Found. Comput. Math. 269–308 (2006)

DOI: 10.1007/s10208-004-0148-8

FOUNDATIONS OF COMPUTATIONAL MATHEMATICS
The Journal of the Society for the Foundations of Computational Mathematics

Efficient Construction of Contact Coordinates for Partial Prolongations

Peter J. Vassiliou

School of Mathematics and Statistics
University of Canberra
Canberra, ACT 2601, Australia
Peter.Vassiliou@canberra.edu.au

Abstract. Let $\mathcal{V}$ be a vector field distribution or Pfaffian system on manifold M. We give an efficient algorithm for the construction of local coordinates on M such that $\mathcal{V}$ may be locally expressed as some partial prolongation of the contact distribution $\mathcal{C}_q^{(1)}$, on the first-order jet bundle of maps from $\mathbb{R}$ to $\mathbb{R}^q$, $q \geq 1$. It is proven that if $\mathcal{V}$ is locally equivalent to a partial prolongation of $\mathcal{C}_q^{(1)}$, then the explicit construction of contact coordinates algorithmically depends upon the determination of certain first integrals in a sequence of geometrically defined and algorithmically determined integrable Pfaffian systems on M. The number of these first integrals that must be computed satisfies a natural minimality criterion. These results provide a full and constructive generalisation of the Goursat normal form from the theory of exterior differential systems.

1. Introduction

Let M be a smooth manifold and $\mathcal{V}$ a vector field distribution or a Pfaffian system on M. It is rare for the local structure of $\mathcal{V}$ to be uniquely determined by its derived type. The classical theorems we possess along these lines form the basic results of local differential geometry and are far from numerous: the Frobenius theorem, the

Date received: July 7, 2004. Final version received: June 13, 2005. Date accepted: August 17, 2005. Communicated by Peter Olver. Online publication February 24, 2006.
AMS classification: 58J60, 34H05, 93B18, 93B27.
Key words and phrases: Generalised Goursat normal form, Partial prolongation, Algorithm, Nonlinear control theory, Contact coordinates, Brunovský normal form.

Pfaff–Darboux theorem and the Goursat normal form [**4**], [**21**]. More recently, the works of R. Bryant [**3**] and K. Yamaguchi [**26**] have considerably extended this list, providing a local geometric characterisation of the contact distribution on jet space $J^k(\mathbb{R}^n, \mathbb{R}^q)$ for each $k, n, q \geq 1$; these being *total prolongations* of $J^1(\mathbb{R}^n, \mathbb{R}^q)$.

The corresponding problem for *partial* prolongations of $J^1(\mathbb{R}^n, \mathbb{R}^q)$ is much more difficult and has hardly been touched in the general case. An exception to this state of affairs occurs for the simplest case, namely, partial prolongations of the contact distribution $\mathcal{C}_q^{(1)}$ on $J^1(\mathbb{R}, \mathbb{R}^q)$, $q \geq 1$. This is principally due to their central role in nonlinear geometric control theory [**8**]. Since the celebrated result of P. Brunovský [**2**], which holds that a controllable, linear control system is equivalent to a partial prolongation of $\mathcal{C}_q^{(1)}$, for some $q \geq 1$, a goal of the field has been to find means of characterising those controllable nonlinear control systems that can be identified with partial prolongations of $\mathcal{C}_q^{(1)}$, also known as Brunovský normal forms, by a suitable transformation of the manifold of states and controls, the so-called *static feedback transformations*. This has led to many interesting results for various classes of nonlinear control systems and control theorists now regard the so-called 'feedback linearisation problem' to be essentially solved. A partial list of relevant works include Krener [**10**], Jakubczyk and Respondek [**9**], Hunt, Su, and Meyer [**7**] and van der Schaft [**22**].

On the mathematical side we mention the work of Libermann [**12**], and Respondek and Pasillas-Lepine [**15**]–[**18**]. In these works, the latter two authors establish a characterisation of the contact distribution on $J^k(\mathbb{R}, \mathbb{R}^q)$, under the full diffeomorphism pseudogroup of the ambient manifold (herein called *general equivalence*), these being the total prolongations of $\mathcal{C}_q^{(1)}$. They also describe an efficient method for constructing the corresponding contact coordinates. The problem of geometrically characterising the partial prolongations of $\mathcal{C}_q^{(1)}$ under general equivalence is considered by Respondek and Pasillas-Lepine in [**16**]. In that paper the authors announce a theorem, the extended Goursat normal form, in which necessary and sufficient conditions on the derived flag of a distribution, in order that it be locally equivalent to a partial prolongation of $\mathcal{C}_q^{(1)}$, are stated. This theorem is based on concomitants of the derived flag and is therefore in the spirit of the present paper; however, I have not been able to locate its proof. From a different direction, Kumpera and Rubin [**11**] study very related questions in the theory of underdetermined ordinary differential equations in possession of the so-called Monge property, establishing a sufficient condition for this property to hold.

Another solution of the static feedback linearisation problem for a fully nonlinear control system was discovered by Gardner and Shadwick [**6**], in which an efficient algorithm was also established whereby the feedback equivalence could be explicitly constructed using the minimum number of integrations. This so-called *GS algorithm* relies on knowledge of a distinguished 1-form leading to a set of structure equations that have to be determined before a static feedback equivalence can be found. Knowledge of this 1-form and the corresponding structure equations was thought to be problematic until the paper of Aranda-Bricaire and Pomet [**1**]. Though presented in the context of control systems, this work may also be viewed as establishing a geometric characterisation, up to general equivalence, of the partial prolongations of $\mathcal{C}_q^{(1)}$, the extended Goursat normal form, as the authors call it. They approach the problem via the notion of infinitesimal Brunovský normal forms which provide an alternative to the derived type of a distribution. The authors also state that their considerations lead to a method for constructing an equivalence to the extended Goursat normal form, whenever such an equivalence exists.

In [**23**] a geometric characterisation of the partial prolongations of $\mathcal{C}_q^{(1)}$, up to general equivalence, in terms of the derived type of a distribution was established, the *generalised Goursat normal form*. Specifically, simple geometric conditions, expressed in terms of numerical invariants associated to the derived flag of a subbundle $\mathcal{V} \subset TM$ or $\Omega \subset T^*M$ were obtained, guaranteeing the existence of a local diffeomorphism from M which identifies $\mathcal{V}$ with some partial prolongation of $\mathcal{C}_q^{(1)}$ or $\mathcal{C}_q^{(1)\perp}$. This characterisation is quite different from what is available in the literature to date and forms the basis of the algorithm established in this paper.

The areas to which potential applications of this result can be made include explicitly integrable partial differential equations and, more generally, Pfaffian systems, nonlinear control theory, sub-Riemannian geometry, differential invariants of curves in a homogeneous space and integrable curve dynamics. For these and related applications, it is desirable not only to settle the recognition problem for partial prolongations but, additionally, to find a method for explicitly *constructing* an equivalence between a differential system and a partial prolongation, whenever one is known to exist.

The main aim of this paper is to establish an efficient, practical algorithm, based on the generalised Goursat normal form, for this very construction problem.

We identify canonical and algorithmically determined integrable Pfaffian systems over the ambient manifold whose first integrals determine local equivalences. The number of these first integrals that must be constructed, according to our algorithm, satisfies a natural minimality criterion. In the special case when $\mathcal{V}$ arises from an autonomous nonlinear control system, the complexity of our algorithm agrees with that of the GS algorithm [**6**]. However, even here, the method described in this paper enjoys a number of advantages over the former. For instance, there is no requirement to construct a distiguished 1-form and concomitant structure equations before an equivalence can be found; nor is it restricted to autonomous control systems and static feedback transformations. This point is briefly discussed in Section 4. We go on in Section 5 to give an illustrative example of a nonlinear control system where, in addition, a necessary condition for feedback linearisation is derived in the nonautonomous case. We also present a number of other examples in Section 5, among these a method for constructing the differential invariants of curves in the homogeneous space of a Lie group. All these examples are mainly presented for the pupose of illustrating the algorithm `Contact`, which is the main result of this paper and for hinting at the possibility of future applications, rather than attempting to present specifically new results in the areas from which the examples are drawn.

Throughout the present work we will rely on the results of the aforementioned paper [**23**] and, wherever possible, adhere closely to its notational conventions. In Sections 2 and 3, we will recall briefly the salient details of [**23**] that are required to establish the main result which is given in Section 4.

2. Preliminaries

2.1. *The Derived Flag*

Suppose M is a smooth manifold and that $\mathcal{V} \subset TM$ is a smooth sub-bundle of its tangent bundle. The structure tensor is the homomorphism of vector bundles $\delta : \Lambda^2\mathcal{V} \to TM/\mathcal{V}$ defined by

$$\delta(X, Y) = [X, Y] \mod \mathcal{V} \qquad \text{for } X, Y \in \Gamma(M, \mathcal{V}).$$

If δ has constant rank, we define the *first derived bundle* $\mathcal{V}^{(1)}$ as the inverse image of $\delta(\Lambda^2\mathcal{V})$ under the canonical projection $TM \to TM/\mathcal{V}$. Informally,

$$\mathcal{V}^{(1)} = \mathcal{V} + [\mathcal{V}, \mathcal{V}].$$

The derived bundles $\mathcal{V}^{(i)}$ are defined inductively

$$\mathcal{V}^{(i+1)} = \mathcal{V}^{(i)} + [\mathcal{V}^{(i)}, \mathcal{V}^{(i)}],$$

assuming that at each iteration, this defines a vector bundle, in which case we shall say that $\mathcal{V}$ is *regular*. For regular $\mathcal{V}$, by dimension reasons, there will be a smallest k for which $\mathcal{V}^{(k+1)} = \mathcal{V}^{(k)}$. This k is called the *derived length* of $\mathcal{V}$ and the whole sequence of sub-bundles

$$\mathcal{V} \subset \mathcal{V}^{(1)} \subset \mathcal{V}^{(2)} \subset \cdots \subset \mathcal{V}^{(k)}$$

the *derived flag* of $\mathcal{V}$. Though this is not essential, we shall in this paper restrict ourselves to sub-bundles which are "maximally nonintegrable" meaning that the final bundle in the derived flag is equal to the tangent bundle of the ambient manifold: $\mathcal{V}^{(k)} = TM$, where k is the derived length of $\mathcal{V}$.

2.2. *Cauchy Bundles*

Let us define

$$\sigma : \mathcal{V} \to \mathrm{Hom}(\mathcal{V}, TM/\mathcal{V}) \qquad \text{by} \qquad \sigma(X)(Y) = \delta(X, Y).$$

Even if $\mathcal{V}$ is regular, the homomorphism σ need not have constant rank. If it does, let us write Char $\mathcal{V}$ for its kernel. The Jacobi identity shows that Char $\mathcal{V}$ is always integrable. It is called the *Cauchy bundle* or *characteristic bundle* of $\mathcal{V}$. If $\mathcal{V}$ is regular and each $\mathcal{V}^{(i)}$ has a Cauchy bundle then, we say that $\mathcal{V}$ is *totally regular*. Then by the *derived type* of $\mathcal{V}$ we shall mean the list $\{\mathcal{V}^{(i)}, \text{Char } \mathcal{V}^{(i)}\}$ of subundles.

2.3. *The Singular Variety*

For each $x \in M$, let

$$\mathcal{S}_x = \{v \in \mathcal{V}_x \backslash 0 \mid \sigma(v) \text{ has less than generic rank}\}.$$

Then $\mathcal{S}_x$ is the zero set of homogeneous polynomials and so defines a subvariety of the projectivisation $\mathbb{P}\mathcal{V}_x$ of $\mathcal{V}_x$. We shall denote by $\mathrm{Sing}(\mathcal{V})$ the fibre bundle over M with fibre over $x \in M$ equal to $\mathcal{S}_x$ and we refer to it as the *singular variety* of $\mathcal{V}$. For $X \in \mathcal{V}$ the matrix of the homomorphism $\sigma(X)$ will be called the *polar matrix* of $[X] \in \mathbb{P}\mathcal{V}$. There is a map $\deg_{\mathcal{V}} : \mathbb{P}\mathcal{V} \to \mathbb{N}$ well defined by

$$\deg_{\mathcal{V}}([X]) = \operatorname{rank} \sigma(X) \qquad \text{for } [X] \in \mathbb{P}\mathcal{V}.$$

We shall call $\deg_{\mathcal{V}}([X])$ the *degree* of $[X]$. The singular variety $\mathrm{Sing}(\mathcal{V})$ is a diffeomorphism invariant in the sense that if $\mathcal{V}_1, \mathcal{V}_2$ are sub-bundles over M_1, M_2, respectively and there is a diffeomorphism $\varphi : M_1 \to M_2$ that identifies them, then $\mathrm{Sing}(\mathcal{V}_2)$ and $\mathrm{Sing}(\varphi_* \mathcal{V}_1)$ are equivalent as projective subvarieties of $\mathbb{P}\mathcal{V}_2$. That is, for each $x \in M_1$, there is an element of the projective linear group $PGL(\mathcal{V}_{2|_{\varphi(x)}}, \mathbb{R})$ that identifies $\mathrm{Sing}(\mathcal{V}_2)(\varphi(x))$ and $\mathrm{Sing}(\varphi_* \mathcal{V}_1)(\varphi(x))$.

We hasten to point out that the computation of the singular variety for any given sub-bundle $\mathcal{V} \subset TM$ is algorithmic. That is, it involves only differentiation and commutative algebra operations. One computes the determinantal variety of the polar matrix for generic $[X]$.

2.4. *The Singular Variety in Positive Degree*

If $X \in \mathrm{Char}\, \mathcal{V}$, then $\deg_{\mathcal{V}}([X]) = 0$. For this reason we pass to the quotient $\widehat{\mathcal{V}} := \mathcal{V}/\mathrm{Char}\, \mathcal{V}$. We have structure tensor $\widehat{\delta} : \Lambda^2 \widehat{\mathcal{V}} \to \widehat{TM}/\widehat{\mathcal{V}}$, well defined by

$$\widehat{\delta}(\widehat{X}, \widehat{Y}) = \pi([X, Y]) \bmod \widehat{\mathcal{V}},$$

where $\widehat{TM} = TM/\mathrm{Char}\, \mathcal{V}$ and

$$\pi : TM \to \widehat{TM}$$

is the canonical projection. The notion of degree descends to this quotient giving a map

$$\deg_{\widehat{\mathcal{V}}} : \mathbb{P}\widehat{\mathcal{V}} \to \mathbb{N}$$

well defined by

$$\deg_{\widehat{\mathcal{V}}}([\widehat{X}]) = \operatorname{rank} \widehat{\sigma}(\widehat{X}) \qquad \text{for } [\widehat{X}] \in \mathbb{P}\widehat{\mathcal{V}},$$

where $\widehat{\sigma}(\widehat{X})(\widehat{Y}) = \widehat{\delta}(\widehat{X}, \widehat{Y})$ for $\widehat{Y} \in \widehat{\mathcal{V}}$. Note that all definitions go over mutatis mutandis when the structure tensor δ is replaced by $\widehat{\delta}$. In particular, we have notions of polar matrix and singular variety, as before. However, if the singular variety of $\widehat{\mathcal{V}}$ is not empty, then each point of $\mathbb{P}\widehat{\mathcal{V}}$ has degree one or more.

2.5. *The Resolvent Bundle*

Suppose $\mathcal{V} \subset TM$ is totally regular of rank $c+q+1$, $q \geq 2$, $c \geq 0$, $\dim M = c+2q+1$. Suppose further that $\mathcal{V}$ satisfies:

(i) $\dim \operatorname{Char} \mathcal{V} = c$, $\mathcal{V}^{(1)} = TM$; and
(ii) $\widehat{\Sigma}_{|x} := \operatorname{Sing}(\widehat{\mathcal{V}})_{|x} = \mathbb{P}\widehat{\mathcal{B}}_{|x} \approx \mathbb{RP}^{q-1}$, for each $x \in M$ and some rank q sub-bundle $\widehat{\mathcal{B}} \subset \widehat{\mathcal{V}}$.

Then we call $(\mathcal{V}, \mathbb{P}\widehat{\mathcal{B}})$ (or $(\mathcal{V}, \widehat{\Sigma})$) a *Weber structure* of rank q on M.

Given a Weber structure $(\mathcal{V}, \mathbb{P}\widehat{\mathcal{B}})$, let $\mathcal{R}_{\widehat{\Sigma}}(\mathcal{V}) \subset \mathcal{V}$ denote the largest sub-bundle such that

$$\pi(\mathcal{R}_{\widehat{\Sigma}}(\mathcal{V})) = \widehat{\mathcal{B}}. \tag{2.1}$$

We call the rank $q+c$ bundle $\mathcal{R}_{\widehat{\Sigma}}(\mathcal{V})$ defined by (2.1) the *resolvent bundle* associated to the Weber structure $(\mathcal{V}, \widehat{\Sigma})$. The bundle $\widehat{\mathcal{B}}$ determined by the singular variety of $\widehat{\mathcal{V}}$ will be called the *singular sub-bundle* of the Weber structure. A Weber structure will be said to be *integrable* if its resolvent bundle is integrable.

An *integrable* Weber structure descends to the quotient of M by the leaves of $\operatorname{Char} \mathcal{V}$ to be the contact bundle on $J^1(\mathbb{R}, \mathbb{R}^q)$. The term honours Eduard von Weber (1870–1934) who was the first to publish a proof of the Goursat normal form [**24**]. We record the following properties of the resolvent bundle of a Weber structure.

