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On Hamiltonian geometry of PDEs

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Plan

1. Model
2. Jets
3. Differential equations

Model: a space

Σ is a "space"
$\mathcal{F}(\Sigma)$ is a vector space (of functions on Σ)
$\mathcal{F}(\Sigma) = \Omega^0(\Sigma) \xrightarrow{\delta} \Omega^1(\Sigma) \xrightarrow{\delta} \Omega^2(\Sigma) \xrightarrow{\delta} \cdots$ is a complex (of differential forms)
$\mathcal{D}(\Sigma)$ is a Lie algebra (of vector fields)
$i_X \colon \Omega^k(\Sigma) \to \Omega^{k-1}(\Sigma)$, $X \in \mathcal{D}(\Sigma)$ (interior product)
$L_X = \delta i_X + i_X \delta \colon \Omega^k(\Sigma) \to \Omega^k(\Sigma)$, $X \in \mathcal{D}(\Sigma)$ (Lie derivative)

- $\delta^2 = 0$
- $i_X i_Y = -i_Y i_X$
- $[L_X, i_Y] = i_{[X,Y]}$

These imply

- $\delta L_X = L_X \delta$
- $[L_X, L_Y] = L_{[X,Y]}$
- $\delta\omega(X_1, \ldots, X_k) = \sum_i (-1)^{i+1} L_{X_i}(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_k))$
$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)$

Model: a cotangent space

$\mathcal{T}^*_\Sigma$ is the "cotangent space" to Σ

Grading: $\Omega^p(\mathcal{T}^*_\Sigma) = \sum_{q\geqslant 0} \Omega^p_q(\mathcal{T}^*_\Sigma)$, $\mathcal{D}(\mathcal{T}^*_\Sigma) = \sum_{q\geqslant -1} \mathcal{D}_q(\mathcal{T}^*_\Sigma)$

$\delta(\Omega^p_q(\mathcal{T}^*_\Sigma)) \subset \Omega^p_q(\mathcal{T}^*_\Sigma)$, $i_X(\Omega^p_q(\mathcal{T}^*_\Sigma)) \subset \Omega^p_{q+r}(\mathcal{T}^*_\Sigma)$, $X \in \mathcal{D}_r(\mathcal{T}^*_\Sigma)$

$\mathcal{T}^*_\Sigma$ is a super space:

$i_X i_Y = -(-1)^{XY} i_Y i_X$, $[L_X, i_Y] = L_X i_Y - (-1)^{XY} i_Y L_X$,

$\mathcal{D}(\mathcal{T}^*_\Sigma)$ is a Lie superalgebra, etc.

Symplectic structure: $S \in \Omega^2_1(\mathcal{T}^*_\Sigma)$

- $\delta S = 0$
- $\alpha\colon \mathcal{D}(\mathcal{T}^*_\Sigma)[1] \to \Omega^1(\mathcal{T}^*_\Sigma)$, $X \mapsto i_X S$ is an isomorphism

$X_f = \alpha^{-1}(\delta f)$, $f \in \mathcal{F}(\mathcal{T}^*_\Sigma)$

$\{f, g\} = L_{X_f}(g)$ is the Poisson superbracket on $\mathcal{F}(\mathcal{T}^*_\Sigma)$

$\{\cdot\,,\cdot\}\colon \mathcal{F}_q(\mathcal{T}^*_\Sigma) \times \mathcal{F}_r(\mathcal{T}^*_\Sigma) \to \mathcal{F}_{q+r-1}(\mathcal{T}^*_\Sigma)$

Relation to Σ:

- $\Omega^p_0(\mathcal{T}^*_\Sigma) = \Omega^p(\Sigma)$, $\delta|_{\Omega^p_0(\mathcal{T}^*_\Sigma)}$ coincides with δ on Σ
- $\mathcal{F}_1(\mathcal{T}^*_\Sigma) = \mathcal{D}(\Sigma)$ a Lie algebra isomorphism
- $f \in \mathcal{F}_1(\mathcal{T}^*_\Sigma) = \mathcal{D}(\Sigma)$, $g \in \mathcal{F}_0(\mathcal{T}^*_\Sigma) = \mathcal{F}(\Sigma)$,
  $\{f, g\}_{\text{on } \mathcal{T}^*_\Sigma} = L_f(g)_{\text{on } \Sigma}$

