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Application of Lie transformation group methods to classical linear theories of rods and plates

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Abstract

In the present paper, a class of partial differential equations governing various rod and plate theories of Bernoulli–Euler and Poisson–Kirchhoff type is studied by Lie transformation group methods. A system of equations determining the generators of the admitted point Lie groups (symmetries) is derived and the general statement of the associated group-classification problem is given. A simple relation is deduced allowing to recognize easily the variational symmetries among the "ordinary" symmetries of a self-adjoint equation of the class examined. Explicit formulae for the conserved currents of the corresponding (via Bessel-Hagen's extension of Noether's theorem) conservation laws are suggested. Solutions of group-classification problems are given for subclasses of equations of the foregoing type governing stability and vibration of rods, fluid conveying pipes and plates resting on variable elastic foundations. The obtained group-classification results are used to derive conservation laws and group-invariant solutions readily applicable in rod dynamics and plate statics and dynamics. New generalized symmetries and conservation laws for the theories of Timoshenko beams, Reissner–Mindlin plates and three-dimensional elastostatics are presented.

Keywords: Rods; Plates; Bernoulli–Euler beams; Timoshenko beams; Poisson–Kirchhoff plates; Reissner–Mindlin plates; Elastostatics; Lie symmetry groups; Generalized symmetries; Conservation laws; Group-invariant solutions

1. Introduction

The main objective of the present paper is to analyze from group-theoretical point of view several well known and acknowledged linear theories for elastic rods [1] and plates which could be called "classical", though some of them have been suggested quite recently, in the sense to be clear below.

Historically, interest in the development of theories describing the mechanical behaviour of slender or thin solid bodies (such as rods and plates) from three-dimensional models by a dimensional reduction can be traced more than three centuries back to Leibniz who introduced the idea to use the average of the stress over the cross-sections of the body for constructing one-dimensional theories for rods. The first complete

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[1] In this work, following Antman (1984) we use "rod" as a generic name for "arch", "bar", "beam", "ring", "column", "tube", "pipe", etc. We employ "rod" in the intuitive sense of a slender solid body.

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theory of that kind was the Bernoulli–Euler beam theory based on Hooke's law (as the constitutive relation) and the (kinematical) hypotheses that a plane cross-section normal to the rod axis, which is supposed to be inextensible and initially straight, remains undeformed and normal to the bent rod axis, the latter being assumed planar. Approximately one century later, the Poisson–Kirchhoff theory for thin elastic plates emerged out of linear elastostatics and the well known Kirchhoff's hypotheses that a straight fiber normal to the plate middle-plane remains undeformed and normal to the bent plate middle-plane, the latter being assumed inextensible. Starting from Euler's celebrated study on the stability of an axially compressed column, the elementary Bernoulli–Euler beam theory has been subjected to various modifications within the framework of the aforementioned hypotheses in order to cover the effects of elastic foundations, follower forces, fluid flow, etc.; similarly, the Poisson–Kirchhoff plate theory has been extended within Kirchhoff's hypotheses in numerous works to describe, for instance, the stability of plates, the influence of elastic foundations, fluid-structure interaction, etc. (see, e.g., Bolotin, 1961; Vlasov and Leont'ev, 1966; Paidoussis, 1998; Elishakoff, 2002). In the present paper, such contributions to the classical theories of Bernoulli–Euler and Poisson–Kirchhoff are also referred to as "classical", for easy reference and to distinguish them from more sophisticated, in general nonlinear, theories for rods and plates (see, e.g., Antman, 1984, 1995; Naghdi, 1984; Simo et al., 1988; Dichmann et al., 1996 and the references therein).

In the classical theories, the state of equilibrium of an elastic rod or plate is fully determined in terms of the transversal displacement of the rod axis or plate middle-plane. The corresponding governing equations are linear fourth-order partial differential equations in one dependent variable, the transversal displacement function, and one or two independent variables—the coordinates of the rod axis or plate middle-plane, respectively. By introducing the inertial force in transversal direction, according to d'Alembert principle, all these theories are recognized and employed to describe the dynamic behaviour of rods and plates as well. In this case, the time appears as an additional independent variable in the governing equations. To complete the description of the theories that are in the focus of attention of the present work, it is to be noticed, that many of them can be set in a variational statement involving only one dependent variable, the corresponding governing equations being the Euler–Lagrange equations associated with an appropriate action functional.

Each of the classical rod or plate theories can be viewed as a certain one- or two-dimensional approximation of three-dimensional elasticity achieved by a systematic use of projection methods (see, e.g., Antman, 1984; Naghdi, 1984; Niordson, 1985). However, during such a dimensional reduction procedure many details are necessarily lost either for the highly restrictive kinematical hypotheses adopted or since only certain averages of the stresses over the rod cross-sections or plate thickness, respectively, are taken into account. In consequence of that, as a rule, the invariance properties (symmetries) inherent to such a "proper" rod or plate theory can neither be reduced to, nor be derived in full from the invariance properties of one-, two- or three-dimensional elasticity. Therefore, in order to analyze a proper rod or plate theory of the forgoing type from group-theoretical point of view, one should thoroughly study the symmetries of its governing equations, resisting the temptation to merely take advantage of the allied results established in the theory of elasticity, which are at disposal in a long series of papers: Chirkunov (1973, 1975), Fletcher (1976), Olver (1984a,b, 1988), Şuhubi (1987, 1989), Honein and Herrmann (1997), Hatfield and Olver (1998), etc. In our opinion, the study of the invariance properties of Bernoulli–Euler type rod equations and Poisson–Kirchhoff type plate equations is far away from its completion; there are only a few works in which results in this field can be found, namely those by Ovsiannikov (1972), Ibragimov (1985), Kienzler (1986), Sosa et al. (1988), Vassilev (1988, 1997), Chien et al. (1993, 1994), Tabarrok et al. (1994) and Vassilev et al. (2000). That is why, in the present work, bearing in mind the general form of the governing equations specified above, we shall apply Lie transformation group methods [2] to examine the invariance of a generic

[2] The foundations of the Lie transformation group methods, including the basic notions, statements, techniques and many applications of the symmetries of differential equations and variational problems, can be found in Ovsiannikov (1982), Ibragimov (1985) and Olver (1993) (see also the references therein).

linear fourth-order partial differential equation in one dependent and several independent variables with respect to local Lie groups of point transformations of the involved independent and dependent variables. Then, we shall use the results obtained to analyze several rod and plate theories of particular interest. The work is motivated both by the wide applicability of the theories in question in structural mechanics, and by the remarkable efficiency demonstrated by the symmetry methods, especially when applied to differential equations arising in physics and engineering.

Actually, once the invariance properties of a given differential equation are established, several important applications of its symmetries arise. First, it is possible to distinguish classes of solutions to this equation which are invariant under the transformations of symmetry groups admitted. The determination of such a group-invariant solution assumes solving a reduced equation involving less independent variables than the original one. Typical examples of group-invariant solutions are axisymmetric solutions, self-similar solutions, travelling waves, etc., which have proved to be quite useful in many branches of physics and engineering. For a self-adjoint differential equation another substantial application of its symmetries is available. As it is well known, the self-adjoint equations are the Euler–Lagrange equations of a certain action functional. If a one-parameter symmetry group of such an equation turns out to be its variational symmetry as well, that is a symmetry of the associated action functional, then Bessel-Hagen's (1921) extension of Noether's theorem (1918) guarantees the existence of a conservation law for the smooth solutions of this equation. Needless to recall or discuss here the fundamental role that the conserved quantities and conservation laws (or the corresponding integral relations, i.e. the balance laws) have played in natural sciences, but it is worth pointing out that the available conservation laws (balance laws) should not be overlooked (as it is often done) in the numerical analysis (when constructing finite difference schemes or verifying numerical results, for instance) or in the examination of discontinuous solutions (acceleration waves, shock waves, etc.) of any system of differential equations of physical interest.

As a matter of fact, the conservation laws have attracted much attention in fracture analysis and defect mechanics of solids and structures just because they have proved useful in the analysis of jump-discontinuities such as notches and cracks. Here, the conservation laws appeared as path-independent integrals, for the first time in Cherepanov (1967) and then, independently, in Rice (1968) who introduced the so-called J-integral and showed its utility in the asymptotic analysis of the stress field near notches and cracks in linearly elastic solids. Later, Budiansky and Rice (1973) introduced next two path-independent integrals (L and M) in linear elastostatics and showed their applicability for calculation of the energy release rate resulting from cavity motion. It should be remarked however, that the J-integral could be derived using the conservative properties of the energy–momentum tensor proposed earlier by Eshelby (1956) as well as be identified, together with L- and M-integrals, among the conservation laws for isotropic homogeneous linear elastostatics established by Günther (1962) and Knowles and Sternberg (1972).

Apparently, Günther (1962) initiated the analysis of the theory of elasticity from group-theoretical point of view as far as his study is based on Noether's theorem. However, his work remained unnoticed and ten years later Knowles and Sternberg (1972) starting anew presented the conservation laws associated through Noether's theorem with translational and rotational invariance of the equations of homogeneous isotropic linear and finite elastostatics. They also derived another conservation law associated with the scale-invariance of the equations of linear homogeneous isotropic elastostatics. The case of linear homogeneous elastodynamics is examined in Fletcher (1976). The aforementioned works concern only geometric symmetries of the equations considered. In a series of papers, Olver (1984a,b, 1988) and Hatfield and Olver (1998) presented a comprehensive classification of the conservation laws related to the invariance under the geometric and first-order generalized symmetries of the equations of elastostatics. All these contributions show the advantages that one can gain from the study of the invariance properties of a theory of physical interest and the successive derivation of conservation laws using Noether's theorem instead of using for that purpose ad hoc techniques or physical arguments.

Unfortunately, as underlined before, the knowledge of the invariance properties of the theory of elasticity is of little usage when the symmetries of a proper rod or plate theory are to be established. The present paper is intended to gain an insight into this latter problem.

The layout of the paper is as follows. A detailed description of the differential equations to be studied as well as the variational statement for the self-adjoint equations among them are given in Section 2. In Section 3, a system of equations determining the generators of the symmetry groups admitted by the equations of the class considered is derived and the general statement of the associated group-classification problem is given. Then, the variational symmetries of the self-adjoint equations of the examined class are investigated. A simple relation allowing to recognize easily the variational symmetries among the "ordinary" point Lie symmetries of such an equation is deduced, and explicit formulae for the conserved currents of the conservation laws corresponding to the variational symmetries via Bessel-Hagen's extension of Noether's theorem are suggested. Group-classification results, conservation laws and group-invariant solutions are presented in Section 4 for differential equations governing vibration of rods on a variable elastic foundation and dynamic stability of fluid conveying pipes. Similar results are displayed in Section 5 for the equations governing stability and vibration of plates of Poisson–Kirchhoff type. In Section 6, new generalized symmetries and conservation laws for the theories of Timoshenko beams, Reissner–Mindlin plates and three-dimensional elastostatics are presented. Finally, in Section 7, one can find an extended summary of the results obtained as well as practical hints on how to use this article without going into detail concerning the Lie group analysis of differential equations.

2. Basic equations

Consider the class of fourth-order linear homogeneous partial differential equations

$$A^{\alpha\beta\gamma\delta}(x)w_{\alpha\beta\gamma\delta} + A^{\alpha\beta\gamma}(x)w_{\alpha\beta\gamma} + A^{\alpha\beta}(x)w_{\alpha\beta} + A^{\alpha}(x)w_{\alpha} + A(x)w = 0, \tag{1}$$

in n independent variables $x = (x^1, \ldots, x^n)$ and one dependent variable $w(x)$. Here and throughout: Greek indices have the range $1, 2, \ldots, n$, unless explicitly stated otherwise; the usual summation convention over a repeated index is employed; $w_{\alpha_1\alpha_2\cdots\alpha_k}$ $(k = 1, 2, \ldots)$ denote (as it is accepted in the group analysis of differential equations) the kth order partial derivatives of the dependent variable with respect to the independent variables, i.e.

$$w_{\alpha_1\alpha_2\cdots\alpha_k} = \frac{\partial^k w}{\partial x^{\alpha_1}\partial x^{\alpha_2}\cdots\partial x^{\alpha_k}} \quad (k = 1, 2, \ldots).$$

Further, a similar notation will be used for the partial derivatives of any other function of the variables $x^1, \ldots, x^n$ but, in this case, the indices indicating the differentiation will be preceded by a comma, e.g.,

$$A^{\alpha\beta\gamma\delta}_{,\alpha_1\alpha_2\cdots\alpha_k} = \frac{\partial^k A^{\alpha\beta\gamma\delta}}{\partial x^{\alpha_1}\partial x^{\alpha_2}\cdots\partial x^{\alpha_k}} \quad (k = 1, 2, \ldots).$$

The coefficients of Eq. (1) are supposed to be smooth functions possessing as many derivatives as may be required on a certain domain of interest, and to be symmetric under any permutation of their indices, i.e.

$$A^{\alpha\beta\gamma\delta} = A^{\beta\alpha\gamma\delta} = A^{\gamma\delta\alpha\beta} = A^{\alpha\gamma\beta\delta}, \quad A^{\alpha\beta\gamma} = A^{\beta\alpha\gamma} = A^{\gamma\beta\alpha} = A^{\alpha\gamma\beta}, \quad A^{\alpha\beta} = A^{\beta\alpha}.$$

Using the total derivative operators

$$D_\alpha = \frac{\partial}{\partial x^\alpha} + w_\alpha\frac{\partial}{\partial w} + w_{\alpha\mu}\frac{\partial}{\partial w_\mu} + w_{\alpha\mu\nu}\frac{\partial}{\partial w_{\mu\nu}} + w_{\alpha\mu\nu\sigma}\frac{\partial}{\partial w_{\mu\nu\sigma}} + \cdots,$$

Eq. (1) may be written in the form

$$\mathscr{D}[w] = 0, \tag{2}$$

where $\mathscr{D}$ is the linear differential operator given by the expression

$$\mathscr{D} = A^{\alpha\beta\gamma\delta} D_\alpha D_\beta D_\gamma D_\delta + A^{\alpha\beta\gamma} D_\alpha D_\beta D_\gamma + A^{\alpha\beta} D_\alpha D_\beta + A^\alpha D_\alpha + A. \tag{3}$$

An equation of form (2) is the Euler–Lagrange equation associated with a certain variational problem involving only one dependent variable if and only if the differential operator $\mathscr{D}$ is self-adjoint, that is

$$\mathscr{D} = \mathscr{D}^*, \tag{4}$$

where $\mathscr{D}^*$ is the (formal) adjoint operator of $\mathscr{D}$ (cf. Olver, 1993). The explicit form of $\mathscr{D}^*$ is

$$\mathscr{D}^* = D_\alpha D_\beta D_\gamma D_\delta A^{\alpha\beta\gamma\delta} - D_\alpha D_\beta D_\gamma A^{\alpha\beta\gamma} + D_\alpha D_\beta A^{\alpha\beta} - D_\alpha A^\alpha + A. \tag{5}$$

In such a case, Eq. (2) can be associated with the variational problem for the functional

$$A[w] = \int \frac{1}{2} w \mathscr{D}[w] \, \mathrm{d}x^1 \cdots \mathrm{d}x^n,$$

since the application of the Euler operator

$$\mathsf{E} = \frac{\partial}{\partial w} - D_\mu \frac{\partial}{\partial w_\mu} + D_\mu D_\nu \frac{\partial}{\partial w_{\mu\nu}} - D_\mu D_\nu D_\sigma \frac{\partial}{\partial w_{\mu\nu\sigma}} + D_\mu D_\nu D_\sigma D_\tau \frac{\partial}{\partial w_{\mu\nu\sigma\tau}} - \cdots$$

on the Lagrangian density

$$L = \tfrac{1}{2} w \mathscr{D}[w] \tag{6}$$

yields

$$\mathscr{D}[w] = \mathsf{E}(L) \tag{7}$$

due to relations (4) and (5).