Proposition 2.1 [**23**]. *Let $(\mathcal{V}, \widehat{\Sigma})$ be a Weber structure on M and $\widehat{\mathcal{B}}$ its singular sub-bundle. If $q \geq 3$, then the following are equivalent*:

(1) *Its resolvent bundle $\mathcal{R}_{\widehat{\Sigma}}(\mathcal{V}) \subset \mathcal{V}$ is integrable.*
(2) *Each point of $\widehat{\Sigma} = \operatorname{Sing}(\widehat{\mathcal{V}})$ has degree one.*
(3) *The structure tensor $\widehat{\delta}$ of $\widehat{\mathcal{V}}$ vanishes on $\widehat{\mathcal{B}}$: $\widehat{\delta}(\widehat{\mathcal{B}}, \widehat{\mathcal{B}}) = 0$.*

Proposition 2.2 [**23**]. *Let $(\mathcal{V}, \widehat{\Sigma})$ be an integrable Weber structure on M. Then its resolvent bundle $\mathcal{R}_{\widehat{\Sigma}}(\mathcal{V})$ is the unique, maximal, integrable sub-bundle of $\mathcal{V}$.*

Checking the integrability of the resolvent bundle is algorithmic. One computes the singular variety $\operatorname{Sing}(\widehat{\mathcal{V}}) = \mathbb{P}\widehat{\mathcal{B}}$. In turn, the singular bundle $\widehat{\mathcal{B}}$ algorithmically determines $\mathcal{R}_{\widehat{\Sigma}}(\mathcal{V})$.

A word on conventions. First, we work exclusively in the smooth (C^∞) category and all objects and maps will be assumed to be smooth without further notice. Second, we will often denote sub-bundles $\mathcal{V} \subset TM$ by a list of vector fields $X, Y, Z, \ldots$ on M enclosed by braces, $\mathcal{V} = \{X, Y, Z, \ldots\}$. This will always denote the bundle $\mathcal{V}$ whose space of sections is the $C^\infty(M)$-module generated by vector fields $X, Y, Z, \ldots$. Unless explicitly stated otherwise the annihilator $\mathcal{V}^\perp$ of $\mathcal{V}$ will be denoted by Ω. That is, $\Omega := \mathcal{V}^\perp$.

3. Partial Prolongations and Goursat Bundles

In this section we give a brief coordinate description of partial prolongations and introduce the notion of a Goursat bundle.

The *contact distribution* on the first-order jet bundle of maps from $\mathbb{R} \to \mathbb{R}^q$, $J^1(\mathbb{R}, \mathbb{R}^q)$, $q \geq 1$, will be denoted by the symbol $\mathcal{C}_q^{(1)}$ and locally expressed in contact coordinates as

$$\mathcal{C}_q^{(1)} = \left\{ \partial_x + \sum_{j=1}^{q} z_1^j \partial_{z^j}, \partial_{z_1^j} \right\}. \tag{3.1}$$

A *partial prolongation* of $\mathcal{C}_q^{(1)}$ may be expressed in contact coordinates as a distribution on $J^k(\mathbb{R}, \mathbb{R}^q)$ of the form

$$\mathcal{C}(\tau) = \left\{ \partial_x + \sum_{j\in\{1,\dots,k\}} \sum_{a_j=1}^{q_j} \sum_{l_j=0}^{j-1} z_{l_j+1}^{a_j,j} \partial_{z_{l_j}^{a_j,j}}, \ \partial_{z_j^{a_j,j}} \right\}_{a_j=1}^{q_j}, \qquad j \in \{1, 2, \dots, k\}, \tag{3.2}$$

for some positive integers q_j, where k is the derived length of $\mathcal{C}(\tau)$ and, as indicated, j takes values in a subset of $\{1, \dots, k\}$. If $\mathcal{C}(\tau)$ contains the q_j elements

$$\partial_{z_j^{1,j}}, \dots, \partial_{z_j^{q_j,j}}$$

for some $j \in \{1, \dots, k\}$, then we shall say it contains or possesses q_j *dependent variables of order* j.

Dually, we can express the partial prolongations in contact coordinates as

$$\mathcal{C}(\tau)^\perp = \bigcup_{j\in\{1,\dots,k\}} \left\{ dz_{l_j}^{a_j,j} - z_{l_j+1}^{a_j,j}\, dx \right\}_{a_j=1,l_j=0}^{q_j,j-1}. \tag{3.3}$$

Here and elsewhere in this paper, the symbol τ denotes the *type* of the partial prolongation which is specified by an ordered list of k non-negative integers

$$\tau = \langle \rho_1, \rho_2, \dots, \rho_k \rangle, \tag{3.4}$$

where the jth element ρ_j indicates the number of variables of order j. Note that if a partial prolongation possesses q_j variables of order j, then $\rho_j = q_j$ and $\rho_j = 0$, otherwise. If $\mathcal{C}(\tau)$ has derived length k and only possesses dependent variables of order k, then (3.2) and (3.3) have a type of the form, $\langle 0, 0, \dots, 0, q \rangle$, where $k - 1$ zeros precede entry $\rho_k = q$. Such a contact system is a *total prolongation* of $\mathcal{C}_q^{(1)}$, denoted $\mathcal{C}_q^{(k)}$, an instance of a partial prolongation. The well-known Goursat normal form [**4**], [**21**] is a characterisation of $\mathcal{C}_1^{(k)}$ in terms of derived type.

Goursat Normal Form. *Let $\mathcal{V} \subset TM$ be a rank* 2, *totally regular sub-bundle with derived length k such that*

$$\dim \mathcal{V}^{(i)} = \dim \mathcal{V}^{(i-1)} + 1, \qquad i = 1, 2, \dots, k.$$

Then there is an open, dense subset $\hat{M} \subseteq M$ such that the restriction of $\mathcal{V}$ to $\hat{M}$ is locally equivalent to $C_1^{(k)}$.

In terms of the notion of type introduced above, observe that the Goursat normal form is a characterisation of totally regular sub-bundles whose type has the form $\langle 0, 0, \ldots, 0, 1\rangle$. The main result of [**23**] is a similar geometric characterisation for arbitrary partial prolongations of $\mathcal{C}_q^{(1)}$, the contact distribution on the first-order jet bundle $J^1(\mathbb{R}, \mathbb{R}^q)$. Such bundles can have completely arbitary type (3.4). Though the proof of this result in [**23**] is constructive, the construction is certainly not optimal. Here we establish an algorithm that is optimal in the sense that the amount of integration that need be performed to construct an equivalence is minimal. Before launching into the details we give an example that illustrates the foregoing considerations.

Example 3.1. The contact distribution on the 'hybrid' jet bundle $J^\tau(\mathbb{R}, \mathbb{R}^{21})$, $\tau = \langle 0, 11, 7, 0, 3\rangle$ has the form

$$\begin{aligned}\mathcal{V} &= \mathcal{C}\langle 0, 11, 7, 0, 3\rangle \\ &= \Bigg\{\partial_x + \sum_{a_2=1}^{11}\sum_{l_2=0}^{1} z_{l_2+1}^{a_2,2}\partial_{z_{l_2}^{a_2,2}} + \sum_{a_3=1}^{7}\sum_{l_3=0}^{2} z_{l_3+1}^{a_3,3}\partial_{z_{l_3}^{a_3,3}} \\ &\qquad + \sum_{a_5=1}^{3}\sum_{l_5=0}^{4} z_{l_5+1}^{a_5,5}\partial_{z_{l_5}^{a_5,5}},\ \partial_{z_2^{a_2,2}},\ \partial_{z_3^{a_3,3}},\ \partial_{z_5^{a^5,5}}\Bigg\}_{a_2=1,a_3=1,a_5=1}^{11,7,3}.\end{aligned}$$

This partial prolongation has 11 dependent variables of order 2; 7 dependent variables of order 3 and 3 dependent variables of order 5. The dual is

$$\begin{aligned}\Omega &= \mathcal{C}\langle 0, 11, 7, 0, 3\rangle^\perp \\ &= \{dz_{l_2}^{a_2,2} - z_{l_2+1}^{a_2,2}dx\}_{a_2=1,l_2=0}^{11,1} \cup \{dz_{l_3}^{a_3,3} - z_{l_3+1}^{a_3,3}dx\}_{a_3=1,l_3=0}^{7,2} \\ &\qquad \cup \{dz_{l_5}^{a_5,5} - z_{l_5+1}^{a_5,5}dx\}_{a_5=1,l_5=0}^{3,4}.\end{aligned}$$

For any totally regular sub-bundle $\mathcal{V} \subset TM$, we have the notion of its derived type. In Section 2 we defined the *derived type* of a bundle as the list of all derived bundles together with their corresponding Cauchy bundles. We shall frequently abuse notation by using the term 'derived type of $\mathcal{V}$' for the ordered list of ordered pairs of the form

$$[[m_0, \chi^0], [m_1, \chi^1], \ldots, [m_k, \chi^k]]$$

where $m_j = \dim \mathcal{V}^{(j)}$ and $\chi^j = \dim \mathrm{Char}\, \mathcal{V}^{(j)}$.

It is important to relate the type of a partial prolongation to its derived type. For this it's convenient to introduce the notions of *velocity*, *acceleration* and *deceleration* of a sub-bundle.

Definition 3.1. Let $\mathcal{V} \subset TM$ be a totally regular sub-bundle with derived type

$$[[m_0, \chi^0], [m_1, \chi^1], \ldots, [m_k, \chi^k]]. \tag{3.5}$$

The *velocity* of $\mathcal{V}$ is the ordered list of k integers

$$\text{vel}(\mathcal{V}) = \langle \Delta_1, \Delta_2, \ldots, \Delta_k \rangle, \qquad \text{where} \quad \Delta_j = m_j - m_{j-1}, \quad 1 \le j \le k.$$

The *acceleration* of $\mathcal{V}$ is the ordered list of k integers

$$\begin{aligned} \text{accel}(\mathcal{V}) \;&=\; \langle \Delta_2^2, \Delta_3^2, \ldots, \Delta_k^2, \Delta_k \rangle, \\ &\qquad \text{where} \quad \Delta_i^2 = \Delta_i - \Delta_{i-1}, \quad 2 \le i \le k. \end{aligned}$$

The *deceleration* of $\mathcal{V}$ is the ordered list of k integers

$$\text{decel}(\mathcal{V}) = \langle -\Delta_2^2, -\Delta_3^2, \ldots, -\Delta_k^2, \Delta_k \rangle.$$

Note that total prolongations $\mathcal{C}_q^{(k)}$ have decelerations of the form $\langle 0, 0, \ldots, 0, q \rangle$, $q \ge 1$, where there are $k-1$ zeros before the final entry, q. The Goursat normal form is the case $q = 1$ in this family of decelerations. The main aim of [**23**] is to generalise this classical result to completely arbitrary decelerations.

Alternatively, one can formulate the theory dually, in terms of a sub-bundle $\Omega \subset T^*M$. In this case we have an analogue of the structure tensor discussed in Section 2. If $\Omega \subset T^*M$ is a sub-bundle, then we define a tensor $\varrho : \Omega \to \Lambda^2 M/(\Lambda^1 M \wedge \Omega)$ defined by

$$\varrho(\omega) = d\omega \mod \Lambda^1 M \wedge \Omega.$$

The kernel $\ker \varrho$ of ϱ is called the derived-bundle $\Omega^{(1)}$ of Ω. In the constant rank case we iterate to obtain the derived flag of Ω,

$$\Omega \supset \Omega^{(1)} \supset \Omega^{(2)} \supset \cdots \supset \Omega^{(k)}.$$

We note that if $\Omega^\perp = \mathcal{V} \subset TM$ and $\mathcal{V}^{(i)}$ is the ith element of the derived flag of $\mathcal{V}$, then $\mathcal{V}^{(i)} = \Omega^{(i)\perp}$. The Cauchy bundle Char Ω of Ω is the Cauchy bundle of the $\Omega^\perp$. The *Cartan system* $\Xi(\Omega)$ of Ω is the annnihilator of Char Ω:

$$\Xi(\Omega) = (\text{Char } \Omega)^\perp.$$

We then define the *derived type* of $\Omega \subset T^*M$ to be the list of sub-bundles $\{\Omega^{(i)}, \Xi(\Omega^{(i)})\}_{i=0}^k$ of T^*M. As in the case of vector field distributions, for totally regular sub-bundles $\Omega \subset T^*M$ we often use the term "derived type of Ω" to denote the list of ordered pairs of non-negative integers

$$[[h_0, \xi^0], [h_1, \xi^1], \ldots, [h_k, \xi^k]]$$

where $h_i = \dim \Omega^{(i)}$, $\xi^i = \dim \Xi(\Omega^{(i)})$, $0 \le i \le k$, and k is the derived length of Ω.

Definition 3.2. Let $\Omega \subset T^*M$ be a totally regular sub-bundle with derived type

$$[[h_0, \xi^0], [h_1, \xi^1], \ldots, [h_k, \xi^k]].$$

The *velocity* of Ω is the ordered list of k integers

$$\mathrm{vel}(\Omega) = \langle \Gamma^1, \Gamma^2, \ldots, \Gamma^k \rangle, \qquad \text{where} \quad \Gamma^j = h^j - h^{j-1}, \quad 1 \le j \le k.$$

The *acceleration* of Ω is the ordered list of k integers

$$\begin{aligned} \mathrm{accel}(\Omega) \; &= \; \langle \Gamma_2^2, \Gamma_2^3, \ldots, \Gamma_2^k, -\Gamma^k \rangle, \\ & \qquad \text{where} \quad \Gamma_2^j = \Gamma^j - \Gamma^{j-1}, \quad 2 \le j \le k. \end{aligned}$$

To recognise when a given sub-bundle has or has not the derived type of a partial prolongation (3.2) we introduce one further canonically associated sub-bundle that plays a crucial role.

Definition 3.3. If $\mathcal{V} \subset TM$ is a totally regular sub-bundle of derived length k we let Char $\mathcal{V}_{j-1}^{(j)}$ denote the intersections

$$\mathrm{Char}\, \mathcal{V}_{j-1}^{(j)} = \mathcal{V}^{(j-1)} \cap \mathrm{Char}\, \mathcal{V}^{(j)}, \qquad 1 \le j \le k-1.$$

If $\Omega \subset T^*M$ is a totally regular sub-bundle of derived length k we let $\Xi(\Omega)_{(j-1)}^j$ denote the unions

$$\Xi(\Omega)_{(j-1)}^j = \Omega^{(j-1)} \cup \Xi(\Omega^{(j)}), \qquad 1 \le j \le k-1.$$

Let

$$\chi_{j-1}^j = \dim \mathrm{Char}\, \mathcal{V}_{j-1}^{(j)}, \qquad \xi_{j-1}^j = \dim \Xi(\Omega)_{(j-1)}^j, \qquad 1 \le j \le k-1.$$

We shall call the integers $\{\chi^0, \chi^j, \chi_{j-1}^j\}_{j=1}^{k-1}$ the *type numbers* of $\mathcal{V} \subset TM$ and $\{\xi^0, \xi^j, \xi_{j-1}^j\}_{j=1}^{k-1}$ the *type numbers* of $\Omega \subset T^*M$. We shall refer to the ordered list of lists

$$[[m_0, \chi^0], [m_1, \chi_0^1, \chi^1], [m_2, \chi_1^2, \chi^2], \ldots, [m_{k-1}, \chi_{k-2}^{k-1}, \chi^{k-1}], [m_k, \chi^k]]$$

as the *refined derived type* of $\mathcal{V}$ and to the ordered list of lists

$$[[h_0, \xi^0], [h_1, \xi_0^1, \xi^1], [h_2, \xi_1^2, \xi^2], \ldots, [h_{k-1}, \xi_{k-2}^{k-1}, \xi^{k-1}], [h_k, \xi^k]]$$

as the *refined derived type* of Ω.

It is easy to see that in every partial prolongation (3.2) these sub-bundles are non-trivial and integrable.

Proposition 3.1. *Let sub-bundle $\mathcal{V} \subset TM$ be totally regular, of derived length k, with velocity and acceleration $\langle \Delta_1, \Delta_2, \ldots, \Delta_k \rangle$ and $\langle \Delta_2^2, \ldots, \Delta_k^2, \Delta_k \rangle$, respectively. Then $\mathcal{V}$ has the derived type of a partial prolongation if and only if the type numbers of $\mathcal{V}$ satisfy*

$$\begin{aligned} \chi^j &= 2m_j - m_{j+1} - 1, \qquad && 0 \le j \le k-1, \\ \chi^i_{i-1} &= m_{i-1} - 1, && 1 \le i \le k-1. \end{aligned} \tag{3.6}$$

The type τ of $\mathcal{V}$ is given by its deceleration, $\tau = \operatorname{decel}(\mathcal{V})$.

Proof. From the local normal form (3.2) we deduce that χ^j satisfies the recurrence relation $\chi^{l+1} = \chi^l + \Delta_{l+1} - \Delta^2_{l+2}$, $0 \le l \le k-2$, with $\chi^0 = 0$; and that χ^l_{l-1} satisfies $\chi^{l+1}_l = \chi^l_{l-1} + \Delta_{l+1} - \Delta^2_{l+1}$, $1 \le l \le k-2$ with $\chi^1_0 = \Delta_1$. The equations in (3.6) are readily deduced from these and the recurrence relations $m_{j+1} = \Delta^2_{j+1} + 2m_j - m_{j-1}$, $m_0 = 1 + P$, $m_1 = 1 + 2P$, $P = \sum_{i=1}^k \rho_l$. Observe that the number of variables of order l in (3.2) is given by $\chi^l - \chi^l_{l-1} = -\Delta^2_{l+1}$ for $1 \le l \le k-1$, and that there are Δ_k variables of order k. This shows that the type of a partial prolongation is given by $\tau = \operatorname{decel}(\mathcal{V})$. □

Dually, we have

Proposition 3.2. *Let sub-bundle $\Omega \subset T^*M$ be totally regular, of derived length k with velocity and acceleration $\langle \Gamma^1, \Gamma^2, \ldots, \Gamma^k \rangle$ and $\langle \Gamma^2_2, \Gamma^3_2 \ldots, \Gamma^k_2, -\Gamma^k \rangle$, respectively. Then Ω has the derived type of a partial prolongation if and only if the type numbers of Ω satisfy*

$$\begin{aligned} \xi^j &= 2h_j - h_{j+1} + 1, \qquad && 0 \le j \le k-1, \\ \xi^i_{i-1} &= h_{i-1} + 1, && 1 \le i \le k-1. \end{aligned}$$

The type τ of Ω is given by its acceleration, $\tau = \operatorname{accel}(\Omega)$.

