# Representation of Lie Groups and Special Functions

## Volume 2: Class I Representations, Special Functions, and Integral Transforms


*by*

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## SERIES EDITOR'S PREFACE

'Et moi, ..., si j'avait su comment en revenir, je n'y serais point allé.'

Jules Verne

The series is divergent; therefore we may be able to do something with it.

O. Heaviside

One service mathematics has rendered the human race. It has put common sense back where it belongs, on the topmost shelf next to the dusty canister labelled 'discarded non-sense'.

Eric T. Bell

Mathematics is a tool for thought. A highly necessary tool in a world where both feedback and non-linearities abound. Similarly, all kinds of parts of mathematics serve as tools for other parts and for other sciences.

Applying a simple rewriting rule to the quote on the right above one finds such statements as: 'One service topology has rendered mathematical physics ...'; 'One service logic has rendered computer science ...'; 'One service category theory has rendered mathematics ...'. All arguably true. And all statements obtainable this way form part of the raison d'être of this series.

This series, *Mathematics and Its Applications*, started in 1977. Now that over one hundred volumes have appeared it seems opportune to reexamine its scope. At the time I wrote

> "Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the 'tree' of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related. Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as 'experimental mathematics', 'CFD', 'completely integrable systems', 'chaos, synergetics and large-scale order', which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics."

By and large, all this still applies today. It is still true that at first sight mathematics seems rather fragmented and that to find, see, and exploit the deeper underlying interrelations more effort is needed and so are books that can help mathematicians and scientists do so. Accordingly MIA will continue to try to make such books available.

If anything, the description I gave in 1977 is now an understatement. To the examples of interaction areas one should add string theory where Riemann surfaces, algebraic geometry, modular functions, knots, quantum field theory, Kac-Moody algebras, monstrous moonshine (and more) all come together. And to the examples of things which can be usefully applied let me add the topic 'finite geometry'; a combination of words which sounds like it might not even exist, let alone be applicable. And yet it is being applied: to statistics via designs, to radar/sonar detection arrays (via finite projective planes), and to bus connections of VLSI chips (via difference sets). There seems to be no part of (so-called pure) mathematics that is not in immediate danger of being applied. And, accordingly, the applied mathematician needs to be aware of much more. Besides analysis and numerics, the traditional workhorses, he may need all kinds of combinatorics, algebra, probability, and so on.

In addition, the applied scientist needs to cope increasingly with the nonlinear world and the extra mathematical sophistication that this requires. For that is where the rewards are. Linear models are honest and a bit sad and depressing: proportional efforts and results. It is in the non-linear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I am hinting at: if electronics were linear we would have no fun with transistors and computers; we would have no TV; in fact you would not be reading these lines.

There is also no safety in ignoring such outlandish things as nonstandard analysis, superspace and anticommuting integration, $p$-adic and ultrametric space. All three have applications in both electrical engineering and physics. Once, complex numbers were equally outlandish, but they frequently proved the shortest path between 'real' results. Similarly, the first two topics named have already provided a number of 'wormhole' paths. There is no telling where all this is leading - fortunately.

Thus the original scope of the series, which for various (sound) reasons now comprises five sub-series: white (Japan), yellow (China), red (USSR), blue (Eastern Europe), and green (everything else), still applies. It has been enlarged a bit to include books treating of the tools from one subdiscipline which are used in others. Thus the series still aims at books dealing with:

- a central concept which plays an important role in several different mathematical and/or scientific specialization areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have, and have had, on the development of another.

Special functions are - well - special. They turn up all over the place in both theoretical and practical investigations and their importance is well illustrated by the fact that scores of them have received special names. For instance, Bessel functions; Jacobi, Legendre, Gegenbauer, Laguerre polynomials, Hamkel and Macdonald functions; Whittaker functions; Krawtchouk and Meixner polynomials; Chebyshev polynomials; Hahn and Racah polynomials; etc.

Both the ubiquity and the special properties of these functions were something of a mystery until the great discovery of Wigner, Miller, and Vilenkin, one of the authors of the present volume, that, especially, these functions arise as the coefficients of representations of groups. This tied two apparently rather disparate parts of mathematics tightly together and enormously stimulated developments in both fields. Since then ('the 1960's) very much has happened: for instance, orthogonal polynomials in several variables, discrete analogues of special functions, and, quite recently, the discovery that $q$-special functions relate to quantum groups (Hopf algebras) in the same way as special functions to (Lie) groups.

The authors have undertaken the monumental task to survey and describe in three volumes all that is known in this area. This is the second volume of this complete, self-contained, and encyclopaedic treatise.

The shortest path between two truths in the real domain passes through the complex domain.

J. Hadamard

La physique ne nous donne pas seulement l'occasion de résoudre des problèmes ... elle nous fait pressentir la solution.

H. Poincaré

Never lend books, for no one ever returns them; the only books I have in my library are books that other folk have lent me.

Anatole France

The function of an expert is not to be more right than other people, but to be wrong for more sophisticated reasons.

David Butler

Amsterdam

Michiel Hazewinkel

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# List of the Most Important Notations

| | |
|---|---|
| $(a;q)_n$ | expressions appearing in the definition of basic hypergeometric functions (Sec. 13.2.2). |
| $(a;q)_\infty$ | limit of $(a;q)_n$ for $n \to \infty$ (Sec. 13.2.2). |
| $\mathfrak{B}^{n\sigma}$ | space of smooth homogeneous functions on $C_+^{n-1}$ (Sec. 9.2.1). |
| $\mathfrak{B}^{n\sigma k}$ | space of smooth homogeneous functions on $C_C^{n-1}$ (Sec. 11.2.4). |
| $C_+^{n-1}$ | cone in $E_{n-1,1}$ (Sec. 9.1.1). |
| $C_C^{n-1}$ | cone in $E_{n-1,1}^C$ (Sec. 11.1.1). |
| $\tilde{C}^{n-1}$ | complex cone in $\mathbb{C}^n$ (Sec. 9.2.2). |
| $C^{pq}$ | cone in $E_{pq}$ (Sec. 9.1.4). |
| $E_n$ | $n$-dimensional Euclidean space (Sec. 9.1.1). |
| $E_n^C$ | $n$-dimensional unitary space (Sec. 11.1.1). |
| $E_n(x;a,b,c)$ | polynomials (Sec. 13.1.6). |
| $E_{pq}$ | $p+q$-dimensional space of signature $(p,q)$ (Sec. 9.1.1). |
| $E_{n-1,1}^C$ | complex space of signature $(n-1,1)$ (Sec. 11.1.1). |
| $GF(q)$ | Galois field (Sec. 13.2.1). |
| $g_i(\theta)$ | rotation in the plane $(i,i+1)$ (Sec. 9.1.1). |
| $g_i'(\theta)$ | hyperbolic rotation in the plane $(i,i+1)$ (Sec. 9.1.1). |
| $g_{ij}(\theta)$ | rotation in the plane $(i,j)$ (Sec. 9.1.1). |
| $g_{ij}'(\theta)$ | hyperbolic rotation in the plane $(i,j)$ (Sec. 9.1.1). |
| $G_{pq}^{mn}$ | Maijer $G$-function (Sec. 10.5.1). |
| $H_\pm^{n-1}$ | hyperboloids $[\mathbf{x},\mathbf{x}]=\pm 1$ in $E_{n-1,1}$ (Sec. 9.1.1). |
| $H_C^{n-1}$ | hyperboloid $[\mathbf{z},\mathbf{z}]=1$ in $E_{n-1,1}^C$ (Sec. 11.1.1). |
| $H_\pm^{pq}$ | hyperboloids $[\mathbf{x},\mathbf{x}]_{pq}=\pm 1$ in $E_{pq}$ (Sec. 9.1.4). |
| $\mathfrak{H}^{n\ell}$ | space of homogeneous harmonic polynomials of degree $\ell$ on $E_n$ (Sec. 9.2.3). |
| $\tilde{\mathfrak{H}}^{n\ell}$ | space of restrictions of polynomials from $\mathfrak{H}^{n\ell}$ onto $S^{n-1}$ (Sec. 9.2.3). |
| $\mathfrak{H}_C^{n\ell\ell'}$ | space of homogeneous harmonic polynomials in $z$ and $\bar{z}$ (Sec. 11.2.1). |
| $IO(\infty)$ | infinite dimensional motion group (Sec. 9.6.10). |
| $K_j(q^{-x};c,N;q)$ | $q$-Krawtchouk polynomials (Sec. 13.2.2). |
| $\begin{bmatrix} n \\ m \end{bmatrix}_q$ | $q$-analog of binomial coefficients (Sec. 13.2.2). |

| | |
|---|---|
| $0(\infty)$ | infinite dimensional rotation group (Sec. 9.6.1) |
| $\mathcal{P}_\sigma$ | Poisson transform (Sec. 10.3.1). |
| $Q_n(q^{-x}; a, b, N; q)$ | $Q$-Hahn polynomials (Sec. 13.2.2). |
| $\mathfrak{R}^{n\ell}$ | space of homogeneous polynomials of degree $\ell$ on $E_n$ (Sec. 9.2.3). |
| $\mathfrak{R}_C^{n\ell\ell'}$ | space of homogeneous polynomials on $E_n^C$ (Sec. 11.2.1). |
| $S_n$ | symmetric group (Sec. 13.1.1). |
| $\lvert X \rvert$ | number of elements of a set $X$ (Sec. 13.1.1). |
| $[\mathbf{x}, \mathbf{y}]$ | bilinear form of signature $(n-1, 1)$ (Sec. 9.1.1). |
| $[\mathbf{x}, \mathbf{y}]_{pq}$ | bilinear form of signature $(p, q)$ (Sec. 9.1.4). |
| $\Delta$ | Laplace operator on $E_n$ and $E_n^C$ (Sec. 9.1.8). |
| $\Delta_0$ | Laplace operator on $S^{n-1}$ (Sec. 9.1.8). |
| $\Box$ | wave operator on $E_{n-1,1}$ (Secs. 9.1.8, 11.1.7). |
| $\Box_\pm$ | Laplace operator on $H_\pm^{n-1}$ (Sec. 9.1.8). |
| $\Box_0$ | Laplace operator on $H_+^{pq}$ and on $H_C^{n-1}$ (Secs. 9.1.9, 11.1.7). |
| $\Xi_K^{n\ell}$ | basis functions in $\mathfrak{L}^2(S^{n-1})$ (Sec. 9.3.1). |
| $\Xi_K^{n\ell\ell'}$ | basis functions in $\mathfrak{L}^2(S_C^{n-1})$ (Sec. 11.3.1). |
| $\Xi_M^{nR}$ | functions on $\mathbb{R}^{n-1}$ (Sec. 9.4.2). |
| $\Xi_M^{n\sigma}$ | functions on $H_+^{n-1}$ (Sec. 9.4.2). |

# Chapter 9.
# Special Functions Connected with $SO(n)$ and with Related Groups

## 9.1. Groups Related to $SO(n)$ and Corresponding Homogeneous Spaces

**9.1.1. The groups $SO(n)$, $SO_0(n-1,1)$ and corresponding spaces.** Let $E_n$ be the $n$-dimensional real Euclidean space with the scalar product $(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{n} x_k y_k$. Let us recall that $O(n)$ denotes the group of orthogonal transformations in $E_n$, that is, linear transformations in $E_n$ preserving $(\mathbf{x}, \mathbf{y})$, and $SO(n)$ denotes the subgroup of unimodular transformations from $O(n)$. The groups $O(n)$ and $SO(n)$ are compact, $SO(n)$ is connected and $O(n)$ consists of two connected components $SO(n)$ and $g_0 SO(n)$, where $g_0 = \mathrm{diag}(1, \ldots, 1, -1)$.

The action of $SO(n)$ splits $E_n$ into orbits. They are spheres of radius $r$, $0 \le r < \infty$. The sphere of unit radius will be denoted by $S^{n-1}$. Vectors of $E_n$, which have unit length, will be denoted by $\boldsymbol{\xi}, \boldsymbol{\eta}$ and so on. The stabilizer of the basis vector $\mathbf{e}_n = (0, \ldots, 0, 1)$ is isomorphic to $SO(n-1)$. Hence, $S^{n-1} = SO(n)/SO(n-1)$. Because of compactness of $SO(n)$, the sphere $S^{n-1}$ is a symmetric Riemannian space of compact type (see Section 1.2.3). The action of $SO(n)$ on $SO(n)/SO(n-1)$ by left shifts defines rotations of $S^{n-1}$.

The rotation by the angle $\theta$ in the plane $(x_i, x_j)$ will be denoted by $g_{ij}(\theta)$ and $g_{i,i+1}(\theta) \equiv g_i(\theta)$. The tangent matrices to the subgroups $\{g_{ij}(\theta)\}$ at $\theta = 0$ are of the form

$$\left.\frac{d}{d\theta} g_{ij}(\theta)\right|_{\theta=0} = e_{ij} - e_{ji} \equiv I_{ij}, \tag{1}$$

where $e_{ij}$ is the matrix with entries $(e_{ij})_{st} = \delta_{is}\delta_{jt}$. The matrices $I_{ij}$, $i < j$, form a basis of the Lie algebra $\mathfrak{so}(n)$ of the group $SO(n)$. Hence, $\mathfrak{so}(n)$ consists of skew-symmetric matrices of order $n$ (see Example 2 of Section 1.1.3). One has the commutation relations

$$[I_{ij}, I_{ks}] = \delta_{jk} I_{is} - \delta_{js} I_{ik} - \delta_{ik} I_{js} + \delta_{is} I_{jk}, \tag{2}$$

where we set $I_{qp} = -I_{pq}$ if $q > p$.

The element

$$C = \sum_{i<j} I_{ij}^2 \tag{3}$$

of the universal enveloping algebra $\mathfrak{U}(\mathfrak{so}(n))$ belongs to the center of this algebra:

$$[C, I_{st}] = 0, \qquad 1 \le s < t \le n.$$

It is called the *quadratic Casimir operator of the Lie algebra* $\mathfrak{so}(n)$.

Let us denote by $E_{n-1,1}$ the $n$-dimensional real linear space with the bilinear form

$$[\mathbf{x},\mathbf{y}] = -x_1y_1 - \ldots - x_{n-1}y_{n-1} + x_ny_n.$$

With the help of $[\mathbf{x},\mathbf{y}]$ one can define a distance in $E_{n-1,1}$ by setting

$$r^2(\mathbf{x},\mathbf{y}) = [\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y}]. \tag{4}$$

It is evident that $r^2(\mathbf{x},\mathbf{y})$ can be positive, negative or zero.

The set $C^{n-1}$ of points from $E_{n-1,1}$, which are at a distance zero from the origin $O$, forms a cone. The upper sheet $C_+^{n-1}$ of this cone consists of points $\mathbf{x}$ for which $[\mathbf{x},\mathbf{x}] = 0$ and $x_n > 0$. Inside of $C^{n-1}$ we have $[\mathbf{x},\mathbf{x}] > 0$ and outside of $C^{n-1}$ we have $[\mathbf{x},\mathbf{x}] < 0$.

The group of linear transformations of $E_{n-1,1}$, preserving $[\mathbf{x},\mathbf{y}]$, is denoted by $O(n-1,1)$. The subgroup, consisting of unimodular transformations from $O(n-1,1)$, is denoted by $SO(n-1,1)$. The subgroup of transformations, preserving both sheets of the cone $C^{n-1}$, is denoted by $O_0(n-1,1)$. One sets

$$SO_0(n-1,1) = SO(n-1,1) \cap O_0(n-1,1).$$

The group $SO_0(n-1,1)$ is connected and locally compact.

The action of $SO_0(n-1,1)$ splits the space $E_{n-1,1}$ into orbits of the following forms: 1) upper sheets of two-sheeted hyperboloids, 2) lower sheets of the same hyperboloids, 3) one-sheeted hyperboloids, 4) the upper sheet of the cone, 5) the lower sheet of the cone, 6) the origin.

Orbits for $O(n-1,1)$ are obtained by joining orbits for $SO_0(n-1,1)$, symmetric with respect to the plane $x_n = 0$.

The upper sheet of the hyperboloid $[\mathbf{x},\mathbf{x}] = 1$ will be denoted by $H_+^{n-1}$ and points of $H_+^{n-1}$ by $\boldsymbol{\xi}, \boldsymbol{\eta}, \ldots$. The subgroup of $SO_0(n-1,1)$, which leaves the basis vector $\mathbf{e}_n = (0,\ldots,0,1)$ fixed, is isomorphic to $SO(n-1)$. It is a maximal compact subgroup of $SO_0(n-1,1)$. Therefore, $H_+^{n-1} = SO_0(n-1,1)/SO(n-1)$ is a symmetric Riemannian space of noncompact type (see Section 1.2.3). The symmetric Riemannian spaces $S^{n-1}$ and $H_+^{n-1}$ are dual by Cartan.

Since the invariance subgroup of $\mathbf{e}_{n-1} = (0,\ldots,0,1,0)$ is isomorphic to $SO_0(n-2,1)$, then the one-sheet hyperboloid

$$H_-^{n-1} = \{\mathbf{x} \in H_{n-1,1} \mid [\mathbf{x},\mathbf{x}] = -1\}$$

is the homogeneous space $SO_0(n-1,1)/SO_0(n-2,1)$. This space is pseudo-Riemannian. The action of $SO_0(n-1,1)$ by left shifts in $SO_0(n-1,1)/SO(n-1)$ defines motions of $H_+^{n-1}$. In the same way the action of $SO_0(n-1,1)$ in $SO_0(n-1,1)/SO_0(n-2,1)$ defines motions of $H_-^{n-1}$.

Let us choose the point $\mathbf{a} = (0, \ldots, 0, 1, 1)$ on $C_+^{n-1}$. The stabilizer of $\mathbf{a}$ in $SO_0(n-1,1)$ consists of transformations of the form $g = mn$, $m \in M \equiv SO(n-2)$, $n \in N$, where $N$ is a subgroup of matrices

$$n \equiv n(\mathbf{t}) = \begin{pmatrix} I_{n-2} & -\mathbf{t}^T & \mathbf{t}^T \\ \mathbf{t} & 1-\frac{t^2}{2} & \frac{t^2}{2} \\ \mathbf{t} & -\frac{t^2}{2} & 1+\frac{t^2}{2} \end{pmatrix}, \tag{5}$$

$$\mathbf{t} = (t_1, \ldots, t_{n-2}), \quad t_j \in \mathbf{R}, \quad t^2 = t_1^2 + \ldots + t_{n-2}^2.$$

Therefore,

$$C_+^{n-1} = SO_0(n-1,1)/MN.$$

Replacing $\mathbf{a}$ by $\mathbf{b} = (0, \ldots, 0, -1, 1)$, we find that

$$C_+^{n-1} = SO_0(n-1,1)/M\bar{N},$$

where the subgroup $\bar{N}$ consists of matrices

$$\bar{n} \equiv \bar{n}(\mathbf{t}) = \begin{pmatrix} I_{n-2} & -\mathbf{t}^T & -\mathbf{t}^T \\ \mathbf{t} & 1-\frac{t^2}{2} & -\frac{t^2}{2} \\ \mathbf{t} & \frac{t^2}{2} & 1+\frac{t^2}{2} \end{pmatrix}.$$

The action of the group $SO_0(n-1,1)$ on $H_+^{n-1}$ determines its action on the set $X_+^{n-1}$ of rays going out from the origin and lying inside of $C_+^{n-1}$. The action of $SO_0(n-1,1)$ on any manifold, lying inside of $C_+^{n-1}$ and intersecting every ray from $X_+^{n-1}$ at one point, is defined also. In particular, the action is determined of $SO_0(n-1,1)$ on the section of the interior of the cone $C_+^{n-1}$ by the plane $x_n = 1$, that is, on the unit ball

$$D^{n-1} = \left\{ \mathbf{y} = (y_1, \ldots, y_{n-1}) \;\middle|\; \sum_{k=1}^{n-1} y_k^2 < 1 \right\}.$$

In the same way one defines the action of $SO_0(n-1,1)$ on the exterior of this ball and on the sphere $S^{n-2}$. We recommend to the reader to write down related formulas.

The invariance subgroup for the point $\mathbf{O} \in D^{n-1}$ is isomorphic to $SO(n-1)$. This means that $D^{n-1} \sim SO_0(n-1,1)/SO(n-1)$. The sphere $S^{n-2}$ is the boundary of $D^{n-1}$. Thus, the compact Riemannian space $SO(n-1)/SO(n-2)$ is the boundary of the noncompact Riemannian space $SO_0(n-1,1)/SO(n-1)$.

**9.1.2. The Lie algebra of $SO_0(n-1,1)$.** Let us denote by $\{g'_{in}(\theta)\}$, $i = 1,\dots,n-1$, the one-parameter subgroups of $SO_0(n-1,1)$ consisting of hyperbolic rotations in the planes $(x_i, x_n)$, that is, of transformations of the form

$$x'_j = x_j,\ j \neq i,n;\ x'_j = x_i \cosh\theta + x_n \sinh\theta,\ x'_n = x_i \sinh\theta + x_n \cosh\theta, \tag{1}$$

and put $g'_{n-1}(\theta) = g'_{n-1,n}(\theta)$. The tangent matrices to the subgroups $\{g'_{in}(\theta)\}$ are

$$\frac{d}{d\theta} g'_{in}(\theta)\bigg|_{\theta=0} = I'_{in} \equiv e_{in} + e_{ni}.$$

The matrices $I_{ij}$, $1 \le i < j \le n-1$ (see Section 2.1.1) and $I'_{in}$, $1 \le i \le n-1$, form a basis of the Lie algebra $\mathfrak{so}(n-1,1)$ of the group $SO_0(n-1,1)$. The commutation relations for them are

$$[I_{ij}, I'_{kn}] = \delta_{jk} I'_{in} - \delta_{ik} I'_{jn},\ j \le n-1, \tag{2}$$

$$[I'_{in}, I'_{jn}] = I_{ij},\ 1 \le i < j \le n-1. \tag{2'}$$

The element

$$C = \sum_{1 \le i < j < n} I_{ij}^2 - \sum_{1 \le i < n} I'^2_{in} \tag{3}$$

of the universal enveloping algebra $\mathfrak{U}(\mathfrak{so}(n-1,1))$ is the quadratic Casimir operator for $\mathfrak{so}(n-1,1)$:

$$[C, I_{ij}] = [C, I'_{in}] = 0.$$

The mapping $\theta: X \to JXJ$, where $J = \mathrm{diag}(-1,\dots,-1,1)$, is an involutive automorphism of $\mathfrak{so}(n-1,1)$. We have

$$\theta(I_{ij}) = I_{ij},\ 1 \le i < j \le n-1,\ \theta(I'_{in}) = -I'_{in},\ 1 \le i < n. \tag{4}$$

Therefore, the space of the Lie algebra $\mathfrak{so}(n-1,1)$ decomposes into the direct sum of eigenspaces corresponding to the eigenvalues 1 and $-1$ of $\theta$:

$$\mathfrak{so}(n-1,1) = \mathfrak{so}(n-1) + \mathfrak{p}. \tag{5}$$

The subspace $\mathfrak{p}$ is generated by the vectors $I'_{in}$, $1 \le i < n$.

The Killing form (see Section 1.1.6) of $\mathfrak{so}(n-1,1)$ has the form

$$B(X,X) = \mathrm{Tr}(\mathrm{ad}X\,\mathrm{ad}X) = (2n-4)\left(-\sum \alpha_{ij}^2 + \sum \alpha_i^2\right), \tag{6}$$

where $X = \sum \alpha_{ij} I_{ij} + \sum \alpha_i I'_{in}$. It is strictly positive on $\mathfrak{p}$ and strictly negative on $\mathfrak{so}(n-1)$. The formula

$$\langle X, Y\rangle = -B(X, \theta Y),\quad X, Y \in \mathfrak{so}(n-1,1), \tag{7}$$

gives a strictly positive scalar product on $\mathfrak{so}(n-1,1)$.

The set $\mathfrak{a} = \mathbf{R}I'_{n-1,n}$ is a maximal commutative subalgebra in $\mathfrak{p}$. The dimension of $\mathfrak{a}$, which is equal to 1, is called the ***real rank*** of $\mathfrak{so}(n-1,1)$ and of $SO_0(n-1,1)$.

The operator $\operatorname{ad}I'_{n-1,n}$ (see Section 1.1.3) is symmetric with respect to the scalar product (7):

$$\langle (\operatorname{ad}I'_{n-1,n})X, Y\rangle = \langle X, (\operatorname{ad}I'_{n-1,n})Y\rangle .$$

Therefore, $\mathfrak{so}(n-1,1)$ decomposes into the orthogonal sum of eigenspaces of $\operatorname{ad}I'_{n-1,n}$:

$$\mathfrak{so}(n-1,1) = \mathfrak{n} + (\mathfrak{m} + \mathfrak{a}) + \bar{\mathfrak{n}}, \tag{8}$$

where $\mathfrak{m}$ is the Lie algebra of the subgroup $M \sim SO(n-2)$ consisting of matrices $m = \operatorname{diag}(g,1,1)$, $g \in SO(n-2)$, $\mathfrak{n}$ is the Lie algebra of the subgroup $N$, consisting of matrices

$$\begin{pmatrix} 0_{n-2} & -\mathbf{y}^T & \mathbf{y}^T \\ \mathbf{y} & 0 & 0 \\ \mathbf{y} & 0 & 0 \end{pmatrix}, \ \mathbf{y} = (y_1, \ldots, y_{n-2}),\ y_j \in \mathbf{R}, \tag{9}$$

and $\bar{\mathfrak{n}}$ is the Lie algebra of the subgroup $\bar{N}$, consisting of matrices

$$\begin{pmatrix} 0_{n-2} & -\mathbf{y}^T & -\mathbf{y}^T \\ \mathbf{y} & 0 & 0 \\ \mathbf{y} & 0 & 0 \end{pmatrix}. \tag{9'}$$

In the decomposition (8) $\mathfrak{n}$, $\bar{\mathfrak{n}}$ and $\mathfrak{m} + \mathfrak{a}$ correspond to the eigenvalues 1, $-1$ and 0 respectively.

The subspaces $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are maximal nilpotent subalgebras in $\mathfrak{so}(n-1,1)$, and $\mathfrak{so}(n-1)$ is a maximal subalgebra on which the form $\langle\,.\,,\,.\,\rangle$ is strictly positive. The commutative one-parameter subgroup $A' = \{g'_{n-1}(\theta)\}$ of $SO_0(n-1,1)$ corresponds to the subalgebra $\mathfrak{a}$. The subgroup $M = SO(n-2)$ is the centralizer of $A'$ in $SO(n-1)$. The normalizer of $A'$ in $SO(n-1)$ coincides with $M' = SO(n-2) \cup g_w SO(n-2)$, where $g_w = \operatorname{diag}(I_{n-3}, w, 1)$, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

It is easy to see that

$$n(\mathbf{t})n(\mathbf{s}) = n(\mathbf{t}+\mathbf{s}), \tag{10}$$

$$\bar{n}(\mathbf{t})\bar{n}(\mathbf{s}) = \bar{n}(\mathbf{t}+\mathbf{s}). \tag{10'}$$

These relations show that the subgroups $N$ and $\bar{N}$ are isomorphic to $\mathbf{R}^{n-2}$. Since

$$g'_{n-1}(\theta)n(\mathbf{t})g'_{n-1}(-\theta) = n(e^{\theta}\mathbf{t}), \tag{11}$$

then $A'N = NA'$ is a subgroup in $SO_0(n-1,1)$ which is the semidirect product of $A'$ and $N$. Analogous statement is valid for $A'\tilde{N} = \tilde{N}A'$.

The relation

$$mn(\mathbf{t})m^{-1} = n(g\mathbf{t}), \tag{12}$$

where $m = \operatorname{diag}(g,1,1) \in M$, can be also directly verified. It implies that $MN = NM$ is a subgroup in $SO_0(n-1,1)$ which is the semidirect product of $M$ and $N$. The analogous statement is valid for $M$ and $\tilde{N}$.

**9.1.3. The groups $ISO(n-1)$ and $ISO_0(n-2,1)$.** Let us denote by $ISO(n-1)$ the group of motions of the space $E_{n-1}$, that is, the group of nonhomogeneous linear transformations in $E_{n-1}$, preserving distances between points and the orientation. Every $g \in ISO(n-1)$ has the form

$$g\mathbf{x} = k\mathbf{x} + \mathbf{a}, \tag{1}$$

where $k \in SO(n-1)$ is a rotation of $E_{n-1}$ about the origin and $\mathbf{a} \in E_{n-1}$. We shall write $g = g(k,\mathbf{a})$.

The group $ISO(n-1)$ is isomorphic to the group of real matrices of the form

$$g(k,\mathbf{a}) = \begin{pmatrix} k & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}, \tag{2}$$

where $\mathbf{0} = (0,\dots,0)$. We have

$$g(k_1,\mathbf{a}_1)g(k_2,\mathbf{a}_2) = g(k_1k_2, \mathbf{a}_1 + k_1\mathbf{a}_2). \tag{3}$$

Therefore,

$$g(k,\mathbf{a}) = g(e,\mathbf{a})g(k,\mathbf{0}) = g(k,\mathbf{0})g(e,k^{-1}\mathbf{a}), \tag{4}$$

$$g^{-1}(k,\mathbf{a}) = g(k^{-1}, -k^{-1}\mathbf{a}). \tag{5}$$

The group $ISO(n-1)$ is the semidirect product of the subgroup $SO(n-1)$ and the invariant subgroup $T_{n-1} \equiv \{g(e,\mathbf{a})\}$, isomorphic to $\mathbb{R}^{n-1}$. The space $\mathbb{R}^{n-1} \sim ISO(n-1)/SO(n-1)$ is triple to the symmetric spaces $S^{n-1}$ and $H_+^{n-1}$ (see Section 1.2.3).

The subgroups $MN$ and $M\tilde{N}$ of $SO_0(n-1,1)$ are isomorphic to $ISO(n-2)$. The isomoprhism $MN \sim ISO(n-2)$ is given by the formula

$$MN \ni mn(\mathbf{t}) \longleftrightarrow g(k,\mathbf{t}) \in ISO(n-2),$$

where $m = \operatorname{diag}(k,1,1)$, $k \in SO(n-2)$.