3. Symmetries and conservation laws

Consider a local one-parameter Lie group of point transformations acting on some open subset Ω of the space $\mathbf{R}^{n+1}$ representing the independent and dependent variables $x^1, \ldots, x^n$ and w involved in our basic equation (2). The infinitesimal generator of such a group is a vector field X on the space $\mathbf{R}^{n+1}$,

$$X = \xi^\mu(x, w) \frac{\partial}{\partial x^\mu} + \eta(x, w) \frac{\partial}{\partial w}, \tag{8}$$

whose components $\xi^\mu(x, w)$ and $\eta(x, w)$ are supposed to be functions of class C^∞ on Ω. By virtue of Theorem 2.31 (Olver, 1993), a vector field X of form (8) generates a point Lie symmetry group of Eq. (2) (or, in other words, the equation admits this vector field) if and only if there exists a function λ depending on x, w and derivatives of w (that is a differential function) such that the following infinitesimal criterion of invariance,

$$\underset{4}{X}(\mathscr{D}[w]) - \lambda \mathscr{D}[w] = 0, \tag{9}$$

holds; here $\underset{k}{X}$ denotes the k-th prolongation of the vector field X (Ovsiannikov, 1982).

The invariance criterion (9) leads, through the standard computational procedure (see, e.g., Ovsiannikov, 1982 or Olver, 1993), to the following results:

(i) each equation of form (2), being linear and homogeneous, is invariant under the point Lie groups generated by the vector fields

$$X_0 = w\frac{\partial}{\partial w}, \quad X_u = u(x)\frac{\partial}{\partial w}, \tag{10}$$

where $u(x)$ is an arbitrary solution of the equation considered, the invariance criterion (9) being fulfilled with $\lambda = 1$ and $\lambda = 0$ for the generators X_0 and X_u, respectively;

(ii) an equation of form (2) admits other vector fields (8), in addition to the aforementioned (10), if and only if they have the special form

$$X = \xi^\mu(x)\frac{\partial}{\partial x^\mu} + \sigma(x)w\frac{\partial}{\partial w}, \tag{11}$$

the functions $\xi^\mu(x)$ and $\sigma(x)$ being nontrivial solutions of the following system of determining equations (called further the DE system for easy reference):

$$\xi^\mu A^{\alpha\beta\gamma\delta}_{,\mu} + (\sigma - \lambda)A^{\alpha\beta\gamma\delta} - A^{\alpha\beta\gamma\mu}\xi^\delta_{,\mu} - A^{\alpha\beta\mu\delta}\xi^\gamma_{,\mu} - A^{\alpha\mu\gamma\delta}\xi^\beta_{,\mu} - A^{\mu\beta\gamma\delta}\xi^\alpha_{,\mu} = 0, \tag{12}$$

$$4A^{\alpha\beta\gamma\mu}\sigma_{,\mu} - 2A^{\alpha\beta\mu\nu}\xi^\gamma_{,\mu\nu} - 2A^{\alpha\gamma\mu\nu}\xi^\beta_{,\mu\nu} - 2A^{\beta\gamma\mu\nu}\xi^\alpha_{,\mu\nu} + \xi^\mu A^{\alpha\beta\gamma}_{,\mu} + (\sigma - \lambda)A^{\alpha\beta\gamma} - A^{\alpha\beta\mu}\xi^\gamma_{,\mu} - A^{\alpha\mu\gamma}\xi^\beta_{,\mu} - A^{\mu\beta\gamma}\xi^\alpha_{,\mu} = 0, \tag{13}$$

$$\begin{aligned} &6A^{\alpha\beta\mu\nu}\sigma_{,\mu\nu} - 2A^{\alpha\mu\nu\sigma}\xi^\beta_{,\mu\nu\sigma} - 2A^{\beta\mu\nu\sigma}\xi^\alpha_{,\mu\nu\sigma} + 3A^{\alpha\beta\mu}\sigma_{,\mu} - (3/2)A^{\alpha\mu\nu}\xi^\beta_{,\mu\nu} - (3/2)A^{\beta\mu\nu}\xi^\alpha_{,\mu\nu} + \xi^\mu A^{\alpha\beta}_{,\mu} \\ &\quad + (\sigma - \lambda)A^{\alpha\beta} - A^{\alpha\mu}\xi^\beta_{,\mu} - A^{\mu\beta}\xi^\alpha_{,\mu} = 0, \end{aligned} \tag{14}$$

$$4A^{\alpha\mu\nu\sigma}\sigma_{,\mu\nu\sigma} - A^{\mu\nu\sigma\tau}\xi^\alpha_{,\mu\nu\sigma\tau} + 3A^{\alpha\mu\nu}\sigma_{,\mu\nu} - A^{\mu\nu\sigma}\xi^\alpha_{,\mu\nu\sigma} + 2A^{\alpha\mu}\sigma_{,\mu} - A^{\mu\nu}\xi^\alpha_{,\mu\nu} + \xi^\mu A^\alpha_{,\mu} + (\sigma - \lambda)A^\alpha - A^\mu\xi^\alpha_{,\mu} = 0, \tag{15}$$

$$A^{\alpha\beta\gamma\delta}\sigma_{,\alpha\beta\gamma\delta} + A^{\alpha\beta\gamma}\sigma_{,\alpha\beta\gamma} + A^{\alpha\beta}\sigma_{,\alpha\beta} + A^\alpha\sigma_{,\alpha} + \xi^\mu A_{,\mu} + (\sigma - \lambda)A = 0, \tag{16}$$

for a certain function λ depending on x only. (Here, by a trivial solution we mean not only $\xi^\mu = 0$, $\sigma = 0$, but also $\xi^\mu = 0$, $\sigma = c = \text{const} \neq 0$, since the latter leads to the vector field cX_0 generating the same group as X_0 which is already identified to be admitted by each equation of the type considered.)

Thus, given an equation of form (2), the question is whether there exist vector fields $X \neq cX_0$ of form (11) which leave it invariant, and the answer depends on whether the respective DE system has at least one nontrivial solution. In this context the coefficients of Eq. (2) are supposed to be known functions, and thereby Eqs. (12)–(16) constitute an over-determined system of linear homogeneous partial differential equations with respect to the unknowns ξ^μ and σ. Therefore, as a rule, it turns out possible to find in an explicit form some (or even all) nontrivial solutions of the DE system, and thus to determine several (all) additional point Lie symmetry groups inherent to the equation in question.

It should be remarked that various equations of form (2) admit only the point Lie groups generated by the vector fields (10) with $u(x)$ being any solution of the respective equation. For instance, it is easy to check that all equations of the form (35) such that $\chi^{\alpha\beta} = \delta^{\alpha\beta}$ and $\kappa(x) = p(x)$, where $p(x)$ is an arbitrary polynomial of x^1 and x^2, belong to this variety. Without too much difficulties one can ascertain that the same holds true for the equations of the form (44) with $N^{\alpha\beta} = \delta^{\alpha\beta}$ and $k(x) = p(x)$.

On the other hand, there are equations of the foregoing type which are invariant under a larger group; an immediate example is the biharmonic equation in two independent variables, $\Delta^2 w = 0$, which admits the seven-parameter group generated by the linear combinations of X_0 and the following six additional basic vector fields (cf. Ovsiannikov, 1972):

$$\frac{\partial}{\partial x^1}, \quad x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}, \quad 2x^1x^2\frac{\partial}{\partial x^1} - [(x^1)^2 - (x^2)^2]\frac{\partial}{\partial x^2} + 2x^2w\frac{\partial}{\partial w},$$
$$\frac{\partial}{\partial x^2}, \quad x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}, \quad [(x^1)^2 - (x^2)^2]\frac{\partial}{\partial x^1} + 2x^1x^2\frac{\partial}{\partial x^2} + 2x^1w\frac{\partial}{\partial w}.$$

An important problem naturally arises in the light of the above note. It may be placed in the category of the so-called group-classification problems (see Ovsiannikov, 1982) and consists in determining all those equations of the type considered that admit a larger group together with this group itself. Its most general statement assumes all functions $A^{\alpha\beta\gamma\delta}(x)$, $A^{\alpha\beta\gamma}(x)$, $A^{\alpha\beta}(x)$, $A^{\alpha}(x)$, $A(x)$, $\xi^{\alpha}(x)$ and $\sigma(x)$ involved in the determining equations (12)–(16) to be regarded as unknown variables and to find all solutions of this system. Here we are not going to study this rather complicated nonlinear problem in general. However, in the subsequent sections we examine the group-classification problems for several subclasses of equations of form (2) widely used in mechanics of solids and structures.

Let us now specialize to the case of self-adjoint equations of form (2). Suppose that

$$\mathscr{D}[w] = 0, \quad \mathscr{D} = \mathscr{D}^* \tag{17}$$

is such an equation. Then, a particular interest exists for its variational symmetries—the Lie groups generated by the so-called infinitesimal divergence symmetries (see Definition 4.33 in Olver, 1993) of any variational functional with (17) as the associated Euler–Lagrange equation. (Note that if two functionals lead to the same Euler–Lagrange equation, then they have the same collection of infinitesimal divergence symmetries.) This interest is motivated by the fact that, in virtue of Bessel-Hagen's extension of Noether's theorem, each variational symmetry of a given self-adjoint equation corresponds to a conservation law admitted by the smooth solutions of the equation. Thus, if a vector field X of form (8) is found to generate a variational symmetry of Eq. (17), then Bessel-Hagen's extension of Noether's theorem implies the existence of a conserved current, which, in the present case, is a n-tuple of differential functions P^{α} such that

$$D_{\alpha}P^{\alpha} = Q\mathscr{D}[w], \tag{18}$$

where Q is the characteristic of the vector field X; by definition

$$Q = \eta - w_{\mu}\xi^{\mu}. \tag{19}$$

The total divergence of the conserved current P^{α} vanishes on the smooth solutions of Eq. (17) and so we have the conservation law

$$D_{\alpha}P^{\alpha} = 0, \tag{20}$$

Eq. (18) being its expression in characteristic form, and Q—its characteristic. Therefore, to derive the conservation laws of the foregoing type, one can proceed by first determining the variational symmetries of Eq. (17), and then using their characteristics (19) to find, from equality (18), explicit expressions for the corresponding conserved currents P^{α}.

Having analyzed earlier the invariance properties of the whole class of Eq. (2), it is convenient to base the determination of the variational symmetries of Eq. (17) on the following observation. A vector field X of form (8) generates a variational symmetry of Eq. (17) if and only if X is an infinitesimal symmetry of this equation, that is the infinitesimal criterion of invariance (9) holds, and

$$\underset{4}{X}(\mathscr{D}[w]) + \left(\frac{\partial\eta}{\partial w} + D_{\mu}\xi^{\mu}\right)\mathscr{D}[w] = 0. \tag{21}$$

This is a consequence of Lemma 4.34 and Proposition 5.55 (Olver, 1993), see also Lemma 7.46 (Olver, 1995). Subtracting expression (9) from (21) we can replace the latter with

$$\left(\frac{\partial \eta}{\partial w} + D_\mu \xi^\mu + \lambda\right)\mathscr{D}[w] = 0,$$

and as $\mathscr{D}[w]$ is not supposed to vanish identically we arrive at the conclusion that

$$\frac{\partial \eta}{\partial w} + D_\mu \xi^\mu + \lambda = 0 \tag{22}$$

is a necessary and sufficient condition for an infinitesimal symmetry admitted by a self-adjoint equation of form (2) to be its infinitesimal variational symmetry as well. It should be remarked that the same holds true for any self-adjoint partial differential equation in one dependent variable. For a vector field of form (11) relation (22) simplifies, and reads

$$\sigma + \xi^\mu_{,\mu} + \lambda = 0. \tag{23}$$

Thus to find the variational symmetries of an equation of form (17), it suffices to check which of its "ordinary" symmetries satisfy the additional requirement (22). For instance, the result (i) implies that $X_0 = w\partial/\partial w$ does not generate a variational symmetry of any equation of form (17), while a vector field $X_u = u(x)\partial/\partial w$ generates a variational symmetry of an equation of form (17) whenever $u(x)$ is its solution (this is a common property of all systems of linear homogeneous partial differential equations, see Section 5.3 in Olver, 1993).

