{-1}(\delta_1 u\,\delta_2 v-\delta_2 u\,\delta_1 v) - u^{-1}v^{-1}(\delta_1 v\,\delta_2 u - \delta_2 v\,\delta_1 u)\big) \\
&= \frac{2\pi i}{4\pi^2(\tau-\bar\tau)}\tau_0(v^{-1}u^{-1}uv + u^{-1}v^{-1}vu) = \frac{i}{\pi(\tau-\bar\tau)} = ⨍ D_\tau^{-2}.
\end{aligned}$$

K-theory and Poincaré duality. The noncommutative torus provides an example of a pre-C^*-algebra, which is neither commutative nor approximately finite but has an interesting and computable K-theory [98]. The group $K_1(\mathcal{A}_\theta)$ is fairly easy to find. There are two generating unitaries, u and v, and all the $u^m v^n$ are mutually non-homotopic in $U(A_\theta)$. Indeed, since $\tau_0(1) = 1$ and $\tau_0(u^m v^n) = 0$ for $(m,n) \neq (0,0)$, there cannot be a continuous path in $U(A_\theta)$ from 1 to $u^m v^n$. Passing to matrices in $M_k(A_\theta)$ cannot remedy this, since the same argument works, with τ_0 replaced by $\tau_0 \otimes \mathrm{tr}$. Thus $K_1(\mathcal{A}_\theta) = K_1(A_\theta) = \mathbb{Z}[u] \oplus \mathbb{Z}[v]$.

To determine $K_0(\mathcal{A}_\theta)$, we seek projectors in $M_k(\mathcal{A}_\theta)$ not equivalent to $e := 1 \oplus 0_{k-1}$. In fact, due to the irrationality of θ, such projectors may be found in $\mathcal{A}_\theta$ itself: the *Powers–Rieffel projector* $p \in \mathcal{A}_\theta$ has the characteristic property that $\tau_0(p) = \theta$. Given this projector, the map $m[e] + n[p] \mapsto m\tau_0(e) + n\tau_0(p) = m + n\theta$ defines a map from $K_0(\mathcal{A}_\theta)$ to $\mathbb{Z} + \mathbb{Z}\theta$ which, by a theorem of Pimsner and Voiculescu [94], is an isomorphism of ordered groups.

The projector p is constructed as follows. Write elements of $\mathcal{A}_\theta$ as $a = \sum_s f_s v^s$, where $f_s = \sum_r a_{rs} u^r$ is a Fourier series expansion of $f_s(t) = \sum_r a_{rs} e^{2\pi i r t}$ in $C^\infty(\mathbb{T})$. Now we look for p of the form

$$p = gv + f + hv^{-1}.$$

Since $p^* = p$, the function f is real and $\overline{h(t)} = g(t+\theta)$. Assuming $\frac{1}{2} < \theta < 1$, as we may, we choose f to be a smooth increasing function on $[0, 1-\theta]$, define $f(t) := 1$ if $t \in [1-\theta, \theta]$, and $f(t) := 1 - f(t-\theta)$ if $t \in [\theta, 1]$; then let g be the smooth bump function supported in $[\theta, 1]$ given by $g(t) := \sqrt{f(t) - f(t)^2}$ for $\theta \leq t \leq 1$. One checks that these conditions guarantee $p^2 = p$ (look at the coefficients of v^2, v and 1 in the expansion of p^2). Moreover,

$$\tau_0(p) = a_{00} = \int_0^1 f(t)\,dt = \int_0^{1-\theta} f(t)\,dt + (2\theta - 1) + \int_\theta^1 f(t)\,dt = \theta.$$

The existence of these projectors (variation of f on the interval $[0, 1-\theta]$ gives rise to many homotopic projectors) shows that the topology of the noncommutative torus is very disconnected, in contrast to the ordinary torus $\mathcal{A}_0$, whose only projectors are 0 and 1.

[The commutative torus $C^\infty(\mathbb{T}^2)$ has the same K-groups: $K_j(\mathcal{A}_0) = \mathbb{Z} \oplus \mathbb{Z}$ for $j = 1, 2$. However, this algebra has no Powers–Rieffel projector: the second generator of $K_0(\mathcal{A}_0)$ is obtained by pulling back the Bott projector from $K_0(C^\infty(\mathbb{S}^2))$.]

Thus, $K_\bullet(\mathcal{A}_\theta)$ has four generators: $[e]$, $[p]$, $[u]$ and $[v]$. The intersection form is antisymmetric (this is typical of dimension two) and the nonzero pairings of the generators are [25]:

$$\langle [e],[p] \rangle = -\langle [p],[e] \rangle = 1, \qquad \langle [u],[v] \rangle = -\langle [v],[u] \rangle = 1.$$

The matrix of the form is a direct sum of two $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ blocks, so it is nondegenerate, and Poincaré duality holds.

We have constructed a geometry $(\mathcal{A}_\theta, \mathcal{H}, D_\tau, \Gamma, J)$, which may be denoted by $\mathbb{T}^2_{\theta,\tau}$. When $\theta = 0$, τ plays the rôle of the modular parameter of an elliptic curve. Whether or not this provides an opening to a noncommutative theory of elliptic curves is a tantalizing speculation; it remains to be seen whether $\mathbb{T}^2_{\theta,\tau}$ can yield useful arithmetic information.

5. The Noncommutative Integral

One of the striking features of noncommutative geometry is how it ties together many mathematical and physical approaches in a single unifying theme. So far, our discussion of the mathematical underpinnings of the geometry has been quite "soft", emphasizing the algebraic character of the overall picture. However, when we examine more closely the interpretation of differential and integral calculus, we see that the theory requires a considerable amount of analysis, of a fairly delicate nature. In this chapter we take up the matter of how to best define the noncommutative integral and relate it to conventional integration on manifolds.

In the course of the initial development of noncommutative geometry, integration came first, beginning with Segal's early work with traces on operator algebras [109] and continuing with Connes' work on foliations [18]. The introduction of universal graded differential algebras [19] shifted the emphasis to differential calculus based on derivations, which formed the backdrop for the first applications to particle physics [28, 39]. The pendulum has recently swung back to integral methods, due to the realization [14, 25, 65] that the Yang–Mills functionals could be obtained in this way. The primary tool for that is the noncommutative integral.

The Dixmier trace on infinitesimals

Early attempts at noncommutative integration [109] used the ordinary trace Tr of operators on a Hilbert space as an ersatz integral, where the traceclass operators play the rôle of integrable functions. For example, in Moyal quantization one computes expectation values by $\mathrm{Tr}(AB) = \int W_A W_B \, d\mu$, where W_A, W_B are Wigner functions of operators A, B and μ is the normalized Liouville measure on phase space [12]. However, on the representation space of a noncommutative geometry one needs an integral that suppresses infinitesimals of order higher than 1, so Tr will not do; moreover, Tr diverges for positive first-order infinitesimals, since

$$\mathrm{Tr}\,|T| = \sum_{k=0}^{\infty} \mu_k(T) = \lim_{n\to\infty} \sigma_n(T) = \infty \qquad \text{if} \quad \sigma_n(T) = O(\log n).$$

Dixmier [37] found other (non-normal) tracial functionals on compact operators that are more suitable for our purposes: they are finite on first order infinitesimals and vanish on those of higher order. To find them, we must look more closely at the fine structure of infinitesimal operators; our treatment follows [29, Appendix A].

The algebra $\mathcal{K}$ of compact operators on a separable, infinite-dimensional Hilbert space contains the ideal $\mathcal{L}^1$ of traceclass operators, on which $\|T\|_1 := \mathrm{Tr}\,|T|$ is a norm —not to be confused with the operator norm $\|T\| = \mu_0(T)$. Each partial sum of singular values σ_n is a norm on $\mathcal{K}$. In fact,

$$\sigma_n(T) = \sup\{\, \|TP_n\|_1 : P_n \text{ is a projector of rank } n \,\}.$$

Notice that $\sigma_n(T) \le n\,\mu_0(T) = n\|T\|$. There is a very cute formula [29], coming from real interpolation theory of Banach spaces, that combines the last two relations:

$$\sigma_n(T) = \inf\{\, \|R\|_1 + n\|S\| : R, S \in \mathcal{K},\ R + S = T\,\}. \tag{5.1}$$

This is worth checking. It is clear that if $T = R + S$, then $\sigma_n(T) \le \sigma_n(R) + \sigma_n(S) \le \|R\|_1 + n\|S\|$. To show that the infimum is attained, we can assume that T is a positive operator, since both sides of (5.1) are unchanged if R, S, T are multiplied on the left by a unitary operator V such that $VT = |T|$. Now let P_n be the projector of rank n whose range is spanned by the eigenvectors of T corresponding to the eigenvalues $\mu_0, \dots, \mu_{n-1}$. Then $R := (T - \mu_n)P_n$ and $S := \mu_n P_n + T(1 - P_n)$ satisfy $\|R\|_1 = \sum_{k<n}(\mu_k - \mu_n) = \sigma_n(T) - n\mu_n$ and $\|S\| = \mu_n$.

We can think of $\sigma_n(T)$ as the trace of $|T|$ with a *cutoff* at the scale n. This scale does not have to be an integer; for any scale $\lambda > 0$, we can *define*

$$\sigma_\lambda(T) := \inf\{\, \|R\|_1 + \lambda\|S\| : R, S \in \mathcal{K},\ R + S = T\,\}.$$

If $0 < \lambda \le 1$, then $\sigma_\lambda(T) = \lambda\|T\|$. If $\lambda = n + t$ with $0 \le t < 1$, one checks that

$$\sigma_\lambda(T) = (1 - t)\sigma_n(T) + t\sigma_{n+1}(T), \tag{5.2}$$

so $\lambda \mapsto \sigma_\lambda(T)$ is a piecewise linear, increasing, concave function on $(0, \infty)$.

Each σ_λ is a norm by (5.2), and so satisfies the triangle inequality. For *positive* compact operators, there is a triangle inequality in the opposite direction:

$$\sigma_\lambda(A) + \sigma_\mu(B) \le \sigma_{\lambda+\mu}(A + B) \qquad \text{if} \quad A, B \ge 0. \tag{5.3}$$

It suffices to check this for integral values $\lambda = m$, $\mu = n$. If P_m, P_n are projectors of respective ranks m, n, and if $P = P_m \vee P_n$ is the projector with range $P_m\mathcal{H} + P_n\mathcal{H}$, then

$$\|AP_m\|_1 + \|BP_n\|_1 = \mathrm{Tr}(P_m A P_m) + \mathrm{Tr}(P_n B P_n) \le \mathrm{Tr}(P(A + B)P) = \|(A + B)P\|_1,$$

and (5.3) follows by taking suprema over P_m, P_n. Thus we have a sandwich of norms:

$$\sigma_\lambda(A + B) \le \sigma_\lambda(A) + \sigma_\lambda(B) \le \sigma_{2\lambda}(A + B) \qquad \text{if} \quad A, B \ge 0. \tag{5.4}$$

The Dixmier ideal. The *first-order infinitesimals* can now be defined precisely as the following normed ideal of compact operators:

$$\mathcal{L}^{1+} := \Big\{\, T \in \mathcal{K} : \|T\|_{1+} := \sup_{\lambda \ge e} \frac{\sigma_\lambda(T)}{\log \lambda} < \infty \,\Big\},$$

that obviously includes the traceclass operators $\mathcal{L}^1$. (On the other hand, if $p > 1$ we have $\mathcal{L}^{1+} \subset \mathcal{L}^p$, where the latter is the ideal of those T such that $\mathrm{Tr}\,|T|^p < \infty$, for which $\sigma_\lambda(T) = O(\lambda^{1-1/p})$.)

If $T \in \mathcal{L}^{1+}$, the function $\lambda \mapsto \sigma_\lambda(T)/\log\lambda$ is *continuous and bounded* on the interval $[e,\infty)$, i.e., it lies in the C^*-algebra $C_b[e,\infty)$. We can then form the following *Cesàro mean* of this function:

$$\tau_\lambda(T) := \frac{1}{\log\lambda}\int_e^\lambda \frac{\sigma_u(T)}{\log u}\,\frac{du}{u}.$$

Then $\lambda \mapsto \tau_\lambda(T)$ lies in $C_b[e,\infty)$ also, with upper bound $\|T\|_{1+}$. From (5.4) we can derive that

$$0 \le \tau_\lambda(A) + \tau_\lambda(B) - \tau_\lambda(A+B) \le \big(\|A\|_{1+} + \|B\|_{1+}\big)\,\log 2\,\frac{\log\log\lambda}{\log\lambda},$$

so that τ_λ is "asymptotically additive" on positive elements of $\mathcal{L}^{1+}$.

We get a true additive functional in two more steps. Firstly, let $\dot\tau(A)$ be the class of $\lambda \mapsto \tau_\lambda(A)$ in the quotient C^*-algebra $\mathcal{B} := C_b[e,\infty)/C_0[e,\infty)$. Then $\dot\tau$ is an additive, positive-homogeneous map from the positive cone of $\mathcal{L}^{1+}$ into $\mathcal{B}$, and $\dot\tau(UAU^{-1}) = \dot\tau(A)$ for any unitary U; therefore it extends to a *linear* map $\dot\tau\colon \mathcal{L}^{1+} \to \mathcal{B}$ such that $\dot\tau(ST) = \dot\tau(TS)$ for $T \in \mathcal{L}^{1+}$ and any S.

Secondly, we follow $\dot\tau$ with any state (i.e., normalized positive linear form) $\omega\colon \mathcal{B} \to \mathbb{C}$. The composition is a *Dixmier trace*:

$$\operatorname{Tr}_\omega(T) := \omega(\dot\tau(T)).$$

The noncommutative integral. Unfortunately, the C^*-algebra $\mathcal{B}$ is not separable and there is no way to *exhibit* any particular state. This problem can be finessed by noticing that a function $f \in C_b[e,\infty)$ has a limit $\lim_{\lambda\to\infty} f(\lambda) = c$ if and only if $\omega(f) = c$ does not depend on ω. Let us say that an operator $T \in \mathcal{L}^{1+}$ is *measurable* if the function $\lambda \mapsto \tau_\lambda(T)$ converges as $\lambda \to \infty$, in which case any $\operatorname{Tr}_\omega(T)$ equals its limit. We denote by $⨍ T$ the common value of the Dixmier traces:

$$⨍ T := \lim_{\lambda\to\infty} \tau_\lambda(T) \quad \text{if this limit exists.}$$

We call this value the *noncommutative integral* of T.

Note that if $T \in \mathcal{K}$ and $\sigma_n(T)/\log n$ converges as $n \to \infty$, then T lies in $\mathcal{L}^{1+}$ and is measurable. This was shown to be the case for the operators $D\!\!\!/^{-2}$ on the Riemann sphere and D^{-2} on the noncommutative torus, whose integrals we have already computed.

We need to do at least one integral calculation in an n-dimensional context. Suppose we try the (commutative!) torus $\mathbb{T}^n := \mathbb{R}^n/(2\pi\mathbb{Z})^n$ and consider its Laplacian

$$\Delta := -\Big(\frac{\partial}{\partial x^1}\Big)^2 - \cdots - \Big(\frac{\partial}{\partial x^n}\Big)^2. \qquad 5.5$$

Its eigenfunctions are $\phi_l(x) := e^{il\cdot x}$ for $l \in \mathbb{Z}^n$. We discard the zero mode ϕ_0 and regard Δ as an invertible operator on the orthogonal complement of the constants in $L^2(\mathbb{T}^n)$. The multiplicity m_λ of the eigenvalue $\lambda = |l|^2$ is the number of lattice points in $\mathbb{Z}^n$ of length $|l|$. Thus the operator Δ^{-s} is compact for any $s > 0$; let us compute $⨍ \Delta^{-s}$.

If N_r is the total number of lattice points with $|l| \leq r$, then $N_{r+dr} - N_r \sim \Omega_n \, r^{n-1} \, dr$ and $N_r \sim n^{-1} \Omega_n \, r^n$, where

$$\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \operatorname{vol}(\mathbb{S}^{n-1})$$

is the volume of the unit sphere. For $N = N_R$ we estimate

$$\sigma_N(\Delta^{-s}) = \sum_{1 \leq |l| \leq R} |l|^{-2s} \sim \int_1^R r^{-2s} \, (N_{r+dr} - N_r) \sim \Omega_n \int_1^R r^{n-2s-1} \, dr.$$

Since $\log N_R \sim n \log R$, we see that $\sigma_\lambda(\Delta^{-s})/\log\lambda$ diverges if $s < n/2$ and converges to 0 if $s > n/2$. For the borderline case $s = n/2$, we get $\sigma_\lambda(\Delta^{-n/2})/\log\lambda \sim \Omega_n/n$, so $\tau_\lambda(\Delta^{-n/2}) \sim \Omega_n/n$ also, and thus

$$⨍ \Delta^{-n/2} = \frac{\Omega_n}{n} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}. \tag{5.6}$$

The Dirac operator $D\!\!\!\!/ = \gamma(dx^j)\,\partial/\partial x^j$ on $\mathbb{T}^n$ satisfies $D\!\!\!\!/^2 = \Delta$, so we may rewrite (5.6) as $⨍ ds^n = ⨍ |D\!\!\!\!/|^{-n} = \Omega_n/n$.