Replacing rotations by transformations from $SO_0(n-2,1)$ in the definition of $ISO(n-1)$, we obtain the group $ISO_0(n-2,1)$. It consists of transformations of the form $g = g(h, \mathbf{a})$, where $h \in SO_0(n-2,1)$, $\mathbf{a} \in E_{n-2,1}$. We have $g(h,\mathbf{a})\mathbf{x} = h\mathbf{x} + \mathbf{a}$. Equation (3) (where $k$ is replaced by $h$) remains valid. The group $ISO_0(n-2,1)$ is isomorphic to the group of matrices $\begin{pmatrix} h & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}$, where $h \in SO_0(n-2,1)$, $\mathbf{a} \in \mathbf{R}^{n-1}$ and $\mathbf{0} = (0,\ldots,0)$. Matrices $\begin{pmatrix} h & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$ form a subgroup, isomorphic to $SO_0(n-2,1)$, and matrices $\begin{pmatrix} I_{n-1} & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}$ form a subgroup isomorphic to $\mathbf{R}^{n-1}$. The group $ISO_0(n-2,1)$ is the semidirect product of these subgroups.

Let us denote by $\Omega_{n-1}$ the interior of the upper sheet $C_+^{n-2}$ of the cone $C^{n-2}$. If $\mathbf{a}, \mathbf{b} \in \Omega_{n-1}$ and $h \in SO_0(n-2,1)$, then $h\mathbf{b} + \mathbf{a} \in \Omega_{n-1}$. It follows from here that elements $g(h,\mathbf{a})$, $\mathbf{a} \in \Omega_{n-1}$, form a semi-group. Any element of this semi-group is represented as

$$g = g(h',\mathbf{0})g(e,\mathbf{a}_r)g(h'',\mathbf{0}),\ \ h', h'' \in SO_0(n-2,1),\ \ \mathbf{a}_r = (0,\ldots,0,r),\ \ r \geq 0$$

This follows from the facts that $g(h,\mathbf{a}) = g(e,\mathbf{a})g(h,\mathbf{0})$ and that any element $g(e,\mathbf{a})$, $\mathbf{a} \in \Omega_{n-1}$, is represented in the form

$$g(e,\mathbf{a}) = g(h',\mathbf{0})g(e,\mathbf{a}_r)g(h'^{-1},\mathbf{0}).$$

Another semi-group consists of transformations $g(h,\mathbf{a})$, where $-\mathbf{a} \in \Omega_{n-1}$.

The Lie algebra $\mathfrak{iso}(n-1)$ of the group $ISO(n-1)$ is the sum of the Lie algebras of the subgroups $SO(n-1)$ and $T_{n-1}$. The Lie algebra $\mathfrak{t}_{n-1}$ of $T_{n-1}$ is spanned by the matrices $J_k$, $k = 0,1,\ldots,n-1$, with entries $(J_k)_{kn} = 1$ and $(J_k)_{st} = 0$ if $(s,t) \neq (k,n)$. The matrix $J_k$ is tangent to the one-parameter subgroup of shifts along the axis $x_i$.

The element

$$C = \sum_{k=1}^{n} J_k^2$$

of the universal enveloping algebra $\mathfrak{U}(\mathfrak{iso}(n-1))$ has the property

$$[C, I_{ij}] = [C, J_k] = 0,$$

that is, it belongs to the center of $\mathfrak{U}(\mathfrak{iso}(n-1))$.

**9.1.4. The group $SO_0(p,q)$ and related homogeneous spaces.** The real $p+q$-dimensional vector space equipped with the bilinear form

$$[\mathbf{x},\mathbf{y}]_{pq} = -x_1y_1 \ldots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_{p+q}y_{p+q} \tag{1}$$

is called the pseudo-Euclidean space of signature $(p,q)$. We shall denote it by $E_{pq}$. Form (1) determines the distance between points in $E_{pq}$: $r^2(\mathbf{x},\mathbf{y}) = [\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y}]_{pq}$. Points $\mathbf{x} \in E_{pq}$, $\mathbf{x} \neq \mathbf{0}$, for which $[\mathbf{x},\mathbf{x}]_{pq} = 0$ form the cone $C^{pq}$ in $E_{pq}$. The equalities $[\mathbf{x},\mathbf{x}]_{pq} = \pm 1$ define the hyperboloids $H_\pm^{pq}$ in $E_{pq}$.

The set of linear transformations of $E_{pq}$ preserving distances between points and the orientation of $E_{pq}$ forms the group $SO_0(p,q)$. The action of $SO_0(p,q)$ splits the space $E_{pq}$ into transitive orbits consisting of the points, for which $[\mathbf{x},\mathbf{x}]_{pq}=r$, $-\infty<r<\infty$. The origin $\mathbf{x}=\mathbf{0}$ is a separate orbit.

Let us choose the point $\mathbf{e}_n=(0,\ldots,0,1)$, $n=p+q$, on $H_+^{pq}$. The subgroup of $SO_0(p,q)$, leaving this point fixed, is isomorhic to $SO_0(p,q-1)$. Therefore, $H_+^{pq}$ is identified with $SO_0(p,q)/SO_0(p,q-1)$. it is pseudo-Riemannian symmetric space (see Section 1.2.4). In the same way we have $H_-^{pq}=SO_0(p,q)/SO_0(p-1,q)$.

The maximal compact subgroup of $SO_0(p,q)$ coincides with $K=SO(p)\times SO(q)$. It consists of matrices $\mathrm{diag}(k,k')$, $k\in SO(p)$, $k'\in SO(q)$. Let us select in $SO_0(p,q)$ the one-parameter subgroups $\{g_{ij}(\theta)\}$, $1\le i<j\le p$ or $p+1\le i<j\le p+q$, of $SO(p)\times SO(q)$ and the one-parameter subgroups $\{g'_{ij}(t)\}$, $1\le i\le p$, $p+1\le j\le p+q$ (the definition of $\{g_{ij}(\theta)\}$ and $\{g'_{ij}(t)\}$ see in Section 9.1.2). The tangent matrices

$$I_{ij}=e_{ij}-e_{ji},\ 1\le i<j\le p \text{ or } p+1\le i<j\le p+q, \tag{2}$$

$$I'_{ij}=e_{ij}+e_{ji},\ 1\le i\le p<j\le p+q, \tag{3}$$

to these one-parameter subgroups at the identity element form a basis of the Lie algebra $\mathfrak{so}(p,q)$ of the group $SO_0(p,q)$.

It is easy to see that the mapping

$$\theta\colon X\to JXJ,\quad X\in\mathfrak{so}(p,q), \tag{4}$$

where $J=\mathrm{diag}(-1,\ldots,-1,1,\ldots,1)$ ($p$ elements are equal to $-1$ and $q$ elements are equal to 1), is an involutive automorphism of $\mathfrak{so}(p,q)$. Matrices (2) and (3) correspond to the eigenvalues 1 and $-1$, respectively. We have

$$\mathfrak{so}(p,q)=(\mathfrak{so}(p)+\mathfrak{so}(q))+\mathfrak{p},$$

where $\mathfrak{p}$ is spanned by matrices (3).

If $B(X,Y)$ is the Killing form on $\mathfrak{so}(p,q)$ (see Section 1.1.5), then with the help of bases (2)–(3) we derive that for

$$X=\sum\alpha_{ij}I_{ij}+\sum\beta_{ij}I'_{ij}$$

we have

$$B(X,Y)\equiv\mathrm{Tr}(\mathrm{ad}X\mathrm{ad}Y)=-(p+q-2)\left(\sum\alpha_{ij}^2-\sum\beta_{ij}^2\right).$$

It is seen that the Killing form is strictly positive on $\mathfrak{p}$ and strictly negative on $\mathfrak{so}(p)+\mathfrak{so}(q)$. The form

$$\langle X,Y\rangle=-B(X,\theta Y), \tag{5}$$

where $\theta$ is defined by (4), is strictly positive on the whole algebra $\mathfrak{so}(p,q)$ and, therefore, is a scalar product on $\mathfrak{so}(p,q)$.

Let us select the element $I'_{1n}$, $n=p+q$, in $\mathfrak{p}$. It generates the one-parameter subalgebra $\mathfrak{a} = \mathbf{R}I'_{1n}$. The maximal connected subgroup in $SO_0(p,q)$, whose elements commute with $I'_{1n}$, coincides with $SO_0(p-1,q-1)$. Matrices of this subgroup are located on intersections of rows and columns with numbers $2,3,\ldots,n-1$.

The operator $\mathrm{ad}I'_{1n}$ is symmetric with respect to the scalar product (5). Therefore, $\mathfrak{so}(p,q)$ decomposes into the orthogonal sum of its eigenspaces:

$$\mathfrak{so}(p,q) = (\mathfrak{a} + \mathfrak{so}(p-1,q-1)) + \mathfrak{n} + \bar{\mathfrak{n}}, \tag{6}$$

where $\mathfrak{a} + \mathfrak{so}(p-1,q-1)$ corresponds to the eigenvalue 0, $\mathfrak{n}$ consists of matrices

$$\begin{pmatrix} 0 & \mathbf{a} & 0 \\ -\mathbf{a}^T & 0_{n-2} & \mathbf{a}^T \\ 0 & \mathbf{a} & 0 \end{pmatrix}, \qquad \begin{matrix} n=p+q, \\ \mathbf{a} \equiv (a_1,\ldots,a_{n-2}) \in \mathbf{R}^{n-2}, \end{matrix} \tag{7}$$

and corresponds to the eigenvalue 1, and $\bar{\mathfrak{n}}$ consists of matrices

$$\begin{pmatrix} 0 & \mathbf{a} & 0 \\ -\mathbf{a}^T & 0_{n-2} & -\mathbf{a}^T \\ 0 & \mathbf{a} & 0 \end{pmatrix}, \quad \mathbf{a} \equiv (a_1,\ldots,a_{n-2}) \in \mathbf{R}^{n-2}, \tag{8}$$

and corresponds to the eigenvalue $-1$. The subspaces $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are commutative subalgebras of $\mathfrak{so}(p,q)$.

By means of the subalgebra $\mathfrak{n}$ we construct the subgroup $N = \exp\mathfrak{n}$. Exponents of matrices (7) are easily evaluated, and we obtain that $N$ consists of matrices

$$n(\mathbf{a}) \equiv n(a_1,\ldots,a_{n-2}) = \begin{pmatrix} 1-\frac{1}{2}[\mathbf{a},\mathbf{a}]^2 & \mathbf{a} & \frac{1}{2}[\mathbf{a},\mathbf{a}]^2 \\ -\mathbf{a}^T & I_{n-1} & \mathbf{a}^T \\ -\frac{1}{2}[\mathbf{a},\mathbf{a}] & \mathbf{a} & 1+\frac{1}{2}[\mathbf{a},\mathbf{a}]^2 \end{pmatrix}, \tag{9}$$

where $[\mathbf{a},\mathbf{a}]^2 \equiv [\mathbf{a},\mathbf{a}]^2_{p-1,q-1}$. The subgroup $N$ is commutative: $n(\mathbf{a})n(\mathbf{b}) = n(\mathbf{a}+\mathbf{b})$. For $g_{1n}(t) \in A' \equiv \exp\mathfrak{a}$ we have

$$g_{1n}(t)n(\mathbf{a})g_{1n}(-t) = n(e^t\mathbf{a}). \tag{10}$$

Therefore, $A'N = NA'$ is a subgroup of $SO_0(p,q)$. It is the semidirect product of $A'$ and $N$, where $N$ is an invariant subgroup.

One can easily verify that for $h \in SO_0(p-1,q-1)$ the relation

$$hn(\mathbf{a})h^{-1} = n(h\mathbf{a}) \tag{11}$$

holds. Therefore, $SO_0(p-1,q-1)N = NSO_0(p-1,q-1)$ is a subgroup in $SO_0(p,q)$. It is isomorphic to the group $ISO_0(p-1,q-1)$ consisting of matrices

$$g(h,\mathbf{a}) = \begin{pmatrix} h & \mathbf{a} \\ 0 & 1 \end{pmatrix}, \tag{12}$$

where $h \in SO_0(p-1,q-1)$ and $\mathbf{a}$ is a column vector.

The subgroup $N^- = \exp \bar{\mathfrak{n}}$ is also commutative and consists of matrices

$$n(\mathbf{a}) \equiv n(a_1,\ldots,a_{n-2}) = \begin{pmatrix} 1+\frac{1}{2}[\mathbf{a},\mathbf{a}]^2 & \mathbf{a} & \frac{1}{2}[\mathbf{a},\mathbf{a}]^2 \\ -\mathbf{a}^T & I_{n-2} & -\mathbf{a}^T \\ -\frac{1}{2}[\mathbf{a},\mathbf{a}]^2 & \mathbf{a} & 1-\frac{1}{2}[\mathbf{a},\mathbf{a}]^2 \end{pmatrix}. \tag{13}$$

Let $\boldsymbol{\xi}_0$ be the point from $C^{pq}$ with coordinates $(1,0,\ldots,0,1)$. Elements of the subgroups $SO_0(p-1,q-1)$ and $N$ leave $\boldsymbol{\xi}_0$ fixed. Moreover, if an element $g \in SO_0(p,q)$ has the property $g\boldsymbol{\xi}_0 = \boldsymbol{\xi}_0$, then $g \in SO_0(p-1,q-1)N$. Therefore,

$$C^{pq} \sim SO_0(p,q)/SO_0(p-1,q-1)N. \tag{14}$$

**9.1.5. Coordinate systems on $S^{n-1}$ and $H_+^{n-1}$.** Every point $\boldsymbol{\xi}$ of $S^{n-1}$ is obtained from $\mathbf{e}_n = (0,\ldots,0,1)$ by some rotation $g \in SO(n)$. This rotation can be given in the form

$$g = g_1(\theta_1)g_2(\theta_2)\ldots g_{n-1}(\theta_{n-1}). \tag{1}$$

Namely, if $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n) \in S^{n-1}$, then

$$\left.\begin{aligned} \xi_1 &= \sin\theta_{n-1}\sin\theta_{n-2}\ldots\sin\theta_2\sin\theta_1, \\ \xi_2 &= \sin\theta_{n-1}\sin\theta_{n-2}\ldots\sin\theta_2\cos\theta_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \xi_{n-1} &= \sin\theta_{n-1}\cos\theta_{n-2}, \\ \xi_n &= \cos\theta_{n-1}, \end{aligned}\right\} \tag{2}$$

where

$$\cos\theta_k = \frac{\xi_{k+1}}{r_{k+1}},\ \sin\theta_k = \frac{r_k}{r_{k+1}},\ r_k^2 = \xi_1^2+\cdots+\xi_k^2,\ r_k \geq 0. \tag{3}$$

It is easy to see that when the parameters $\theta_1,\ldots,\theta_{n-1}$ vary in the region

$$0 \leq \theta_1 < 2\pi,\ 0 \leq \theta_k < \pi,\ 2 \leq k \leq n-1, \tag{4}$$

then the point $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n)$ runs over the sphere $S^{n-1}$. In addition, for almost all points of the sphere (that is, for all points except for a manifold of smaller dimension) values $\theta_1,\ldots,\theta_{n-1}$ are uniquely determined. They give *spherical coordinates* on $S^{n-1}$. Since every vector $\mathbf{x} \in E_n$ is represented as $\mathbf{x} = r\boldsymbol{\xi}$, where $r \geq 0$ and $\boldsymbol{\xi} \in S^{n-1}$, spherical coordinates $r, \theta_1, \ldots, \theta_{n-1}$ are introduced in $E_n$.

Since $S^{n-1} = SO(n)/SO(n-1)$, then any element $g \in SO(n)$ is represented in the form

$$g = g_1(\theta_1)\ldots g_{n-1}(\theta_{n-1})k, \quad k \in K \equiv SO(n-1). \tag{5}$$

Applying similar factorization to $k$ and continuing this procedure, we obtain the factorization

$$g = g^{(n-1)}g^{(n-2)}\ldots g^{(1)}, \tag{6}$$

where

$$g^{(k)} = g_1(\theta_1^k)g_2(\theta_2^k)\ldots g_k(\theta_k^k), \quad 1 \leq k \leq n-1, \tag{7}$$

$g_i(\theta)$ is the rotation in the plane $(x_i, x_{i+1})$ by the angle $\theta$ and

$$0 \leq \theta_k^1 < 2\pi, \quad 0 \leq \theta_k^j < \pi, \quad 2 \leq j \leq k. \tag{8}$$

The parameters $\theta_k^j$ are uniquely determined for almost all elements of the group $SO(n)$. They are called *Euler angles* on $SO(n)$.

It follows from (5) that $g = k'g_{n-1}(\theta)k$, where $k, k' \in SO(n-1)$. The decomposition

$$SO(n) = KAK, \tag{9}$$

where $K = SO(n-1)$ and $A$ is the one-parameter subgroup of elements $g_{n-1}(\theta)$, is called the *Cartan decomposition* of $SO(n)$. This decomposition is not unique for elements $g \in SO(n)$ since

$$k'g_{n-1}(\theta)k = (k'm)g_{n-1}(\theta)(m^{-1}k), \ m \in SO(n-2),$$

and

$$g_w g_{n-1}(\theta)g_w^{-1} = g_{n-1}(2\pi - \theta),$$

where $g_w = \mathrm{diag}(I_{n-3}, w, 1)$, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

There is a coordinate system on$H_+^{n-1}$, analogous to spherical one. Every point $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in H_+^{n-1}$ has the form

$$\boldsymbol{\xi} = g_1(\theta_1)\ldots g_{n-2}(\theta_{n-2})g'_{n-1}(\theta_{n-1})\mathbf{e}_n, \tag{10}$$

where $g_i(\theta_i)$, $1 \leq i \leq n-2$, are Euclidean rotations in the planes $(x_i, x_{i+1})$ and $g'_{n-1}(\theta_{n-1})$ is a hyperbolic rotation in the plane $(x_{n-1}, x_n)$. Therefore,

$$\left.\begin{aligned}
\xi_1 &= \sinh\theta_{n-1}\sin\theta_{n-2}\ldots\sin\theta_2\sin\theta_1, \\
\xi_2 &= \sinh\theta_{n-1}\sin\theta_{n-2}\ldots\sin\theta_2\cos\theta_1, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\xi_{n-1} &= \sinh\theta_{n-1}\cos\theta_{n-2}, \\
\xi_n &= \cosh\theta_{n-1},
\end{aligned}\right\} \tag{11}$$

where $0 \le \theta_1 < 2\pi$, $0 \le \theta_j < \pi$, $1 < j \le n-2$, $0 \le \theta_{n-1} < +\infty$.

Since $H_+^{n-1} = SO_0(n-1,1)/SO(n-1)$, then any element $g \in SO_0(n-1,1)$ is represented in the form

$$g = g_1(\theta_1) \dots g_{n-2}(\theta_{n-2}) g'_{n-1}(\theta_{n-1}) k, \tag{12}$$

where $k \in SO(n-1)$. Taking the Euler angles of the element $k$ together with the parameters $\theta_1, \dots, \theta_{n-2}, \theta_{n-1}$, we obtain a parametrization of the group $SO_0(n-1,1)$, analogous to Euler angles on $SO(n)$.

The decomposition

$$SO_0(n-1,1) = KA'K, \tag{13}$$

where $K = SO(n-1)$ and $A'$ is the one-parameter subgroup of elements $g'_{n-1}(\theta)$, is an analog of the Cartan decomposition for $SO_0(n-1,1)$. This decomposition is not unique for elements $g$ from $SO_0(n-1,1)$. The Cartan decompositions of $SO(n)$ and $SO_0(n-1,1)$ show that the passage from the first group to the second one is fulfilled by means of the passage from $A \sim SO(2)$ to $A' \sim SO_0(1,1)$, described in Section 3.1.5.

Other coordinate systems on $S^{n-1}$ are obtained in the following way. We represent the space $E_n$ as the direct sum $E_n = E_s + E_{n-s}$, where $E_s$ is spanned by the vectors $\mathbf{e}_1, \dots, \mathbf{e}_s$ and $E_{n-s}$ is spanned by $\mathbf{e}_{s+1}, \dots, \mathbf{e}_n$. Every vector $\boldsymbol{\xi} \in S^{n-1}$ is represented in the form $\boldsymbol{\xi} = \boldsymbol{\eta} \sin\theta + \boldsymbol{\zeta} \cos\theta$, where

$$\boldsymbol{\eta} \in S^{n-1} \cap E_s = S^{s-1}, \quad \boldsymbol{\zeta} \in S^{n-1} \cap E_{n-s} = S^{n-s-1}.$$

The vector $\boldsymbol{\xi}$ is uniquely given by a number $\theta$, $0 \le \theta \le \frac{\pi}{2}$, and by vectors $\boldsymbol{\eta} \in S^{s-1}$, $\boldsymbol{\zeta} \in S^{n-s-1}$. If $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ are given by their spherical coordinates, then we obtain *bispherical coordinates* on $S^{n-1}$.

There is a similar coordinate system on $H_+^{n-1}$. Every vector $\boldsymbol{\xi} \in H_+^{n-1}$ is representable as $\boldsymbol{\xi} = \boldsymbol{\eta} \sinh\theta + \boldsymbol{\zeta} \cosh\theta$, where

$$\boldsymbol{\eta} \in H_+^{n-1} \cap E_s = S^{s-1}, \quad \boldsymbol{\zeta} \in H_+^{n-1} \cap E_{n-s-1,1} = H_+^{n-s-1}$$

and $0 \le \theta < +\infty$. If $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ are given by coordinates according to (2) and (11), then we obtain a coordinate system on $H_+^{n-1}$, analoguos to bispherical one.

Coordinate systems on $S^{n-1}$ and $H_+^{n-1}$, constructed here, are connected with the *generalized Cartan decompositions*

$$SO(n) = K_s A_s K, \tag{14}$$

$$SO_0(n-1,1) = K'_s A'_s K, \tag{15}$$

where

$$\begin{gathered} K_s = SO(s) \times SO(n-s),\ K'_s = SO(s) \times SO_0(n-s-1,1), \\ K = SO(n-1),\ A_s = \{g_{sn}(\theta)\},\ A'_s = \{g'_{sn}(\theta)\}. \end{gathered} \tag{15'}$$

It is clear that the subgroups $K_{n-1}$ and $K$ coincide and the subgroups $K_1$ and $K$ are isomorphic. Factorizations of elements, corresponding to formulas (14) and (15), are not unique, since, for example,

$$k_1 g_{sn}(\theta)k_2 = (k_1 m_s)g_{sn}(\theta)(m_s^{-1}k_2),$$
$$m_s \in SO(s-1)\times SO(n-s-1),\; k_1 \in K_s, k_2 \in K.$$

Instead of giving the vectors $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ from the equalities $\boldsymbol{\xi} = \boldsymbol{\eta}\sin\theta + \boldsymbol{\zeta}\cos\theta$ and $\boldsymbol{\xi} = \boldsymbol{\eta}\sinh\theta + \boldsymbol{\zeta}\cosh\theta$ by spherical coordinates, we can apply to them similar factorizations. Continuing this procedure, we obtain ***polyspherical coordinates*** on $S^{n-1}$ (respectively, on $H_+^{n-1}$). Every one of these coordinate systems is given by some tree, describing successive partitions of the set of the coordinates $\xi_1,\ldots,\xi_n$. The tree method will be described in Section 10.5.

Let us introduce now orispherical coordinates on $H_+^{n-1}$. Every $\boldsymbol{\xi} \in H_+^{n-1}$ can be represented as

$$\boldsymbol{\xi} = n(\mathbf{t})g'_{n-1}(\theta)\mathbf{e}_n, \tag{16}$$

where $n(\mathbf{t})$ is matrix (5) from Section 9.1.1 and $g'_{n-1}(\theta) = g'_{n-1,n}(\theta)$. If $\mathbf{t} = (t_1.\ldots,t_{n-2})$, then $t_1,\ldots,t_{n-2},\theta$ are coordinates on $H_+^{n-1}$:

$$\left.\begin{aligned} \xi_1 &= t_1 e^{-\theta},\\ &\cdots\cdots\cdots\cdots\\ \xi_{n-2} &= t_{n-2}e^{-\theta},\\ \xi_{n-1} &= \sinh\theta + \tfrac{1}{2}t^2 e^{-\theta},\\ \xi_n &= \cosh\theta + \tfrac{1}{2}t^2 e^{-\theta},\\ t^2 &= t_1^2+\cdots+t_{n-2}^2. \end{aligned}\right\} \tag{16$'$}$$

Instead of $t_1,\ldots,t_{n-2}$ one can take spherical coordinates on $\mathbb{R}^{n-2}$. Coordinates on $H_+^{n-1}$ obtained will be called *orispherical*. The parameters $t_1,\ldots,t_{n-2},\theta$ take any real values; moreover, if $(t_1,\ldots,t_{n-2},\theta) \neq (t'_1,\ldots,t'_{n-2},\theta')$, then the corresponding points from $H_+^{n-1}$ are distinct.

Since the stabilizer subgroup of the basis vector $\mathbf{e}_n$ coincides with $SO(n-1)$, then (16) implies that any element $g \in SO_0(n-1,1)$ is representable in the form

$$g = n(\mathbf{t})g'_{n-1}(\theta)k, \quad k \in SO(n-1). \tag{17}$$

Therefore,

$$SO_0(n-1,1) = NA'K, \tag{18}$$

where $A' = \{g'_{n-1}(\theta)\}$ and $K = SO(n-1)$. It is easy to show that factorization (17) is uniquely determined for all $g \in SO_0(n-1,1)$. Formula (18) is called the *Iwasawa decomposition* of the group $SO_0(n-1,1)$.

Denoting $SO_0(n-1,1)$ by $G$ we have

$$G = G^{-1} = (NA'K)^{-1} = K^{-1}A'^{-1}N^{-1} = KA'N. \tag{19}$$

Since $A'N = NA'$, then $G = A'NK = KA'N$. We also have

$$SO_0(n-1,1) = \bar{N}A'K = A'\bar{N}K = KA'\bar{N} = K\bar{N}A'. \tag{20}$$

The coordinate systems on $H_+^{n-1}$, analogous to orispherical one, correspond to the decompositions $A'NK$, $\bar{N}A'K$, $A'\bar{N}K$.

**9.1.6. Coordinate systems on $H_-^{n-1}$ and $C_+^{n-1}$.** Every point $\xi$ of the one-sheeted hyperboloid $H_-^{n-1}$ is representble as

$$\xi = g_1(\theta_1)\dots g_{n-2}(\theta_{n-2})g'_{n-1}(\theta_{n-1})\mathbf{e}_{n-1} \tag{1}$$

and every point $\xi$ of the upper sheet $C_+^{n-1}$ of the cone is representable as

$$\xi = g_1(\theta_1)\dots g_{n-2}(\theta_{n-2})g'_{n-1}(\theta_{n-1})(\mathbf{e}_{n-1}+\mathbf{e}_n). \tag{2}$$

From here we obtain coordinates on $H_-^{n-1}$ and $C_+^{n-1}$. They are of the form

$$\left.\begin{array}{l} \xi_1 = \cosh\theta_{n-1}\sin\theta_{n-2}\dots\sin\theta_2\sin\theta_1,\\ \xi_2 = \cosh\theta_{n-1}\sin\theta_{n-2}\dots\sin\theta_2\cos\theta_1,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \xi_{n-1} = \cosh\theta_{n-1}\cos\theta_{n-2},\\ \xi_n = \sinh\theta_{n-1} \end{array}\right\} \tag{3}$$

on $H_-^{n-1}$ and of the form

$$\left.\begin{array}{l} \xi_1 = r\sin\theta_{n-2}\dots\sin\theta_2\sin\theta_1,\\ \xi_2 = r\sin\theta_{n-2}\dots\sin\theta_2\cos\theta_1,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \xi_{n-1} = r\cos\theta_{n-2},\\ \xi_n = r \end{array}\right\} \tag{4}$$

on $C_+^{n-1}$ (we have set $e^{\theta_{n-1}} = r$).

Since the invariance subgroups of the basis vector $\mathbf{e}_{n-1}$ and of the point $\mathbf{e}_{n-1}+\mathbf{e}_n$ coincide with $K' = SO_0(n-2,1)$ and $MN$, respectively, and $MA' = A'M$, then to these coordiante systems there correspond the decompositions

$$SO_0(n-1,1) = KA'K', \tag{5}$$
$$SO_0(n-1,1) = KA'N \tag{6}$$

of $SO_0(n-1,1)$, where $K = SO(n-1)$ and $A' = \{g'_{n-1}(\theta)\}$. Essentially (5) coincides with decomposition (15) of Section 9.1.5 for $s=1$.

The spherical coordinate system (4) on $C_+^{n-1}$ is obtained by sections of $C_+^{n-1}$ by the hyperplanes $x_n = \exp\theta_{n-1}$, $-\infty < \theta_{n-1} < +\infty$. Let us consider sections of the cone $C_+^{n-1}$ by the hyperplanes $x_{n-1} = \exp\theta_{n-1}$ and $x_{n-1} = -\exp\theta_{n-1}$, $-\infty < \theta_{n-1} < +\infty$. They are upper sheets of hyperboloids in $E_{n-2,1}$. Introducing coordinates (11) of Section 9.1.5 on them, we obtian ***hyperbolic coordinates*** $\theta_1, \ldots, \theta_{n-2}, \theta_{n-1}$ on $C_+^{n-1}$:

$$\left.\begin{aligned} \xi_1 &= r\sinh\theta_{n-2}\sin\theta_{n-3}\ldots\sin\theta_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \xi_{n-2} &= r\sinh\theta_{n-2}\cos\theta_{n-3}, \\ \xi_{n-1} &= r\,\mathrm{sign}\,\xi_{n-1}, \\ \xi_n &= r\cosh\theta_{n-2}, \end{aligned}\right\} \tag{7}$$

where $r = e^{\theta_{n-1}}$. The point $\boldsymbol{\xi}$ with coordinates (7) is obtained from $\boldsymbol{\xi}_0 = \mathbf{e}_{n-1} + \mathbf{e}_n$ as follows

$$\boldsymbol{\xi} = g_1(\theta_1)\ldots g_{n-3}(\theta_{n-3})g'_{n-2,n}(\theta_{n-2})g'_{n-1}(\theta_{n-1})\boldsymbol{\xi}_0 \text{ if } \xi_{n-1} > 0,$$
$$\boldsymbol{\xi} = g_1(\theta_1)\ldots g_{n-3}(\theta_{n-3})g'_{n-2,n}(\theta_{n-2})\tilde{g}g'_{n-1}(\theta_{n-1})\boldsymbol{\xi}_0 \text{ if } \xi_{n-1} < 0,$$

where $\tilde{g} = \mathrm{diag}(-1,1,\ldots,1,-1,1)$. Therefore, with the hyperbolic coordinate system on $C_+^{n-1}$ the *generalized Iwasawa decomposition*

$$SO_0(n-1,1) = K'A'N \cup K'\tilde{g}A'N,\ K' = SO_0(n-2,1) \tag{8}$$

is connected ($K'A'N$ and $K'\tilde{g}A'N$ are disjoint).