Suppose one has established that a vector field X with characteristic Q generates a variational symmetry of a given equation of form (17), and now wishes to find the conserved current P^α of the corresponding conservation law (20). For this purpose one can use formulae (5.150) and (5.151) given by Olver (1993) which express (in an explicit form) a null Lagrangian as a divergence. Indeed, in this case the right-hand side of equality (18) is a total divergence or, in other words, a null Lagrangian. However, bearing in mind the recommendation of Olver (1993) to use these formulae only as a last resort since "the homotopy formula (5.151) can rapidly become unmanageable", in the present paper we suggest another way for determination of the sought conserved current.

Our starting point is the so-called Noether identity (cf. Ibragimov, 1985):

$$\underset{\infty}{X}(\mathscr{L}) + (D_\alpha \xi^\alpha)\mathscr{L} = Q\mathsf{E}(\mathscr{L}) + D_\alpha N^\alpha(\mathscr{L}), \tag{24}$$

which holds for any differential function $\mathscr{L}$ and vector field X of the types considered here, N^α being the differential operators given by the expressions

$$\begin{aligned} N^\alpha = \xi^\alpha + Q\left\{\frac{\partial}{\partial w_\alpha} + \sum_{s\geqslant 1}(-1)^s D_{v_1}\cdots D_{v_s}\frac{\partial}{\partial w_{\alpha v_1\cdots v_s}}\right\} \\ + \sum_{r\geqslant 1}(D_{\mu_1}\cdots D_{\mu_r}Q)\left\{\frac{\partial}{\partial w_{\alpha\mu_1\cdots\mu_r}} + \sum_{s\geqslant 1}(-1)^s D_{v_1}\cdots D_{v_s}\frac{\partial}{\partial w_{\alpha\mu_1\cdots\mu_r v_1\cdots v_s}}\right\}, \end{aligned} \tag{25}$$

where $Q = \eta - \xi^\alpha w_\alpha$ is the characteristic of the vector field X. By setting $\mathscr{L} = L$ in identity (24) and taking into account expressions (6) and (7), one obtains (after a little manipulation) the identity

$$D_\mu N^\mu(-w\mathscr{D}[w]) = -w\underset{4}{X}(\mathscr{D}[w]) - \{\eta + (D_\mu\xi^\mu)w - 2Q\}\mathscr{D}[w], \tag{26}$$

valid for any self-adjoint differential operator $\mathscr{D}$ of form (3) and vector field of form (8).

In particular, for $X_v = v(x)\partial/\partial w$, where $v(x)$ is an arbitrary smooth function, we have

$$\xi^\alpha = 0, \quad Q = \eta = v, \tag{27}$$

and hence

$$\underset{4}{X_v}(\mathscr{D}[w]) = \mathscr{D}[v], \tag{28}$$

since $\mathscr{D}$ is a linear differential operator. Substituting expressions (27) and (28) into identity (26) we obtain

$$D_\mu N^\mu(-w\mathscr{D}[w]) = v\mathscr{D}[w] - w\mathscr{D}[v], \tag{29}$$

which is nothing but the reciprocity relation associated with the equation $\mathscr{D}[w] = 0$. Under the additional assumption $v = u(x)$, where $u(x)$ is an arbitrary smooth solution of the latter equation, the reciprocity relation (29) becomes

$$D_\mu N^\mu(-w\mathscr{D}[w]) = u\mathscr{D}[w]. \tag{30}$$

Taking into account expression (30), we can give now the following general formula for the conserved currents P^α of the conservation laws with characteristics $Q = u$ corresponding to the infinitesimal variational symmetries $X_u = u\,\partial/\partial w$ of Eq. (17):

$$P^\alpha = P^\alpha_{(u)} + G^\alpha,$$

where

$$P^\alpha_{(u)} = N^\alpha(-w\mathscr{D}[w]), \tag{31}$$

and G^α is a current of a trivial conservation law. Of course,

$$D_\mu P^\mu_{(u)} = u\mathscr{D}[w]$$

and

$$D_\mu P^\mu_{(u)} = 0 \tag{32}$$

on the smooth solutions of Eq. (17).

Next, let X be an infinitesimal variational symmetry of an equation of form (17) with characteristic $Q = w\sigma - w_\mu\xi^\mu$. Then, on account of equality (21), identity (26) takes the form

$$D_\mu N^\mu(-\tfrac{1}{2}w\mathscr{D}[w]) = Q\mathscr{D}[w],$$

and hence we can write down the following explicit formula for the conserved currents P^α of the conservation laws with characteristics $Q = w\sigma - w_\mu\xi^\mu$ corresponding to the aforementioned variational symmetries of the considered equation of form (17), namely

$$P^\alpha = B^\alpha + G^\alpha, \tag{33}$$

$$B^\alpha = N^\alpha(-\tfrac{1}{2}wD[w]) + \tfrac{1}{2}D_\mu(w\xi^\alpha A^{\mu\beta\gamma\delta}D_\beta D_\gamma D_\delta w - w\xi^\mu A^{\alpha\beta\gamma\delta}D_\beta D_\gamma D_\delta w), \tag{34}$$

where, as before, G^α is a current of a trivial conservation law. Of course,

$$D_\mu B^\mu = Q\mathscr{D}[w],$$

and on the smooth solutions of the considered equation of form (17) we have

$$D_\mu B^\mu = 0.$$

Let us remark that the special null divergence,

$$\tfrac{1}{2}D_\mu\left(w\xi^\alpha A^{\mu\beta\gamma\delta}D_\beta D_\gamma D_\delta w - w\xi^\mu A^{\alpha\beta\gamma\delta}D_\beta D_\gamma D_\delta w\right),$$

is used in expression (34) for the conserved current B^α to cut away the fourth-order derivatives of the dependent variable w since in practice one is usually interested in conserved currents which involve derivatives of order not higher than $k - 1$, where k is the order of the equation considered. Using the operators (25) it is easy to check that the right-hand side of expression (34) incorporates derivatives of the variable w of order less than fourth. In the subsequent sections just (31) and (34) will be referred to as the expressions for the conserved currents of the conservation laws with characteristics $Q = u$ and $Q = w\sigma - w_\mu \xi^\mu$, respectively, derived for equations of the form (17).

To summarize, given an equation of form (17), the crucial point on the way of deriving conservation laws admitted by its smooth solutions is to find vector fields of form (11) generating "ordinary" point Lie symmetries of the given equation. For that purpose, we should look for solutions of the respective DE system (12)–(16). Once such vector fields are found, it is easy to check which of their linear combinations satisfy the requirement (22) and hence generate variational symmetries of the equation considered. Now, using the characteristics of these symmetries we first construct the operators N^α from formulae (25) and then calculate from formula (34) the conserved currents of the corresponding conservation laws.

4. Symmetries, conservation laws and group-invariant solutions of rod equations within Bernoulli–Euler theory

Consider a subclass of Eqs. (1), with $n = 2$, consisting of the self-adjoint partial differential equations

$$\gamma w_{1111} + \chi^{\alpha\beta} w_{\alpha\beta} + \kappa(x) w = 0, \tag{35}$$

where $\gamma = \mathrm{const} \neq 0$, $\chi^{\alpha\beta}$ are arbitrary constants (but $(\chi^{12})^2 + (\chi^{22})^2 \neq 0$, otherwise Eq. (35) degenerates and becomes ordinary differential equation), and $\kappa(x)$ is an arbitrary function. Equations of this special type are used by many authors to study applied engineering problems concerning dynamics and stability of both elastic beams resting on elastic foundations (see, e.g., Smith and Herrmann, 1972) and pipes conveying fluid (see, e.g., Paidoussis, 1998). In this context, the dependent variable w is the transversal displacement of the rod axis, x^1—the coordinate along this axis and x^2—the time.

First of all, the group-classification problem is considered. In view of the results (i) and (ii) of Section 3, it is clear that each equation of form (35) is invariant under the point Lie groups generated by the vector fields $X_0 = w\partial/\partial w$ and $X_u = u(x)\partial/\partial w$, where $u(x)$ is any smooth solution of the foregoing equation and the objective is to find those equations of the type considered which admit vector fields X of form (11), $X \neq cX_0$, $c = \mathrm{const} \neq 0$.

The system of determining equations (12)–(16) simplifies considerably for the equations of form (35). The solution of this simplified system involves lengthy computations which are omitted here (for details we refer to Vassilev et al., 2000). The results of the group-classification analysis of the class of Eq. (35) are summarized in Table 1, where the equations invariant under larger groups are given through their coefficients together with the generators of the admitted symmetry groups. In Table 1, β, β^1 and β^2 are arbitrary constants and for convenience the following vector fields are introduced:

$$Y_\alpha = \frac{\partial}{\partial x^\alpha}, \quad Y_3 = \left(x^1 + \frac{\chi^{12}}{\chi^{22}} x^2\right)\frac{\partial}{\partial x^1} + 2x^2\frac{\partial}{\partial x^2}, \quad Y_4 = \left(x^1 + \frac{\chi^{11}}{\chi^{12}} x^2\right)\frac{\partial}{\partial x^1} + 3x^2\frac{\partial}{\partial x^2}.$$

Having completely solved the group-classification problem, our next step is to identify the variational symmetries of those equations of form (35) which are found to admit larger symmetry groups. For this purpose, we are to apply condition (22) to the linear combinations of the vector field $X_0 = w\partial/\partial w$ and the vector fields presented in Table 1 with $\lambda = \sigma - 4\xi^1_{,1}$. Omitting the details, we found that all vector fields quoted under numbers 1, 3, 5, 7, 8, 9 and 11 generate variational symmetries of the respective equations. In case #2, the variational symmetries are generated by the vector field $[\beta^1 + 2(\chi^{12}/\chi^{22})\beta^2]Y_1 + 2\beta^2 Y_2 + Y_3 + (1/2)X_0$, in case #4—by Y_1 and $2\beta Y_2 + Y_3 + (1/2)X_0$, in case #6—by $(\chi^{12}/\chi^{22})Y_1 + Y_2$ and $\beta Y_1 + Y_3 + (1/2)X_0$, and in case #10—by Y_1, Y_2 and $Y_3 + (1/2)X_0$.

Once the variational symmetries of the differential equations of form (35) are identified, we are ready to derive the corresponding conservation laws. The conserved currents of the conservation laws for the

Table 1
Equations of form (35) invariant under larger symmetry groups

#	Coefficients	Generators
1	$\kappa(x) = f(\beta^2 x^1 - \beta^1 x^2)$	$\beta^1 Y_1 + \beta^2 Y_2$
2	$\chi^{22} \neq 0,\ \det(\chi^{\alpha\beta}) = 0,\ \kappa(x) = (\beta^2 + x^2)^{-2} f(y),$ $y = (\beta^2 + x^2)^{-1/2}[\beta^1 + x^1 - (\chi^{12}/\chi^{22})x^2]$	$[\beta^1 + 2(\chi^{12}/\chi^{22})\beta^2]Y_1 + 2\beta^2 Y_2 + Y_3$
3	$\chi^{22} = 0,\ \det(\chi^{\alpha\beta}) \neq 0,\ \kappa(x) = (\beta^2 + x^2)^{-4/3} f(y),$ $y = (\beta^2 + x^2)^{-1/3}[\beta^1 + 2x^1 - (\chi^{11}/\chi^{12})x^2]$	$[\beta^1 + 3(\chi^{11}/\chi^{12})\beta^2]Y_1 + 6\beta^2 Y_2 + 2Y_4$
4	$\chi^{22} \neq 0,\ \det(\chi^{\alpha\beta}) = 0,\ \kappa(x) = \kappa_0(\beta + x^2)^{-2},$	$Y_1,\ 2\beta Y_2 + Y_3$
5	$\chi^{22} = 0,\ \det(\chi^{\alpha\beta}) \neq 0,\ \kappa(x) = \kappa_0(\beta + x^2)^{-4/3}$	$Y_1,\ 3\beta Y_2 + Y_4$
6	$\chi^{22} \neq 0,\ \det(\chi^{\alpha\beta}) = 0,\ \kappa(x) = \kappa_0(\beta + x^1 - (\chi^{12}/\chi^{22})x^2)^{-4}$	$\beta Y_1 + Y_3,\ (\chi^{12}/\chi^{22})Y_1 + Y_2$
7	$\chi^{22} = 0,\ \det(\chi^{\alpha\beta}) \neq 0,\ \kappa(x) = \kappa_0(\beta + 2x^1 - (\chi^{11}/\chi^{12})x^2)^{-4}$	$\beta Y_1 + 2Y_4,\ (\chi^{11}/\chi^{12})Y_1 + 2Y_2$
8	$\chi^{22} \det(\chi^{\alpha\beta}) \neq 0,\ \kappa(x) = \mathrm{const}$	$Y_1,\ Y_2$
9	$\chi^{22} \det(\chi^{\alpha\beta}) = 0,\ \kappa(x) = \mathrm{const} \neq 0$	$Y_1,\ Y_2$
10	$\chi^{22} \neq 0,\ \det(\chi^{\alpha\beta}) = 0,\ \kappa(x) = 0$	$Y_1,\ Y_2,\ Y_3$
11	$\chi^{22} = 0,\ \det(\chi^{\alpha\beta}) \neq 0,\ \kappa(x) = 0$	$Y_1,\ Y_2,\ Y_4$

equations given in Table 1 are computed using formula (34), the above notes concerning the variational symmetries being taken into account. The obtained conservation laws are listed in Table 2 (in the same order as in Table 1) in terms of the differential functions:

$$B^1_{(1)} = -\tfrac{1}{2}[\gamma(2w_1 w_{111} - w_{11}^2) + \chi^{11} w_1^2 - \chi^{22} w_2^2 + \kappa w^2] - \tfrac{1}{2}(\chi^{2\mu} w w_\mu)_{,2},$$

$$B^2_{(1)} = -\chi^{2\mu} w_1 w_\mu + \tfrac{1}{2}(\chi^{2\mu} w w_\mu)_{,1},$$

$$B^1_{(2)} = -\chi^{1\mu} w_2 w_\mu + \gamma(w_{11} w_{12} - w_2 w_{111}) - \tfrac{1}{2}(\gamma w_1 w_{11} - \chi^{1\mu} w w_\mu)_{,2},$$

$$B^2_{(2)} = -\tfrac{1}{2}(\gamma w_{11}^2 + \chi^{22} w_2^2 - \chi^{11} w_1^2 + \kappa w^2) + \tfrac{1}{2}(\gamma w_1 w_{11} - \chi^{1\mu} w w_\mu)_{,1},$$

$$B^\alpha_{(3)} = [x^1 + (\chi^{12}/\chi^{22})x^2]B^\alpha_{(1)} + 2x^2 B^\alpha_{(2)} + \chi^{\alpha\mu} w w_\mu + \tfrac{1}{2}\gamma\delta^{1\alpha}(w w_{111} - w_1 w_{11}),$$