Pseudodifferential operators

In all cases considered up to now, the computation of $⨍ T$ has required a complete determination of the spectrum of T. This is usually a fairly onerous task and is most suited to fairly simple examples. An alternative approach is needed, which may allow us to calculate $⨍ T$ by a general procedure.

For the commutative case, such a procedure is available: it is the pseudodifferential operator calculus. The extension of this calculus to the noncommutative case has already begun and is undergoing rapid development [23, 26, 29], but we cannot report on it here. We shall confine our attention to a fairly familiar case: elliptic classical pseudodifferential operators (ΨDOs) on compact Riemannian manifolds.

Definition. A *pseudodifferential operator A of order d* on a manifold M is an operator between two Hilbert spaces of sections of Hermitian vector bundles over M, that can be written in local coordinates as

$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a(x,\xi) u(y) \, d^n y \, d^n \xi,$$

where the *symbol* $a(x,\xi)$ is a matrix of smooth functions whose derivatives satisfy the growth conditions $|\partial_x^\alpha \partial_\xi^\beta a_{ij}(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{d-|\beta|}$. We simplify a little by assuming that a is scalar-valued, i.e., that the bundles are line bundles. We then say that A is a *classical* ΨDO, written $A \in \Psi^d(M)$, if its symbol has an asymptotic expansion of the form

$$a(x,\xi) \sim \sum_{j=0}^{\infty} a_{d-j}(x,\xi) \tag{5.7}$$

where each $a_r(x,\xi)$ is *r-homogeneous* in the variable ξ, that is, $a_r(x,t\xi) \equiv t^r\, a_r(x,\xi)$. We refer to [83, 113] for the full story of these operators. Although the terms of the expansion (5.7) are generally coordinate-dependent, the *principal symbol* $a_d(x,\xi)$ is globally defined as a function on the cotangent bundle T^*M, except possibly on the zero section M. We call the operator A *elliptic* if $a_d(x,\xi)$ is invertible for $\xi \neq 0$.

The spaces $\Psi^d(M)$ are decreasingly nested, the intersection being the *smoothing* operators $\Psi^\infty(M)$. Clearly, the symbol a determines the operator A up to a smoothing operator. The quotient algebra $\mathcal{P} := \Psi^{-\infty}(M)/\Psi^\infty(M)$ is called, a little improperly, the *algebra of classical pseudodifferential operators* on M. The product $AB = C$ of ΨDOs corresponds to the composition of symbols given by the expansion

$$c(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{(-i)^{|\alpha|}}{\alpha!}\, \partial_\xi^\alpha a\, \partial_x^\alpha b.$$

From the leading term, we see that if $A \in \Psi^d(M)$, $B \in \Psi^r(M)$, then $AB \in \Psi^{d+r}(M)$. Also, if $P = [A,B]$ is a commutator in $\Psi^{-\infty}(M)$, then

$$p(x,\xi) \sim \sum_{|\alpha|>0} \frac{(-i)^{|\alpha|}}{\alpha!}\, \big(\partial_\xi^\alpha a\, \partial_x^\alpha b - \partial_\xi^\alpha b\, \partial_x^\alpha a\big). \tag{5.8}$$

The Wodzicki residue

If M is an n-dimensional manifold, the term of order $(-n)$ of the expansion (5.7) has a special significance. It is coordinate-dependent, so let us fix a coordinate domain $U \subset M$ over which the cotangent bundle is trivial, and consider $a_{-n}(x,\xi)$ as a smooth function on $T^*U \setminus U$ (i.e., we omit the zero section). Then

$$\alpha := a_{-n}(x,\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n \wedge dx^1 \wedge \cdots \wedge dx^n$$

is invariant under the dilations $\xi \mapsto t\xi$ of the cotangent spaces. Thus, if $R = \sum_j \xi_j\, \partial/\partial\xi_j$ is the radial vector field on $T^*U \setminus U$ that generates these dilations, then

$$d\,\iota_R\alpha = \mathcal{L}_R\, \alpha = 0,$$

so $\iota_R\alpha$ is a closed $(2n-1)$-form on $T^*U \setminus U$. (Here $\mathcal{L}_R = \iota_R\, d + d\,\iota_R$ denotes the Lie derivative.) On abbreviating $d^nx := dx^1 \wedge \cdots \wedge dx^n$, we find that

$$\iota_R\alpha = a_{-n}(x,\xi)\, \sigma_\xi \wedge d^nx, \quad \text{with} \quad \sigma_\xi := \sum_{j=1}^n (-1)^{j-1} \xi_j\, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n.$$

Of course, σ_ξ restricts to the volume form on the unit sphere $|\xi| = 1$ in each T_x^*M. On integrating $\iota_R\alpha$ over these spheres, we get a quantity that transforms under coordinate changes $x \mapsto y = \phi(x)$, $\xi \mapsto \eta = \phi'(x)^t\xi$, $a_{-n}(x,\xi) \mapsto \tilde{a}_{-n}(y,\eta)$ as follows [47]:

$$\int_{|\eta|=1} \tilde{a}_{-n}(y,\eta)\, \sigma_\eta = |\det\phi'(x)| \int_{|\xi|=1} a_{-n}(x,\xi)\, \sigma_\xi. \tag{5.9}$$

The absolute value of the Jacobian $\det\phi'(x)$ appears here because if $\phi'(x)^t$ reverses the orientation on the unit sphere in T_x^*M then the integral over the sphere also changes sign.

The Wresidue density. As a consequence of (5.9), we get a 1-*density* on M, denoted $\operatorname{wres} A$, whose local expression on any coordinate chart is

$$\operatorname{wres}_x A := \left(\int_{|\xi|=1} a_{-n}(x,\xi)\, \sigma_\xi \right) dx^1 \wedge \cdots \wedge dx^n.$$

This is the *Wodzicki residue density*. By integrating this 1-density over M, we get the **Wodzicki residue** [47, 70, 122]:

$$\operatorname{Wres} A := \int_M \operatorname{wres} A = \int_{\mathbb{S}^* M} \iota_R \alpha = \int_{\mathbb{S}^* M} a_{-n}(x,\xi)\, \sigma_\xi \, d^n x. \tag{5.10}$$

Here $\mathbb{S}^*M := \{\, (x,\xi) \in T^*M : |\xi| = 1 \,\}$ is the "cosphere bundle" over M. The integral (5.10) may diverge for some A; we shall shortly identify its domain.

(In the literature, this Wresidue is commonly written $\operatorname{res} A$; we adjoin the W —for Wodzicki— to distinguish the density from the functional.)

The tracial property of the Wresidue. It turns out that Wres is a *trace* on the algebra $\mathcal{P}$ of classical pseudodifferential operators, i.e., that $\operatorname{Wres}[A,B] = 0$ always, provided that $\dim M > 1$. The reason is that each term in the expansion is a finite sum of derivatives $\partial p/\partial x^j + \partial q/\partial \xi_j$. For instance, the leading term of (5.8) is

$$-i\left(\frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x^j} - \frac{\partial b}{\partial \xi_j}\frac{\partial a}{\partial x^j} \right) = \frac{\partial}{\partial x^j}\left(ia\frac{\partial b}{\partial \xi_j} \right) - \frac{\partial}{\partial \xi_j}\left(ia\frac{\partial b}{\partial x^j} \right).$$

We can assume that a, b are supported on a compact subset of a chart domain U of M (since we can later patch together with a partition of unity), so that all $(-n)$-homogeneous terms of type $\partial p/\partial x^j$ have zero integral over $\mathbb{S}^*M$. Since we are integrating a closed $(2n-1)$-form over $\mathbb{S}^{n-1} \times U$, we get the same result by integration over the cylinder $\mathbb{S}^{n-2} \times \mathbb{R} \times U$: these are cohomologous cycles in $(\mathbb{R}^n \setminus \{0\}) \times U$. For any term of the form $\partial q/\partial \xi_j$, where q is $(-n+1)$-homogeneous, we then get

$$\int_{|\xi|=1} \frac{\partial q}{\partial \xi_j} = \pm \int_{|\xi'|=1} \int_{-\infty}^{\infty} \frac{\partial q}{\partial \xi_j}\, d\xi_j\, \sigma_{\xi'} = 0$$

if $\xi' := (\xi_1, \dots, \widehat{\xi_j}, \dots, \xi_n)$, since $q(x,\xi) \to 0$ as $\xi_j \to \pm\infty$ because $-n+1 < 0$.

The crucial property of Wres is that, up to scalar multiples, it is the **unique** trace on the algebra $\mathcal{P}$. We give the gist of the beautiful elementary proof of this by Fedosov *et al* [47]. From the symbol calculus, derivatives are commutators, since

$$[x^j, a] = i\frac{\partial a}{\partial \xi_j}, \qquad [\xi_j, a] = -i\frac{\partial a}{\partial x^j}$$

in view of (5.8). Hence any trace T on symbols must vanish on derivatives. For $r \neq -n$, each r-homogeneous term $a_r(x,\xi)$ is a derivative, since $\partial/\partial \xi_j(\xi_j a_r) = (n+r)a$ by Euler's theorem. Furthermore, after averaging over spheres, $\overline{a}_{-n}(x) := \Omega_n^{-1} \int_{|\xi|=1} a_{-n}(x,\xi)\,\sigma_\xi$, one can show that the centred $(-n)$-homogeneous term

$$a_{-n}(x,\xi) - \overline{a}_{-n}(x)\,|\xi|^{-n}$$

is a finite sum of derivatives. The upshot is that $T(a) = T(\overline{a}_{-n}(x)\,|\xi|^{-n})$ is a linear functional of $\overline{a}_{-n}(x)$ that kills derivatives, so it must be of the form

$$T(a) = C \int_U \overline{a}_{-n}(x)\,d^n x = C\ \operatorname{Wres} A.$$

For more general classical ΨDOs with matrix-valued symbols, the same arguments are applicable, provided we replace $a_{-n}(x,\xi)$ by its matrix trace $\operatorname{tr} a_{-n}(x,\xi)$ throughout. The formula

$$\operatorname{Wres} A := \int_{\mathbb{S}^* M} \operatorname{tr} a_{-n}(x,\xi)\,\sigma_\xi\,dx^1 \wedge \cdots \wedge dx^n \tag{5.11}$$

then defines a unique (up to multiples) trace on the algebra of classical ΨDOs whose coefficients are endomorphisms of a given vector bundle over M.

The trace theorem

This uniqueness of the trace was exploited by Connes [20], who saw how the Dixmier traces fit into this picture. The point is that pseudodifferential operators of low enough order over a compact manifold are already compact operators [113], so that any Dixmier trace Tr_ω defines a trace on $\Psi^{-n}(M)$; and they all define the same trace, since they coincide on measurable operators. Thus all ΨDOs of order $(-n)$ are measurable, and the noncommutative integral is a multiple of Wres. It remains only to compute the proportionality constant. Moreover, since we can reduce to local calculations by patching with partitions of unity, this constant must be the same for all manifolds of a given dimension.

To find it, we can use the power $\Delta^{-n/2}$ of the Laplacian on the torus $\mathbb{T}^n$, whose noncommutative integral we already know (5.6). Now Δ is a differential operator (5.5), with symbol $b(x,\xi) = |\xi|^2$. Thus $\Delta^{-n/2}$ is a ΨDO with symbol $|\xi|^{-n}$, which is of course $(-n)$-homogeneous; and better yet, $|\xi|^{-n} \equiv 1$ on the cosphere bundle $\mathbb{S}^*M$. Thus

$$\operatorname{Wres} \Delta^{-n/2} = \int_{\mathbb{T}^n} \int_{|\xi|=1} \sigma_\xi\,d^n x = (2\pi)^n\,\Omega_n.$$

On comparing (5.6), we see that the proportionality constant is $1/n(2\pi)^n$, that is,

$$⨍ A = \frac{1}{n\,(2\pi)^n} \operatorname{Wres} A, \tag{5.12}$$

for any ΨDO of order $(-n)$ or lower. This is Connes' trace theorem [20].

We remark that the Wodzicki residue is sometimes written with a factor $n\,(2\pi)^n$ so that the noncommutative and the adjusted Wresidue coincide; that was the convention adopted in [117].

The commutative integral. Suppose that M is an n-dimensional spin manifold, with Dirac operator $D\!\!\!/$. The operator $|D\!\!\!/|^{-n}$ is a first-order infinitesimal and is also a pseudodifferential operator of order $(-n)$. Indeed, $D\!\!\!/$ acts on spinors with the symbol $i\,\gamma(\xi)$ where γ denotes the spin representation, so $D\!\!\!/^2$ has symbol $g^{-1}(\xi,\xi)\,1_N$, where g denotes the Riemannian metric; this is a scalar matrix whose size N is the rank of the spinor bundle. Recall that $N = 2^m$ if $n = 2m$ or $n = 2m+1$ (see the discussion of spin^c structures in §1). The symbol of $|D\!\!\!/|^{-n} = (D\!\!\!/^2)^{-n/2}$ is then given locally by

$$\sqrt{\det g(x)}\,|\xi|^{-n}\,1_N.$$

More generally, when $a \in C^\infty(M)$ is represented as a multiplication operator on the spinor space $\mathcal{H}$, the operator $a\,|D\!\!\!/|^{-n}$ is also pseudodifferential of order $(-n)$, with symbol $a_{-n}(x,\xi) := a(x)\sqrt{\det g(x)}\,|\xi|^{-n}\,1_N$. Let us be mindful that the Riemannian volume form is $\Omega = \sqrt{\det g(x)}\,dx^1 \wedge \cdots \wedge dx^n$. Invoking (5.12) and (5.11), we end up with

$$\begin{aligned} ⨍ a\,|D\!\!\!/|^{-n} &= \frac{1}{n\,(2\pi)^n}\,\mathrm{Wres}\,a\,|D\!\!\!/|^{-n} = \frac{1}{n\,(2\pi)^n}\int_{\mathbb{S}^*M} \mathrm{tr}\,a_{-n}(x,\xi)\,\sigma_\xi\,d^nx \\ &= \frac{2^{\lfloor n/2\rfloor}\,\Omega_n}{n\,(2\pi)^n}\int_M a(x)\,\Omega. \end{aligned}$$

Thus the *commutative integral* on functions, determined by the orientation $[\Omega]$, is

$$\int_M a\,\Omega = \begin{cases} m!\,(2\pi)^m ⨍ a\,D\!\!\!/^{-2m} & \text{if } \dim M = 2m \text{ is even,} \\ (2m+1)!!\,\pi^{m+1} ⨍ a\,|D\!\!\!/|^{-2m-1} & \text{if } \dim M = 2m+1 \text{ is odd.} \end{cases}$$

In particular, since orientable 2-dimensional manifolds always admit a spin structure [54], the area of a surface ($m = 1$) is $2\pi ⨍ D\!\!\!/^{-2}$, as we had previously claimed.

Integrals and zeta residues

We have not explained why the Wodzicki functional is called a residue. It was originally discovered as the Cauchy residue of a zeta function: see Wodzicki's introductory remarks in [122]. Indeed, the following formula can be established [117] with the help of Seeley's symbol calculus for complex powers of an elliptic ΨDO [108]:

$$\mathrm{Res}_{s=1}\,\zeta_A(s) = \frac{1}{n\,(2\pi)^n}\,\mathrm{Wres}\,A,$$

where the zeta function of a positive compact operator A with eigenvalues $\lambda_k(A)$ may be defined as

$$\zeta_A(s) := \sum_{k=1}^\infty \lambda_k(A)^s \qquad \text{for} \quad \Re s > \text{some } s_0$$

and extended to a meromorphic function on $\mathbb{C}$ by analytic continuation.

In view of (5.12), it should be possible to prove directly that this zeta residue actually coincides with the noncommutative integral of measurable infinitesimals:

$$\operatorname{Res}_{s=1} \zeta_A(s) = \fint A. \tag{5.13}$$

This can indeed be achieved by known Tauberian theorems; see, for instance, [22, IV.2.β, Proposition 4] or [117, §2]. Rather than repeat these technical proofs here, we give instead a heuristic argument based on the delta-function calculus of [45], that shows why (5.13) is to be expected.

Let us take A to be a compact positive operator whose eigenvalues satisfy $\lambda_k(A) \sim L/k$ as $k \to \infty$. Then $\sigma_n(A) \sim L \log n$, so that A is measurable with $\fint A = L$. In the particular case of the operator R for which $\lambda_k(R) = 1/k$ for all k, $\zeta_R(s)$ is precisely the Riemann zeta function.