The formulas

$$\left.\begin{aligned} \xi_1 &= r\sin\varphi_{s-1}\ldots\sin\varphi_2\sin\varphi_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \xi_{s-1} &= r\sin\varphi_{s-1}\cos\varphi_{s-2}, \\ \xi_s &= r\cos\varphi_{s-1}, \\ \xi_{s+1} &= r\sinh\psi_{n-s-1}\sin\psi_{n-s-2}\ldots\sin\psi_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \xi_{n-1} &= r\sinh\psi_{n-s-1}\cos\psi_{n-s-2}, \\ \xi_n &= r\cosh\psi_{n-s-1} \end{aligned}\right\} \tag{9}$$

define the coordinate system $r, \varphi_1, \ldots, \varphi_{s-1}, \psi_1, \ldots, \psi_{n-s-1}$ on $C_+^{n-1}$ which corresponds to the representation of points $\boldsymbol{\xi} \in C_+^{n-1}$ in the form

$$\boldsymbol{\xi} = k_s k'_{n-s} g'_{sn}(\theta)(\mathbf{e}_s + \mathbf{e}_n), \tag{10}$$

where

$$k_s = g_1(\varphi_1)\dots g_{s-1}(\varphi_{s-1}) \in SO(s),$$
$$k'_{n-s} = g_{s+1}(\psi_1)\dots g_{n-2}(\psi_{n-s-2})g'_{n-1}(\psi_{n-s-1}) \in SO_0(n-s-1,1).$$

Transposing the coordinates $\xi_s$ and $\xi_{n-1}$ we shall see that to this coordinate system there corresponds the decomposition

$$SO_0(n-1,1) = [SO(s) \times SO_0(n-s-1,1)]A'MN. \tag{11}$$

Introducing polyspherical coordinates on $S^{s-1}$ instead of $\varphi_1,\dots,\varphi_{s-1}$ and polyspherical coordinates on $H_+^{n-s-1}$ instead of $\psi_1,\dots,\psi_{n-s-1}$, we obtain polyspherical coordinates on $C_+^{n-1}$.

In order to construct polyspherical coordinates on $H_-^{n-1}$ one has to divide $H_-^{n-1}$ into domains. Let $1 \le s \le n-2$. We denote by $H_{-,\varepsilon}^{n-1,s}$, $\varepsilon \in \{+,-\}$, the subsets in $H_-^{n-1}$, given by the inequalities[1] $\varepsilon(\xi_1^2+\dots+\xi_s^2-1) > 0$. Every point $\boldsymbol{\xi} \in H_{-,+}^{n-1,s}$ has the form $\boldsymbol{\xi} = \boldsymbol{\eta}\cosh\theta + \boldsymbol{\zeta}\sinh\theta$, where

$$\boldsymbol{\eta} \in S^{s-1}, \quad \boldsymbol{\zeta} \in H_+^{n-s-1}, \quad -\infty < \theta < +\infty,$$

and every $\boldsymbol{\xi} \in H_{-,-}^{n-1,s}$ has the form $\boldsymbol{\xi} = \boldsymbol{\eta}\cos\theta + \boldsymbol{\zeta}\sin\theta$, where

$$\boldsymbol{\eta} \in S^{s-1}, \quad \boldsymbol{\zeta} \in H_-^{n-s-1}, \quad 0 \le \theta \le \frac{\pi}{2}.$$

Giving $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ by polyspherical coordinates, we obtain polyspherical coordinates on $H_-^{n-1}$. They can be also described by means of trees. We recommend that the reader writes out decompositions of $SO_0(n-1,1)$, corresponding to coordinates introduced.

Almost every $\boldsymbol{\xi} \in H_-^{n-1}$ can be represented as

$$\boldsymbol{\xi} = \varepsilon n(\mathbf{t})g'_{n-1}(\theta)\mathbf{e}_{n-1}, \quad \varepsilon \in \{+,-\}, \tag{12}$$

where $n(\mathbf{t})$ and $g'_{n-1}(\theta)$ are the same as in formula (16) of Section 9.1.5. Replacing the parameters $t_1,\dots,t_{n-2}$ by spherical coordinates in $\mathbb{R}^{n-2}$, we obtain the orispherical coordinate system $\varepsilon, r, \varphi_1,\dots,\varphi_{n-3},\theta$ on $H_-^{n-1}$. It is connected with the generalized Iwasawa decomposition

$$SO_0(n-1,1) = NA'K' \cup NA'\tilde{g}K' \tag{13}$$

[1] Note that $H_{-,+}^{n-1,s}$ for $s \ge 2$ consists of two connected parts, corresponding to the sheets of the hyperboloids $H^{n-s-1}$. For $s=1$ the number of parts is equal to 4 since one has to take into account the sign of $\xi_1$. For $s=1$ and for $s=n-2$ the subset $H_{-,-}^{n-1,s}$ consists of two connected parts.

of the group $SO_0(n-1,1)$, where $K' = SO_0(n-2,1)$ and $\tilde{g} = \mathrm{diag}(-1,1,\ldots,1,-1,1)$ (see decomposition (8)).

Almost every point $\boldsymbol{\xi} \in C_+^{n-1}$ is representable in the form

$$\boldsymbol{\xi} = n(\mathbf{t})g'_{n-1}(\theta)(\mathbf{e}_n - \mathbf{e}_{n-1}). \tag{14}$$

This factorization defines orispherical coordinates $t_1,\ldots,t_{n-2},\theta$ (or $r,\varphi_1,\ldots,\varphi_{n-3},\theta$) on $C_+^{n-1}$. They are connected with the decomposition

$$SO_0(n-1,1) = NA'M\bar{N} \tag{15}$$

(the equality holds almost everywhere). It is called the ***Gauss decomposition*** of $SO_0(n-1,1)$.

For almost every $\boldsymbol{\xi} \in C_+^{n-1}$ one has the equality

$$\boldsymbol{\xi} = n(\mathbf{t})g_w g'_{n-1}(\theta)(\mathbf{e}_n + \mathbf{e}_{n-1}) \tag{16}$$

defining the coordinate system $(t_1,\ldots,t_{n-2},\theta)$ on $C_+^{n-1}$. It is connected with the decomposition

$$SO_0(n-1,1) = NA'MN \cup Ng_wA'MN, \tag{17}$$

where the dimensionality of the first term is less than that of the whole group. It is called the ***Bruhat decomposition*** of $SO_0(n-1,1)$.

Every $\mathbf{x} \in E_{n-1,1}$ can be represented either as $\mathbf{x} = \pm r\boldsymbol{\xi}$, $r \geq 0$, $\boldsymbol{\xi} \in H_+^{n-1}$, or as $\mathbf{x} = r\boldsymbol{\eta}$, $r \geq 0$, $\boldsymbol{\eta} \in H_-^{n-1}$. Choosing a corresponding coordinate system on $H_+^{n-1}$ and on $H_-^{n-1}$, we obtain coordinate systems in $E_{n-1,1}$, analogous to the spherical system in $E_n$.

**9.1.7. Spherical coordinates on $H_\pm^{pq}$ and $C^{pq}$.** Let $\Omega^{pq}$ be the set of points $\mathbf{x} \in E_{pq}$ lying inside of the cone $C^{pq}$ (that is, such that $[\mathbf{x},\mathbf{x}]_{pq} > 0$). Spherical coordintes on $\Omega^{pq}$ are introduced by the formulas

$$\left.\begin{aligned} x_1 &= r\sinh t\cos\varphi_{p-1}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_p &= r\sinh t\sin\varphi_{p-1}\ldots\sin\varphi_1, \\ x_{p+1} &= r\cosh t\sin\theta_{q-1}\ldots\sin\theta_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{p+q} &= r\cosh t\cos\theta_{q-1}, \end{aligned}\right\} \tag{1}$$

where $r = [\mathbf{x},\mathbf{x}]_{pq}$ and

$$0 \leq t < \infty, \quad 0 \leq \varphi_1,\theta_1 < 2\pi, \quad 0 \leq \varphi_j,\theta_j \leq \pi, \quad j \neq 1. \tag{2}$$

By putting $r = 1$ we obtain the orispherical coordinate system on the hyperboloid $H_+^{pq}$. It is obtained by splitting $H_+^{pq}$ into spheres

$$x_{p+1}^2 + \cdots + x_{p+q}^2 = 1 + x_1^2 + \cdots + x_p^2 = \cosh^2 t. \tag{3}$$

The point $\mathbf{x} \in H_+^{pq}$ with coordinates (1) at $r = 1$ is obtained from $\mathbf{x}_n = (0, \dots, 0, 1) \in H_+^{pq}$ as

$$\mathbf{x} \equiv \mathbf{x}(t, \boldsymbol{\varphi}, \boldsymbol{\theta}) = g^p(\varphi_1, \dots, \varphi_{p-1}) g^q(\theta_1, \dots, \theta_{q-1}) g'_{1n}(t) \mathbf{e}_n, \tag{4}$$

where $g^p(\boldsymbol{\varphi})$ and $g^q(\boldsymbol{\theta})$ are elements of the subgroups $SO(p)$ and $SO(q)$ which are of the form

$$g^p(\boldsymbol{\varphi}) = g_{p-1}(-\varphi_1) g_{p-2}(-\varphi_2) \dots g_1(-\varphi_{p-1}), \tag{5}$$
$$g^q(\boldsymbol{\theta}) = g_{p+1}(\theta_1) g_{p+2}(\theta_2) \dots g_{n-1}(\theta_{q-1}). \tag{6}$$

In the same way one constructs spherical coordinates on $H_-^{pq}$.

It follows from (4) that elements $g \in SO_0(p, q)$ are representable in the form

$$g = g^p(\boldsymbol{\varphi}) g^q(\boldsymbol{\theta}) g'_{1n}(t) h, \quad h \in SO_0(p, q-1). \tag{7}$$

This implies that

$$SO_0(p, q) = SO(p) SO(q) \cdot A' \cdot SO_0(p, q-1). \tag{8}$$

The formulas

$$\left.\begin{array}{l} y_1 = e^t \cos\varphi_{p-1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ y_p = e^t \sin\varphi_{p-1} \dots \sin\varphi_1, \\ y_{p+1} = e^t \sin\theta_{q-1} \dots \sin\theta_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ y_{p+q} = e^t \cos\theta_{q-1} \end{array}\right\} \tag{9}$$

with $-\infty < t < +\infty$ and with restrictions (2) on remaining parameters define spherical coordinates on the cone $C^{pq}$. They are obtained by splitting the cone into spheres

$$y_1^2 + \dots + y_p^2 = y_{p+1}^2 + \dots + y_{p+q}^2 = e^t.$$

The point $\mathbf{y} \in C^{pq}$ with coordinates (9) is obtained from $\boldsymbol{\xi}_0 = (1, 0, \dots, 0, 1) \in C^{pq}$ in the following way:

$$\mathbf{y} = g^p(\varphi_1, \dots, \varphi_{p-1}) g^q(\theta_1, \dots, \theta_{q-1}) g'_{1n}(t) \boldsymbol{\xi}_0. \tag{10}$$

Another coordinate system on $C^{pq}$ is constructed by cutting $C^{pq}$ by the hyperplanes $y_{p+q} = c$, $c \in \mathbb{R}$. For a fixed $c$ this section is the hyperboloid

$$y_1^2 + \dots + y_p^2 - y_{p+1}^2 - \dots - y_{p+q-1}^2 = c^2.$$

It is identified with the quotient space $SO_0(p,q-1)/SO_0(p-1,q-1)$. Introducing coordinates of the type (1) on these sections, we obtain the coordinate system on $C^{pq}$:

$$\left.\begin{aligned} y_1 &= c\cosh\beta\cos\psi_{p-1}, \\ &\dots\dots\dots\dots\dots\dots \\ y_p &= c\cosh\beta\sin\psi_{p-1}\dots\sin\psi_1, \\ y_{p+1} &= c\sinh\beta\sin\gamma_{q-2}\dots\sin\gamma_1, \\ &\dots\dots\dots\dots\dots\dots \\ y_{n-1} &= c\sinh\beta\cos\gamma_{q-2}, \\ y_n &= c. \end{aligned}\right\} \qquad (11)$$

**9.1.8. The Laplace operators.** Let $\mathfrak{H}_n$ be the space of infinitely differentiable functions on $E_n$. The formula

$$(L(g)f)(\mathbf{x}) = f(g^{-1}\mathbf{x}), \quad g \in ISO(n), \tag{1}$$

defines the quasi-regular representation of the group $ISO(n)$ in $\mathfrak{H}_n$. The infinitesimal operators $I_{ij}$, $J_i$ (see Section 9.1.3) of this representation have the form

$$I_{ij} = \left.\frac{dL(g_{ij}(\theta))}{d\theta}\right|_{\theta=0} = x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}, \tag{2}$$

$$J_i = \left.\frac{dL(g^i(t))}{dt}\right|_{t=0} = -\frac{\partial}{\partial x_i}, \tag{3}$$

where $g^i(t)$ are shifts along the axis $x_i$.

To the element $C$ (see formula (6) of Section 9.1.3) of the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{iso}(n))$ in the representation $L$ there corresponds the operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}, \tag{4}$$

called the *Laplace operator* on $E_n$. By virtue of formula (7) of Section 9.1.3 it commutes with $I_{ij}$ and $J_i$. Since

$$L(g_{ij}(\theta)) = \exp\theta I_{ij}, \quad L(g^i(t)) = \exp tJ_i,$$

then $\Delta$ commutes with $L(g_{ij}(\theta))$ and $L(g^i(t))$ and, therefore, with all operators $L(g)$, $g \in ISO(n)$.

In order to find expressions for the Laplace operator in different coordinate systems we shall use the following assertion: *if the differential of an arc length in the coordinate system $u_1, \dots, u_n$ of $E_n$ has the form*

$$ds^2 = \sum_{i,j=1}^{n} A^{ij}(u_1, \dots, u_n)du_i du_j, \tag{5}$$

*then the Laplace operator* $\Delta$ *in this coordinate system is given by the formula*

$$\Delta = \frac{1}{A}\sum_{i,j=1}^{n}\frac{\partial}{\partial u_i}A_{ij}\frac{\partial}{\partial u_j}, \tag{6}$$

*where* $A = \det(A^{ij})$, $(A_{ij}) = (A^{ij})^{-1}$.

If $\mathbf{x} = r\boldsymbol{\xi}$, $\mathbf{x} \in E_n$, $\boldsymbol{\xi} \in S^{n-1}$, and $d\sigma$ is the differential of an arc length on $S^{n-1}$, then

$$ds^2 = dr^2 + r^2 d\sigma^2. \tag{7}$$

Further, if one chooses $\theta$ and $\boldsymbol{\eta} \in S^{s-1}$, $\boldsymbol{\zeta} \in S^{n-s-1}$ (see Section 9.1.5) as coordinates on $S^{n-1}$ and if $d\tau$ and $d\omega$ are the differentials of arc lengths on $S^{s-1}$ and $S^{n-s-1}$, respectively, then

$$d\sigma^2 = d\theta^2 + \sin^2\theta d\tau^2 + \cos^2\theta d\omega^2.$$

Substituting this expression for $d\sigma^2$ into (7) and computing $A_{ij}$, in accordance with (6) we obtain that

$$\Delta = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0, \tag{8}$$

where $\Delta_0$ is the Laplace operator on $S^{n-1}$ which is of the form

$$\begin{aligned}\delta_0 \equiv \Delta_0^{(n-1)} &= \sin^{-s+1}\theta\cos^{-n+s+1}\theta\frac{\partial}{\partial\theta}\sin^{s-1}\theta\cos^{n-s-1}\theta\frac{\partial}{\partial\theta} \\ &+ \sin^{-2}\theta\Delta_0^{(s-1)} + \cos^{-2}\theta\Delta_0^{(n-s-1)}.\end{aligned} \tag{9}$$

Here $\Delta_0^{(s-1)}$ and $\Delta_0^{(n-s-1)}$ are the Laplace operators on $S^{s-1}$ and $S^{n-s-1}$, respectively. For the spherical coordinate system $\theta_1, \dots, \theta_{n-1}$ on $S^{n-1}$ we have

$$\begin{aligned}\Delta_0^{(n-1)} &= \frac{1}{\sin^{n-2}\theta_{n-1}}\frac{\partial}{\partial\theta_{n-1}}\sin^{n-2}\theta_{n-1}\frac{\partial}{\partial\theta_{n-1}} + \frac{1}{\sin^2\theta_{n-1}}\Delta_0^{(n-2)} \\ &= \frac{1}{\sin^{n-2}\theta_{n-1}}\frac{\partial}{\partial\theta_{n-1}}\sin^{n-2}\theta_{n-1}\frac{\partial}{\partial\theta_{n-1}} + \cdots \\ &\quad + \frac{1}{\sin^2\theta_{n-1}\dots\sin^2\theta_2}\frac{\partial^2}{\partial\theta_1^2}.\end{aligned} \tag{10}$$

Instead of the Laplace operator $\Delta$ on $E_n$, in the space $E_{n-1,1}$ we have the *wave operator*

$$\Box = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} - \frac{\partial^2}{\partial x_n^2} \tag{11}$$

commuting with operators of the quasi-regular representation of the group $SO_0(n-1,1)$. By the above reasonings, one proves that inside of the cone $C_+^{n-1}$ (that is, at points $\mathbf{x} \in E_{n-1,1}$ such that $[\mathbf{x},\mathbf{x}] > 0$, $x_n \geq 0$) we have

$$\Box = -\frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\Box_+, \tag{12}$$

where $\Box_+$ is the Laplace operator on $H_+^{n-1}$. In addition, if $\boldsymbol{\xi} \in H_+^{n-1}$ is given by the parameters $\theta$ and $\boldsymbol{\eta} \in S^{s-1}$, $\boldsymbol{\zeta} \in H_+^{n-s-1}$ (see Section 9.1.5), then

$$\begin{aligned}\Box_+ \equiv \Box_+^{(n-1)} &= \sinh^{-s+1}\theta\cosh^{-n+s+1}\theta\frac{\partial}{\partial\theta}\sinh^{s-1}\theta\cosh^{n-s-1}\theta\frac{\partial}{\partial\theta}\\ &+\sinh^{-2}\theta\Delta_0^{(s-1)} - \cosh^{-2}\theta\,\Box_+^{(n-s-1)},\end{aligned} \tag{13}$$

where $\Delta_0^{(s-1)}$ is the Laplace operator on $S^{s-1}$ and $\Box_+^{(n-s-1)}$ is the Laplace operator on $H_+^{n-s-1}$.

In the orispherical coordinates on $H_+^{n-1}$ we have

$$\Box_+^{(n-1)} = e^{(n-2)\theta}\frac{\partial}{\partial\theta}e^{-(n-2)\theta}\frac{\partial}{\partial\theta} + e^{2\theta}\Delta^{(n-2)}, \tag{14}$$

where $\Delta^{(n-2)}$ is the Laplace operator on $E_{n-2}$.

Outside of the cone $C^{n-1}$ we have

$$\Box = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\Box_-, \tag{15}$$

where $\Box_-$ is the Laplace operator on $H_-^{n-1}$. In addition, if $\boldsymbol{\xi} \in H_-^{n-1}$ is given by the parameters $\theta$ and $\boldsymbol{\eta} \in S^{s-1}$, $\boldsymbol{\zeta} \in H_+^{n-s-1}$ or by the parameters $\theta$ and $\boldsymbol{\eta} \in S^{s-1}$, $\boldsymbol{\zeta} \in H_-^{n-s-1}$, then we have

$$\begin{aligned}\Box_- \equiv \Box_-^{(n-1)} &= \cosh^{-s+1}\theta\sinh^{-n+s+1}\theta\frac{\partial}{\partial\theta}\cosh^{s-1}\theta\sinh^{n-s-1}\theta\frac{\partial}{\partial\theta}\\ &+\cosh^{-2}\theta\Delta_0^{(s-1)} + \sinh^{-2}\theta\,\Box_+^{(n-s-1)}\end{aligned} \tag{16}$$

or

$$\begin{aligned}\Box_- \equiv \Box_-^{(n-1)} &= \sin^{-s+1}\theta\cos^{-n+s+1}\theta\frac{\partial}{\partial\theta}\sin^{s-1}\theta\cos^{n-s-1}\theta\frac{\partial}{\partial\theta}\\ &+\sin^{-2}\theta\Delta_0^{(s-1)} + \cos^{-2}\theta\,\Box_-^{(n-s-1)},\end{aligned} \tag{17}$$

respectively.

In the orispherical coordinates on $H_-^{n-1}$ we have

$$\Box_-^{(n-1)} = e^{(n-2)\theta}\frac{\partial}{\partial\theta}e^{-(n-2)\theta}\frac{\partial}{\partial\theta} + e^{2\theta}\Box_+^{(n-2)}. \tag{18}$$

**9.1.9. Invariant measures.** The measure on $S^{n-1}$, invariant with respect to $SO(n)$, is given by the formula

$$d\boldsymbol{\xi} = c\delta(\|\mathbf{x}\| - 1)\, d\mathbf{x}, \tag{1}$$

where $c > 0$, $d\mathbf{x} = dx_1 \dots dx_n$ and $\delta(y)$ is the delta-function. Indeed, both $\|\mathbf{x}\|$ and $d\mathbf{x}$ are invariant under rotations.

In the Cartesian coordiantes formula (1) takes the form

$$d\boldsymbol{\xi} = c\frac{dx_2 \dots dx_n}{|x_1|} = \dots = c\frac{dx_1 \dots dx_{n-1}}{|x_n|}. \tag{2}$$

The normalizing factor $c$ is chosen such that the measure of the whole sphere is equal to 1.

The invariant measure on $E_n$ in an orthogonal coordinate system $u_1, \dots u_n$, in which $ds^2$ has the form (5) of Section 9.1.8, is given by the formula

$$d\mathbf{x} = A du_1 \dots du_n. \tag{3}$$

Therefore, if $\boldsymbol{\xi} = \boldsymbol{\eta} \sin\theta + \boldsymbol{\zeta} \cos\theta$, where $\boldsymbol{\eta} \in S^{s-1}$, $\boldsymbol{\zeta} \in S^{n-s-1}$, then the invariant measures $d\boldsymbol{\xi}$ on $S^{n-1}$ is expressed in terms of the invariant measure $d\boldsymbol{\eta}$ on $S^{s-1}$ and $d\boldsymbol{\zeta}$ on $S^{n-s-1}$:

$$d\boldsymbol{\xi} = c \sin^{s-1}\theta \cos^{n-s-1}\theta d\boldsymbol{\eta} d\boldsymbol{\zeta} d\theta. \tag{4}$$

Applying formula (1) of Section 3.4.6, we see that

$$c = 2\Gamma\left(\frac{n}{2}\right) \Big/ \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{n-s}{2}\right). \tag{5}$$

In particular, formula (4) implies that[2]

$$d\boldsymbol{\xi} = \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \sin^{n-2}\theta\, d\boldsymbol{\zeta}\, d\theta, \quad \boldsymbol{\eta} \in S^{n-2}.$$

Hence it follows by induction that

$$d\boldsymbol{\xi} = \frac{\Gamma(n/2)}{2\pi^{n/2}} \sin^{n-2}\theta_{n-1} \sin^{n-3}\theta_{n-2} \dots \sin\theta_2 d\theta_1 \dots d\theta_{n-1} \tag{6}$$

in the spherical coordinates on $S^{n-1}$.

[2]Since $S^0$ consists of two points we have the factor $\frac{1}{2}$ in normalization of the measure.

Now let us find the invariant measure on $SO(n)$. Factorization (5) of Section 9.1.5 allows us to establish that $dg = dkd\boldsymbol{\xi}$, where $dk$ and $d\boldsymbol{\xi}$ are the invariant measures on $SO(n-1)$ and $S^{n-1}$, respectively. Therefore,

$$dg = \frac{\prod_{i=1}^{n} \Gamma(i/2)}{2^n \pi^{n(n+1)/2}} \prod_{i=1}^{n-1} \prod_{j=1}^{i} \sin^{j-1} \theta_j^i d\theta_j^i, \tag{7}$$

where $\theta_j^i$ are the Euler angles on $SO(n)$.

The invariant measures on $H+-^{n-1}$ and $SO_0(n-1,1)$ are of the forms analogous to (4) and (7). For example, for invariant measure $d\boldsymbol{\xi}$ on $H_+^{n-1}$ in coordinates (11) of Section 9.1.5 we have

$$d\boldsymbol{\xi} = \frac{dx_1 \ldots dx_{n-1}}{|x_n|} = \sinh^{n-2} \theta_{n-1} \sin^{n-3} \theta_{n-2} \ldots \sin \theta_2 d\theta_1 \ldots d\theta_{n-1}. \tag{8}$$

Hence,

$$d\boldsymbol{\xi} = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \sinh^{n-2} \theta_{n-1} d\eta d\theta_{n-1}, \tag{8'}$$

where $d\eta$ is the normalized invariant measure on $S^{n-2}$.

Every vector $\boldsymbol{\xi} \in H_+^{n-1}$ is represented as $\boldsymbol{\xi} = \boldsymbol{\eta} \sinh \theta + \boldsymbol{\zeta} \cosh \theta$, where $0 \leq \theta < +\infty$, $\boldsymbol{\eta} \in S^{s-1}$, $\boldsymbol{\zeta} \in H_+^{n-s-1}$ (see Section 9.1.5). One has the formula

$$d\boldsymbol{\xi} = c \sinh^{s-1} \theta \cosh^{n-s-1} \theta d\boldsymbol{\eta} d\boldsymbol{\zeta} d\theta, \tag{9}$$

analogous to formula (4) for $S^{n-1}$. Here $c = 2\pi^{s/2} / \Gamma\left(\frac{s}{2}\right)$.

For $SO_0(n-1,1)$ we have

$$dg = c_1 d\boldsymbol{\xi} dk, \tag{10}$$

where $d\boldsymbol{\xi}$ and $dk$ are the invariant measures on $H_+^{n-1}$ and $SO(n-1)$, respectively, and $c_1$ is a constant depending on normalization of the measure on $SO_0(n-1,1)$.

For the orispherical coordinate system (17) of Section 9.1.5 on $H_+^{n-1}$ we have

$$d\boldsymbol{\xi} = e^{-(n-2)\theta} d\theta dt_1 \ldots dt_{n-2}. \tag{11}$$

Therefore, to the decomposition (18) of Section 9.1.5 there corresponds the invariant measure

$$dg = c_2 e^{-(n-2)\theta} dk d\theta dn \tag{12}$$

on $SO_0(n-1,1)$, where $dk$ and $dn$ are the invariant measures on $SO(n-1)$ and $N$, respectively, and $c_2$ is a constant.

In coordinates (4) of Section 9.1.6 the invariant measure on the cone $C_+^{n-1}$ is given by the formula

$$d\boldsymbol{\xi} = \frac{dx_1 \ldots dx_{n-1}}{|x_n|} = r^{n-3} \sin^{n-3} \theta_{n-2} \ldots \sin \theta_2 dr d\theta_1 \ldots d\theta_{n-2}. \tag{13}$$

Let $\Gamma$ be a contour on $C_+^{n-1}$ intersecting every generatrix of the cone at one point. We denote by $d\boldsymbol{\eta}$ a measure on this contour such that

$$d\boldsymbol{\xi} = t^{n-3} dt d\boldsymbol{\eta}, \tag{14}$$

where $\boldsymbol{\xi} = t\boldsymbol{\eta}$, $t > 0$, $\boldsymbol{\eta} \in \Gamma$. One can easily verify that if a function $f(\boldsymbol{\xi})$ on $C_+^{n-1}$ is homogeneous of degree $2-n$ (that is, $f(t\boldsymbol{\xi}) = t^{2-n} f(\boldsymbol{\xi})$, $t > 0$), then the integral $\int_\Gamma f(\boldsymbol{\eta}) d\boldsymbol{\eta}$ does not depend on the choice of $\Gamma$. In particular, if $\Gamma_0$ is the section of $C_+^{n-1}$ by the plane $\boldsymbol{\xi}_n = 1$, then $d\boldsymbol{\eta}$ is the non-normalized measure on the sphere $S^{n-2}$. If $\Gamma_1^\varepsilon$, $\varepsilon \in \{+,-\}$, is the section of $C_+^{n-1}$ by the plane $\varepsilon \xi_{n-1} = 1$, then $d\boldsymbol{\eta}$ is measure (8) on the hyperboloid $H_+^{n-2}$. Finally, if $\Gamma_2$ is the section of $C_+^{n-1}$ by the plane $\xi_{n-1} + \xi_n = 1$, then $d\boldsymbol{\eta}$ is the Euclidean measure on $\mathbb{R}^{n-2}$. Namely, this section consists of points of the form

$$\boldsymbol{\eta} = \left( \mathbf{t}, \frac{1}{2}(1 - (\mathbf{t}, \mathbf{t})), \frac{1}{2}(1 + (\mathbf{t}, \mathbf{t})) \right), \quad \mathbf{t} \in \mathbb{R}^{n-2}, \tag{15}$$

and $d\boldsymbol{\eta} = d\mathbf{t}$.

Above we have introduced the Gauss decomposition of the group $SO_0(n-1,1)$ (see formula (15) of Section 9.1.6). In the parameters $n$, $m$, $\theta$, $\bar{n}$ the invariant measure $dg$ on $SO_0(n-1,1)$ is of the form

$$dg = c_2 e^{(n-2)\theta} dn dm d\theta d\bar{n}, \tag{16}$$

where $dn$, $dm$, $d\bar{n}$ are the invariant measures on $N$, $M$, $\bar{N}$, respectively, and $\theta$ is determined by the element $g'_{n-1}(\theta) \in A'$. The invariant measure $dn$ on $N$ has the form

$$dn \equiv dn(\mathbf{t}) = d\mathbf{t}. \tag{17}$$

The invariant measure on $\bar{N}$ has the same form.