Table 2
Conservation laws for equations of form (35)

#	Conservation laws
1	$D_\alpha[\beta^1 B^\alpha_{(1)} + \beta^2 B^\alpha_{(2)}] = 0$
2	$D_\alpha[(\beta^1 + 2(\chi^{12}/\chi^{22})\beta^2)B^\alpha_{(1)} + 2\beta^2 B^\alpha_{(2)} + B^\alpha_{(3)}] = 0$
3	$D_\alpha[(\beta^1 + 3(\chi^{11}/\chi^{12})\beta^2)B^\alpha_{(1)} + 6\beta^2 B^\alpha_{(2)} + 2B^\alpha_{(4)}] = 0$
4	$D_\alpha B^\alpha_{(1)} = 0,\ D_\alpha[2\beta B^\alpha_{(2)} + B^\alpha_{(3)}] = 0$
5	$D_\alpha B^\alpha_{(1)} = 0,\ D_\alpha[3\beta B^\alpha_{(2)} + B^\alpha_{(4)}] = 0$
6	$D_\alpha[\beta B^\alpha_{(1)} + B^\alpha_{(3)}] = 0,\ D_\alpha[(\chi^{12}/\chi^{22})B^\alpha_{(1)} + B^\alpha_{(2)}] = 0$
7	$D_\alpha[\beta B^\alpha_{(1)} + 2B^\alpha_{(4)}] = 0,\ D_\alpha[(\chi^{11}/\chi^{12})B^\alpha_{(1)} + 2B^\alpha_{(2)}] = 0$
8	$D_\alpha B^\alpha_{(1)} = 0,\ D_\alpha B^\alpha_{(2)} = 0$
9	$D_\alpha B^\alpha_{(1)} = 0,\ D_\alpha B^\alpha_{(2)} = 0$
10	$D_\alpha B^\alpha_{(1)} = 0,\ D_\alpha B^\alpha_{(2)} = 0,\ D_\alpha B^\alpha_{(3)} = 0$
11	$D_\alpha B^\alpha_{(1)} = 0,\ D_\alpha B^\alpha_{(2)} = 0,\ D_\alpha B^\alpha_{(4)} = 0$

$$B^{\alpha}_{(4)} = [x^1 + (\chi^{11}/\chi^{12})x^2]B^{\alpha}_{(1)} + 3x^2 B^{\alpha}_{(2)} + \tfrac{1}{2}[\chi^{\alpha\mu} w w_{\mu} + \delta^{1\alpha}(\chi^{11} w w_1 + 2\chi^{12} w w_2 - \gamma w_1 w_{11})].$$

According to the general results of Section 3, each equation of form (35) admits conservation laws with characteristics $Q = u(x)$ as well, where $u(x)$ is any smooth solution of the equation considered. These conservation laws are of form (32), that is

$$D_{\alpha} P^{\alpha}_{(u)} = 0,$$

the corresponding conserved currents $P^{\alpha}_{(u)}$ being given by formula (31), which in the case under consideration simplifies and reads

$$P^{\alpha}_{(u)} = \chi^{\alpha\mu}(u w_{\mu} - u_{,\mu} w) + \delta^{1\alpha}\gamma(u w_{111} + u_{,11} w_1 - u_{,111} w - u_{,1} w_{11}). \tag{36}$$

Let us now specialize to the differential equations governing the small transversal vibration of elastic pipes of uniform thickness and outer radius of the pipe cross-section, conveying inviscid fluid of flow velocity $U = \text{const}$, compressed by an axial end force $p = \text{const}$ and lying on a Winkler foundation with stiffness $c = \text{const}$:

$$EJ w_{1111} + (p + MU^2) w_{11} + 2MU w_{12} + (m + M) w_{22} + cw = 0, \tag{37}$$

where EJ is the pipe bending rigidity while m and M are the masses of the pipe and the fluid per unit length, respectively (see Paidoussis, 1998). Obviously, Eq. (37) belongs to the class (35) with coefficients

$$\gamma = EJ, \quad \chi^{11} = p + MU^2, \quad \chi^{22} = m + M, \quad \chi^{12} = \chi^{21} = MU, \quad \kappa(x) = c. \tag{38}$$

In this case $\chi^{22} \neq 0$, $\kappa = \text{const}$ and hence, according to the above results concerning the whole class (35), Eq. (37) admits two infinitesimal variational symmetries Y_1 and Y_2 for arbitrary values of its coefficients, and an additional one, $Y_3 + (1/2)X_0$, in the special case $\det(\chi^{\alpha\beta}) = 0$ and $\kappa = 0$, that is when

$$p(m + M) + mMU^2 = 0, \quad c = 0. \tag{39}$$

Since the independent variables x^1 and x^2 are the spatial variable along the pipe axis and the time, respectively, each conservation law admitted by the smooth solutions of Eq. (37) may be written in the more familiar form

$$\frac{\partial \Psi}{\partial x^2} + \frac{\partial P}{\partial x^1} = 0, \tag{40}$$

where Ψ is the density and P is the flux of the conservation law. The densities and fluxes of the conservation laws for Eq. (37) related to the vector fields Y_1 and Y_2 are

$$\Psi_1 = MU w_1 w_1 + (m + M) w_1 w_2, \quad P_1 = EJ w_1 w_{111} + cw^2 - \mathscr{E},$$

$$\Psi_2 = \mathscr{E}, \quad P_2 = EJ(w_2 w_{111} - w_{11} w_{12}) + (p + MU^2) w_1 w_2 + MU w_2 w_2,$$

$$\mathscr{E} = \tfrac{1}{2}[EJ w_{11}^2 + (m + M) w_2^2 - (p + MU^2) w_1^2 + cw^2].$$

The conservation law with density Ψ_1 and flux P_1 corresponds to conservation of wave momentum, while that with density $\Psi_2 = \mathscr{E}$ and flux P_2 represents conservation of energy. These two conservation laws hold for arbitrary values of the pipe parameters EJ, M, m, U, p and c. In the special case when equalities (39) hold (note that this may happen only if the force p is negative, that is if the pipe is extended by it, as the constants U, m, M should be positive due to their physical meaning), Eq. (37) admits an additional conservation law related to the vector field $Y_3 + (1/2)X_0$ with density and flux

$$\Psi_3 = \left(x^1 + \frac{MU}{m+M}x^2\right)\left\{\Psi_1 - \frac{1}{2}[MUww_1 + (m+M)ww_2]_{,1}\right\}$$
$$+ x^2\left\{2\Psi_2 - \left[EJw_1w_{11} - \frac{M^2U^2}{m+M}ww_1 - MUww_2\right]_{,1}\right\} - w[MUw_1 + (m+M)w_2],$$

$$P_3 = \left(x^1 + \frac{MU}{m+M}x^2\right)\left\{P_1 + \frac{1}{2}[MUww_1 + (m+M)ww_2]_{,2}\right\} - \frac{1}{2}EJ(ww_{111} - w_1w_{11})$$
$$+ x^2\left\{2P_2 + \left[EJw_1w_{11} - \frac{M^2U^2}{m+M}ww_1 - MUww_2\right]_{,2}\right\} - \frac{M^2U^2}{m+M}ww_1 - MUww_2.$$

In order to clarify the significance and applicability of these results in the theory of fluid conveying pipes, let us emphasize the following substantial feature of the suggested density Ψ_2 and flux P_2 of the energy conservation law. Recall that, according to formula (33) given in Section 3, the energy conservation law

$$\frac{\partial \Psi_2}{\partial x^2} + \frac{\partial P_2}{\partial x^1} = 0, \tag{41}$$

could be modified by means of a trivial conservation law with current G^α, say taking $G^1 = \partial F/\partial x^1$ and $G^2 = -\partial F/\partial x^2$, where F is a smooth differential function, to the form

$$\frac{\partial}{\partial x^2}\left(\Psi_2 + \frac{\partial F}{\partial x^1}\right) + \frac{\partial}{\partial x^1}\left(P_2 - \frac{\partial F}{\partial x^2}\right) = 0.$$

Thus, one has at disposal a multitude of mathematically equivalent densities $\Psi_2 + \partial F/\partial x^1$ and fluxes $P_2 - \partial F/\partial x^2$ for the energy conservation law. Under these circumstances, a natural question arises: which of the aforesaid densities is to be referred to as "proper" energy density of the fluid conveying pipe. The answer relies on a purely physical argument: the flux corresponding to this "proper" energy density should represent the rate of work done on the pipe by the external forces and couples. Integrating the law (41) over the pipe length ($a \leqslant x^1 \leqslant b$) one obtains

$$\frac{\partial}{\partial x^2}\int_a^b \mathscr{E}\,\mathrm{d}x^1 = EJw_{11}w_{12}|_a^b - EJw_2w_{111}|_a^b + pw_1w_2|_a^b + (MUw_2w_2 + MU^2w_1w_2)|_a^b.$$

The left-hand side of this equality is the rate of change of total energy of the pipe, while the right-hand side represents the rate of work done on the pipe by the external forces and couples. Indeed, $EJw_{11}w_{12}$ represents the rate of work done by the bending moment EJw_{11} over the angular velocity w_{12} of the pipe cross-section, EJw_2w_{111} is the rate of work done by the shear force EJw_{111} over the transversal velocity w_2 of the pipe cross-section, pw_1w_2 is the rate of work done by the transversal projection pw_1 of the axial force p over the transversal velocity w_2, and the terms MUw_2w_2 and $MU^2w_1w_2$ represent the rate of work done by forces and couples due to the fluid flow (see Benjamin, 1961; Paidoussis, 1998). On the basis of this argument, the suggested expressions for Ψ_2 and P_2 are recognized as the "proper" energy density and the rate of work done on the pipe, respectively.

Substituting $\gamma = EJ$, $\chi^{11} = \chi^{12} = 0$, $\chi^{22} = m$ and $\kappa(x) = 0$ in Eq. (35), one arrives at the well known differential equation

$$EJw_{1111} + mw_{22} = 0, \tag{42}$$

governing the dynamics of a classical homogeneous Bernoulli–Euler beam of bending rigidity EJ and mass per unit length m. Eq. (42) admits the following five linearly independent infinitesimal variational symmetries: Y_1, Y_2, $Y_3 + (1/2)X_0$, $Y_5 = \partial/\partial w$ and $Y_6 = x^2\partial/\partial w$, where Y_5 and Y_6 are vector fields of the type $X_u = u(x)\partial/\partial w$ corresponding to the solutions $u = 1$ and $u = x^2$ of Eq. (42), respectively. The densities and fluxes of the conservation laws for Eq. (42) associated with these vector fields together with their physical interpretation are presented in Table 3.

Conservation laws in the dynamics of rods are considered in many papers (see, e.g., Antman, 1984; Chien et al., 1993; Maddocks and Dichmann, 1994; Tabarrok et al., 1994; Djondjorov, 1995) with or without reference to the symmetries of the respective governing equations. In order to identify the novelties in this field achieved in the present contribution, the results presented in some of the aforementioned works are discussed below.

Chien et al. (1993), employing a technique called by these authors neutral action method, derive conservation laws for the dynamics of nonhomogeneous Bernoulli–Euler beams of bending rigidity $B(x^1)$ and inertia term $H(x^1)$ governed by the equation

$$Bw_{1111} + 2B_{,1}w_{111} + B_{,11}w_{11} + Hw_{22} = 0. \tag{43}$$

This equation is of form (1) and hence, for the same purpose, the results of Section 3 can be applied. Thus, given an equation of form (43), one can easily ascertain that the general solution (ξ^1, ξ^2, σ) of the determining equations (12)–(16), provided that condition (23) holds, is of the form $\xi^1 = -f^1$, $\xi^2 = -f^2$ and $\sigma = f^3$, where f^i are given by the expressions (33) in Chien et al. (1993); at that, Eqs. (34) in Chien et al. (1993) remain the only conditions (necessary and sufficient) for the existence of a nontrivial solution of the foregoing form. Hence, both approaches lead to the same variety of conservation laws with characteristics $Q = w\sigma - w_\mu\xi^\mu$. The corresponding conserved currents (35) in Chien et al. (1993) coincide (up to a null divergence terms) with those obtainable through our formula (34). As for the conservation laws with characteristics of the form $Q = u(x)$, only a part of them are identified in Chien et al. (1993), namely those with $u(x) = f^4(x)$, f^4 as in (33) in Chien et al. (1993), while any solution $u(x)$ of Eq. (43) gives rise to a conservation law whose current is given explicitly by the general formula (31).

Conservation laws for the dynamics of rods of variable length within the Bernoulli–Euler theory are derived by Tabarrok et al. (1994). Rods, whose length is a linear function of the time, are governed by an equation of form (35) that belongs to the case #10 in Table 1. For such an equation, Tabarrok et al. (1994) suggest three conservation laws which (in our notation) are $D_\alpha B^\alpha_{(1)} = 0$ and $D_\alpha B^\alpha_{(2)} = 0$ given in Table 2 as well as the conservation law with characteristic $Q = u(x)$ and current (36) corresponding to the solution $u(x) = \text{const}$ of this equation. In addition to these, the conservation law $D_\alpha B^\alpha_{(3)} = 0$ from Table 2 and the set of conservation laws with characteristics $Q = u(x)$ and currents (36) corresponding to any solution of the governing equation are presented here.