On the other hand, let us examine an interesting distribution on $\mathbb{R}$, the "Dirac comb" $\sum_{k\in\mathbb{Z}} \delta(x-k)$. It is periodic and its mean value is 1; therefore $f(x) := \sum_{k\in\mathbb{Z}} \delta(x-k) - 1$ is a periodic distribution of mean zero. It can then be shown [44] that the *moments* $\mu_m := \int_{-\infty}^{\infty} f(x)\, x^m \, dx$ exist for all m. The same is true if we cut off the x-axis at $x = 1$: the distribution

$$f_R(x) := \sum_{k=1}^{\infty} \delta(x-k) - \theta(x-1)$$

has moments of all orders, and the function

$$Z_R(s) := \int_{\mathbb{R}} f_R(x)\, x^{-s} \, dx = \sum_{k=1}^{\infty} \frac{1}{k^s} - \int_1^{\infty} x^{-s} \, dx = \zeta_R(s) - \frac{1}{s-1}$$

is an entire analytic function of s. (Notice how this argument shows that ζ_R is meromorphic with a single simple pole at $s = 1$, whose residue is 1.)

Now since $L/\lambda_k(A) \sim k$ as $k \to \infty$, we replace $\delta(x-k)$ by $\delta(x - L/\lambda_k)$ and define

$$f_A(x) := \sum_{k=1}^{\infty} \delta(x - L/\lambda_k) - \theta(x-1).$$

This suggests that

$$Z_A(s) := \int_{\mathbb{R}} f_A(x)\, x^{-s} \, dx = L^{-s} \sum_{k=1}^{\infty} \lambda_k^s - \int_1^{\infty} x^{-s} \, dx = L^{-s} \zeta_A(s) - \frac{1}{s-1}.$$

From this we conclude that $\zeta_A(s) = L^s Z_A(s) + L^s/(s-1)$ is meromorphic, analytic for $\Re s > 1$, and has a simple pole at $s = 1$ with residue

$$\operatorname{Res}_{s=1} \zeta_A(s) = L = \fint A.$$

This surprising nexus between the Wodzicki functional, the noncommutative integral and the zeta-function residue suggests that the first two may be profitably used in quantum field theory; see [40] for a recent example of that.

6. Quantization and the Tangent Groupoid

Before embarking on the classification of geometries, let us first explore an issue of a more topological nature, namely, the extent to which noncommutative methods allow us to achieve contact with the quantum world. To begin with, the facile but oft-repeated phrase, "noncommutative = quantum", must be disregarded. As the story of the Connes–Lott model shows, a perfectly noncommutative geometry may be employed to produce Lagrangians for physical models at the classical level only [22, VI.5]; quantization must then proceed from this starting point. Nevertheless, the foundations of quantum mechanics do throw up noncommutative geometries, as we have seen with the torus. Also, the integrality features of the pairing of cyclic cohomology with K-theory give genuine examples of quantizing. So, it is worthwhile to ask: what is the relation of known quantization procedures with noncommutative geometry?

The first step in the quantization of a system with finitely many degrees of freedom is to place the classical and quantum descriptions of the system on the same footing; the second step is to draw an unbroken line between these descriptions. In conventional quantum mechanics, the simplest such method is the Wigner–Weyl or Moyal quantization, which consists in "deforming" an algebra of *functions on phase space* to an algebra of *operator kernels*. In noncommutative geometry, there is a device that accomplishes this in a most economical manner, namely the *tangent groupoid* of a configuration space.

Moyal quantizers and the Heisenberg deformation

Since [4] and [6] at least, deformations of algebras have been related to the physics of quantization. We first sketch the general scheme [56], and then illustrate it with the simplest possible example, namely, the Moyal quantization in terms of the Schrödinger representation of ordinary quantum mechanics for spinless, nonrelativistic particles.

Let X be a smooth symplectic manifold, μ (an appropriate multiple of) the associated Liouville measure, and $\mathcal{H}$ a Hilbert space. A Moyal **quantizer** for the triple $(X, \mu, \mathcal{H})$ is a map Δ of the phase space X into the space of selfadjoint operators on $\mathcal{H}$ satisfying

$$\operatorname{Tr} \Delta(u) = 1, \tag{6.1a}$$

$$\operatorname{Tr}\big[\Delta(u)\Delta(v)\big] = \delta(u-v), \tag{6.1b}$$

for $u, v \in X$, at least in a distributional sense; and such that $\{\, \Delta(u) : u \in X \,\}$ spans a weakly dense subspace of $\mathcal{L}(\mathcal{H})$. More precisely, the notation "$\delta(u-v)$" means the (distributional) reproducing kernel for the measure μ. An essentially equivalent definition, in the equivariant context, was introduced first in [116]. For the proud owner of a Moyal quantizer, all quantization problems are solved in principle. *Quantization* of any function a on X is effected by

$$a \mapsto Q(a) := \int_X a(u)\Delta(u)\, d\mu(u), \tag{6.2}$$

and *dequantization* of any operator $A \in \mathcal{L}(\mathcal{H})$ by:

$$A \mapsto W_A, \qquad \text{with} \quad W_A(u) := \operatorname{Tr}\big[A\Delta(u)\big].$$

This makes automatic $W_I = 1$. Moreover, we have $W_{Q(a)} = a$ by (6.1b) and therefore $Q(W_A) = A$ by irreducibility, from which it follows that $Q(1) = I$, i.e., $\int_X \Delta(u)\, d\mu(u) = I$.

We can reformulate (6.1) as

$$\operatorname{Tr} Q(a) = \int_X a(u)\, d\mu(u), \tag{6.3a}$$

$$\operatorname{Tr}\big[Q(a)Q(b)\big] = \int_X a(u)b(u)\, d\mu(u), \tag{6.3b}$$

for *real* functions $a, b \in L^2(X, d\mu)$.

Moyal quantizers are essentially unique and understandably difficult to come by [56]. The standard example is given by the phase space $T^*(\mathbb{R}^n)$ with the canonical symplectic structure. With Planck constant $\hbar > 0$ and coordinates $u = (q, \xi)$, we take $d\mu(q, \xi) := (2\pi\hbar)^{-n}\, d^n q\, d^n \xi$. We also take $\mathcal{H} = L^2(\mathbb{R}^n)$. Then the Moyal quantizer on $T^*(\mathbb{R}^n)$ is given by a family of symmetries $\Delta^\hbar(q, \xi)$; explicitly, in the Schrödinger representation:

$$\big[\Delta^\hbar(q,\xi) f\big](x) = 2^n\, e^{2i\xi(x-q)/\hbar}\, f(2q - x). \tag{6.4}$$

The *twisted product* $a \times_\hbar b$ of two elements a, b of $\mathcal{S}(T^*\mathbb{R}^n)$, say, is defined by the requirement that $Q(a \times_\hbar b) = Q(a)Q(b)$. We obviously have:

$$a \times_\hbar b(u) = \iint L^\hbar(u,v,w) a(v) b(w)\, d\mu(v)\, d\mu(w), \qquad \text{with}$$

$$L^\hbar(u,v,w) := \operatorname{Tr}\big[\Delta^\hbar(u)\Delta^\hbar(v)\Delta^\hbar(w)\big] = 2^{2n} \exp\Big\{-\frac{2i}{\hbar}\big(s(u,v) + s(v,w) + s(w,u)\big)\Big\},$$

where $s(u, u') := q\xi' - q'\xi$ is the linear symplectic form on $T^*\mathbb{R}^n$. Note that $\int a \times_\hbar b = \int ab$. By duality, then, the quantization rule can be extended to very large spaces of functions and distributions [57, 102, 115].

Moyal quantization has other several interesting uniqueness properties; for instance, it can be also uniquely obtained by demanding equivariance with respect to the linear symplectic group (upon introducing the metaplectic representation) [110].

Asymptotic morphisms.

Due to its intrinsic homotopy invariance, K-theory is fairly rigid under deformations of algebras. K_0 is a functor, so to any $*$-homomorphism of C^*-algebras $\phi : A \to B$ there corresponds a group homomorphism $\phi_* : K_0(A) \to K_0(B)$. However, there is no need to ask for $*$-homomorphisms; is order to have K-theory maps, it is enough to construct *asymptotic* morphisms of the C^*-algebras. These were introduced by Connes and Higson in [27].

Definition. Let A, B be C^*-algebras and $\hbar_0$ a positive real number. An **asymptotic morphism** from A to B is a family of maps $T = \{\, T_\hbar : 0 < \hbar \leq \hbar_0 \,\}$, such that $\hbar \mapsto T_\hbar(a)$ is norm-continuous on $(0, \hbar_0]$ for each $a \in A$, and such that, for $a, b \in A$ and $\lambda \in \mathbb{C}$, the following norm limits apply:

$$\lim_{\hbar \downarrow 0} T_\hbar(a) + \lambda T_\hbar(b) - T_\hbar(a + \lambda b) = 0, \qquad \lim_{\hbar \downarrow 0} T_\hbar(a)^* - T_\hbar(a^*) = 0,$$

$$\lim_{\hbar \downarrow 0} T_\hbar(ab) - T_\hbar(a)T_\hbar(b) = 0.$$

Two asymptotic morphisms T, S are equivalent if $\lim_{\hbar\downarrow 0}(T_\hbar(a) - S_\hbar(a)) = 0$ for all $a \in A$. Thus, the equivalence classes of asymptotic morphisms from A to B corresponds to the morphisms from A to the quotient C^*-algebra $\widetilde{B} := C_b((0,\hbar_0],B)/C_0((0,\hbar_0],B)$ by setting $\widetilde{T}(a)_\hbar := T_\hbar(a)$. We remark that our $T_\hbar$ is the $\phi_{1/\hbar}$ of [22, 27].

In most cases we shall have only a *preasymptotic morphisms*. This is a family of maps $T = \{\, T_\hbar : 0 < \hbar \leq \hbar_0 \,\}$ from a pre-C^*-algebra $\mathcal{A}$ to a C^*-algebra B, such that the previous conditions hold. Such a family gives rise to a $*$-homomorphism from $\mathcal{A}$ into the C^*-algebra $\widetilde{B}$. Assume this homomorphism is continuous in the sup norm. Then, composing it with a section $\widetilde{B} \to C_b((0,\hbar_0],B)$ (that need not be linear nor multiplicative), we can get an asymptotic morphism. A preasymptotic morphism is *real* if $T_\hbar(a)^* = T_\hbar(a^*)$ for all $\hbar$; these are easier to handle.

In order to define maps of K-theory groups, it is enough to have asymptotic morphisms. This works as follows [63]: first extend $T_\hbar$ entrywise to an asymptotic morphism from $M_m(A)$ to $M_m(B)$. If p is a projector in $M_m(A)$, then by functional calculus there is a continuous family of projectors $\{\, q_\hbar : 0 < \hbar \leq \hbar_0 \,\}$, whose K-theory class is well defined, such that $\|T_\hbar(p) - q_\hbar\| \to 0$ as $\hbar \downarrow 0$. Define $T_*\colon K_0(A) \to K_0(B)$ by $T_*[p] := [q_{\hbar_0}]$.

If A and B are two C^*-algebras, a **strong deformation** from A to B is a *continuous field of C^*-algebras* $\{\, A_\hbar : 0 \leq \hbar \leq \hbar_0 \,\}$ in the sense of [38], such that $A_0 = A$ and $A_\hbar = B$ for $\hbar > 0$. The definition of a continuous field involves specifying the space Γ of norm-continuous sections $\hbar \mapsto s(\hbar) \in A_\hbar$, and guarantees that for any $a \in A_0$ there is such a section s_a with $s_a(0) = a$. Such a deformation gives rise to an asymptotic morphism from A to B by setting $T_\hbar(a) := s_a(\hbar)$.

The Moyal preasymptotic morphism. A very important asymptotic morphism, from $C_0(T^*\mathbb{R}^n)$ to $\mathcal{K}(L^2(\mathbb{R}^n))$, is the *Moyal deformation*, given in terms of integral kernels by:

$$\big[T_\hbar(a)f\big](x) := \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} a\Big(\frac{x+y}{2}, \xi\Big) e^{i\xi(x-y)/\hbar} f(y)\, dy\, d\xi. \tag{6.5}$$

on substituting the quantizer (6.4) in (6.2). Here a is an element of $C_c^\infty(T^*\mathbb{R}^n)$ or $\mathcal{S}(T^*\mathbb{R}^n)$, to begin with, so the integral is well defined and the operators $T_\hbar(a)$ are in fact trace-class; they are uniformly bounded in $\hbar$. It is clear that $T_\hbar(a)$ is the adjoint of $T_\hbar(a^*)$. Connes calls it the *Heisenberg deformation* [22, II.B.ϵ].

In order to check the continuity of the deformation at $\hbar = 0$, one can use, for instance, the following distributional identity:

$$\lim_{\epsilon\downarrow 0} \epsilon^{-n} e^{ixy/\epsilon} = (2\pi)^n\, \delta(x)\, \delta(y), \qquad (x, y \in \mathbb{R}^n),$$

from the theory of Fresnel integrals. It follows that $\lim_{\hbar\downarrow 0} L^\hbar(u,v,w) = \delta(u-v)\,\delta(u-w)$, so the twisted product reduces to the ordinary product in the limit $\hbar \downarrow 0$.

Now, any $*$-homomorphism from $C_c(T^*\mathbb{R}^n)$ or $\mathcal{S}(T^*\mathbb{R}^n)$ into any C^*-algebra B is continuous in the sup norm. When $B = \mathbb{C}$, this is true since any positive distribution is a measure [53]; the general case follows from automatic continuity theorems for positive mappings: see [105, V.5.6], for instance. Thus one has a real asymptotic morphism from $C_0(T^*\mathbb{R}^n)$ to $\mathcal{K}(L^2(\mathbb{R}^n))$. (Note that it is not claimed that the extension of Moyal quantization to elements of $C_0(T^*\mathbb{R}^n)$ yield compact —or even bounded— operators.)

Groupoids

By definition, a **groupoid** $G \rightrightarrows U$ is a small category in which every morphism has an inverse. Its set of objects is U (often written $G^{(0)}$) and its set of morphisms is G.

In practice, this means that we have a set G, a set U of "units" with an inclusion $i : U \hookrightarrow G$, two maps $r, s \colon G \to U$, and a composition law $G^{(2)} \to G$ with domain

$$G^{(2)} := \{\, (g,h) : s(g) = r(h) \,\} \subseteq G \times G,$$

subject to the following rules:

(i) $r(gh) = r(g)$ and $s(gh) = s(h)$ if $(g,h) \in G^{(2)}$;
(ii) if $u \in U$ then $r(u) = s(u) = u$;
(iii) $r(g)g = g = gs(g)$;
(iv) $(gh)k = g(hk)$ if $(g,h) \in G^{(2)}$ and $(gh,k) \in G^{(2)}$;
(v) each $g \in G$ has an "inverse" g^{-1}, satisfying $gg^{-1} = r(g)$ and $g^{-1}g = s(g)$.

Any group G is a groupoid, with $U = \{e\}$. On the other hand, any set X is a groupoid, with $G = U = X$ and trivial composition law $x \cdot x = x$. Less trivial examples include graphs of equivalence relations, group actions, and vector bundles with fibrewise addition. A basic and important example is the **double groupoid** of a set X. Take $G = X \times X$ and $U = X$, included in $X \times X$ as the diagonal subset $\Delta_X := \{\, (x,x) : x \in X \,\}$. Define $r(x,y) := x$, $s(x,y) := y$. Then $(x,y)^{-1} = (y,x)$ and the composition law is

$$(x,y) \cdot (y,z) = (x,z).$$

On the strength of this example, we shall call U the *diagonal* of G.

Notice that a disjoint union of groupoids is itself a groupoid.

Definition. A **smooth groupoid** is a groupoid $G \rightrightarrows U$ where G, U and $G^{(2)}$ are manifolds (possibly with boundaries), such that the inclusion $i : U \hookrightarrow G$ and the composition and inversion operations are smooth maps, and the maps $r, s \colon G \to U$ are *submersions*.

Thus, the tangent maps $T_g r$ and $T_g s$ are surjective at each $g \in G$. In particular, this implies that $\operatorname{rank} r = \operatorname{rank} s = \dim U$. Relevant examples are a Lie group, a vector bundle over a smooth manifold, and the double $M \times M$ of a smooth manifold.

One can add more structure, if desired. For example, there are symplectic groupoids, where G is a symplectic manifold and U is a Lagrangian submanifold. These can be used to connect the Kostant–Kirillov–Souriau theory of geometric quantization with Moyal quantization [58, 121].

