

Undergraduate Lecture Notes in De Rham–Hodge Theory

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Abstract

These lecture notes in the De Rham–Hodge theory are designed for a 1–semester undergraduate course (in mathematics, physics, engineering, chemistry or biology). This landmark theory of the 20th Century mathematics gives a rigorous foundation to modern field and gauge theories in physics, engineering and physiology. The only necessary background for comprehensive reading of these notes is Green’s theorem from multivariable calculus.

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1 Exterior geometrical machinery

To grasp the essence of Hodge–De Rham theory, we need first to familiarize ourselves with exterior differential forms and Stokes’ theorem.

1.1 From Green’s to Stokes’ theorem

Recall that *Green’s theorem* in the region C in (x, y) –plane \mathbb{R}^2 connects a line integral $\oint_{\partial C}$ (over the boundary ∂C of C) with a double integral \iint_C over C (see e.g., [1])

$$\oint_{\partial C} Pdx + Qdy = \iint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

In other words, if we define two differential forms (integrands of $\oint_{\partial C}$ and \iint_C) as

$$\begin{aligned} \text{1-form} & : \mathbf{A} = Pdx + Qdy, & \text{and} \\ \text{2-form} & : \mathbf{dA} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy, \end{aligned}$$

(where \mathbf{d} denotes the *exterior derivative* that makes a $(p + 1)$ –form out of a p –form, see next subsection), then we can rewrite Green’s theorem as *Stokes’ theorem*:

$$\int_{\partial C} \mathbf{A} = \int_C \mathbf{dA}.$$

The integration domain C is in topology called a *chain*, and ∂C is a 1D boundary of a 2D chain C . In general, *the boundary of a boundary is zero* (see [11, 12]), that is, $\partial(\partial C) = 0$, or formally $\partial^2 = 0$.

1.2 Exterior derivative

The exterior derivative \mathbf{d} is a generalization of ordinary vector differential operators (*grad*, *div* and *curl* see [9, 10]) that transforms p –forms ω into $(p + 1)$ –forms $\mathbf{d}\omega$ (see next subsection), with the main property: $\mathbf{d}\mathbf{d} = \mathbf{d}^2 = 0$, so that in \mathbb{R}^3 we have (see Figures 1 and 2)

- any scalar function $f = f(x, y, z)$ is a 0–form;
- the gradient $\mathbf{d}f = \omega$ of any smooth function f is a 1–form

$$\omega = \mathbf{d}f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz;$$

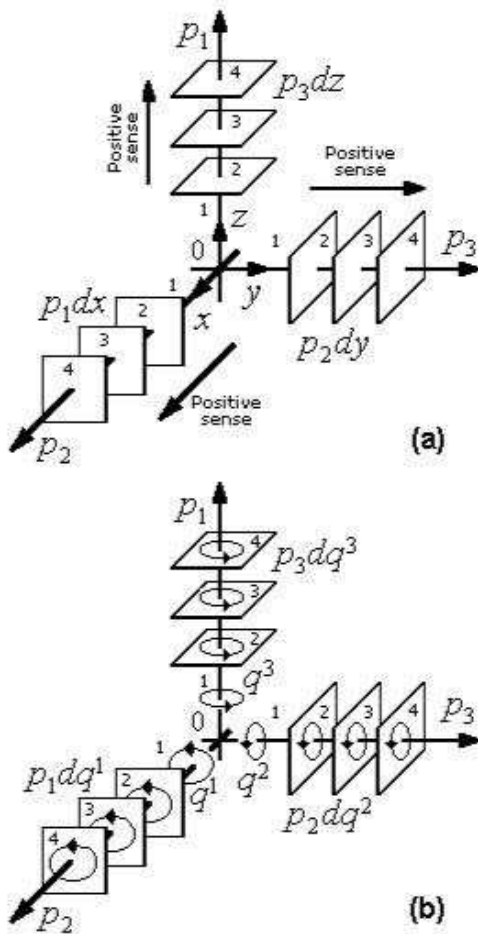


Figure 1: Basis vectors and one-forms in Euclidean \mathbb{R}^3 -space: (a) Translational case; and (b) Rotational case [2]. For the same geometry in \mathbb{R}^3 , see [11].

- the curl $\alpha = \mathbf{d}\omega$ of any smooth 1-form ω is a 2-form

$$\alpha = \mathbf{d}\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy;$$

if $\omega = \mathbf{d}f \Rightarrow \alpha = \mathbf{d}\mathbf{d}f = 0$.

- the divergence $\beta = \mathbf{d}\alpha$ of any smooth 2-form α is a 3-form

$$\beta = \mathbf{d}\alpha = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) dxdydz; \quad \text{if } \alpha = \mathbf{d}\omega \Rightarrow \beta = \mathbf{d}\mathbf{d}\omega = 0.$$

In general, for any two smooth functions $f = f(x, y, z)$ and $g = g(x, y, z)$, the exterior derivative \mathbf{d} obeys the *Leibniz rule* [3, 4]:

$$\mathbf{d}(fg) = g \mathbf{d}f + f \mathbf{d}g,$$

and the *chain rule*:

$$\mathbf{d}(g(f)) = g'(f) \mathbf{d}f.$$

1.3 Exterior forms

In general, given a so-called 4D *coframe*, that is a set of coordinate differentials $\{dx^i\} \in \mathbb{R}^4$, we can define the space of all p -forms, denoted $\Omega^p(\mathbb{R}^4)$, using the exterior derivative $\mathbf{d} : \Omega^p(\mathbb{R}^4) \rightarrow \Omega^{p+1}(\mathbb{R}^4)$ and Einstein's summation convention over repeated indices (e.g., $A_i dx^i = \sum_{i=0}^3 A_i dx^i$), we have:

1-form – a generalization of the Green's 1-form $Pdx + Qdy$,

$$\mathbf{A} = A_i dx^i \in \Omega^1(\mathbb{R}^4).$$

For example, in 4D electrodynamics, \mathbf{A} represents electromagnetic (co)vector potential.