Table 3
Conservation laws for Bernoulli–Euler beams

Generators	Densities and fluxes of the corresponding conservation laws
Space translations Y_1	*Wave momentum* $\Psi_{(1)} = mw_1w_2$ $P_{(1)} = (1/2)[EJ(2w_1w_{111} - w_{11}^2) - mw_2^2]$
Time translations Y_2	*Energy* $\Psi_{(2)} = (1/2)(EJw_{11}^2 + mw_2^2)$ $P_{(2)} = EJ(w_2w_{111} - w_{11}w_{12})$
Scaling $Y_3 + (1/2)X_0$	$\Psi_3 = x^1\Psi_1 + 2x^2\Psi_2 - EJx^2(w_1w_{11})_{,1} - \frac{1}{2}x^1m(ww_2)_{,1} - mww_2$ $P_{(3)} = x^1P_1 + 2x^2P_2 + EJx^2(w_1w_{11})_{,2} - \frac{1}{2}EJ(ww_{111} - w_1w_{11}) + \frac{1}{2}x^1m(ww_2)_{,2}$
Y_5	*Linear momentum* $\Psi_{(5)} = mw_2$, $P_{(5)} = EJw_{111}$
Galilean boost Y_7	*Center-of-mass theorem* $\Psi_{(7)} = m(x^2w_2 - w)$, $P_{(7)} = EJx^2w_{111}$

Five conservation laws in the dynamics of rods are reported in Maddocks and Dichmann (1994) within a general nonlinear direct theory. The restricted version of this theory describing small planar bending of an uniform inextensible unshearable isotropic elastic rod with a linear constitutive law, the rotatory inertia of the rod cross-section being neglected, is exactly the classic Bernoulli–Euler theory for homogeneous beams whose governing equation is (42). Rewriting the conservation laws in Maddocks and Dichmann (1994) taking into account the aforementioned restrictions we observe that: (1) the conservation law for the total angular momentum (formula 2.14 in Maddocks and Dichmann, 1994) degenerates to the well known basic relation of Bernoulli–Euler theory $Q = \partial M/\partial x^1$ (here Q and M denote shear force and bending moment, respectively, see Washizu, 1982); (2) the density and flux of the conservation law associated with material isotropy (formula 4.5 in Maddocks and Dichmann, 1994) vanish identically; (3) the conservation law corresponding to material homogeneity (formula 3.2 in Maddocks and Dichmann, 1994) reduces to conservation of the wave momentum (see Table 3); (4) the expressions for the densities and fluxes of energy (formula 2.19 in Maddocks and Dichmann, 1994) and linear momentum (formula 2.12 in Maddocks and Dichmann, 1994) conservation laws coincide with the respective ones presented in Table 3. The set of conservation laws with characteristics $Q = u(x)$, where $u(x)$ is any solution of Eq. (42), as well as the conservation law associated with the variational scaling symmetry $Y_3 + (1/2)X_0$ (see Table 3) have no analogues in Maddocks and Dichmann (1994).

Three interesting kinds of group-invariant solutions to certain equations of class (35) are identified below. The first of them corresponds to vector fields $cY_1 \mp Y_2$, where $c = \text{const}$. These group-invariant solutions are travelling waves

$$w = U(s), \quad s = x^1 \pm cx^2,$$

admissible only for Eq. (35) with $\kappa(x^1, x^2) = f(s)$. The reduced equations determining such group-invariant solutions are

$$\gamma \frac{\mathrm{d}^4 U}{\mathrm{d}s^4} + (\chi^{11} \pm 2\chi^{12}c + \chi^{22}c^2)\frac{\mathrm{d}^2 U}{\mathrm{d}s^2} + f(s)U = 0.$$

The second one corresponds to the vector field Y_3 and is of the form

$$w = U(s), \quad s = x^1 (x^2)^{-1/2} - \frac{\chi^{12}}{\chi^{22}} (x^2)^{1/2}.$$

The vector field Y_3 is admitted only if $\kappa(x^1, x^2) = (x^2)^{-2} f(s)$ (see cases #2, 4, 6 and 10 in Table 1). The reduced equations for these invariant solutions are

$$4\gamma \frac{\mathrm{d}^4 U}{\mathrm{d}s^4} + \chi^{22} s^2 \frac{\mathrm{d}^2 U}{\mathrm{d}s^2} + 3\chi^{22} s \frac{\mathrm{d}U}{\mathrm{d}s} + 4f(s)U = 0.$$

The third kind of group-invariant solutions corresponds to the vector field Y_4:

$$w = U(s), \quad s = 2x^1 (x^2)^{-1/3} - \frac{\chi^{11}}{\chi^{12}} (x^2)^{2/3}.$$

The vector field Y_4 is admitted only if $\kappa(x^1, x^2) = (x^2)^{-4/3} f(s)$ (see cases #3, 5, 7 and 11 in Table 1). The reduced equations for the invariant solution under consideration are

$$48\gamma \frac{\mathrm{d}^4 U}{\mathrm{d}s^4} - 4\chi^{12} s \frac{\mathrm{d}^2 U}{\mathrm{d}s^2} - 4\chi^{12} \frac{\mathrm{d}U}{\mathrm{d}s} + 3f(s)U = 0.$$

Obviously, the latter two kinds of group-invariant solutions could be reduced to self-similar solutions if $\chi^{12} = 0$ or $\chi^{11} = 0$, respectively.

5. Symmetries, conservation laws and group-invariant solutions of Poisson–Kirchhoff type plate equations

Consider a thin isotropic elastic plate of bending rigidity $D = \mathrm{const}$ resting on an elastic foundation of Winkler type with variable modulus $k(x^1, x^2)$ and subjected to an edge loading leading to the appearance of nonuniform membrane stresses $N^{\alpha\beta}(x^1, x^2)$. In the framework of Poisson–Kirchhoff plate theory, the equation governing the small bending of the plate is

$$D\Delta^2 w + N^{\alpha\beta} w_{\alpha\beta} + kw = 0, \tag{44}$$

the membrane stress tensor $N^{\alpha\beta}$ being symmetric, $N^{\alpha\beta} = N^{\beta\alpha}$, and divergence free, i.e. $N^{\alpha\mu}_{,\mu} = 0$. Here, the independent variables x^1, x^2 are the coordinates of the plate middle-plane; the dependent variable w represents the transversal displacement field; ν is Poisson's ratio; Δ is the Laplace operator, that is $\Delta \equiv \delta^{\alpha\beta}\partial^2/\partial x^\alpha \partial x^\beta$, where $\delta^{\alpha\beta}$ is the Kronecker delta symbol; throughout this section Greek indices are supposed to take values 1, 2. There is a vast amount of papers in which problems concerning stability and vibration of isotropic thin elastic plates are studied on the ground of this type of equations. Eq. (44) is self-adjoint and belongs to class (1), $n = 2$, with coefficients

$$A^{\alpha\beta\gamma\delta} = \tfrac{1}{3}D(\delta^{\alpha\beta}\delta^{\gamma\delta} + \delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}), \quad A^{\alpha\beta\gamma} = 0, \quad A^{\alpha\beta} = N^{\alpha\beta}, \quad A^{\alpha} = 0, \quad A = k. \tag{45}$$

The aim of this section is to establish the group properties of Eq. (44). First, in view of the general results of Section 3, it is clear that $X_u = u(x)\partial/\partial w$ generates a variational symmetry of any equation of form (44) whenever $u(x)$ is its solution, while $X_0 = w\partial/\partial w$ alone could never generate a variational symmetry of an equation of form (44), though it is always its infinitesimal point Lie symmetry. Substituting expressions (45) into the determining equations (12)–(16) one arrives, after a straightforward computation, at the conclusion that an equation of form (44) is invariant under a point Lie group generated by a vector field X of form (11), $X \neq cX_0$ ($c = \mathrm{const}$), if and only if

$$\sigma = \tfrac{1}{2}\xi^{\mu}_{,\mu}, \tag{46}$$

$$\delta^{\alpha\mu}\xi^{\beta}_{,\mu} + \delta^{\mu\beta}\xi^{\alpha}_{,\mu} - \delta^{\alpha\beta}\xi^{\mu}_{,\mu} = 0, \tag{47}$$

$$\xi^{\mu}N^{\alpha\beta}_{,\mu} - N^{\alpha\mu}\xi^{\beta}_{,\mu} - N^{\mu\beta}\xi^{\alpha}_{,\mu} + 2\xi^{\mu}_{,\mu}N^{\alpha\beta} + 2D\delta^{\alpha\tau}\delta^{\beta\nu}\xi^{\mu}_{,\mu\tau\nu} = 0, \tag{48}$$

$$N^{\alpha\nu}\xi^{\mu}_{,\mu\nu} - N^{\mu\nu}\xi^{\alpha}_{,\mu\nu} = 0, \tag{49}$$

$$2\xi^{\mu}k_{,\mu} + 4\xi^{\mu}_{,\mu}k + N^{\mu\nu}\xi^{\tau}_{,\tau\mu\nu} = 0, \tag{50}$$

$$\lambda = -\tfrac{3}{2}\xi^{\mu}_{,\mu}. \tag{51}$$

Substituting equality (46) into expression (11), and equalities (46) and (51) into condition (23), one can conclude that the generator of such a group is a vector field of form

$$X = \xi^{\mu}\frac{\partial}{\partial x^{\mu}} + \frac{1}{2}\xi^{\mu}_{,\mu}w\frac{\partial}{\partial w}, \tag{52}$$

each symmetry of that kind of Eq. (44) being variational symmetry of the latter equation as well. Hence, there exists a conservation law with characteristic

$$Q = \tfrac{1}{2}\xi^{\mu}_{,\mu}w - w_{\mu}\xi^{\mu}$$

and conserved current B^{α} given by formula (34) admitted by the smooth solutions of the equation considered. Thus, to derive the conservation laws, which correspond to the variational symmetries of an equation of form (44), it suffices to know the results of the group classification of the class of equations in question; of course, the same holds true for the derivation of group-invariant solutions to Eq. (44). This group-classification problem is studied in Vassilev (1988, 1991, 1997). The classification results presented below are based on results obtained in these works.

The scalar fields,

$$s_{(1)} = N^{\mu\nu}\delta_{\mu\nu}, \quad s_{(2)} = (8k - \delta_{\alpha\mu}\delta_{\beta\nu}N^{\alpha\beta}N^{\mu\nu})^{1/2}, \tag{53}$$

are found to be of key importance for the group classification of the considered class of equations. These scalar fields are called invariants of Eq. (44) since here they play a role similar to the role that Laplace's and Cotton's invariants play in the group classification of the second-order linear partial differential equations (see Ovsiannikov, 1982; Ibragimov, 1985). The following two properties of the scalar fields (53) give us both an additional reason to call them invariants of Eq. (44) and explicit expressions for the invariants of groups admitted by Eq. (44). First, if an equation of form (44) admits a vector field of form (52), then

$$\xi^{\mu}_{,\mu}s_{(j)} + \xi^{\mu}s_{(j),\mu} = 0 \quad (j = 1, 2),$$

and hence $U_{(j)} = w\sqrt{s_{(j)}}$ are invariants of the corresponding Lie group whenever $s_{(j)} \not\equiv 0$. Second, if an equation of form (44) admits a vector field of form (52) and is such that its both invariants (53) are not identically equal to zero, then $s_{(1)}/s_{(2)}$ is an invariant of the corresponding symmetry group. Note, that the invariants $s_{(1)}$ and $s_{(2)}$ of such an equation of form (44) provide two couples of functionally independent invariants, namely $U_{(1)} = w\sqrt{s_{(1)}}$ and $s_{(1)}/s_{(2)}$ as well as $U_{(2)} = w\sqrt{s_{(2)}}$ and $s_{(1)}/s_{(2)}$, of the admitted symmetry group, both couples being readily applicable for constructing group-invariant solutions to the respective equation. However, if even one of the invariants (53) of an equation of form (44) is not identically equal to zero, then this equation admits at most a three-parameter group with generators of form (52). On the other hand, if all invariants (53) of an equation of form (44) are identically equal to zero, then this equation admits a six-parameter group with generators of form (52). Below, the latter case is set out in detail.

Let $\omega(z) \not\equiv \text{const}$ be an analytic function of the complex variable $z = x^1 + \mathrm{i}x^2$, and let E_ω be the equation of the form (44) with coefficients

$$N^{11} = -N^{22} = 4\mathrm{Re}\{\phi\}, \quad N^{12} = N^{21} = -4\mathrm{Im}\{\phi\}, \quad k = 4\phi\bar{\phi}, \tag{54}$$

where ϕ is the Schwarzian derivative of the function ω, that is

$$\phi = \left(\frac{\omega''}{\omega'}\right)' - \frac{1}{2}\left(\frac{\omega''}{\omega'}\right)^2, \tag{55}$$

$\bar{\phi}$ is the complex conjugated of ϕ, and the prime is used to denote differentiation with respect to the variable z. Substituting expressions (54) into formulae (53) one can see that all invariants of equation E_ω are identically equal to zero. Then, taking into account the DE system (47)–(50), (54) and (55), one can verify by direct computing that equation E_ω admits the six-parameter group generated by the vector fields

$$Z_{(j)} = \xi^{\mu}_{(j)}\frac{\partial}{\partial x^{\mu}} + \frac{1}{2}\xi^{\mu}_{(j),\mu}w\frac{\partial}{\partial w} \quad (j = 1, \dots, 6),$$

the functions $\xi^{\mu}_{(j)}$ being given by the expressions

$$\xi^1_{(1)} = \mathrm{Re}\{\omega_1\}, \quad \xi^2_{(1)} = \mathrm{Im}\{\omega_1\},$$

$$\xi^1_{(2)} = \mathrm{Re}\{\mathrm{i}\omega_1\}, \quad \xi^2_{(2)} = \mathrm{Im}\{\mathrm{i}\omega_1\},$$

$$\xi^1_{(3)} = \mathrm{Re}\{\omega_2\}, \quad \xi^2_{(3)} = \mathrm{Im}\{\omega_2\},$$

$$\xi^1_{(4)} = \mathrm{Re}\{\mathrm{i}\omega_2\}, \quad \xi^2_{(4)} = \mathrm{Im}\{\mathrm{i}\omega_2\},$$

$$\xi^1_{(5)} = \mathrm{Re}\{\omega_3\}, \quad \xi^2_{(5)} = \mathrm{Im}\{\omega_3\},$$

$$\xi^1_{(6)} = \mathrm{Re}\{\mathrm{i}\omega_3\}, \quad \xi^2_{(6)} = \mathrm{Im}\{\mathrm{i}\omega_3\},$$

where

$$\omega_1 = \frac{1}{\omega'}, \quad \omega_2 = \frac{\omega}{\omega'}, \quad \omega_3 = \frac{\omega^2}{\omega'}. \tag{56}$$

It should be remarked that each equation of form (44) which admits a six-parameter group with generators of form (52) is of type E_ω, meaning that it can be generated in the above manner using a suitable analytic function ω. The coefficients of each equation of this type are of the form

$$N^{\alpha\beta} = \delta^{\alpha\mu}\delta^{\beta\nu}\varphi_{,\mu\nu}, \quad k = \tfrac{1}{8}\delta^{\alpha\mu}\delta^{\beta\nu}\varphi_{,\alpha\beta}\varphi_{,\mu\nu},$$

where φ is a harmonic function, that is $\delta^{\alpha\beta}\varphi_{,\alpha\beta} = 0$, and vice versa. It is noteworthy that each equation with variable coefficients of type E_ω can be mapped to an equation with constant coefficients belonging to the same family. It is easy to verify by direct computing that the equation E_ω, corresponding to an analytic function ω whose Schwarzian derivative is not constant, transforms to a constant coefficients equation under the following change of the variables:

$$y^\alpha = f^\alpha(x^1, x^2), \quad W = wU(x^1, x^2), \tag{57}$$

$$f^1(x^1, x^2) = \mathrm{Re}\left\{\int f^{-1}\,\mathrm{d}z\right\}, \quad f^2(x^1, x^2) = \mathrm{Im}\left\{\int f^{-1}\,\mathrm{d}z\right\}, \quad U(x^1, x^2) = (f\bar{f})^{-1/2},$$

where f is any linear combination of the functions (56) such that $f \not\equiv 0$, i.e.

$$f = k_1\omega_1 + k_2\omega_2 + k_3\omega_3, \tag{58}$$

where k_1, k_2 and k_3 are complex constants such that $k_1^2 + k_2^2 + k_3^2 \neq 0$.