Convolution on groupoids. Functions on groupoids can be convolved in the following way [73, 97]. Suppose that on each r-fibre $G^u := \{\, g \in G : r(g) = u \,\}$ there is given a measure λ^u so that $\lambda^{r(g)} = g\lambda^{s(g)}$ for all $g \in G$; such a family of measures is called a "Haar system" for G. The inversion map $g \mapsto g^{-1}$ carries each λ^u to a measure λ_u on the s-fibre $G_u := \{\, g \in G : s(g) = u \,\}$. The *convolution* of two functions a, b on G is then defined by

$$(a * b)(g) := \int_{hk=g} a(h)\, b(k) := \int_{G^{r(g)}} a(h)\, b(h^{-1}g)\, d\lambda^{r(g)}(h).$$

Definition. The (reduced) C^*-algebra of the smooth groupoid $G \rightrightarrows U$ with a given Haar system is the algebra $C^*_r(G)$ obtained by completing $C^\infty_c(G)$ in the norm $\|a\| := \sup_{u\in U} \|\pi_u(a)\|$, where π_u is the representation of $C^\infty_c(G)$ on the Hilbert space $L^2(G_u, \lambda_u)$:

$$\bigl[\pi_u(a)\xi\bigr](g) := \int_{hk=g} a(h)\,\xi(k) := \int_{G^{r(g)}} a(h)\xi(h^{-1}g)\,d\lambda^{r(g)}(h) \quad \text{for } g \in G_u.$$

There is a more canonical procedure to define convolution, if one wishes to avoid hunting for suitable measures λ^u, which is to take a, b to be not functions but *half-densities* on G [22, II.5]. These form a complex line bundle $\Omega^{1/2} \to G$ and one replaces $C^\infty_c(G)$ by the compactly supported smooth sections $C^\infty_c(G, \Omega^{1/2})$ in defining $C^*_r(G)$. However, for the examples considered here, the previous definition will do.

When $G = M \times M$, with M an oriented Riemannian manifold, we obtain just the convolution of kernels:

$$(a * b)(x, z) := \int_M a(x, y)\,b(y, z)\,d\nu(y)$$

where $d\nu(y) = \sqrt{\det g(y)}\,d^n y$ is the measure given by the volume form on M. Here $C^\infty_c(M \times M)$ is the usual algebra of kernels of smoothing operators on $L^2(M)$, and the C^*-algebra $C^*_r(M \times M)$ is the completion of $C^\infty_c(M \times M)$ acting as integral kernels on $L^2(M)$, so that $C^*_r(M \times M) \simeq \mathcal{K}$.

When $G = TM$ is the tangent bundle, with the operation of addition of tangent vectors, and $U = M$ is included in TM as the zero section, r and s being of course the fibering $\tau\colon TM \to M$, then $C^*_r(TM)$ is the completion of the convolution algebra

$$(a * b)(q, v) := \int_{T_q M} a(q, u)\,b(q, v - u)\,\sqrt{\det g(q)}\,d^n u,$$

where we may take $a(q, \cdot)$ and $b(q, \cdot)$ in $C^\infty_c(T_q M)$. The *Fourier transform*

$$\mathcal{F}a(q, \xi) := \int_{T_q M} e^{-i\xi v} a(q, v)\,\sqrt{\det g(q)}\,d^n v$$

replaces convolution by the ordinary product on the total space T^*M of the cotangent bundle. This gives an isomorphism from $C^*_r(TM)$ to $C_0(T^*M)$, also called $\mathcal{F}$, with inverse:

$$\mathcal{F}^{-1}b(q, v) = (2\pi)^{-n} \int_{T^*_q M} e^{i\xi v} b(q, \xi)\,\det\nolimits^{-1/2} g(q)\,d^n\xi.$$

We shall write $d\mu_q(v) := \det^{1/2} g(q)\,dv$ and $d\mu_q(\xi) := (2\pi)^{-n} \det^{-1/2} g(q)\,d\xi$.

The tangent groupoid

The asymptotic morphism involved in Moyal quantization can be given a concrete geometrical realization and a far-reaching generalization, by the concept of a tangent groupoid.

To build the tangent groupoid of a manifold M, we first form the disjoint union $G' := M \times M \times (0,1]$ of copies of the double groupoid of M parametrized by $0 < \hbar \leq 1$. Its diagonal is $U' = M \times (0,1]$. We also take $G'' := TM$, whose diagonal is $U'' = M$. The **tangent groupoid** of M is the disjoint union $G_M := G' \uplus G''$, with $U_M := U' \uplus U''$ as diagonal, whose composition law is given by

$$\begin{aligned} (x,y,\hbar)\cdot(y,z,\hbar) &:= (x,z,\hbar) &\quad& \text{for } \hbar > 0 \text{ and } x,y,z \in M, \\ (q,v_q)\cdot(q,w_q) &:= (q, v_q + w_q) && \text{for } q \in M \text{ and } v_q, w_q \in T_zM. \end{aligned}$$

Also, $(x,y,\hbar)^{-1} := (y,x,\hbar)$ and $(q,v_q)^{-1} := (q,-v_q)$.

The tangent groupoid can be given a structure of smooth groupoid in such a way that G_M is a manifold with boundary, G' contains the interior of the manifold and G'' is the nontrivial boundary.

In order to see that, let us first recall what is meant by the *normal bundle* over a submanifold R of a manifold M [35]. If $j\colon R \to M$ is the inclusion map, the tangent bundle of M restricts to R as the pullback $j^*(TM) = TM\big|_R$; this is a vector bundle over R, of which TR is a subbundle, and the normal bundle is the quotient $N^j := j^*(TM)/TR$. When M has a Riemannian metric, we may identify the fibre N^j_q to the orthogonal complement of T_qR in T_qM and thus regard N^j as a subbundle of $TM\big|_R$.

At each $q \in R$, the exponential map $\exp_q$ is one-to-one from a small enough ball in N^j_q into M. If the submanifold R is compact, then for some $\epsilon > 0$, the map $(q,v_q) \mapsto \exp_q(v_q)$ with $|v_q| < \epsilon$ is a diffeomorphism from a neighbourhood of the zero section in N^j to a neighbourhood of R in M (this is the tubular neighbourhood theorem).

Now consider the normal bundle N^Δ associated to the diagonal embedding of $\Delta\colon M \to M \times M$. We can identify $\Delta^*T(M \times M)$ to $TM \oplus TM$, and thereby the normal bundle over M is identified with

$$N^\Delta = \{(\Delta(q), \tfrac12 v_q, -\tfrac12 v_q) : (q,v_q) \in TM\},$$

which gives an obvious isomorphism between TM and N^Δ.

As in the tubular neighborhood theorem, we can (if M is compact, at any rate) find a diffeomorphism $\phi\colon V_1 \to V_2$ between an open neighbourhood V_1 of M in N^Δ (considering $M \subset N^\Delta$ as the zero section) and an open neighborhood V_2 of $\Delta(M)$ in $M \times M$. Explicitly, we can find $r_0 > 0$ so that

$$\phi(\Delta(q), v_q, -v_q) := \big(\exp_q(\tfrac12 v_q), \exp_q(-\tfrac12 v_q)\big)$$

is a diffeomorphism provided $v_q \in N^\Delta_q$ and $|v_q| < r_0$; take V_1 to be the union of these open balls of radius r_0.

Now we can define the manifold structure of G_M. The set G' is given the usual product manifold structure; it has an "outer" boundary $M \times M \times \{1\}$. In order to attach G'' to it as an "inner" boundary, we consider

$$U_1 := \{\, (q, v_q, \hbar) : (q, \hbar v_q) \in V_1 \,\},$$

which is an open subset of $TM \times [0,1]$; indeed, it is the union of $TM \times \{0\}$ and the tube of radius $r_0/\hbar$ around $\Delta(M) \times \{\hbar\}$ for each $\hbar \in (0,1]$. Therefore, the map $\Phi\colon U_1 \to G_M$ given by

$$\begin{aligned} &\Phi(q, v_q, \hbar) := \big(\exp_q(\tfrac12 \hbar v_q), \exp_q(-\tfrac12 \hbar v_q), \hbar\big) \qquad \text{for } \hbar > 0, \\ &\Phi(q, v_q, 0) := (q, v_q) \qquad \text{for } \hbar = 0, \end{aligned} \tag{6.6}$$

is one-to-one and maps the boundary of U_1 onto G''. The restriction of Φ to $U_1' := \{\,(q, v_q, \hbar) \in U_1 : 0 < \hbar \leq 1\,\} \subset TM \times (0,1]$ is a local diffeomorphism between U_1' and its image $U_2' \subset M \times M \times (0,1]$. One checks that changes of charts are smooth; thus, even if M is not compact, we can construct maps (6.6) locally and patch them together to transport the smooth structure from sets like U_1 to the inner boundary of the groupoid G_M.

To prove that G_M is a smooth groupoid, one also has to check the required properties of the inclusion $U_M \hookrightarrow G_M$, the maps r and s, the inversion and the product. Actually, the present construction is a particular case of the *tangent groupoid to a given groupoid* given by [64] and [88]. They remark that, given a smooth groupoid $G \rightrightarrows U$ (in our case the double groupoid of M), then if N is the normal bundle to U in G, the set $N \times \{0\} \uplus G \times (0,1]$ is a smooth groupoid Γ_U^G with diagonal $U \times [0,1]$, the construction (and therefore the correspondence $M \mapsto G_M$) being functorial. The smoothness properties are proven by repeated application of the following elementary result: if X, X' be smooth manifolds and Y, Y' respective closed submanifolds, and if $f\colon X \to X'$ a smooth map such that $f(Y) \subset Y'$, then the induced map from Γ_Y^X to $\Gamma_{Y'}^{X'}$ is smooth.

Moyal quantization as a continuity condition

Let us now bring into play the Gelfand–Naĭmark cofunctor C on tangent groupoids. A function on G_M is first of all a pair of functions on G' and G'' respectively. The first one is essentially a kernel, the second is the (inverse) Fourier transform of a function on the cotangent bundle T^*M. The condition that both match to give a continuous function on G_M can be seen precisely as the quantization rule. For clarity, we consider first the case $M = \mathbb{R}^n$. Let $a(x,\xi)$ be a function on $T^*\mathbb{R}^n$. Its inverse Fourier transform gives us a function on $T\mathbb{R}^n$:

$$\mathcal{F}^{-1}a(q,v) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi v} a(q,\xi)\, d\xi.$$

On $\mathbb{R}^n$, the exponentials are given by

$$x := \exp_q(\tfrac12 \hbar v) = q + \tfrac12 \hbar v; \qquad y := \exp_q(-\tfrac12 \hbar v) = q - \tfrac12 \hbar v. \tag{6.7}$$

Thus we can solve for (q,v):

$$q = \frac{x+y}{2}; \qquad v = \frac{x-y}{\hbar}. \tag{6.8}$$

To the function a we associate the following family of kernels:

$$k_a(x,y;\hbar) := \hbar^{-n} \mathcal{F}^{-1} a(q,v) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} a\Big(\frac{x+y}{2}, \xi\Big) e^{i(x-y)\xi/\hbar}\, d\xi,$$

that is, precisely the Moyal quantization formula (6.5). The factor $\hbar^{-n}$ is the Jacobian of the transformation (6.8).

We get the dequantization rule by Fourier inversion:

$$a(q,\xi) = \int_{\mathbb{R}^n} k_a(q + \tfrac12\hbar v, q - \tfrac12\hbar v)\, e^{iv\xi}\, dv.$$

Here k_a is the kernel of the operator $Q(a)$ of (6.2).

The general Moyal asymptotic morphism. If M is a Riemannian manifold, we can now quantize any function a on T^*M such that $\mathcal{F}^{-1}a$, its inverse Fourier transform in the second variable, is smooth and has compact support, say K_a. For $\hbar_0$ small enough, the map Φ of (6.6) is defined on $K_a \times [0,\hbar_0)$. For $\hbar < \hbar_0$, the formulae (6.7), (6.8) must be generalized to a transformation between TM and $M \times M \times \{\hbar\}$ whose Jacobian must be determined. We follow the treatment in [82].

Let $\gamma_{q,v}$ be the geodesic on M starting at q with velocity v, with an affine parameter s, i.e., $\gamma_{q,tv}(s) \equiv \gamma_{q,v}(ts)$. Locally, we may write

$$\begin{aligned} x &:= \gamma_{q,v}(s), \\ y &:= \gamma_{q,v}(-s), \end{aligned} \qquad \text{with Jacobian matrix} \quad \frac{\partial(x,y)}{\partial(q,v)}(s). \tag{6.9}$$

The Jacobian matrix can be computed from the equations of geodesic deviation [82]. Introduce

$$J(q,v;s) := s^{-n}\, \frac{\sqrt{\det g(\gamma_{q,v}(s))}\sqrt{\det g(\gamma_{q,v}(-s))}}{\det g(q)} \left| \frac{\partial(x,y)}{\partial(q,v)} \right|(s).$$

Then we have the change of variables formula:

$$\int_{M\times M} F(x,y)\, \mathrm{d}\nu(x)\, \mathrm{d}\nu(y) = \int_M \int_{T_qM} F(\gamma_{q,v}(\tfrac12), \gamma_{q,v}(-\tfrac12))\, J(q,v;\tfrac12)\, \mathrm{d}\mu_q(v)\, \mathrm{d}\nu(q).$$

The quantization/dequantization recipes are now given by

$$\begin{aligned} k_a(x,y;\hbar) &:= \hbar^{-n} J^{-1/2}(q,v,\tfrac12\hbar)\, \mathcal{F}^{-1}a(q,v), \\ a(q,\xi) &= \mathcal{F}\big[J^{1/2}(q,v,\tfrac12\hbar)\, k_a(x,y;\hbar)\big], \end{aligned}$$

where (x,y) and (q,v) are related by (6.9) with $s = \frac12\hbar$.

One can check that

$$J(q,v,\tfrac12\hbar) = 1 + O(\hbar^2);$$

a long but straightforward computation then shows that we have defined an (obviously real) preasymptotic morphism from $C_c^\infty(T^*M)$ to $\mathcal{K}(L^2(M))$. Moreover the "tracial property" (6.3b) for the associated quantization rule is satisfied:

$$\operatorname{Tr}\big[T_\hbar(a)T_\hbar(b)\big] = \int_{T^*M} a(u)b(u)\, d\mu_\hbar(u).$$

The corresponding map in K-theory: $T_*\colon K^0(T^*M) \to \mathbb{Z}$ is an *analytical index map*, that in fact [88] is just the analytical index map of Atiyah–Singer theory [3].

The hexagon and the analytical index

An essentially equivalent argument is done by Connes in the language of C^*-algebra theory [22, II.5]. In effect, given a smooth groupoid $G = G' \uplus G''$ which is a disjoint union of two smooth groupoids with G' open and G'' closed in G, there is a short exact sequence of C^*-algebras

$$0 \longrightarrow C^*(G') \longrightarrow C^*(G) \xrightarrow{\sigma} C^*(G'') \longrightarrow 0$$

where σ is the homomorphism defined by restriction from $C_c^\infty(G)$ to $C_c^\infty(G'')$: it is enough to notice that σ is continuous for the C^*-norms because one takes the supremum of $\|\pi_u(a)\|$ over the closed subset $u \in U''$, and it is clear that $\ker\sigma \simeq C^*(G')$.

There is a short exact sequence of C^*-algebras

$$0 \longrightarrow C_0(0,1] \otimes \mathcal{K} \longrightarrow C^*(G_M) \xrightarrow{\sigma} C_0(T^*M) \longrightarrow 0$$

that yields isomorphisms in K-theory:

$$K_j(C^*(G_M)) \xrightarrow{\sigma_*} K_j(C_0(T^*M)) = K^j(T^*M). \qquad (j = 0,1). \tag{6.10}$$

This is seen as follows: since $C^*(M \times M) = \mathcal{K}$, the C^*-algebra $C^*(G')$, obtained by completing the (algebraic) tensor product $C_c^\infty(0,1] \otimes C_c^\infty(M \times M)$, is $C_0(0,1] \otimes \mathcal{K}$, which is contractible, via the homotopy $\alpha_t(f \otimes A) := f(t\,\cdot) \otimes A$ for $f \in C_0(0,1]$, $0 \le t \le 1$; in particular, $K_j(C_0(0,1] \otimes \mathcal{K}) = 0$. At this point, we appeal to the six-term cyclic exact sequence in K-theory of C^*-algebras [119]:

$$\begin{array}{ccccc}
K_1(C_0(0,1] \otimes \mathcal{K}) & \longrightarrow & K_1(C^*(G_M)) & \xrightarrow{\sigma_*} & K_1(C_0(T^*M)) \\
{\scriptstyle\delta}\big\uparrow & & & & \big\downarrow{\scriptstyle\delta} \\
K_0(C_0(T^*M)) & \xleftarrow{\sigma_*} & K_0(C^*(G_M)) & \longleftarrow & K_0(C_0(0,1] \otimes \mathcal{K})
\end{array}$$

The two trivial groups break the circuit and leave the two isomorphisms (6.10).

The restriction of elements of $C^*(G)$ to the outer boundary $M \times M \times \{1\}$ gives a homomorphism

$$\rho\colon C^*(G_M) \to C^*(M \times M \times \{1\}) \simeq \mathcal{K},$$

and in K-theory this yields a homomorphism $\rho_*\colon K_0(C^*(G_M)) \to K_0(\mathcal{K}) = \mathbb{Z}$. Finally, we have the composition $\rho_*(\sigma_*)^{-1}\colon K^0(T^*M) \to \mathbb{Z}$, which is just the analytical index map.

Remarks on quantization and the index theorem

From the point of view of quantization theory, this is not the whole story. Certainly, in the previous argument, we could have substituted any interval $[0,\hbar_0]$ for $[0,1]$. But for a given value of $\hbar_0$, not every reasonable function on T^*M can be successfully quantized. For exponential manifolds, like flat phase space or the Poincaré disk, everything should work fine. However, when one tries to apply a similar procedure in compact symplectic manifolds, one typically finds cohomological obstructions. This is dealt with in [46], leading back to the standard results in geometric quantization *à la* Kostant–Kirillov–Souriau.