Model: multivectors

By definition, multivectors on Σ are functions on $\mathcal{T}^*_\Sigma$

$$\boxed{\mathcal{D}^q(\Sigma) = \mathcal{F}_q(\mathcal{T}^*_\Sigma)}$$

$\mathcal{D}^0(\Sigma) = \mathcal{F}(\Sigma)$, $\mathcal{D}^1(\Sigma) = \mathcal{D}(\Sigma)$
The Poisson superbracket on $\mathcal{T}^*_\Sigma$ induces *Schouten bracket* on multivectors
$[\![\cdot\,,\cdot]\!] \colon \mathcal{D}^q(\Sigma) \times \mathcal{D}^r(\Sigma) \to \mathcal{D}^{q+r-1}(\Sigma)$
$[\![\theta_1, \theta_2]\!] = -(-1)^{(\theta_1-1)(\theta_2-1)}[\![\theta_2, \theta_1]\!]$
$(-1)^{(\theta_1-1)(\theta_3-1)}[\![\theta_1, [\![\theta_2, \theta_3]\!]]\!] + (-1)^{(\theta_1-1)(\theta_2-1)}[\![\theta_2, [\![\theta_3, \theta_1]\!]]\!] + (-1)^{(\theta_2-1)(\theta_3-1)}[\![\theta_3, [\![\theta_1, \theta_2]\!]]\!] = 0, \quad \theta_1,\ \theta_2,\ \theta_3 \in \mathcal{D}^*(\Sigma)$

$\theta \in \mathcal{D}^2(\Sigma) \quad [\![\theta, \theta]\!] = 0$
Poisson bracket: $\{f, g\}_\theta = [\![[\![\theta, f]\!], g]\!], \quad f,\ g \in \mathcal{F}(\Sigma)$
$H \in \mathcal{F}(\Sigma) \quad X_H = [\![\theta, H]\!] \in \mathcal{D}(\Sigma)$ is a Hamiltonian vector field
$[X_{H_1}, X_{H_2}] = X_{\{H_1,H_2\}} \qquad [\![\theta_1, \theta_2]\!] = 0$ bi-Hamiltonian structure
Magri hierarchy: $[\![\theta_1, H_q]\!] = [\![\theta_2, H_{q+1}]\!]$
$\{H_q, H_s\}_{\theta_1} = \{H_q, H_s\}_{\theta_2} = 0 \qquad [X_{H_q}, X_{H_s}] = 0$

Infinite jet space: notation

The jet space J^∞ with coordinates x^i, u^j_σ

$D_i = \partial_{x^i} + \sum_{j,\sigma} u^j_{\sigma i} \partial_{u^j_\sigma}$ are total derivatives

D_i span the Cartan distribution

$E_\varphi = \sum_j \varphi^j \partial_{u^j} + \sum_{ji} D_i(\varphi^j)\partial_{u^j_i} + \dots$ is an evolutionary field,
$\varphi = (\varphi^1, \dots, \varphi^m)$ is a vector function on J^∞

$\ell_f = \left\| \sum_\sigma \partial_{u^j_\sigma}(f_i) D_\sigma \right\|$ is the linearization
of a vector function f on J^∞, $\ell_f(\varphi) = E_\varphi(f)$

$\Delta^* = \| \sum_\sigma (-1)^\sigma D_\sigma a^{ji}_\sigma \|$, if $\Delta = \| \sum_\sigma a^{ij}_\sigma D_\sigma \|$,
the adjoint $\mathcal{C}$-differential operator