For the invariant measure on $H_-^{n-1}$ we have

$$d\boldsymbol{\xi} = \delta([\mathbf{x}, \mathbf{x}] + 1) dx_1 \ldots dx_n, \tag{18}$$

that is,

$$d\boldsymbol{\xi} = \frac{dx_2 \ldots dx_n}{|x_1|} = \ldots = \frac{dx_1 \ldots dx_{n-1}}{|x_n|}. \tag{19}$$

In parameters (3) of Section 9.1.6 we have

$$d\xi = \cosh^{n-2}\theta_{n-1}\sin^{n-3}\theta_{n-2}\dots\sin\theta_2 d\theta_1 d\theta_2\dots d\theta_{n-1}. \tag{20}$$

We recommend to the reader to write down invariant measures on $H_-^{n-1}$ in other coordinate systems and corresponding invariant measures on $SO_0(n-1,1)$.

In the same way as in the case of $H_+^{n-1}$, we show that the invariant measure on the hyperboloid $H_+^{pq}$ is given by the formula

$$d\mathbf{x} = \frac{4\pi^{(p+q)/2}}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}\sinh^{p-1}t\cosh^{q-1}t\,d\xi d\eta, \tag{21}$$

where $d\xi$ and $d\eta$ are measures on $S^{p-1}$ and $S^{q-1}$, respectively.

Let us consider on $\Omega^{pq}$ the wave operator

$$\square_{pq} = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2}. \tag{22}$$

In spherical coordinates (1) of Section 9.1.9 on $\Omega^{pq}$ it takes the form

$$\square_{pq} = -\frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\square_0, \tag{23}$$

where

$$\square_0 = \frac{1}{\sinh^{p-1}t\cosh^{q-1}t}\frac{\partial}{\partial t}\sinh^{p-1}t\cosh^{q-1}t\frac{\partial}{\partial t} + \frac{\Delta_0^{(p-1)}}{\sinh^2 t} - \frac{\Delta_0^{(q-1)}}{\cosh^2 t}. \tag{24}$$

Here $\Delta_0^{(p-1)}$ and $\Delta_0^{(q-1)}$ are the Laplace operators on $S^{p-1}$ and $S^{q-1}$, respectively, in spherical coordinates. The operator $\square_0$ is called the *Laplace operator* on $H_+^{pq}$.

If $\xi$ is a fixed point on the cone $C^{pq}$ and $\sigma \in \mathbb{C}$, then $f_{\sigma\xi}$ will denote a function on $H_+^{pq}$ defined as

$$f_{\sigma\xi}(\mathbf{x}) = |-x_1\xi_1 - \dots - x_p\xi_p + x_{p+1}\xi_{p+1} + \dots + x_{p+q}\xi_{p+q}|^\sigma \equiv |[\mathbf{x},\xi]_{pq}|^\sigma. \tag{25}$$

Direct differentiation shows that

$$\square_0 f_{\sigma\xi}(\mathbf{x}) = \sigma(\sigma+p+q-2)f_{\sigma\xi}(\mathbf{x}). \tag{26}$$

## 9.2. Class 1 Representations of $SO(n)$ and of Related Groups

**9.2.1. The representations $T^{n\sigma}$ of the group $SO_0(n-1,1)$.** Let $\sigma$ be a complex number. Let us denote by $\mathfrak{B}^{n\sigma}$ the space of smooth functions on $C_+^{n-1}$

such that $f(t\boldsymbol{\xi}) = t^{\sigma}f(\boldsymbol{\xi})$, $t > 0$. It is obvious that this space is invariant with respect to the shift operators

$$(T(g)f)(\boldsymbol{\xi}) = f(g^{-1}\boldsymbol{\xi}), \quad g \in SO_0(n-1,1). \tag{1}$$

Therefore, restricting these operators onto $\mathfrak{B}^{n\sigma}$, we obtain a representation of $SO_0(n-1,1)$, denoted by $T^{n\sigma}$.

Since homogeneous functions on the cone are uniquely defined by their values on any contour $\Gamma$ intersecting every generatrix at one point, the representations $T^{n\sigma}$ can be realized in spaces of functions on these contours. In particular, they can be realized on the contours $\Gamma_0$, $\Gamma_1^{\pm}$, $\Gamma_2$ described in Section 9.1.9.

Let $\boldsymbol{\xi} \in \Gamma$. We put $\boldsymbol{\xi}' = g^{-1}\boldsymbol{\xi}$ and denote by $\alpha(\boldsymbol{\xi}, g)$ a factor such that $\hat{\boldsymbol{\xi}} = \alpha^{-1}(\boldsymbol{\xi}, g)\boldsymbol{\xi}' \in \Gamma$. Then we have

$$(T^{n\sigma}(g)f)(\boldsymbol{\xi}) = f(\boldsymbol{\xi}') = (\alpha(\boldsymbol{\xi}, g))^{\sigma} f(\hat{\boldsymbol{\xi}}) \tag{2}$$

(for the sake of simplicity, the realization of $T^{n\sigma}$ in the space of functions on $\Gamma$ is denoted by the same symbol $T^{n\sigma}$ and the restriction of a function $f \in \mathfrak{B}^{n\sigma}$ onto $\Gamma$ is denoted by the same symbol $f$).

In particular, let $\Gamma = \Gamma_0$ be the section of $C_+^{n-1}$ by the plane $\xi_n = 1$. For $\boldsymbol{\xi}' = g^{-1}\boldsymbol{\xi}$ we have $\alpha(\boldsymbol{\xi}, g) = \xi_n'$, where $\boldsymbol{\xi}' = (\xi_1', \dots, \xi_n')$. We obtain the realization of $T^{n\sigma}$ in the space of smooth functions on $S^{n-2}$. In addition, to elements $k \in SO(n-1)$ there correspond the operators

$$(T^{n\sigma}(k)f)(\boldsymbol{\xi}) = f(k^{-1}\boldsymbol{\xi}) \tag{3}$$

and to the element $g_{n-1}'(\varphi)$ there corresponds the operator

$$(T^{n\sigma}(g_{n-1}'(\varphi))f)(\boldsymbol{\xi}) = (\xi_n')^{\sigma} f\left(\frac{\boldsymbol{\xi}'}{\xi_n'}\right), \tag{4}$$

where $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{n-1}, 1)$, $\xi_j' = \xi_j$, $1 \le j \le n-2$,

$$\xi_{n-1}' = \xi_{n-1}\cosh\varphi - \sinh\varphi, \quad \xi_n' = \cosh\varphi - \xi_{n-1}\sinh\varphi \tag{5}$$

(since $|\xi_{n-1}| \le 1$, then $\cosh\varphi \ge |\sinh\varphi\ \xi_{n-1}|$, and $(\xi_n')^{\sigma}$ is uniquely defined). Thus, by setting

$$f(\xi_1, \dots, \xi_{n-1}, 1) = F(\xi_1, \dots, \xi_{n-1}), \ (\xi_1, \dots, \xi_{n-1}) \in S^{n-2},$$

we have

$$\begin{aligned}(T^{n\sigma}(g_{n-1}'(\varphi))F)(\xi_1, \dots, \xi_{n-1}) &= (\cosh\varphi - \xi_{n-1}\sinh\varphi)^{\sigma} \\ \times F\left(\frac{\xi_1}{\cosh\varphi - \xi_{n-1}\sinh\varphi}, \dots, \frac{\xi_{n-2}}{\cosh\varphi - \xi_{n-1}\sinh\varphi}, \frac{\xi_{n-1}\cosh\varphi - \sinh\varphi}{\cosh\varphi - \xi_{n-1}\sinh\varphi}\right)&. \end{aligned} \tag{4'}$$

By virtue of decomposition (13) of Section 9.1.5, formulas (3) and (4) or (4′) are sufficient for giving the representation $T^{n\sigma}$ in the space of functions on $S^{n-2}$.

Taking into account the connection between Cartesian and spherical coordinates on $S^{n-2}$ (see Section 9.1.5), we find that

$$(T^{n\sigma}(g'_{n-1}(\varphi))F)(\theta_1,\ldots,\theta_{n-2}) \\ = (\cosh\varphi - \sinh\varphi\cos\theta_{n-2})^\sigma F(\theta_1,\ldots,\theta_{n-3},\theta'_{n-2}), \quad (6)$$

where

$$\cos\theta'_{n-2} = \frac{\cosh\varphi\cos\theta_{n-2} - \sinh\varphi}{\cosh\varphi - \sinh\varphi\cos\theta_{n-2}} \quad (7)$$

(it is easy to show that $|\cos\theta'_{n-2}| \le 1$). Formulas for $T^{n\sigma}$ on other contours can be written down in the similar way.

We equip the space $\mathfrak{D}$ of smooth functions on $S^{n-2}$ with the scalar product

$$(F_1, F_2) = \int_{S^{n-2}} F_1(\xi)\overline{F_2(\xi)}d\xi, \quad (8)$$

where $d\xi$ is the invariant measure on $S^{n-2}$. Closing the space $\mathfrak{D}$ we obtain the Hilbert space $\mathfrak{L}^2(S^{n-2})$ with the scalar product (8). The operators $T^{n\sigma}(g)$, $g \in SO_0(n-1,1)$, are continued to bounded operators in $\mathfrak{L}^2(S^{n-2})$. As a result we obtain representations of $SO_0(n-1,1)$ in $\mathfrak{L}^2(S^{n-2})$ also denoted by $T^{n\sigma}$.

**9.2.2. Finite dimensional representations of the group $SO(n)$.** If $\sigma = \ell$ is a non-negative integer, then in $\mathfrak{B}^{n\sigma}$ there exists the finite dimensional subspace $\widehat{\mathfrak{B}}^{n\ell}$, invariant with respect to $SO_0(n-1,1)$. It consists of restrictions of homogeneous polynomials of degree $\ell$ in $n$ variables $x_1,\ldots,x_n$ onto $C_+^{n-1}$. We shall denote the restriction of the representation $T^{n\sigma}$, $\sigma = \ell$, onto $\widehat{\mathfrak{B}}^{n\ell}$ by $\widehat{T}^{n\ell}$. In order to go over from the representations $\widehat{T}^{n\ell}$ of the group $SO_0(n-1,1)$ to representations of the group $SO(n)$, we consider the "complex cone" $\widetilde{C}^{n-1}$, that is, the subset $\{\zeta \mid (\zeta,\zeta) = 0\}$ in $\mathbb{C}^n$, where

$$(\zeta,\zeta) = \zeta_1^2 + \ldots + \zeta_n^2.$$

It is obvious that the space of smooth functions on $\widetilde{C}^{n-1}$ is invariant with respect to the shift operators

$$(T(g)f)(\zeta) = f(g^{-1}\zeta), \quad g \in SO(n,\mathbb{C}),$$

and, in particular, with respect to the operators $T(g)$, $g \in SO(n)$. The space of homogeneous polynomials of degree $\ell$ also remains invariant.

Let us denote by $\check{\mathfrak{B}}^{n\ell}$ the space of restrictions of homogeneous polynomials in $\zeta_1, \ldots, \zeta_n$ of degree $\ell$ onto $\widetilde{C}^{n-1}$. The equality

$$\check{T}^{n\ell}(g)P(\boldsymbol{\zeta}) = P(g^{-1}\boldsymbol{\zeta}), \quad g \in SO(n, \mathbb{C}), \tag{1}$$

defines a finite dimensional representation of $SO(n,\mathbb{C})$ in $\check{\mathfrak{B}}^{n\ell}$ which is equivalent to the analytic continuation of the representation $\widehat{T}^{n\ell}$ of $SO_0(n-1,1)$ to the group $SO(n,\mathbb{C})$.

We denote by $\widetilde{\Gamma}_0$ the section of $\widetilde{C}^{n-1}$ by the plane $\zeta_n = i \equiv \sqrt{-1}$. It is the complex sphere $\Sigma^{n-2} = \{\boldsymbol{\zeta}' \in \mathbb{C}^{n-1} \mid (\boldsymbol{\zeta}', \boldsymbol{\zeta}') = 1\}$. Values of any polynomial of $\zeta_1, \ldots, \zeta_n$ on $\Sigma^{n-2}$ are uniquely defined by its values on $S^{n-2}$. We shall denote values of a polynomial $P$ on $\Sigma^{n-2}$ by $P(\boldsymbol{\zeta}', i)$. Then for $k \in SO(n-1)$ and for $g_{n-1}(\varphi) \in SO(n)$ we have

$$\check{T}^{n\ell}(k)P(\boldsymbol{\zeta}', i) = P(k^{-1}\boldsymbol{\zeta}', i), \tag{2}$$

$$\begin{aligned} &\check{T}^{n\ell}(g_{n-1}(\varphi))P(\zeta_1, \ldots, \zeta_{n-1}, i) \\ &\quad = P(\zeta_1, \ldots, \zeta_{n-2}, \zeta_{n-1}\cos\varphi - i\sin\varphi, \zeta_{n-1}\sin\varphi + i\cos\varphi) \\ &\qquad = (\cos\varphi - i\zeta_{n-1}\sin\varphi)^{\ell} \\ &\times P\left(\frac{\zeta_1}{\cos\varphi - i\zeta_{n-1}\sin\varphi}, \ldots, \frac{\zeta_{n-2}}{\cos\varphi - i\zeta_{n-1}\sin\varphi}, \frac{\zeta_{n-1}\cos\varphi - i\sin\varphi}{\cos\varphi - i\zeta_{n-1}\sin\varphi}\right). \end{aligned} \tag{2'}$$

In the spherical coordinates on $S^{n-2}$ the operator $\check{T}^{n\ell}(g_{n-1}(\varphi))$ is given as

$$\check{T}^{n\ell}(g_{n-1}(\varphi))P(\theta_1, \ldots, \theta_{n-2}) = (\cos\varphi - i\cos\theta_{n-2}\sin\varphi)^{\ell} P(\theta_1, \ldots, \theta_{n-3}, \theta'_{n-2}), \tag{3}$$

where

$$\cos\theta'_{n-1} = \frac{\cos\theta_{n-2}\cos\varphi - i\sin\varphi}{\cos\varphi - i\cos\theta_{n-2}\sin\varphi}. \tag{4}$$

Thus, we obtain the realization of the representation $\check{T}^{n\ell}$ of $SO(n)$ in the space $\mathfrak{D}^{n-1,\ell}$ which consists of restrictions of polynomials of $x_1, \ldots, x_{n-1}$ of degree $\leq \ell$ onto $S^{n-2}$.

**9.2.3. Realizations of representations of the groups $SO(n)$ and $SO_0(n-1,1)$ in spaces of harmonic and □-harmonic functions.** Another realization of representations of the group $SO(n)$ (respectively, of $SO_0(n-1,1)$) is based on the fact that the operator $\Delta$ (respectively, $\Box$) commutes with the action of $SO(n)$ (respectively, of $SO_0(n-1,1)$) in the space of functions on $E_n$ (respectively, inside of $C_+^{n-1}$ or outside of $C^{n-1}$). Therefore, if $\Delta f(\mathbf{x}) = 0$ (respectively,

$\Box f(\mathbf{x}) = 0$) then $\Delta f(g^{-1}\mathbf{x}) = 0$ (respectively, $\Box f(g^{-1}\mathbf{x}) = 0$), where $g \in SO(n)$ (respectively, $g \in SO_0(n-1,1)$). A function $f$ will be called ***harmonic*** (respectively, $\Box$-***harmonic***), if $\Delta f = 0$ (respectively, $\Box f = 0$).

As an example of a function, harmonic in $E_n$ at all points except for the origin, one can regard $r^{2-n}$, where $r^2 = x_1^2 + \ldots + x_n^2$. The direct calculation shows that $\Delta(r^{2-n}) = 0$ for $\mathbf{x} \neq \mathbf{0}$. Making use of the Green formula, one can easily prove that

$$\Delta(r^{2-n}) = -\frac{2\pi^{n/2}(n-2)}{\Gamma(n/2)}\delta(\mathbf{x}).$$

By virtue of the invariance of the Laplace operator with respect to shifts, the function $\|\mathbf{x} - \mathbf{a}\|^{2-n}$ is also harmonic in $E_n$ except for the point $\mathbf{a} \in E_n$.

Functions of the form $f(\mathbf{x}) = \varphi((\mathbf{x}, \boldsymbol{\zeta}))$, where $\varphi$ is twice differentiable, $(\mathbf{x}, \boldsymbol{\zeta}) = x_1\zeta_1 + \ldots + x_n\zeta_n$ and $\boldsymbol{\zeta}$ is a point on the complex cone $\widetilde{C}^{n-1}$, also are examples of harmonic functions. In particular, polynomials of the form $(\mathbf{x}, \boldsymbol{\zeta})^\ell$, $\boldsymbol{\zeta} \in \widetilde{C}^{n-1}$, are harmonic.

The function $f(\mathbf{x}) = [\mathbf{x}, \mathbf{x}]^{(2-n)/2}$ is an example of a $\Box$-harmonic function, for which $\Box f = 0$ in $\mathbb{R}^n \backslash C^{n-1}$. One can show that functions of the form $\varphi([\mathbf{x}, \boldsymbol{\zeta}])$, where $\varphi$ is twice differentiable and $[\boldsymbol{\zeta}, \boldsymbol{\zeta}] = 0$, also are $\Box$-harmonic. In particular, the function $[\mathbf{x}, \boldsymbol{\zeta}]^\sigma$, where $[\boldsymbol{\zeta}, \boldsymbol{\zeta}] = 0$ are $\Box$-harmonic in $\Omega_n$.

Note that harmonicity in $E_n$ of a homogeneous function $f(\mathbf{x})$ of degree $\ell$ implies that the function $r^{2-2\ell-n} f(\mathbf{x})$ is also harmonic. Analogous statement holds for $\Box$-harmonic functions, if we replace $r$ by $[\mathbf{x}, \mathbf{x}]^{1/2}$. This statement is directly verified by making use of the Euler formula. The passage from $f(\mathbf{x})$ to $r^{2-2\ell-n} f(\mathbf{x})$ is called the *Kelvin transform.*

Since the Laplace operator $\Delta$ (respectively, the operator $\Box$) is invariant with respect to the action of $SO(n)$ (respectively, of $SO_0(n-1,1)$), then the space of harmonic (respectively, $\Box$-harmonic) functions is invariant with respect to operators of the quasi-regular representation of $SO(n)$ (respectively, of $SO_0(n-1,1)$). Since the space of homogeneous functions is also invariant with respect to shift operators, then we obtain the following realizations of representations of the groups $SO(n)$ and $SO_0(n-1,1)$.

Let us denote by $\mathfrak{H}^{n\ell}$ the space of homogeneous harmonic polynomials of degree $\ell$ in variables $x_1, \ldots, x_n$. Then the equality $T^{n\ell}(g)f(\mathbf{x}) = f(g^{-1}\mathbf{x})$ defines a representation of the group[3] $SO(n)$ in $\mathfrak{H}^{n\ell}$. We denote by $\mathfrak{H}^+(n,\sigma)$ (respectively, by $\mathfrak{H}^-(n,\sigma)$) the space of $\Box$-harmonic functions of degree $\sigma$, defined inside of the upper sheet of (respectively, outside of) the cone $C^{n-1}$. The equality $R^{n\sigma}(g)f(\mathbf{x}) = f(g^{-1}\mathbf{x})$ defines a representation of the group $SO_0(n-1,1)$ in the spaces $\mathfrak{H}^+(n,\sigma)$ and $\mathfrak{H}^-(n,\sigma)$.

We shall prove that the representation $T^{n\ell}$ of $SO(n)$ is equivalent to the representation $\check{T}^{n\ell}$ of this group, constructed in Section 9.2.2. For this we note

[3]It should be noted that the representation $T^{n\ell}$ does not coincide with $T^{n\sigma}$, $\sigma = \ell$, since these representations are related to different groups.

that by virtue of the statement of Section 8.1.1 the decomposition

$$\mathfrak{N}^{n\ell} = \mathfrak{H}^{n\ell} + r^2\mathfrak{N}^{n,\ell-2} \tag{1}$$

holds, where $\mathfrak{N}^{n\ell}$ denotes the space of homogeneous polynomials of degree $\ell$ in $x_1, \ldots, x_n$, $r^2 = x_1^2 + \ldots + x_n^2$ and $r^2\mathfrak{N}^{n,\ell-2}$ is the space of polynomials of the form $r^2P(\mathbf{x})$, where $P(\mathbf{x}) \in \mathfrak{N}^{n,\ell-2}$. We have $r^2 = 0$ on the cone $\widetilde{C}^{n-1}$ and, therefore, the value of any polynomial $P(\mathbf{x})$ from $\mathfrak{N}^{n\ell}$ on $\widetilde{C}^{n-1}$ coincides with the value on $\widetilde{C}^{n-1}$ of uniquely defined harmonic polynomial $h_\ell(\mathbf{x}) \in \mathfrak{H}^{n\ell}$. This determines an isomorphism between the spaces $\mathfrak{N}^{n\ell}/r^2\mathfrak{N}^{n,\ell-2}$ and $\mathfrak{H}^{n\ell}$, which gives an equivalence of the representations $\check{T}^{n\ell}$ and $T^{n\ell}$.

An equivalence of the representations $T^{n\sigma}$ and $R^{n\sigma}$ of the group $SO_0(n-1,1)$ is proved in the same way.

Note that decomposition (1) implies the following statement. *Any homogeneous polynomial $P(\mathbf{x})$ of degree $\ell$ in $n$ variables can be represented in the form*

$$P(\mathbf{x}) = \sum_{j=0}^{[\ell/2]} r^{2j} h_{\ell-2j}(\mathbf{x}), \tag{2}$$

*where $h_{\ell-2j} \in \mathfrak{H}^{n,\ell-2j}$ are harmonic polynomials and $[\ell/2]$ is the integral part of the number $\ell/2$.*

This expansion of $P(\mathbf{x})$ will be called *canonical.* The polynomial $h_\ell(\mathbf{x})$ from (2) will be called the *harmonic projection* of the polynomial $P(\mathbf{x})$. One can easily verify that

$$h_\ell(\mathbf{x}) = \sum_{j=0}^{[\ell/2]} \frac{(-1)^j r^{2j} \Delta^j P(\mathbf{x})}{2^j j!(n+2\ell-4)(n+2\ell-6)\ldots(n+2\ell-2j-2)}. \tag{3}$$

We have $\dim \mathfrak{N}^{n\ell} = \frac{(n-\ell-1)!}{(n-1)!\ell!}$. Let us denote $\dim \mathfrak{H}^{n\ell}$ by $h(n,\ell)$. Equality (1) implies that

$$h(n,\ell) = \dim \mathfrak{N}^{n\ell} - \dim \mathfrak{N}^{n,\ell-2} = \frac{(2\ell+n-2)(n+\ell-3)!}{(n-2)!\ell!}. \tag{4}$$

Since in $\mathfrak{H}^{n\ell}$ the representation $T^{n\ell}$ of $SO(n)$, equivalent to $\check{T}^{n\ell}$, is realized, then

$$\dim T^{n\ell} = \dim \check{T}^{n\ell} = \frac{(\ell+n-3)!(2\ell+n-2)}{(n-2)!\ell!}. \tag{5}$$

We denote by $\widetilde{\mathfrak{H}}^{n\ell}$ the space of functions on $S^{n-1}$ which are values on $S^{n-1}$ of polynomials from $\mathfrak{H}^{n\ell}$. By Weierstrass's theorem, restrictions of polynomials, defined in $\mathbf{R}^n$, onto $S^{n-1}$ form an everywhere dense set of $C(S^{n-1})$ and, therefore, of $\mathfrak{L}^2(S^{n-1})$. By formula (2) any such restriction is a harmonic polynomial. Hence, we obtain that the spaces $\widetilde{\mathfrak{H}}^{n\ell}$, $\ell = 0, 1, 2, \ldots$, generate an everywhere dense subset in $\mathfrak{L}^2(S^{n-1})$. The correspondence $\widetilde{\mathfrak{H}}^{n\ell} \ni P(\boldsymbol{\xi}) \leftrightarrow P(\mathbf{x}) \in \mathfrak{H}^{n\ell}$ is one-to-one.

The following statements will be used below: a) *if* $P \in \mathfrak{R}^{n,\ell-2}$, *then*

$$\Delta(r^2 P) = (2n + 4\ell - 8)P + r^2 \Delta P; \tag{6}$$

b) *if* $h(\mathbf{x}) \in \mathfrak{H}^{n\ell}$, *then for any* $j$, $1 \le j \le n$, *we have* $\partial h/\partial x_j \in \mathfrak{H}^{n,\ell-1}$ *and*

$$\hat{h}(\mathbf{x}) \equiv x_j h(\mathbf{x}) - \frac{r^2}{n + 2\ell - 2} \frac{\partial h}{\partial x_j} \in \mathfrak{H}^{n,\ell+1}.$$

They are proved in the following way. Direct computation shows that

$$\Delta(r^2 P) = 2nP + 4 \sum_{k=1}^{n} x_k \frac{\partial P}{\partial x_k} + r^2 \Delta P. \tag{7}$$

Since $P$ is a homogeneous polynomial of degree $\ell - 2$, then, by Euler's theorem, $\sum\limits_{k=1}^{n} x_k \frac{\partial P}{\partial x_k} = (\ell - 2)P$ and, therefore, (7) implies (6). The fact, that $\partial h/\partial x_j \in \mathfrak{H}^{n,\ell-1}$, follows directly from permutability of $\Delta$ and $\partial/\partial x_j$. Further, it follows from $\Delta h = 0$ that $\Delta(x_j h) = 2\partial h/\partial x_j$. On the other hand, since $\partial h/\partial x_j \in \mathfrak{H}^{n,\ell-1}$, then

$$\Delta \left( r^2 \frac{\partial h}{\partial x_j} \right) = (2n + 4\ell - 4) \frac{\partial h}{\partial x_j} = (n + 2\ell - 2)\Delta(x_j h(\mathbf{x})).$$

Thus,

$$\Delta \hat{h}(\mathbf{x}) = \Delta \left[ x_j h(\mathbf{x}) - \frac{r^2}{n + 2\ell - 2} \frac{\partial h}{\partial x_j} \right] = 0.$$

The statement b) is proved.

**9.2.4. The representations $T^{nR}$ of the groups $ISO(n-1)$ and $ISO_0(n-2,1)$.** The group $ISO(n-1)$ can be obtained by a limit passage from the group $SO_0(n-1,1)$ in analogous way as it was done in Section 6.1.3 for $n = 3$. This allows us to obtain representations of $ISO(n-1)$ by means of a limit passage from the representations $T^{n\sigma}$ of $SO_0(n-1,1)$. Namely, let us replace in formulas (6) and (7) of Section 9.2.1 $\varphi$ by $\varphi/t$ and $\sigma$ by $t\sigma$, then take the limit $t \to +\infty$. We obtain the representations $T^{n\sigma}$, $\sigma \in \mathbb{C}$, of the group $ISO(n-1)$. They are realized in the space of smooth functions on $S^{n-2}$ or in $\mathfrak{L}^2(S^{n-2})$. For $g = k \in SO(n-1)$ they are given by the formula

$$(T^{n\sigma}(k)f)(\boldsymbol{\xi}) = f(k^{-1}\boldsymbol{\xi}) \tag{1}$$

and for $g \equiv g_r = g(e, \mathbf{a}_r)$, $\mathbf{a}_r = (0, \ldots, 0, r)$, by the formula

$$(T^{n\sigma}(g_r)f)(\boldsymbol{\xi}) = e^{r\sigma\cos\theta_{n-2}} f(\boldsymbol{\xi}), \tag{2}$$

where $\theta_{n-2}$ is the last spherical coordinate of $\boldsymbol{\xi} \in S^{n-2}$. These formulas can be given by the common equality

$$(T^{nR}(g(k, \mathbf{a}))f)(\boldsymbol{\xi}) = e^{R(\mathbf{a}, \boldsymbol{\xi})} f(k^{-1}\boldsymbol{\xi}), \quad R \in \mathbb{C}, \tag{3}$$

(we have replaced $\sigma$ by $R$), where $(\mathbf{a}, \boldsymbol{\xi}) = a_1\xi_1 + \ldots + a_{n-1}\xi_{n-1}$. One can also directly verify that formula (3) defines a representation of the group $ISO(n-1)$.

Analogously, the representations $T^{nR}$, $R \in \mathbb{C}$, of $ISO_0(n-2,1)$ are constructed in the space of smooth functions on $H_+^{n-2}$ and are given by the formula

$$(T^{nR}(g(h, \mathbf{a}))f)(\boldsymbol{\xi}) = e^{R[\mathbf{a}, \boldsymbol{\xi}]} f(h^{-1}\boldsymbol{\xi}), \tag{4}$$

where $[\mathbf{a}, \boldsymbol{\xi}] = -a_1\xi_1 \ldots - a_{n-2}\xi_{n-2} + a_{n-1}\xi_{n-1}$. One can construct representations of the group $ISO_0(n-2,1)$ on the cone $C_+^{n-2}$ and on the hyperboloid $H_-^{n-2}$. They are defined by formula (4) provided $\boldsymbol{\xi}$ belongs to the corresponding manifold.

We describe another realization of irreducible representations of $ISO(n-1)$, analogous to that of the representations $T^{n\sigma}$ of $SO_0(n-1,1)$. Let us denote by $\mathfrak{H}^{n,iR}$ the space of functions on $\mathbb{R}^{n-1}$ satisfying the differential equation $\Delta f(\mathbf{x}) = -R^2 f(\mathbf{x})$. The equality

$$(T^{n,iR}(g)f)(\mathbf{x}) = f(g^{-1}\mathbf{x}) \tag{5}$$

defines a representation of the group $ISO(n-1)$ in $\mathfrak{H}^{n,iR}$, which is irreducible for $R \neq 0$. It is unitary if $R \in \mathbb{R}$.