2-form – generalizing the Green's 2-form $(\partial_x Q - \partial_y P) dxdy$ (with $\partial_j = \partial/\partial x^j$),

$$\begin{aligned} \mathbf{B} &= \mathbf{d}\mathbf{A} \in \Omega^2(\mathbb{R}^4), & \text{with components} \\ \mathbf{B} &= \frac{1}{2} B_{ij} dx^i \wedge dx^j, & \text{or} \\ \mathbf{B} &= \partial_j A_i dx^j \wedge dx^i, & \text{so that} \\ B_{ij} &= -2\partial_j A_i = \partial_i A_j - \partial_j A_i = -B_{ji}. \end{aligned}$$

where \wedge is the anticommutative exterior (or, 'wedge') product of two differential forms; given a p -form $\alpha \in \Omega^p(\mathbb{R}^4)$ and a q -form $\beta \in \Omega^q(\mathbb{R}^4)$, their exterior product

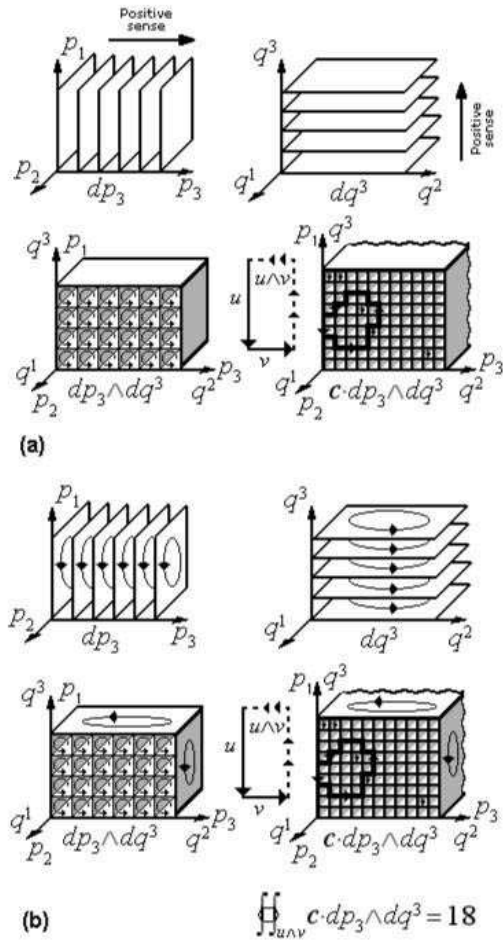


Figure 2: Fundamental two-form and its flux in \mathbb{R}^3 : (a) Translational case; (b) Rotational case. In both cases the flux through the plane $u \wedge v$ is defined as $\int \int_{u \wedge v} c dp_i dq^i$ and measured by the number of tubes crossed by the circulation oriented by $u \wedge v$ [2]. For the same geometry in \mathbb{R}^3 , see [11].

is a $(p + q)$ -form $\alpha \wedge \beta \in \Omega^{p+q}(\mathbb{R}^4)$; e.g., if we have two 1-forms $\mathbf{a} = a_i dx^i$, and $\mathbf{b} = b_i dx^i$, their wedge product $\mathbf{a} \wedge \mathbf{b}$ is a 2-form α given by

$$\alpha = \mathbf{a} \wedge \mathbf{b} = a_i b_j dx^i \wedge dx^j = -a_i b_j dx^j \wedge dx^i = -\mathbf{b} \wedge \mathbf{a}.$$

The exterior product \wedge is related to the exterior derivative $\mathbf{d} = \partial_i dx^i$, by

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathbf{d}\beta.$$

3-form

$$\mathbf{C} = \mathbf{d}\mathbf{B} (= \mathbf{d}\mathbf{d}\mathbf{A} \equiv 0) \in \Omega^3(\mathbb{R}^4), \quad \text{with components}$$

$$\mathbf{C} = \frac{1}{3!} C_{ijk} dx^i \wedge dx^j \wedge dx^k, \quad \text{or}$$

$$\mathbf{C} = \partial_k B_{[ij]} dx^k \wedge dx^i \wedge dx^j, \quad \text{so that}$$

$$C_{ijk} = -6\partial_k B_{[ij]}, \quad \text{where } B_{[ij]} \text{ is the skew-symmetric part of } B_{ij}.$$

For example, in the 4D electrodynamics, \mathbf{B} represents the field 2-form *Faraday*, or the Liénard–Wiechert 2-form (in the next section we will use the standard symbol \mathbf{F} instead of \mathbf{B}) satisfying the sourceless magnetic Maxwell's equation,

$$\text{Bianchi identity : } \mathbf{d}\mathbf{B} = 0, \quad \text{in components } \partial_k B_{[ij]} = 0.$$