Consider, as a simple example, the equation E_ω corresponding to the analytic function $\omega = z^{\sqrt{\varkappa/2}}$, where $\varkappa$ is a positive real constant. Then, formula (55) gives $\phi = 4(2-\varkappa)z^2$ and hence, according to expressions (54), the coefficients of equation E_ω read

$$\begin{aligned} N^{11} &= -N^{22} = (2-\varkappa)\frac{(x^1)^2 - (x^2)^2}{[(x^1)^2 + (x^2)^2]^2}, \\ N^{12} &= N^{21} = (2-\varkappa)\frac{2x^1x^2}{[(x^1)^2 + (x^2)^2]^2}, \quad k = (2-\varkappa)^2\frac{1}{4[(x^1)^2 + (x^2)^2]^2}. \end{aligned} \tag{59}$$

Using the function $f = z$ obtained from (58) for $\omega = z^{\sqrt{\varkappa/2}}$, $k_1 = k_3 = 0$, $k_2 = 1 + \sqrt{\varkappa/2}$, we introduce, according to formulae (57), the new independent and dependent variables

$$y^1 = \frac{1}{2}\ln\left[(x^1)^2 + (x^2)^2\right], \quad y^2 = \arctan\left(\frac{x^2}{x^1}\right), \quad W = w\left[(x^1)^2 + (x^2)^2\right]^{-1/2}. \tag{60}$$

Note that the inverse transformations are given by the expressions

$$x^1 = \mathrm{e}^{y^1}\cos y^2, \quad x^2 = \mathrm{e}^{y^1}\sin y^2, \quad w = W\left[(x^1)^2 + (x^2)^2\right]^{1/2}. \tag{61}$$

Under the change of the variables of form (60), the considered equation E_ω transforms to the following one,

$$\delta^{\alpha\beta}\delta^{\mu\nu}\frac{\partial^4 W}{\partial y^\alpha \partial y^\beta \partial y^\mu \partial y^\nu} - \varkappa\frac{\partial^2 W}{\partial y^1 \partial y^1} + \varkappa\frac{\partial^2 W}{\partial y^2 \partial y^2} + \frac{1}{4}\varkappa^2 W = 0, \tag{62}$$

which belongs to the same class (since it corresponds to the analytic function $\omega = \mathrm{e}^{z\sqrt{\varkappa/2}}$) but whose coefficients are constant.

Eq. (62) admits the six-parameter group of variational symmetries generated by the basic vector fields

$$V_\alpha = \frac{\partial}{\partial y^\alpha},$$

$$V_3 = \mathrm{e}^{\theta y^1}\cos(\theta y^2)\frac{\partial}{\partial y^1} + \mathrm{e}^{\theta y^1}\sin(\theta y^2)\frac{\partial}{\partial y^2} + \theta \mathrm{e}^{\theta y^1} w\cos(\theta y^2)\frac{\partial}{\partial w},$$

$$V_4 = -\mathrm{e}^{\theta y^1}\cos(\theta y^2)\frac{\partial}{\partial y^1} + \mathrm{e}^{\theta y^1}\cos(\theta y^2)\frac{\partial}{\partial y^2} - \theta \mathrm{e}^{\theta y^1} w\sin(\theta y^2)\frac{\partial}{\partial w},$$

$$V_5 = \mathrm{e}^{-\theta y^1}\cos(\theta y^2)\frac{\partial}{\partial y^1} - \mathrm{e}^{-\theta y^1}\sin(\theta y^2)\frac{\partial}{\partial y^2} - \theta \mathrm{e}^{-\theta y^1} w\cos(\theta y^2)\frac{\partial}{\partial w},$$

$$V_6 = \mathrm{e}^{-\theta y^1}\cos(\theta y^2)\frac{\partial}{\partial y^1} + \mathrm{e}^{-\theta y^1}\sin(\theta y^2)\frac{\partial}{\partial y^2} - \theta \mathrm{e}^{\theta y^1} w\cos(\theta y^2)\frac{\partial}{\partial w},$$

where $\theta = \sqrt{\varkappa/2}$. These vector fields give rise to six linearly independent conservation laws for Eq. (62). The characteristics of these conservation laws are

$$Q_{(j)} = \frac{1}{2}W\frac{\partial}{\partial y^\mu}V_j(y^\mu) - W_\mu V_j(y^\mu) \quad (j = 1, \ldots, 6).$$

Here, V_j are regarded as operators acting on the functions $\zeta : R^2 \to R$. The corresponding conserved currents can be easily calculated from formula (34).

Finally, let us remark that each one-parameter group generated by a linear combination of the basic vector fields V_j can be used for constructing group-invariant solutions of Eq. (62). Consider, for instance, the group $H(V_3 + V_5)$ generated by the vector field $V_3 + V_5$. The functions $s = \sin(\theta y^2)/\cosh(\theta y^1)$ and $u = W/\cosh(\theta y^1)$ constitute a complete set of invariants for this group and hence, following the well known algorithm (Ovsiannikov, 1982; Olver, 1993), we seek the $H(V_3 + V_5)$-invariant solutions of Eq. (62) in the form

$$W = u(s)\cosh(\theta y^1), \quad s = \frac{\sin(\theta y^2)}{\cosh(\theta y^1)}. \tag{63}$$

Substituting expressions (63) into Eq. (62), we get the reduced equation

$$(s^2 - 1)^2\frac{\mathrm{d}^4 u}{\mathrm{d}s^4} + 8s(s^2 - 1)\frac{\mathrm{d}^3 u}{\mathrm{d}s^3} + 4(3s^2 - 1)\frac{\mathrm{d}^2 u}{\mathrm{d}s^2} = 0.$$

The general solution to this ordinary differential equation is

$$u(s) = C_1 + C_2\ln\left(\frac{s+1}{s-1}\right) + C_3 s + C_4 s\ln\left(\frac{s+1}{s-1}\right),$$

where C_1, C_2, C_3 and C_4 are real constants. Hence, the $H(V_3 + V_5)$-invariant solutions of Eq. (62) are given by the expression

$$W(y^1, y^2) = C_1 \cosh(\theta y^1) + C_2 \cosh(\theta y^1) \ln\left[\frac{\sin(\theta y^2) + \cosh(\theta y^1)}{\sin(\theta y^2) - \cosh(\theta y^1)}\right] + C_3 \sin(\theta y^2) + C_4 \frac{\sin(\theta y^2)}{\cosh(\theta y^1)} \ln\left[\frac{\sin(\theta y^2) + \cosh(\theta y^1)}{\sin(\theta y^2) - \cosh(\theta y^1)}\right].$$

Using the inverse transformations (61) one can convert the above solutions of Eq. (62) into solutions of the equation E_ω, $\omega = z\sqrt{\varkappa/2}$, with variable coefficients (59).

Actually, Eq. (44) describes the state of equilibrium of a plate but introducing, according to d'Alembert principle, the inertia force $-\rho w_{33}$ in its right-hand side, w_{33} being the second derivative of the displacement field with respect to the time x^3 and $\rho(x^1, x^2)$—the mass per unit area of the plate middle-plane, one can extend (44) to the equation

$$D\Delta^2 w + N^{\alpha\beta} w_{\alpha\beta} + kw + \rho w_{33} = 0, \tag{64}$$

describing the dynamic behaviour of the plate, provided that the rotatory inertia is negligible. Eq. (64) is evidently self-adjoint and belongs to the class (1), $n = 3$, with coefficients given by expressions (45) and the following ones:

$$A^{3333} = A^{\alpha 333} = A^{\alpha\beta 33} = A^{\alpha\beta\gamma 3} = A^{333} = A^{\alpha 33} = A^{\alpha\beta 3} = A^{\alpha 3} = A^3 = 0, \quad A^{33} = \rho. \tag{65}$$

Substitution of expressions (45) and (65) into the determining equations (12)–(16) implies that an equation of form (64) is invariant under a point Lie group if and only if its generator is a linear combination of the vector field $X = w\partial/\partial w$ and one of the form

$$X = \xi^\mu \frac{\partial}{\partial x^\mu} + \xi^3 \frac{\partial}{\partial x^3} + \sigma w \frac{\partial}{\partial w}, \tag{66}$$

where

$$\xi^\mu = \xi^\mu(x^1, x^2), \quad \xi^3 = C_1 x^3 + C_2 \quad (C_1, C_2 = \text{const}), \quad \sigma = \tfrac{1}{2}\xi^\mu_{,\mu}, \tag{67}$$

and

$$2\rho\xi^\mu_{,\mu} + \xi^\mu \rho_{,\mu} = 2C_1\rho, \tag{68}$$

holds together with Eqs. (46)–(50), the function λ being given by expression (51).

An ordinary symmetry of the equation considered is either its variational symmetry, as well, or one that can be completed to a variational symmetry, by adding to its generator a term of form $Cw\partial/\partial w$ ($C = \text{const}$). Indeed, in the present case condition (23) reads $\sigma + \lambda + \xi^\mu_{,\mu} + \xi^3_{,3} = 0$, so substituting here expressions (51) and (67) one obtains all symmetries with $C_1 = 0$ to be variational ones. If $C_1 \neq 0$, then the corresponding symmetry is not a variational one but when the term $-C_1 w\partial/\partial w$ is added to its generator, it becomes a variational one. Hence, for each symmetry of Eq. (64) generated by a vector field of form (66) there exists a conservation law with characteristic

$$Q = (\tfrac{1}{2}\xi^\mu_{,\mu} - C_1)w - w_\mu \xi^\mu - w_3 \xi^3 \tag{69}$$

and conserved current B^α given by formula (34) admitted by the smooth solutions of the equation considered. Thus, as in the time-independent case, to derive the conservation laws, which correspond to the variational symmetries of an equation of form (64), it suffices to know the results of the group classification of the class of equations in question; of course, the same holds true for the derivation of group-invariant solutions to Eq. (64). This group-classification problem is the same as in the time-independent case provided that the function ρ satisfies Eq. (68).

Consider now as an example the equations of form (64) with constant coefficients. They govern the dynamics of plates of constant bending rigidity and mass density, lying on a Winkler foundation of constant modulus and subjected to edge loading resulting in constant membrane stresses. Omitting the details, the solution of the corresponding group-classification problem is as follows: (a) each equation of this type admits the translations generated by the vector fields

$$X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^2}, \quad T = \frac{\partial}{\partial x^3};$$

(b) the equations of form

$$D\Delta^2 w + N^{11} w_{11} + N^{22} w_{22} + kw + \rho w_{33} = 0, \quad N^{11} = N^{22},$$

admit additionally the rotation group with generator

$$X_3 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2};$$

(c) the equations of form

$$D\Delta^2 w + \rho w_{33} = 0, \tag{70}$$

admit additionally the scaling group with generator

$$S = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + 2x^3 \frac{\partial}{\partial x^3}.$$

In each of the aforementioned three cases, the characteristics and currents of the corresponding conservation laws can easily be computed from formulae (69) and (34), respectively. Let us remark, that the conservation laws for Eq. (70) associated with the translational (X_1, X_2, T) and rotational (X_3) symmetries are obtained by Ibragimov (1985). Here, these equations are found to admit an additional conservation law with characteristic $Q = -w_\mu x^\mu - 2w_3 x^3$, associated with the scaling symmetry S.

6. Generalized symmetries and conservation laws in theories of Timoshenko beams, Reissner–Mindlin plates and 3D elastostatics

The results obtained in Section 3 are readily applicable in many other theories of solids and structures provided that such theories involve linear differential equations in one dependent variable of order less than or equal to four. The present section comprises three examples illustrating the application of these results in the nonclassical theories of Timoshenko beams and Reissner–Mindlin plates as well as in three-dimensional elastostatics.

