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# A SET OF AXIOMS FOR DIFFERENTIAL GEOMETRY

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1. *Introductory.*—The axioms set forth in this note are intended to describe the class of manifolds of $n$ dimensions to which the theories nowadays grouped together under the heading of differential geometry are applicable. The manifolds are classes of elements called points, having a structure which is characterized by means of coördinate systems. A *coördinate system* is a (1–1) correspondence, $P \longrightarrow x$, between a set of points, $[P]$, of the manifold, and a set, $[x]$, of ordered sets of $n$ real numbers, $x = (x^1, \ldots, x^n)$. For convenience we call any ordered set of $n$ real numbers an arithmetic point and the totality of arithmetic points, for a fixed $n$, the arithmetic space of $n$ dimensions. Each point $P$ which corresponds in a coördinate system, $P \longrightarrow x$, to an arithmetic point $x$, is said to be represented by $x$. The set, $[P]$, of all points represented in a given coördinate system is called the *domain* of the coördinate system, and the set, $[x]$, of the arithmetic points which represent them is called its *arithmetic domain.*

If $P \longrightarrow x$ and $P \longrightarrow y$ are two coördinate systems having the same domain, the transformation $x \longrightarrow y$ of the arithmetic space, which is the resultant of $x \longrightarrow P$ followed by $P \longrightarrow y$, is called the transformation of coördinates from $P \longrightarrow x$ to $P \longrightarrow y$, or the transformation between $P \longrightarrow x$ and $P \longrightarrow y$. The general scheme of the axioms is to characterize a class of "allowable coördinate systems" by means of the analytic properties of the transformations between them. In order to state these properties clearly we recall a few arithmetic theorems and definitions:

A set of arithmetic points given by

$$| x^i - x^i_0 | < \delta,$$

for some positive $\delta$, will be called a *box,* and the point $x_0$ will be called its center. A set of points $[x]$ will be called a region if each $x$ is the center of a box which is contained in $[x]$.

A function $F(x^1, \ldots, x^n)$, defined over a region, $[x]$, will be described as of class $u$ if it and its first $u$ derivatives exist and are continuous at each point of $[x]$. Here $u$ can be any positive integer. A function will be described as belonging to the class $\infty$ if all its derivatives exist. Continuous functions will be described as belonging to the class 0, and analytic functions, i.e., functions which can be expanded in a power series about each point in $[x]$, will be described as of class $\omega$. We deal with single valued functions only.

Let $y^1(x), \ldots, y^n(x)$ be $n$ functions of class $u > 0$, defined over a region $[x]$. The equations

$$y^i = y^i(x) \tag{1.1}$$

define a transformation, $x \longrightarrow y$, of $[x]$ into a set of points $[y]$. If the Jacobian

$$\frac{\partial(y^1, \ldots, y^n)}{\partial(x^1, \ldots, x^n)}$$

does not vanish in $[x]$ it follows from the implicit function theorem that $[y]$ is also a region. If the transformation $x \longrightarrow y$ is non-singular, i.e., has a single-valued inverse, it will be called a *regular transformation of class u*. Unless otherwise stated it is to be assumed that any transformation to which we refer is of this type. We assume $u$ to be fixed (either as $0, 1, \ldots, \infty$ or $\omega$), and shall often omit the words "of class $u$," as applied to functions, transformations and, later on, to $n$-cells.

A regular transformation, $x \longrightarrow y$, of a region $[x]$ into a region $[y]$, followed by a regular transformation, $y \longrightarrow z$, of $[y]$ into $[z]$ is a regular transformation, $x \longrightarrow z$, of $[x]$ into $[z]$. Moreover, the inverse of a regular transformation is regular.

The last two sentences assert that the totality of regular transformations between regions is what we call a *pseudo-group*. A pseudo-group is any set of transformations which satisfy the conditions:

(1). *If the resultant of any two exists, it is in the set.*

(2). *The inverse of each transformation in the set is also in the set.*

The pseudo-group of regular transformation between regions will be called the *pseudo-group of class u*.

An *arithmetic n-cell of class u* is any set of arithmetic points obtained from a box by a regular transformation of class $u$. Thus the arithmetic space is itself an $n$-cell. Moreover, any two $n$-cells of class $u$ are equivalent under the pseudo-group of class $u$. If $u \leqq u'$, any function, transformation or $n$-cell of class $u'$ is also a function, transformation or $n$-cell of class $u$.