Example 3.2. We compute the refined derived type and relevant bundles associated with the partial prolongation

$$\mathcal{V} = \mathcal{C}\langle 4,3,2\rangle = \Bigg\{ X = \partial_x + \sum_{a_1=1}^{4} z_1^{a_1,1}\partial_{z^{a_1,1}} + \sum_{a_2=1}^{3}\sum_{l_2=0}^{1} z_{l_2+1}^{a_2,2}\partial_{z_{l_2}^{a_2,2}} + \sum_{a_3=1}^{2}\sum_{l_3=0}^{2} z_{l_3+1}^{a_3,3}\partial_{z_{l_3}^{a_3,3}},\ \partial_{z_1^{a_1,1}}, \partial_{z_2^{a_2,2}}, \partial_{z_3^{a_3,3}} \Bigg\}$$

on $J^\tau(\mathbb{R}, \mathbb{R}^9)$, $\tau = \langle 4, 3, 2\rangle$. The refined derived type is

$$[[10, 0], [19, 9, 13], [24, 18, 21], [26, 26]],$$

and the derived length is $k = 3$. The Cauchy bundles and intersections are

$$\begin{aligned}\text{Char } \mathcal{V}^{(1)} &= \{\partial_{z_1^{a_1,1}}, \partial_{z_2^{a_2,2}}, \partial_{z_3^{a_3,3}}, \partial_{z^{a_1,1}}\},\\ \text{Char } \mathcal{V}_0^{(1)} &= \{\partial_{z_1^{a_1,1}}, \partial_{z_2^{a_2,2}}, \partial_{z_3^{a_3,3}}\},\end{aligned}$$

and

$$\begin{aligned}\text{Char } \mathcal{V}^{(2)} &= \{\partial_{z_1^{a_2,2}}, \partial_{z_2^{a_3,3}}, \partial_{z^{a_2,2}}, \partial_{z_1^{a_1,1}}, \partial_{z_2^{a_2,2}}, \partial_{z_3^{a_3,3}}, \partial_{z^{a_1,1}}\},\\ \text{Char } \mathcal{V}_1^{(2)} &= \{\partial_{z_1^{a_2,2}}, \partial_{z_2^{a_3,3}}, \partial_{z_1^{a_1,1}}, \partial_{z_2^{a_2,2}}, \partial_{z_3^{a_3,3}}, \partial_{z^{a_1,1}}\}.\end{aligned}$$

The reader will find it easy to verify that the type numbers are in agreement with Proposition 3.1. The singular variety of $\widehat{\mathcal{V}}^{(2)} = \mathcal{V}^{(2)}/\text{Char } \mathcal{V}^{(2)}$ consists of lines $E = [e^1\pi(X) + e^2\pi(\partial_{z_1^{1,3}}) + e^3\pi(\partial_{z_1^{2,3}})]$ whose degree is less than the generic degree which is 2. The polar matrix of E, defined in Section 2.3, is

$$\begin{pmatrix}-e^2 & e^1 & 0\\ -e^3 & 0 & e^1\end{pmatrix}$$

whose rank is less than 2 if and only if $e^1 = 0$. According to section 2.3, we deduce that

$$\text{Sing}(\widehat{\mathcal{V}}^{(2)})_{|\mathbf{z}} = \mathbb{P}\mathcal{B}_{|\mathbf{z}} = \mathbb{P}\{\partial_{z_1^{1,3}}, \partial_{z_1^{2,3}}\}_{|\mathbf{z}} \approx \mathbb{RP}^1, \qquad \forall\, \mathbf{z} \in J^\tau(\mathbb{R}, \mathbb{R}^9).$$

Consequently, $\mathcal{V}^{(2)}$ is a Weber structure of rank 2 with resolvent bundle,

$$\mathcal{R}_{\widehat{\Sigma}_2}(\mathcal{V}^{(2)}) = \text{Char } \mathcal{V}^{(2)} \oplus \{\partial_{z_1^{1,3}}, \partial_{z_1^{2,3}}\},$$

which is integrable. We note that $\mathcal{V}$ has $\chi^1 - \chi^1_0 = 4$ dependent variables of order 1, $\chi^2 - \chi^2_1 = 3$ dependent variables of order 2, and $\rho_3 := \Delta_3 = 2$ dependent variables of order 3. Finally, we observe that $\text{decel}(\mathcal{V}) = \langle 4, 3, 2\rangle$.

Definition 3.4. A totally regular sub-bundle $\mathcal{V} \subset TM$ $\big(\Omega \subset T^*M\big)$ of derived length k will be called a *Goursat bundle of type* τ if

(i) $\mathcal{V}$ $\big(\Omega\big)$ has the derived type of a partial prolongation whose type is $\tau = \text{decel}(\mathcal{V})$ ($\tau = \text{accel}(\Omega)$).
(ii) Each intersection Char $\mathcal{V}^{(i)}_{i-1}$ (union $\Xi(\Omega)^j_{(j-1)}$) is an integrable sub-bundle whose rank, assumed to be constant on M, agrees with the corresponding rank in $\mathcal{C}(\tau)$ ($\mathcal{C}(\tau)^\perp$).
(iii) In case $\Delta_k > 1$ ($\Gamma^k < -1$), then $\mathcal{V}^{(k-1)}$ ($\Omega^{(k-1)\perp}$) determines an integrable Weber structure of rank $\Delta_k = -\Gamma^k$ on M.

4. Efficient Construction of Contact Coordinates

In this section we establish our main result, an efficient, practical algorithm for the construction of contact coordinates for any smooth sub-bundle $\mathcal{V} \subset TM$ ($\Omega \subset T^*M$) locally equivalent to a partial prolongation of the contact bundle $\mathcal{C}_q^{(1)}$ ($(\mathcal{C}_q^{(1)})^\perp$) on $J^1(\mathbb{R}, \mathbb{R}^q)$, $q \geq 1$. Specifically, we show how to algorithmically construct certain canonical Pfaffian systems over the ambient manifold and appropriate first integrals of these Pfaffian systems whose derivatives generate a local equivalence, identifying a given Goursat bundle with some partial prolongation. The type of $\mathcal{V}$ or Ω is given by the deceleration vector, decel($\mathcal{V}$), or by the acceleration vector, accel(Ω), respectively.

The main result upon which our algorithm is based is the generalised Goursat normal form established in [**23**].

Theorem 4.1 (Generalised Goursat Normal Form [**23**]). *Let $\mathcal{V} \subset TM$ ($\Omega \subset T^*M$) be a Goursat bundle over manifold M, of derived length $k > 1$ and type $\tau = \text{decel}(\mathcal{V}) = \text{accel}(\Omega)$. Then there is an open, dense subset $\hat{M} \subseteq M$ such that the restriction of $\mathcal{V}$ (Ω) to $\hat{M}$ is locally equivalent to $\mathcal{C}(\tau)$ ($\mathcal{C}(\tau)^\perp$). Conversely, any partial prolongation of $\mathcal{C}_q^{(1)}$ ($\mathcal{C}_q^{(1)\perp}$) is a Goursat bundle.*

This theorem settles the recognition problem for a distribution in terms of simple constraints on its derived type. While the proof in [**23**] is constructive, it is extravagant with respect to the number of integrations that are carried out. The aim here is to show that the number of integrations that must be carried out, to actually *construct* an equivalence for any Goursat bundle, is comparatively small. In fact we show that the number of first integrals that must be computed in the general case is equal to the number of 'dependent variables' featured in $\mathcal{C}(\tau)$, plus one, where $\tau = \langle \rho_1, \rho_2, \ldots, \rho_k \rangle$ is the bundle's deceleration vector, decel($\mathcal{V}$), or acceleration vector, accel(Ω). That is, $\sum_{j=1}^{k} \rho_j + 1 = P + 1$ first integrals must be found in the general case. The remaining coordinates are computed by differentiation. This is the 'natural' minimality criterion alluded to in the Introduction. Our algorithm therefore performs as well as the GS algorithm [**6**] in the special case when distribution $\mathcal{V}$ or Ω happens to arise from an autonomous control system.

However, our approach doesn't involve the construction of a distiguished 1-form and concomitant structure equations before an equivalence can be found; nor is it restricted to autonomous control systems and static feedback transformations. Moreover, the geometric data that must be computed to settle the recognition problem is expressed more naturally in terms of the bundle's derived type. We now proceed to the description of our method.

Let $\mathcal{V} \subset TM$ be a Goursat bundle over M of derived length k. Recall that there is a distinction between the cases $\rho_k > 1$ and $\rho_k = 1$. In the former case we have the filtration

$$\begin{aligned}\mathrm{Char}\,\mathcal{V}_0^{(1)} \;&\subseteq\; \mathrm{Char}\,\mathcal{V}^{(1)} \subset \cdots \subset \mathrm{Char}\,\mathcal{V}_{j-1}^{(j)} \subseteq \mathrm{Char}\,\mathcal{V}^{(j)} \subset \cdots \\ &\subset\; \mathrm{Char}\,\mathcal{V}_{k-2}^{(k-1)} \subseteq \mathrm{Char}\,\mathcal{V}^{(k-1)} \subset \mathcal{R}_{\widehat{\Sigma}_{k-1}}(\mathcal{V}^{(k-1)}) \subset TM, \quad (4.1)\end{aligned}$$

where $\mathcal{R}_{\widehat{\Sigma}_{k-1}}(\mathcal{V}^{(k-1)})$ is the resolvent bundle of the integrable Weber structure $(\mathcal{V}^{(k-1)}, \widehat{\Sigma}_{k-1})$. Note that for j in the range $1 \le j \le k-1$, $\mathrm{Char}\,\mathcal{V}_{j-1}^{(j)} = \mathrm{Char}\,\mathcal{V}^{(j)}$ if and only if $\Delta_{j+1}^2 = 0$.

Dually, we let $\Omega = \mathcal{V}^{\perp}$ and use the convenient notation

$$\begin{aligned}\nu_j &= \Delta_j, && 1 \le j \le k,\\ n_i &= \nu_{i+1}, && 0 \le i \le k-1,\\ N_l &= \dim M - m_l, && 0 \le l \le k. \qquad (4.2)\end{aligned}$$

We have

$$\Xi(\Omega^{(j)}) = \mathrm{Char}\,\mathcal{V}^{(j)\perp}, \qquad \Xi(\Omega)^{j}_{(j-1)} = \mathrm{Char}\,\mathcal{V}_{j-1}^{(j)\ \perp}, \qquad 1 \le j \le k-1,$$

and we let

$$\Upsilon_{\widehat{\Sigma}_{k-1}}(\Omega^{(k-1)}) = \mathcal{R}_{\widehat{\Sigma}_{k-1}}(\mathcal{V}^{(k-1)})^{\perp}.$$

Then we have a filtration of the cotangent bundle T^*M,

$$\begin{aligned}\Upsilon_{\widehat{\Sigma}_{k-1}}(\Omega^{(k-1)}) \;&\subset\; \Xi(\Omega^{(k-1)}) \subseteq \Xi(\Omega)^{k-1}_{(k-2)} \subset \cdots \subset \Xi(\Omega^{(1)}) \\ &\subseteq\; \Xi(\Omega)^{1}_{(0)} \subset T^*M \qquad (4.3)\end{aligned}$$

by integrable sub-bundles. It follows easily from Proposition 3.1 and (4.2) that

$$\begin{aligned}\dim \Upsilon_{\widehat{\Sigma}_{k-1}}(\Omega^{(k-1)}) &= N_{k-1} + 1,\\ \dim \Xi(\Omega^{(j)}) &= N_j + \Delta_{j+1} + 1,\\ \dim \Xi(\Omega)^{j}_{(j-1)} &= N_{j-1} + 1, \quad 1 \le j \le k-1, \qquad (4.4)\end{aligned}$$

and

$$\dim \Xi(\Omega)^{j}_{(j-1)} - \dim \Xi(\Omega^{(j)}) = \Gamma_2^{j+1} = -\Delta_{j+1}^2 = \rho_j, \qquad 1 \le j \le k-1. \qquad (4.5)$$

We can therefore construct a filtered basis for sub-bundle $\Xi(\Omega)^{1}_{(0)} \subset T^*M$ as follows

$$\begin{aligned}\omega_0, \omega_1, \ldots, \omega_{N_{j-1}} &\quad \text{for } \Xi(\Omega)^{j}_{(j-1)},\\ \omega_0, \omega_1, \ldots, \omega_{N_j+n_j} &\quad \text{for } \Xi(\Omega^{(j)}),\\ \omega_0, \omega_1, \ldots, \omega_{N_{k-1}} &\quad \text{for } \Upsilon_{\widehat{\Sigma}_{k-1}}(\Omega^{(k-1)}), \quad 1 \le j \le k-1. \qquad (4.6)\end{aligned}$$

Definition 4.1. For each $j \in \{1, \ldots, k-1\}$ we have the quotient bundles

$$\mathcal{F}^j(\Omega) := \Xi(\Omega)^j_{(j-1)} / \Xi(\Omega^{(j)})$$

with natural projections

$$\mathbf{p}_j : \Xi(\Omega)^j_{(j-1)} \to \Xi(\Omega)^j_{(j-1)} / \Xi(\Omega^{(j)}).$$

If for some j it happens that $\rho_j > 0$, then $\mathcal{F}^j(\Omega)$ is nontrivial and we call it the *fundamental bundle of order j.*

In view of (4.2) and (4.6), we see that a basis for $\mathcal{F}^j(\Omega)$ is

$$\mathcal{F}^j(\Omega) = \{\omega_{N_j+n_j+1}, \ldots, \omega_{N_j+\nu_j}\} \quad \mathrm{mod}\ \Xi(\Omega^{(j)}) = \mathbf{p}_j(\Xi(\Omega)^j_{(j-1)}).$$

By the Frobenius theorem, there are functions $\{\varphi^{l_j,j}\}_{l_j=1}^{\rho_j}$ and nonsingular matrices M_j such that

$$\begin{pmatrix} d\varphi^{1,j} \\ d\varphi^{2,j} \\ \cdot \\ \cdot \\ d\varphi^{\rho_j,j} \end{pmatrix} \equiv M_j \begin{pmatrix} \omega_{N_j+n_j+1} \\ \omega_{N_j+n_j+2} \\ \cdot \\ \cdot \\ \omega_{N_j+\nu_j} \end{pmatrix} \quad \mathrm{mod}\ \Xi(\Omega)^{(j)} \tag{4.7}$$

and $d\varphi^{1,j}, d\varphi^{2,j}, \ldots, d\varphi^{\rho_j,j}$ determine a basis for $\mathcal{F}^j(\Omega)$.

Definition 4.2. We refer to the functions $\varphi^{l_j,j}$, $j \in \{1, 2, \ldots, k-1\}$, defined by (4.7) as *fundamental functions of order j*. Let $\varphi^{0,k}, \varphi^{1,k}, \ldots, \varphi^{N_{k-1},k}$ span the first integrals of the integrable sub-bundle $\Upsilon_{\widehat{\Sigma}_{k-1}}(\Omega^{(k-1)})$. We refer to these as *fundamental functions of order k*.

We now prove that the construction of fundamental functions of all orders is the only integration that need be carried out in order to construct contact coordinates for any Goursat bundle $\mathcal{V} \subset TM$ or $\Omega \subset T^*M$.

Theorem 4.2. *Let $\mathcal{V} \subset TM$ ($\Omega \subset T^*M$) be a Goursat bundle of derived length k with type $\tau = \mathrm{decel}(\mathcal{V}) = \mathrm{accel}(\Omega) = \langle \rho_1, \rho_2, \ldots, \rho_k \rangle$, $\rho_k \geq 2$. Let $\{x, \varphi^{1,k}, \varphi^{2,k}, \ldots, \varphi^{\rho_k,k}\}$ denote the fundamental functions of order k and for each j in the range $1 \leq j \leq k-1$ for which $\rho_j > 0$, let $\{\varphi^{1,j}, \ldots, \varphi^{\rho_j,j}\}$ denote the fundamental functions of order j defined on some open subset $\mathcal{U} \subseteq M$.*

Then there is an open, dense subset $\widehat{\mathcal{U}} \subseteq \mathcal{U}$ and a section Y of $\mathcal{V}$ (Y of $\Omega^\perp$) such that on $\widehat{\mathcal{U}}$, $Yx \neq 0$ and the fundamental functions $x, \varphi_0^{l_j,j} := \varphi^{l_j,j}$, together with the functions

$$\varphi_{s_j+1}^{l_j,j} = \frac{Y\varphi_{s_j}^{l_j,j}}{Yx}, \qquad j \in \{1, \ldots, k\}, \quad 1 \leq l_j \leq \rho_j, \quad 0 \leq s_j \leq j-1, \tag{4.8}$$

are contact coordinates for $\mathcal{V}$ (Ω) *identifying it with the partial prolongation* $\mathcal{C}(\tau)$ $(\mathcal{C}(\tau)^{\perp})$.