Infinite jet space: the model

$\mathcal{D}(J^\infty) = \varkappa =$ the Lie algebra of evolutionary fields
$\Lambda^q(J^\infty) \supset \mathcal{C}\Lambda^q(J^\infty) \supset \mathcal{C}^2\Lambda^q(J^\infty) \supset \mathcal{C}^3\Lambda^q(J^\infty) \supset \cdots$

$$
\begin{array}{l}
E_1^{0,n} \xrightarrow{d_1^{0,n}} E_1^{1,n} \xrightarrow{d_1^{1,n}} E_1^{2,n} \xrightarrow{d_1^{2,n}} E_1^{3,n} \xrightarrow{d_1^{3,n}} \cdots \\
E_1^{0,n-1} \\
\vdots \\
E_1^{0,0}
\end{array}
$$

n is number of x's

$E_1^{0,n}$ consists of all "actions" $\int L(x^i, u_\sigma^j)\, dx^1 \wedge \cdots \wedge dx^n$
$E_1^{1,n} = \hat{\varkappa}, \quad \hat{\varkappa} = \mathrm{Hom}_{C^\infty(J^\infty)}(\varkappa, \Lambda^n(J^\infty)/\mathcal{C}\Lambda^n(J^\infty))$
$d_1^{0,n}$ is the Euler operator
$E_1^{2,n} = \mathcal{C}^{\mathrm{skew}}(\varkappa, \hat{\varkappa})$
$d_1^{1,n}(\psi) = \ell_\psi - \ell_\psi^*$
$\varphi \in \varkappa \quad \psi \in \hat{\varkappa} \quad i_\varphi(\psi) = \int \psi(\varphi)$
$\varphi \in \varkappa \quad \Delta \in \mathcal{C}^{\mathrm{skew}}(\varkappa, \hat{\varkappa}) \quad i_\varphi(\Delta) = \Delta(\varphi)$

Infinite jet space: the cotangent space

B. A. KUPERSHMIDT, *Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalisms*, Lect. Notes Math. 775, 1980, pp. 162–218

$$\boxed{\mathcal{T}^*_{J^\infty} = J^\infty_h(\hat{\varkappa})}$$

$$S \in \Omega^2(\mathcal{T}^*_{J^\infty}) = \mathcal{C}(\varkappa \oplus \hat{\varkappa}, \varkappa \oplus \hat{\varkappa}) \qquad S(\varphi, \psi) = (-\psi, \varphi)$$

$$\mathcal{D}^2(J^\infty) = \mathcal{C}^{\mathrm{skew}}(\hat{\varkappa}, \varkappa) \qquad A_1,\ A_2 \in \mathcal{D}^2(J^\infty)$$

$$\begin{aligned} [\![A_1, A_2]\!](\psi_1, \psi_2) &= \ell_{A_1,\psi_1}(A_2(\psi_2)) - \ell_{A_1,\psi_2}(A_2(\psi_1)) \\ &+ \ell_{A_2,\psi_1}(A_1(\psi_2)) - \ell_{A_2,\psi_2}(A_1(\psi_1)) \\ &\quad - A_1(\ell^*_{A_2,\psi_2}(\psi_1)) - A_2(\ell^*_{A_1,\psi_2}(\psi_1)), \end{aligned}$$

where $\ell_{A,\psi} = \ell_{A(\psi)} - A\ell_\psi$

$u_t = u_{xxx} + 6uu_x = D_x\delta(u^3 - u_x^2/2) = [\![D_x, \int(u^3 - u_x^2/2)dx]\!]$

Differential equations: notation

Let $F_k(x^i, u^j_\sigma) = 0$, $k = 1, \ldots, l$, be a system of equations

Relations $F = 0$, $D_\sigma(F) = 0$ define its infinite prolongation $\mathcal{E} \subset J^\infty$

$\ell_\mathcal{E} = \ell_F|_\mathcal{E}$ is the linearization of the equation $\mathcal{E}$

E_φ is a symmetry of $\mathcal{E}$ if $E_\varphi(F)|_\mathcal{E} = \ell_\mathcal{E}(\varphi) = 0$, $\mathrm{Sym}(\mathcal{E}) = \ker \ell_\mathcal{E}$
φ is its generating function