We are left with the impression that the rôle of the apparatus of Moyal quantization in the foundations of noncommutative (topology and) geometry cannot be fortuitous.

Conversely, one can ask what noncommutative geometry can do for quantization theory. Of course, one has to agree first on the meaning (at least, on the mathematical side) of the word "quantization". The nearly perfect match afforded by the Moyal machinery in its particular realm is not to be expected in general. For any symplectic manifold, some kind of "quantum" deformation or star-product can always be found. However, that is mostly formal and of little use; in general a Moyal quantizer is missing.

The modern temperament (see for instance [50]), that we readily adopt, is to consider that *quantization is embodied in the index theorem*. This dictum goes well with the original meaning of the word "quantization" in Bohr's old quantum theory. In the rare instances where it works well, though, it appears to give much more information than just a few integers (the indices of a certain Fredholm operator). The two more successful examples of quantization are Moyal quantization of finite-dimensional symplectic vector spaces and Kirillov–Kostant–Souriau geometric quantization of flag manifolds. With respect to the latter, Vergne [118] (see also [8]) has suggested to replace the concept of polarization, central to Kostant's work, by the use of Dirac-type operators —which of course takes us back to the spectral triples of noncommutative geometry.

The new scheme for quantization runs more or less as follows. Let M be a smooth manifold endowed with a spin^c structure (starting from a symplectic manifold, one can introduce a compatible almost complex structure in order to produce the spin^c structure; the results only depend weakly of the choice made, as the set of almost complex structures is contractible). Construct prequantum line bundles L over M according to the KKS recipe, or some improved version like that of [103]. Let D_L be a twisted Dirac operator for L. A quantization of M is the (virtual) Hilbert space

$$\mathcal{H}_{D,L} := \ker D_L^+ - \ker D_L^-.$$

The index theorem gives precisely the dimension of such a space. Under favourable circumstances, we can do better. In the case of flag manifolds, one quantizes G-bundles and obtains G-Hilbert spaces. Then the G-index theorem gives us the character of the Kirillov representation associated to L [7], which contains all the quantum information we seek.

Moyal quantization, on the other hand, is a tool of choice for the proof of the index theorem, as indicated here. The "logical" (though not the historical) way to go about the Index Theorem would be to prove the theorem in the flat case first using Moyal quantum mechanics —see [41] or [46]— and then go to analytically simpler but geometrically more involved cases. Conversely, one is left with the problem of how to recover the whole of Moyal theory from the index theorem.

7. Equivalence of Geometries

We wish to classify geometries and to form some idea of how many geometries of a given type are available to us. When modelling physical systems that have an underlying geometry, we naturally wish to select the most suitable geometry from several plausible candidates. The first question to ask, then, is: when are two geometries the same?

Unitary equivalence of geometries

In order to compare two geometries $(\mathcal{A}, \mathcal{H}, D, \Gamma, J)$ and $(\mathcal{A}', \mathcal{H}', D', \Gamma', J')$, we focus first of all on the algebras $\mathcal{A}$ and $\mathcal{A}'$. It is natural to ask that these be isomorphic, that is to say, that there be an involutive isomorphism $\phi: A \to A'$ between their C^*-algebra closures, such that $\phi(\mathcal{A}) = \mathcal{A}'$. Since these algebras define geometries only through their representations on the Hilbert spaces, we lose nothing by assuming that they are the *same* algebra $\mathcal{A}$. We can also assume that the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ are the same, so that $\mathcal{A}$ acts on $\mathcal{H}$ with two possibly different (faithful) representations.

One must then match the operators D and D', etc., on the Hilbert space $\mathcal{H}$. We are thus led to the notion of *unitary* equivalence of geometries.

Definition. Two geometries $\mathcal{G} = (\mathcal{A}, \mathcal{H}, D, \Gamma, J)$ and $\mathcal{G}' = (\mathcal{A}, \mathcal{H}, D', \Gamma', J')$ with the same algebra and Hilbert space are **unitarily equivalent** if there is a unitary operator $U: \mathcal{H} \to \mathcal{H}$ such that

(a) $UD = D'U$, $U\Gamma = \Gamma' U$ and $UJ = J'U$;

(b) $U\pi(a)U^{-1} \equiv \pi(\sigma(a))$ for an automorphism σ of $\mathcal{A}$.

By "automorphism of $\mathcal{A}$" is meant an involutive automorphism of the C^*-algebra A that maps $\mathcal{A}$ into itself. Since $UJ = JU$, we also get

$$U\pi^0(b)U^{-1} = UJ\pi(b^*)J^{-1}U^{-1} = J\pi(\sigma(b^*))J^{-1} = J\pi(\sigma(b)^*)J^{-1} = \pi^0(\sigma(b)).$$

To be sure that this definition is consistent, let us check the following statement: given a geometry $\mathcal{G} := (\mathcal{A}, \mathcal{H}, D, \Gamma, J)$ and a unitary operator on $\mathcal{H}$ such that $U\pi(\mathcal{A})U^{-1} = \pi(\mathcal{A})$, then $\mathcal{G}' := (\mathcal{A}, \mathcal{H}, UDU^{-1}, U\Gamma U^{-1}, UJU^{-1})$ is also a geometry.

Firstly, $\pi(a) \mapsto U\pi(a)U^{-1} =: \pi(\sigma(a))$ determines an automorphism σ of $\mathcal{A}$, since π is faithful.

The operator $D' := UDU^{-1}$ has the same spectral properties as D, so the dimension is unchanged and Poincaré duality remains valid for $\mathcal{G}'$. The order-one condition is satisfied, since

$$[[D', \sigma(a)], \sigma(b)^0] = U[[D, a], b^0]U^{-1} = 0.$$

Also, $U\pi_D(c)U^{-1} = \pi_{D'}(\sigma(c))$ for $c \in C_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0)$, where the action of σ on $\mathcal{A}$ and $\mathcal{A}^0$ is extended to Hochschild cochains in the obvious way. In particular, if c is the orientation cycle, then

$$\pi_{D'}(\sigma(c)) = U\Gamma U^{-1} = \Gamma \qquad \text{or} \qquad \pi_{D'}(\sigma(c)) = UU^{-1} = 1,$$

according as the dimension is even or odd. Thus $\sigma(c)$ is the orientation cycle for $\mathcal{G}'$.

For the finiteness property, the space of smooth vectors $\mathcal{H}'_\infty = \bigcap_k \mathrm{Dom}(D')^k$ equals $U\mathcal{H}_\infty$, and we may define $(U\xi \mid U\eta)' := \sigma(\xi \mid \eta)$ for $\xi, \eta \in \mathcal{H}_\infty$. This is the appropriate hermitian structure on $\mathcal{H}'_\infty$, since (3.7) shows that

$$\begin{aligned} ⨍ \sigma(a)\,(U\xi \mid U\eta)'\,ds'^n &= ⨍ Ua(\xi \mid \eta)U^{-1}|D'|^{-n} = ⨍ Ua(\xi \mid \eta)|D|^{-n}U^{-1} \\ &= ⨍ a(\xi \mid \eta)|D|^{-n} = ⨍ a\,(\xi \mid \eta)\,ds^n = \langle \xi \mid a\eta\rangle = \langle U\xi \mid \sigma(a)U\eta\rangle. \end{aligned}$$

Unitary equivalence of toral geometries. Let us now consider the effect of the "hyperbolic" automorphism (4.6) of the algebra $\mathcal{A}_\theta$ on the geometry $\mathbb{T}^2_{\theta,\tau}$ of the noncommutative torus.

The mapping $\underline{a} \mapsto \underline{\sigma(a)}$ determined by $\sigma(u) := u^a v^b$, $\sigma(v) := u^c v^d$ extends to a unitary operator U_σ on $L^2(\mathcal{A}_\theta, \tau_0)$, since it just permutes the orthonormal basis $\{\, \underline{u^m v^n} : m, n \in \mathbb{Z}\,\}$ (actually, each basis vector is also multiplied by a phase factor of absolute value 1). Let $U = U_\sigma \oplus U_\sigma$ be the corresponding unitary operator on $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$; it is evident that $U\pi(a)U^{-1} = \pi(\sigma(a))$ for $a \in \mathcal{A}_\theta$. By construction, $U\Gamma = \Gamma U$. Also, $UJ = JU$ on account of $U_\sigma J_0 \underline{a} = U_\sigma \underline{a^*} = \underline{\sigma(a)^*} = J_0 U_\sigma \underline{a}$ since σ is involutive.

The inverse-length operator transforms as

$$D_\tau = \begin{pmatrix} 0 & \underline{\partial}_\tau^\dagger \\ \underline{\partial}_\tau & 0 \end{pmatrix} \longmapsto UD_\tau U^{-1} = \begin{pmatrix} 0 & \underline{\tilde\partial}_\tau^\dagger \\ \underline{\tilde\partial}_\tau & 0 \end{pmatrix}$$

where $\underline{\tilde\partial}_\tau = U_\sigma \underline{\partial}_\tau U_\sigma^{-1}$ is given by $\tilde\partial_\tau = \sigma \circ \partial_\tau \circ \sigma^{-1}$. Since

$$\sigma^{-1}(u) = \lambda^{bd(a-c-1)/2} u^d v^{-b}, \quad \sigma^{-1}(v) = \lambda^{ac(d-b-1)/2} u^{-c} v^a,$$

we get at once $\tilde\partial_\tau u = 2\pi i(d - b\tau)u$, $\tilde\partial_\tau v = 2\pi i(a\tau - c)v$. Since $\partial_\tau = \delta_1 + \tau\delta_2$, we arrive at

$$\tilde\partial_\tau = (d - b\tau)\,\delta_1 + (a\tau - c)\,\delta_2.$$

This is tantamount to replacing τ by $\sigma^{-1}\cdot\tau = (a\tau - c)/(d - b\tau)$ in the definition of D_τ, together with a rescaling in order to preserve the area given by the orientation cycle (4.12). [It is easy to check that $\langle \phi, c\rangle = \langle \phi, \sigma(c)\rangle$ owing to $ad - bc = 1$.] Harking back to elliptic curves for a moment, we see that the period parallelograms for the period pairs $(1, \tau)$ and $(d - b\tau, a\tau - c)$ have the same area.

Thus, for each geometry $(\mathcal{A}_\theta, \mathcal{H}, D_\tau, \Gamma, J)$, there is a family of unitarily equivalent geometries $(\mathcal{A}_\theta, \mathcal{H}, UD_\tau U^{-1}, \Gamma, J)$. If we replace the particular derivation ∂_τ of (4.10) by the most general derivation $\partial = \alpha\delta_1 + \beta\delta_2$ where $\Im(\beta/\alpha) > 0$, we obtain a family of geometries over $\mathcal{A}_\theta$, parametrized up to unitary equivalence by the fundamental domain of the modular group $PSL(2, \mathbb{Z})$.

Action of inner automorphisms. If u is a unitary element of the algebra $\mathcal{A}$, i.e., $u^*u = uu^* = 1$, consider the unitary operator on $\mathcal{H}$ given by we write

$$U := \pi(u)\pi^0(u^{-1}) = uJuJ^\dagger = JuJ^\dagger u.$$

Since $J^2 = \pm 1$, we get

$$UJ = uJu = \pm uJ^\dagger u = J^2 uJ^\dagger u = JU.$$

The grading operator Γ commutes with $\pi(u)$ and $\pi^0(u^{-1})$, so we also have $U\Gamma = \Gamma U$. Furthermore, if $a \in \mathcal{A}$, then $UaU^{-1} = uau^{-1}$ since $JuJ^\dagger$ commutes with a, so U implements the *inner* automorphism of $\mathcal{A}$:

$$\sigma_u(a) := u\,a\,u^{-1}.$$

Such operators U provide unitary equivalences of the geometries $(\mathcal{A}, \mathcal{H}, D, \Gamma, J)$ and $(\mathcal{A}, \mathcal{H}, {}^uD, \Gamma, J)$, where

$$\begin{aligned} {}^uD &:= UDU^{-1} = UDU^* = JuJ^\dagger uDu^*Ju^*J^\dagger = JuJ^\dagger(D + u[D,u^*])Ju^*J^\dagger \\ &= JuJ^\dagger DJu^*J^\dagger + u[D,u^*] = D + JuJ^\dagger[D, Ju^*J^\dagger] + u[D,u^*] \\ &= D + u[D,u^*] \pm Ju[D,u^*]J^\dagger. \end{aligned} \tag{7.1}$$

Here we have used the order-one condition and the relation $JD = \pm DJ$; the latter gives the $\pm$ sign on the right hand side, which is negative iff $n \equiv 1 \bmod 4$.

Notice that the operator $u[D,u^*] = uDu^* - D$ is bounded and selfadjoint in $\mathcal{L}(\mathcal{H})$.

Morita equivalence and Hermitian connections

The unitary equivalence of geometries helps to eliminate obvious redundancies, but it is not by any means the only way to compare geometries. For one thing, the metric (1.6) is unchanged —if we think of the right hand side of (1.6) as defining the distance between pure states $\hat{p}$, $\hat{q}$ of the algebra $\mathcal{A}$.

We need a looser notion of equivalence between geometries that allows to vary not just the operator data but also the algebra and the Hilbert space. Here the Morita equivalence of algebras gives us a clue as to how to proceed. We can change the algebra $\mathcal{A}$ to a Morita-equivalent algebra $\mathcal{B}$, which also involves changing the representation space according to well-defined rules. How should we then adapt the remaining data Γ, J and most importantly D, in order to obtain a *Morita equivalence of geometries*?

We start with any geometry $(\mathcal{A}, \mathcal{H}, D, \Gamma, J)$ and a finite projective right $\mathcal{A}$-module $\mathcal{E}$. Using the representation $\pi\colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and the antirepresentation $\pi^0\colon b \mapsto J\pi(b^*)J^\dagger$, we can regard the space $\mathcal{H}$ as an $\mathcal{A}$-bimodule. This allows us to introduce the vector space

$$\widetilde{\mathcal{H}} := \mathcal{E} \otimes_\mathcal{A} \mathcal{H} \otimes_\mathcal{A} \overline{\mathcal{E}}. \tag{7.2}$$

If $\mathcal{E} = p\mathcal{A}^m$, then $\overline{\mathcal{E}} = \overline{\mathcal{A}}^m\, p$ and $\widetilde{\mathcal{H}} = \pi(p)\pi^0(p)[\mathcal{H} \otimes \mathbb{C}^{m^2}]$, so that $\widetilde{\mathcal{H}}$ becomes a Hilbert space under the scalar product

$$\langle r \otimes \eta \otimes \bar{q} \mid s \otimes \xi \otimes \bar{t}\,\rangle := \langle \eta \mid \pi(r \mid s)\, \pi^0(t \mid q)\, \xi \rangle.$$

If $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is $\mathbb{Z}_2$-graded, there is an obvious $\mathbb{Z}_2$-grading of $\widetilde{\mathcal{H}}$.

The antilinear correspondence $s \mapsto \bar{s}$ between $\mathcal{E}$ and $\overline{\mathcal{E}}$ also gives an obvious way to extend J to $\widetilde{\mathcal{H}}$:

$$\widetilde{J}(s \otimes \xi \otimes \bar{t}\,) := t \otimes J\xi \otimes \bar{s}. \tag{7.3}$$

Let $\mathcal{B} := \operatorname{End}_{\mathcal{A}} \mathcal{E}$, and recall that $\mathcal{E}$ is a left $\mathcal{B}$-module. Then

$$\rho(b) : s \otimes \xi \otimes \bar{t} \longmapsto b\,s \otimes \xi \otimes \bar{t}$$

yields a representation ρ of $\mathcal{B}$ on $\widetilde{\mathcal{H}}$, satisfying

$$\rho^0(b) := \widetilde{J}\rho(b^*)\widetilde{J}^\dagger : s \otimes \xi \otimes \bar{t} \longmapsto s \otimes \xi \otimes \bar{t}\,b,$$

where $\bar{t}\,b := \overline{b^* t}$, of course. The action ρ, ρ^0 of $\mathcal{B}$ on $\widetilde{\mathcal{H}}$ obviously commute.

Where connections come from. The nontrivial part of the construction of the new geometries $(\mathcal{B}, \widetilde{\mathcal{H}}, \widetilde{D}, \widetilde{\Gamma}, \widetilde{J})$ is the determination of an appropriate operator $\widetilde{D}$ on $\widetilde{\mathcal{H}}$. Guided by the differential properties of Dirac operators, the most suitable procedure is to postulate a *Leibniz rule*:

$$\widetilde{D}(s \otimes \xi \otimes \bar{t}\,) := (\nabla s)\xi \otimes \bar{t} + s \otimes D\xi \otimes \bar{t} + s \otimes \xi(\overline{\nabla t}), \tag{7.4}$$

where ∇s, ∇t belong to some space whose elements can be represented on $\mathcal{H}$ by suitable extensions of π and π^0.