Proof. We will prove this theorem with reference to a sub-bundle $\mathcal{V} \subset TM$. One needs to make only minor adjustments to prove it for $\Omega \subset T^*M$. Fix a point $\bar{y}_0 \in M$ in a neighbourhood $\mathcal{U}$ of which $\mathcal{V}$ is a Goursat bundle. The proof of Theorem 4.1 in [**23**] shows that we may extend the fundamental functions of order $j \in \{1, \ldots, k\}$, for which $\rho_j > 0$, namely, $x = \varphi^{0,k}$, $z^{l_j,j} = \varphi^{l_j,j}$, $1 \le l_j \le \rho_j$, to a system of contact coordinates

$$\bar{z} = (x, z^{l_j,j}, z_1^{l_j,j}, z_2^{l_j,j}, \ldots, z_j^{l_j,j})_{j=1,l_j=1}^{k,\rho_j} \tag{4.9}$$

on an open set $\widehat{\mathcal{U}} \subseteq \mathcal{U}$. Since Char $\mathcal{V}_0^{(1)}$ has codimension 1 in $\mathcal{V}$, it follows that there is a section Z of $\mathcal{V}$ such that $Zx = 1$ on a dense open subset of $\widehat{\mathcal{U}}$, which need not, in fact, contain $\bar{y}_0$, and which, for simplicity, we denote by the same symbol. Let

$$\psi : \bar{y} \mapsto \bar{z}$$

be the local diffeomorphism defined by the change of variable from the original coordinates $\bar{y}$ to the contact coordinates (4.9). Then we have that $\psi_*\mathcal{V} = \mathcal{C}(\tau)$, where

$$\mathcal{C}(\tau) = \left\{ X = \partial_x + \sum_{j\in\{1,\ldots,k\}} \sum_{l_j=1}^{\rho_j} \sum_{h_j=0}^{j-1} z_{h_j+1}^{l_j,j} \partial_{z_{h_j}^{l_j,j}}, \partial_{z_j^{l_j,j}} \right\},$$

and $z_0^{l_j,j} := z^{l_j,j}$. Consequently, we have, for some functions α, $\alpha^{l_j,j}$ on $\widehat{\mathcal{U}}$,

$$\psi_* Z = \alpha X + \sum_{j\in\{1,2,\ldots,k\}} \sum_{l_j=1}^{\rho_j} \alpha^{l_j,j} \partial_{z_j^{l_j,j}}.$$

Since

$$\alpha = (\psi_* Z)x = Z(x \circ \psi) = Z\varphi^{0,k} = 1,$$

we have

$$\psi_* Z = X + \sum_{j\in\{1,2,\ldots,k\}} \sum_{l_j=1}^{\rho_j} \alpha^{l_j,j} \partial_{z_j^{l_j,j}}. \tag{4.10}$$

For each $j \in \{1, 2, \ldots, k\}$ such that $\rho_j > 0$ define functions $\varphi_1^{l_j,j}, \varphi_2^{l_j,j}, \ldots, \varphi_j^{l_j,j}$ by

$$x = \varphi^{0,k}, \qquad \varphi_{s_j+1}^{l_j,j} = Z\varphi_{s_j}^{l_j,j}, \qquad 1 \le l_j \le \rho_j, \quad 0 \le s_j \le j-1, \tag{4.11}$$

where $\varphi_0^{l_j,j} := \varphi^{l_j,j}$. By (4.10) we have

$$\varphi_1^{l_j,j} = Z(\varphi^{l_j,j})(\bar{y}) = Z(\psi^* z^{l_j,j})(\bar{y}) = (\psi_* Z)(z^{l_j,j})(\psi(\bar{y})) = \psi^*(z_1^{l_j,j}). \tag{4.12}$$

The calculation in (4.12) can be repeated so that for each r_j in the range $1 \le r_j \le j$, we have $\varphi_{r_j}^{l_j,j} = \psi^*(z_{r_j}^{l_j,j})$ showing that the functions defined in (4.11) are independent on $\widehat{\mathcal{U}}$. Observe that if Y is any section of $\mathcal{V}$ such that $Yx \neq 0$ on $\widehat{\mathcal{U}}$, then we may take Z to be the vector field $(Yx)^{-1}Y$.

We complete the proof by showing that Char $\mathcal{V}_0^{(1)}$ has the correct local form. First, because Char $\mathcal{V}_{j-1}^{(j)}$ is integrable, of codimension one in $\mathcal{V}^{(j-1)}$ for each $1 \le j \le k-1$ and $Zx \neq 0$, we deduce that it is spanned as

$$\text{Char } \mathcal{V}_{j-1}^{(j)} = \{C_\beta, \text{ad}(Z)C_\beta, \text{ad}^2(Z)C_\beta, \dots, \text{ad}^{j-1}(Z)C_\beta\},$$

where the C_β form a basis for Char $\mathcal{V}_0^{(1)}$. Similarly, it may be deduced that the resolvent bundle is spanned as

$$\mathcal{R}_{\Sigma_{k-1}}(\mathcal{V}^{(k-1)}) = \{C_\beta, \text{ad}(Z)C_\beta, \text{ad}^2(Z)C_\beta, \dots, \text{ad}^{k-1}(Z)C_\beta\}.$$

From their definition each fundamental function of order j is an invariant of Char $\mathcal{V}_{j-1}^{(j)}$ whose local form above allows us to deduce, for all β, l_j, that

$$C_\beta \varphi_s^{l_j,j} = (-1)^i \text{ad}^i(Z) C_\beta \varphi_{s-i}^{l_j,j}, \qquad 0 \le s \le j-1, \quad 0 \le i \le s,$$

for each j such that $\rho_j > 0$. In particular, we deduce that

$$C_\beta \varphi_s^{l_j,j} = \text{ad}^s(Z) C_\beta \varphi^{l_j,j} = 0, \qquad 0 \le s \le j-1.$$

It follows from this that

$$\psi_* \text{ Char } \mathcal{V}_0^{(1)} = \bigcup_{j \in \{1,2,\dots,k\}} \{\partial_{z_j^{l_j,j}}\}_{l_j=1}^{\rho_j},$$

as required. □

The only case remaining is $\rho_k = 1$. Here $(M/\text{Char } \mathcal{V}^{(k-1)}, \mathcal{V}/\text{Char } \mathcal{V}^{(k-1)})$ is a three-dimensional contact manifold so there is no canonical maximal integrable sub-bundle of $\mathcal{V}^{(k-1)}$ as there is in the case $\rho_k \ge 2$. The role of the resolvent bundle when $\rho_k = 1$ is played by a locally defined bundle, Π^k, whose construction we now describe. Let x denote any first integral of Char $\mathcal{V}^{(k-1)}$ and seek any section Z of $\mathcal{V}$ such that $Zx = 1$.[1] Define a sub-bundle $\Pi^k \subset \mathcal{V}^{(k-1)}$ inductively by

$$\Pi^{l+1} = [\Pi^l, Z], \qquad \Pi^1 = \text{Char } \mathcal{V}_0^{(1)}, \qquad 1 \le l \le k-1. \tag{4.13}$$

The proof of Theorem 4.1 shows that Π^k is integrable, has codimension 2 in TM and first integral x. In this case, filtration (4.3) is replaced by

$$\Pi^{k\perp} \subset \Xi(\Omega^{(k-1)}) \subseteq \Xi(\Omega)_{(k-2)}^{k-1} \subset \cdots \subset \Xi(\Omega^{(1)}) \subseteq \Xi(\Omega)_{(0)}^1 \subset T^*M, \tag{4.14}$$

[1] This is an open condition and is easily satisfied since $Xx \neq 0$ for generic sections X of $\mathcal{V}$.

and the filtered basis (4.6) for $\Xi(\Omega)^1_{(0)}$ is replaced by

$$\begin{aligned} &\omega_0, \omega_1, \ldots, \omega_{N_{j-1}} && \text{for } \Xi(\Omega)^j_{(j-1)}, \\ &\omega_0, \omega_1, \ldots, \omega_{N_j+n_j} && \text{for } \Xi(\Omega^{(j)}), \quad 1 \le j \le k-1, \\ &\omega_0, \omega_1, \ldots, \omega_{N_{k-1}} && \text{for } \Pi^{k\perp}. \end{aligned} \tag{4.15}$$

Then by an argument similar to that of Theorem 4.2, we have

Theorem 4.3. *Let $\mathcal{V} \subset TM$ ($\Omega \subset T^*M$) be a Goursat bundle of derived length k and type $\tau = \mathrm{decel}(\mathcal{V}) = \mathrm{accel}(\Omega) = \langle \rho_1, \rho_2, \ldots, \rho_k \rangle$, $\rho_k = 1$. Let Π^k be the bundle locally defined in (4.13). Let $\varphi^{1,k}$ be any other first integral of Π^k such that $dx \wedge d\varphi^{1,k} \neq 0$ on an open set $\mathcal{U} \subseteq M$, where x is any invariant of Char $\mathcal{V}^{(k-1)}$ $(\Xi(\Omega^{(k-1)}))$.*

Then there is an open, dense subset $\widehat{\mathcal{U}} \subseteq \mathcal{U}$ upon which is defined a section Z of $\mathcal{V}$ (of $\Omega^\perp$) satisfying $Zx = 1$ such that the fundamental functions $x, \varphi_0^{l_j,j} := \varphi^{l_j,j}$ together with the functions

$$\varphi^{l_j,j}_{s_j+1} = Z\varphi^{l_j,j}_{s_j}, \qquad j \in \{1, \ldots, k\}, \quad 0 \le s_j \le j-1, \quad 1 \le l_j \le \rho_j,$$

are contact coordinates for $\mathcal{V}$ on $\widehat{\mathcal{U}}$, indentifying it with the partial prolongation $\mathcal{C}(\tau)$ ($\mathcal{C}(\tau)^\perp$).

Theorems 4.2 and 4.3 prove the correctness of algorithm `Contact`.

Algorithm Contact A

`INPUT`: Goursat bundle $\mathcal{V} \subset TM$ (or $\Omega \subset T^*M$) of derived length k and type $\tau = \mathrm{decel}(\mathcal{V}) = \mathrm{accel}(\Omega) = \langle \rho_1, \ldots, \rho_k \rangle$, $\rho_k > 1$.

1. Build filtration (4.3) of T^*M.
2. Build filtered basis (4.6) of bundle $\Xi(\Omega)^1_{(0)}$ constructed in *step* 1.
3. For each j, $1 \le j \le k-1$ such that $\rho_j > 0$, compute a basis for the fundamental bundle $\Xi(\Omega)^j_{(j-1)}/\Xi(\Omega^{(j)})$ of order j.
4. Compute a basis for fundamental bundle of order k: $\Upsilon_{\widehat{\Sigma}_{k-1}}(\Omega^{(k-1)}) = \mathcal{R}_{\widehat{\Sigma}_{k-1}}(\mathcal{V}^{(k-1)})^\perp$
5. For each j, $1 \le j \le k-1$, such that is $\Xi(\Omega)^j_{(j-1)}/\Xi(\Omega^{(j)})$ nontrivial, compute the fundamental functions $\{\varphi^{l_j,j}\}^{\rho_j}_{l_j=1}$ of order j and the fundamental functions of order k from $\Upsilon_{\widehat{\Sigma}_{k-1}}(\Omega^{(k-1)})$
6. Fix any fundamental function of order k, denoted x and any section Z of $\mathcal{V}$ (Z of $\Omega^\perp$) such that $Zx = 1$. [*This and step* 5 *are the only ones requiring integration. The remaining steps require differentiation and linear algebra, alone*.]

7. For each j, such that $\rho_j > 0$ let $z^{l_j,j} = \varphi^{l_j,j}$, $1 \le l_j \le \rho_j$.
8. For each j, such that $\rho_j > 0$ define functions

$$x, z_0^{l_j,j} := z^{l_j,j} = \varphi^{l_j,j}, \quad z_{s_j+1}^{l_j,j} = Zz_{s_j}^{l_j,j}, \quad 0 \le s_j \le j-1,\ 1 \le l_j \le \rho_j.$$

`OUTPUT`: Contact coordinates for $\mathcal{V}$ (Ω) identifying it with $\mathcal{C}(\tau)$ ($\mathcal{C}(\tau)^{\perp}$).

Algorithm Contact B

`INPUT`: Goursat bundle $\mathcal{V} \subset TM$ ($\Omega \subset T^*M$) of derived length k and type $\tau = \mathrm{decel}(\mathcal{V}) = \mathrm{accel}(\Omega) = \langle \rho_1, \dots, \rho_k \rangle$, $\rho_k = 1$.

1. Compute filtration (4.1) of TM up to Char $\mathcal{V}^{(k-1)}$ (filtration (4.14) of T^*M down to $\Xi(\Omega^{(k-1)})$).
2. Fix any first integral of Char $\mathcal{V}^{(k-1)}$ ($\Xi(\Omega^{(k-1)})$), denoted x, and any section Z of $\mathcal{V}$ (Z of $\Omega^{\perp}$) such that $Zx = 1$.
3. Build distribution Π^k, defined by (4.13), giving refinement (4.14) of the filtration constructed in *step* 1.
4. Let $z^k := \varphi^{1,k}$ be any first integral of Π^k such that $dx \wedge d\varphi^{1,k} \neq 0$.
5. Build filtered basis (4.15) of bundle $\Xi(\Omega)^1_{(0)}$.
6. For each j, such that $\rho_j > 0$, compute the fundamental bundle $\Xi(\Omega)^j_{(j-1)}/$ $\Xi(\Omega^{(j)})$ of order j.
7. For each j, such that $\Xi(\Omega)^j_{(j-1)}/\Xi(\Omega^{(j)})$ is non-trivial, compute the fundamental functions $\{\varphi^{l_j,j}\}_{l_j=1}^{\rho_j}$ of order j. [*This and step 2 are the only ones requiring integration. The remaining steps require differentiation and linear algebra, alone.*]
8. For each j, such that $\rho_j > 0$ let $z^{l_j,j} = \varphi^{l_j,j}$, $1 \le l_j \le \rho_j$.
9. For each j, such that $\rho_j > 0$ define functions

$$x, z_0^{l_j,j} := z^{l_j,j} = \varphi^{l_j,j}, \quad z_{s_j+1}^{l_j,j} = Zz_{s_j}^{l_j,j}, \quad 0 \le s_j \le j-1,\ 1 \le l_j \le \rho_j.$$

`OUTPUT`: Contact coordinates for $\mathcal{V}$ (Ω) identifying it with $\mathcal{C}(\tau)$ ($\mathcal{C}(\tau)^{\perp}$).

5. Examples

In this section we illustrate our algorithm for finding contact coordinates for vector field distributions or Pfaffian systems which are locally equivalent to partial prolongations. That is, those systems that determine Goursat bundles.

The data required to construct an equivalence is built out of those canonical geometric structures required to settle the recognition question. In outline recognition the procedure is as follows. For a given bundle $\mathcal{V} \subset TM$ or $\Omega \subset T^*M$ of derived length k, one computes the refined derived type (Definition 3.3) from which the type numbers may be read off. If these numbers agree with those given in Propositions 3.1 or 3.2 then we declare the bundle to have the derived type of a partial prolongation of type $\mathrm{decel}(\mathcal{V}) = \mathrm{accel}(\Omega) = \langle \rho_1, \rho_2, \ldots, \rho_k \rangle$. We then check the integrability of the intersections $\mathrm{Char}\,\mathcal{V}^{(j)}_{j-1}$ or unions $\Xi(\Omega)^{j}_{(j-1)}$, as appropriate, according to Definition 3.4. If they are all integrable and $\rho_k = 1$, then the bundle in question is a Goursat bundle and a general equivalence to a partial prolongation exists by Theorem 4.1. The partial prolongation to which the bundle is equivalent is specified by $\mathrm{decel}(\mathcal{V}) = \mathrm{accel}(\Omega)$. If $\rho_k > 1$, then we must go on to check that $\mathcal{V}^{(k-1)} = (\Omega^{(k-1)})^{\perp}$ determines an integrable Weber structure as described in Section 2.5. If so, then $\mathcal{V}$ or Ω is a Goursat bundle and an equivalence to a partial prolongation exists, as in the previous case.

The examples below are mainly designed to illustrate all the steps in algorithm `Contact`, which is the main result of this paper. Beyond that, we have attempted to provide some indication of the ease and range of possible applications. Indeed, a suit of `Maple` procedures, `<derived>`, built on the differential geometry package `Vessiot`[2] has been written that implements the algorithm. All calculations below were performed using `<derived>`.

Example 5.1. We begin with a pedagogical example that is sufficiently nontrivial to illustrate all the steps in algorithm `Contact A`. Consider the sub-bundle of $T\mathbb{R}^{21}$ defined by

$$\begin{aligned}
\mathcal{V} = \{ & e^{-x_1}\partial_{x_1} + (x_4 - x_8 - e^{x_1})\partial_{x_2} + (x_5 - x_{21} - x_{20} + x_4 - x_8 - e^{x_1})\partial_{x_3} \\
& + (x_9 + 2x_{21} + 2x_{20} - 4x_4 + 4x_8 + 3e^{x_1})(\partial_{x_4} + \partial_{x_8}) \\
& + (2 - 2x_{12} + x_7)\partial_{x_6} \\
& + (x_8 + e^{x_1})\partial_{x_7} - \partial_{x_8} + \partial_{x_9} + (1 - x_{13} - x_{14} - x_{12})\partial_{x_{10}} + \\
& + (\tfrac{1}{2}(x_{13} + x_{14}) + x_4 - x_8 - e^{x_1})\partial_{x_{11}} + \tfrac{1}{2}(x_{13} - x_{14})\partial_{x_{12}} \\
& + (x_{15} - x_{16})\partial_{x_{13}} + (x_{15} + x_{16})\partial_{x_{14}} + x_{17}\partial_{x_{15}} + (x_{18} - 2x_2)\partial_{x_{16}} \\
& + (x_{19} - x_8 - e^{x_1})\partial_{x_{17}} + x_{20}\partial_{x_{18}} \\
& + (3x_{21} + 3x_{20} - 6x_4 + 6x_8 + 5e^{x_1} + x_9)\partial_{x_{19}}, \\
& \partial_{x_4} + 2\partial_{x_{20}}, \partial_{x_5}, \partial_{x_9}, \partial_{x_{20}} - \partial_{x_{21}}, \partial_{x_5} - 2\partial_{x_9} + \partial_{x_{21}} \} \\
= \{ & X_1, X_2, \ldots, X_6 \}.
\end{aligned}$$

The *refined* derived type (Definition 3.3) is

$$[[6, 0], [11, 5, 7], [14, 10, 10], [17, 13, 14], [19, 16, 16], [21, 21]],$$

where for $i = 0$ and $i = 5$ the two-element lists record $[m_i, \chi^i]$ and for $1 \le i \le 4$ the 3-element lists record $[m_i, \chi^i_{i-1}, \chi^i]$. Hence the derived length is $k = 5$. From

[2] I used `Vessiot`, courtesy of its key developer Ian M. Anderson. Available at: `http://www.math.usu.edu/~fg_mp`

the dimensions $m_0 = 6, m_1 = 11, \ldots, m_5 = 21$ we compute the corresponding *type numbers* as described in Proposition 3.1,

$$\begin{array}{llll} \chi^1 = 7, & \chi^2 = 10, & \chi^3 = 14, & \chi^4 = 16, \\ \chi^1_0 = 5, & \chi^2_1 = 10, & \chi^3_2 = 13, & \chi^4_3 = 16. \end{array}$$