6.1. Timoshenko beam equations

The small vibration of homogeneous Timoshenko beams is described (see, e.g., Washizu, 1982) by the following system of two coupled second-order partial differential equations:

$$\begin{aligned} \mathscr{D}_1[\varphi, u] &\equiv EJ\varphi_{11} + kGA(u_1 - \varphi) - \rho J\varphi_{22} = 0, \\ \mathscr{D}_2[\varphi, u] &\equiv kGA(u_{11} - \varphi_1) - \rho A u_{22} = 0, \end{aligned} \tag{71}$$

where the rotation angle φ and the transversal displacement u are the dependent variables; the coordinate x^1 along the rod axis and the time x^2 are the independent variables; ρ, G and E are the mass density, shear and Young's moduli of the beam material; A and J are the cross-section area and inertia moment of the beam; k is the shear correction factor.

Given a solution (φ, u) of system (71), both the rotation angle φ and the transversal displacement u satisfy the single fourth-order partial differential equation

$$\mathscr{D}_0[w] \equiv EJw_{1111} - \rho J\left(1 + 2\frac{1+v}{k}\right)w_{1122} + \frac{2\rho^2 J(1+v)}{kE}w_{2222} + \rho A w_{22} = 0, \tag{72}$$

in one dependent variable w and two independent variables; here w stands for both φ and u, v denotes Poisson's ratio and the well known relation $E = 2G(1+v)$ is taken into account. Indeed, by direct computation, one can verify that

$$\mathscr{D}_0[\varphi] = -D_1\mathscr{D}_2[\varphi, u] + D_1D_1\mathscr{D}_1[\varphi, u] - \frac{\rho}{kG}D_2D_2\mathscr{D}_1[\varphi, u],$$

$$\mathscr{D}_0[u] = -\mathscr{D}_2[\varphi, u] + D_1\mathscr{D}_1[\varphi, u] + \frac{2J(1+v)}{kA}D_1D_1\mathscr{D}_2[\varphi, u] - \frac{\rho J}{kGA}D_2D_2\mathscr{D}_2[\varphi, u].$$

It does not mean, however, that the Timoshenko beam equations (71) are thus decoupled since the opposite of the above assertion is not always true, namely: given two solutions $w = \varphi$ and $w = u$ of Eq. (72), the couple (φ, u) is not necessarily a solution of system (71). Therefore, Eq. (72) cannot replace system (71) entirely, but nevertheless it is of considerable importance for the theory of shearable beams and so the exploration of its symmetries is of interest.

Eq. (72) is self-adjoint and belongs to the class (1), $n = 2$, with nonzero coefficients

$$A^{1111} = EJ, \quad A^{1122} = -\frac{\rho J}{6}\left(1 + 2\frac{1+v}{k}\right), \quad A^{2222} = \frac{2\rho^2 J(1+v)}{kE}, \quad A^{22} = \rho A. \tag{73}$$

Substituting expressions (73) in the determining equations (12)–(16) and solving the over-determined system obtained in this way, one can see that Eq. (72) admits only two vector fields

$$X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^2},$$

in addition to the vector fields (10). Moreover, condition (23) implies that the vector fields X_1 and X_2 are infinitesimal variational symmetries of Eq. (72) and hence, two conservation laws with characteristics $Q_1 = -w_1$ and $Q_2 = -w_2$, respectively, are associated with them. The densities and fluxes of these conservation laws, calculated through formulae (34), read

$$\begin{aligned}
\Psi_{(1)} &= -\frac{\rho J}{12}\left(1 + 2\frac{1+v}{k}\right)(3w_{11}w_{12} - 8w_1w_{112} - w_2w_{111}) \\
&\quad + \frac{\rho^2 J(1+v)}{kE}(w_{12}w_{22} - 2w_1w_{222} - w_2w_{122}) + \frac{1}{2}\rho A(ww_{12} - w_1w_2), \\
P_{(1)} &= \frac{1}{2}EJ(w_{11}w_{11} - 2w_1w_{111}) + \frac{\rho^2 J(1+v)}{kE}w_2w_{222} \\
&\quad - \frac{1}{2}\rho A w w_{22} - \frac{\rho J}{12}\left(1 + 2\frac{1+v}{k}\right)(w_{11}w_{22} + 2w_{12}w_{12} + w_2w_{112} - 4w_1w_{122}), \\
\Psi_{(2)} &= \frac{1}{2}EJw_1w_{111} + \frac{\rho^2 J(1+v)}{kE}(w_{22}w_{22} - 2w_2w_{222}) \\
&\quad - \frac{1}{2}\rho A w_2w_2 - \frac{\rho J}{12}\left(1 + 2\frac{1+v}{k}\right)(w_{11}w_{22} + 2w_{12}w_{12} + w_1w_{122} - 4w_2w_{112}), \\
P_{(2)} &= \frac{1}{2}EJ(w_{11}w_{12} - w_1w_{112} - 2w_2w_{111}) - \frac{\rho J}{12}\left(1 + 2\frac{1+v}{k}\right)(3w_{12}w_{22} - w_1w_{222} - 8w_2w_{122}).
\end{aligned} \tag{74}$$

Each of the foregoing two conservation laws established for the single fourth-order equation (72) gives rise, upon replacing w by φ or u, to two conservation laws for the Timoshenko beam equations (71). In the case when w stands for u, these conservation laws can be written in the characteristic form

$$D_1\widetilde{P}_{(\alpha)} + D_2\widetilde{\Psi}_{(\alpha)} = \widetilde{Q}^{\mu}_{(\alpha)}\mathscr{D}_\mu[\varphi, u] + D_\mu R^{\mu}_{(\alpha)}, \quad \alpha = 1, 2,$$

with characteristics

$$\widetilde{Q}^{1}_{(\alpha)} = u_{1\alpha}, \quad \widetilde{Q}^{2}_{(\alpha)} = u_\alpha - \frac{2J(1+v)}{kA}u_{11\alpha} + \frac{\rho J}{kGA}u_{\alpha 22}, \quad \alpha = 1, 2, \tag{75}$$

and

$$R^{1}_{(\alpha)} = -u_\alpha\mathscr{D}_1[\varphi, u] + \frac{2J(1+v)}{kA}(u_{1\alpha}\mathscr{D}_2[\varphi, u] - u_\alpha D_1\mathscr{D}_2[\varphi, u]),$$

$$R^{2}_{(\alpha)} = \frac{\rho J}{kGA}(u_\alpha D_2\mathscr{D}_2[\varphi, u] - u_{\alpha 2}\mathscr{D}_2[\varphi, u]), \quad \alpha = 1, 2.$$

Here, $\widetilde{\Psi}_{(\alpha)}$ and $\widetilde{P}_{(\alpha)}$ denote the densities and fluxes (74) with u instead of w. Evidently, $R^{\mu}_{(\alpha)}$ are currents of trivial conservation laws as they vanish on the solutions of Eq. (71). In a similar way, conservation laws for the rotation angle φ could be established.

The invariance properties of Timoshenko beam equations (71) are considered in Djondjorov (1995). In that paper, the vector fields X_1 and X_2 are identified to be infinitesimal variational symmetries of system (71) and the corresponding conservation laws are derived therein. The conservation laws with characteristics (75) which are found here to hold on the solutions of system (71) are new. They differ from the conservation laws in Djondjorov (1995), because the latter correspond to geometric symmetries of system (71) while expressions (75) imply that the conservation laws presented here correspond to generalized symmetries of this system.

Neglecting the shear deformation of the rod cross-section one arrives at another rod theory, still accounting for the rotatory inertia of the cross-section, governed by the equation

$$EJw_{1111} - \rho Jw_{1122} + \rho Aw_{22} = 0,$$

which follows from Eq. (72) when $k \to \infty$. Here, w denotes the transversal displacement of the rod axis. Omitting the details, this equation is found to admit the same variational symmetries as Eq. (72), the densities and fluxes of the associated conservation laws being limit cases of those given by expressions (74) when $k \to \infty$.

6.2. Reissner–Mindlin plate equations

Within the framework of the Reissner–Mindlin plate theory, the small vibration of a homogeneous elastic plate of mass density ρ and thickness h is governed (see, e.g., Washizu, 1982) by the following system of partial differential equations:

$$\begin{aligned}
&\delta^{\alpha\beta}\varphi_{\alpha\beta} + \frac{1+v}{2}(\psi_{12} - \varphi_{22}) - \frac{6k(1-v)}{h^2}(w_1 + \varphi) - \frac{\rho(1-v^2)}{E}\varphi_{33} = 0,\\
&\delta^{\alpha\beta}\psi_{\alpha\beta} + \frac{1+v}{2}(\varphi_{12} - \psi_{11}) - \frac{6k(1-v)}{h^2}(w_2 + \psi) - \frac{\rho(1-v^2)}{E}\psi_{33} = 0,\\
&\frac{k(1-v)}{2}(\delta^{\alpha\beta}w_{\alpha\beta} + \varphi_1 + \psi_2) - \frac{\rho(1-v^2)}{E}w_{33} = 0,
\end{aligned} \tag{76}$$

where (x^1, x^2) are Cartesian coordinates of the plate middle-plane, x^3 is the time, w is the transverse displacement of the plate middle-plane, φ and ψ are the rotation angles of a straight element which is normal to the middle-plane in the reference state, k is the shear correction factor, v is Poisson's ratio, and E is Young's modulus. This system can easily be solved with respect to the derivatives of the variable w, namely

$$\begin{aligned} w_1 &= -\varphi + \frac{h^2}{6k(1-v)}\left[\delta^{\alpha\beta}\varphi_{\alpha\beta} + \frac{1+v}{2}(\psi_{12} - \varphi_{22}) - \frac{\rho(1-v^2)}{E}\varphi_{33}\right], \\ w_2 &= -\psi + \frac{h^2}{6k(1-v)}\left[\delta^{\alpha\beta}\psi_{\alpha\beta} + \frac{1+v}{2}(\varphi_{12} - \psi_{11}) - \frac{\rho(1-v^2)}{E}\psi_{33}\right], \\ w_{33} &= \frac{D}{\rho h}\left[\delta^{\alpha\beta}(\varphi_{1\alpha\beta} + \psi_{2\alpha\beta}) - \frac{\rho(1-v^2)}{E}(\varphi_{133} + \psi_{233})\right], \end{aligned} \tag{77}$$

where $D = Eh^3/(12(1-v^2))$ is the bending rigidity of the plate. The left-hand sides of expressions (77) meet the compatibility conditions $(w_1)_{,33} = (w_{33})_{,1}$ and $(w_2)_{,33} = (w_{33})_{,2}$ that are actually trivial conservation laws. Nevertheless, substituting here the derivatives of the variable w from expressions (77), one could obtain two nontrivial conservation laws for the solutions of the Reissner–Mindlin plate equations (76) whose currents involve derivatives of the rotation angles φ and ψ only. The left-hand sides of expressions (77) also satisfy the compatibility condition $(w_1)_{,2} = (w_2)_{,1}$, which leads to the equation

$$\delta^{\alpha\beta}\phi_{\alpha\beta} - \frac{2\rho(1+v)}{E}\phi_{33} - \frac{12k}{h^2}\phi = 0, \tag{78}$$

for the function $\phi = \psi_1 - \varphi_2$. This is a self-adjoint equation of form (1) with $n = 3$ and nonzero coefficients:

$$A^{\alpha\beta} = \delta^{\alpha\beta}, \quad A^{33} = -\frac{2\rho(1+v)}{E}, \quad A = -\frac{12k}{h^2}.$$

Substituting these coefficients in system (12)–(16) and solving it, one obtains, taking into account condition (23), that the following six vector fields

$$\begin{aligned} &X_k = \frac{\partial}{\partial x^k} \quad (k = 1, 2, 3), \quad X_4 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}, \\ &X_5 = x^3\frac{\partial}{\partial x^1} + \frac{2\rho(1+v)}{E}x^1\frac{\partial}{\partial x^3}, \quad X_6 = x^3\frac{\partial}{\partial x^2} + \frac{2\rho(1+v)}{E}x^2\frac{\partial}{\partial x^3}, \end{aligned}$$

generate variational symmetries of Eq. (78). The currents of the associated conservation laws could easily be derived from formula (34) in terms of the dependent variable ϕ and its first derivatives. Then, using the relation $\phi = \psi_1 - \varphi_2$, it is a simple matter to rewrite these currents in terms of the rotation angles φ and ψ and their derivatives. In this manner, one will obtain six conservation laws that are valid on the solutions of system (76) because each solution (φ, ψ, w) of this system transforms (via the relation $\phi = \psi_1 - \varphi_2$) to a solution of Eq. (78). Thus, we find that there exist eight conservation laws for the solutions of the Reissner–Mindlin plate equations (76) that involve only derivatives of the rotation angles φ and ψ.

On the other hand, it turns out that there exist conservation laws for system (76) involving only derivatives of the transversal displacement w. Indeed, eliminating φ and ψ from system (76) one arrives at the well known equation

$$D\Delta\Delta w - \frac{\rho h^3}{12(1-v)}\left(\frac{2}{k} + 1 - v\right)\Delta w_{33} + \frac{\rho^2 h^3(1+v)}{6kE}w_{3333} + \rho h w_{33} = 0 \tag{79}$$

for the transversal displacement w. This equation is self-adjoint and belongs to class (1) with $n = 3$ and nonzero coefficients

$$A^{1111} = A^{2222} = 3A^{1122} = D, \quad A^{3333} = \frac{\rho^2 h^3(1+\nu)}{6kE}, \quad A^{33} = \rho h, \quad A^{1133} = A^{2233} = \frac{\rho h^3}{72(1-\nu)}\left(\frac{2}{k} + 1 - \nu\right).$$

Substituting these coefficients in determining equations (12)–(16) and solving the over-determined system, obtained in this way, one finds that Eq. (79) admits the vector fields

$$Z_j = \frac{\partial}{\partial x^j}, \quad j = 1, 2, 3, \quad Z_4 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}. \tag{80}$$

These four vector fields generate variational symmetries of Eq. (79) and this is easily confirmed using condition (23). Thus, given an equation of form (79), the characteristics of its conservation laws associated with the vector fields (80) are

$$Q_{(j)} = -w_j, \quad j = 1, 2, 3, \quad Q_{(4)} = x^1 w_2 - x^2 w_1,$$

and the corresponding conserved currents could be computed from formula (34). Each solution of the Reissner–Mindlin plate equations (76) is a solution of Eq. (79) too. For this reason, the conservation laws for the solutions of Eq. (79) are conservation laws for the solutions of system (76) as well. In this manner, the conservation laws derived here for the single fourth-order equation (79) are conservation laws for the Reissner–Mindlin plate equations (76) too, but in this case they are with characteristics

$$\begin{aligned} Q^{\alpha}_{(j)} &= D\delta^{\alpha\beta}D_{\beta}Q_{(j)} \quad (\alpha = 1, 2; j = 1, 2, 3, 4), \\ Q^{3}_{(j)} &= -\frac{12D}{h^2}Q_{(j)} + \frac{2D}{k(1-\nu)}\delta^{\alpha\beta}D_{\alpha}D_{\beta}Q_{(j)} - \frac{\rho h^3}{6k(1-\nu)}D_3D_3Q. \end{aligned} \tag{81}$$

A number of conservation laws for nonhomogeneous Reissner–Mindlin plates are derived by Chien et al. (1994). As for the homogeneous plates considered in this subsection, the general results of Chien et al. (1994) imply that the vector fields Z_j, $j = 1$, 2 and 3 are generators of variational symmetries of system (76). However, the vector field Z_4 is not recognized as infinitesimal variational symmetry of system (76) by Chien et al. (1994) and hence, the corresponding conservation law of characteristic $(Q^1_{(4)}, Q^2_{(4)}, Q^3_{(4)})$ is new in the Reissner–Mindlin plate theory. The conservation laws for the solutions of system (76) associated with Z_1, Z_2 and Z_3 whose currents are obtainable through our formula (34) are also new in the Reissner–Mindlin plate theory because the form of their characteristics (81) implies that they correspond to generalized symmetries of system (76) whereas the conservation laws in Chien et al. (1994) correspond to geometric symmetries of system (76).