**9.2.5. Infinitesimal operators of representations.** Let us compute infinitesimal operators of the representations $T^{n\sigma}$ of $SO_0(n-1,1)$ in the realization on the contour $\Gamma_0 = S^{n-2}$. According to formula (3) of Section 9.2.1 to the one-parameter subgroups $\{g_{jk}(\theta)\}$, $1 \le j < k \le n-1$, there correspond operators of rotations in the planes $(x_j, x_k)$. Corresponding infinitesimal operators are given by formula (2) of Section 9.1.8. In order to find the infinitesimal operator $I'^{n\sigma}_{n-1,n}$ corresponding to the subgroup $\{g'_{n-1}(t)\}$ we differentiate the right hand side of equality (6) of Section 9.2.1 with respect to $\varphi$ and put $\varphi = 0$. We obtain

$$I'^{n\sigma}_{n-1,n} = -\sigma\cos\theta_{n-2} + \sin\theta_{n-2}\frac{\partial}{\partial\theta_{n-2}}. \tag{1}$$

The infinitesimal operators $I^{n\ell}_{ij}$, $1 \le i < j \le n-1$, for the representations $\check{T}^{n\ell}$ of $SO(n)$ have the same form as for $T^{n\sigma}$. The infinitesimal operator $I^{n\ell}_{n-1,n}$ is found with the help of formulas (3) and (4) of Section 9.2.2. It is given by the formula

$$I^{n\ell}_{n-1,n} = i\,\ell\cos\theta_{n-2} - i\cos\theta_{n-2}\frac{\partial}{\partial\theta_{n-2}}. \tag{2}$$

We similarly find infinitesimal operators for the representations $T^{nR}$ of the group $ISO(n-1)$. For the subgroups $\{g_{jk}(\varphi)\}$, $1 \le j < k \le n-1$, they are of the form (2) of Section 9.1.8. To the subgroup $\{g_r \equiv g(e, \mathbf{a}_r)\}$, $\mathbf{a}_r = (0, \ldots, 0, r)$, of $ISO(n-1)$ there corresponds the infinitesimal operator

$$J_{n-1}^{nR} = R \cos \theta_{n-2}. \tag{3}$$

To the subgroups $\{g_{ij}(\varphi)\}$, $1 \le i < j \le n-2$, of $ISO_0(n-2,1)$ there correspond in the representations $T^{nR}$ the same infinitesimal operators as in the case of representations of $ISO(n-1)$. To the one-parameter subgroups of hyperbolic rotations $\{g'_{j,n-1}(\varphi)\}$, $1 \le j \le n-2$, there correspond the infinitesimal operators $I'_{j,n-1}$, analogous to operators (2) of Section 9.1.8:

$$I'_{j,n-1} = x_{n-1} \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_{n-1}}. \tag{4}$$

For the infinitesimal operator $J'_{n-1}$ corresponding to shifts along $x_{n-1}$ we have

$$J'_{n-1} = R \cosh \theta_{n-2}. \tag{5}$$

**9.2.6. Irreducibility.** Let us prove that the representations $T^{n\ell}$ of $SO(n)$, constructed in Section 9.2.2, are irreducible. We use induction with respect to $n$. For $n = 3$ the statement follows from irreducibility of the representations $T_\ell$ of the group $SU(2)$ (and of the group $SO(3)$, locally isomorphic to $SU(2)$, in the case of integral $\ell$), proved in Section 6.2.2.

Assume that the representations $T^{n-1,k}$ of $SO(n-1)$, $n \ge 4$, are irreducible. In order to prove irreducibility of $T^{n\ell}$ let us realize this representation in the space of homogeneous polynomials of degree $\ell$ on the complex cone $\widetilde{C}^{n-1}$ (see Section 9.2.2). Restricting $T^{n\ell}$ onto $SO(n-1)$, we obtain the representation of $SO(n-1)$ by shifts in the space $\mathfrak{F}^{n\ell}$ of functions on $S^{n-2}$ which are restrictions onto $S^{n-2}$ of harmonic polynomials of degree $\le \ell$ in $\boldsymbol{\xi}' = (\xi_1, \ldots, \xi_{n-1})$ (see formula (2) of Section 9.2.2). Decomposing the space of this representation into the subspaces $\mathfrak{A}^{n\ell k}$ of homogeneous harmonic polynomials in $\boldsymbol{\xi}'$ of degree $k$, $0 \le k \le \ell$, we obtain on every $\mathfrak{A}^{n\ell k}$ the representation of $SO(n-1)$, equivalent to $T^{n-1,k}$. By the assumption, these representations are irreducible. In addition, if $k \ne m$, then $T^{n-1,k}$ and $T^{n-1,m}$ are nonequivalent, since $\dim T^{n-1,k} \ne \dim T^{n-1,m}$ (see formula (5) of Section 9.2.3).

Thus, we have proved that

$$T^{n\ell} \downarrow \begin{matrix} SO(n) \\ SO(n-1) \end{matrix} = \sum_{k=0}^{\ell} T^{n-1,k}. \tag{1}$$

By virtue of irreducibility of $T^{n-1,k}$ any subspace $\mathfrak{T}$ of $\mathfrak{F}^{n\ell}$, invariant with respect to $T^{n\ell}$, is the direct sum of some of the subspaces $\mathfrak{A}^{n\ell k}$.

It remains to prove that if $\mathfrak{A}^{n\ell m} \subset \mathfrak{T}$, then $\mathfrak{A}^{n\ell,m-1} \subset \mathfrak{T}$ and $\mathfrak{A}^{n\ell,m+1} \subset \mathfrak{T}$ (here we assume that $\mathfrak{A}^{n,\ell,-1} = \mathfrak{A}^{n,\ell,\ell+1} = 0$). For this we note that the infinitesimal operator $I_{n-1,n}$ of the quasi-regular representation of $SO(n)$, corresponding to the subgroup $\{g_{n-1}(\varphi)\}$, has the form $x_n\partial/\partial x_{n-1} - x_{n-1}\partial/\partial x_n$. But the space $\mathfrak{A}^{n\ell m}$ consists of cosets with respect to $r^2\mathfrak{R}^{n,\ell-2}$ (see Section 9.2.3), containing functions of the form $x_n^{\ell-m}h_m(\mathbf{x}')$, where $\mathbf{x}' = (x_1, \ldots, x_{n-1})$, $h_m(\mathbf{x}') \in \mathfrak{H}^{n-1,m}$. We have

$$I_{n-1,n}(x_n^{\ell-m}h_m(\mathbf{x}')) = x_n^{\ell-m+1}\frac{\partial h_m(\mathbf{x}')}{\partial x_{n-1}} - (\ell-m)x_n^{\ell-m-1}x_{n-1}h_m(\mathbf{x}').$$

Let us prove that the expression on the right hand side of this equality can be represented in the form of a linear combination of functions from the subspaces

$$x_n^{\ell-m+1}\mathfrak{H}^{n-1,m-1},\quad x^{\ell-m-1}\mathfrak{H}^{n-1,m+1},\quad r^2\mathfrak{R}^{n,\ell-2}.$$

It was shown in Section 9.2.3 that

$$\hat{h}_{m+1}(\mathbf{x}') \equiv x_{n-1}h_m(\mathbf{x}') - \frac{r_{n-1}^2}{n+2m-3}\frac{\partial h_m(\mathbf{x}')}{\partial x_{n-1}} \in \mathfrak{H}^{n-1,m+1}.$$

We have $r_{n-1}^2 = r^2 - x_n^2$. Therefore,

$$\begin{aligned} I_{n-1,n}\left(x_n^{\ell-m}h_m(\mathbf{x}')\right) &= \frac{\ell+m+n-3}{2m+n-3}x_n^{\ell-m+1}\frac{\partial h_m(\mathbf{x}')}{\partial x_{n-1}} \\ &\quad - (\ell-m)x_n^{\ell-m-1}\hat{h}_{m+1}(\mathbf{x}') - \frac{\ell-m}{2m+n-3}r^2x_n^{\ell-m-1}\frac{\partial h_m(\mathbf{x}')}{\partial x_{n-1}}. \qquad (2)\end{aligned}$$

Since $\partial h_m/\partial x_{n-1} \in \mathfrak{H}^{n-1,m-1}$, $\hat{h}_{m+1} \in \mathfrak{H}^{n-1,m+1}$, then expansion (2) is the desired linear combination for $I_{n-1,n}(x_n^{\ell-m}h_m(\mathbf{x}'))$.

If $m \neq 0$, then the harmonic polynomial $h_m(\mathbf{x}')$ can be chosen in such a way that $\partial h_m/\partial x_{n-1} \neq 0$, $\hat{h}_{m+1} \neq 0$ (it was shown in Section 9.2.3 that $h_m(\mathbf{x}')$ and, therefore, $x_{n-1}h_m(\mathbf{x}')$ are indivisible by $r^2$). Hence, for $m \neq 0$, $m \neq \ell$ the first and the second summands in (2) are nonvanishing. For $m = 0$ we have $\hat{h}_{m+1} \neq 0$ and for $m = \ell$ we have $\partial h_\ell/\partial x_{n-1} \neq 0$. therefore, the image of the polynomial $I_{n-1,n}(x_n^{\ell-m}h_m(\mathbf{x}'))$ under the mapping from $\mathfrak{R}^{n\ell}$ onto $\mathfrak{R}^{n\ell}/r^2\mathfrak{R}^{n,\ell-2}$ lies in the sum of the subspaces $\mathfrak{A}^{n\ell,m-1}$ and $\mathfrak{A}^{n\ell,m+1}$. Since the image of this polynomial belongs to the invariant subspace $\mathfrak{T}$, then $\mathfrak{T}$ has to contain both $\mathfrak{A}^{n\ell,m-1}$ and $\mathfrak{A}^{n\ell,m+1}$ (for $m = 0$ only $\mathfrak{A}^{n\ell 1}$ and for $m = \ell$ only $\mathfrak{A}^{n\ell,\ell-1}$). Therefore, $\mathfrak{T}$ contains all the subspaces $\mathfrak{A}^{n\ell m}$ and, consequently, coincides with $\mathfrak{F}^{n\ell}$. Irreducibility of $T^{n\ell}$ is proved.

Now we show that $T^{n\ell}$ are *representations of class 1 with respect to the subgroup* $SO(n-1)$. For this it is sufficient to observe that the coset $x_n^\ell + r^2\mathfrak{N}^{n,\ell-2}$ is invariant under the action of the operators $T^{n\ell}(k)$, $k \in SO(n-1)$. One can easily show that any coset in $\mathfrak{N}^{n\ell}$, invariant with respect to all $T^{n\ell}(k)$, $k \in SO(n-1)$, is proportional to $x_n^\ell + r^2\mathfrak{N}^{n,\ell-2}$.

It was shown in Section 9.2.3 that the subspaces $\widetilde{\mathfrak{H}}^{n\ell}$, $0 \le \ell < \infty$, generate $\mathfrak{L}^2(S^{n-1})$. Since nonequivalent representations of $SO(n)$ are realized in these subspaces, then every pair of them is orthogonal. Consequently,

$$\mathfrak{L}^2(S^{n-1}) = \sum_{\ell=0}^{\infty} \oplus \widetilde{\mathfrak{H}}^{n\ell}. \tag{2'}$$

The shifts $L(g)f(\boldsymbol{\xi}) = f(g^{-1}\boldsymbol{\xi})$ realize the representations $T^{n\ell}$ in the spaces $\widetilde{\mathfrak{H}}^{n\ell}$. Therefore, the *quasi-regular representation $L$ of the group $SO(n)$ in $\mathfrak{L}^2(S^{n-1})$ is the direct sum of the irreducible representations $T^{n\ell}$*, $\ell = 0, 1, 2, \dots$ .

Irreducibility of the representations $T^{n\sigma}$ of $SO_0(n-1,1)$ for non-integral values of $\sigma$ is proved in the same way as in the case of the representations $T^{n\ell}$ of $SO(n)$. At first we restrict $T^{n\sigma}$ onto the subgroup $SO(n-1)$ and obtain the quasi-regular representation of this subgroup in $\mathfrak{L}^2(S^{n-2})$. Then we decompose $\mathfrak{L}^2(S^{n-2})$ into irreducible subspaces $\widetilde{\mathfrak{H}}^{n-1,\ell}$, $\ell = 0, 1, 2, \dots$ . Finally, making use of formula (1) of Section 9.2.5 for the operator $I'^{n\sigma}_{n-1,n}$ we prove that if some component $\widetilde{\mathfrak{H}}^{n-1,\ell}$ belongs to the subspace $\mathfrak{T} \subset \mathfrak{L}^2(S^{n-2})$, invariant with respect to $SO_0(n-1,1)$, then $\mathfrak{T}$ contains also neighboring components $\widetilde{\mathfrak{H}}^{n-1,\ell-1}$ and $\widetilde{\mathfrak{H}}^{n-1,\ell+1}$. We recommend to the reader to carry out the details of this proof.

In the same way one proves irreducibility of the representations $T^{n\sigma}$ for integral $\sigma$ from the interval $-n+1 \le \sigma < 0$. When $\sigma$ is an integer, lying outside of the interval $-n+1 \le \sigma < 0$, the representation $T^{n\sigma}$ is reducible. Namely, if $\sigma = \ell \in \mathbf{Z}_+ \cup \{0\}$, then the finite dimensional subspace $\widetilde{\mathfrak{B}}^{n\sigma}$ (see Section 9.2.2) is invariant. If $\sigma = -n - \ell + 2$, $\ell \in \mathbf{Z}_+ \cup \{0\}$, then we have the invariant subspace, consisting of functions $f(\boldsymbol{\xi})$ on the cone, such that $f(\lambda\boldsymbol{\xi}) = \lambda^{-n-\ell+2} f(\boldsymbol{\xi})$, $\lambda > 0$, and

$$\int_{S^{n-2}} f(\boldsymbol{\xi}) P(\boldsymbol{\xi}) d\boldsymbol{\xi} = 0$$

for all homogeneous polynomials $P(\boldsymbol{\xi})$ of degree $\ell$.

The representations $T^{nR}$, $R \neq 0$, of $ISO(n-1)$ are irreducible. The proof is carried out in the same way as in the case of the group $SO_0(n-1,1)$.

Now we show that the collection of the representations $T^{n\ell}$, $\ell = 0, 1, 2, \dots$ , of $SO(n)$ is complete in the set of irreducible representations of $SO(n)$, which have class 1 with respect to $SO(n-1)$. This means that *any irreducible representation $T$ of class 1 is equivalent to one of the representations $T^{n\ell}$*.

Let $\mathbf{e}_1$ be the normalized vector in the space $\mathfrak{A}$ of the representation $T$, which is invariant with respect to $SO(n-1)$. With every vector $\mathbf{f} \in \mathfrak{A}$ we associated the function $f(g) = (T(g^{-1})\mathbf{f}, \mathbf{e}_1)$ on $SO(n)$, where $(\,.\,,\,.\,)$ is the invariant scalar product on $\mathfrak{A}$. The function $f(g)$ is constant on left cosets with respect to $SO(n-1)$. Therefore, $f(g)$ can be considered as a function on $S^{n-1}$: $f(g) = \varphi(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in S^{n-1}$.

The passage from vectors $\mathbf{f}$ to corresponding functions $\varphi(\boldsymbol{\xi})$ transforms the representation $T$ into the representation $L'$: $L'(g)\varphi(\boldsymbol{\xi}) = \varphi(g^{-1}\boldsymbol{\xi})$. Since the representation $T$ is continuous, then $\varphi(\boldsymbol{\xi}) \in \mathfrak{L}^2(S^{n-1})$. Hence, $L'$ is an irreducible component of the quasi-regular representation $L$ of the group $SO(n)$. This means that $L'$ is equivalent to one of the representations $T^{n\ell}$. Our statement is proved.

If $f(\mathbf{x}) \in \mathfrak{H}^{n\ell}$, then it can be represented in the form

$$f(\mathbf{x}) = r^{\ell} f\left(\frac{\mathbf{x}}{r}\right), \quad \frac{\mathbf{x}}{r} \in S^{n-1}. \tag{3}$$

We have $\Delta f(\mathbf{x}) = 0$. Taking into account formula (8) of Section 9.1.8 we obtain that

$$\Delta_0 f\left(\frac{\mathbf{x}}{r}\right) = -\ell(\ell + n - 2) f\left(\frac{\mathbf{x}}{r}\right). \tag{4}$$

Thus, *restrictions of polynomials* $f \in \mathfrak{H}^{n\ell}$ *onto* $S^{n-1}$ *are eigenfunctions of the operator* $\Delta_0$*, corresponding to the eigenvalue* $-\ell(\ell + n - 2)$.

**9.2.7. Intertwining operators for the representations $T^{n\sigma}$ of the group $SO_0(n-1,1)$.** Let us find for what values of $\sigma$ and $\tau$ there exists a nonzero operator $Q^{\sigma\tau}$, intertwining the representations $T^{n\sigma}$ and $T^{n\tau}$. We realize these representations in the spaces $\mathfrak{B}^{n\sigma}$ and $\mathfrak{B}^{n\tau}$, respectively. Since elements of these spaces are uniquely determined by functions on some contour $\Gamma$ lying on the cone $C_+^{n-1}$, then, by virtue of the theorem on kernel, the operator $Q^{\sigma\tau}$ has the form

$$(Q^{\sigma\tau} f)(\boldsymbol{\xi}) = \int_{\Gamma} Q(\boldsymbol{\xi}, \boldsymbol{\eta}) f(\boldsymbol{\eta}) d\boldsymbol{\eta}, \quad \boldsymbol{\xi}, \boldsymbol{\eta} \in \Gamma. \tag{1}$$

Here $Q(\boldsymbol{\xi}, \boldsymbol{\eta})$ is a generalized function in the space of smooth functions on $\Gamma \times \Gamma$. We continue this function by homogeneity onto $C_+^{n-1} \times C_+^{n-1}$ in such a way that integral (1) is independent of the choice of $\Gamma$ and the function $Q^{\sigma\tau} f$ belongs to $\mathfrak{B}^{n\tau}$. This can be done if $Q(\boldsymbol{\xi}, \boldsymbol{\eta})$ has homogeneity degrees $-\sigma - n + 2$ and $\tau$ in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, respectively.

Without losing generality, we can assume that $\Gamma$ is section of $C_+^{n-1}$ by the plane $\xi_n = 1$. The equality

$$Q^{\sigma\tau} T^{n\sigma}(g) = T^{n\tau}(g) Q^{\sigma\tau}$$

implies that

$$\int_{\Gamma} Q(\boldsymbol{\xi}, \boldsymbol{\eta}) f(g^{-1}\boldsymbol{\eta}) d\boldsymbol{\eta} = \int_{\Gamma} Q(g^{-1}\boldsymbol{\xi}, \boldsymbol{\eta}) f(\boldsymbol{\eta}) d\boldsymbol{\eta}.$$

Making the substitutions $g^{-1}\boldsymbol{\eta} = \hat{\boldsymbol{\eta}}$, $g^{-1}\boldsymbol{\xi} = \hat{\boldsymbol{\xi}}$ and taking into account that the integral is independent of the choice of contour, we obtain the equality $Q(g\boldsymbol{\xi}, g\boldsymbol{\eta}) = Q(\boldsymbol{\xi}, \boldsymbol{\eta})$. Therefore, there exists a generalized function $Q_0$ of one variable such that $Q(\boldsymbol{\xi}, \boldsymbol{\eta}) = Q_0([\boldsymbol{\xi}, \boldsymbol{\eta}])$. Taking into account homogeneity of $Q(\boldsymbol{\xi}, \boldsymbol{\eta})$ we conclude that if the generalized function $Q(\boldsymbol{\xi}, \boldsymbol{\eta})$ is not singular, then the equality $\tau = -n - \sigma + 2$ holds, where $Q(\boldsymbol{\xi}, \boldsymbol{\eta}) = \lambda[\boldsymbol{\xi}, \boldsymbol{\eta}]^\tau$. In the singular case we have $\sigma = \tau$ and[4] $Q(\boldsymbol{\xi}, \boldsymbol{\eta}) = \lambda\delta(\boldsymbol{\xi} - \boldsymbol{\eta})$.

Thus, *a non-zero intertwining operator for* $T^{n\sigma}$ *and* $T^{n\tau}$ *exists if and only if either* $\tau = -\sigma - n + 2$ *or* $\tau = \sigma$. *It is given by formula (1), where for* $\tau = -\sigma - n + 2$ *we have* $Q(\boldsymbol{\xi}, \boldsymbol{\eta}) = \lambda[\boldsymbol{\xi}, \boldsymbol{\eta}]^\tau$ *and for* $\tau = \sigma$ *we have* $Q(\boldsymbol{\xi}, \boldsymbol{\eta}) = \lambda\delta(\boldsymbol{\xi} - \boldsymbol{\eta})$. This operator is defined up to a constant. Let us choose $\lambda = (\Gamma(\tau + 1))^{-1} = (\Gamma(-\sigma - n + 3))^{-1}$ in the first case and $\lambda = 1$ in the second case.

Irreducibility of $T^{n\sigma}$ for non-integral $\sigma$ implies that the representations $T^{n\sigma}$ and $T^{n,-\sigma-n+2}$, $\sigma \bar{\in} \mathbf{Z}$, are equivalent. This statement is also valid for $\sigma \in \mathbf{Z}$ such that $-n + 1 \leq \sigma < 0$. Equivalence of the irreducible representations $T^{n\sigma}$ and $T^{n,-\sigma-n+2}$ can be also established by making use of their realizations $R^{n\sigma}$ and $R^{n,-\sigma-n+2}$ in the spaces of homogeneous $\Box$-harmonic functions. Namely, the *Kelvin transform*

$$(Kf)(\mathbf{x}) = [\mathbf{x}, \mathbf{x}]^{(-n-\sigma+2)/2} f\left(\frac{\mathbf{x}}{[\mathbf{x}, \mathbf{x}]^{1/2}}\right)$$

intertwines $R^{n\sigma}$ and $R^{n,-\sigma-n+2}$. The formula

$$(Qf)(\mathbf{x}) = \frac{1}{\Gamma(-\sigma - n + 3)} \int_\Gamma [\mathbf{x}, \boldsymbol{\xi}]^{-\sigma-n+2} f(\boldsymbol{\xi}) d\boldsymbol{\xi} \tag{2}$$

gives an intertwining operator for the representations $T^{n\sigma}$ and $R^{n,-\sigma-n+2}$. Formula (2) is an analogue of the well known Poisson formula for harmonic functions.

The operator $A = Q^{\tau\sigma}Q^{\sigma\tau}$ commutes with operators of the representation $T^{n\sigma}$. Therefore, it is a multiple of the identity operator, $A = \lambda I$, for irreducible representations $T^{n\sigma}$. In order to find $\lambda$ we apply $A$ to the function, equal identically to 1 on $S^{n-2}$. We have

$$(Q^{\sigma\tau}1)(\boldsymbol{\xi}) = \frac{1}{\Gamma(\tau + 1)} \int_{S^{n-2}} [\boldsymbol{\xi}, \boldsymbol{\eta}]^\tau d\boldsymbol{\eta}, \quad \tau = -\sigma - n + 2.$$

Since $[\boldsymbol{\xi}, \boldsymbol{\eta}]$ and $d\boldsymbol{\eta}$ are invariant with respect to rotations from $SO(n - 1)$, the function $Q^{\sigma\tau}1$ is constant on $S^{n-2}$. Therefore, it is sufficient to find its value at

[4] Let us note that $[\boldsymbol{\xi}, \boldsymbol{\eta}] = 0$ if and only if $\boldsymbol{\xi} = \mu\boldsymbol{\eta}$. By virtue of homogeneity, one can set $\mu = 1$.

the point $\boldsymbol{\xi}_0 = (0, \ldots, 0, 1, 1)$. We have

$$(Q^{\sigma\tau}1)(\boldsymbol{\xi}_0) = \frac{1}{\Gamma(\tau+1)} \int\limits_{S^{n-2}} (1-\eta_{n-1})^\tau d\boldsymbol{\eta} = \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\tau+1)\Gamma(\frac{n-2}{2})\sqrt{\pi}}$$

$$\times \int\limits_0^\pi (1-\cos\theta)^\tau \sin^{n-3}\theta d\theta = \frac{2^{\tau+n-3}\Gamma(\frac{n-1}{2})\Gamma\left(\tau+\frac{n-2}{2}\right)}{\Gamma(\tau+1)\Gamma(\tau+n-2)\sqrt{\pi}}. \quad (3)$$

It follows from here that

$$\lambda = \frac{2^{n-4}\Gamma\left(-\sigma-\frac{n-2}{2}\right)\Gamma\left(\sigma+\frac{n-2}{2}\right)\Gamma^2\left(\frac{n-1}{2}\right)}{\pi\Gamma(-\sigma)\Gamma(\sigma+1)\Gamma(\sigma+n-2)\Gamma(-\sigma-n+3)}.$$

Since the operator $Q^{\tau\sigma}Q^{\sigma\tau}$ is scalar, we have the following statement: If $\sigma \bar{\in} \mathbf{Z}$, then the transform

$$\check{f}(\boldsymbol{\xi}) = \frac{\pi\Gamma(-\sigma)\Gamma(\sigma+n-2)\Gamma(-\sigma-n+3)}{2^{n-4}\Gamma\left(-\sigma-\frac{n-2}{2}\right)\Gamma\left(\sigma+\frac{n-2}{2}\right)\Gamma^2\left(\frac{n-1}{2}\right)} \int\limits_{S^{n-2}} [\boldsymbol{\xi}, \boldsymbol{\eta}]^\sigma f(\boldsymbol{\eta}) d\boldsymbol{\eta}$$

is inverse to the integral transform

$$\hat{f}(\boldsymbol{\xi}) = \frac{1}{\Gamma(\tau+1)} \int\limits_{S^{n-2}} [\boldsymbol{\xi}, \boldsymbol{\eta}]^\tau f(\boldsymbol{\eta}) d\boldsymbol{\eta}, \quad \tau = -\sigma - n + 2.$$

It is clear that $T^{n\sigma}(h)$ and $T^{n\tau}(h)$, $h \in SO(n-1)$, are shift operators in $\mathfrak{L}^2(S^{n-2})$. Hence, they coincide. For $h \in SO(n-1)$ we have $Q^{\sigma\tau}T^{n\sigma}(h) = T^{n\tau}(h)Q^{\sigma\tau}$. By virtue of Schur's lemma it follows from here that the operator $Q^{\sigma\tau}$ is scalar on every subspace, which is irreducible with respect to the subgroup $SO(n-1)$. Values of corresponding factors will be evaluated in Section 9.5.14.

Now we consider the case when $\sigma$ and $\tau$ are integers. If $\ell \in \mathbf{Z}_+ \cup \{0\}$, then in $\mathfrak{B}^{n\ell}$ there exists the finite dimensional invariant subspace $\widehat{\mathfrak{B}}^{n\ell}$ (see Section 9.2.2), consisting of polynomials of degree $\ell$. Since

$$\left.\frac{x_+^\lambda}{\Gamma(\lambda+1)}\right|_{\lambda=-n-1} = \delta^{(n)}(x),$$

then for $\ell \in \mathbf{Z}_+ \cup \{0\}$ the operator $Q^{-\ell-n+2,\ell}$ becomes differential:

$$(Q^{-\ell-n+2,\ell}f)(\boldsymbol{\xi}) = \int\limits_{S^{n-2}} \delta^{(\ell)}([\boldsymbol{\xi}, \boldsymbol{\eta}]) f(\boldsymbol{\eta}) d\boldsymbol{\eta}.$$

It annuls $\widehat{\mathfrak{B}}^{n\ell}$. We denote by $\check{T}^{n\ell}$ the representation induced by $T^{n\sigma}$, $\sigma = \ell$, in $\mathfrak{B}^{n\ell}/\widehat{\mathfrak{B}}^{n\ell}$ and by $\widehat{\mathfrak{B}}^{n,-\ell-n+2}$ the space $Q^{-\ell-n+2,\ell}\mathfrak{B}^{n\ell}$. This space is invariant with respect to the operators $T^{n,-\sigma-n+2}$, $\sigma = \ell$. One can show that there are no other invariant subspaces in $\mathfrak{B}^{n,-\ell-n+2}$ and that the restriction of $T^{n,-\sigma-n+2}$, $\sigma = \ell$, onto $\widehat{\mathfrak{B}}^{n,-\ell-n+2}$ is irreducible. The operator $Q^{-\ell-n+2,\ell}$ intertwines $\widehat{T}^{n\ell}$ and $\check{T}^{n,-\ell-n+2}$. The operator $Q^{\ell,-\ell-n+2}$ annuls $\widehat{\mathfrak{B}}^{n,-\ell-n+2}$ and intertwines the finite dimensional representation $\check{T}^{n,-\ell-n+2}$ induced by $T^{n,-\sigma-n+2}$, $\sigma = \ell$, in $\mathfrak{B}^{n,-\ell-n+2}/\widehat{\mathfrak{B}}^{n,-\ell-n+2}$ and the representation $\widehat{T}^{n\ell}$ which is the restriction of $T^{n\sigma}$, $\sigma = \ell$, onto $\widehat{\mathfrak{B}}^{n\ell}$.

Thus we have established the following equivalences:

1) $T^{n\sigma} \sim T^{n,-\sigma-n+2}$, $\sigma \bar{\in} \mathbb{Z}$ or $-n+1 \le \sigma < 0$,

2) $\check{T}^{n\ell} \sim \widehat{T}^{n,-\ell-n+2}$, $\widehat{T}^{n\ell} \sim \check{T}^{n,-\ell-n+2}$, $\ell \in \mathbb{Z}_+ \cup \{0\}$.

They exhaust all equivalences in the set, consisting of the irreducible representations $T^{n\sigma}$ and of the irreducible representations obtained from the reducible representations $T^{n\sigma}$ by restrictions onto subspaces and by the passage to quotient representations.

**9.2.8. Unitary representations.** The representation, adjoint to the representation $T^{n\sigma}$ of the group $SO_0(n-1,1)$ in the spaces $\mathfrak{D}$ of smooth functions on $S^{n-2}$, is given by the same formula as in the space $\mathfrak{D}'$. This statement follows from independence of the bilinear functional

$$B(\varphi,\psi) = \int_\Gamma \varphi(\xi)\psi(\xi)d\xi, \quad \varphi \in \mathfrak{B}^{n\sigma},\ \psi \in \mathfrak{B}^{n,-\sigma-n+2},$$

on the choice of contour $\Gamma$. Therefore, the representation $T^{n,-\bar{\sigma}-n+2}$ is Hermitian adjoint to $T^{n\sigma}$. If the representation $T^{n,-\bar{\sigma}-n+2}$ is irreducible then it is equivalent to $T^{n\bar{\sigma}}$.