Comparing these with the refined derived type of $\mathcal{V}$ we conclude that it has the derived type of a partial prolongation whose type is

$$\mathrm{decel}(\mathcal{V}) = \langle 2, 0, 1, 0, 2 \rangle.$$

Since $\rho_5 := \Delta_5 = 2 > 1$, we check (according to Definition 3.4) the singular variety of $\widehat{\mathcal{V}}^{(4)} = \mathcal{V}^{(4)}/\mathrm{Char}\,\mathcal{V}^{(4)}$. We compute that

$$\widehat{\mathcal{V}}^{(4)} = \{\widehat{X}_1 = \pi(X_1),\ \widehat{X}_2 = \pi(\partial_{x_{12}}),\ \widehat{X}_3 = \pi(\partial_{x_{13}} + \partial_{x_{14}})\}$$

whose nonzero structure is

$$\widehat{\delta}(\widehat{X}_1, \widehat{X}_2) = \langle \pi(2\partial_{x_6} + \partial_{x_{10}}) \rangle, \qquad \widehat{\delta}(\widehat{X}_1, \widehat{X}_3) = \langle \pi(2\partial_{x_{10}} - \partial_{x_{11}}) \rangle.$$

As described in Section 2, $\widehat{\delta}$ is the structure tensor of $\widehat{\mathcal{V}}^{(4)}$,

$$\pi : T\mathbb{R}^{21} \to T\mathbb{R}^{21}/\mathrm{Char}\,\mathcal{V}^{(4)} =: \widehat{T}\mathbb{R}^{21}$$

is the natural projection and for any $X \in T\mathbb{R}^{21}$, $\langle \pi(X) \rangle$ denotes the element of the quotient bundle

$$\widehat{T}\mathbb{R}^{21}/\widehat{\mathcal{V}}^{(4)}$$

with representative $\pi(X)$. From this structure, we easily compute that an arbitrary point $E = [a^1\widehat{X}_1 + a^2\widehat{X}_2 + a^3\widehat{X}_3] \in \mathbb{P}\widehat{\mathcal{V}}^{(4)}$ has polar matrix

$$\begin{pmatrix} -a^2 & a^1 & 0 \\ -a^3 & 0 & a^1 \end{pmatrix}.$$

This matrix has less than generic rank if and only if $a^1 = 0$ and hence the point E belongs to the singular variety if and only if $a^1 = 0$ and a^2, a^3 do not vanish simultaneously. It follows that the set of lines in $\widehat{\mathcal{V}}^4$ with singular degree is $[a^2\widehat{X}_2 + a^3\widehat{X}_3] \approx \mathbb{RP}^1$ (see Sections 2.3 and 2.4). Thus we deduce that

$$\mathrm{Sing}(\widehat{\mathcal{V}}^{(4)}) = \mathbb{P}\{\pi(\partial_{x_{12}}), \pi(\partial_{x_{13}} + \partial_{x_{14}})\}$$

and the resolvent bundle is therefore

$$\mathcal{R}_{\widehat{\Sigma}_4}(\mathcal{V}^{(4)}) = \mathrm{Char}\,\mathcal{V}^{(4)} \oplus \{\partial_{x_{12}}, \partial_{x_{13}} + \partial_{x_{14}}\}.$$

The filtration induced on $T\mathbb{R}^{21}$ by $\mathcal{V}$ is

$$\begin{aligned} \mathrm{Char}\,\mathcal{V}^{(1)}_0 \subset \mathrm{Char}\,\mathcal{V}^{(1)} &\subset \mathrm{Char}\,\mathcal{V}^{(2)} \subset \mathrm{Char}\,\mathcal{V}^{(3)}_2 \subset \mathrm{Char}\,\mathcal{V}^{(3)} \\ &\subset \mathrm{Char}\,\mathcal{V}^{(4)} \subset \mathcal{R}_{\widehat{\Sigma}_4}(\mathcal{V}^{(4)}) \subset T\mathbb{R}^{21}, \end{aligned} \tag{5.1}$$

and one can check that Char $\mathcal{V}_0^{(1)}$, Char $\mathcal{V}_2^{(3)}$ and the resolvent bundle $\mathcal{R}_{\widehat{\Sigma}_4}(\mathcal{V}^{(4)})$ are all integrable. This data confirms that $\mathcal{V}$ is a Goursat bundle of type decel$(\mathcal{V}) = \langle 2, 0, 1, 0, 2\rangle$. By Theorem 4.1, $\mathcal{V}$ is locally equivalent to the partial prolongation with this type. We now go on to use algorithm `Contact` to construct an explicit equivalence.

The filtration of $T^*\mathbb{R}^{21}$ induced by $\mathcal{V}$ and dual to (5.1) is, in this case,

$$\Upsilon_{\widehat{\Sigma}_4}(\Omega^{(4)}) \subset \Xi(\Omega^{(4)}) \subset \Xi(\Omega^{(3)}) \subset \Xi(\Omega)^3_{(2)} \subset \Xi(\Omega^{(2)}) \subset \Xi(\Omega^{(1)}) \subset \Xi(\Omega)^1_{(0)} \subset T^*\mathbb{R}^{21}. \tag{5.2}$$

Pursuing Step 2 of algorithm `Contact A`, `<derived>` computed a filtered basis for $\Xi(\Omega)^1_{(0)}$ to be

$$\begin{aligned}
\Upsilon_{\widehat{\Sigma}_4}(\Omega^{(4)}) &= \{dx_1, dx_2 - dx_{11}, dx_{10}\},\\
\Xi(\Omega^{(4)}) &= \{dx_1, dx_2 - dx_{11}, dx_{10}, dx_{12}, dx_{13} + dx_{14}\},\\
\Xi(\Omega^{(3)}) &= \{dx_1, dx_2 - dx_{11}, dx_{10}, dx_{12}, dx_{13} + dx_{14}, dx_{15}, dx_{14}\},\\
\Xi(\Omega)^3_{(2)} &= \{dx_1, dx_2 - dx_{11}, dx_{10}, dx_{12}, dx_{13} + dx_{14}, dx_{15}, dx_{14}, dx_6\},\\
\Xi(\Omega^{(2)}) &= \{dx_1, dx_2 - dx_{11}, dx_{10}, dx_{12}, dx_{13} + dx_{14},\\
&\qquad dx_{15}, dx_{14}, dx_6, dx_7, dx_{16}, dx_{17}\},\\
\Xi(\Omega^{(1)}) &= \{dx_1, dx_2 - dx_{11}, dx_{10}, dx_{12}, dx_{13} + dx_{14}, dx_{15}, dx_{14}, dx_6, dx_7,\\
&\qquad dx_{16}, dx_{17}, dx_{19}, dx_8, 2dx_{11} - dx_{18}\},\\
\Xi(\Omega)^1_{(0)} &= \{dx_1, dx_2 - dx_{11}, dx_{10}, dx_{12}, dx_{13} + dx_{14}, dx_{15}, dx_{14}, dx_6, dx_7,\\
&\qquad dx_{16}, dx_{17}, dx_{19}, dx_8, 2dx_{11} - dx_{18}, dx_3, dx_{18}\}.
\end{aligned}$$

Since ρ_1, ρ_3 and ρ_5 alone are nonzero, we deduce from this that bases for the fundamental bundles are

$$\begin{aligned}
\Xi(\Omega)^1_{(0)}/\Xi(\Omega^{(1)}) &= \{\mathbf{p}_1(dx_3), \mathbf{p}_1(dx_{18})\},\\
\Xi(\Omega)^3_{(2)}/\Xi(\Omega^{(3)}) &= \{\mathbf{p}_3(dx_6)\},
\end{aligned}$$

$$\Upsilon_{\widehat{\Sigma}_4}(\Omega^{(4)}) = \{dx_1, dx_2 - dx_{11}, dx_{10}\},$$

where the projections $\mathbf{p}_i$ are defined in Definition 4.1. The corresponding fundamental functions may therefore be taken to be (for instance)

$$\begin{aligned}
\mathcal{F}_1(\Omega) &= \{z^{1,1} = x_3, z^{2,1} = x_{18}\},\\
\mathcal{F}_3(\Omega) &= \{z^{1,3} = x_6\},\\
\mathcal{F}_5(\Omega) &= \{z^{1,5} = -x_2 + x_{11}, z^{2,5} = x_{10}\},
\end{aligned}$$

and we take the independent variable x to be the fundamental function e^{x_1} of order 5, for then $X_1 e^{x_1} = 1$, and $Z = X_1$ may be taken to be the operator of total differentiation.

Finally, executing Step 7, we compute the remaining components of the equivalence by differentiating *once* the elements of $\mathcal{F}_1(\Omega)$, differentiating *three/* times the elements of $\mathcal{F}_3(\Omega)$ and, finally, differentiating *five/* times the elements of $\mathcal{F}_5(\Omega)$:

$$z_1^{l_1,1} = X_1 z^{l_1,1}, \qquad z_{r_3+1}^{1,3} = X_1 z_{r_3}^{1,3}, \qquad z_{r_5+1}^{l_5,5} = X_1 z_{r_5}^{l_5,5},$$

where

$$\begin{aligned} 1 &\le l_1 \le 2, \qquad 1 \le l_5 \le 2, \\ 0 &\le r_3 \le 2, \qquad 0 \le r_5 \le 4, \end{aligned}$$

and $z_0^{l_j,j} := z^{l_j,j}$. We thereby obtain the functions

$$\begin{aligned}
x &= e^{x_1}, \quad z^{1,1} = x_3, \quad z^{2,1} = x_{18}, \quad z_1^{1,1} = x_5 - x_{21} - x_{20} + x_4 - x_8 - e^{x_1}, \\
z_1^{2,1} &= x_{20}, \\
z^{1,3} &= x_6, \quad z_1^{1,3} = -2x_{12} + x_7 + 2, \quad z_2^{1,3} = x_8 - x_{13} + x_{14} + e^{x_1}, \\
z_3^{1,3} &= x_9 + 2(x_{20} + x_{21} - 2x_4 + 2x_8 + x_{16}) + 3e^{x_1}, \\
z^{1,5} &= x_{10}, \quad z^{2,5} = -x_2 + x_{11}, \quad z_1^{1,5} = 1 - (x_{12} + x_{13} + x_{14}), \\
z_1^{2,5} &= \tfrac{1}{2}(x_{13} + x_{14}), \\
z_2^{1,5} &= \tfrac{1}{2}(x_{14} - x_{13} - 4x_{15}), \quad z_2^{2,5} = x_{15}, \quad z_3^{1,5} = x_{16} - 2x_{17}, \quad z_3^{2,5} = x_{17}, \\
z_4^{1,5} &= x_{18} - 2(x_2 + x_{19} - x_8 - e^{x_1}), \quad z_4^{2,5} = x_{19} - x_8 - e^{x_1}, \\
z_5^{1,5} &= 2(x_4 - x_8 - e^{x_1} - x_{21}) - x_{20}, \quad z_5^{2,5} = 2(-x_4 + x_8 + e^{x_1}) + x_{20} + x_{21},
\end{aligned}$$

in accordance with `Contact A`. By Theorem 4.2, these functions define a local diffeomorphism $\psi : \mathbb{R}^{21} \to J^{\langle 2,0,1,0,2\rangle}$ satisfying

$$\psi_* \mathcal{V} = \mathcal{C}\langle 2, 0, 1, 0, 2\rangle,$$

where $J^{\langle 2,0,1,0,2\rangle}$ denotes the partial prolongation of $J^1(\mathbb{R}, \mathbb{R}5)$ in which and two variables remain at *order* 1, one variable is prolonged to *order* 3, two are prolonged to *order* 5. Finally, the contact distribution on $J^{\langle 2,0,1,0,2\rangle}$ has the form

$$\begin{aligned}
\mathcal{C}\langle 2, 0, 1, 0, 2\rangle = \Bigg\{ &\partial_x + \sum_{l_1=1}^{2} z_1^{l_1,1} \partial_{z^{l_1,1}} + \sum_{h_3=0}^{2} z_{h_3+1}^{1,3} \partial_{z_{h_3}^{1,3}} \\
&+ \sum_{l_5=1}^{2} \sum_{h_5=0}^{4} z_{h_5+1}^{l_5,5} \partial_{z_{h_5}^{l_5,5}}, \partial_{z_1^{1,1}}, \partial_{z_1^{2,1}}, \partial_{z_3^{1,3}}, \partial_{z_5^{1,5}}, \partial_{z_5^{2,5}} \Bigg\}.
\end{aligned}$$

Example 5.2 (Nonlinear Control Theory). A well-known and much studied example of a nonlinear control system is that of a car moving in the xy-plane modelled (see [**8**], [**25**]) as follows. The *state* of the car is described by four variables (x, y, θ, φ). The ordered pair (x, y) gives the coordinates on the xy-plane of the centre of the rear axle. The variable θ is the angle between the x-axis fixed on the plane and the vertical axis V of the car, running perpendicular to the axles. The variable φ is the angle the front wheels make relative to V.

Assuming the wheels do not slip as the car moves in the plane, then one obtains the Pfaffian system

$$-\sin\theta\, dx + \cos\theta\, dy = 0, \qquad L\cos\varphi\, d\theta - \sin\varphi(\cos\theta\, dx + \sin\theta\, dy) = 0.$$

This in turns leads to the control system $\Omega = \{\omega_1 = 0, \dots, \omega_4 = 0\}$, known as the *kinematic car*, where

$$\begin{aligned} \omega_1 &= dx - u^1 \cos\theta\, dt, \\ \omega_2 &= dy - u^1 \sin\theta\, dt, \\ \omega_3 &= d\theta - \frac{u^1}{L}\tan\varphi\, dt, \\ \omega_4 &= d\varphi - u^2\, dt, \end{aligned}$$

and L is the length from the rear to the front axle. Here u^1 models the speed of the point (x, y) and u^2 the speed at which the front wheels swivel. Variables u^1, u^2 are the *controls*, for prescribing these as functions of time t gives a system of ordinary differential equations for the state of the car.

Equivalently, control system Ω defines the sub-bundle $\mathcal{K} = \Omega^\perp \subset T(\mathbb{R}_t \times M)$, given by

$$\mathcal{K} = \left\{\partial_t + u^1\left(\cos\theta\,\partial_x + \sin\theta\,\partial_y + \frac{1}{L}\tan\varphi\,\partial_\theta\right) + u^2\partial_\varphi, \partial_{u^1}, \partial_{u^2}\right\}, \tag{5.3}$$

where $M = \mathbb{R}^2_{(x,y)} \times \mathbb{R}^2_{(u_1,u_2)} \times S^1 \times S^1$ is the manifold of states and controls. An important question in nonlinear control theory is: when can a nonlinear control system be "linearised" by a *static feedback transformation* and more generally a *dynamic feedback transformation*? We will derive the well-known results (see, e.g., Isidori [**8**]) for the kinematic car as an illustration of our general theory and, in particular, algorithm `Contact`.

We begin by showing that $\mathcal{K}$ is, in fact, a Goursat bundle of type $\tau = \langle 1, 0, 1\rangle$. The refined derived type of $\mathcal{K}$ is

$$[[3, 0], [5, 2, 3], [6, 4, 4], [7, 7]].$$

Consequently, the derived length is 3. For dimensions $m_0 = 3, m_1 = 5, m_3 = 6, m_4 = 7$ associated to $\mathcal{K}$, we compare with the type numbers (Proposition 3.1) of a partial prolongation to be

$$\begin{aligned} \chi^1 &= 3, \qquad \chi^2 = 4, \\ \chi_0^1 &= 2, \qquad \chi_1^2 = 4. \end{aligned}$$

Since these type numbers agree with the refined derived type of $\mathcal{K}$, we conclude by Proposition 3.1 that $\mathcal{K}$ has the derived type of a partial prolongation. The filtration induced by $\mathcal{K}$ is

$$\text{Char } \mathcal{K}_0^{(1)} \subset \text{Char } \mathcal{K}^{(1)} \subset \text{Char } \mathcal{K}^{(2)} \subset T\mathbb{R}^7$$

where

$$\begin{aligned} \text{Char } \mathcal{K}_0^{(1)} &= \{\partial_{u^1}, \partial_{u^2}\}, \\ \text{Char } \mathcal{K}^{(1)} &= \{\partial_{u^1}, \partial_{u^2}, \partial_t\}, \\ \text{Char } \mathcal{K}^{(2)} &= \{\partial_{u^1}, \partial_{u^2}, \partial_t, \partial_\varphi\}. \end{aligned} \tag{5.4}$$

Since the distributions in (5.4) are all integrable and $\rho_3 = 1$, we conclude by Definition 3.4 that $\mathcal{K}$ is a Goursat bundle of type

$$\text{decel}(\mathcal{K}) = \langle 1, 0, 1 \rangle.$$

By Theorem 4.1, $\mathcal{K}$ is locally equivalent to the contact distribution $\mathcal{C}\langle 1, 0, 1\rangle$. This settles the recognition problem for $\mathcal{K}$. We go on to find an equivalence using `Contact`.

For this, according to `Contact B`, we compute distribution Π^3, as described in (4.13). The invariants of Char $\mathcal{K}^{(2)}$ are x, y, θ. Any one of these may be taken to be the independent variable. If we choose x for this purpose then we take

$$X = \partial_x + \frac{1}{u^1 \cos\theta}(\partial_t + u^1 \sin\theta \partial_y + L^{-1} u^1 \tan\varphi \partial_\theta + u^2 \partial_\varphi)$$

for the operator of total differentiation. By (4.13), we find that

$$\Pi^3 = \{\partial_{u_1}, \partial_{u_2}, \partial_t, \partial_\varphi, \partial_\theta\}.$$

The filtration of $T^*\mathbb{R}^7$ induced by $\mathcal{K}$ is therefore

$$\Pi^{3^\perp} \subset \Xi(\Omega^{(2)}) \subset \Xi(\Omega^{(1)}) \subset \Xi(\Omega)_0^{(1)} \subset T^*\mathbb{R}^7$$

where

$$\begin{aligned} \Pi^{3^\perp} &= \{dx, dy\}, \quad \Xi(\Omega^{(2)}) = \{dx, dy, d\theta\}, \\ \Xi(\Omega^{(1)}) &= \{dx, dy, d\theta, d\varphi\}, \quad \Xi(\Omega)_{(0)}^1 = \{dx, dy, d\theta, d\varphi, dt\}, \end{aligned}$$

and $\Omega = \mathcal{K}^\perp$. It follows that the fundamental bundles are

$$\Xi(\Omega)_{(0)}^1 \big/ \Xi(\Omega^{(1)}) = \{\mathbf{p}_1(dt)\}, \qquad \Pi^{3^\perp} = \{dy\},$$

and the fundamental functions are therefore

$$\mathcal{F}_1(\Omega) = \{t\}, \qquad \mathcal{F}_3(\Omega) = \{y\}.$$

Consequently, by `Contact B` the map $\psi : \mathbb{R}^7 \to J^{\langle 1,0,1\rangle}$ defined by

$$\begin{aligned} x, z^{1,1} &= t, \quad z_1^{1,1} = Xt, \quad z^{1,3} = y, \\ z_1^{1,3} &= Xy, \quad z_2^{1,3} = X^2 y, \quad z_3^{1,3} = X^3 y, \end{aligned} \tag{5.5}$$

is an equivalence, according to Theorem 4.3. That is, $\psi_* \mathcal{K} = \mathcal{C}\langle 1, 0, 1\rangle$.