4-form

$$\mathbf{D} = \mathbf{d}\mathbf{C} (= \mathbf{d}\mathbf{d}\mathbf{B} \equiv 0) \in \Omega^4(\mathbb{R}^4), \quad \text{with components}$$

$$\mathbf{D} = \partial_l C_{[ijk]} dx^l \wedge dx^i \wedge dx^j \wedge dx^k, \quad \text{or}$$

$$\mathbf{D} = \frac{1}{4!} D_{ijkl} dx^i \wedge dx^j \wedge dx^k \wedge dx^l, \quad \text{so that}$$

$$D_{ijkl} = -24\partial_l C_{[ijk]}.$$

1.4 Stokes theorem

Generalization of the Green's theorem in the plane (and all other integral theorems from vector calculus) is the Stokes theorem for the p -form ω , in an oriented n D domain C (which is a p -chain with a $(p - 1)$ -boundary ∂C , see next section)

$$\int_{\partial C} \omega = \int_C \mathbf{d}\omega.$$

For example, in the 4D Euclidean space \mathbb{R}^4 we have the following three particular cases of the Stokes theorem, related to the subspaces C of \mathbb{R}^4 :

The 2D Stokes theorem:

$$\int_{\partial C^2} \mathbf{A} = \int_{C^2} \mathbf{B}.$$

The 3D Stokes theorem:

$$\int_{\partial C^3} \mathbf{B} = \int_{C^3} \mathbf{C}.$$

The 4D Stokes theorem:

$$\int_{\partial C^4} \mathbf{C} = \int_{C^4} \mathbf{D}.$$

2 De Rham–Hodge theory basics

Now that we are familiar with differential forms and Stokes’ theorem, we can introduce Hodge–De Rham theory.

2.1 Exact and closed forms and chains

Notation change: we drop boldface letters from now on. In general, a p –form β is called *closed* if its exterior derivative $d = \partial_i dx^i$ is equal to zero,

$$d\beta = 0.$$

From this condition one can see that the closed form (the *kernel* of the exterior derivative operator d) is conserved quantity. Therefore, closed p –forms possess certain invariant properties, physically corresponding to the conservation laws (see e.g., [6]).

Also, a p –form β that is an exterior derivative of some $(p - 1)$ –form α ,

$$\beta = d\alpha,$$

is called *exact* (the *image* of the exterior derivative operator d). By Poincaré lemma, exact forms prove to be closed automatically,

$$d\beta = d(d\alpha) = 0.$$

Since $d^2 = 0$, *every exact form is closed*. The converse is only partially true, by Poincaré lemma: every closed form is *locally exact*.

Technically, this means that given a closed p –form $\alpha \in \Omega^p(U)$, defined on an open set U of a smooth manifold M (see Figure 3),¹ any point $m \in U$ has a neighborhood on which there exists a $(p - 1)$ –form $\beta \in \Omega^{p-1}(U)$ such that $d\beta = \alpha|_U$. In particular, there is

¹Smooth manifold is a curved n D space which is locally equivalent to \mathbb{R}^n . To sketch its formal definition, consider a set M (see Figure 3) which is a *candidate* for a manifold. Any point $x \in M^2$ has its *Euclidean chart*, given by a 1–1 and *onto* map $\varphi_i : M \rightarrow \mathbb{R}^n$, with its *Euclidean image* $V_i = \varphi_i(U_i)$. Formally, a chart φ_i is defined by

$$\varphi_i : M \supset U_i \ni x \mapsto \varphi_i(x) \in V_i \subset \mathbb{R}^n,$$

where $U_i \subset M$ and $V_i \subset \mathbb{R}^n$ are open sets.

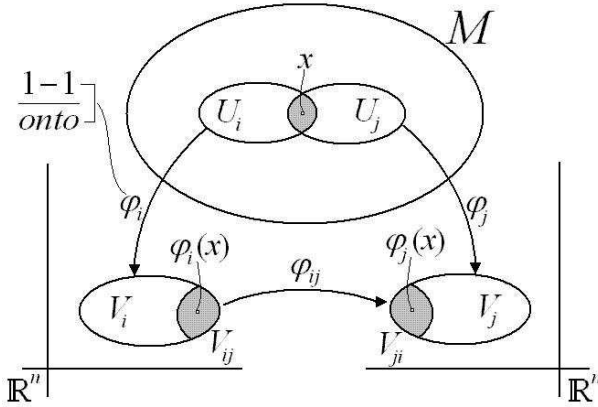


Figure 3: Depicting the manifold concept (see footnote 1).

a Poincaré lemma for contractible manifolds: Any closed form on a smoothly contractible manifold is exact.

The Poincaré lemma is a generalization and unification of two well-known facts in vector calculus:

(i) If $\text{curl } F = 0$, then locally $F = \text{grad } f$; and (ii) If $\text{div } F = 0$, then locally $F = \text{curl } G$.

A *cycle* is a p -chain, (or, an oriented p -domain) $C \in \mathcal{C}_p(M)$ such that $\partial C = 0$. A *boundary* is a chain C such that $C = \partial B$, for any other chain $B \in \mathcal{C}_p(M)$. Similarly, a *cocycle* (i.e., a *closed form*) is a cochain ω such that $d\omega = 0$. A *coboundary* (i.e., an *exact form*) is a cochain ω such that $\omega = d\theta$, for any other cochain θ . All exact forms are closed ($\omega = d\theta \Rightarrow d\omega = 0$) and all boundaries are cycles ($C = \partial B \Rightarrow \partial C = 0$). Converse is true only for smooth contractible manifolds, by Poincaré lemma.

Any point $x \in M$ can have several different charts (see Figure 3). Consider a case of two charts, $\varphi_i, \varphi_j : M \rightarrow \mathbb{R}^n$, having in their images two open sets, $V_{ij} = \varphi_i(U_i \cap U_j)$ and $V_{ji} = \varphi_j(U_i \cap U_j)$. Then we have *transition functions* φ_{ij} between them,

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : V_{ij} \rightarrow V_{ji}, \quad \text{locally given by} \quad \varphi_{ij}(x) = \varphi_j(\varphi_i^{-1}(x)).$$

If transition functions φ_{ij} exist, then we say that two charts, φ_i and φ_j are *compatible*. Transition functions represent a general (nonlinear) *transformations of coordinates*, which are the core of classical *tensor calculus*.