What has been said above implies that a non-degenerate Hermitian-symmetric functional $H(\varphi,\psi)$ can exist for the representation $T^{n\sigma}$ in two cases: $\sigma+\bar{\sigma} = -n+2$ or $\sigma = \bar{\sigma}$. In the first case this functional is of the form

$$H(\varphi,\psi) = \int_{S^{n-2}} \varphi(\xi)\overline{\psi(\xi)}d\xi.$$

It is positive. Hence, for $\sigma = i\rho - \frac{n-2}{2}$, $\rho \in \mathbb{R}$, the representation $T^{n\sigma}$ is unitary. This series of unitary representations is called ***principal.*** For $\sigma = s \in \mathbb{R}$ the functional $H(\varphi,\psi)$ has the form

$$H(\varphi,\psi) = \frac{1}{\Gamma(s+1)} \int_{S^{n-2}} \int_{S^{n-2}} [\xi,\eta]^s \varphi(\xi)\overline{\psi(\eta)}d\xi d\eta.$$

One can show that this functional is positive for $2-n<\sigma<0$. The corresponding series of unitary representations is called *complementary*.

The representations $\check{T}^{n,-\ell-n+2}$, $\ell\in\mathbf{Z}_+\cup\{0\}$, are also unitary. The invariant scalar products for them are constructed as follows. For any $\varphi$ and $\psi$ of $\mathfrak{D}$ we put

$$H_\ell(\varphi,\psi)=\operatorname*{Res}_{\sigma=\ell}\left\{\frac{(-1)^{\ell-1}\sqrt{\pi}\Gamma(-\sigma)}{2^{-\sigma-1}\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(-\sigma-\frac{n-2}{2}\right)}\int_{S^{n-2}}\int_{S^{n-2}}[\xi,\eta]^\sigma\varphi(\xi)\overline{\psi(\eta)}d\xi d\eta\right\}.$$

Then $H_\ell(\varphi,\varphi)\geq 0$ on the space, where $\check{T}^{n,-\ell-n+2}$ is realized, and $H_\ell(\varphi,\psi)$ is invariant with respect to all operators $\check{T}^{n,-\ell-n+2}(g)$, $g\in SO_0(n-1,1)$. The representations $\check{T}^{n,-\ell-n+2}$, $\ell\in\mathbf{Z}_+$, are said to be representations of the *discrete series*.

**9.2.9. Representations of the group $SO_0(p,q)$.** Let $\sigma\in\mathbb{C}$ and $\varepsilon\in\{0,1\}$. We denote by $\mathfrak{B}^{\sigma\varepsilon}_{pq}$ the space of infinitely differentiable functions on the cone $C^{pq}$ (see Section 9.1.4), satisfying the homogeneity condition

$$f(a\mathbf{x})=|a|^\sigma\operatorname{sign}^\varepsilon a\, f(\mathbf{x}),\quad a\in\mathbb{R},\ \mathbf{x}\in C^{pq}. \tag{1}$$

The operators

$$T^{\sigma\varepsilon}_{pq}(g)f(\mathbf{x})=f(g^{-1}\mathbf{x}) \tag{2}$$

define a representation of the group $SO_0(p,q)$ in $\mathfrak{B}^{\sigma\varepsilon}_{pq}$.

Formula (1) shows that a function $f\in\mathfrak{B}^{\sigma\varepsilon}_{pq}$ is uniquely determined by its values on a contour $\Gamma$, intersecting every generatrix of $C^{pq}$ at one point. The contour $\Gamma_1$, consisting of points $\mathbf{x}=(\mathbf{y},\mathbf{z})$, $\mathbf{y}\in\mathbb{R}^p$, $\mathbf{z}\in\mathbb{R}^q$, such that $y_1^2+\ldots+y_p^2+z_1^2+\ldots+z_q^2=1$, satisfies this condition. This contour is the topological product of two spheres: $\Gamma_1=S^{p-1}\times S^{q-1}$. The space $\mathfrak{D}^\varepsilon$ of functions $F$, obtained by restricting funtions of $\mathfrak{B}^{\sigma\varepsilon}_{pq}$ onto $\Gamma_1$, consists of infinitely differentiable functions on $\Gamma_1$ of evenness $\varepsilon$: $F(-\boldsymbol{\xi})=(-1)^\varepsilon F(\boldsymbol{\xi})$, $\boldsymbol{\xi}\in\Gamma_1$. A function $f\in\mathfrak{B}^{\sigma\varepsilon}_{pq}$ is expressed in terms of the corresponding function $F\in\mathfrak{D}^\varepsilon$ by the formula

$$f(\mathbf{x})=r^\sigma F\left(\frac{\mathbf{y}}{r},\frac{\mathbf{z}}{r}\right),\quad\text{where}\quad \mathbf{x}=(\mathbf{y},\mathbf{z}),\ r=(\mathbf{y},\mathbf{y})^{1/2}. \tag{3}$$

Putting $t=0$ into formula (9) of Section 9.1.7 we obtain spherical coordinates $(\boldsymbol{\varphi},\boldsymbol{\theta})=(\varphi_1,\ldots,\varphi_{p-1},\theta_1,\ldots,\theta_{q-1})$ on $\Gamma_1$. It is easy to derive that in these coordinates the operators $T^{\sigma\varepsilon}_{pq}(g'_{1n}(t))$ act upon functions $f\in\mathfrak{D}^\varepsilon$ according to the formula

$$(T^{\sigma\varepsilon}_{pq}(g'_{1n}(t))F)(\boldsymbol{\varphi},\boldsymbol{\theta})=[\sin^2\varphi+(\cosh t\cos\varphi-\sinh t\cos\theta)^2]^{\sigma/2}$$
$$\times F(\varphi_1,\ldots,\varphi_{p-2},\varphi'_{p-1};\theta_1,\ldots,\theta_{q-2},\theta'_{q-1}), \tag{4}$$

where $\varphi \equiv \varphi_{p-1}$, $\theta \equiv \theta_{q-1}$ and

$$\cos \varphi'_{p-1} = \frac{\cosh t \cos \varphi - \sinh t \cos \theta}{[\sin^2 \varphi + (\cosh t \cos \varphi - \sinh t \cos \theta)^2]^{1/2}}, \tag{5}$$

$$\cos \theta'_{q-1} = \frac{\cosh t \cos \theta - \sinh t \cos \varphi}{[\sin^2 \theta + (\cosh t \cos \theta - \sinh t \cos \varphi)^2]^{1/2}}. \tag{6}$$

For the operators $T^{\sigma\epsilon}_{pq}(k)$, $k \in SO(p) \times SO(q)$, we have

$$(T^{\sigma\epsilon}_{pq}(k)F)(\mathbf{y}) = F(k^{-1}\mathbf{y}), \quad \mathbf{y} \in \Gamma_1.$$

Let us introduce the scalar product

$$(F_1, F_2) = \int\limits_{S^{q-1}} \int\limits_{S^{p-1}} F_1(\boldsymbol{\xi}, \boldsymbol{\eta}) \overline{F_2(\boldsymbol{\xi}, \boldsymbol{\eta})} d\boldsymbol{\xi} d\boldsymbol{\eta} \tag{7}$$

into $\mathfrak{D}^\epsilon$ and complete this space to obtain the Hilbert space $\mathfrak{L}^2_\epsilon(S^{p-1} \times S^{q-1})$. One can show that the operators $T^{\sigma\epsilon}_{pq}(g)$, $g \in SO_0(n-1,1)$, are continued to bounded operators in $\mathfrak{L}^2_\epsilon(S^{p-1} \times S^{q-1})$ and we receive the representation of $SO_0(p,q)$ in $\mathfrak{L}^2_\epsilon(S^{p-1} \times S^{q-1})$ which is also denoted by $T^{\sigma\epsilon}_{pq}$.

Another realization of $T^{\sigma\epsilon}_{pq}$ is constructed in the following way. Let us denote by $\Omega^{pq}_\pm$ the domain $\{\mathbf{x} \in E_{pq} \mid [\mathbf{x},\mathbf{x}]_{pq} \gtrless 0\}$ and by $\mathfrak{F}^{\sigma\epsilon,\pm}$ the space of $\Box_{pq}$-harmonic functions $f$ on $\Omega^{pq}_\pm$ of homogeneity degree $(\sigma, \varepsilon)$, i.e. such that $\Box_{pq} f = 0$ and $f(a\mathbf{x}) = |a|^\sigma (\mathrm{sign}^\epsilon a) f(\mathbf{x})$, $a \in \mathbb{R}$. The equality

$$(Q^{\sigma\epsilon}_{pq}(g)f)(\mathbf{x}) = f(g^{-1}\mathbf{x}), \quad \mathbf{x} \in \Omega^{pq}_\pm,$$

gives the representation of $SO_0(p,q)$ in $\mathfrak{F}^{\sigma\epsilon,\pm}$. For non-integral values of $\sigma$ it is equivalent to the representation $T^{-\sigma-n+2,\epsilon}_{pq}$, $n = p+q$.

Since the operators $T^{\sigma\epsilon}_{pq}(k)$, $k \in SO(p) \times SO(q)$, act in $\mathfrak{D}^\epsilon$ as left shifts, formula (1) and the definition of $\mathfrak{D}^\epsilon$ imply that for the restriction of the representation $T^{\sigma\epsilon}_{pq}$ onto $K \equiv SO(p) \times SO(q)$ we have

$$T^{\sigma\epsilon}_{pq} \downarrow^{SO_0(p,q)}_{K} = \sum_{m+m' \equiv \epsilon \pmod 2} \oplus (T^{pm} \times T^{qm'}). \tag{8}$$

To the one-parameter subgroup $\{g_{1n}(\theta)\}$ of $SO_0(p,q)$ there corresponds the infinitesimal generator $I'_{1n} = e_{1n} + e_{n1}$. In the representation $T^{\sigma\epsilon}_{pq}$ it is given by the infinitesimal operator

$$I^{\sigma\epsilon}_{1n} = \left. \frac{dT^{\sigma\epsilon}_{pq}(g_{1n}(\theta))}{d\theta} \right|_{\theta=0}.$$

With the help of formula (4) we find that

$$(I_{1n}^{\sigma\varepsilon}F)(\varphi,\theta) = \left(\sigma\cos\varphi\cos\theta + \sin^2\varphi\cos\theta\frac{\partial}{\partial\cos\varphi} + \cos\varphi\sin^2\theta\frac{\partial}{\partial\cos\theta}\right)F(\varphi,\theta), \tag{9}$$

where $\varphi \equiv \varphi_{p-1}$, $\theta = \theta_{q-1}$.

As in the case of the group $SO_0(n-1,1)$, with the help of infinitesimal operator (9) we derive conditions of irreducibility of $T_{pq}^{\sigma\varepsilon}$. For even $p$ and $q$ the representation $T_{pq}^{\sigma\varepsilon}$ is irreducible if and only if $\sigma$ is not an integer of the same evenness as $\varepsilon$ is. If $p$ and $q$ are of opposite evenness, then $T_{pq}^{\sigma\varepsilon}$ is irreducible for non-integral $\sigma$. If $p$ and $q$ are odd, then $T_{pq}^{\sigma\varepsilon}$ is irreducible for non-integral $\sigma$ as well as for integral $\sigma$ such that $0 < \sigma < \frac{n}{2} - 2$, $n = p + q$.

Let us recall that representations $T_{pq}^{\sigma\varepsilon}$ and $T_{pq}^{\sigma'\varepsilon'}$ are said to be Hermitian-adjoint if for any$g \in SO_0(p,q)$ and for all $F_1$ and $F_2$ from $\mathfrak{L}_\varepsilon^2(S^{p-1} \times S^{q-1})$ the equality

$$(T_{pq}^{\sigma\varepsilon}(g)F_1, T_{pq}^{\sigma'\varepsilon'}(g)F_2) = (F_1, F_2)$$

holds. In the same way as in the case of $SO_0(n-1,1)$ (see Section 9.2.8), one proves that the representations $T_{pq}^{\sigma\varepsilon}$ and $T_{pq}^{\tau\varepsilon}$,$\tau = -\bar\sigma - n + 2$, are Hermitian-adjoint.

For $\sigma = \tau = -\bar\sigma - n + 2$ the condition of Hermitian adjointness becomes the condition of unitarity. Hence, the representations

$$T_{pq}^{\sigma\varepsilon},\ \sigma = i\rho - \frac{n-2}{2},\ \rho \in \mathbb{R},$$

are unitary in the space $\mathfrak{L}_\varepsilon^2(S^{p-1} \times S^{q-1})$. They form the *maximally degenerate principal unitary series* of representations of $SO_0(p,q)$.

The group $SO_0(p,q)$ has also the *complementary series* of irreducible unitary representations. It contains the representations $T_{pq}^{\sigma\varepsilon}$, where

$$-\frac{n}{2} < \sigma < -\frac{n-2}{2} \quad \text{if} \quad p \equiv q \pmod 2,$$
$$-\frac{n-1}{2} < \sigma < -\frac{n-2}{2} \quad \text{if} \quad p \not\equiv q \pmod 2.$$

The invariant scalar product is of the form

$$(f_1, f_2)_{\sigma\varepsilon} = c\int_{\Gamma_1}\int_{\Gamma_2} |[\xi_1,\xi_2]|^{-\sigma-n+2}\,\mathrm{sign}^\varepsilon[\xi_1,\xi_2] f_1(\xi_1)\overline{f_2(\xi_2)}d\xi_1 d\xi_2,$$

where $\Gamma_1 = S^{p-1} \times S^{q-1}$ and $c$ is a constant.

Let $\Gamma$ be a section of the cone $C^{pq}$, intersecting every generatrix at one point, and let $d\xi'$ be the measure on $\Gamma$, related to the invariant measure $d\xi$ on $\Gamma_1$ by the formula $d\xi' = d\xi$, where $\xi' = a\xi$, $a \in \mathbf{R}$. Then formula (7) can be written in the form

$$(f_1, f_2) = \int_\Gamma f_1(\xi)\overline{f_2(\xi)}d\xi. \qquad (10)$$

We equip $\mathfrak{B}_{pq}^{\sigma\varepsilon}$ with the topology of uniform convergence on $\Gamma$ of functions and of their derivatives. One can regard functions from $\mathfrak{B}_{pq}^{\sigma\varepsilon}$ as continuous functionals on $\mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon}$, $n = p+q$, by setting $f_1(f_2) = (f_1, \bar{f}_2)$, where $f_1 \in \mathfrak{B}_{pq}^{\sigma\varepsilon}$, $f_2 \in \mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon}$, and $(.\,,.)$ coincides with the scalar product (10). We denote by $(\mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon})'$ the space of all continuous linear functionals on $\mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon}$. The action of elements from $(\mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon})'$ upon $\mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon}$ will be written by the formula

$$v(f) = \langle v, f\rangle \equiv (v, \bar{f}),\ \ v \in (\mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon})',\ \ f \in \mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon}.$$

In particular, $(\mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon})'$ contains ordinary functions $f_1$ on $\Gamma$, for which the integral $(f_1, \bar{f})$ converges. The action of the representation $T_{pq}^{\sigma\varepsilon}$ can be continued onto the space $(\mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon})'$. We will denote the continued representation by $\widehat{T}_{pq}^{\sigma\varepsilon}$. Then we have

$$\left\langle \widehat{T}_{pq}^{\sigma\varepsilon}(g)v, T_{pq}^{-\sigma-n+2,\varepsilon}(g)f\right\rangle = \langle v, f\rangle,$$
$$v \in (\mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon})',\ \ f \in \mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon}.$$

An element $v \in (\mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon})'$ such that $\widehat{T}_{pq}^{\sigma\varepsilon}(g)v = v$ for all $g \in SO_0(p, q-1)$ is called $SO_0(p,q-1)$-invariant. One can show that for every representation $T_{pq}^{\sigma\varepsilon}$ there exists an $SO_0(p, q-1)$-invariant element. The function $f(\xi) \equiv 1$ is $SO_0(p, q-1)$-invariant element on the section $\Gamma$ of $C^{pq}$ by the plane $x_n = 1$.

**9.2.10. Discrete series representations of the group $SO_0(p,q)$ on $H_\pm^{pq}$.** The restriction of $\Box_{pq}$-harmonic functions of homogeneity degree $(\sigma, \varepsilon)$ onto $H_\pm^{pq}$ leads to functions which, in general, do not belong to $\mathfrak{L}^2(H_\pm^{pq})$. Functions from $\mathfrak{L}^2(H_\pm^{pq})$ can be obtained when $\sigma = \ell \in \mathbf{Z}$. Left shifts by elements $g \in SO_0(p,q)$ in the spaces $\mathfrak{H}_\pm^\ell$ of these functions define the irreducible representations $T_\pm^\ell$ of $SO_0(p,q)$, which are unitary with respect to appropriate scalar products. These representations form the discrete series $\{T_+^\ell\}$ and $\{T_-^\ell\}$ of unitary representations of $SO_0(p,q)$ on $H_+^{pq}$ and $H_-^{pq}$, respectively. They are equivalent to some subrepresentations of quotient representations or to quotient representations of subrepresentations of $T_{pq}^{\sigma\varepsilon}$. Below we give a description of discrete series representations by indicating decompositions of their restrictions onto the subgroup $K \equiv SO(p) \times SO(q)$ into irreducible components. The proofs are based on studying actions of infinitesimal operators upon basis functions and matrix elements of representations (see [230], [231], [308]).

The discrete series representations $T^\ell_\pm$ exist for $\ell > -(p+q-2)/2$. If $p>2$, $q>2$, then we have

$$T^\ell_+ \downarrow^{SO_0(p,q)}_K = \sum_{m'-m>\ell+p-2} \oplus(T^{pm}\times T^{qm'}), \tag{1}$$

$$T^\ell_- \downarrow^{SO_0(p,q)}_K = \sum_{m-m'>\ell+q-2} \oplus(T^{pm}\times T^{qm'}). \tag{2}$$

If $p$ and $q$ are of the same evenness, then $T^\ell_\pm$ are contained in $T^{\ell\varepsilon}_{pq}$ as irreducible components, where $\ell\equiv\varepsilon \pmod 2$ for even $p$ and $q$, and $\ell\equiv(\varepsilon+1) \pmod 2$ for odd $p$ and $q$. If $p$ is even and $q$ is odd, then $T^\ell_+$ is contained in $T^{\ell\varepsilon}_{pq}$, $\ell\equiv\varepsilon \pmod 2$, and $T^\ell_-$ is contained in $T^{\ell\varepsilon}_{pq}$, $\ell\equiv(\varepsilon+1) \pmod 2$. If $p$ is odd and $q$ is even, then $T^\ell_+$ is contained in $T^{\ell\varepsilon}_{pq}$, $\ell\equiv(\varepsilon+1) \pmod 2$, and $T^\ell_-$ is contained in $T^{\ell\varepsilon}_{pq}$, $\ell\equiv\varepsilon \pmod 2$. Let us note that the summations in (1) and (2) are over those values $m$ and $m'$ for which $m+m'\equiv\varepsilon \pmod 2$.

For $SO_0(p,2)$, $p>2$, the representation $T^\ell_+$ is reducible and decomposes into the sum of two irreducible representations $T^{\ell+}_+$ and $T^{\ell-}_+$. We have

$$T^{\ell+}_+ \downarrow^{SO_0(p,2)}_K = \sum_{m'-m>\ell+p-2} \oplus(T^{pm}\times T^{2m'}), \tag{3}$$

$$T^{\ell-}_+ \downarrow^{SO_0(p,2)}_K = \sum_{m+m'<-\ell-p+2} \oplus(T^{pm}\times T^{2m'}). \tag{4}$$

The representation $T^\ell_-$ of $S_0(p,2)$, $p>2$, is irreducible and

$$T^\ell_- \downarrow^{SO_0(p,2)}_K = \sum_{\substack{m+m'\le\ell\\ m-m'\le\ell}} \oplus(T^{pm}\times T^{2m'}). \tag{5}$$

If $p$ is even, then $T^{\ell\pm}_+$ and $T^\ell_-$ are contained in $T^{\ell\varepsilon}_{pq}$, $\ell\equiv\varepsilon \pmod 2$. If $p$ is odd, then $T^{\ell\pm}_+$ are contained in $T^{\ell\varepsilon}_{pq}$, $\ell\equiv\varepsilon \pmod 2$ and $T^\ell_-$ is contained in $T^{\ell\varepsilon}_{pq}$, $\ell\equiv(\varepsilon+1) \pmod 2$. The summations in (3)-(5) are over those values of $m$ and $m'$ for which $m+m'\equiv\varepsilon \pmod 2$.

## 9.3. Zonal Spherical Functions of Representations of $SO(n)$ and of Related Groups

**9.3.1. The orthonormal basis of the space $\mathfrak{L}^2(S^{n-1})$ and spherical functions of representations of $SO(n)$.** It was shown in Section 9.2.6 that

$$\mathfrak{L}^2(S^{n-1}) = \sum_{\ell=0}^{\infty} \oplus\tilde{\mathfrak{H}}^{n\ell},$$

where $\widetilde{\mathfrak{H}}^{n\ell}$ is the space of the irreducible representation $T^{n\ell}$ of the group $SO(n)$, and that the representations $T^{n\ell}$ and $T^{nm}$ are nonequivalent for $\ell \neq m$. It was also shown that the restriction $T^{n\ell}\downarrow^{SO(n)}_{SO(n-1)}$ is the direct sum of pairwise nonequivalent irreducible representations $T^{n-1,m}$, $0 \le m \le \ell$, of the group $SO(n-1)$.

Continuing restrictions of representations and decompositions into irreducible representations with accordance with the chain of groups $SO(n) \supset SO(n-1) \supset \ldots \supset SO(2)$ and taking into account the fact that irreducible representations of $SO(2)$ are one-dimensional, we obtain the decomposition

$$\mathfrak{L}^2(S^{n-1}) = \sum_{\widehat{K}} \oplus \widetilde{\mathfrak{H}}^n_{\widehat{K}}. \tag{1}$$

Here $\widetilde{\mathfrak{H}}^n{}_{\widehat{K}}$ are one-dimensional subspaces, $\widehat{K} = (k_0, k_1, \ldots, k_{n-3}, k_{n-2})$, $k_j \in \mathbf{Z}_+ \cup \{0\}$, $0 \le j \le n-3$, $k_{n-2} \in \mathbf{Z}$ and $k_0 \ge k_1 \ge \ldots \ge k_{n-3} \ge |k_{n-2}|$. We denote $k_0$ by $\ell$ and put $K = (k_1, k_2, \ldots, k_{n-2})$, $\widetilde{\mathfrak{H}}^n_{\widehat{K}} = \widetilde{\mathfrak{H}}^{n\ell}_k$. Then we choose a vector $\Xi^{n\ell}_K$ of a unit length in every subspace $\widetilde{\mathfrak{H}}^{n\ell}{}_K$. This vector (which is a function on $S^{n-1}$) is uniquely defined up to a scalar factor $\alpha$ such that $|\alpha| = 1$.

We fix values $\ell = k_0, k_1, \ldots, k_j$ and denote by $\widetilde{\mathfrak{H}}^{n\ell}_{K_j}$ the subspace of $\mathfrak{L}^2(S^{n-1})$, generated by all vectors $\Xi^{n\ell}_K$ with these values of $\ell$, $k_1, \ldots, k_j$. It is obvious that $\widetilde{\mathfrak{H}}^{n\ell}_{K_j}$ is a carrier space for the representation (by shift operators) of $SO(n-j)$ and that $\widetilde{\mathfrak{H}}^{n\ell}_{K_j}$ is the direct sum of its subspaces $\widetilde{\mathfrak{H}}^{n\ell}{}_{K_{j+1}}$, $K_{j+1} = (K_j, k_{j+1})$.

The functions $\Xi^{n\ell}_K$ form an orthonormal basis in $\mathfrak{L}^2(S^{n-1})$:

$$\left(\Xi^{n\ell}_K, \Xi^{n\ell'}_S\right) = \int_{S^{n-1}} \Xi^{n\ell}_K(\boldsymbol{\xi})\overline{\Xi^{n\ell'}_L(\boldsymbol{\xi})}d\boldsymbol{\xi} = \delta_{KL}\delta_{\ell\ell'}. \tag{2}$$

The function $\Xi^{n0}_O$, $O = (0, \ldots, 0)$, corresponds to the trivial representation of the group $SO(n)$. It is constant on $S^{n-1}$ and invariant with respect to shift operators. The normalization condition implies that $|\Xi^{n0}_O(\boldsymbol{\xi})| = 1$. We set $\Xi^{n0}_O(\boldsymbol{\xi}) = 1$. It follows from the orthogonality relation (2) that

$$\int_{S^{n-1}} \Xi^{n\ell}_M(\boldsymbol{\xi})d\boldsymbol{\xi} = 0$$

if $\ell \neq 0$.

The functions $\Xi^{n\ell}_K(\boldsymbol{\xi})$ with a fixed $\ell$ form an orthonormal basis of $\widetilde{\mathfrak{H}}^{n\ell}$. The matrix elements of the representation $T^{n\ell}$ in the basis $\{\Xi^{n\ell}_K\}$ will be denoted by $t^{n\ell}_{KM}(g)$. For $M = O$ these matrix elements are spherical functions of $T^{n\ell}$. They are zonal spherical functions if $K = O$ and associated spherical functions otherwise. These functions are constant on right cosets with respect to $SO(n-1)$. Hence, they can be considered as functions on $S^{n-1}$. Therefore, we shall write $t^{n\ell}_{KO}(g) = t^{n\ell}_{KO}(\boldsymbol{\xi})$ if $g\mathbf{e}_n = \boldsymbol{\xi}$.

Let us prove that

$$\Xi^{n\ell}_K(\boldsymbol{\xi}) = \left[\frac{(2\ell+n-2)(\ell+n-3)!}{(n-2)!\ell!}\right]^{1/2} \overline{t^{n\ell}_{KO}(\boldsymbol{\xi})}. \tag{3}$$

Indeed, for any $g_0 \in SO(n)$ the equalities

$$\begin{aligned} T^{n\ell}(g_0)\Xi^{n\ell}_M(\boldsymbol{\xi}) &= \Xi^{n\ell}_M(g_0^{-1}\boldsymbol{\xi}) = \sum_K t^{n\ell}_{KM}(g_0)\Xi^{n\ell}_K(\boldsymbol{\xi}), \\ L(g_0)\overline{t^{n\ell}_{MO}(\boldsymbol{\xi})} &= \overline{t^{n\ell}_{MO}(g_0^{-1}\boldsymbol{\xi})} = \sum_K \overline{t^{n\ell}_{MK}(g_0^{-1})t^{n\ell}_{KO}(\boldsymbol{\xi})} \\ &= \sum_K t^{n\ell}_{KM}(g_0)\overline{t^{n\ell}_{KO}(\boldsymbol{\xi})} \end{aligned}$$

hold. They show that in the bases $\{\Xi^{n\ell}_K\}$ and $\{\overline{t^{n\ell}_{KO}}\}$ the matrices of shift operators on $S^{n-1}$ coincide. Since the representations $T^{n\ell}$ are irreducible and the representations $T^{nk}$, $T^{n\ell}$ are nonequivalent for $k \neq \ell$, then by Schur's lemma these bases differ from each other in a scalar factor, common for all vectors. The absolute value of this factor is determined by formula (2) and by the equality

$$\int\limits_{SO(n)} \left|t^{n\ell}_{KO}(g)\right|^2 dg = (\dim T^{n\ell})^{-1} = \frac{(n-2)!\ell!}{(n+\ell-3)!(2\ell+n-2)} \tag{4}$$

(see formula (5) of Section 9.2.3). Taking into account that $dg = d\boldsymbol{\xi}dh$, where $d\boldsymbol{\xi}$ and $dh$ are the normalized invariant measures on $S^{n-1}$ and $SO(n-1)$, respectively, we obtain (3).

Since the functions $\Xi^{n\ell}_K(\boldsymbol{\xi})$ form a basis of the space of $T^{n\ell}$ and this representation can be realized in the space of harmonic polynomials of degree $\ell$ on $\mathbb{R}^n$, then the functions $\Xi^{n\ell}_K(\boldsymbol{\xi})$ are restrictions of harmonic polynomials of degree $\ell$ onto $S^{n-1}$. These polynomials are given by the formula

$$\Xi^{n\ell}_K(\mathbf{x}) = r^\ell \Xi^{n\ell}_K\left(\frac{\mathbf{x}}{r}\right), \tag{5}$$

where $r^2 = (\mathbf{x}, \mathbf{x})$.

**9.3.2. Evaluation of zonal spherical functions.** The zonal spherical functions

$$t^{n\ell}_{OO}(g) = (T^{n\ell}(g)\Xi^{n\ell}_O(\boldsymbol{\xi}), \Xi^{n\ell}_O(\boldsymbol{\xi})) \tag{1}$$

of the representations $T^{n\ell}$ of $SO(n)$ are constant on two-sided cosets with respect to the subgroup $SO(n-1)$. It follows from the Cartan decomposition of $SO(n)$ (see Section 9.1.5) that $t^{n\ell}_{OO}(g)$ is determined by its values on the subgroup $\{g_{n-1}(\theta)\}$. If $g = k_1 g_{n-1}(\theta) k_2$, $k_1, k_2 \in SO(n-1)$, then

$$t^{n\ell}_{OO}(g) = t^{n\ell}_{OO}(g_{n-1}(\theta)).$$

Denote $t^{n\ell}_{OO}(g_{n-1}(\theta))$ by $\varphi^{n\ell}(\theta)$.