While (5.5) is certainly an identification of $\mathcal{K}$ with a partial prolongation, it is not of much use in control theory. This is because it is not a feedback equivalence. That is, (5.5) does not respect the special role played by the time coordinate t in the original control system, nor the distinction between the roles played by the state and control variables. We pause briefly to describe the class of transformations that preserves the set of all control systems.

Let

$$\frac{d\mathbf{x}}{dt} = f(t, \mathbf{x}, \mathbf{u}) \tag{5.6}$$

be a control system, where $\mathbf{x}$ denotes the state variables and $\mathbf{u}$ the control variables. A local diffeomorphism of the form

$$t \mapsto t, \quad \mathbf{x} \mapsto \Phi(\mathbf{x}), \quad \mathbf{u} \mapsto \Psi(\mathbf{x}, \mathbf{u}), \tag{5.7}$$

is said to be a (*static*) *feedback transformation*. A question of interest is: Does there exist a static feedback transformation that identifies a given control system with some partial prolongation of $\mathcal{C}_q^{(1)}$ for some q or, as it is more commonly known in the control theory literature, a Brunovský normal form?

Writing (5.5) out explicitly shows that it is *not* a static feedback equivalence. In fact, being ‘driftless’, it is standard in geometric control theory that no static feedback equivalence exists for the kinematic car [**8**]. For instance, an elegant formulation in Sluis [20, Theorem 33] gives a proof of this based on the GS algorithm and, consequently, applies to autonomous control systems such as the kinematic car. In fact, a stronger necessary condition for feedback equivalence can be derived via Theorems 4.2 and 4.3 that is valid in both the autonomous and nonautonomous cases. That is, in case the diffeomorphisms that identify control systems are more general than static feedback transformations (5.7). In this more general setting we allow Φ and Ψ to depend upon time t, as well as states and controls. We shall call these more general transformations *control morphisms*.

Definition 5.1. A local diffeomorphism of the form

$$t \mapsto t, \quad \mathbf{x} \mapsto \Phi(t, \mathbf{x}), \quad \mathbf{u} \mapsto \Psi(t, \mathbf{x}, \mathbf{u}),$$

identifying a pair of control systems of the form (5.6), where $\mathbf{x}$ represent states and $\mathbf{u}$ represent controls, is said to be a *control morphism*.

Control morphisms generalise static feedback equivalences to the nonautonomous case. Clearly if a contol system cannot be identified with a Brunovský normal form by a control morphism, then it is not static feedback equivalent to one either.

Theorem 5.1. *Let* $\mathcal{K} = \{\partial_t + f(t, \mathbf{x}, \mathbf{u})\partial_{\mathbf{x}}, \partial_{\mathbf{u}}\} \subset T\,(\mathbb{R}_t \times M)$ *arise from any smooth control system* (5.6), *where* M *is the manifold of states and controls. A necessary condition in order that there be a control morphism that identifies* $\mathcal{K}$ *with a Brunovský normal form is that*:

(i) $\mathcal{K}$ *is a Goursat bundle*; *and*
(ii) *If* k *is the derived length of* $\mathcal{K}$, *then* $dt \in \operatorname{Char} \mathcal{K}^{(k-1)\perp}$ *if* $\rho_k = 1$ *or* $dt \in \Upsilon_{\widehat{\Sigma}_{k-1}}(\mathcal{K}^{(k-1)})$ *if* $\rho_k > 1$.

Proof. Suppose there is a control morphism ϑ identifying $\mathcal{K}$ with some Brunovský normal form. Every such normal form is a partial prolongation $\mathcal{C}\langle\tau\rangle$ of $\mathcal{C}_q^{(1)}$, where $\tau = \langle\rho_1, \rho_2, \ldots, \rho_k\rangle$, $\rho_k \geq 1$. In the case $\rho_k > 1$, the independent variable x of $\mathcal{C}\langle\tau\rangle$ is an invariant of its resolvent bundle. Consequently, $\vartheta^* x = t$ is an invariant of the resolvent bundle $\mathcal{R}_{\Sigma_{k-1}}(\mathcal{K}^{(k-1)})$ determined by $\mathcal{K}$. In the case $\rho_k = 1$, x is an invariant of the Cauchy bundle $\operatorname{Char} \mathcal{C}\langle\tau\rangle^{(k-1)}$ of the $(k-1)$th derived bundle $\mathcal{C}\langle\tau\rangle^{(k-1)}$. Hence, $\vartheta^* x = t$ is an invariant of $\operatorname{Char} \mathcal{K}^{(k-1)}$. □

While the kinematic car $\mathcal{K}$ is certainly a Goursat bundle and therefore locally equivalent to a partial prolongation, it does not satisfy condition (ii) of Theorem 5.1. We deduce that $\mathcal{K}$ cannot be identified with a Brunovský normal form by a control morphism and hence a static feedback transformation, in agreement with the standard result.

However, as is well known for this example, a certain *Cartan prolongation* of $\mathcal{K}$ *is* static feedback linearisable. A precise definition is given in Example 5.3. For the present, we merely exhibit a Cartan prolongation of $\mathcal{K}$ and show how to apply algorithm `Contact` to determine a static feedback linearisation of the Cartan prolonged distribution.

We obtain a Cartan prolongation of the kinematic car system as follows. Define a new control system

$$\frac{dx}{dt} = u^1\cos\theta, \quad \frac{dy}{dt} = u^1\sin\theta, \quad \frac{d\theta}{dt} = \frac{u^1}{L}\tan\varphi,$$
$$\frac{d\varphi}{dt} = u^2, \quad \frac{du^1}{dt} = w^2, \quad \frac{dw^2}{dt} = v_1, \tag{5.8}$$

by 'twice differentiating u^1'. In control system (5.8) the coordinate u^1 has become a state variable and the new control variables are v^1 and u^2. To see the significance for control theory of this admittedly ad hoc construction one needs to check for the existence of a static feedback equivalence for system (5.8). We begin by changing notation slightly and examining the sub-bundle $\operatorname{pr} \mathcal{K} \subset T\,(\mathbb{R}_t \times \bar{M})$, defined by (5.8), where $\bar{M}$ is the manifold of new states and controls. Setting $u^1 = w^1$ and

$u^2 = v^2$, we have

$$\mathrm{pr}\,\mathcal{K} \;=\; \left\{T = \partial_t + w^1\left(\cos\theta\,\partial_x + \sin\theta\,\partial_y + \frac{\tan\varphi}{L}\partial_\theta\right)\right.$$
$$\left. + v^2\partial_\varphi + w^2\partial_{w^1} + v^1\partial_{w^2}, \partial_{v^1}, \partial_{v^2}\right\}.$$

The refined derived type is

$$[[3, 0], [5, 2, 2], [7, 4, 4], [9, 9]].$$

Proposition 3.1 shows that this is the derived type of the partial prolongation $\mathcal{C}\langle 0, 0, 2\rangle$, that is, the total prolongation $\mathcal{C}_2^{(3)}$. A calculation reveals that $\mathrm{pr}\,\widehat{\mathcal{K}}^{(2)} := \mathrm{pr}\,\mathcal{K}^{(2)}/\mathrm{Char}\,\mathrm{pr}\,\mathcal{K}^{(2)}$ is spanned by

$$\mathrm{pr}\,\widehat{\mathcal{K}}^{(2)} = \pi\{\Gamma = \partial_t + w^1(\cos\theta\,\partial_x + \sin\theta\,\partial_y), \partial_{w^1}, \partial_\theta\},$$

and that the polar matrix of a line $[a^1\pi(\Gamma) + a^2\pi(\partial_{w^1}) + a^3\pi(\partial_\theta)]$ is

$$\begin{pmatrix} -a^2 & a^1 & 0 \\ -a^3 & 0 & a^1 \end{pmatrix}$$

with respect to the nonzero structure

$$\begin{aligned} \widehat{\delta}(\pi(\partial_{w^1}), \pi(\Gamma)) &= \langle \pi(\cos\theta\,\partial_x + \sin\theta\,\partial_y)\rangle, \\ \widehat{\delta}(\pi(\Gamma), \pi(\partial_\theta)) &= \langle \pi(w^1\sin\theta\,\partial_x - w^1\cos\theta\,\partial_y)\rangle. \end{aligned}$$

Here, again, $\widehat{\delta}$ is the structure tensor of $\mathrm{pr}\,\widehat{\mathcal{K}}^{(2)}$,

$$\pi : T(\mathbb{R}_t \times \bar{M}) \to T(\mathbb{R}_t \times \bar{M})/\mathrm{Char}\,\mathrm{pr}\,\mathcal{K}^{(2)} =: \widehat{T}(\mathbb{R}_t \times \bar{M})$$

is the natural projection and for any $X \in T(\mathbb{R}_t \times \bar{M})$, $\langle\pi(X)\rangle$ denotes the element of the quotient bundle

$$\widehat{T}(\mathbb{R}_t \times \bar{M})/\mathrm{pr}\,\widehat{\mathcal{K}}^{(2)}$$

with representative $\pi(X)$. It follows that $\mathrm{Sing}(\mathrm{pr}\,\widehat{\mathcal{K}}^{(2)}) = \mathbb{P}\{\pi(\partial_{w^1}), \pi(\partial_\theta)\}$ and so, the resolvent bundle of $\mathrm{pr}\,\mathcal{K}^{(2)}$ is

$$\mathcal{R}_{\widehat{\Sigma}_2}(\mathrm{pr}\,\mathcal{K}^{(2)}) = \mathrm{Char}\,\mathcal{K}^{(2)} \oplus \{\partial_{w^1}, \partial_\theta\},$$

which is integrable. The above data show that $\mathrm{pr}\,\mathcal{K}$ is a Goursat bundle of type $\langle 0, 0, 2\rangle$. By Theorem 4.1 there is an equivalence identifying $\mathrm{pr}\,\mathcal{K}$ with the contact distribution $\mathcal{C}\langle 0, 0, 2\rangle$, that is, a Brunovský normal form. But is it a static feedback equivalence? To find out, we use algorithm `Contact`.

The only nonempty fundamental bundle in this case is the one of order 3,

$$\Upsilon_{\widehat{\Sigma}_2}(\mathrm{pr}\,\mathcal{K}^{(2)}) = \mathcal{R}_{\widehat{\Sigma}_2}(\mathrm{pr}\,\mathcal{K}^{(2)})^{\perp} = \{dt, dx, dy\}.$$

So in this case, $dt \in \Upsilon_{\widehat{\Sigma}_2}(\mathrm{pr}\,\mathcal{K}^{(2)})$ and we may choose $z^{1,3} = x$, $z^{2,3} = y$ and

$$z_1^{1,3} = Tz^{1,3}, \quad z_1^{2,3} = Tz^{2,3}, \quad z_2^{1,3} = Tz_1^{1,3},$$
$$z_2^{2,3} = Tz_1^{2,3}, \quad z_3^{1,3} = Tz_2^{1,3}, \quad z_3^{2,3} = Tz_2^{2,3}.$$

It is a simple matter to verify that these are indeed contact coordinates as predicted by Theorem 4.2. Moreover, an inspection of the formulas

$$z_1^{1,3} = w^1 \cos\theta, \quad z_1^{2,3} = w^1 \sin\theta,$$
$$z_2^{1,3} = w^2 \cos\theta - \frac{(w^1)^2}{L} \sin\theta \tan\varphi, \quad z_2^{2,3} = w^2 \sin\theta + \frac{(w^1)^2}{L} \cos\theta \tan\varphi,$$
$$z_3^{1,3} = \frac{\begin{array}{c}-(w^1)^3 \cos\theta \sin 2\varphi - 3w^1 w^2 L \sin\varphi \cos\varphi \sin\theta \\ -(w^1)^2 v^2 L \sin\theta + v^1 L^2 \cos\theta \cos 2\varphi\end{array}}{L^2 \cos 2\varphi},$$
$$z_3^{2,3} = \frac{\begin{array}{c}-(w^1)^3 \sin\theta \sin 2\varphi + 3w^1 w^2 L \sin\varphi \cos\varphi \cos\theta \\ +(w^1)^2 v^2 L \cos\theta + v^1 L^2 \sin\theta \cos 2\varphi\end{array}}{L^2 \cos 2\varphi},$$

shows that they define a static feedback equivalence. Because of the projection $\bar{M} \to M$, any integral manifold of pr $\mathcal{K}$ maps to a unique integral manifold of $\mathcal{K}$. This, together with the fact that there is a static feedback equivalence for pr $\mathcal{K}$ implies that the kinematic car example is *dynamic* feedback linearisable in the language of nonlinear control theory. It is an important and largely open problem to geometrically characterise the class of nonlinear control systems which are dynamic feedback linearisable.

Example 5.3 (Differential Invariants of Curves). A classical problem in differential geometry is to find the differential invariants of submanifolds under the action of a Lie group. This problem has attracted fresh interest in recent years as a result of a new approach [**5**] due to Fels and Olver. Assuming one knows the group action explicitly, the authors give a simple procedure for the construction of the differential invariants of the action that requires no integrations to be performed.[3] In this example, we suggest an alternative procedure, at least in the case where the submanifolds are curves, effectively requiring only an explicit parametrisation of the matrix Lie group. In particular, no integration is required. The method uses the algorithm worked out in this paper and is based on an observation of Shadwick and Sluis [**19**]. These authors noticed that curves in the various Klein geometries can be endowed with a contact structure in a fairly natural way. One begins with a Pfaffian system whose integral submanifolds are lifts of curves in the homogeneous space of a Lie group. Generally, this Pfaffian system is not a contact structure. However,

[3] For very nice applications of the Fels–Olver procedure, see Mari Beffa [**14**] and Mansfield [**13**].

Shadwick and Sluis observed that if it is subjected to a Cartan prolongation, then the resulting Cartan prolonged Pfaffian system on the enlarged space is a contact structure.

As an illustration of algorithm `Contact` we shall, in this example, enlarge upon this idea of Shadwick and Sluis by studying one special case in detail, namely that of curves in $\mathbb{E}^3$. Our aim is to show how the well-known Euclidean invariants of curves, the curvature and torsion, arise from the generalised Goursat normal form and algorithm `Contact`.

We have the diffeomorphism $\mathbb{E}^3 \to E(3)/SO(3)$ given by

$$\mathbf{x} \mapsto \begin{pmatrix} 1 & 0 \\ \mathbf{x} & I_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix},$$

where $A \in SO(3)$, the three-dimensional special orthogonal group. Denoting the left-invariant Maurer–Cartan form on $E(3)$, the (oriented) Euclidean group in dimension 3, by

$$\Omega_{MC} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \theta_1 & 0 & \omega_{12} & \omega_{13} \\ \theta_2 & -\omega_{12} & 0 & \omega_{23} \\ \theta_3 & -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}, \tag{5.9}$$

it is easily shown that the standard Serret–Frenet lift of curves in $\mathbb{E}^3$ are integral submanifolds of the Pfaffian system

$$\mathcal{I} = \{\theta_2 = 0, \theta_3 = 0, \omega_{13} = 0, \ [\theta_1]\}, \tag{5.10}$$

where θ_1 is the independence form. Equivalently, they are integral curves of the sub-bundle

$$\mathcal{V} = \{\partial_{\theta_1}, \partial_{\omega_{12}}, \partial_{\omega_{23}}\} \subset TE(3). \tag{5.11}$$

The structure equations of the Lie algebra $\mathbf{e}(3)$ of left-invariant vector fields on $E(3)$ are

	∂_{θ_1}	∂_{θ_2}	∂_{θ_3}	$\partial_{\omega_{12}}$	$\partial_{\omega_{13}}$	$\partial_{\omega_{23}}$
∂_{θ_1}	0	0	0	$-\partial_{\theta_2}$	$-\partial_{\theta_3}$	0
∂_{θ_2}	0	0	0	∂_{θ_1}	0	$-\partial_{\theta_3}$
∂_{θ_3}	0	0	0	0	∂_{θ_1}	∂_{θ_2}
$\partial_{\omega_{12}}$	∂_{θ_2}	$-\partial_{\theta_1}$	0	0	$\partial_{\omega_{23}}$	$\partial_{\omega_{13}}$
$\partial_{\omega_{13}}$	∂_{θ_3}	0	$-\partial_{\theta_1}$	$-\partial_{\omega_{23}}$	0	$\partial_{\omega_{12}}$
$\partial_{\omega_{13}}$	0	∂_{θ_3}	$-\partial_{\theta_2}$	$-\partial_{\omega_{13}}$	$-\partial_{\omega_{12}}$	0

From these structure equations it is easy to deduce that

$$\begin{aligned} \mathcal{V}^{(1)} &= \{\partial_{\theta_1}, \partial_{\omega_{12}}, \partial_{\omega_{23}}, \partial_{\theta_2}, \partial_{\omega_{13}}\}, \\ \mathcal{V}^{(2)} &= \mathcal{V}^{(1)} \oplus \{\partial_{\theta_3}\} = TE(3), \end{aligned}$$

$$\text{Char}\ \mathcal{V} = \{0\}, \quad \text{Char}\ \mathcal{V}^{(1)} = \text{Char}\ \mathcal{V}_0^{(1)} = \{\partial_{\omega_{12}}\},$$

leading to the refined derived type

$$[[3, 0], [5, 1, 1], [6, 6]]$$

and hence, in particular, $m_0 = 3, m_1 = 5, m_2 = 6$. By Proposition 3.1, in order for $\mathcal{V}$ to have the derived type of a partial prolongation, it is necessary that $\chi^0 = 0$, $\chi_0^1 = 2$ and $\chi^1 = 3$; instead of which we have $\chi^0 = 0$, $\chi_0^1 = 1$ and $\chi^1 = 1$. Hence there can be no local diffeomorphism between $\mathcal{V}$ and a partial prolongation of $\mathcal{C}_q^{(1)}$.