A set of compatible charts $\varphi_i : M \rightarrow \mathbb{R}^n$, such that each point $x \in M$ has its Euclidean image in at least one chart, is called an *atlas*. Two atlases are *equivalent* iff all their charts are compatible (i.e., transition functions exist between them), so their union is also an atlas. A *manifold structure* is a class of equivalent atlases.

Finally, as charts $\varphi_i : M \rightarrow \mathbb{R}^n$ were supposed to be 1-1 and onto maps, they can be either *homeomorphisms*, in which case we have a *topological* (C^0) manifold, or *diffeomorphisms*, in which case we have a *smooth* (C^k) manifold.

2.2 De Rham duality of forms and chains

Integration on a smooth manifold M should be thought of as a nondegenerate bilinear pairing \langle, \rangle between p -forms and p -chains (spanning a finite domain on M). Duality of p -forms and p -chains on M is based on the De Rham's 'period', defined as [9, 7]

$$\text{Period} := \int_C \omega := \langle C, \omega \rangle,$$

where C is a cycle, ω is a cocycle, while $\langle C, \omega \rangle = \omega(C)$ is their inner product $\langle C, \omega \rangle : \Omega^p(M) \times \mathcal{C}_p(M) \rightarrow \mathbb{R}$. From the Poincaré lemma, a closed p -form ω is exact iff $\langle C, \omega \rangle = 0$.

The fundamental topological duality is based on the Stokes theorem,

$$\int_{\partial C} \omega = \int_C d\omega \quad \text{or} \quad \langle \partial C, \omega \rangle = \langle C, d\omega \rangle,$$

where ∂C is the boundary of the p -chain C oriented coherently with C on M . While the *boundary operator* ∂ is a global operator, the coboundary operator d is local, and thus more suitable for applications. The main property of the exterior differential,

$$d \circ d \equiv d^2 = 0 \quad \implies \quad \partial \circ \partial \equiv \partial^2 = 0, \quad (\text{and converse}),$$

can be easily proved using the Stokes' theorem as

$$0 = \langle \partial^2 C, \omega \rangle = \langle \partial C, d\omega \rangle = \langle C, d^2 \omega \rangle = 0.$$

2.3 De Rham cochain and chain complex

In the Euclidean 3D space \mathbb{R}^3 we have the following De Rham *cochain complex*

$$0 \rightarrow \Omega^0(\mathbb{R}^3) \xrightarrow[\text{grad}]{d} \Omega^1(\mathbb{R}^3) \xrightarrow[\text{curl}]{d} \Omega^2(\mathbb{R}^3) \xrightarrow[\text{div}]{d} \Omega^3(\mathbb{R}^3) \rightarrow 0.$$

Using the *closure property* for the exterior differential in \mathbb{R}^3 , $d \circ d \equiv d^2 = 0$, we get the standard identities from vector calculus

$$\text{curl} \cdot \text{grad} = 0 \quad \text{and} \quad \text{div} \cdot \text{curl} = 0.$$

As a duality, in \mathbb{R}^3 we have the following *chain complex*

$$0 \leftarrow \mathcal{C}_0(\mathbb{R}^3) \xleftarrow{\partial} \mathcal{C}_1(\mathbb{R}^3) \xleftarrow{\partial} \mathcal{C}_2(\mathbb{R}^3) \xleftarrow{\partial} \mathcal{C}_3(\mathbb{R}^3) \leftarrow 0,$$

(with the closure property $\partial \circ \partial \equiv \partial^2 = 0$) which implies the following three boundaries:

$$C_1 \xrightarrow{\partial} C_0 = \partial(C_1), \quad C_2 \xrightarrow{\partial} C_1 = \partial(C_2), \quad C_3 \xrightarrow{\partial} C_2 = \partial(C_3),$$

where $C_0 \in \mathcal{C}_0$ is a 0-boundary (or, a point), $C_1 \in \mathcal{C}_1$ is a 1-boundary (or, a line), $C_2 \in \mathcal{C}_2$ is a 2-boundary (or, a surface), and $C_3 \in \mathcal{C}_3$ is a 3-boundary (or, a hypersurface). Similarly, the De Rham complex implies the following three coboundaries:

$$C^0 \xrightarrow{d} C^1 = d(C^0), \quad C^1 \xrightarrow{d} C^2 = d(C^1), \quad C^2 \xrightarrow{d} C^3 = d(C^2),$$

where $C^0 \in \Omega^0$ is 0-form (or, a function), $C^1 \in \Omega^1$ is a 1-form, $C^2 \in \Omega^2$ is a 2-form, and $C^3 \in \Omega^3$ is a 3-form.

In general, on a smooth n D manifold M we have the following De Rham cochain complex [9]

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \rightarrow 0,$$

satisfying the closure property on M , $d \circ d \equiv d^2 = 0$.