Consistency of (7.4) with the actions of $\mathcal{A}$ on $\mathcal{E}$ and $\mathcal{H}$ demands that ∇ itself comply with a Leibniz rule. Indeed, since

$$sa \otimes \xi \otimes \bar{t} = s \otimes a\xi \otimes \bar{t} \qquad \text{for all} \quad a \in \mathcal{A},$$

we get from (7.4)

$$\nabla(sa)\xi \otimes \bar{t} + s \otimes a\,D\xi \otimes \bar{t} = (\nabla s)a\xi \otimes \bar{t} + s \otimes D\,a\xi \otimes \bar{t},$$

so we infer that

$$\nabla(sa) = (\nabla s)a + [D, a], \tag{7.5}$$

or more pedantically, $\nabla(sa) = (\nabla s)\pi(a) + [D, \pi(a)]$ as operators on $\mathcal{H}$.

To satisfy these requirements, we introduce the space of bounded operators

$$\Omega^1_D := \operatorname{span}\{\, a\,[D, b] : a, b \in \mathcal{A}\,\} \subseteq \mathcal{L}(\mathcal{H}),$$

which is evidently an $\mathcal{A}$-bimodule, the right action of $\mathcal{A}$ being given by $a\,[D, b] \cdot c := a\,[D, bc] - ab\,[D, c]$. The notation is chosen to remind us of differential 1-forms; indeed, for the commutative geometry $(C^\infty(M), L^2(M, S), \not{D}, \chi, J)$, we get

$$\Omega^1_{\not{D}} = \{\, \gamma(\alpha) : \alpha \in \mathcal{A}^1(M)\,\},$$

i.e., conventional 1-forms on M, represented on spinor space as (Clifford) multiplication operators.

Definition. We can now form the right $\mathcal{A}$-module $\mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$. A **connection** on $\mathcal{E}$ is a linear mapping

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$$

that satisfies the Leibniz rule (7.5).

It is worth mentioning that only *projective* modules admit connections [32]. In the present case, if we define linear maps

$$0 \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D \xrightarrow{j} \mathcal{E} \otimes_{\mathbb{C}} \mathcal{A} \xrightarrow{m} \mathcal{E} \longrightarrow 0$$

by $j(s\,[D,a]) := sa \otimes 1 - s \otimes a$ and $m(s \otimes a) := sa$, we get a short exact sequence of right $\mathcal{A}$-modules (think of $\mathcal{E} \otimes_{\mathbb{C}} \mathcal{A}$ as a free $\mathcal{A}$-module generated by a vector-space basis of $\mathcal{E}$). Any linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ gives a linear section of m by $f(s) := s \otimes 1 - j(\nabla s)$. Then $f(sa) - f(s)a = j(s\,[D,a] - \nabla(sa) + (\nabla s)a)$, so f is an $\mathcal{A}$-module map precisely when ∇ satisfies the Leibniz rule (7.5). If that happens, f splits the exact sequence and embeds $\mathcal{E}$ as a direct summand of the free $\mathcal{A}$-module $\mathcal{E} \otimes_{\mathbb{C}} \mathcal{A}$, so $\mathcal{E}$ is projective.

Hermitian connections. The operator $\widetilde{D}$ must be selfadjoint on $\widetilde{\mathcal{H}}$. If $\xi, \eta \in \operatorname{Dom}(D)$, we get

$$\begin{aligned}
\langle r \otimes \eta \otimes \bar{q} \mid \widetilde{D}(s \otimes \xi \otimes \bar{t}\,)\rangle &= \langle \eta \mid \pi_D(r \mid \nabla s)\, \pi^0(t \mid q)\, \xi\rangle + \langle \eta \mid \pi(r \mid s)\, \pi^0(t \mid q)\, D\xi\rangle \\
&\quad + \langle \eta \mid \pi(r \mid s)\, \pi^0_D(\nabla t \mid q)\, \xi\rangle, \\
\langle \widetilde{D}(r \otimes \eta \otimes \bar{q}) \mid s \otimes \xi \otimes \bar{t}\,\rangle &= \langle \eta \mid \pi_D(\nabla r \mid s)\, \pi^0(t \mid q)\, \xi\rangle + \langle \eta \mid D\pi(r \mid s)\, \pi^0(t \mid q)\, \xi\rangle \\
&\quad + \langle \eta \mid \pi(r \mid s)\, \pi^0_D(t \mid \nabla q)\, \xi\rangle.
\end{aligned}$$

This reduces to the condition that

$$(r \mid \nabla s) - (\nabla r \mid s) = [D, (r \mid s)] \qquad \text{for all} \quad r, s \in \mathcal{E}. \tag{7.6}$$

where the order one condition ensures commutation of $\pi_D(\Omega^1_D)$ with $\pi^0(\mathcal{A})$.

We call the connection ∇ *Hermitian* (with respect to D) if (7.6) holds. The minus sign is due to the presence of the selfadjoint operator D where a skewadjoint differential operator is used in the standard definition of a metric-preserving connection [7, 83].

To sum up: two *geometries* $(\mathcal{A}, \mathcal{H}, D, \Gamma, J)$ and $(\mathcal{B}, \widetilde{\mathcal{H}}, \widetilde{D}, \widetilde{\Gamma}, \widetilde{J})$ are *Morita-equivalent* if there exist a finite projective right $\mathcal{A}$-module $\mathcal{E}$ and an Ω^1_D-valued Hermitian connection ∇ on $\mathcal{E}$, such that: $\mathcal{B} = \operatorname{End}_{\mathcal{A}} \mathcal{E}$, $\widetilde{\mathcal{H}}$ and $\widetilde{\Gamma}$ are given by (7.2), $\widetilde{J}$ by (7.3), and $\widetilde{D}$ by (7.4).

Vector bundles over the noncommutative torus

The finite projective modules over the torus C^*-algebra A_θ were defined in [16] and fully classified in [101]. (Indeed, [101] also constructs vector bundles over $\mathbb{T}^2$ that represent distinct classes in $K_0(C^\infty(\mathbb{T}^2)) \simeq \mathbb{Z} \oplus \mathbb{Z}$; but the projective modules so obtained are unlike those of the irrational case.) To describe the latter, we return to the Weyl operators (4.1). The translation and multiplication operators $W_\theta(a, 0)$ and $W_\theta(0, b)$ are generated by $i\,d/dt$ and t; the space of smooth vectors for these derivations is just the Schwartz space $\mathcal{S}(\mathbb{R})$.

Clearly $\mathcal{S}(\mathbb{R})$ is a left $\mathcal{A}_\theta$-module; but it can also be made a *right* $\mathcal{A}_\theta$-module by making the generators act in another way. If p is any integer, one can define

$$\begin{aligned}\psi \cdot u &:= W_\theta(p-\theta, 0)\psi : t \mapsto \psi(t - p + \theta),\\ \psi \cdot v &:= W_\theta(0, 1/\theta)\psi : t \mapsto e^{2\pi i t}\psi(t).\end{aligned}$$

Therefore

$$\psi \cdot vu := e^{-\pi i(p-\theta)} W_\theta(p-\theta, 1/\theta)\psi, \quad \psi \cdot uv := e^{+\pi i(p-\theta)} W_\theta(p-\theta, 1/\theta)\psi,$$

so $\psi \cdot vu = e^{2\pi i\theta}\psi \cdot uv$. Since the generators act compatibly with the commutation relation (4.3), this defines a right action of $\mathcal{A}_\theta$ on $\mathcal{S}(\mathbb{R})$. This right module will be denoted $\mathcal{E}_p$.

One can define more $\mathcal{A}_\theta$-modules by a simple trick. Let q be a positive integer; the Weyl operators act on $\mathcal{S}(\mathbb{R}^q) = \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^q$ as $W_\theta(a,b) \otimes 1_q$. If $z \in M_q(\mathbb{C})$ is the cyclic shift $(x_1, \dots, x_q) \mapsto (x_2, \dots, x_q, x_1)$ and $w \in M_q(\mathbb{C})$ is the diagonal operator $(x_1, \dots, x_q) \mapsto (\zeta x_1, \zeta^2 x_2, \dots, x_q)$ for $\zeta := e^{2\pi i/q}$, then $zw = e^{2\pi i/q} wz$, so that

$$\psi \cdot u := \big(W_\theta(\tfrac{p}{q} - \theta, 0) \otimes z^p\big)\psi, \quad \psi \cdot v := \big(W_\theta(0, 1/\theta) \otimes w\big)\psi$$

satisfy $\psi \cdot vu = \lambda \psi \cdot uv$ with $\lambda = e^{2\pi i(\theta - p/q)} e^{2\pi i p/q} = e^{2\pi i\theta}$. This right action of $\mathcal{A}_\theta$ on $\mathcal{S}(\mathbb{R}^q)$ defines a right module $\mathcal{E}_{p,q}$.

It turns out that the free modules $\mathcal{A}^m$ and these $\mathcal{E}_{p,q}$ (with $p, q \in \mathbb{Z}$, $q > 0$) are mutually nonisomorphic and any finite projective right $\mathcal{A}_\theta$-module is isomorphic to one of them. Actually, it is perhaps not obvious that the $\mathcal{E}_{p,q}$ are finitely generated and projective. This is proved in [101], using the following Hermitian structure [30] that makes $\mathcal{E}_{p,q}$ a pre-C^*-module over $\mathcal{A}_\theta$:

$$(\phi \mid \psi) := \textstyle\sum_{r,s} u^r v^s \, \langle \phi \cdot u^r v^s \mid \psi \rangle_{L^2(\mathbb{R}^q)} \qquad \text{for} \quad \phi, \psi \in \mathcal{S}(\mathbb{R}^q),$$

where the coefficients, in the case $q = 1$, are:

$$\langle \phi \cdot u^r v^s \mid \psi \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} e^{-2\pi i s t}\, \overline{\phi(t - r(p-\theta))}\, \psi(t)\, dt. \tag{7.7}$$

We shall soon verify projectiveness in another way, by introducing connections.

The endomorphism algebras. To reduce notational complications, let us take $q = 1$. The algebra $\mathcal{B} := \operatorname{End}_{\mathcal{A}_\theta} \mathcal{E}_p$ is generated by Weyl operators that commute with $W_\theta(p-\theta, 0)$ and $W_\theta(0, 1/\theta)$. In view of (4.2), we can take as generators the operators

$$U := W_\theta(1, 0), \qquad V := W_\theta(0, 1/\theta(p-\theta)).$$

Then $VU = \mu\, UV$ where $\mu = \exp(2\pi i/(p-\theta))$, so that $\mathcal{B} \simeq \mathcal{A}_{1/(p-\theta)}$.

For the simplest case $p = 0$, $q = 1$, we have

$$U\psi(t) = \psi(t-1), \qquad V\psi(t) = e^{-2\pi i t/\theta}\psi(t), \tag{7.8}$$

so that $\mathcal{A}_\theta$ and $\mathcal{A}_{-1/\theta}$ are Morita equivalent via $\mathcal{E}$.

It is known [98] that $\mathcal{A}_\theta$ and $\mathcal{A}_\phi$ are Morita equivalent if and only if either ϕ or $-\phi$ lies in the orbit of θ under the action of $SL(2,\mathbb{Z})$, i.e., if and only if $\pm\phi = (a\theta+b)/(c\theta+d)$. The proof is K-theoretic: since $\tau_{0*}(\mathcal{K}_0(\mathcal{A}_\theta)) = \mathbb{Z}+\mathbb{Z}\theta$, a necessary condition is that $\mathbb{Z}+\mathbb{Z}\phi = r(\mathbb{Z}+\mathbb{Z}\theta)$ for some $r>0$. Sufficiency is proved by exhibiting an appropriate equivalence bimodule $\mathcal{E}_{p,q}$.

Morita-equivalent toral geometries

Let us now construct a Hermitian connection ∇ (with respect to the operator D_τ) on the $\mathcal{A}_\theta$-module $\mathcal{E}_0 = \mathcal{S}(\mathbb{R})$. To do so, we must first determine the bimodule $\Omega^1_{D_\tau}$. Clearly

$$\pi(a)\,[D_\tau, \pi(b)] = \begin{pmatrix} 0 & a\,\partial_\tau^* b \\ a\,\partial_\tau b & 0 \end{pmatrix},$$

so that $\Omega^1_{D_\tau} \simeq \mathcal{A}_\theta \oplus \mathcal{A}_\theta$ as $\mathcal{A}_\theta$-bimodules. Thus $\mathcal{E}_0 \otimes_{\mathcal{A}} \Omega^1_{D_\tau} \simeq \mathcal{E}_0 \oplus \mathcal{E}_0$.

Therefore, $\nabla\psi = (\nabla'\psi, \nabla''\psi)$ where ∇', ∇'' are two derivations on $\mathcal{S}(\mathbb{R})$. The corresponding Leibniz rules are given by (7.5):

$$\nabla'(\psi\cdot a) = (\nabla'\psi)\cdot a + \psi\cdot\partial_\tau a, \quad \nabla''(\psi\cdot a) = (\nabla''\psi)\cdot a - \psi\cdot\partial_{\bar\tau} a.$$

This implies that $\nabla' = \nabla_1 + \tau\,\nabla_2$ and $\nabla'' = -\nabla_1 - \bar\tau\,\nabla_2$, where ∇_1, ∇_2 comply with Leibniz rules involving the basic derivations:

$$\nabla_j(\psi\cdot a) = (\nabla_j\psi)\cdot a + \psi\cdot\delta_j a,$$

and it is enough to check these relations for $a = u, v$.

It will come as no surprise that ∇_1 and ∇_2 are just the position and momentum operators of quantum mechanics (with a scale factor of $i\theta/2\pi = i/2\pi\hbar$); in fact,

$$\nabla_1\psi(t) := -\frac{2\pi i t}{\theta}\psi(t), \qquad \nabla_2\psi(t) := \psi'(t). \tag{7.9}$$

One immediately checks that

$$\begin{aligned}
\nabla_1(\psi\cdot u) - (\nabla_1\psi)\cdot u &= [t \mapsto 2\pi i\,\psi(t+\theta)] = \psi\cdot\delta_1 u,\\
\nabla_1(\psi\cdot v) - (\nabla_1\psi)\cdot v &= 0 = \psi\cdot\delta_1 v,\\
\nabla_2(\psi\cdot u) - (\nabla_2\psi)\cdot u &= 0 = \psi\cdot\delta_2 u,\\
\nabla_2(\psi\cdot v) - (\nabla_2\psi)\cdot v &= [t \mapsto 2\pi i\,e^{2\pi i t}\psi(t)] = \psi\cdot\delta_2 v.
\end{aligned}$$

Thus ∇ is a connection satisfying (7.5) with $D = D_\tau$.

To see that ∇ is Hermitian, it is enough to observe that (7.6) is equivalent to

$$(\phi \mid \nabla'\psi) - (\nabla''\phi \mid \psi) = \partial_\tau(\phi \mid \psi) \qquad \text{for all} \quad \phi,\psi\in\mathcal{S}(\mathbb{R}),$$

or equivalently

$$(\phi \mid \nabla_j \psi) + (\nabla_j \phi \mid \psi) = \delta_j(\phi \mid \psi) \qquad (j = 1, 2), \tag{7.10}$$

where the $\mathcal{A}_\theta$-valued Hermitian structure on $\mathcal{S}(\mathbb{R})$ is the special case of (7.7):

$$(\phi \mid \psi) := \textstyle\sum_{r,s} a_{rs}\, u^r v^s, \quad a_{rs} := \displaystyle\int_{\mathbb{R}} e^{-2\pi i s t}\, \overline{\phi(t + r\theta)}\, \psi(t)\, dt.$$

This can be verified by direct calculation. For instance, when $j = 2$, the left hand side of (7.10) equals

$$\textstyle\sum_{r,s} u^r v^s \displaystyle\int_{\mathbb{R}} e^{-2\pi i s t} \frac{d}{dt}\Big[\overline{\phi(t + r\theta)}\psi(t)\Big]\, dt = \textstyle\sum_{r,s} 2\pi i s\, a_{rs}\, u^r v^s = \delta_2(\phi \mid \psi).$$

The geometry on $\mathcal{A}_{-1/\theta}$. Let us take stock of the new geometry. The algebra is $\mathcal{A}_{-1/\theta}$, with generators U, V of (7.8). The Hilbert space is $\mathbb{Z}_2$-graded, with $\widetilde{\mathcal{H}}^+ = \mathcal{E}_0 \otimes_{\mathcal{A}} \mathcal{H}^+ \otimes_{\mathcal{A}} \overline{\mathcal{E}}_0$, that we can identify with $L^2(\mathcal{A}_{-1/\theta}, \tau_0)$. Under this identification, $\widetilde{J}$ becomes $\underline{a} \mapsto \underline{a^*}$, as before.