2. *The First Group of Axioms.*—The axioms are arranged in three groups, the first being:

$A_1$. *The transformation between two allowable coördinate systems which have the same domain is regular if the arithmetic domain of one of them is a region.*

$A_2$. *Any coördinate system obtained by a regular transformation of coördinates from an allowable coördinate system is allowable.*

Definition: The image in an allowable coordinate system of a box will be called an *n-cell of class u.*

$A_3$. *The correspondence in which each point of an n-cell corresponds to its image in an allowable coördinate system is an allowable coördinate system.*

3. *The Second Group of Axioms.*—Let $[K_\alpha]$ be any set of allowable coördinate systems, finite or infinite. Let $K$ be the correspondence in which each point in the domain of at least one $K_\alpha$ corresponds to every arithmetic point by which it is represented in at least one $K_\alpha$. If the correspondence $K$ is (1–1) it will be called the *union* of $[K_\alpha]$. It can be shown that the union of $[K_\alpha]$ exists if the union of each pair $K_\alpha$ and $K_\beta$ exists.

The axioms of the second group are:

$B_1$. *Any coördinate system which is the union of a set of allowable coördinate systems whose domains are n-cells is allowable.*

$B_2$. *Each allowable coördinate system is the union of a set of allowable coördinate systems whose domains are n-cells.*

4. *The Third Group of Axioms.*—The axioms of the third group are:

$C_1$. *If two n-cells have a point in common they have in common an n-cell containing this point.*

$C_2$. *If $P$ and $Q$ are any two distinct points there is an n-cell $C_P$, containing $P$, and an n-cell $C_Q$, containing $Q$, such that $C_P$ and $C_Q$ have no point in common.*

$C_3$. *There are at least two points.*

5. *Regular Manifolds.*—Any space satisfying the axioms A, B and C will be called an *n-dimensional manifold of class u,* or a *regular manifold.* When $u = 0$ the theory of a regular manifold is a branch of analysis situs. When $u = 1$ it is possible to define a tangent space of differentials at each point of the manifold, and in addition to pure continuity considerations we can apply some of the formal machinery of the differential calculus. When $u = 2$ we have second differentials, affine connections and so on.

It would obviously be impossible in the space here available to show how differential geometry is built out of these axioms. We have tried to do this in a small book called "Foundations of Different Geometry," which we hope to publish as a Cambridge Tract, and will not deal further with it here. Instead, we make a few remarks about the special peculiarities of our axioms, and prove their independence.

6. *Topological Considerations.*—From the axioms C it follows that a regular manifold satisfies the axioms given by F. Hausdorff (Mengenlehre, Leipzig, 1914, p. 213) for a topological space. Indeed, if C be taken as a separate set of axioms with points and $n$-cells as the undefined elements,

the spaces which they determine satisfy Hausdorff's axioms, provided each $n$-cell is taken as a neighborhood of each point in it. In the presence of the axioms A and B, a regular manifold is what some writers call a homogeneous topological space of $n$ dimensions, that is, a topological space, each of whose neighborhoods is homeomorphic with the arithmetic space of $n$ dimensions. This statement is equivalent to the theorem:

*An allowable coördinate system is a homeomorphism between its domain and its arithmetic domain.*

Let $[x]$ be a set of arithmetic points having a limit point $x'$, where $[x]$ and $x'$ are both in the arithmetic domain of an allowable coördinate system $P \longrightarrow x$. Let $[P]$ and $P'$ be the respective images of $[x]$ and $x'$. It will follow that $P'$ is a limit point of $[P]$, if every $n$-cell C contained in the domain of $P \longrightarrow x$, and containing $P'$, contains points of $[P]$ other than $P'$. From the axioms A it follows that the image of C in $P \longrightarrow x$ is an arithmetic $n$-cell containing $x'$, and since $x'$ is a limit point of $[x]$, a point of $[x]$ other than $x'$. Therefore C contains points of $[P]$ other than $P'$, and $P'$ is a limit point of $[P]$. Therefore the transformation $x \longrightarrow P$ is continuous.

Interchanging the parts played by the regular manifold and the arithmetic space, it follows by a similar argument that the transformation $P \longrightarrow x$ is continuous. It is, therefore, a homeomorphism.