Vector function $R = (R^1, \ldots, R^n)$ on $\mathcal{E}$ is a conserved current if $\sum_i D_i(R^i) = 0$ on $\mathcal{E}$
A conserved current is trivial if
$R^i = \sum_{j<i} D_j(T^{ji}) - \sum_{i<j} D_j(T^{ij})$
Conservation laws of $\mathcal{E}$ are the conserved currents modulo trivial ones.
Generating function of a conservation law:
$\psi = (\psi_1, \ldots, \psi_m) = \Delta^*(1)$, where $\sum_i D_i(R^i) = \Delta(F)$ on J^∞

$$\ell^*_\mathcal{E}(\psi) = 0, \qquad \mathrm{CL}(\mathcal{E}) \subset \ker \ell^*_\mathcal{E}$$

Differential equations: the model

$\mathcal{D}(\mathcal{E}) = \mathrm{Sym}(\mathcal{E}) =$ the Lie algebra of symmetries of $\mathcal{E}$

$\Lambda^q(\mathcal{E}) \supset \mathcal{C}\Lambda^q(\mathcal{E}) \supset \mathcal{C}^2\Lambda^q(\mathcal{E}) \supset \mathcal{C}^3\Lambda^q(\mathcal{E}) \supset \cdots$

$$
\begin{array}{lllllllll}
E_1^{0,n} & \xrightarrow{d_1^{0,n}} & E_1^{1,n} & \xrightarrow{d_1^{1,n}} & E_1^{2,n} & \xrightarrow{d_1^{2,n}} & E_1^{3,n} & \xrightarrow{d_1^{3,n}} & \cdots \\
E_1^{0,n-1} & \xrightarrow{d_1^{0,n-1}} & E_1^{1,n-1} & \xrightarrow{d_1^{1,n-1}} & E_1^{2,n-1} & \xrightarrow{d_1^{2,n-1}} & E_1^{3,n-1} & \xrightarrow{d_1^{3,n-1}} & \cdots \\
E_1^{0,n-2} & & & & & & & & \\
\vdots & & & & & & & & \\
E_1^{0,0} & & & & & & & &
\end{array}
$$

$$E_1^{0,n-1} = \text{space of conservation laws}$$
$$E_1^{1,n-1} = \mathrm{Cosym}\,\mathcal{E} = \ker \ell_{\mathcal{E}}^*$$
$$E_1^{2,n-1} = \{\,\Delta \mid \ell_{\mathcal{E}}^*\Delta = \Delta^*\ell_{\mathcal{E}}\,\}\big/\{\,\nabla\,\ell_{\mathcal{E}} \mid \nabla^* = \nabla\,\}$$

Differential equations: the cotangent space

$\mathcal{T}^*_{\mathcal{E}}$: $\quad F = 0, \quad \ell^*_{\mathcal{E}}(\boldsymbol{p}) = 0$

$\mathcal{L} = \langle F, \boldsymbol{p} \rangle \qquad \ell^*_{\mathcal{T}^*_{\mathcal{E}}} = \ell_{\mathcal{T}^*_{\mathcal{E}}} \qquad S = \mathrm{id}$

Variational multivectors on $\mathcal{E}$ are conservation laws on $\mathcal{T}^*_{\mathcal{E}}$.

Theorem

A variational bivector on $\mathcal{E}$ can be identified with the equivalence class of operators A on $\mathcal{E}$ that satisfy the condition

$$\ell_{\mathcal{E}} A = A^* \ell^*_{\mathcal{E}},$$

with two operators being equivalent if they differ by an operator of the form $\square \, \ell^*_{\mathcal{E}}$.

If A is a bivector and $\mathcal{E}$ is written in evolution form then $A^* = -A$.