In order to evaluate $\varphi^{n\ell}(\theta)$ we make use of the realization of $T^{n\ell}$ in the space $\mathfrak{D}^{n-1,\ell}$ of polynomials of degree $\le \ell$ on $S^{n-2}$ (see Section 9.2.2). In this realization to elements $h \in SO(n-1)$ there correspond rotations of the sphere $S^{n-2}$. Keeping in mind irreducibility of the representations $T^{n-1,k}$, their pairwise nonequivalence and Schur's lemma, we find that to the orthonormal basis $\{\Xi^{n\ell}_K\}$ of $\widetilde{\mathfrak{H}}^{n\ell}$ there corresponds in $\mathfrak{D}^{n-1,\ell}$ an orthonormal basis of the form $\lambda_k \Xi^{n-1,k}_{\widetilde{K}}$, where $K = (k, \widetilde{K})$ and the value of $\lambda_k$ depends on $k$ only. The values of $\lambda_k$ will be indicated in Section 9.4.2. We set $\lambda_0 = 1$, that is, we associate with $\Xi^{n\ell}_O$ the function $\Xi^{n-1,0}_O$ on $S^{n-2}$, identically equal to 1.

Making use of formulas (3) and (4) of Section 9.2.2, for $\varphi^{n\ell}(\theta)$ we obtain the formula

$$\varphi^{n\ell}(\theta) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)} \int_0^\pi (\cos\theta - i\sin\theta\cos\psi)^\ell \sin^{n-3}\psi\, d\psi. \tag{2}$$

Removing the parentheses and integrating term by term, we obtain

$$\begin{aligned}\varphi^{n\ell}(\theta) &= \cos^\ell\theta\, F\left(-\frac{\ell}{2}, -\frac{\ell-1}{2}; \frac{n-1}{2}; -\tan^2\theta\right)\\ &= F\left(-\frac{\ell}{2}, \frac{\ell+n-2}{2}; \frac{n-1}{2}; \sin^2\theta\right)\end{aligned} \tag{3}$$

(see formula (4) of Section 3.5.3). With the help of formula (1) of Section 3.5.8 we express $\varphi^{n\ell}(\theta)$ in terms of the Jacobi polynomial. With the help of formulas (17) and (18) of Section 6.3.9 this Jacobi polynomial is written in terms of the Gegenbauer polynomial. We have

$$\varphi^{n\ell}(\theta) = \frac{(2p-1)!\,\ell!}{(2p+\ell-1)!} C^p_\ell(\cos\theta), \tag{4}$$

where $p = (n-2)/2$.

In the same way one evaluates zonal spherical functions for representations of the groups $SO_0(N-1,1)$ and $ISO(n-1)$. For $SO_0(n-1,1)$ we derive from formulas (6) and (7) of Section 9.2.1 that

$$t^{n\ell}_{OO}(g'_{n-1}(\theta)) \equiv \varphi^{n\sigma}(\theta) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)} \int_0^\pi (\cosh\theta - \cos\varphi\sinh\theta)^\sigma \sin^{n-3}\varphi\, d\varphi. \tag{5}$$

This integral is evaluated in the same way as the integral in (2). We obtain

$$\varphi^{n\sigma}(\theta) = \cosh^\sigma\theta F\left(-\frac{\sigma}{2}, -\frac{\sigma-1}{2}; \frac{n-1}{2}; \tanh^2\theta\right)$$
$$= F\left(-\frac{\sigma}{2}, \frac{\sigma+n-2}{2}; \frac{n-1}{2}; -\sinh^2\theta\right). \tag{6}$$

Comparing formulas (3) and (6) we see that (6) is obtained from (3) by replacing $\ell$ by $\sigma$ and $\theta$ by $i\theta$.

Making use of formula (6) of Section 7.4.4 we obtain

$$\varphi^{n\sigma}(\theta) = 2^{p-1/2}\Gamma\left(p+\frac{1}{2}\right)\sinh^{-p+1/2}\theta\mathfrak{P}^{-p+1/2}_{\sigma+p-1/2}(\cosh\theta),\ p = \frac{n-2}{2}. \tag{7}$$

In the same way, it follows from formula (2) of Section 9.2.4 that for the representation $T^{nR}$ of $ISO(n-1)$ we have

$$t^{nR}_{OO}(g_r) \equiv \varphi^{nR}(r) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)}\int_0^\pi e^{rR\cos\varphi}\sin^{n-3}\varphi d\varphi. \tag{8}$$

Expanding $e^{rR\cos\varphi}$ in powers of $r$ and integrating term by term we find

$$\varphi^{nR}(r) = \Gamma\left(p+\frac{1}{2}\right)\left(-\frac{iRr}{2}\right)^{-p+1/2} J_{p-1/2}(-iRr), \tag{9}$$

where $p = (n-2)/2$. For $R = i\rho$, $\rho \in \mathbb{R}$, we have

$$\varphi^{n,i\rho}(r) = \Gamma\left(p+\frac{1}{2}\right)\left(\frac{\rho r}{2}\right)^{-p+1/2} J_{p-1/2}(\rho r). \tag{10}$$

We propose to the reader to prove that

1) zonal spherical functions of equivalent representations coincide,
2) a zonal spherical function does not depend on the choice of a normalized vector $\Xi^{n\ell}_O$.

**9.3.3. Integral representations of special functions.** Formulas (2)-(5), (7)-(9) of Section 9.3.2 imply the following integral representations of special functions:

$$C^p_\ell(\cos\theta) = \frac{\Gamma(2p+\ell)}{2^{2p-1}\Gamma^2(p)\ell!}\int_0^\pi(\cos\theta - i\sin\theta\cos\varphi)^\ell\sin^{2p-1}\varphi d\varphi, \tag{1}$$

$$\mathfrak{P}^{-p+1/2}_{\sigma+p-1/2}(\cosh\theta) = \frac{\sinh^{p-1/2}\theta}{2^{p-1/2}\sqrt{\pi}\Gamma(p)}\int_0^\pi(\cosh\theta - \cos\varphi\sinh\theta)^\sigma\sin^{2p-1}\varphi d\varphi, \tag{2}$$

$$J_{p-1/2}(r) = \frac{r^{p-1/2}}{2^{p-1/2}\sqrt{\pi}\Gamma(p)}\int_0^\pi e^{\pm ir\cos\varphi}\sin^{2p-1}\varphi d\varphi. \tag{3}$$

These formulas are established for integral and half-integral values of $p$. However, since their derivation is based on expanding into series and making use of the formula for the Beta-function, then formulas (1)-(3) are valid for all $p$ such that $\operatorname{Re} p > 0$.

Further, equivalence of the representations $T^{n\sigma}$ and $T^{n,-\sigma-n+2}$ of $SO_0(n-1,1)$ (see Section 9.2.7) implies that their zonal functions coincide, that is, $\varphi^{n\sigma}(\theta) = \varphi^{n,-\sigma-n+2}(\theta)$. Therefore,

$$\mathfrak{P}^{-p+1/2}_{\sigma+p-1/2}(\cosh\theta) = \mathfrak{P}^{-p+1/2}_{-\sigma-p+1/2}(\cosh\theta). \tag{4}$$

This equality can be obtained from (2) by means of the substitution

$$\cosh\psi = \frac{\cosh\theta\cos\varphi - \sinh\theta}{\cosh\theta - \cos\varphi\sinh\theta}.$$

Analogously, one proves that

$$C^p_\ell(\cos\theta) = \frac{\Gamma(2p+\ell)}{2^{2p-1}\Gamma^2(p)\ell!}\int_0^\pi (\cos\theta - i\sin\theta\cos\varphi)^{-\ell-2p}\sin^{2p-1}\varphi\, d\varphi. \tag{5}$$

We make the substitution $x = \cos\theta - i\sin\theta\cos\varphi$ in (1). As a result we obtain

$$C^p_\ell(\cos\theta) = -\frac{i\Gamma(2p+\ell)\sin^{1-2p}\theta}{2^{2p-1}\Gamma^2(p)\ell!}\int_{e^{-i\theta}}^{e^{i\theta}} z^\ell(1-2z\cos\theta+z^2)^{p-1}dz. \tag{6}$$

Substituting $z = e^{i\psi}$ into (6) and deforming the interval of circle arc we obtain

$$C^p_\ell(\cos\theta) = \frac{\Gamma(2p+\ell)\sin^{1-2p}\theta}{2^{p-1}\Gamma^2(p)\ell!}\int_0^\theta \cos(\ell+p)\psi(\cos\psi-\cos\theta)^{p-1}d\psi. \tag{7}$$

For values of $z$, lying outside of $[-1,1]$, one defines $C^p_\ell(z)$ by analytic continuation.

Analogously, (2) implies

$$\begin{aligned}\mathfrak{P}^{-p+1/2}_{\sigma+p-1/2}(\cosh\theta) &= \frac{\sinh^{-p+1/2}\theta}{\sqrt{\pi}2^{p-1/2}\Gamma(p)}\int_{e^{-i\theta}}^{e^{i\theta}} z^\sigma(2z\cosh\theta - z^2-1)^{p-1}dz \\ &= \frac{\sqrt{2}\sinh^{-p+1/2}\theta}{\sqrt{\pi}\Gamma(p)}\int_0^\theta \cosh(\sigma+p)\psi\ (\cosh\theta-\cosh\psi)^{p-1}d\psi.\end{aligned} \tag{8}$$

This formula can be rewritten in the form

$$(z^2-1)^{-\mu/2}\mathfrak{P}_\nu^\mu(z)=\sqrt{\frac{2}{\pi}}I_{+,1}^{-\mu+1/2}\cosh\left(\nu+\frac{1}{2}\right)\theta,$$

where $\cosh\theta = z$. Here $I_{+,1}^{\lambda}$ is the operator of fractional differentiation with the initial point 1 (not 0 as in formula (2) of Section 3.5.10). Making use of the semigroup property of this operator we get the equality

$$\int_1^u (u-z)^{\tau-1}(z^2-1)^{-\mu/2}\mathfrak{P}_\nu^\mu(z)dz=\Gamma(\tau)(u^2-1)^{(\tau-\mu)/2}\mathfrak{P}_\nu^{\mu-\tau}(u),$$

where $u>1$, $\operatorname{Re}\tau>0$, $\operatorname{Re}\mu<1$.

**9.3.4. The connections with other functions.** It follows from formula (2) of Section 9.3.3 that the function

$$\sinh^{-p+1/2}\theta\mathfrak{P}_{\sigma+p-1/2}^{-p+1/2}(\cosh\theta)$$

for $\theta=0$ is equal to $2^{-p+1/2}/\Gamma(p+1/2)$. We also have

$$\mathfrak{P}_{p-1/2}^{-p+1/2}(\cosh\theta)=\frac{\sinh^{p-1/2}\theta}{2^{p-1/2}\Gamma\left(p+\frac{1}{2}\right)}.$$

In the same way we have from formula (3) of Section 9.3.3 that for $r=0$ the function $r^{-p+1/2}J_{p-1/2}(r)$ is equal to $2^{-p+1/2}/\Gamma\left(p+\frac{1}{2}\right)$, and

$$J_{1/2}(r)=\sqrt{\frac{2}{\pi r}}\sin r.$$

For $n=3$ we have $p=\frac{1}{2}$. Hence, $C_\ell^{1/2}(x)$ coincides with the zonal spherical function of the representation $T_\ell$ of $SO(3)$ which is equal to the Legendre polynomial $P_\ell(x)$.

Now we put $n=4$ into formula (1) of Section 9.3.3. By formula (2) of Section 6.9.1 we obtain

$$t_{OO}^{4\ell}(g_3(\theta))=\frac{1}{2i}\int_{e^{-i\theta}}^{e^{i\theta}}z^\ell dz=\frac{\sin(\ell+1)\theta}{(\ell+1)\sin\theta}=\frac{U_\ell(\cos\theta)}{\ell+1},$$

where $U_\ell(z)$ is the Chebyshev polynomial of the second kind.

Similarly, formula (2) of Section 9.3.3 implies that

$$t^{4\sigma}_{OO}(g'_3(\theta)) = \frac{\sinh(\sigma+1)\theta}{(\sigma+1)\sinh\theta}.$$

The function $\mathfrak{U}_\sigma(z)$, defined by the equality

$$\mathfrak{U}_\sigma(\cosh\theta) = \frac{\sinh(\sigma+1)\theta}{\sinh\theta}, \tag{1}$$

is said to be the Chebyshev function of the second kind. Thus,

$$\sinh^{-1/2}\theta\,\mathfrak{P}^{-1/2}_{\sigma+1/2}(\cosh\theta) = \frac{\sqrt{2}}{\sqrt{\pi}(\sigma+1)}\mathfrak{U}_\sigma(\cosh\theta). \tag{2}$$

We pass in formula (1) of Section 9.3.3 to the limit $\theta \to 0$, $\ell \to \infty$, where $\theta\ell = r$. By virtue of formula (3) of Section 9.3.3 we obtain the equality

$$\lim_{\ell\to\infty} \frac{\ell!}{\Gamma(2p+\ell)} C^p_\ell\left(\cos\frac{r}{\ell}\right) = \frac{\sqrt{\pi}}{(2p)^{p-1/2}\Gamma(p)} J_{p-1/2}(r). \tag{3}$$

Analogously, it follows from formulas (2) and (3) of Section 9.3.3 that

$$\lim_{t\to\infty} t^{p-1/2}\mathfrak{P}^{-p+1/2}_{\sigma+p-1/2}\left(\cosh\frac{r}{t}\right) = J_{p-1/2}(r), \tag{4}$$

where $\sigma = s + it$.

Zonal spherical functions of the groups $SO(n)$, $SO_0(n-1,1)$ and $ISO(n-1)$ are related by formulas (3) and (4). These relations are corollaries of the fact that the group $ISO(n-1)$ is obtained by "rectification" of the groups $SO(n)$ and $SO_0(n-1,1)$.

**9.3.5. Differential equations and integral representations.** The funtion $C^p_\ell(\cos\theta)$, $p=(n-2)/2$, differs from $\Xi^{n\ell}_O(\xi)$, $\xi_n = \cos\theta$, in a constant factor only. It folllows from here that the function $r^\ell C^p_\ell(\frac{x_n}{r})$, $r^2 = x_1^2 + \dots, x_n^2$, is harmonic in $\mathbb{R}^n$ (see Section 9.2.6). This means that $\Delta(r^\ell C^p_\ell(x_n/r)) = 0$. Writing this equality in spherical coordinates (see Section 9.1.5) we obtain the differential equation

$$\left[\frac{d^2}{d\theta^2} + 2p\cot\theta\frac{d}{d\theta} + \ell(2p+\ell)\right] C^p_\ell(\cos\theta) = 0. \tag{1}$$

We derive from here the differential equation for $C^p_\ell(t)$:

$$\left[(1-t^2)\frac{d^2}{dt^2} - (2p+1)t\frac{d}{dt} + \ell(2p+\ell)\right] C^p_\ell(t) = 0. \tag{2}$$

In the same way one derives the differential equation

$$\left[\frac{d^2}{d\theta^2} + 2p\tanh^{-1}\theta\frac{d}{d\theta} - \sigma(\sigma+2p)\right] y = 0 \tag{3}$$

for the function $y = \sinh^{-p+1/2}\theta\mathfrak{P}^{-p+1/2}_{\sigma+p-1/2}(\cosh\theta)$. Another solution of (3) is $\sinh^{-p+1/2}\theta\mathfrak{Q}^{-p+1/2}_{\sigma+p-1/2}(\cosh\theta)$. One can easily obtain from (3) equation (10) of Section 7.4.4 for the functions $\mathfrak{P}^{\lambda}_{\mu}(t)$ and $\mathfrak{Q}^{\lambda}_{\mu}(t)$.

We suggest to the reader to prove that the function $r^{-p+1/2}J_{p-1/2}(r)$ satisfies the equation

$$\left[\frac{d^2}{dr^2} + \frac{2p}{r}\frac{d}{dr} - r^2\right] y = 0 \tag{4}$$

and that the funtion $(z^2-1)^{\mu/2}\mathfrak{P}^{\mu}_{\nu}(z)$ satisfies the equation

$$\left[(z^2-1)\frac{d^2}{dz^2} - (2\mu-2)z\frac{d}{dz} - (\mu+\nu)(\nu-\mu+1)\right] y = 0. \tag{5}$$

With the help of the equations derived we shall find new integral representations of special functions. Twice differentiating the function

$$A(z) = \int_0^\infty (z+\cosh t)^{\mu-\nu-1}\sinh^{2\nu+1} t\,dt, \quad -\mathrm{Re}\mu > \mathrm{Re}\nu > -1, \tag{6}$$

with respect to $z$ and integrating by parts, we find that this function satisfies equation (5). The general solution of this equation has the form

$$C_1(z^2-1)^{\mu/2}\mathfrak{P}^{\mu}_{\nu}(z) + C_2(z^2-1)^{\mu/2}\mathfrak{Q}^{\mu}_{\nu}(z)$$

and for $z=0$ the first summand is equal to

$$C_1 2^{\mu}\sqrt{\pi}\Big/\Gamma\left(\frac{1-\nu-\mu}{2}\right)\Gamma\left(\frac{2+\nu-\mu}{2}\right)$$

(see formula (5) of Section 7.4.4) and the second one has a singularity at $z=0$. Since

$$A(0) = \int_0^\infty \cosh^{\mu-\nu-1} t\sinh^{2\nu+1} t\,dt = \frac{\Gamma(\nu+1)\Gamma\left(\frac{-\nu-\mu}{2}\right)}{2\Gamma\left(\frac{\nu-\mu}{2}+1\right)},$$

then $A(z) = 2^{\nu}\Gamma(-\mu-\nu)\Gamma(\nu+1)(z^2-1)^{\mu/2}\mathfrak{P}^{\mu}_{\nu}(z)$, and we have

$$\mathfrak{P}^{\mu}_{\nu}(z) = \frac{2^{-\nu}(z^2-1)^{-\mu/2}}{\Gamma(-\mu-\nu)\Gamma(\nu+1)}\int_0^\infty (z+\cosh t)^{\mu-\nu-1}\sinh^{2\nu+1} t\,dt, \tag{7}$$

where $-\mathrm{Re}\,\mu > \mathrm{Re}\,\nu > -1$. Analogously, one proves the equality

$$\mathfrak{P}_\nu^{-\mu}(z) = \sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\mu+\frac{1}{2}\right)(z^2-1)^{\mu/2}}{\Gamma(\nu+\mu+1)\Gamma(\mu-\nu)}\int_0^\infty (z+\cosh t)^{-\mu-1/2}\cosh\left(\nu+\frac{1}{2}\right)t\,dt, \quad (8)$$

where $\mathrm{Re}\,(\mu-\nu) > 0$, $\mathrm{Re}\,(\mu+\nu+1) > 0$, and the integral representation for the Macdonald function

$$K_\nu(z) = \frac{\sqrt{\pi}z^\nu}{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty e^{-z\cosh t}\sinh^{2\nu} t dt, \quad (9)$$

where either $\mathrm{Re}\,\nu > -1/2$, $\mathrm{Re}\,z > 0$ or $-1/2 < \mathrm{Re}\,\nu < 1/2$, $\mathrm{Re}\,z = 0$. In the last case instead of consideration of the function at $z = 0$ one has to compare asymptotics for $z \to \infty$ of the integrals

$$\int_0^\infty e^{-z\cosh t}\sinh^{2\nu} t\,dt, \qquad \int_0^\infty e^{-z\cosh t}\cosh\nu t\,dt.$$

Let us give integral representations for $\mathfrak{Q}_\nu^\mu(z)$. The equality

$$\Gamma(\nu-\mu+1)\Gamma\left(\mu+\frac{1}{2}\right)\mathfrak{Q}_\nu^\mu(\cosh\theta)$$
$$= e^{i\mu\pi}\sqrt{\pi}2^{-\mu}\Gamma(\nu+\mu+1)\sinh^\mu\theta\int_0^\infty(\cosh\theta+\sinh\theta\cosh t)^{-\nu-\mu-1}\sinh^{2\mu} t dt, \quad (10)$$

where $\mathrm{Re}\,(\nu\pm\mu+1) > 0$, $\mathrm{Re}\,\theta > 0$, is an analog of formula (2) of Section 9.3.3. In order to prove (10) it is sufficient to use formula (19) of Section 7.4.4 and integral representation (7) for $\mathfrak{P}_\nu^\mu(z)$.

Making the substitution $e^v = \cosh\theta + \sinh\theta\cosh t$ in (10) we obtain

$$\mathfrak{Q}_\nu^\mu(\cosh\theta) = \sqrt{\frac{\pi}{2}}e^{\mu\pi i}\frac{\sinh^\mu\theta}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_\theta^\infty e^{-v\left(\nu+\frac{1}{2}\right)}(\cosh v-\cosh\theta)^{-\mu-\frac{1}{2}}dv, \quad (11)$$

where $\theta > 0$, $\mathrm{Re}\,(\nu+\mu+1) > 0$,$\mathrm{Re}\,\mu < 1/2$.

By similar methods one proves the formulas

$$\mathfrak{Q}_\nu^\mu(z) = \frac{e^{\mu\pi i}\Gamma(\nu+\mu+1)(z^2-1)^{-\mu/2}}{2^{\nu+1}\Gamma(\nu+1)}\int_0^\pi(z-\cos t)^{\mu-\nu-1}\sin^{2\nu+1} t dt, \quad (12)$$

where $\operatorname{Re}\nu > -1$, $\operatorname{Re}(\nu+\mu+1) > 0$,

$$\mathfrak{Q}_\nu^\mu(z) = \frac{e^{i\mu\pi}\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)} \int_0^\infty \left(z + \sqrt{z^1 - 1}\cosh t\right)^{-\nu-1} \cosh\mu t\, dt, \tag{13}$$

where $\operatorname{Re}(\nu+\mu) > -1$, $\nu \neq 0, -1, -2, \dots$,

$$\mathrm{P}_\nu^\mu(\cos\theta) = \frac{2^\mu \sin^{-\mu}\theta}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\mu\right)} \int_0^\pi (\cos\theta + i\sin\theta\cos t)^{\nu+\mu} \sin^{-2\mu} t dt, \tag{14}$$

where $\operatorname{Re}\mu < 1/2$. Making the substitution $\cos\theta + i\sin\theta\cos t = e^{iv}$ into (14) we derive that

$$\mathrm{P}_\nu^\mu(\cos\theta) = \sqrt{\frac{2}{\pi}} \frac{\sin^\mu\theta}{\Gamma\left(\frac{1}{2}-\mu\right)} \int_0^\theta (\cos v - \cos\theta)^{-\mu-1/2} \cos\left[\left(\nu + \frac{1}{2}\right) v\right] dv. \tag{14'}$$

We also give the equalities

$$\int_0^{\arccos x} \cos\left(\nu + \frac{1}{2}\right) t\, (\cos t - x)^{\mu-1/2} dt = \sqrt{\frac{\pi}{2}} \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}$$
$$\times (1-x^2)^{\mu/2} \left[\cos\mu\pi\, \mathrm{P}_\nu^\mu(x) - \frac{2}{\pi}\sin\mu\pi \mathrm{Q}_\nu^\mu(x)\right], \tag{15}$$

where $-1 \le x \le 1$, $\operatorname{Re}\mu > -1/2$,

$$\int_{\arccos x}^\pi \frac{\cos\left[\left(\nu+\frac{1}{2}\right)(t-\pi)\right] dt}{(x - \cos t)^{-\mu+1/2}} = \sqrt{\frac{\pi}{2}} \frac{\Gamma(\mu+1)\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} (1-x^2)^{\mu/2}$$
$$\times \left[\cos\nu\pi\, \mathrm{P}_\nu^\mu(x) - \frac{2}{\pi}\sin\nu\pi\, \mathrm{Q}_\nu^\mu(x)\right], \tag{16}$$

where $-1 \le x \le 1$, $\operatorname{Re}\mu > -\frac{1}{2}$,

$$\int_{\arccos x}^\pi \frac{\cos\left[\left(\nu+\frac{1}{2}\right)(t-\pi)\right] dt}{(x - \cos t)^{\mu+1/2}} = \sqrt{\frac{\pi}{2}}\Gamma\left(\frac{1}{2}-\mu\right)(1-x^2)^{-\mu/2}$$
$$\times \left[\cos(\mu+\nu)\pi\, \mathrm{P}_\nu^\mu(x) - \frac{2}{\pi}\sin(\mu+\nu)\pi\, \mathrm{Q}_\nu^\mu(x)\right], \tag{17}$$

where $-1 \le x \le 1$, $\operatorname{Re}\mu < \frac{1}{2}$,

$$\int_0^\pi \frac{\sin^{2\mu} t dt}{\left(x \pm i\sqrt{1-x^2}\cos t\right)^{\nu-\mu}} = \frac{2^\mu \sqrt{\pi}\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}(1-x^2)^{-\mu/2} \times \left[\cos\mu\pi\, \mathrm{P}_\nu^\mu(x) - \frac{2}{\pi}\sin\mu\pi\, \mathrm{Q}_\nu^\mu(x)\right], \quad (18)$$

where $-1 \le x \le 1$, $\operatorname{Re}\mu > -\frac{1}{2}$.

The function $\mathfrak{P}_\sigma^\tau$ is related to the associated Legendre functions of the second kind $\mathfrak{Q}_\nu^\mu$ by the Whipple formula

$$\mathfrak{Q}_\nu^\mu(z) = \sqrt{\frac{\pi}{2}} e^{i\mu\pi}\Gamma(\mu+\nu+1)(z^2-1)^{-1/4}\mathfrak{P}_{-\mu-1/2}^{-\nu-1/2}\left(\frac{z^2}{\sqrt{z^2-1}}\right), \quad \operatorname{Re} z > 0 \quad (19)$$

(see Section 7.4.4). This formula can be written down as

$$\Gamma(-\nu-\mu)\mathfrak{P}_\nu^\mu(z) = ie^{i\nu\pi}\left(\frac{\pi}{2}\right)^{-1/2}(z^2-1)^{-1/4}\mathfrak{Q}_{-\mu-1/2}^{-\nu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right), \quad \operatorname{Re} z > 0. \quad (20)$$

It follows from the definition of associated Legendre functions on the cut (see Section 7.4.4) and from (19) and (20) that

$$\mathfrak{Q}_\nu^\mu(ix) = \left(\frac{\pi}{2}\right)^{1/2}\exp\left[i\pi\left(\nu - \frac{\mu+1}{2}\right)\right]\Gamma(\nu+\mu+1) \times (1+x^2)^{-1/4}\mathrm{P}_{-\mu-1/2}^{-\nu-1/2}\left(\frac{x}{\sqrt{1+x^2}}\right), \quad x > 0, \quad (21)$$

$$\mathrm{P}_\nu^\mu(x) = \left(\frac{\pi}{2}\right)^{-1/2}\exp\left[i\pi\left(\nu + \frac{\mu}{2} + \frac{1}{4}\right)\right]\frac{(1-x^2)^{1/4}}{\Gamma(-\nu-\mu)} \times \mathfrak{Q}_{-\mu-1/2}^{-\nu-1/2}\left(\frac{ix}{\sqrt{1-x^2}}\right), \quad 0 \le x < 1. \quad (22)$$

**9.3.6. Analogs of the Rodrigues formula.** By setting $\cos\theta = t$ into formula (6) of Section 9.3.3 we obtain

$$(1-t^2)^{p-1/2}C_\ell^p(t) = -\frac{i\Gamma(2p+\ell)}{2^{2p-1}\Gamma^2(p)\ell!}\int_{t-i\sqrt{1-t^2}}^{t+i\sqrt{1-t^2}} z^\ell(z^2-2tz+1)^{p-1}dz. \quad (1)$$

In particular, for $\ell = 0$ we obtain

$$(1-t^2)^{p-1/2} = -\frac{i\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(p)} \int_{t-i\sqrt{1-t^2}}^{t+i\sqrt{1-t^2}} (z^2-2tz+1)^{p-1}dz. \tag{2}$$

Let us replace $p$ by $p+\ell$ in (2), differentiate $\ell$ times with respect to $t$ and compare the result with formula (6) of Section 9.3.3. After simple transformations we obtain the Rodrigues formula for Gegenbauer polynomials:

$$C_\ell^p(t) = \left(-\frac{1}{2}\right)^\ell \frac{\Gamma(2p+\ell)\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(2p)\Gamma\left(p+\ell+\frac{1}{2}\right)\ell!}(1-t^2)^{-p+1/2}\frac{d^\ell}{dt^\ell}[(1-t^2)^{p+\ell-1/2}]. \tag{3}$$

One can use this formula for evaluation of the integrals

$$I = \int_{-1}^{1} F(t)C_\ell^p(t)(1-t^2)^{p-1/2}dt. \tag{4}$$

Namely, replacing $C_\ell^p(t)$ by expression (3) and integrating $\ell$ times by parts we find that

$$I = \frac{\Gamma(2p+\ell)\Gamma\left(p+\frac{1}{2}\right)}{2^\ell\Gamma(2p)\Gamma\left(p+\ell+\frac{1}{2}\right)\ell!}\int_{-1}^{1} F^{(\ell)}(t)(1-t^2)^{p+\ell-1/2}dt. \tag{4'}$$

In particular, setting $t=\cos\varphi$, $F(\cos\varphi) = (\cos\theta - i\sin\theta\cos\varphi)^\ell$, $p=(n-2)/2$ and making use of formula (1) of Section 9.3.3 we derive

$$\int_0^\pi (\cos\theta - i\sin\theta\cos\varphi)^\ell C_m^{p-1/2}(\cos\varphi)\sin^{2p-1}\varphi d\varphi$$
$$= \frac{2^{m+1}\sqrt{\pi}\Gamma(p+m)\ell!\Gamma(2p+m-1)}{m!\Gamma\left(p-\frac{1}{2}\right)\Gamma(2p+m+\ell)} i^{-m}\sin^m\theta\, C_{\ell-m}^{p+m}(\cos\theta). \tag{5}$$

In the same way one proves the equalities

$$\int_0^\pi (\cosh\theta - \sinh\theta\cos\varphi)^\sigma C_m^{p-1/2}(\cos\varphi)\sin^{2p-1}\varphi d\varphi$$
$$= \frac{(-1)^m\pi 2^{p-1/2}\Gamma(2p+m-1)\Gamma(p)\Gamma(\sigma+1)}{m!\Gamma(2p-1)\Gamma(\sigma-m+1)}\sinh^{-p+1/2}\theta\,\mathfrak{P}_{\sigma+p-1/2}^{-p-m+1/2}(\cosh\theta), \tag{6}$$

$$\int_0^\pi (\sinh\theta - \cosh\theta\cos\varphi)_\delta^\sigma C_m^{p-1/2}(\cos\varphi)\sin^{2p-1}\varphi d\varphi = \frac{(-1)^m 2^{p-1}}{m!\Gamma(2p-1)}$$
$$\times\Gamma(2p+m-1)\Gamma(p)\Gamma(\sigma+1)\cosh^{-p}\theta \mathrm{P}_{-m-p}^{-\sigma-p}(-\delta\tanh\theta),\ \delta\in\{+,-\}, \quad (6')$$

$$\int_0^\pi e^{ix\cos\varphi} C_m^{p-1/2}(\cos\varphi)\sin^{2p-1}\varphi d\varphi$$
$$= \frac{\sqrt{\pi}2^{p-1/2}i^m\Gamma(2p+m-1)\Gamma(p)}{m!\Gamma(2p-1)} x^{-p+1/2} J_{m+p-1/2}(x). \quad (7)$$

Another relation of the form (4′) follows from formula (8) of Section 9.3.3. Replacing in this formula $\frac{1}{2}-p$ by $\mu$, $\sigma+p-\frac{1}{2}$ by $\nu$ and $\cosh\theta$ by $z$, we obtain

$$J \equiv \int_1^\infty F(z)(z^2-1)^{-\mu/2}\mathfrak{P}_\nu^\mu(z)dz$$
$$= \sqrt{\frac{2}{\pi}}\frac{1}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_1^\infty F(z)\int_1^z \frac{\cosh\left(\nu+\frac{1}{2}\right)\psi}{\sinh\psi}(z-u)^{-\mu-1/2}dudz,$$

where $u=\cosh\psi$. Changing the order of integration, we derive

$$J = \sqrt{\frac{2}{\pi}}\frac{1}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_1^\infty \frac{\cosh\left(\nu+\frac{1}{2}\right)\psi}{\sinh\psi}\int_u^\infty F(z)(z-u)^{-\mu-1/2}dzdu$$
$$= \sqrt{\frac{2}{\pi}}\int_1^\infty \frac{\cosh\left(\nu+\frac{1}{2}\right)\psi}{\sinh\psi}(I_-^{-\mu+1/2}F)(u)du, \quad (8)$$

where $u=\cosh\psi$. Since

$$I_-^{-\mu+1/2}e^{-uz} = u^{\mu-1/2}e^{-uz}$$

for $\operatorname{Re} u > 0$, then

$$\int_0^\infty e^{-uz}(z^2-1)^{-\mu/2}\mathfrak{P}_\nu^\mu(z)dz$$
$$= \sqrt{\frac{2}{\pi}}u^{\mu-1/2}\int_0^\infty e^{-u\cosh\psi}\cosh\left(\nu+\frac{1}{2}\right)\psi\, d\psi = \sqrt{\frac{2}{\pi}}u^{\mu-1/2}K_{\nu+1/2}(u), \quad (9)$$

where $\operatorname{Re}\mu < 1$, $\operatorname{Re} u > 0$ (see formula (23) of Section 3.5.6).