According to the theorem just proved, the domain of any allowable coördinate system is topologically equivalent to some arithmetic region. No arithmetic region is self-compact (a region $U$ is said to be self-compact if any infinite set of points in $U$ has at least one limit point in $U$) and we have as a corollary: *The domain of an allowable coördinate system is not self-compact.* Therefore a manifold such as the surface of a sphere or an anchor ring cannot be represented completely in a single allowable coördinate system.

7. *Consistency and Independence Examples.*—As a consistency example take the arithmetic space of $n$ dimensions with regular transformations between regions for allowable coördinate systems. The axioms A, B and C are obviously satisfied by this set of coördinate systems. We shall show that the axioms are independent by giving consistency examples for the sets obtained by denying each one in turn.

*Denying* $A_1$. Take the arithmetic space with all non-singular transformations which operate on regions for allowable coördinate systems. Then the domain of an allowable coördinate system is a region, but the arithmetic domain may be any set of points with the power of the continuum.

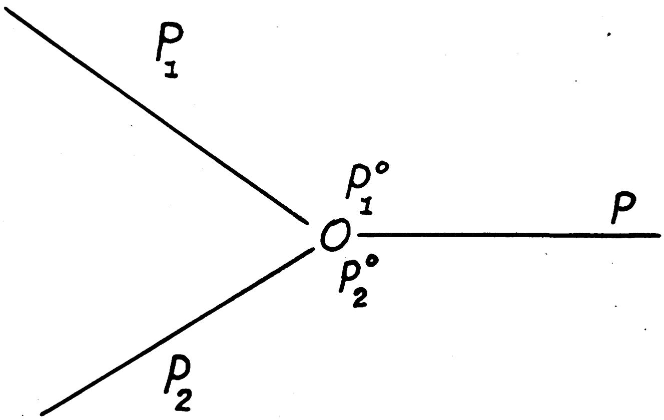
*Denying* $A_2$. Take the arithmetic space with any correspondence in which each point of a region corresponds to itself, and only such correspondences, as allowable coördinate systems.

*Denying* $A_3$. Limit the allowable coördinate systems in the arithmetic space to those regular transformations in which the whole space corresponds to the whole or a part of itself.

*Denying* $B_1$. Limit the allowable coördinate systems in the arithmetic space to regular correspondences between $n$-cells.

*Denying* $B_2$. The allowable coördinate systems are to be those given in the consistency proof together with all those in which a single point corresponds to a single point.

*Denying* $C_1$. The space ($n = 1$) is to be the arithmetic space of two dimensions, and an allowable coördinate system is to be a regular correspondence between a set (finite or infinite) of 1-cells in this space, no two of which have a common point, and a set of segments in the arithmetic




space of one dimension. Any such correspondence is to be an allowable coördinate system. Thus the correspondence given by

$$x = t,\ y = 0,$$

for $-1 < t < 1$, will be one allowable coördinate system and that given by

$$x = 0,\ y = t,$$

for $-1 < t < 1$, will be another. The domains of these two coördinate systems will have the origin (0, 0) in common, and no other point. Therefore $C_1$ is not satisfied.

*Denying* $C_2$. Let $[P]$ be a set of points in a (1–1) correspondence, $K$, with the interval $0 < t < 1$. Let $[P_1]$ and $[P_2]$ be sets of points in (1–1) correspondences $K_1$ and $K_2$, respectively, with the real numbers which are not greater than zero. The space is to consist of these three sets of points with the allowable coördinate systems defined as follows: The correspondence $K'_1$ in which each $P$ corresponds to its image in $K$, and each $P_1$ corresponds to its image in $K_1$, is to be an allowable coördinate system. The correspondences obtained from $K'_1$ according to the axioms A are also to be allowable coördinate systems. The analogous correspondences for the set of points consisting of $[P]$ together with $[P_2]$ are to be allowable coördinate systems, and the remaining allowable coördinate systems are the unions of these.

Let $P_1^0$ and $P_2^0$ be the points which correspond to zero under $K_1$ and $K_2$, respectively. Any two $n$-cells containing $P_1^0$ and $P_2^0$ have a point in common, contradicting $C_2$.

*Denying* $C_3$. The space shall have no points.