Differential equations: the Schouten bracket of bivectors

$$
\begin{aligned}
&[\![A_1, A_2]\!](\psi_1, \psi_2) \\
&\quad = \ell_{A_1,\psi_1}(A_2(\psi_2)) - \ell_{A_1,\psi_2}(A_2(\psi_1)) \\
&\quad + \ell_{A_2,\psi_1}(A_1(\psi_2)) - \ell_{A_2,\psi_2}(A_1(\psi_1)) \\
&\qquad - A_1(B_2^*(\psi_1, \psi_2)) - A_2(B_1^*(\psi_1, \psi_2)),
\end{aligned}
$$

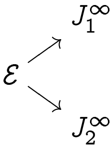
where $\ell_F A_i - A_i^* \ell_F^* = B_i(F, \cdot)$ on J^∞,

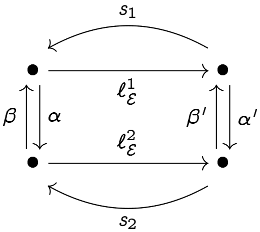
$B_i^*(\psi_1, \psi_2) = B_i^{*_1}(\psi_1, \psi_2)|_{\mathcal{E}}$.

B_i^* are skew-symmetric and skew-adjoint in each argument.

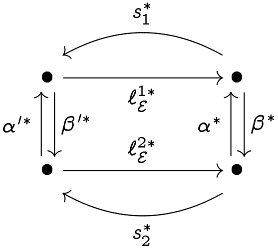
If $\mathcal{E}$ is in evolution form then $B_i^*(\psi_1, \psi_2) = \ell^*_{A_i,\psi_2}(\psi_1)$

Invariance of the cotangent equation





$$\ell_{\mathcal{E}}^1\,\beta = \beta'\,\ell_{\mathcal{E}}^2, \quad \ell_{\mathcal{E}}^2\,\alpha = \alpha'\,\ell_{\mathcal{E}}^1, \qquad \beta\,\alpha = \mathrm{id} + s_1\,\ell_{\mathcal{E}}^1, \quad \alpha\,\beta = \mathrm{id} + s_2\,\ell_{\mathcal{E}}^2.$$



$$\alpha'^*\,\beta'^* = \mathrm{id} + s_1^*\,\ell_{\mathcal{E}}^{1*}, \quad \beta'^*\,\alpha'^* = \mathrm{id} + s_2^*\,\ell_{\mathcal{E}}^{2*}.$$

Invariance of the cotangent equation

Theorem

If $\ell_{\mathcal{E}}^{1}$ is equivalent to $\ell_{\mathcal{E}}^{2}$ then $\ell_{\mathcal{E}}^{1*}$ is equivalent to $\ell_{\mathcal{E}}^{2*}$.

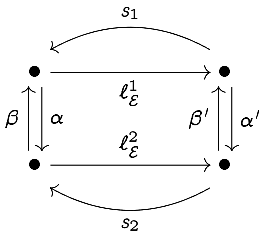
Corollary

$\mathcal{T}_{\mathcal{E}}^{*}$ doesn't depend on the inclusion $\mathcal{E} \to J^{\infty}$.

$$A^2 = \alpha\, A^1\, \alpha'^*$$

$$A^1 = \beta\, A^2\, \beta'^*$$

Example: KdV



$$F_1 = u_t - u_{xxx} - 6uu_x = 0$$

$$F_2 = \begin{pmatrix} u_x - v \\ v_x - w \\ w_x - u_t + 6uv \end{pmatrix} = 0$$

$$\ell^1_{\mathcal{E}} = D_t - D_{xxx} - 6uD_x - 6u_x \qquad \ell^2_{\mathcal{E}} = \begin{pmatrix} D_x & -1 & 0 \\ 0 & D_x & -1 \\ -D_t + 6v & 6u & D_x \end{pmatrix}$$

$$\alpha = \begin{pmatrix} 1 \\ D_x \\ D_{xx} \end{pmatrix} \quad \alpha' = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad \begin{aligned} \beta &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \\ \beta' &= \begin{pmatrix} -D_{xx} - 6u & -D_x & -1 \end{pmatrix} \end{aligned}$$

$$s_1 = 0 \quad s_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ D_x & 1 & 0 \end{pmatrix}$$