It remains to identify the operator $\widetilde{D}$, whose general form has been determined in §4. We find that

$$[\widetilde{D}, U](\psi \otimes \xi \otimes \bar{\phi}) = \big([\nabla, U]\psi\big)\, \xi \otimes \bar{\phi}$$

for $\psi, \phi \in \mathcal{S}(\mathbb{R})$, $\xi \in \mathcal{H}$,where

$$[\nabla, U] = \begin{pmatrix} 0 & [\nabla'', U] \\ [\nabla', U] & 0 \end{pmatrix}.$$

It is immediate from the definitions (7.8), (7.9) that

$$[\nabla_1, U] = -\frac{2\pi i}{\theta} U = -\frac{1}{\theta}\delta_1 U, \quad [\nabla_2, V] = -\frac{2\pi i}{\theta} V = -\frac{1}{\theta}\delta_2 V, \quad [\nabla_1, V] = [\nabla_2, U] = 0.$$

Thus

$$\widetilde{D} = -\frac{1}{\theta}\begin{pmatrix} 0 & \underline{\partial}_\tau^\dagger \\ \underline{\partial}_\tau & 0 \end{pmatrix}.$$

Setting aside the overall scale factor $-1/\theta$, we see that the modulus τ is unchanged. We conclude that the geometries $\mathbb{T}^2_{\theta,\tau}$ and $\mathbb{T}^2_{-1/\theta,\tau}$ are Morita equivalent.

Gauge potentials

Let us examine what Morita equivalence entails when the algebra $\mathcal{A}$ is unchanged, and the equivalence bimodule is $\mathcal{A}$ itself. The algebra $\mathcal{A}$, regarded as a right $\mathcal{A}$-module, carries a standard Hermitian connection with respect to D, namely

$$\mathrm{Ad}_D \colon \mathcal{A} \to \Omega^1_D : b \mapsto [D, b],$$

and by the Leibniz rule (7.5), any connection differs from Ad_D by an operator in Ω^1_D:

$$\nabla b =: [D, b] + \mathbb{A} b, \tag{7.11}$$

where

$$\mathbb{A} := \textstyle\sum_j a_j [D, b_j] \qquad \text{(finite sum)}$$

lies in Ω^1_D. We call it a *gauge potential* if it is selfadjoint: $\mathbb{A}^* = \mathbb{A}$. Hermiticity of the connection for the product $(a \mid b) := a^* b$ demands that $a^* \nabla b - (\nabla a)^* b = [D, a^* b]$, that is, $a^*(\mathbb{A} - \mathbb{A}^*) b = 0$ for all $a, b \in \mathcal{A}$, so a Hermitian connection on $\mathcal{A}$ is given by a gauge potential $\mathbb{A}$.

On substituting the connection (7.11) in the recipe (7.4) for an extended Dirac operator, one obtains

$$\begin{aligned} \widetilde{D}(b\xi) &= ([D, b] + \mathbb{A} b)\xi + b\, D\xi \pm bJ(\nabla 1)J^\dagger \xi \\ &= (D + \mathbb{A} \pm J\mathbb{A}J^\dagger)(b\xi), \end{aligned} \tag{7.12}$$

where the signs are as in (7.1). Therefore, the gauge transformation $D \mapsto D + \mathbb{A} \pm J\mathbb{A}J^\dagger$ yields a geometry that is *Morita-equivalent* to the original. Another way of saying this is that the geometries whose other data $(\mathcal{A}, \mathcal{H}, \Gamma, J)$ are fixed form an affine space modelled on the selfadjoint part of Ω^1_D.

In summary, we have shown how the classification of geometries up to Morita equivalence allows a first-order differential calculus to enter the picture, via the Hermitian connections. In the next section, we shall explore the various geometries on a noncommutative manifold from a variational point of view.

8. Action Functionals

On a differential manifold, one may use many Riemannian metrics; on a spin manifold with a given Riemannian metric, there may be many distinct (i.e., unitarily inequivalent) geometries. An important task, already in the commutative case, is to select, if possible, a particular geometry by some general criterion, such as minimization of an action functional, a time-honoured tradition in physics. In the noncommutative case, the minimizing geometries are often not unique, leading to the phenomenon of spontaneous symmetry breaking, an important motivation for physical applications [39].

Automorphisms of the algebra

In order to classify geometries, we fix the data $(\mathcal{A}, \mathcal{H}, \Gamma, J)$ and consider how the inverse distance operator D may be modified under the actions of automorphisms of the algebra $\mathcal{A}$.

The point at issue here is that the *automorphism group* of the algebra is just the noncommutative version of the *group of diffeomorphisms* of a commutative manifold. For instance, if $\mathcal{A} = C^\infty(M)$ for a smooth manifold M, and if $\alpha \in \mathrm{Aut}(\mathcal{A})$, then each character $\hat{x}$ of $\mathcal{A}$ is the image under α of a unique character $\hat{y}$ (that is, $\alpha^{-1}(\hat{x})$ is also a character, so it equals $\hat{y}$ for some $y \in M$). Write $\phi(x) := y$; then ϕ is a continuous bijection on M satisfying $\alpha(f)(x) = f(\phi^{-1}(x))$, and the chain rule for derivatives shows that ϕ is itself smooth and hence is a diffeomorphism of M. In fine, $\alpha \leftrightarrow \phi$ is a group isomorphism from $\mathrm{Aut}(C^\infty(M))$ onto $\mathrm{Diff}(M)$.

On a noncommutative algebra, there are many *inner automorphisms*

$$\sigma_u(a) := u\,a\,u^{-1},$$

where u lies in the unitary group $\mathcal{U}(\mathcal{A})$; these are of course trivial when $\mathcal{A}$ is commutative. We adopt the attitude that these inner automorphisms are hence forth to be regarded as *internal diffeomorphisms* of our algebra $\mathcal{A}$.

Already in the commutative case, diffeomorphisms change the metric on a manifold. To select a particular metric, some sort of *variational principle* may be used. In general relativity, one works with the Einstein–Hilbert action

$$I_{\mathrm{EH}} \propto \int_M r(x)\,\sqrt{g(x)}\,d^n x = \int_M r(x)\,\Omega,$$

where r is the scalar curvature of the metric g, in order to select a metric minimizing this action. In Yang–Mills theories of particle physics, the bosonic action functional is of the form $I_{\mathrm{YM}} \propto \int F(\star F)$ where F is a gauge field, i.e., a curvature form.

The question then arises as to what is the *general prescription* for appropriate action functionals in noncommutative geometry.

Inner automorphisms and gauge potentials. Let us first recall how inner automorphisms act on geometries. If $u \in \mathcal{U}(\mathcal{A})$, the operator $U := uJuJ^\dagger$ implements a unitary equivalence (7.1) between the geometries determined by D and by

$${}^uD = D + u[D, u^*] \pm Ju[D, u^*]J^\dagger.$$

More generally, any selfadjoint $\mathbb{A} \in \Omega^1_D$ gives rise to a Morita equivalence (7.12) between the geometries determined by D and by $D + \mathbb{A} \pm J\mathbb{A}J^\dagger$.

The slaying of abelian gauge fields. It is important to observe that these gauge transformations are *trivial* when the algebra $\mathcal{A}$ is commutative. Recall that $\pi^0(b) = \pi(b)$ in the commutative case (since the action by J on spinors takes multiplication by a function to multiplication by its complex conjugate). Therefore we can write $a = Ja^*J^\dagger$ when $\mathcal{A}$ is commutative. But then,

$$\begin{aligned} Ja[D,b]J^\dagger &= a^* J[D,b]J^\dagger = J[D,b]J^\dagger a^* \\ &= [JDJ^\dagger, JbJ^\dagger]a^* = \pm[D,b^*]a^* = \mp\big(a[D,b]\big)^* \end{aligned}$$

since $JDJ^\dagger = \pm D$. Hence $J\mathbb{A}J^\dagger = \mp\mathbb{A}^*$ for $\mathbb{A} \in \Omega^1_D$, and thus $\mathbb{A} \pm J\mathbb{A}J^\dagger = \mathbb{A} - \mathbb{A}^*$; for a selfadjoint gauge potential, $\mathbb{A} \pm J\mathbb{A}J^\dagger$ vanishes.

As pointed out in [86], this means that, within our postulates, a commutative manifold could support gravity but not electromagnetism; in other words, even to get abelian gauge fields we need that the underlying manifold be noncommutative!

Exercise. Show that the gauge potentials $\mathbb{A} = u^m v^n [D_\tau, v^{-n}u^{-m}]$ for the toral geometry $\mathbb{T}^2_{\theta,\tau}$ satisfy $\mathbb{A} + J\mathbb{A}J^\dagger = 0$, but this is not so for $\mathbb{A} := v[D_\tau, u^*] - [D_\tau, u]v^*$. $\diamondsuit$

In [25], Connes also raised the issue of whether $\mathcal{A}$ might admit further symmetries arising from Hopf algebras. We cannot go into this here, but we should mention the recent investigations [31, 33, 75] on a 27-dimensional Hopf algebra closely related to the gauge group of the Standard Model.

The fermionic action

In the Standard Model of particle physics, the following prescription defines the fermionic action functional:

$$I(\xi, \mathbb{A}) := \langle \xi \mid (D + \mathbb{A} \pm J\mathbb{A}J^\dagger)\xi \rangle \tag{8.1}$$

(with the $\pm$ sign as before). Here ξ may be interpreted as a multiplet of spinors representing elementary particles and antiparticles [22, 86, 107].

The gauge group $\mathcal{U}(\mathcal{A})$ acts on potentials in the following way. If $u \in \mathcal{A}$ is unitary and if $\nabla = \operatorname{Ad}_D + \mathbb{A}$ is a hermitian connection, then so is

$$u\nabla u^* = u \operatorname{Ad}_D u^* + u\mathbb{A}u^* = \operatorname{Ad}_D + u[D,u^*] + u\mathbb{A}u^*,$$

so that ${}^u\mathbb{A} := u\mathbb{A}u^* + u[D,u^*]$ is the gauge-transformed potential. With $U = uJuJ^\dagger$, we get $U\mathbb{A}U^{-1} = u\mathbb{A}u^{-1}$ since $JuJ^\dagger$ commutes with Ω^1_D, and so

$$\begin{aligned} D + {}^u\mathbb{A} \pm J\,{}^u\mathbb{A}J^\dagger &= D + u[D,u^*] \pm Ju[D,u^*]J^\dagger + u\mathbb{A}u^* \pm Ju\mathbb{A}u^*J^\dagger \\ &= U(D + \mathbb{A} \pm J\mathbb{A}J^\dagger)U^{-1}. \end{aligned}$$

The gauge invariance of (8.1) under the group $\mathcal{U}(\mathcal{A})$ is now established by

$$I(U\xi, {}^u\mathbb{A}) = \langle U\xi \mid (D + {}^u\mathbb{A} \pm J\,{}^u\mathbb{A}J^\dagger)U\xi \rangle = \langle U\xi \mid U(D + \mathbb{A} \pm J\mathbb{A}J^\dagger)\xi \rangle = I(\xi, \mathbb{A}).$$

A remark on curvature. In Yang–Mills models, the fermionic action is supplemented by a *bosonic action* that is a quadratic functional of the gauge fields or curvatures associated to the gauge potential $\mathbb{A}$. One may formulate the curvature of a connection in noncommutative geometry and obtain a Yang–Mills action; indeed, this is the main component of the Connes–Lott models [28]. One can formally introduce the curvature as $\mathbb{F} := \mathrm{d}\mathbb{A} + \mathbb{A}^2$, where the notation means

$$\mathrm{d}\mathbb{A} := \textstyle\sum_j [D, a_j]\,[D, b_j] \qquad \text{whenever} \quad \mathbb{A} = \sum_j a_j[D, b_j]. \tag{8.2}$$

Regrettably, this definition is flawed, since the first sum may be nonzero in cases where the second sum vanishes [22, VI.1]. For instance, in the commutative case, one may have $a[\not{D}, a] - [\not{D}, \frac{1}{2}a^2] = \gamma(a\,da - d(\frac{1}{2}a^2)) = 0$ but $[\not{D}, a]\,[\not{D}, a] = \gamma(da)^2 = -(da \mid da) < 0$. If we push ahead anyway, we can make a formal check that $\mathbb{F}$ transforms under the gauge group $\mathcal{U}(\mathcal{A})$ by ${}^u\mathbb{F} = u\mathbb{F}u^*$. Indeed,

$$\begin{aligned} \mathrm{d}({}^u\mathbb{A}) &= [D, u]\,[D, u^*] + \textstyle\sum_j [D, ua_j]\,[D, b_j u^*] - \sum_j [D, ua_j b_j]\,[D, u^*] \\ &= [D, u]\,[D, u^*] + [D, u]\mathbb{A}u^* - u\mathbb{A}[D, u^*] + \textstyle\sum_j u[D, a_j]\,[D, b_j]u^*, \end{aligned}$$

whereas, using the identity $u[D, u^*]u = -[D, u]$, we have

$$\begin{aligned} ({}^u\mathbb{A})^2 &= u[D, u^*]u[D, u^*] + u[D, u^*]u\mathbb{A}u^* + u\mathbb{A}[D, u^*] + u\mathbb{A}^2u^* \\ &= -[D, u]\,[D, u^*] - [D, u]\mathbb{A}u^* + u\mathbb{A}[D, u^*] + u\mathbb{A}^2u^*, \end{aligned}$$

and consequently

$${}^u\mathbb{F} := \mathrm{d}({}^u\mathbb{A}) + ({}^u\mathbb{A})^2 = u(\mathrm{d}\mathbb{A} + \mathbb{A}^2)u^* = u\mathbb{F}u^*.$$

Provided that the definition (8.2) can be corrected, one can then define a gauge-invariant action [59] as the symmetrized Yang–Mills type functional

$$⨍ (\mathbb{F} + J\mathbb{F}J^\dagger)^2\,ds^n = ⨍ (\mathbb{F} + J\mathbb{F}J^\dagger)^2\,|D|^{-n},$$

since

$$⨍ ({}^u\mathbb{F} + J\,{}^u\mathbb{F}J^\dagger)^2\,|{}^uD|^{-n} = ⨍ U(\mathbb{F} + J\mathbb{F}J^\dagger)^2\,|D|^{-n}U^* = ⨍ (\mathbb{F} + J\mathbb{F}J^\dagger)^2\,|D|^{-n}.$$

The ambiguity in (8.2) can be removed by introducing the $\mathcal{A}$-bimodule $(\Omega^1_D)^2/J_2$, where the subbimodule J_2 consists of the so-called "junk" terms $\sum_j [D, a_j]\,[D, b_j]$ for which $\sum_j a_j[D, b_j] = 0$. Then, by redefining $\mathbb{F}$ as the orthogonal projection of $\mathrm{d}\mathbb{A} + \mathbb{A}^2$ on the orthogonal complement of J_2 in $(\Omega^1_D)^2$, one gets a well-defined curvature and the noncommutative integral of its square gives the desired Yang–Mills action.

The spectral action principle

This Yang–Mills action, evaluated on a suitable geometry, achieves the remarkable feat of reproducing the classical Lagrangian of the Standard Model. This is discussed at length in [22, VI] and in several other places [10, 13, 66, 76, 86]. However, its computation leads to fearsome algebraic manipulations and very delicate handling of the junk terms, leading one to question whether this action is really fundamental.

The seminal paper [25] makes an alternative proposal. The unitary equivalence $D \mapsto D + u[D, u^*] \pm Ju[D, u^*]J^\dagger$ is a perturbation by internal diffeomorphisms, and one can regard the Morita equivalence $D \mapsto D + \mathbb{A} \pm J\mathbb{A}J^\dagger$ as an *internal perturbation* of D. The correct bosonic action functional should not merely be *diffeomorphism invariant* (where by diffeomorphisms we mean automorphisms of $\mathcal{A}$), that is to say, "of purely gravitational nature", but one can go further and ask that it be *spectrally invariant*. As stated unambiguously by Chamseddine and Connes [14]:

"The physical action only depends upon $\operatorname{sp}(D)$."

The fruitfulness of this viewpoint has been exemplified by Landi and Rovelli [80, 81], who consider the eigenvalues of the Dirac operator as dynamical variables for general relativity.

Since quantum corrections must still be provided for [1], the particular action chosen should incorporate a cutoff scale Λ (roughly comparable to inverse Planck length, or Planck mass, where the commutative spacetime geometry must surely break down), and some suitable cutoff function: $\phi(t) \geq 0$ for $t \geq 0$ with $\phi(t) = 0$ for $t \gg 1$. Therefore, Chamseddine and Connes propose a bosonic action of the form

$$B_\phi(D) = \operatorname{Tr} \phi(D^2/\Lambda^2). \tag{8.3}$$

This spectral action turns out to include not only the Standard Model bosonic action but also the Einstein–Hilbert action for gravity, plus some higher-order gravitational terms, thereby establishing it firmly as an action for an effective field theory at low energies. We refer to [14, 65, 106] for the details of how all these terms emerge in the calculation. Most of these terms can also be recovered by an alternative procedure involving Quillen's superconnections [48], which seems to suggest that the Chamseddine–Connes action is in the nature of things. Here we must limit ourselves to the humble computational task of explaining the general method of extracting such terms from (8.3), by a spectral asymptotic development in the cutoff parameter Λ.