8. *Other Pseudo-groups.*—Let $g$ be any pseudo-group of continuous transformations between regions in the arithmetic space, such that the axioms A and B are consistent when the space is the arithmetic space and the transformations of $g$ are taken as allowable coördinate systems. We obtain a consistent set of axioms if we substitute "transformation of $g$" for "regular transformation" in the axioms A and leave the axioms B and C unchanged. The spaces which satisfy these axioms constitute a sub-class of the manifolds of class 0. It is a special case of this remark that the manifolds of class 0 include those of class 1, which include those of class 2, and so on.

Let us consider a few examples of other pseudo-groups which give rise to such spaces.

(1). Let $g$ be the pseudo group of regular transformations $x \longrightarrow y$ with a constant Jacobian $\left|\frac{\partial y}{\partial x}\right|$. In a space determined by $g$ there is a definition of ratios of volume. This pseudo-group has an invariant sub-pseudo-group, $g'$, of transformations whose Jacobian is unity. In the spaces $g'$ there will be an invariant unit of volume. The relation between the spaces $g$ and the spaces $g'$ is analogous to that between affine and equiaffine spaces.

(2). Let $g^+$ be the pseudo-group of direct transformations of class $u$, that is, regular transformations with a positive Jacobian. The spaces $g^+$ are oriented manifolds of class $u$.

(3). Let $g$ be the pseudo-group of linear transformations between regions. This may be called the affine pseudo-group, as apart from the affine group whose transformations carry the arithmetic space into itself as a whole. The spaces defined by the affine pseudo-group may be called locally flat affine spaces.

(4). Let $n = 2r$, and let $g$ be the pseudo-group of regular contact transformations. That is to say, any transformation of $g$ is given by equations of the form

$$\begin{cases} \bar{x}^\lambda = \bar{x}^\lambda (x^1, \ldots, x^r, p_1, \ldots, p_r), \\ \bar{p}_\lambda = \bar{p}_\lambda (x^1, \ldots, x^r, p_1, \ldots, p_r), \end{cases} (\lambda = 1, \ldots, r)$$

where $\bar{x}^\lambda(x, p)$ and $\bar{p}_\lambda(x, p)$ are functions of class $u$ such that

$$\bar{p}_\lambda d\bar{x}_\lambda = p_\lambda dx^\lambda + dW(x, p),$$

$W(x, p)$ being an arbitrary function of class $u$.

(5). Let $n = 2r$, and let $g$ be the pseudo-group of transformations given by

$$w^\lambda = w^\lambda(z^1, \ldots, z^r), \quad (\lambda = 1, \ldots, r),$$

where $z$ and $w$ are complex variables, and $w^\lambda(z)$ are analytic functions. If

$$z^\lambda = x^\lambda + iy^\lambda,$$
$$w^\lambda = u^\lambda + iv^\lambda,$$

the transformations of $g$ may also be written

$$u^\lambda = u^\lambda(x, y),$$
$$v^\lambda = v^\lambda(x, y).$$

If $n = 2$ the spaces $g$ are Riemann surfaces (cf. H. Weyl, "Die Idee der Riemannschen Flache," Leipzig, 1913, p. 36) and their geometry is two dimensional conformal geometry.

In a 2-dimensional manifold of class $\omega$, a scalar, $f(P)$, of highest class has as its component in any allowable coördinate system an analytic function of two real variables. But in a space $g$ it has a harmonic function as its component, i.e., a function which is the real or imaginary part of an analytic function of a complex variable. That is to say, we can define complex point-functions, $f(P)$, such that

$$F(x, y) = f(P),$$

is an analytic function of $x + iy$, where $P \longrightarrow (x, y)$ is any one of the class of coördinate systems which satisfies the axioms $g$.

In general any pseudo-group, $G$, of continuous transformations between regions in the arithmetic $n$-space determines a unique pseudo-group which defines a special class of regular manifolds. For $g$ is that pseudo-group which contains $G$ and satisfies the axioms $A_3$, $B_1$ and $B_2$, when the transformations of $g$ are taken as allowable coördinate systems.

9. *Two Existence Theorems.*—The use of the pseudo-group of transformations of coördinates of class $<\omega$ raises a question as to the status of operations which are ordinarily described by means of power series. A case in point is the transformation to normal coördinates in the affine geometry of paths. The space of this geometry is a regular manifold with a symmetric affine connection defined at every point. In any allowable coördinate system the paths are given by the differential equations

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{9.1}$$

where $\Gamma^i_{jk}$ are the components of the affine connection. The transformation law of an affine connection involves the second but not the higher derivatives, and therefore it is an invariant condition to require $\Gamma^i_{jk}$ to be functions of class $u'$ if $u' \leqq u - 2$, but not if $u' > u - 2$.