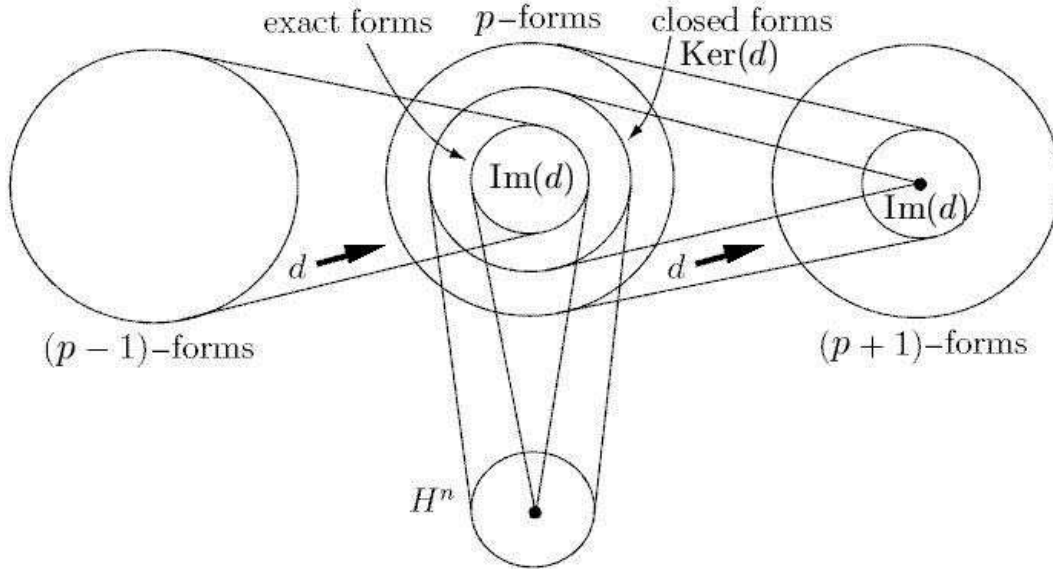


Figure 4: A small portion of the De Rham cochain complex, showing a homomorphism of cohomology groups.

2.4 De Rham cohomology vs. chain homology

Briefly, the De Rham cohomology is the (functional) space of closed differential p -forms modulo exact ones on a smooth manifold.

More precisely, the subspace of all closed p -forms (cocycles) on a smooth manifold M is the kernel $\text{Ker}(d)$ of the De Rham d -homomorphism (see Figure 4), denoted by

$Z^p(M) \subset \Omega^p(M)$, and the sub-subspace of all exact p -forms (coboundaries) on M is the image $\text{Im}(d)$ of the De Rham homomorphism denoted by $B^p(M) \subset Z^p(M)$. The *quotient space*

$$H_{DR}^p(M) := \frac{Z^p(M)}{B^p(M)} = \frac{\text{Ker}(d : \Omega^p(M) \rightarrow \Omega^{p+1}(M))}{\text{Im}(d : \Omega^{p-1}(M) \rightarrow \Omega^p(M))}, \quad (1)$$

is called the p th De Rham *cohomology group* of a manifold M . It is a topological invariant of a manifold. Two p -cocycles $\alpha, \beta \in \Omega^p(M)$ are *cohomologous*, or belong to the same *cohomology class* $[\alpha] \in H^p(M)$, if they differ by a $(p-1)$ -coboundary $\alpha - \beta = d\theta \in \Omega^{p-1}(M)$. The dimension $b_p = \dim H^p(M)$ of the De Rham cohomology group $H_{DR}^p(M)$ of the manifold M is called the Betti number b_p .

Similarly, the subspace of all p -cycles on a smooth manifold M is the kernel $\text{Ker}(\partial)$ of the ∂ -homomorphism, denoted by $Z_p(M) \subset \mathcal{C}_p(M)$, and the sub-subspace of all p -boundaries on M is the image $\text{Im}(\partial)$ of the ∂ -homomorphism, denoted by $B_p(M) \subset \mathcal{C}_p(M)$. Two p -cycles $C_1, C_2 \in \mathcal{C}_p$ are *homologous*, if they differ by a $(p-1)$ -boundary $C_1 - C_2 = \partial B \in \mathcal{C}_{p-1}(M)$. Then C_1 and C_2 belong to the same *homology class* $[C] \in H_p(M)$, where $H_p(M)$ is the homology group of the manifold M , defined as

$$H_p(M) := \frac{Z_p(M)}{B_p(M)} = \frac{\text{Ker}(\partial : \mathcal{C}_p(M) \rightarrow \mathcal{C}_{p-1}(M))}{\text{Im}(\partial : \mathcal{C}_{p+1}(M) \rightarrow \mathcal{C}_p(M))},$$

where Z_p is the vector space of cycles and $B_p \subset Z_p$ is the vector space of boundaries on M . The dimension $b_p = \dim H_p(M)$ of the homology group $H_p(M)$ is, by the De Rham theorem, the same Betti number b_p .

If we know the Betti numbers for all (co)homology groups of the manifold M , we can calculate the *Euler-Poincaré characteristic* of M as

$$\chi(M) = \sum_{p=1}^n (-1)^p b_p.$$

For example, consider a small portion of the De Rham cochain complex of Figure 4 spanning a space-time 4-manifold M ,

$$\Omega^{p-1}(M) \xrightarrow{d_{p-1}} \Omega^p(M) \xrightarrow{d_p} \Omega^{p+1}(M)$$

As we have seen above, cohomology classifies topological spaces by comparing two subspaces of Ω^p : (i) the space of p -cocycles, $Z^p(M) = \text{Ker } d_p$, and (ii) the space of p -coboundaries, $B^p(M) = \text{Im } d_{p-1}$. Thus, for the cochain complex of any space-time 4-manifold we have,

$$d^2 = 0 \quad \Rightarrow \quad B^p(M) \subset Z^p(M),$$

that is, every p -coboundary is a p -cocycle. Whether the converse of this statement is true, according to Poincaré lemma, depends on the particular topology of a space-time