If the shear deformation is neglected, but the rotatory inertia of the straight element is retained in the description, another plate theory arises whose governing equation is

$$D\Delta\Delta w - \frac{\rho h^3}{12}\Delta w_{33} + \rho h w_{33} = 0, \tag{82}$$

which follows from Eq. (79) when $k \to \infty$. Omitting the details, Eq. (82) is found to admit the same variational symmetries as Eq. (79). The currents of the conservation laws for Eq. (82) are limits of the currents of the conservation laws for Eq. (79) when $k \to \infty$.

6.3. Three-dimensional elasticity

The three-dimensional homogeneous isotropic elastostatics is governed by the equations

$$E^{\gamma} \equiv \mu\delta^{\alpha\beta}u^{\gamma}_{\alpha\beta} + (\lambda + \mu)\delta^{\gamma\beta}u^{\alpha}_{\alpha\beta} = 0, \quad \alpha, \beta, \gamma = 1, 2, 3, \tag{83}$$

where (u^1, u^2, u^3) is the displacement vector in the Cartesian frame (x^1, x^2, x^3). In this last subsection we (following the tradition) use the notation λ and μ for the Lamé moduli. It is well known (see, e.g., Landau and Lifshitz, 1970), that each component u^γ $(\gamma = 1, 2, 3)$ of the displacement vector satisfies the three-dimensional biharmonic equation

$$\delta^{\alpha\beta}\delta^{\sigma\tau}w_{\alpha\beta\sigma\tau} = 0, \tag{84}$$

where w stands for any of the displacement components u^γ $(\gamma = 1, 2, 3)$. Apparently, Eq. (84) is self-adjoint and belongs to the class (1) with $n = 3$.

Solving the corresponding determining equations (12)–(16) one arrives, after a straightforward computation, at the conclusion that the three-dimensional biharmonic equation (84) is invariant under the Lie group generated by the vector fields

$$\begin{aligned}
&F_1 = \frac{\partial}{\partial x^1}, \quad F_2 = \frac{\partial}{\partial x^2}, \quad F_3 = \frac{\partial}{\partial x^3}, \\
&F_4 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}, \quad F_5 = x^1\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^1}, \quad F_6 = x^3\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^3}, \\
&F_7 = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + \frac{1}{2}w\frac{\partial}{\partial w}, \\
&F_8 = [(x^1)^2 - (x^2)^2 - (x^3)^2]\frac{\partial}{\partial x^1} + 2x^1x^2\frac{\partial}{\partial x^2} + 2x^1x^3\frac{\partial}{\partial x^3} + x^1w\frac{\partial}{\partial w}, \\
&F_9 = 2x^1x^2\frac{\partial}{\partial x^1} + [-(x^1)^2 + (x^2)^2 - (x^3)^2]\frac{\partial}{\partial x^2} + 2x^2x^3\frac{\partial}{\partial x^3} + x^2w\frac{\partial}{\partial w}, \\
&F_{10} = 2x^1x^3\frac{\partial}{\partial x^1} + 2x^2x^3\frac{\partial}{\partial x^2} + [-(x^1)^2 - (x^2)^2 + (x^3)^2]\frac{\partial}{\partial x^3} + x^3w\frac{\partial}{\partial w}.
\end{aligned} \tag{85}$$

In addition, condition (23) implies that all these symmetries are variational ones. Thus, in virtue of Noether's theorem, the characteristics $Q_{(s)}$ of the vector field F_s $(s = 1, 2, \ldots, 10)$,

$$\begin{aligned}
&Q_{(1)} = -w_1, \quad Q_{(2)} = -w_2, \quad Q_{(3)} = -w_3, \\
&Q_{(4)} = x^1w_2 - x^2w_1, \quad Q_{(5)} = x^3w_1 - x^1w_3, \quad Q_{(6)} = x^2w_3 - x^3w_2, \\
&Q_{(7)} = \tfrac{1}{2}w - x^1w_1 - x^2w_2 - x^3w_3, \\
&Q_{(8)} = 2x^1Q_{(7)} - [(x^1)^2 + (x^2)^2 + (x^3)^2]w_1, \\
&Q_{(9)} = 2x^2Q_{(7)} - [(x^1)^2 + (x^2)^2 + (x^3)^2]w_2, \\
&Q_{(10)} = 2x^3Q_{(7)} - [(x^1)^2 + (x^2)^2 + (x^3)^2]w_{3,}
\end{aligned} \tag{86}$$

are simultaneously characteristics of 10 linearly independent conservation laws for the solutions of Eq. (84). The corresponding conserved currents $P^\alpha_{(s)}$ can be calculated from the general formula (34) taking into account the particular form of the respective characteristic $Q_{(s)}$.

Evidently, each of the foregoing 10 conservation laws established for the single fourth-order equation (84) gives rise, upon replacing w by u^γ $(\gamma = 1, 2, 3)$, to three conservation laws for Eq. (83). These conservation laws can be written in the characteristic form

$$D_\alpha(P^{(\gamma)\alpha}_{(s)} - R^{(\gamma)\alpha}_{(s)}) = Q^{(\gamma)}_{(s)\alpha}E^\alpha, \quad s = 1, 2, \ldots, 10, \ \ \gamma = 1, 2, 3 \tag{87}$$

with characteristics

$$Q^{(\gamma)}_{(s)\alpha} = \frac{1}{\mu}\delta^{\sigma\beta}(D_\sigma D_\beta Q^{(\gamma)}_{(s)})\delta^\gamma_\alpha - \frac{\lambda + \mu}{\mu(\lambda + 2\mu)}\delta^{\beta\gamma}(D_\alpha D_\beta Q^{(\gamma)}_{(s)}) \tag{88}$$

and

$$R_{(s)}^{(\gamma)\alpha} = \frac{1}{\mu}\left(\frac{\lambda+\mu}{\lambda+2\mu}\delta^{\beta\gamma}E^{\alpha} - \delta^{\alpha\beta}E^{\gamma}\right)D_{\beta}Q_{(s)}^{(\gamma)} + \frac{1}{\mu}\delta^{\alpha\beta}Q_{(s)}^{(\gamma)}D_{\beta}E^{\gamma}. \tag{89}$$

Here, $P_{(s)}^{(\gamma)\alpha}$ and $Q_{(s)}^{(\gamma)}$ denote the conserved currents $P_{(s)}^{\alpha}$ and the characteristics $Q_{(s)}$ with u^{γ} instead of w; no summation is assumed over the repeated index γ. Expressions (89) show that $R_{(s)}^{(\gamma)\alpha}$ are currents of trivial conservation laws since they vanish on the solutions of Eq. (83).

To the best of our knowledge, the conservation laws (87) are new for three-dimensional homogeneous isotropic elastostatics since, as it follows by expressions (86) and (88), they correspond to generalized symmetries of Eq. (83) of order higher than the one considered previously in the literature (cf. Olver, 1984b; Hatfield and Olver, 1998).

7. Concluding remarks

In this paper, Lie transformation group methods have been applied to the class of partial differential equations (1). This class is of interest to structural mechanics since the governing equations of various classical rod and plate theories belong to it. In the context of structural mechanics, the results of the group analysis of Eq. (1) give a number of attractive possibilities. Here, the established point Lie symmetries of Eq. (1) are used to construct group-invariant solutions to the governing equations of several rod and plate models, to derive conservation laws revealing important features of such models and to find transformations simplifying the differential structure of equations associated with particular plate problems. Several nonclassical structural theories as well as three-dimensional elastostatics involve important equations belonging to the class (1) which provides the opportunity to achieve new knowledge in these theories; the examples given in Section 6 illustrate this fact.

First of all, the well known computational procedure for finding the most general point Lie symmetry group has been applied to the foregoing class of equations. As a result, the system of equations (12)–(16) is derived determining the equations of the type considered that admit a larger group together with the generators of this group; naturally, all equations of this class, being linear and homogeneous, admit the point Lie groups generated by the vector fields (10). System (12)–(16) allows the associated group-classification problem to be stated and examined.

The group-classification problem for Eq. (35) governing stability and vibration of rods and fluid conveying pipes resting on variable elastic foundations within the classical Bernoulli–Euler theory is completely solved in Section 4. All equations of that kind admitting point Lie symmetry groups, in addition to the ones generated by the vector fields (10), are determined and presented in Table 1 together with the generators of the respective groups. The largest symmetry groups are admitted by the equations of form (35) whose coefficients are such that $\chi^{22}\det(\chi^{\alpha\beta}) = 0$, $\kappa(x) \equiv 0$. The most interesting group-invariant solutions for Eq. (35) are identified and the corresponding reduced equations are presented at the end of Section 4.

In Section 5, this problem is solved for Eq. (44) governing bending and stability of plates resting on variable elastic foundations within Poisson–Kirchhoff theory in terms of the invariants $s_{(1)}$ and $s_{(2)}$ defined by expressions (53). The equations of form (44) with $s_{(1)} \equiv s_{(2)} \equiv 0$ are found to admit the largest symmetry groups. It is noteworthy that each equation of this kind with variable coefficients can be transformed, using a suitable change of variables, to an equation with constant coefficients belonging to the same class. Next, an example of such a transformation is given, and, in addition, a class of group-invariant solutions to the equation considered is presented. The group-classification problem for the differential equation (64) governing the dynamics of Poisson–Kirchhoff plates of constant bending rigidity, mass density and membrane stresses is also solved in Section 5.

Once the "ordinary" point Lie symmetries of an equation of form (1) are determined, one can easily find, using the general criterion (22), which of them are variational symmetries of this equation. Then, formulae (31) and (34) provide explicit expressions for the conserved currents of the conservation laws associated through Bessel-Hagen's extension of Noether's theorem with the established variational symmetries. The conserved currents obtained in this way involve derivatives of the dependent variable of lowest possible order. This is important in view of their application in mechanics and engineering. The reciprocity relation valid for each equation of form (1) is given explicitly by formula (29).

The conservation laws for the rod equations listed in Table 1 are given in Table 2. Inspecting these results one can see that the equations for rods without foundation and for rods on Winkler foundations admit two independent conservation laws associated with the wave momentum ($D_\alpha B^\alpha_{(1)} = 0$) and energy ($D_\alpha B^\alpha_{(2)} = 0$). Equations of form (37) governing the stability of rods and fluid conveying pipes belong to this class. Rod equations with $\kappa(x) = 0$ and $\det(\chi^{\alpha\beta}) = 0$ admit a supplementary conservation law $D_\alpha B^\alpha_{(3)} = 0$ associated with the scaling symmetry. Such are Eq. (37), governing pipes conveying fluid and extended by end force (39), and Eq. (42), governing the vibration of the classical Bernoulli–Euler beam. The conservation laws for the rod equations derived here are discussed in the light of the relevant results obtained by Chien et al. (1993), Tabarrok et al. (1994) and Maddocks and Dichmann (1994).

In Section 5, it is shown, using the consequence (23) of the general criterion (22), that each point Lie symmetry of a plate equation of form (44) generated by a vector field of form (52) is a variational symmetry of this equation. Therefore, it gives rise to a conservation law with characteristic $Q = (1/2)\xi^\mu_{,\mu} w - w_\mu \xi^\mu$ and conserved current given by formula (34) admitted by the smooth solutions of the respective equation. Similarly, each symmetry of the corresponding dynamic equations (64), except the one associated with the vector field $X_0 = w\partial/\partial w$, is shown to generate a conservation law with characteristic $Q = Cw - w_\mu \xi^\mu - w_3 \xi^3$, where C is an appropriate constant, the corresponding conserved current being given by formula (34).

It should be remarked that the applicability of the general results presented in Section 3 exceeds the classical Bernoulli–Euler and Poisson–Kirchhoff type theories. Indeed, many other theories of solids and structures involve differential equations of form (1) that are satisfied by the solutions of the respective governing equations. Then, the geometric symmetries of such equations of form (1) turn out to be generalized symmetries of the governing equations of the theories in question. Besides, the conservation laws for the equations of form (1) are admitted by the solutions of the respective governing equations as well, and involve only a part of the dependent variables. This wider applicability of the general results achieved in Section 3 is illustrated in Section 6 by three examples—the theories of Timoshenko beams, Reissner–Mindlin plates and three-dimensional elastostatics. The physical interpretation of the new conservation laws from Section 6 is not yet clarified and will be considered in a forthcoming paper.

Finally, it should be underlined that a reader who is interested in particular rod or plate equations belonging to the class considered could use the results obtained here without going into detail concerning the Lie group analysis of differential equations. He could profit from the paper by following the procedure given at the end of Section 3.

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