*If $u \geqq 3$ and if $\Gamma^i_{jk}$ are of class $u - 2$ at a point $Q$, there exists an n-cell $C_Q$, containing $Q$, which is represented in a coördinate system $P \longrightarrow y$, in which the paths through $Q$ are given by linear equations of the form*

$$y^i = p^i s. \tag{9.2}$$

*The transformation of coördinates between $P \longrightarrow y$ and any allowable coordinate system in which $Q$ is represented, is of class $u - 2$.*

The equations (9.1) admit a unique set of solutions (see, for instance, L. Bieberbach, "Theorie der Differentialgleichungen," Berlin, 1926, pp. 115–116),

$$f^i(q, p, s),$$

which satisfy the initial conditions

$$\begin{cases} f^i(q, p, 0) = q^i, \\ \left(\dfrac{\partial f^i}{\partial s}\right)_{s=0} = p_i, \end{cases}$$

where $Q \longrightarrow q$ in a given allowable coordinate system, $P \longrightarrow x$. If $\lambda$ is any constant it follows from the form of the equations (9.1) that

$$f^i(q, p, \lambda s)$$

are the solutions which satisfy the initial conditions

$$\begin{cases} f^i(q, p, 0) = q^i, \\ \left(\dfrac{\partial f^i}{\partial s}\right)_{s=0} = \lambda p^i. \end{cases}$$

That is to say,

$$f^i(q, p, \lambda s) = f^i(q, \lambda p, s),$$

and the solutions may therefore be written

$$F^i(q, y),$$

where

$$y^i = p^i s$$

It also follows that $F^i(q, y)$ are of class $u - 2$ for values of $y$ in some arithmetic $n$-cell containing the origin (see L. Bieberbach, loc. cit., pp. 39–41).

Since

$$\left(\frac{\partial x^i}{\partial y^j}\right)_0 p^j = p^i$$

for all values of $p$, we have

$$\left(\frac{\partial x^i}{\partial y^j}\right)_0 = \delta^i_j, \tag{9.3}$$

and therefore the Jacobian $\left|\frac{\partial x}{\partial y}\right|$ does not vanish for values of $y$ near the origin. Therefore the equations

$$x^i = F^i(q, y) \tag{9.4}$$

can be solved to give a regular transformation $x \longrightarrow y$ of class $u - 2$, which carries some arithmetic $n$-cell contained in the arithmetic domain of $P \longrightarrow x$, and containing $q$, into an arithmetic $n$-cell containing the origin. Therefore, $x \longrightarrow y$ is a transformation between $P \longrightarrow x$ and a coördinate system, $P \longrightarrow y$, satisfying the required conditions. Because $P \longrightarrow y$ also satisfies the conditions (9.3) it is called a normal coördinate system at the point $Q$, for the coördinate system $P \longrightarrow x$. Clearly the union of all normal coördinate systems at $Q$, for the same coördinate system, $P \longrightarrow x$, exists, and may be called *the* normal coördinate system at $Q$ for $P \longrightarrow x$. Normal coördinate systems are not necessarily allowable except when $u = \infty$, and when $u = \omega$.

10. When the curvature tensor or one of its generalizations arises as an integrability condition, an existence theorem is in the background which, for $u$ arbitrary, takes the following form:

*Let*

$$F^\alpha(x^1, \ldots, x^n, z^1, \ldots, z^m)$$

(Greek indices will run from 1 to $m$, Roman indices from 1 to $n$) *be functions of class 1 in the region*

$$|x^i| < 1, \; |z^\alpha| < 1. \tag{10.1}$$

*If, and only if,*

$$\frac{\partial F^\alpha_i}{\partial x^j} - \frac{\partial F^\alpha_j}{\partial x^i} + \frac{\partial F^\alpha_i}{\partial z^\beta} F^\beta_j - \frac{\partial F^\alpha_j}{\partial z^\beta} F^\beta_i = 0, \tag{10.2}$$

*the differential equations*

$$\frac{\partial z^\alpha}{\partial x^i} = F^\alpha_i\,(x, z), \tag{10.3}$$

*admit a unique set of solutions*

$$z^\alpha(x_0, z_0, x),$$

*which satisfy given initial conditions*

$$z^\alpha(x_0, z_0, x_0) = z_o^\alpha, \tag{10.4}$$

*where* $x_0$ *and* $z_0$ *are any values of the variables* $x$ *and* $z$ *which satisfy* (10.1).