4-manifold. If every p -cocycle is a p -coboundary, so that B^p and Z^p are equal, then the cochain complex is exact at $\Omega^p(M)$. In topologically interesting regions of a space-time manifold M , exactness may fail [13], and we measure the failure of exactness by taking the p th cohomology group

$$H^p(M) = Z^p(M)/B^p(M).$$

2.5 Hodge star operator

The *Hodge star* operator $\star : \Omega^p(M) \rightarrow \Omega^{n-p}(M)$, which maps any p -form α into its *dual* $(n-p)$ -form $\star\alpha$ on a smooth n -manifold M , is defined as (see, e.g. [8])

$$\begin{aligned} \alpha \wedge \star\beta &= \beta \wedge \star\alpha = \langle \alpha, \beta \rangle \mu, & \text{for } \alpha, \beta \in \Omega^p(M), \\ \star\star\alpha &= (-1)^{p(n-p)}\alpha, \\ \star(c_1\alpha + c_2\beta) &= c_1(\star\alpha) + c_2(\star\beta), \\ \alpha \wedge \star\alpha &= 0 \Rightarrow \alpha \equiv 0. \end{aligned}$$

The \star operator depends on the Riemannian metric $g = g_{ij}$ on M^3 and also on the orientation (reversing orientation will change the sign)⁴ [4]. The *volume form* μ is defined in local coordinates on an n -manifold M as (compare with Hodge inner product below)

$$\mu = \text{vol} = \star(1) = \sqrt{\det(g_{ij})} dx^1 \wedge \dots \wedge dx^n, \quad (2)$$

and the total volume on M is given by

$$\text{vol}(M) = \int_M \star(1).$$

For example, in Euclidean \mathbb{R}^3 space with Cartesian (x, y, z) coordinates, we have:

$$\star dx = dy \wedge dz, \quad \star dy = dz \wedge dx, \quad \star dz = dx \wedge dy.$$

The Hodge dual in this case clearly corresponds to the 3D cross-product.

In the 4D-electrodynamics, the dual 2-form *Maxwell* $\star F$ satisfies the electric Maxwell equation with the source [11],

$$\text{Dual Bianchi identity : } d\star F = \star J,$$

where $\star J$ is the 3-form dual to the charge-current 1-form J .

³In local coordinates on a smooth manifold M , the metric $g = g_{ij}$ is defined for any orthonormal basis $(\partial_i = \partial_{x^i})$ in M by

$$g_{ij} = g(\partial_i, \partial_j) = \delta_{ij}, \quad \partial_k g_{ij} = 0.$$

⁴Hodge \star operator is defined locally in an orthonormal basis (coframe) of 1-forms $e_i dx^i$ on a smooth manifold M as:

$$\star(e_i \wedge e_j) = e_k, \quad (\star)^2 = 1.$$

2.6 Hodge inner product

For any two p -forms $\alpha, \beta \in \Omega^p(M)$ with compact support on an n -manifold M , we define bilinear and positive-definite Hodge L^2 -inner product as

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle \star(1) = \int_M \alpha \wedge \star \beta, \quad (3)$$

where $\alpha \wedge \star \beta$ is an n -form. We can extend the product (\cdot, \cdot) to $L^2(\Omega^p(M))$; it remains bilinear and positive-definite, because as usual in the definition of L^2 , functions that differ only on a set of measure zero are identified. The inner product (3) is evidently linear in each variable and symmetric, $(\alpha, \beta) = (\beta, \alpha)$. We have: $(\alpha, \alpha) \geq 0$ and $(\alpha, \alpha) = 0$ iff $\alpha = 0$. Also, $(\star \alpha, \star \beta) = (\alpha, \beta)$. Thus, operation (3) turns the space $\Omega^p(M)$ into an infinite-dimensional inner-product space.

From (3) it follows that for every p -form $\alpha \in \Omega^p(M)$ we can define the *norm functional*

$$\|\alpha\| = \int_M \langle \alpha, \alpha \rangle \star(1) = \int_M \alpha \wedge \star \alpha,$$

for which the *Euler-Lagrangian equation* becomes the Laplace equation (see Hodge Laplacian below),

$$\Delta \alpha = 0.$$

For example, the standard *Lagrangian* for the free Maxwell electromagnetic field, $F = dA$ (where $A = A_i dx^i$ is the electromagnetic potential 1-form), is given by [3, 4, 5]

$$\mathcal{L}(A) = \frac{1}{2}(F \wedge \star F),$$

with the corresponding action

$$S(A) = \frac{1}{2} \int F \wedge \star F.$$

Using the Hodge L^2 -inner product (3), we can rewrite this electrodynamic action as

$$S(A) = \frac{1}{2}(F, F). \quad (4)$$

2.7 Hodge codifferential operator

The Hodge dual (or, formal adjoint) to the exterior derivative $d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ on a smooth manifold M is the *codifferential* δ , a linear map $\delta : \Omega^p(M) \rightarrow \Omega^{p-1}(M)$, which is a generalization of the divergence, defined by [9, 8]

$$\delta = (-1)^{n(p+1)+1} \star d \star, \quad \text{so that} \quad d = (-1)^{np} \star \delta \star.$$

That is, if the dimension n of the manifold M is even, then $\delta = -\star d\star$.

Applied to any p -form $\omega \in \Omega^p(M)$, the codifferential δ gives

$$\delta\omega = (-1)^{n(p+1)+1} \star d\star\omega, \quad \delta d\omega = (-1)^{np+1} \star d\star d\omega.$$

If $\omega = f$ is a 0-form, or function, then $\delta f = 0$. If a p -form α is a codifferential of a $(p+1)$ -form β , that is $\alpha = \delta\beta$, then β is called the *coexact* form. A p -form α is *coclosed* if $\delta\alpha = 0$; then $\star\alpha$ is closed (i.e., $d\star\alpha = 0$) and conversely.