Example: KdV

$$u_t = u_{xxx} + 6uu_x = D_x\delta(u^3 - u_x^2/2)$$
$$= (D_{xxx} + 4uD_x + 2u_x)\delta(u^2/2)$$

$$u_x = v, \quad v_x = w, \quad w_x = u_t - 6uv$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_x = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -6u \\ 0 & 6u & D_t \end{pmatrix} \delta\left(uw - v^2/2 + 2u^3\right)$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_x = \begin{pmatrix} 0 & -2u & -D_t - 2v \\ 2u & D_t & -12u^2 - 2w \\ -D_t + 2v & 12u^2 + 2w & 8uD_t + 4u_t \end{pmatrix} \delta\left(-3u^2/2 - w/2\right)$$

S. P. Tsarev, *The Hamilton property of stationary and inverse equations of condensed matter mechanics and mathematical physics*, Math. Notes **46** (1989), 569–573

Example: Camassa-Holm equation

$$u_t - u_{txx} - uu_{xxx} - 2u_x u_{xx} + 3uu_x = 0$$

$$A_1 = D_x \qquad A_2 = -D_t - uD_x + u_x.$$

$$m_t + um_x + 2u_x m = 0,$$
$$m - u + u_{xx} = 0$$

$$u = (1 - D_x^2)^{-1} m$$

Example: Camassa-Holm equation

$$u_t - u_{txx} - uu_{xxx} - 2u_x u_{xx} + 3uu_x = 0$$

$$A_1 = D_x \qquad A_2 = -D_t - uD_x + u_x.$$

$$m_t + um_x + 2u_x m = 0,$$
$$m - u + u_{xx} = 0$$

$$u = (1 - D_x^2)^{-1} m$$

$$A_1' = \begin{pmatrix} D_x & 0 \\ D_x - D_x^3 & 0 \end{pmatrix} \qquad A_2' = \begin{pmatrix} 0 & -1 \\ 2mD_x + m_x & 0 \end{pmatrix}$$

Example: Kupershmidt deformation

B.A.Kupershmidt, *KdV6: An integrable system*, Phys. Lett. A **372** (2008), 2634–2639

$$u_t = f(t, x, u, u_x, u_{xx}, \dots)$$

A_1, A_2 are compatible Hamiltonian operators
H_1, H_2, ... is a Magri hierarchy of conserved densities
$D_t(H_i) = 0$, $A_1\,\delta(H_i) = A_2\,\delta(H_{i+1})$.

$$u_t = f - A_1(w), \quad A_2(w) = 0 \qquad (*)$$

The KdV6 equation
(A. Karasu-Kalkanlı, A. Karasu, A. Sakovich, S. Sakovich, and R. Turhan, *A new integrable generalization of the Korteweg-de Vries equation*, J. Math. Phys. **49** (2008) 073516)

$$u_t = u_{xxx} + 6uu_x - w_x, \quad w_{xxx} + 4uw_x + 2u_x w = 0$$

Theorem (Kupershmidt)

H_1, H_2, ... *are conserved densities for* $(*)$.

Example: Kupershmidt deformation

Let $\mathcal{E}$ be a bi-Hamiltonian equation given by $F = 0$

Definition

The Kupershmidt deformation $\tilde{\mathcal{E}}$ has the form

$$F + A_1^*(w) = 0, \qquad A_2^*(w) = 0,$$

where $w = (w^1, \dots, w^l)$ are new dependent variables

Theorem

The Kupershmidt deformation $\tilde{\mathcal{E}}$ is bi-Hamiltonian.

Proof.

The following two bivectors define a bi-Hamiltonian structures:

$$\tilde{A}_1 = \begin{pmatrix} A_1 & -A_1 \\ 0 & \ell_{F+A_1^*(w)+A_2^*(w)} \end{pmatrix} \quad \tilde{A}_2 = \begin{pmatrix} A_2 & -A_2 \\ -\ell_{F+A_1^*(w)+A_2^*(w)} & 0 \end{pmatrix}$$

□

Example: Baran-Marvan equation

H. Baran and M. Marvan, *On integrability of Weingarten surfaces: a forgotten class*, J. Phys. A: Math. Theor. **42** (2009), 404007

$z_{yy} + (1/z)_{xx} + 2 = 0$

D_x^2

$2zD_{xy} - z_y D_x + z_x D_y.$

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