Spectral densities and asymptotics

We consider the general problem of providing the functional (8.3) with an asymptotic expansion as $\Lambda \to \infty$, without prejudging the particular cutoff function ϕ. In any case, as we shall see, the dependence of the final results on ϕ is very weak. There is, of course, a great deal of accumulated experience with the related heat kernel expansion for pseudodifferential operators [54]. One can adapt the heat kernel expansion [14] to develop (8.3), under the tacit assumption that ϕ is a Laplace transform. However, we take a more direct route, avoiding the detour through the heat kernel expansion.

The basic idea, expounded in detail in [44], is to develop *distributional* asymptotics directly from the **spectral density** of the positive selfadjoint operator $A = D^2$. If the spectral projectors of A are $\{\, E(\lambda) : \lambda \geq 0 \,\}$, the spectral density is the derivative

$$\delta(\lambda - A) := \frac{dE(\lambda)}{d\lambda}$$

that makes sense as a distribution with operatorial values in $\mathcal{L}(\mathcal{H}_\infty, \mathcal{H})$. For instance, when A has discrete spectrum $\{\lambda_j\}$ (in increasing order) with a corresponding orthonormal basis of eigenfunctions u_j, then

$$E(\lambda) = \sum_{\lambda_j \leq \lambda} |u_j\rangle\langle u_j|, \qquad \text{and so} \quad d_A(\lambda) = \sum_{j=1}^{\infty} |u_j\rangle\langle u_j|\, \delta(\lambda - \lambda_j).$$

A functional calculus may be defined by setting

$$f(A) := \int_0^\infty f(\lambda)\, \delta(\lambda - A)\, d\lambda.$$

For instance,

$$A^k = \int_0^\infty \lambda^k\, \delta(\lambda - A)\, d\lambda; \qquad e^{-tA} = \int_0^\infty e^{-t\lambda}\, \delta(\lambda - A)\, d\lambda.$$

For further details of this calculus and the conditions for its validity, we refer to [43, 44].

Spectral densities of pseudodifferential operators. The algebra of the Standard Model spectral triple is of the form $C^\infty(M) \otimes \mathcal{A}_F$, where M is a compactified (Euclidean) spacetime and $\mathcal{A}_F$ is an algebra with a finite basis; in fact, $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, acting on a spinor multiplet space $L^2(M, S) \otimes \mathcal{H}_F \simeq L^2(M, S \otimes \mathcal{H}_F)$ through a finite-dimensional real representation [106]. Thus the operator D is of the form $D\!\!\!/ \otimes 1 + \gamma_5 \otimes D_F$, where D_F is a matrix of Yukawa mass terms and $D\!\!\!/$ is the Dirac operator on the spinor space of M. By Lichnerowicz' formula [7],

$$D\!\!\!/^2 = \Delta^S + \tfrac{1}{4} r,$$

where Δ^S is the spinor Laplacian and r is the scalar curvature. After incorporating the terms from D_F, one finds [14, 65] that D^2 is a generalized Laplacian [7] with matrix-valued coefficients. Thus the task is to compute an expansion for (8.3) under the assumption that $A = D^2$ is a pseudodifferential operator of order $d = 2$.

We suppose, then, that A is a positive, elliptic, classical pseudodifferential operator of order d on an n-dimensional manifold M. If A has symbol $\sigma(A) = a(x, \xi)$ in local coordinates, we ask what the symbol $\sigma(\delta(\lambda - A))$ might be. If A has constant coefficients, the symbol of A^k is just

$$a(x, \xi)^k = \int \lambda^k\, \delta(\lambda - a(x, \xi))\, d\lambda,$$

so $\delta(\lambda - a(x, \xi))$ is the symbol of $\delta(\lambda - A)$ in that particular case. In general, the symbol of A^k depends also on the derivatives of $a(x, \xi)$, so we arrive at the prescription [44]:

$$\begin{aligned} \sigma\big(\delta(\lambda - A)\big) \sim \delta(\lambda - \sigma(A)) - q_1\, \delta'(\lambda - \sigma(A)) + q_2\, \delta''(\lambda - \sigma(A)) \\ - \cdots + (-1)^k q_k\, \delta^{(k)}(\lambda - \sigma(A)) + \cdots \qquad \text{as } \lambda \to \infty. \end{aligned} \tag{8.4}$$

By computing $\int \lambda^k \, \sigma(\delta(\lambda - A)) \, d\lambda$ for $k = 0, 1, 2, \dots$, we get $q_1 = 0$,

$$q_2(x, \xi) = \tfrac{1}{2}\bigl(\sigma(A^2) - \sigma(A)^2\bigr), \qquad q_3(x, \xi) = \tfrac{1}{6}\bigl(\sigma(A^3) - 3\sigma(A^2)\sigma(A) + 2\sigma(A)^3\bigr),$$

and so on. The order of the symbol q_2 is $\leq (2d - 1)$, the order of q_3 is $\leq (3d - 2)$, etc.

Cesàro calculus. An important technical issue is how to interpret the distributional development (8.4). On subtracting the first N terms on the right from the left hand side, one needs a distribution that falls off like λ^{α_N} as $\lambda \to \infty$, with exponents α_N that decrease to $-\infty$. It turns out that this holds, in a *Cesàro-averaged* sense [44]. To be precise, a distribution f is of order λ^α at infinity, in the Cesàro sense:

$$f(\lambda) = O(\lambda^\alpha) \quad (C) \qquad \text{as } \lambda \to \infty,$$

if there is, for some N, a function f_N whose Nth distributional derivative equals f, such that $f_N(\lambda) = p(\lambda) + O(\lambda^{\alpha+N})$ as $\lambda \to \infty$ with p a polynomial of degree $< N$. If $\sum_{n=1}^\infty a_n$ is a Cesàro-summable series, then the distribution $f(\lambda) := \sum_{n=1}^\infty a_n \, \delta(\lambda - n)$ satisfies

$$\int_0^\infty f(\lambda) \, d\lambda \sim \sum_{n=1}^\infty a_n \quad (C).$$

Let us recall that the symbol of A is defined by writing $Au(x) = \int k_A(x, y) u(y) \, d^n y$ where the kernel is the distribution

$$k_A(x, y) := (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x, \xi) \, d^n \xi,$$

and in particular, on the diagonal:

$$k_A(x, x) = (2\pi)^{-n} \int a(x, \xi) \, d^n \xi.$$

The kernel for the spectral density $\delta(\lambda - A)$ is then given on the diagonal by

$$d_A(x, x; \lambda) \sim (2\pi)^{-n} \int \bigl[\delta(\lambda - a(x, \xi)) + q_2(x, \xi) \, \delta''(\lambda - a(x, \xi)) - \cdots\bigr] \, d^n \xi \quad (C). \tag{8.5}$$

By the functional calculus, the action functional (8.3) may then be expressed as

$$\operatorname{Tr} \phi(D^2/\Lambda^2) = \int_M \int_0^\infty \phi(\lambda/\Lambda^2) d_A(x, x; \lambda) \, d\lambda \, \sqrt{g(x)} \, d^n x,$$

provided one learns the trick of integrating a Cesàro development to get a parametric development in $t = \Lambda^{-2}$ as $t \downarrow 0$.

Parametric developments. Some distributions have zero Cesàro expansion, namely those f for which $f(\lambda) = o(\lambda^{-\infty})$ (C) as $|\lambda| \to \infty$. These coincide with the dual space $\mathcal{K}'$

of the space $\mathcal{K}$ of GLS symbols [61]: elements of $\mathcal{K}$ are smooth functions ϕ such that for some α, $\phi^{(k)}(\lambda) = O(|\lambda|^{\alpha-k})$ as $|\lambda| \to \infty$. The space $\mathcal{K}$ includes all polynomials, so any $f \in \mathcal{K}'$ has *moments* $\mu_k := \int \lambda^k f(\lambda)\,d\lambda$ of all orders. Indeed, $\mathcal{K}'$ is precisely the space of distributions that satisfy the *moment asymptotic expansion* [45]:

$$f(\sigma\lambda) \sim \sum_{k=0}^{\infty} \frac{(-1)^k \mu_k\, \delta^{(k)}(\lambda)}{k!\,\sigma^{k+1}} \qquad \text{as } \sigma \to \infty.$$

This a parametric development of $f(\lambda)$. For a general distribution, we may have a Cesàro expansion in falling powers of λ:

$$f(\lambda) \sim \sum_{k\geq 1} c_k \lambda^{\alpha_k} \quad (C) \qquad \text{as } \lambda \to \infty,$$

and the corresponding parametric development is of the form [45]:

$$f(\sigma\lambda) \sim \sum_{k\geq 1} c_k (\sigma\lambda)^{\alpha_k} + \sum_{m\geq 0} \frac{(-1)^m \mu_m\, \delta^{(m)}(\lambda)}{m!\,\sigma^{m+1}} \qquad \text{as } \sigma \to \infty. \tag{8.6}$$

(This is an oversimplification, valid only if no α_k is a negative integer: the general case is treated in [45] and utilized in [44].)

The moral is this: if one knows the Cesàro development, the parametric development is available also, assuming that one can compute the moments that appear in (8.6). Then one can evaluate on a test function by a change of variable, obtaining an ordinary asymptotic expansion in a new parameter:

$$\int f(\lambda)\phi(t\lambda)\,d\lambda \sim \sum_{k\geq 1} c_k\, t^{-\alpha_k-1} \int_0^\infty \lambda^{\alpha_k}\phi(\lambda)\,d\lambda + \sum_{m\geq 0} \frac{\mu_m\,\phi^{(m)}(0)}{m!}\, t^m \qquad \text{as } t \downarrow 0. \tag{8.7}$$

(The integral on the right is to be regarded as a finite-part integral; also, when some α_k are negative integers, there are extra terms in $t^r \log t$.) The heat kernel development may be obtained by taking $\phi(\lambda) := e^{-\lambda}$ for $\lambda \geq 0$.

The spectral coefficients. The coefficients of the spectral density kernel (8.5), after integration over ξ, have an intrinsic meaning: in fact, they are all Wodzicki residues! More precisely, it has been argued in [44] that (8.5) simplifies to

$$\begin{aligned} d_A(x,x;\lambda)\,d^n x \sim \frac{1}{d\,(2\pi)^n}\big[&\mathrm{wres}_x(A^{-n/d})\lambda^{(n-d)/d} + \mathrm{wres}_x(A^{(1-n)/d})\lambda^{(n-d-1)/d} \\ &+ \cdots + \mathrm{wres}_x(A^{(k-n)/d})\lambda^{(n-d-k)/d} + \cdots\big]\ (C) \qquad \text{as } \lambda \to \infty. \end{aligned} \tag{8.8}$$

It is worth indicating briefly how this comes about: we shall compute the leading term in (8.8). To integrate (8.5), we use polar coordinates $\xi = r\omega$ with $|\omega| = 1$. Since the integrand involves $\delta(\lambda - a(x, r\omega))$ and its derivatives, we must solve $a(x, r\omega) = \lambda$ for r; denote the unique positive solution by $r = r(x, \omega; \lambda)$. To solve, we revert the expansion

$$\lambda = a(x, r\omega) \sim \sum_{j\geq 0} a_{d-j}(x,\omega)\, r^{d-j} \qquad \text{as } r \to \infty \tag{8.9}$$

to get a development in falling powers of λ:

$$r = r(x,\omega;\lambda) \sim \sum_{k\geq 0} r_k(x,\omega)\, \lambda^{(1-k)/d} \qquad \text{as } \lambda \to \infty. \tag{8.10}$$

Now we unpack the distribution

$$\delta(\lambda - a(x, r\omega)) = \frac{\delta(r(x,\omega;\lambda))}{a'(x, r(x,\omega;\lambda)\omega)}.$$

Since $d^n\xi = r^{n-1}\, dr\, \sigma_\omega$, the first term in (8.5) yields

$$(2\pi)^{-n} \int_{|\omega|=1} \frac{r^{n-1}(x,\omega;\lambda)}{a'(x, r(x,\omega;\lambda)\omega)}\, \sigma_\omega. \tag{8.11}$$

If we retain only the first terms in (8.9) and (8.10), this integrand becomes

$$\frac{r^{n-1}}{d\, r^{d-1} a_d(x,\omega)} \sim \frac{r_0(x,\omega)^{n-d} \lambda^{(n-d)/d}}{d\, a_d(x,\omega)} \sim \frac{\lambda^{(n-d)/d}}{d} a_d(x,\omega)^{-n/d},$$

since $r_0(x,\omega) = a_d(x,\omega)^{-1/d}$. Thus the leading term in the λ-development of (8.11) is

$$\frac{\lambda^{(n-d)/d}}{d\,(2\pi)^n} \int_{|\omega|=1} a_d(x,\omega)^{-n/d}\, \sigma_\omega,$$

and it remains only to notice that $a_d(x,\omega)^{-n/d}$ is the principal symbol, of order $(-n)$, of the operator $A^{-n/d}$.

Spectral densities of generalized Laplacians. We can apply this general machinery to the case where A is a generalized Laplacian, with a symbol of the form

$$a(x,\xi) = -g^{ij}(x)\,\xi_i\xi_j + b^k(x)\xi_k + c(x),$$

where b^k, c are scalar functions on M. Rewriting (8.8) as

$$d\,(2\pi)^n\, d_A(x,x;\lambda) \sim a_0(x)\lambda^{(n-d)/d} + a_1(x)\lambda^{(n-d-1)/d} + a_2(x)\lambda^{(n-d-2)/d} + \cdots,$$

we see that $a_0(x)$ is constant with value Ω_n. Also, $a_k = 0$ for odd k since their computation involves integrating odd powers of the ω_j over the sphere $|\omega| = 1$. For $a_2(x)$, one can express the metric in normal coordinates [7]:

$$g_{ij}(x) \sim \delta_{ij} + \tfrac{1}{3} R_{iklj}(x_0)\,(x-x_0)^k (x-x_0)^l + \cdots$$

where R_{iklj} is the Riemann curvature tensor. The q_2 term of (8.5) extracts from this a Ricci-tensor term $\frac{1}{3}R_{kj}(x)\,\xi_k\xi_j$ and integration over the unit sphere leaves the scalar curvature $r(x)$. The upshot is that

$$a_2(x) = \frac{(n-2)\Omega_n}{2}\big(\tfrac{1}{6}r(x) - c(x)\big). \tag{8.12}$$

On the other hand, this gives the Wresidue density of $A^{(2-n)/2}$. We thus arrive at one of the most striking results in noncommutative geometry, derived independently by Kastler [74] and Kalau and Walze [69], namely that the Einstein–Hilbert action of general relativity is a multiple of the Wodzicki residue of $\not{D}^{-2}$ on a 4-dimensional manifold:

$$\operatorname{Wres} \not{D}^{-2} \propto \int_M r(x)\,\sqrt{g(x)}\,d^4x,$$

on combining (8.12) with the Lichnerowicz formula $c = \frac{1}{4}r$. The computation of $\operatorname{Wres} D^{-2}$ for the Standard Model is given in [14, 65].

The Chamseddine–Connes action. Pulling all the threads together, we apply the expansion (8.12) to the action functional (8.3). For the Standard Model plus gravity, we take $n = 4$ and $A = D^2$, a generalized Laplacian, acting on a space of sections of a vector bundle E over M. From (8.8) we get

$$d_{D^2}(x, x; \lambda) \sim \frac{\operatorname{rank} E}{16\pi^2}\lambda + \frac{1}{32\pi^4}\operatorname{wres}_x D^{-2} \quad (C) \qquad \text{as } \lambda \to \infty$$

since the nonnegative powers of the differential operator D^2 have zero Wresidue. Applying (8.7) with $t = \Lambda^{-2}$ gives an expansion of the form

$$\operatorname{Tr}\phi(D^2/\Lambda^2) \sim \frac{\operatorname{rank} E}{16\pi^2}\phi_0\,\Lambda^4 + \frac{1}{32\pi^4}\operatorname{Wres} D^{-2}\phi_2\,\Lambda^2 + \sum_{m\geq 0} b_{2m+4}(D^2)\phi_{2m+4}\,\Lambda^{-2m}$$

as $\Lambda \to \infty$, where $\phi_0 = \int_0^\infty \lambda\phi(\lambda)\,d\lambda$, $\phi_2 = \int_0^\infty \phi(\lambda)\,d\lambda$ and $\phi_{2m+4} = (-1)^m\phi^{(m)}(0)$ for $m = 0, 1, 2, \ldots$. Thus the cutoff function ϕ plays only a minor rôle, and these integrals and derivatives may be determined from experimental data.

In [14] detailed results are given for the spectral triple associated to the Standard Model. The bosonic parts of the SM appear in the Λ^2 and Λ^0 terms; the Einstein–Hilbert action appears in the Λ^2 term, as expected; and other gravity pieces and a gravity-Higgs coupling in the Λ^0 term; the Λ^4 term is cosmological. The Λ^0 term is conformally invariant. Higher-order terms may be neglected.

Thus the stage is set for a theory that encompasses gravity and matter fields on the same footing. However, it must, when we find it, be a fully quantum theory; and that is for the future.

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