The Hodge codifferential δ satisfies the following set of rules:

- $\delta\delta = \delta^2 = 0$, the same as $dd = d^2 = 0$;
- $\delta\star = (-1)^{p+1} \star d$; $\star\delta = (-1)^p \star d$;
- $d\delta\star = \star\delta d$; $\star d\delta = \delta d\star$.

Standard example is classical electrodynamics, in which the *gauge field* is an electromagnetic potential 1-form (a *connection* on a $U(1)$ -bundle),

$$A = A_\mu dx^\mu = A_\mu dx^\mu + df, \quad (f = \text{arbitrary scalar field}),$$

with the corresponding electromagnetic field 2-form (the *curvature* of the connection A)

$$\begin{aligned} F &= dA, & \text{in components given by} \\ F &= \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, & \text{with } F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu. \end{aligned}$$

Electrodynamics is governed by the *Maxwell equations*, which in exterior formulation read⁵

$$\begin{aligned} dF &= 0, & \delta F &= -4\pi J, & \text{or in components,} \\ F_{[\mu\nu,\eta]} &= 0, & F_{\mu\nu,\mu} &= -4\pi J_\mu, \end{aligned}$$

where comma denotes the partial derivative and the 1-form of electric current $J = J_\mu dx^\mu$ is conserved, by the electrical *continuity equation*,

$$\delta J = 0, \quad \text{or in components,} \quad J_{\mu,\mu} = 0.$$

⁵The first, sourceless Maxwell equation, $dF = 0$, gives vector magnetostatics and magnetodynamics,

$$\begin{aligned} \text{Magnetic Gauss' law} &: \operatorname{div} \mathbf{B} = 0, \\ \text{Faraday's law} &: \partial_t \mathbf{B} + \operatorname{curl} \mathbf{E} = 0. \end{aligned}$$

The second Maxwell equation with source, $\delta F = J$ (or, $d\star F = -\star J$), gives vector electrostatics and electrostatics,

$$\begin{aligned} \text{Electric Gauss' law} &: \operatorname{div} \mathbf{E} = 4\pi\rho, \\ \text{Ampère's law} &: \partial_t \mathbf{E} - \operatorname{curl} \mathbf{B} = -4\pi\mathbf{j}. \end{aligned}$$

2.8 Hodge Laplacian operator

The codifferential δ can be coupled with the exterior derivative d to construct the *Hodge Laplacian* $\Delta : \Omega^p(M) \rightarrow \Omega^p(M)$, a harmonic generalization of the Laplace–Beltrami differential operator, given by⁶

$$\Delta = \delta d + d\delta = (d + \delta)^2.$$

Δ satisfies the following set of rules:

$$\delta \Delta = \Delta \delta = \delta d\delta; \quad d \Delta = \Delta d = d\delta d; \quad \star \Delta = \Delta \star.$$

A p -form α is called *harmonic* iff

$$\Delta \alpha = 0 \Leftrightarrow (d\alpha = 0, \delta\alpha = 0).$$

Thus, α is harmonic in a compact domain $D \subset M^7$ iff it is both closed and coclosed in D . Informally, every harmonic form is both closed and coclosed. As a proof, we have:

$$0 = (\alpha, \Delta \alpha) = (\alpha, d\delta \alpha) + (\alpha, \delta d \alpha) = (\delta \alpha, \delta \alpha) + (d\alpha, d\alpha).$$

Since $(\beta, \beta) \geq 0$ for any form β , $(\delta \alpha, \delta \alpha)$ and $(d\alpha, d\alpha)$ must vanish separately. Thus, $d\alpha = 0$ and $\delta \alpha = 0$.

All harmonic p -forms on a smooth manifold M form the vector space $H_{\Delta}^p(M)$.

Also, given a p -form λ , there is another p -form η such that the equation

$$\Delta \eta = \lambda$$

is satisfied iff for any harmonic p -form γ we have $(\gamma, \lambda) = 0$.

⁶Note that the difference $d - \delta = \partial_D$ is called the *Dirac operator*. Its square ∂_D^2 equals the Hodge Laplacian Δ .

Also, in his QFT-based rewriting the Morse topology, E. Witten [14] considered also the operators:

$$d_t = e^{-tf} de^{tf}, \quad \text{their adjoints: } d_t^* = e^{tf} de^{-tf}, \\ \text{as well as their Laplacian: } \Delta_t = d_t d_t^* + d_t^* d_t.$$

For $t = 0$, Δ_0 is the *Hodge Laplacian*, whereas for $t \rightarrow \infty$, one has the following expansion

$$\Delta_t = dd^* + d^*d + t^2 \|df\|^2 + t \sum_{k,j} \frac{\partial^2 h}{\partial x^k \partial x^j} [i \partial_{x^k}, dx^j],$$

where $(\partial_{x^k})_{k=1,\dots,n}$ is an orthonormal frame at the point under consideration. This becomes very large for $t \rightarrow \infty$, except at the critical points of f , i.e., where $df = 0$. Therefore, the eigenvalues of Δ_t will concentrate near the critical points of f for $t \rightarrow \infty$, and we get an *interpolation* between De Rham cohomology and Morse cohomology.

⁷A domain D is compact if every open cover of D has a finite subcover.