However, as observed by Shadwick and Sluis, one can make progress by carrying out a Cartan prolongation as we did in Example 5.2. Here we need to discuss this a little more precisely than before.

Definition 5.2. Let $\mathcal{I}$ be a Pfaffian system on manifold M and let $\eta : B \to M$ be a fibred manifold. A Pfaffian system $\mathcal{J}$ on B is said to be a *Cartan prolongation* of $\mathcal{I}$ if:

(i) $\eta^*\mathcal{I} \subset \mathcal{J}$; and
(ii) For every integral manifold $\sigma : U \subseteq \mathbb{R}^m \to M$ of $\mathcal{I}$, there is a unique integral manifold $\tilde{\sigma} : U \to B$ of $\mathcal{J}$ such that $\eta \circ \tilde{\sigma} = \sigma$. We say that $\tilde{\sigma}$ is the *Cartan lift* of σ.

If $\gamma : \mathbb{R} \to E(3)/SO(3) \approx \mathbb{E}^3$ is an immersion into Euclidean 3-space, then we define the Serret-Frenet lift of γ as an integral submanifold $\Gamma : \mathbb{R} \to E(3)$ of Pfaffian system $\mathcal{I}$. We then define a Cartan prolongation $(E(3) \times \mathbb{R}^3_{\kappa,\kappa_1,\tau}, \mathcal{J})$ of $(E(3), \mathcal{I})$ by

$$\mathcal{J} = \{\theta_2, \theta_3, \omega_{13}, \omega_{12} - \kappa\theta_1, d\kappa - \kappa_1\theta_1, \omega_{23} - \tau\theta_1, [\theta_1]\}. \tag{5.12}$$

Equivalently, the Cartan lifts of γ are the integral submanifolds of

$$\text{pr}\ \mathcal{V} := \{\partial_{\theta_1} + \kappa\partial_{\omega_{12}} + \kappa_1\partial_\kappa + \tau\partial_{\omega_{23}}, \partial_{\kappa_1}, \partial_\tau\}. \tag{5.13}$$

We therefore have the commutative diagram

$$\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\tilde{\Gamma}} & (E(3) \times \mathbb{R}^3_{\kappa,\kappa_1,\tau}, \mathcal{J}) \\
\uparrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \tilde{\eta}} \\
\mathbb{R} & \xrightarrow{\Gamma} & (E(3), \mathcal{I}) \\
\uparrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \eta} \\
\mathbb{R} & \xrightarrow{\gamma} & E(3)/SO(3)
\end{array}$$

Proposition 5.2. *Let $\gamma_i : I_t \subseteq \mathbb{R} \to \mathbb{E}^3$ be a pair of immersions with Cartan lifts*

$$\tilde{\Gamma}_i : I \to E(3) \times \mathbb{R}^3_{\kappa,\kappa_1,\tau}, \qquad i = 1, 2,$$

such that $\tilde{\Gamma}_1^*\theta_1 = \tilde{\Gamma}_2^*\theta_1$. Then the curves γ_i are congruent if and only if

$$\tilde{\Gamma}_1^*\kappa = \tilde{\Gamma}_2^*\kappa, \qquad \tilde{\Gamma}_1^*\tau = \tilde{\Gamma}_2^*\tau. \tag{5.14}$$

Proof. By definition, Cartan lifts $\tilde{\Gamma}_i$ of the γ_i are integral submanifolds of the Cartan prolongations $\mathcal{J}$ on $E(3) \times \mathbb{R}^3_{\kappa,\kappa_1,\tau}$ and, consequently,

$$\begin{aligned}\tilde{\Gamma}_i^*\theta_2 &= \tilde{\Gamma}_i^*\theta_3 = \tilde{\Gamma}_i^*\omega_{13} = 0, \quad \tilde{\Gamma}_i^*\omega_{12} = (\tilde{\Gamma}_i^*\kappa)(\tilde{\Gamma}_i^*\theta_1),\\ \tilde{\Gamma}_i^*\omega_{23} &= (\tilde{\Gamma}_i^*\tau)(\tilde{\Gamma}_i^*\theta_1), \quad i = 1, 2.\end{aligned}$$

Since $\tilde{\Gamma}_1^*\theta_1 = \tilde{\Gamma}_2^*\theta_1$, then (5.14) implies that $\tilde{\Gamma}_1^*\Omega_{MC} = \tilde{\Gamma}_2^*\Omega_{MC}$, where Ω_{MC} is the Maurer–Cartan form (5.9). Hence, the $\Gamma_i = \tilde{\eta} \circ \tilde{\Gamma}_i$ satisfy

$$\Gamma_1^*\Omega_{MC} = \Gamma_2^*\Omega_{MC}. \tag{5.15}$$

It follows from the standard theorem about maps into a Lie group that the $\gamma_i = \eta \circ \tilde{\eta} \circ \tilde{\Gamma}_i$, $i = 1, 2$, can be identified by a fixed Euclidean isometry $g \in E(3)$. Conversely, if $\gamma_2 = g \cdot \gamma_1$ for some $g \in E(3)$, then their Serret–Frenet lifts Γ_i of γ_i satisfy (5.15) and are integral submanifolds of $\mathcal{I}$. But since $\mathcal{J}$ is a Cartan prolongation of $\mathcal{I}$, there are Cartan lifts $\tilde{\Gamma}_i$ of the Γ_i which are integral submanifolds $\mathcal{J}$. Equations (5.14) follow from this fact and from equation (5.15). □

The usefulness of this result derives from the following fact.

Proposition 5.3. *The Cartan prolongation $\mathcal{J}$ of $\mathcal{I}$ is locally equivalent to the contact distribution $\mathcal{C}\langle 0, 0, 2\rangle$. Furthermore, the invariants of the resolvent bundle form a local coordinate system on $E(3)/SO(3)$.*

Proof. A calculation using the structure equations of $\mathbf{e}(3)$ shows that the (refined) derived type of $\mathcal{J}^\perp = \text{pr}\,\mathcal{V}$ is

$$[[3, 0], [5, 2, 2], [7, 4, 4], [9, 9]]$$

and hence its type is $\tau = \langle 0, 0, 2\rangle$;[4] the derived length is $k = 3$ and $\Delta_3 = 2$. Indeed, we find that

$$\begin{aligned}\text{pr}\,\mathcal{V}^{(1)} &= \{\partial_{\theta_1} + \kappa\partial_{\omega_{12}}, \partial_\kappa, \partial_{\omega_{23}}, \partial_{\kappa_1}, \partial_\tau\},\\ \text{pr}\,\mathcal{V}^{(2)} &= \{\partial_{\theta_1}, \partial_{\omega_{12}}, \partial_{\omega_{13}}, \partial_\kappa, \partial_{\omega_{23}}, \partial_{\kappa_1}, \partial_\tau\},\end{aligned}$$

[4] We apologise for this notational clash. Symbol τ has been used throughout the paper to denote the type of a partial prolongation but in this example it is also used to denote, as usual, the torsion of a curve in $\mathbb{E}^3$. However, the two uses of τ should be clear from the context.

$$\begin{aligned}\text{Char pr } \mathcal{V}^{(1)} &= \{\partial_{\kappa_1}, \partial_\tau\},\\ \text{Char pr } \mathcal{V}^{(2)} &= \{\partial_{\kappa_1}, \partial_\tau, \partial_\kappa, \partial_{\omega_{23}}\}.\end{aligned}$$

Hence, we have the quotient

$$\text{pr } \mathcal{V}^{(2)}/\text{Char pr } \mathcal{V}^{(2)} = \{\hat{e}_1 = \pi(\partial_{\theta_1}), \hat{e}_2 = \pi(\partial_{\omega_{12}}), \hat{e}_3 = \pi(\partial_{\omega_{13}})\},$$

with structure

$$\widehat{\delta}(\hat{e}_2, \hat{e}_1) = \langle \pi(\partial_{\theta_2})\rangle, \quad \widehat{\delta}(\hat{e}_3, \hat{e}_1) = \langle \widehat{\pi}(\partial_{\theta_3})\rangle, \quad \widehat{\delta}(\hat{e}_2, \hat{e}_3) = 0,$$

where $\widehat{\delta}$ denotes the structure tensor of pr $\widehat{\mathcal{V}}^2 :=$ pr $\mathcal{V}^{(2)}/$Char pr $\mathcal{V}^{(2)}$. Computing the polar matrix of an arbitrary line $[a^1\hat{e}_1 + a^2\hat{e}_2 + a^3\hat{e}_3]$ in pr $\widehat{\mathcal{V}}^2$ we easily deduce that pr $\mathcal{V}^{(2)}$ determines a Weber structure with singular bundle

$$\mathcal{B} = \mathbb{P}\{\hat{e}_2, \hat{e}_3\} \subset \mathbb{P}(\text{pr } \widehat{\mathcal{V}}^{(2)}).$$

The resolvent bundle associated to this Weber structure is therefore

$$\mathcal{R}_{\Sigma_2}(\text{pr } \mathcal{V}^{(2)}) = \mathbf{so}(3) \oplus \{\partial_\kappa, \partial_{\kappa_1}, \partial_\tau\}. \tag{5.16}$$

Since (5.16) is integrable, we've shown that pr $\mathcal{V}$ is a Goursat bundle of type $\langle 0, 0, 2\rangle$. By the generalised Goursat normal form, we can conclude that pr $\mathcal{V}$ is locally equivalent to $\mathcal{C}\langle 0, 0, 2\rangle$. That is, there is a local diffeomorphism

$$\psi : E(3) \times \mathbb{R}^3_{(\kappa,\kappa_1,\tau)} \to J^3(\mathbb{R}, \mathbb{R}^2) \tag{5.17}$$

such that $\psi_*(\text{pr } \mathcal{V}) = \mathcal{C}\langle 0, 0, 2\rangle$, where

$$\mathcal{C}\langle 0, 0, 2\rangle = \{\partial_x + u_1\partial_u + v_1\partial_v + u_2\partial_{u_1} + v_2\partial_{v_1} + u_3\partial_{u_2} + v_3\partial_{v_2}, \partial_{u_3}, \partial_{v_3}\}$$

is the contact distribution on jet bundle $J^3(\mathbb{R}, \mathbb{R}^2)$. □

We can now pass to local coordinates and use `Contact` to construct the differential invariants explicitly. Parametrise the elements of $SO(3)$ by

$$A = \begin{pmatrix} \sqrt{1-b^2}\sqrt{1-a^2} & a\sqrt{1-b^2} & -b \\ -a\sqrt{1-c^2} - bc\sqrt{1-a^2} & \sqrt{1-a^2}\sqrt{1-c^2} - abc & -c\sqrt{1-b^2} \\ b\sqrt{1-a^2}\sqrt{1-c^2} - ac & ab\sqrt{1-c^2} + c\sqrt{1-a^2} & \sqrt{1-b^2}\sqrt{1-c^2} \end{pmatrix}.$$

In these coordinates, the Lie algebra $\mathbf{e}(3)$ is easily computed to have local basis

$$\begin{aligned}\partial_{\theta_1} &= \sqrt{1-a^2}\sqrt{1-b^2}\partial_x - (a\sqrt{1-c^2} + bc\sqrt{1-a^2})\partial_y\\ &\quad + (b\sqrt{1-a^2}\sqrt{1-c^2} - ac)\partial_z,\end{aligned}$$

$$\begin{aligned}
\partial_{\theta_2} &= a\sqrt{1-b^2}\partial_x + (\sqrt{1-a^2}\sqrt{1-c^2} - abc)\partial_y \\
&\quad + (ab\sqrt{1-c^2} + c\sqrt{1-a^2})\partial_z, \\
\partial_{\theta_3} &= -b\partial_x - c\sqrt{1-b^2}\partial_y + \sqrt{1-b^2}\sqrt{1-c^2}\partial_z, \\
\partial_{\omega_{12}} &= -\sqrt{1-a^2}\partial_a, \\
\partial_{\omega_{13}} &= \frac{ab\sqrt{1-a^2}}{\sqrt{1-b^2}}\partial_a + \sqrt{1-a^2}\sqrt{1-b^2}\partial_b - \frac{a\sqrt{1-c^2}}{\sqrt{1-b^2}}\partial_c, \\
\partial_{\omega_{23}} &= -\frac{b(1-a^2)}{\sqrt{1-b^2}}\partial_a + a\sqrt{1-b^2}\partial_b + \frac{\sqrt{1-a^2}\sqrt{1-c^2}}{\sqrt{1-b^2}}\partial_c. \qquad (5.18)
\end{aligned}$$

From the derived type of pr $\mathcal{V}$ we see that $\rho_1 = \rho_2 = 0$ and $\rho_3 = 2$. Hence the only non-empty fundamental bundle is the one of highest order $\Upsilon_{\widehat{\Sigma}_2}(\mathrm{pr}\,\mathcal{V}^{(2)})$. The local basis (5.18) and resolvent bundle (5.14) show that this is spanned by $\{dx, dy, dz\}$.

Fix a parameter value $t_0 \in I$ and let $(x_i(t), y_i(t), z_i(t))$ be the explicit parametrisations of the curves $\gamma_i(t)$, $i = 1, 2$. We may assume, without loss of generality, that $\dot{x}_1(t_0)\dot{x}_2(t_0) \neq 0$, if necessary after acting on the γ_i by Euclidean motions $g_i \in E(3)$. Consequently, near $\mathbf{x}(t_0)$, we can take x as a parameter along the curves and hence the tangents along the integral curves of distribution (5.13) are sections of

$$\begin{aligned}
\mathrm{pr}\mathcal{V} = \Bigg\{ X = \partial_x - \frac{1}{\sqrt{1-a^2}\sqrt{1-b^2}}\Big((a\sqrt{1-c^2} + bc\sqrt{1-a^2})\partial_y \\
+ (b\sqrt{1-a^2}\sqrt{1-c^2} - ac)\partial_z + \kappa\sqrt{1-a^2}\partial_a \\
+ \tau\left(\frac{b(1-a^2)}{\sqrt{1-b^2}}\partial_a - a\sqrt{1-b^2}\partial_b - \frac{\sqrt{1-a^2}\sqrt{1-c^2}}{\sqrt{1-b^2}}\partial_c\right) - \kappa_1\partial_\kappa\Big), \partial_\tau, \partial_{\kappa_1}\Bigg\}.
\end{aligned}$$

Despite its fierce appearance, pr $\mathcal{V}$ is amenable since only differentiation is required in order to construct the local diffeomorphism ψ whose existence is guaranteed by the generalised Goursat normal form as proved in Proposition 5.2. By algorithm `Contact` the total differential operator is X and the fundamental functions of order 3 are x, y and z. By differentiation we obtain the components of ψ to be

$$x = x, \quad u = y, \quad v = z,$$

$$\begin{aligned}
u_1 &= Xu = -\frac{a\sqrt{1-c^2} + bc\sqrt{1-a^2}}{\sqrt{1-a^2}\sqrt{1-b^2}}, \\
u_2 &= Xu_1 = \frac{\kappa\sqrt{1-c^2}}{\sqrt{1-a^2}^3(1-b^2)}, \quad u_3 = Xu_2, \\
v_1 &= Xv = \frac{b\sqrt{1-a^2}\sqrt{1-c^2} - ac}{\sqrt{1-a^2}\sqrt{1-b^2}},
\end{aligned}$$

$$v_2 = Xv_1 = \frac{c\kappa}{\sqrt{1-a^2}^3(1-b^2)}, \quad v_3 = Xv_2,$$

where the u_3, v_3 components of ψ are supressed due to their complexity. The inverse φ of ψ is

$$x = x, \quad y = u, \quad z = v,$$

$$\begin{aligned}
a &= -\frac{v_1v_2 + u_1u_2}{\sqrt{1+u_1^2+v_1^2}\sqrt{u_2^2+v_2^2}}, \\
b &= \frac{u_2v_1 - u_1v_2}{\sqrt{u_2^2(1+v_1^2) - 2u_1u_2v_1v_2 + v_2^2(1+u_1^2)}}, \\
c &= \frac{v_2}{\sqrt{u_2^2+v_2^2}}, \\
\kappa &= \frac{\sqrt{u_2^2(1+v_1^2) - 2u_1u_2v_1v_2 + v_2^2(1+u_1^2)}}{\sqrt{1+u_1^2+v_1^2}^3}, \\
\tau &= \frac{u_2v_3 - u_3v_2}{u_2^2(1+v_1^2) - 2u_1u_2v_1v_2 + v_2^2(1+u_1^2)}, \\
\kappa_1 &= D_x\kappa,
\end{aligned}$$

where D_x is the total differential operator on $J^3(\mathbb{R}, \mathbb{R}^2)$. That the differential functions κ and τ given above are the complete invariants for curves under Eucldean motions follows from Proposition 5.2. We have effectively factored the integral curves of $\mathcal{J}$ through the jet bundle $J^3(\mathbb{R}, \mathbb{R}^2)$. If the corresponding expressions for a, b, c given above are substituted into the general group element of $E(3)$, then we obtain the moving frame as in [**5**]. With a little more effort, we could have predicted that κ is a second-order invariant and τ a third-order invariant, directly from Proposition 5.3.