We give a short proof of this theorem, which is perhaps well known, because we have not seen it proved elsewhere except when the functions $F$ are analytic.

By a suitable transformation we can reduce the problem to the case where $x_0 = z_0 = 0$, and we suppose this to have been done.

Instead of (10.3), let us consider the ordinary differential equations

$$\frac{dz^\alpha}{dt} = p^i F_i^\alpha(p^1 t, \ldots, p^n t, z^1, \ldots, z^m), \tag{10.5}$$

involving the parameters $p^1, \ldots, p^n$. These equations admit a unique set of solutions

$$z^1(p, t), \ldots, z^m(p, t),$$

all of which vanish with $t$. By substituting $\lambda t$ for $t$ it follows from the form of (10.5) that

$$z^\alpha(p, \lambda t) = z^\alpha(\lambda p, t),$$

or that the solutions may be written

$$z^1(x), \ldots, z^m(x), \tag{10.6}$$

where

$$x^i = p^i t.$$

Moreover, $z^\alpha(x)$ are of class 1 in some $n$-cell containing the origin (Bieberbach, loc. cit., pp. 39–41).

We remark in passing that (10.3) have at most one solution satisfying the given initial conditions. For if $u(x)$ and $v(x)$ are any solutions, $u(pt)$ and $v(pt)$ both satisfy (10.5), and vanish with $t$. But (10.5) only admits one such solution for given values of $p^1, \ldots, p^n$. Therefore $u(x) = v(x)$.

We have now to show that if the conditions (10.2) are satisfied, the functions (10.6) satisfy (10.3). We have

$$z^\alpha = \int_0^t p^i F_i^\alpha(ps, z(ps))ds.$$

When $x \neq 0$, therefore,

$$\frac{\partial z^\alpha}{\partial x^j} = \frac{1}{t}\frac{\partial z^\alpha}{\partial p^j}$$

$$= \frac{1}{t}\int_0^t \left\{\left(\frac{\partial F_i^\alpha}{\partial x^j} + \frac{\partial F_i^\alpha}{\partial z^\beta}\frac{\partial z^\beta}{\partial x^j}\right)p^i s + F_j^\alpha\right\} ds,$$

$$= F_j^\alpha + \frac{1}{t}\int_0^t s\left(\frac{\partial F_i^\alpha}{\partial x^j} - \frac{\partial F_j^\alpha}{\partial x^j} + \frac{\partial F_i^\alpha}{\partial z^\beta}\frac{\partial z^\beta}{\partial x^j}\right.$$

$$\left. - \frac{\partial F_j^\alpha}{\partial z^\beta} F_i^\beta\right)p^i\, ds, \qquad (10.7)$$

where we have integrated by parts and used the identities

$$\left(\frac{\partial z^\alpha}{\partial x^i} - F_i^\alpha\right)p^i = 0,$$

to obtain the last equality. From these identities it also follows that (10.3) are satisfied for $x = 0$. From (10.2) and (10.7) we have

$$t\left(\frac{\partial z^\alpha}{\partial x^j} - F_j^\alpha\right) = \int_0^t p^i \frac{\partial F_i^\alpha}{\partial z^\beta} s\left(\frac{\partial z^\beta}{\partial x^j} - F_j^\beta\right) ds.$$

Therefore

$$t\left(\frac{\partial z^1}{\partial x^j} - F_j^1\right., \ldots, t\left(\frac{\partial z^m}{\partial x^j} - F_j^m\right)$$

satisfy the differential equations

$$\frac{dy^\alpha}{dt} = a_\beta^\alpha(t)\, y^\beta, \qquad (10.8)$$

where $$a_\beta^\alpha(t) = p^i \frac{\partial F^\alpha(pt, z(pt))}{\partial z^\beta}.$$

These functions are continuous and therefore there is only one solution which vanishes with $t$, namely, $(0, \ldots, 0)$. Therefore

$$\frac{\partial z^\alpha}{\partial x^j} = F_j^\alpha(x, z),$$

and so (10.2) are sufficient conditions for (10.3) to be completely integrable. They are obviously necessary, and so the theorem is established.