For example, to translate notions from standard 3D vector calculus, we first identify scalar functions with 0-forms, field intensity vectors with 1-forms, flux vectors with 2-forms and scalar densities with 3-forms. We then have the following correspondence:

$$\begin{aligned} \text{grad} &\longrightarrow d : \text{ on 0-forms;} & \text{curl} &\longrightarrow \star d : \text{ on 1-forms;} \\ \text{div} &\longrightarrow \delta : \text{ on 1-forms;} & \text{div grad} &\longrightarrow \Delta : \text{ on 0-forms;} \\ & & \text{curl curl} - \text{grad div} &\longrightarrow \Delta : \text{ on 1-forms.} \end{aligned}$$

We remark here that exact and coexact p -forms ($\alpha = d\beta$ and $\omega = \delta\beta$) are mutually orthogonal with respect to the L^2 -inner product (3). The orthogonal complement consists of forms that are both closed and coclosed: that is, of harmonic forms ($\Delta\gamma = 0$).

2.9 Hodge adjoints and self-adjoints

If α is a p -form and β is a $(p+1)$ -form then we have [9]

$$(d\alpha, \beta) = (\alpha, \delta\beta) \quad \text{and} \quad (\delta\alpha, \beta) = (\alpha, d\beta). \quad (5)$$

This relation is usually interpreted as saying that the two exterior differentials, d and δ , are *adjoint* (or, dual) to each other. This identity follows from the fact that for the volume form μ given by (2) we have $d\mu = 0$ and thus

$$\int_M d(\alpha \wedge \star\beta) = 0.$$

Relation (5) also implies that the Hodge Laplacian Δ is *self-adjoint* (or, self-dual),

$$(\Delta\alpha, \beta) = (\alpha, \Delta\beta),$$

which is obvious as either side is $(d\alpha, d\beta) + (\delta\alpha, \delta\beta)$. Since $(\Delta\alpha, \alpha) \geq 0$, with $(\Delta\alpha, \alpha) = 0$ only when $\Delta\alpha = 0$, Δ is a positive-definite (elliptic) self-adjoint differential operator.

2.10 Hodge decomposition theorem

The celebrated *Hodge decomposition theorem* (HDT) states that, on a compact orientable smooth n -manifold M (with $n \geq p$), any exterior p -form can be written as a unique sum of an *exact* form, a *coexact* form, and a *harmonic* form. More precisely, for any form $\omega \in \Omega^p(M)$ there are unique forms $\alpha \in \Omega^{p-1}(M)$, $\beta \in \Omega^{p+1}(M)$ and a harmonic form $\gamma \in \Omega^p(M)$, such that

$\text{HDT :} \quad \overset{\text{any form}}{\omega} = \overset{\text{exact}}{d\alpha} + \overset{\text{coexact}}{\delta\beta} + \overset{\text{harmonic}}{\gamma}$
--

For the proof, see [8, 9].

In physics community, the exact form $d\alpha$ is called *longitudinal*, while the coexact form $\delta\beta$ is called *transversal*, so that they are mutually orthogonal. Thus any form can be orthogonally decomposed into a harmonic, a longitudinal and transversal form. For example, in fluid dynamics, any vector-field v can be decomposed into the sum of two vector-fields, one of which is divergence-free, and the other is curl-free.

Since γ is harmonic, $d\gamma = 0$. Also, by Poincaré lemma, $d(d\alpha) = 0$. In case ω is a closed p -form, $d\omega = 0$, then the term $\delta\beta$ in HDT is absent, so we have the *short Hodge decomposition*,

$$\omega = d\alpha + \gamma, \quad (6)$$

thus ω and γ differ by $d\alpha$. In topological terminology, ω and γ belong to the same *cohomology class* $[\omega] \in H^p(M)$. Now, by the De Rham theorems it follows that if C is any p -cycle, then

$$\int_C \omega = \int_C \gamma,$$

that is, γ and ω have the same periods. More precisely, if ω is any closed p -form, then there exists a unique harmonic p -form γ with the same periods as those of ω (see [9, 10]).

The *Hodge-Weyl theorem* [8, 9] states that every De Rham cohomology class has a unique harmonic representative. In other words, the space $H_{\Delta}^p(M)$ of harmonic p -forms on a smooth manifold M is isomorphic to the De Rham cohomology group (1), or $H_{\Delta}^p(M) \cong H_{DR}^p(M)$. That is, the harmonic part γ of HDT depends only on the global structure, i.e., the topology of M .

For example, in $(2 + 1)$ D electrodynamics, p -form Maxwell equations in the Fourier domain Σ are written as [15]

$$\begin{aligned} dE &= i\omega B, & dB &= 0, \\ dH &= -i\omega D + J, & dD &= Q, \end{aligned}$$

where H is a 0-form (magnetizing field), D (electric displacement field), J (electric current density) and E (electric field) are 1-forms, while B (magnetic field) and Q (electric charge density) are 2-forms. From $d^2 = 0$ it follows that the J and the Q satisfy the *continuity equation*

$$dJ = iwQ,$$

where $i = \sqrt{-1}$ and w is the field frequency. Constitutive equations, which include all metric information in this framework, are written in terms of Hodge star operators (that fix an isomorphism between p forms and $(2 - p)$ forms in the $(2 + 1)$ case)

$$D = \star E, \quad B = \star H.$$

Applying HDT to the electric field intensity 1-form E , we get [16]

$$E = d\phi + \delta A + \chi,$$

where ϕ is a 0-form (a scalar field) and A is a 2-form; $d\phi$ represents the static field and δA represents the dynamic field, and χ represents the harmonic field component. If domain Σ is contractible, χ is identically zero and we have the short Hodge decomposition,

$$E = d\phi + \delta A.$$

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