We complete the example of curves in $\mathbb{E}^3$ by obtaining the familiar invariants for immersions of curves up to the action of the Euclidean group in terms of an arbitrary parametrisation. By construction, $\dot{x}(t) \neq 0$ for t near t_0 and hence locally we have

$$f_1'(x(t)) = \frac{\dot{y}}{\dot{x}}, \quad f_2'(x(t)) = \frac{\dot{z}}{\dot{x}}, \quad f_1''(x(t)) = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3}, \quad f_2''(x(t)) = \frac{\ddot{z}\dot{x} - \dot{z}\ddot{x}}{\dot{x}^3},$$

$$\begin{aligned}
f_1'''(x(t)) &= \frac{\dddot{y}\,\dot{x}^2 - 2\ddot{y}\ddot{x}\dot{x} + 2\dot{y}\ddot{x}^2 - \dot{y}\,\dddot{x}\,\dot{x}}{\dot{x}^5}, \\
f_2'''(x(t)) &= \frac{\dddot{z}\,\dot{x}^2 - 2\ddot{z}\ddot{x}\dot{x} + 2\dot{z}\ddot{x}^2 - \dot{z}\,\dddot{x}\,\dot{x}}{\dot{x}^5}.
\end{aligned}$$

Substituting these into the differential expressions for κ and τ given by the inverse of ψ above yields the usual formulas

$$\kappa(t) = \frac{|\dot{\gamma}(t) \wedge \ddot{\gamma}(t)|}{|\dot{\gamma}(t)|^3}, \quad \tau(t) = \frac{\dot{\gamma}(t) \wedge \ddot{\gamma}(t) \cdot \dddot{\gamma}(t)}{|\dot{\gamma}(t)|^2},$$

for the curvature and torsion of the curve $\gamma(t) = (x(t)\, y(t)\, z(t))^T$. Note that ψ is a local diffeomorphism away from the hyperplane $\kappa = 0$, as may be verified by a computation. So the results above are valid for any $C3$ immersion of an interval into $\mathbb{E}^3$ if and only if κ is nonzero. That is, if and only if $\dot{\gamma}(t) \wedge \ddot{\gamma}(t)$ is nonzero near t_0.

Curves in the other Klein geometries can be treated in much the same way, yielding the corresponding differential invariants and moving frames. More generally, it is interesting to carry out the above construction for curves in a nonflat Klein geometry, that is, a *Cartan* geometry. We will report on this in subsequent work.

Example 5.4 (Pfaffian Systems). In this final example we illustrate the application of our algorithm to Pfaffian systems. Consider the following Pfaffian system Ω on a generic subset of $\mathbb{R}^{16}$,

$$\Omega : \omega^i = 0, \qquad 1 \leq i \leq 12,$$

where

$$\begin{aligned}
\omega^1 &= x_2 x_{12}\, dx_{16} + (x_{16} - 1)^2 (x_9 \eta - x_3)\, dx_{11} - x_{12}(x_{16} - 1)\, dx_2, \\
\omega^2 &= x_{12}\, dx_9 + (2x_{11} - x_{10})\, dx_{11}, \quad \omega^3 = x_{12}\, dx_{12} - x_{13} dx_{11}, \\
\omega^4 &= x_{12}(x_{16} - 1)\, dx_1 + (x_{16}(x_2 - 1) + 1)\, dx_{11}, \\
\omega^5 &= x_{11} x_{12}\, dx_6 + (x_6 x_{12} - x_7 + \eta)\, dx_{11}, \\
\omega^6 &= x_{12}\, dx_{14} + (4x_{11} - 2x_{10} - x_{15})\, dx_{11}, \\
\omega^7 &= x_{12}(x_{16} - 1)(dx_4 + 3dx_{10}) - ((x_{16} - 1)(x_5 + 6x_{12}) - x_2 x_{16})\, dx_{11}, \\
\omega^8 &= x_{12}\, dx_{13} + (x_9 - x_{14})\, dx_{11}, \quad \omega^9 = x_{12}\, dx_8 + (1 - x_9)\, dx_{11}, \\
\omega^{10} &= x_{12}(x_{16} - 1)(dx_{15} + dx_{10}) - (2x_{12}(x_{16} - 1) - x_2 x_{16})\, dx_{11}, \\
\omega^{11} &= x_{12}\, dx_7 - (1 + x_8 + \eta)\, dx_{11}, \\
\omega^{12} &= x_{12}\, dx_3 + (6x_{11} + 4x_{11}^2 - x_4 - x_9 - 3x_{10} + x_{10}^2 \\
&\quad - 4x_{10}x_{11} + 2x_1 x_{11} - 2x_{11}x_{15} - x_1 x_{10} + x_{10} x_{15})\, dx_{11},
\end{aligned}$$

and $\eta = x_1 - x_{10} - x_{15} + 2x_{11}$.

We seek the integral submanifolds of Ω. The package `<derived>` took a little over a minute to compute the Pfaffian system's refined derived type to be

$$[[12, 16], [9, 13, 13], [6, 10, 10], [3, 7, 6], [1, 4, 3], [0, 16]].$$

The derived length is $k = 5$ and we find that the only nontrivial fundamental bundles are

$$\mathcal{F}^3(\Omega) = \Xi(\Omega)^3_{(2)}/\Xi(\Omega^{(3)}) = \left\{\mathbf{p}_3\left(\frac{dx_{16}}{1-x_{16}} + \frac{dx_2}{x_2}\right)\right\},$$
$$\mathcal{F}^4(\Omega) = \Xi(\Omega)^4_{(3)}/\Xi(\Omega^{(4)}) = \{\mathbf{p}_4(dx_6)\}.$$

Computing the type numbers ξ^j, ξ^i_{i-1}, as defined in Proposition 3.2, we obtain

$$\xi^0 = 16, \quad \xi^1 = 13, \quad \xi^2 = 10, \quad \xi^3 = 6, \quad \xi^4 = 3,$$
$$\xi^1_0 = 13, \quad \xi^2_1 = 10, \quad \xi^3_2 = 7, \quad \xi^4_3 = 4.$$

Since $\rho_5 = 1$ and these data agree with the above refined derived type, we confirm that Ω is a Goursat bundle of type $\tau = \text{accel}(\Omega) = \langle 0, 0, 1, 1, 1\rangle$ and, hence, by Theorem 4.1, there is an equivalence to the contact system $\mathcal{C}\langle 0, 0, 1, 1, 1\rangle^{\perp}$. We go on to find an equivalence and then use it to find the general solution of Ω, explicitly.

From the fundamental bundles we deduce that there is one fundamental function of order 3 and one of order 4, namely

$$\varphi^{1,3} = \frac{x_2}{x_{16}-1}, \qquad \varphi^{1,4} = x_6.$$

We emphasise that fundamental functions are not unique. Their exterior derivatives are defined up to elements of the corresponding Cartan bundles. Thus while the fundamental bundles are canonical, the choice of first integral is not. Some choices lead to more complicated equivalences than others. In this case the two Cartan systems in question are

$$\Xi(\Omega^{(3)}) = \{dx_6, dx_7, dx_{11}, dx_{12}, dx_{13}, -dx_1 + dx_{10} + dx_{15}\},$$
$$\Xi(\Omega^{(4)}) = \{dx_{11}, dx_{12}, -dx_1 + dx_{10} + dx_{15}\},$$

and we see that $d(x_6x_{11}) \in \Xi(\Omega)4_{(3)}/\Xi(\Omega^{(4)})$. Hence we may, if we wish, also take $\varphi^{1,4} = x_6x_{11}$. This freedom in the choice of fundamental functions of each order can be expoited to derive, if possible, simple equivalences. Such choices can only be derived from heuristics, as is the case here.

Now since $\rho_5 = 1$, we turn to `Contact B` according to which we must at this point choose a first integral in $\Xi(\Omega^{(4)})$ that will play the role of the independent variable x in the target contact system. Here again there is no canonical choice of first integral, nor can there be. The target contact system is invariant under an infinite Lie pseudogroup. One makes a choice that delivers a simple equivalence. With a little experimentation it is possible to discover that

$$x = \eta = x_1 - x_{10} - x_{15} + 2x_{11}$$

is a good choice for 'independent variable' x since we have that $Zx = 1$, where[5] $Z \in \Omega^{\perp}$ has the form

$$
\begin{aligned}
Z \;=\; & \left(\frac{1 - x_{16} + x_2 x_{16}}{1 - x_{16}}\right)\partial_{x_1} + (x_3 - x_9\eta)\partial_{x_2} + (x_9 + x_4 + 3x_{10} - 6x_{11} \\
& + x_1 x_{10} - x_{10}x_{15} - x_{10}^2 + 4x_{10}x_{11} - 2x_1 x_{11} + 2x_{11}x_{15} - 4x_{11}^2)\partial_{x_3} \\
& + \left(\frac{x_5 - x_5 x_{16} + x_2 x_{16}}{1 - x_{16}}\right)\partial_{x_4} + \left(\frac{x_7 - x_6 x_{12} - \eta}{x_{11}}\right)\partial_{x_6} \\
& + (1 + x_8 + \eta)\partial_{x_7} + (x_9 - 1)\partial_{x_8} + (x_{10} - 2x_{11})\partial_{x_9} + 2x_{12}\partial_{x_{10}} + x_{12}\partial_{x_{11}} \\
& + x_{13}\partial_{x_{12}} + (x_{14} - x_9)\partial_{x_{13}} + (2x_{10} - 4x_{11} + x_{15})\partial_{x_{14}} + \left(\frac{x_2 x_{16}}{1 - x_{16}}\right)\partial_{x_{15}} \\
& + \left(\frac{x_{16}(x_3 - x_9\eta)(x_{16} - 1)}{x_2}\right)\partial_{x_{16}}.
\end{aligned}
$$

Setting $\mathcal{V} = \Omega^{\perp}$, we compute a basis for Char $\mathcal{V}_0^{(1)}$,

$$
\text{Char } \mathcal{V}_0^{(1)} = \left\{\partial_{x_2} + \frac{x_{16} - 1}{x_2}\partial_{x_{16}},\ \partial_{x_5},\ 3\partial_{x_4} - \partial_{x_{10}} + \partial_{x_{15}}\right\}.
$$

Using the inductive definition of bundle Π^5 given by (4.13) we find that

$$
\Pi^{5^{\perp}} = \{dx_{11}, dx_1 - dx_{10} - dx_{15}\}.
$$

This is the fundamental bundle of highest order 5, and we may take

$$
\varphi^{1,5} = x_{11}.
$$

The remaining contact coordinates are computed via differentiation by Z as in `Contact B`:

$$
\begin{aligned}
z^{1,3} \;&=\; \frac{x_2}{x_{16} - 1}, \quad z_1^{1,3} = Zz^{1,3} = -x_3 + x_9\eta, \\
z_2^{1,3} \;&=\; Zz_1^{1,3} = -x_4 - 3x_{10} + 6x_{11}, \quad z_3^{1,3} = \frac{x_5 - x_{16}(x_2 - x_5)}{x_{16} - 1}, \\
z^{1,4} \;&=\; x_6 x_{11}, \quad z_1^{1,4} = Zz^{1,4} = x_7 - \eta, \quad z_2^{1,4} = Zz_1^{1,4} = x_8 + \eta, \\
z_3^{1,4} \;&=\; Zz_2^{1,4} = x_9, \quad z_4^{1,4} = Zz_3^{1,4} = x_{10} - 2x_{11}, \\
z^{1,5} \;&=\; x_{11}, \quad z_1^{1,5} = Zz^{1,5} = x_{12}, \quad z_2^{1,5} = Zz_1^{1,5} = x_{13}, \\
z_3^{1,5} \;&=\; Zz_2^{1,5} = x_{14} - x_9, \\
z_4^{1,5} \;&=\; Zz_3^{1,5} = x_{10} + x_{15} - 2x_{11}, \quad z_5^{1,5} = Zz_4^{1,5} = \frac{x_2 x_{16}}{1 - x_{16}}.
\end{aligned}
$$

[5] We emphasise that the construction of a vector field Z with this property is algorithmic.

The inverse of this equivalence gives the general solution of Ω in terms of three arbitrary functions $u(x)$, $v(x)$, $w(x)$ and their derivatives $u^{(i)}$, $v^{(j)}$, $w^{(l)}$:

$$\begin{aligned}
x_1 &= x + w^{(4)}, \quad x_2 = u(x) - w^{(5)}, \quad x_3 = u^{(1)} + x v^{(3)},\\
x_4 &= u^{(2)} - 3v^{(4)}, \quad x_5 = u^{(3)} - w^{(5)},\\
x_6 &= \frac{v(x)}{w(x)}, \quad x_7 = x + v^{(1)}, \quad x_8 = v^{(2)} - x, \quad x_9 = v^{(3)},\\
x_{10} &= v^{(4)} + 2w(x), \quad x_{11} = w(x),\\
x_{12} &= w^{(1)}, \ x_{13} = w^{(2)}, \ x_{14} = v^{(3)} + w^{(3)}, \ x_{15} = w^{(4)} - v^{(4)}, \ x_{16} = \frac{w^{(5)}}{u(x)}.
\end{aligned}$$

Acknowledgments

The computations in this paper were greatly assisted by use of the general purpose differential geometry package `Vessiot` developed at Utah State University by Ian M. Anderson and colleagues. Its use is gratefully acknowledged. I'm indebted to the anonymous referees for detailed comments which improved the presentation of this paper.

References

[1] E. Aranda-Bricaire and J.-B. Pomet, Some explicit conditions for a control system to be feedback equivalent to extended Goursat form, *Proceedings of the IFAC Nonlinear Control Systems Design Symposium*, Tahoe City, CA, 1995.

[2] P. Brunovský, A classification of linear controllable systems, *Kybernetika* **3**(6) (1970), 173–188.

[3] R. L. Bryant, *Some Aspects of the Local and Global Theory of Pfaffian Systems*, PhD thesis, University of North Carolina, Chapel Hill, 1979.

[4] R. L. Bryant, S.-S. Chern, R. B. Gardner, H. Goldschmidt, and P. Griffiths, *Exterior Differential Systems*, MSRI, Springer-Verlag, New York, 1991.

[5] M. Fels and P. J. Olver, Moving coframes I: A practical algorithm, *Acta Appl. Math.* **51** (1998), 161–213; Moving coframes II. Regularization and theoretical foundations, *Acta Appl. Math.* **55** (1999), 127–208.

[6] R. B. Gardner and W. F. Shadwick, The GS algorithm for exact linearization to Brunovský normal form, *IEEE Trans. Automat. Control* **37**(2) (1992), 224–230.

[7] L. R. Hunt, R. Su, and G. Meyer, Design for multi-input nonlinear systems, in *Differential Geometric Control Theory*, Birkhäuser, Boston, MA, 1982, pp. 268–298.

[8] A. Isidori, *Nonlinear Control Systems*, 3rd ed., Communications and Control Engineering Series. Springer-Verlag, New York, 1995.

[9] B. Jakubczyk and W. Respondek, On linearization of control systems, *Bull. Acad. Polon. Sci. Math. Astro. Phys.* **28** (1980), 517–522.

[10] A. Krener, On the equivalence of control systems and the linearization of nonlinear systems, *SIAM J. Control* **11** (1973), 670–676.

[11] A. Kumpera and J. L. Rubin, Multi-flag systems and ordinary differential equations, *Nagoya Math. J.* **166** (2002), 1–27.

[12] P. Libermann, Sur le problème d'équivalence des systèmes de Pfaff non complèment integrables, in *Differential Geometry*, Publ. Math. Univ. Paris VII, Vol. 3, pp. 73–109, Université de Paris VII, 1978.

[13] E. L. Mansfield, Algorithms for symmetric differential systems, *Found. Comput. Math.* **1** (2001), 335–383.

[14] G. Mari Beffa, Hamiltonian structures on the space of differential invariants of curves in flat semi-simple homogeneous manifolds; Poisson geometry of differential invariants of curves in some non-semi-simple homogeneous spaces, http://www.math.wisc.edu/~maribeff/

[15] W. Pasillas-Lepine and W. Respondek, Contact systems and corank 1 involutive subdisributions, *Acta Appl. Math.* **69** (2001), 105–128.

[16] W. Respondek and W. Pasillas-Lepine, Extended Goursat normal form: A geometric characterisation, in *Nonlinear Control in the Year* 2000, Vol. 2, pp. 323–338 (A. Isidori, F. Lamnabhi-Laggarigue, and W. Respondek, eds.), Springer-Verlag, New York, 2000.

[17] W. Respondek and W. Pasillas-Lepine, Canonical contact systems for curves: A survey, in *Contemporary Trends in Geometric Control Theory and Applications* (A. Anzaldo-Meneses, B. Bonnard, J. P. Gauthier, and F. Monroy-Pérez, eds.), World Scientific, New York, 2001, pp. 77–112.

[18] W. Respondek, Transforming nonholonomic control systems into the canonical contact form, in *Proceedings of the* 40^{th} *IEEE Conference on Decision and Control*, Orlando, FL, 2001.

[19] W. Shadwick and W. Sluis, Dynamic feedback for classical geometries, *Contemp. Math.* **170** (1994).

[20] W. Sluis, *Absolute Equivalence and its Applications to Control Theory*, PhD Thesis, University of Waterloo, Waterloo, Canada, 1992.

[21] O. Stormark, *Lie's Structural Approach to PDE Systems*, Encyclopedia of Mathematics and its Applications, Vol. 80, Cambridge University Press, Cambridge, 2000.

[22] A. van der Schaft, Linearization and input–output decoupling for general nonlinear systems, *Systems Control Lett.* **5** (1984), 27–33.

[23] P. Vassiliou, A constructive generalised Goursat normal form, *Differential Geom. Appl.*, to appear; Mathematics ArXiv, math.DG/0404377, April 2004.

[24] E. von Weber, Zur invariantentheorie der systeme Pfaff'scher gleichungen, *Berichte Verhandlungen der Koniglich Sachsischen Gesellshaft der Wissenshaften Mathematisch Physikalische Klasse*, *Leipzig*, **50** (1898), 207–229.

[25] G. Wilkens, Finsler geometry in low-dimensional control theory, in D. Bao, S.-S. Chern, and Z. Shen (eds.), *Finsler Geometry*, Contemporary Mathematics, Vol. 196, American Mathematical Society, Providence, RI, 1996.

[26] K. Yamaguchi, Contact geometry of higher order, *Japan J. Math.* **8** (1982), 109–176; Geometrization of jet bundles, *Hokkaido Math. J.* **12** (1983), 27–40.