Let us now evaluate the integrals

$$A = \int_1^\infty (z+u)^\sigma (z^2-1)^{-\mu/2} \mathfrak{P}_\nu^\mu(z)dz, \quad u > 0, \tag{9'}$$

$$B = \int_1^u (u-z)^\sigma (z^2-1)^{-\mu/2} \mathfrak{P}_\nu^\mu(z)dz$$

which will be useful in the next chapter. In (9′) we use the formula (8) of Section 9.3.3. Inverting the order of integration, we have

$$A = \sqrt{\frac{2}{\pi}} \frac{1}{\Gamma\left(\frac{1}{2}-\mu\right)} \int_1^\infty \frac{\cosh(\nu+1/2)t}{\sinh t} dy \int_y^\infty (z+u)^\sigma (z-y)^{-\mu-1} dz,$$

where $y = \cosh t$. Since

$$\int_y^\infty (z+u)^\sigma (z-y)^{-\mu-1} dz = \int_0^\infty (z+u+y)^\sigma z^{-\mu-1} dz$$

$$= \frac{\Gamma(-\mu)\Gamma(\mu-\sigma)}{\Gamma(-\sigma)} (u+y)^{\sigma-\mu},$$

then we obtain

$$A = \sqrt{\frac{2}{\pi}} \frac{\Gamma(-\mu)\Gamma(\mu-\sigma)}{\Gamma\left(\frac{1}{2}-\mu\right)\Gamma(-\sigma)} \int_0^\infty (u+\cosh t)^{\sigma-\mu} \cosh\left(\nu+\frac{1}{2}\right) t\, dt.$$

By virtue of formula (8) of Section 9.3.5, for $u > 1$ we have

$$A = \frac{\Gamma(\mu-\nu-\sigma-1)\Gamma(\mu-\sigma+\nu)}{\Gamma(-\sigma)} (u^2-1)^{(\sigma-\mu+1)/2} \mathfrak{P}_\nu^{\sigma-\mu+1}(u), \tag{10}$$

where $\operatorname{Re}\mu < 1$, $-\operatorname{Re}(\mu-\sigma) < \operatorname{Re}\nu < \operatorname{Re}(\mu-\sigma) - 1$.

If $|u| < 1$, then we obtain

$$A = \frac{\Gamma(\mu-\nu-\sigma-1)\Gamma(\mu-\sigma+\nu)}{\Gamma(-\sigma)} (1-u^2)^{(\sigma-\mu+1)/2} \mathrm{P}_\nu^{\sigma-\mu+1}(u), \tag{10'}$$

where the conditions for $\mu$, $\nu$, $\sigma$ are the same as above.

Analogously we prove that

$$B = \Gamma(\sigma+1)(u^2-1)^{-\sigma+\mu-1}\mathfrak{P}_{\tau-\mu}^{-\sigma+\mu-1}(u). \tag{11}$$

Let us prove the formulas

$$\int_0^b x^{\nu+1}(b^2-x^2)^{\mu} J_\nu(ax)dx = 2^{\mu}\Gamma(\mu+1)\frac{b^{\mu+\nu+1}}{a^{\mu+1}} J_{\mu+\nu+1}(ab), \tag{12}$$

$$\int_0^\infty x^{\nu+1}(x^2+b^2)^{-\mu-1} J_\nu(ax)dx = \frac{a^{\mu}b^{\nu-\mu}}{2^{\mu}\Gamma(\mu+1)} K_{\nu-\mu}(ab), \tag{13}$$

which will be used in the following chapter. In order to prove the first formula, we expand the function $J_\nu(ax)$ into a power series and integrate it term by term with the help of the integral formula for the Beta-function. In order to prove the second formula, we use the integral representation

$$J_\nu(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(-s)x^{\nu+2s}}{2^{\nu+2s}\Gamma(\nu+s+1)} ds$$

(which is proved with the help of the residue theorem) instead of expansion into a power series. Inverting the integration order and using the equality

$$\int_0^\infty x^{2\nu+2s+1}(x^2+b^2)^{-\mu-1}dx = \frac{\Gamma(\nu+s+1)\Gamma(\mu-\nu-s)}{2\Gamma(\mu+1)}$$

we obtain that

$$\int_0^\infty x^{\nu+1}(x^2+b^2)^{-\mu-1} J_\nu(ax)dx$$

$$= \frac{1}{4\pi i\Gamma(\mu+1)}\int_{-i\infty}^{i\infty} \Gamma(-s)\Gamma(\mu-\nu-s)\left(\frac{a}{2}\right)^{\nu+2s} b^{2\nu+2s-2\mu} ds.$$

Using the residue theorem, we derive from there formula (13).

**9.3.7. Generating functions.** We have shown in Section 9.2.3 that the function $r^{-2p} = (x_1^2+\ldots+x_n^2)^{-p}$ is harmonic. Since the Laplace operator is invariant with respect to parallel translations, then the function

$$F(\mathbf{x},h) = (x_1^2+\ldots+x_{n-1}^2+(x_n-h)^2)^{-p}$$

$$= (r^2-2x_nh+h^2)^{-p} = \frac{1}{r^{2p}}\left(1-\frac{2hx_n}{r^2}+\frac{h^2}{r^2}\right)^{-p}, \; p = \frac{n-2}{2},$$

is also harmonic.

We expand this function in powers of $h$:

$$F(\mathbf{x}, h) = \sum_{k=0}^{\infty} \frac{h^k}{r^{2p+k}} R_k\left(\frac{x_n}{r}\right), \tag{1}$$

where $R_k(x)$ is a polynomial of degree $k$. Since for all $h$ the left hand side of (1) is harmonic, then all the functions $r^{-2p-k}R_k(x_n/r)$ are harmonic. They are obtained from the harmonic polynomials $r^k R_k(x_n/r)$ by the Kelvin transform. Since these polynomials are invariant with respect to rotations from $SO(n-1)$, they can differ from the zonal polynomials $r^k C_k^p(x_n/r)$ by a constant factor only. Therefore, by setting $r = 1$, $x_n = t$, we obtain

$$(1 - 2ht + h^2)^{-p} = \sum_{k=0}^{\infty} \lambda_k C_k^p(t) h^k.$$

In order to find $\lambda_k$ we put $t = 1$. Since $C_k^p(1) = \Gamma(2p+k)/k!\Gamma(2p)$, we have

$$(1 - h^2)^{-2p} = \sum_{k=0}^{\infty} \frac{\Gamma(2p+k)}{k!\Gamma(2p)} \lambda_k h^k.$$

Expanding the left hand side in powers of $h$ and comparing coefficients of the same powers of $h$ on the left and on the right, we find that all $\lambda_k$ are equal to 1. Thus,

$$(1 - 2th + h^2)^{-p} = \sum_{k=0}^{\infty} C_k^p(t) h^k. \tag{2}$$

Formula (2) provides the generating function for Gegenbauer polynomials with a fixed $p$. By virtue of (2) one can define $C_k^p$ for arbitrary $p \in \mathbb{C}$ and $k \in \mathbb{Z}_+$. Some properties of Gegenbauer polynomials follow from (2). For example, replace $p$ by $q$ in (2) and multiply this expansion by expansion (2). Applying (2) to $(1 - 2th + h^2)^{-p-q}$ and comparing coefficients on the left and on the right, we obtain the relation

$$C_n^{p+q}(t) = \sum_{k=0}^{n} C_{n-k}^p(t) C_k^q(t). \tag{3}$$

Let us note that

$$1 - 2h\cos\theta + h^2 = (1 - he^{i\theta})(1 - he^{-i\theta}).$$

Applying to the expansion

$$(1 - he^{i\theta})^{-p}(1 - he^{-i\theta})^{-p} = \sum_{n=0}^{\infty} C_n^p(\cos\theta) h^n$$

the formula for coefficients for the Taylor series, we find that

$$\begin{aligned} C_n^p(\cos\theta) &= \frac{1}{n!}\frac{d^n}{dh^n}\left[(1-he^{i\theta})^{-p}(1-he^{-i\theta})^{-p}\right]\Big|_{h=0} \\ &= \frac{1}{\Gamma^2(p)}\sum_{j=0}^{n}\frac{\Gamma(p+j)\Gamma(p+n-j)}{j!(n-j)!}e^{i(2j-n)\theta} \\ &= \frac{\Gamma(p+n)}{\Gamma(p)n!}e^{-in\theta}F(-n,p;1-n-p;e^{2i\theta}) \\ &= \frac{\Gamma(p+n)}{\Gamma(p)n!}e^{in\theta}F(-n,p;1-n-p;e^{-2i\theta}). \end{aligned} \tag{4}$$

One easily derives from $\cos 2\theta = 2\cos^2\theta - 1$ that

$$(1-2\cos 2\theta\, z^2+z^4)^{-p} = (1-2\cos\theta\, z+z^2)^{-p}(1+2\cos\theta z+z^2)^{-p}.$$

Applying (2) and the relation $C_n^p(-x) = (-1)^n C_n^p(x)$, after simple transformations we obtain the equality

$$\sum_{n=0}^{\infty} C_n^p(\cos 2\theta)z^{2n} = \sum_{m=0}^{\infty}\sum_{k=0}^{m}(-1)^k C_{m-k}^p(\cos\theta)C_k^p(\cos\theta)z^m.$$

Comparison of coefficients of $z^{2n}$ yields

$$C_n^p(\cos 2\theta) = \sum_{k=0}^{2n}(-1)^k C_k^p(\cos\theta)C_{2n-k}^p(\cos\theta). \tag{4'}$$

One can easily derive from (2) recurrence formulas for Gegenbauer polynomials. For example, if we differentiate both sides of (2) with respect to $h$, expand the left hand side in powers of $h$ and compare coefficients of powers of $h$ on the left and on the right, we obtain the recurrence relation

$$(k+1)C_{k+1}^p(t) = 2p[tC_k^{p+1}(t) - C_{k-1}^{p+1}(t)]. \tag{5}$$

Put here $k+1=\ell$, multiply both sides by $\Gamma(p)$ and pass to the limit $p\to 0$. Taking into account the equality

$$C_\ell^1(\cos\varphi) = \frac{\sin(\ell+1)\varphi}{\sin\varphi},$$

we conclude that

$$\lim_{p\to 0}\Gamma(p)C_\ell^p(\cos\varphi) = \frac{2\cos\ell\varphi}{\ell}. \tag{6}$$

We set $\cos\varphi = t$ into formula (3) of Section 9.3.3 and apply the inversion formula for the Fourier cosine-transform. We obtain

$$(1-t^2)_+^{p-1} = \frac{2^{p-1/2}\Gamma(p)}{\sqrt{\pi}} \int_0^\infty J_{p-1/2}(r) r^{-p+1/2} \cos rt\, dr, \tag{7}$$

where $t \in \mathbb{R}$. Writing down expansion (7) for $p = q$, multiplying it by (7), then applying (7) to the left hand side of the relation obtained, we derive

$$\int_{-\infty}^{\infty} |R|^{-p-q+3/2} J_{p+q-3/2}(|R|) e^{iRt} dR = \frac{\Gamma(p)\Gamma(q)}{\sqrt{2\pi}\Gamma(p+q-1)}$$
$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{p-1/2}(|r|) J_{q-1/2}(|\rho|) |r|^{-p+1/2} |\rho|^{-q+1/2} e^{i(r+\rho)t} dr\, d\rho. \tag{8}$$

Making the substitution $r + \rho = R$, we obtain the relation

$$\frac{J_{p+q-3/2}(R)}{R^{p+q-3/2}} = \sqrt{\frac{2}{\pi}} \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q-1)} \int_0^\infty \frac{J_{p-1/2}(r)}{r^{p-1/2}}$$
$$\times \left[\frac{J_{q-1/2}(|R-r|)}{|R-r|^{q-1/2}} + \frac{J_{q-1/2}(R+r)}{(R+r)^{q-1/2}}\right] dr. \tag{9}$$

Formula (8) of Section 9.3.3 leads to the continual generating function for Legendre function:

$$(\cosh\theta - \cosh\psi)_+^{p-1} = \frac{\Gamma(p)\sinh^{p-1/2}\theta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathfrak{P}_{i\lambda-1/2}^{-p+1/2}(\cosh\theta) e^{i\lambda\psi} d\lambda. \tag{10}$$

Writing down this expansion for $p = q$, multiplying it by (10), then applying (10) to the left hand side of the result, we derive the relation

$$\sinh^{-1/2}\theta \mathfrak{P}_{i\nu-1/2}^{-p-q+3/2}(\cosh\theta) = \frac{\Gamma(p)\Gamma(q)}{\sqrt{2\pi}\Gamma(p+q-1)}$$
$$\times \int_{-\infty} 6\infty \mathfrak{P}_{i\nu-i\mu-1/2}^{-p+1/2}(\cosh\theta) \mathfrak{P}_{i\mu-1/2}^{-q+1/2}(\cosh\theta) d\mu. \tag{11}$$

**9.3.8. Orthogonality relations for Gegenbauer polynomials.** It follows from the results of Section 2.3.5 that the zonal spherical function $t_{OO}^{n\ell}(g)$ obey the orthogonality relation

$$\int_{SO(n)} t_{OO}^{nm}(g) \overline{t_{OO}^{n\ell}(g)} dg = \frac{\delta_{m\ell}}{\dim T^{n\ell}} = \frac{(n-2)!\ell!}{(\ell+n-3)!(2\ell+n-2)} \delta_{m\ell}.$$

Replacing the integration over $SO(n)$ by the integrations over $SO(n-1)$ and $S^{n-1}$, taking into account two-sided invariance of $t^{n\ell}_{OO}(g)$ with respect to $SO(n-1)$, we derive

$$\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})}\int_0^{\pi} t^{nm}_{OO}(g_{n-1}(\theta))\overline{t^{n\ell}_{OO}(g_{n-1}(\theta))}\sin^{n-2}\theta d\theta = \frac{(n-2)!\ell!\delta_{m\ell}}{(\ell+n-3)!(2\ell+n-2)}. \tag{1}$$

But the Gegenbauer polynomials $C_\ell^{(n-2)/2}(\cos\theta)$, $\ell = 0,1,\ldots$, differ from $t^{n\ell}_{00}(g_{n-1}(\theta))$ by a scalar factor only (see formula (4) of Section 9.3.2). Therefore, we have the orthogonality relation

$$\int_{-1}^{1} C^p_m(t)C^p_\ell(t)(1-t^2)^{p-1/2}dt = \frac{\pi\Gamma(2p+\ell)}{2^{2p-1}\ell!(\ell+p)\Gamma^2(p)}\delta_{m\ell}. \tag{2}$$

Thus, the polynomials

$$\left[\frac{2^{2p-1}\ell!(p+\ell)}{\pi\Gamma(2p+\ell)}\right]^{1/2}\Gamma(p)C^p_\ell(t), \quad \ell = 0,1,2,\ldots, \tag{3}$$

form an orthonormal system on the interval $[-1,1]$ with respect to the weight $(1-t^2)^{p-1/2}$. The results of Section 2.3.9 and the fact that the representations $T^{n\ell}$, $\ell = 0,1,2,\ldots$, exhaust all representations of class 1 of the group $SO(n)$ imply completeness of the system of polynomials $C^p_\ell(t)$, $\ell = 0,1,2,\ldots$, with respect to the weight indicated above (see Section 6.10.2).

Orthogonality of the system $\{C^p_\ell(t)\}$ implies that if for degree $s$ of a polynomial $P(t)$ we have $s < \ell$, then

$$\int_{-1}^{1} P(t)C^p_\ell(t)(1-t^2)^{p-1/2}dt = 0. \tag{4}$$

This integral also vanishes in the case when evenness of $P(t)$ is different from that of $C^p_\ell(t)$.

By means of orthogonality relations one evaluates integrals containing Gegenbauer polynomials. For example, it follows from expansion (2) of Section 9.3.7 that

$$\int_{-1}^{1}(1-2th+h^2)^{-p}C^p_k(t)(1-t^2)^{p-1/2}dt = \frac{\pi\Gamma(2p+k)h^k}{2^{2p-1}k!(p+k)\Gamma^2(p)}. \tag{5}$$

In the same way, formulas (5), (6) and (7) of Section 9.3.6 give

$$(\cos\theta - i\sin\theta\cos\varphi)^{\ell} = \frac{\Gamma(2p-1)\ell!}{\Gamma(p)}$$

$$\times \sum_{m=0}^{\ell} \frac{(-2i)^m(2m+2p-1)\Gamma(p+m)}{\Gamma(2p+\ell+m)} \sin^m\theta\, C_m^{p-1/2}(\cos\varphi) C_{\ell-m}^{p+m}(\cos\theta), \quad (6)$$

$$(\cosh\theta - \sinh\theta\cos\varphi)^{\sigma} = 2^{p-3/2}\Gamma(\sigma+1)\Gamma\left(p-\frac{1}{2}\right)$$

$$\times \sum_{m=0}^{\infty} \frac{(-1)^m(2p+2m-1)}{\Gamma(\sigma-m+1)} \sinh^{-p+1/2}\theta \mathfrak{P}_{\sigma+p-1/2}^{-p-m+1/2}(\cosh\theta) C_m^{p-1/2}(\cos\varphi), \quad (7)$$

$$e^{iR\cos\varphi} =$$

$$= 2^{p-3/2}\Gamma\left(p-\frac{1}{2}\right) R^{-p+1/2} \sum_{m=0}^{\infty} i^m(2p+2m-1) J_{p+m-1/2}(R) C_m^{p-1/2}(\cos\varphi). \quad (8)$$

Putting $\theta = \pi/2$ into (6) we obtain the expansin of $\cos^{\ell}\varphi$ in Gegenbauer polynomials, and putting $\varphi = 0$ we obtain the expansion of $e^{-i\ell\theta}$ (we suggest to the reader to write down corresponding formulas). Further, let us multiply out expansions of the form (6) for $\ell = \ell_1$ and $\ell = \ell_2$ and apply (6) to the left hand side of the expansion obtained. Making use of the orthogonality relation for Gegenbauer polynomials, we obtain the relation

$$C_{\ell_1+\ell_2}^{p}(\cos\varphi) = \frac{\ell_1!\ell_2!\Gamma(2p+\ell_1+\ell_2)}{(\ell_1+\ell_2)!\Gamma^2(p)} \sum_{j=0}^{\min(\ell_1,\ell_2)} \frac{(-4)^j\Gamma^2(p+j)\Gamma(j+2p-1)}{j!\Gamma(2p+\ell_1+j)}$$

$$\times \frac{2j+2p-1}{\Gamma(2p+\ell_2+j)} \sin^{2j}\varphi C_{\ell_1-j}^{p+j}(\cos\varphi) C_{\ell_2-j}^{p+j}(\cos\varphi). \quad (9)$$

Similarly, formula (7) gives the expansions of $\cosh^{\sigma}\varphi$ and of $e^{\sigma\theta}$ in Legendre functions and the identity

$$\sinh^{p-1/2}\theta \mathfrak{P}_{\sigma+\tau+p-1/2}^{-p+1/2}(\cosh\theta) = \frac{\Gamma(\sigma+1)\Gamma(\tau+1)}{2^{p-1/2}\Gamma\left(p+\frac{1}{2}\right)}$$

$$\times \sum_{j=0}^{\infty} \frac{(2p+2j-1)\Gamma(2p+j-1)}{j!\Gamma(\sigma-j+1)\Gamma(\tau-j+1)} \mathfrak{P}_{\sigma+p-1/2}^{-p-j+1/2}(\cosh\theta) \mathfrak{P}_{\tau+p-1/2}^{-p-j+1/2}(\cosh\theta). \quad (10)$$

The formula (8) implies the expansion of $e^{iR}$ in Bessel functions and the identity

$$(R_1+R_2)^{-p+1/2}J_{p-1/2}(R_1+R_2)=\sqrt{\pi}(2R_1R_2)^{-p+1/2}$$
$$\times\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(2p+2j-1)\Gamma(2p+j-1)J_{p+j-1/2}(R_1)J_{p+j-1/2}(R_2). \quad (11)$$

## 9.4. Associated Spherical Functions and Their Properties

**9.4.1. The matrices of the representations $T^{n\ell}$, $T^{n\sigma}$, and $T^{nR}$.** We realize the representation $T^{n\ell}$ of the group $SO(n)$ in the space $\mathfrak{D}^{n-1,\ell}$ of functions on the sphere $S^{n-2}$ (see Sectin 9.2.2). This space is the direct sum of the subspaces $\widetilde{\mathfrak{H}}^{n-1,m}$, $0\le m\le \ell$, in which the shift operators $\varphi(\boldsymbol{\xi}')\to\varphi(g^{-1}\boldsymbol{\xi}')$ define the irreducible unitary representations $T^{n-1,m}$, $0\le m\le\ell$, of $SO(n-1)$. It follows from this that the restriction of the invariant scalar product $(\,\cdot\,,\,\cdot\,)_\ell$ in $\mathfrak{D}^{n-1,\ell}$ to $\widetilde{\mathfrak{H}}^{n-1,m}$ differs from the scalar product in $\mathfrak{L}^2(S^{n-2})$ in a scalar factor only. In other words, if

$$\varphi(\boldsymbol{\xi})=\sum_{k=0}^{\ell}\varphi_k(\boldsymbol{\xi}),\quad \psi(\boldsymbol{\xi})=\sum_{k=0}^{\ell}\psi_k(\boldsymbol{\xi}),\quad \boldsymbol{\xi}\in S^{n-2}, \quad (1)$$

where $\varphi_k,\psi_k\in\widetilde{\mathfrak{H}}^{n-1,k}$, then

$$(\varphi,\psi)_\ell=\sum_{k=0}^{\ell}\lambda_k^2\int_{S^{n-2}}\varphi_k(\boldsymbol{\xi})\overline{\psi_k(\boldsymbol{\xi})}d\boldsymbol{\xi}, \quad (2)$$

where $d\boldsymbol{\xi}$ is the normalized invariant measure on $S^{n-2}$.

The functions $\Xi_{K'}^{n-1,k}$, $K'=(k_1,\dots,k_{n-3})$ form a basis in $\widetilde{\mathfrak{H}}^{n-1,k}$ which is orthogonal with respect to the scalar product in $\mathfrak{L}^2(S^{n-2})$. Therefore, the functions $\lambda_k^{-1}\Xi_{K'}^{n-1,k}(\boldsymbol{\xi})$, $0\le k\le\ell$, form an orthonormal basis in $\mathfrak{D}^{n-1,\ell}$. It follows from here that matrix elements of the representation $T^{n\ell}$ in this basis are given by the formula

$$t_{KM}^{n\ell}(g)=\frac{\lambda_k}{\lambda_m}\int_{S^{n-2}}(T^{n\ell}(g)\Xi_{M'}^{n-1,m}(\boldsymbol{\xi}))\overline{\Xi_{K'}^{n-1,k}(\boldsymbol{\xi})}d\boldsymbol{\xi}, \quad (3)$$

where $K=(k,K')$, $M=(m,M')$. Taking into account formula (3) of Section 9.3.1, we see that the basis elements $\Xi_K^{n\ell}$ are evaluated by induction in $n$. We shall find explicit expressions for them by using expressions for zonal spherical functions from Section 9.3.2.

First of all, we establish the structure of matrices of the representation $T^{n\ell}$. The restriction of $T^{n\ell}$ onto $SO(n-1)$ decomposes into the direct sum of representations $T^{n-1,k}$, $0\le k\le\ell$. The basis vectors $\Xi_{K'}^{n-1,k}$ of the subspace $\widetilde{\mathfrak{H}}^{n-1,k}$

form an orthogonal basis in this subspace, and the normalization factor is common for all vectors. It follows from here that in the basis $\{\Xi_{K'}^{n-1,k}\}$ the matrix $T^{n\ell}(h)$, $h \in SO(n-1)$, is block-diagonal and the main diagonal consists of the matrices of the irreducible representations $T^{n-1,k}$ of $SO(n-1)$:

$$T^{n\ell}(h) = \mathrm{diag}(T^{n-1,0}(h), T^{n-1,1}(h), \ldots, T^{n-1,\ell}(h)), \; h \in SO(n-1). \tag{4}$$

To this decomposition there corresponds the decomposition of $T^{n\ell}(g)$, $g \in SO(n)$, into the blocks $T^{n\ell}_{km}(g)$, where $k$ and $m$ enumerate the representations $T^{n-1,r}$, $0 \le r \le \ell$, of $SO(n-1)$. For $g = g_1 g_2$ we have $T^{n\ell}(g) = T^{n\ell}(g_1)T^{n\ell}(g_2)$ and, hence,

$$T^{n\ell}_{km}(g_1 g_2) = \sum_{r=0}^{\ell} T^{n\ell}_{kr}(g_1) T^{n\ell}_{rm}(g_2). \tag{5}$$

In particular, if $h_1, h_2 \in SO(n-1)$, then

$$T^{n\ell}_{km}(h_1 g h_2) = T^{n-1,k}(h_1) T^{n\ell}_{km}(g) T^{n-1,m}(h_2). \tag{6}$$

By virtue of the Cartan decomposition (see Section 9.1.5) we conclude from (6) that evaluation of matrix elements of the representation $T^{n\ell}$ of $SO(n)$ is reduced to evaluations of matrix elements of representations of $SO(n-1)$ and of the matrix $T^{n\ell}(g_{n-1}(\theta))$.

In order to find the structure of $T^{n\ell}(g_{n-1}(\theta))$ let us restrict $T^{n\ell}$ onto $SO(n-2)$. Then every block $T^{n\ell}_{km}(g)$ decomposes into blocks $T^{n\ell}_{(k,k_1)(m,m_1)}(g)$, where the indices $k_1$ and $m_1$ numerate the irreducible representations $T^{n-2,k_1}$, $0 \le k_1 \le k$, and $T^{n-2,m_1}$, $0 \le m_1 \le m$, of $SO(n-2)$. The equality

$$h g_{n-1}(\theta) = g_{n-1}(\theta) h, \quad h \in SO(n-2),$$

implies that

$$T^{n\ell}(h) T^{n\ell}(g_{n-1}(\theta)) = T^{n\ell}(g_{n-1}(\theta)) T^{n\ell}(h)$$

and, therefore,

$$T^{n-2,k_1}(h) T^{n\ell}_{(k,k_1)(m,m_1)}(g_{n-1}(\theta)) = T^{n\ell}_{(k,k_1)(m,m_1)}(g_{n-1}(\theta)) T^{n-2,m_1}(h).$$

By virtue of irreducibility of $T^{n-2,k_1}$ and $T^{n-2,m_1}$ we conclude from Schur's lemma that the block $T^{n\ell}_{(k,k_1)(m,m_1)}(g_{n-1}(\theta))$ consists of zero elements if $k_1 \ne m_1$ and it has the form $\mu_{k_1} I$ if $k_1 = m_1$. Here $I$ is the unit matrix.

Our reasonings lead to the statement: *If*

$$K = (k, k_1, \ldots, k_{n-3}) \quad M = (m, m_1, \ldots, m_{n-3}),$$
$$K' = (k_1, \ldots, k_{n-3}), \quad M' = (m_1, \ldots, m_{n-3}),$$

*then for $K' \neq M'$ we have $t^{n\ell}_{KM}(g_{n-1}(\theta)) = 0$ and for $K' = M'$ the value of $t^{n\ell}_{KM}(g_{n-1}(\theta))$ depends on $\ell$, $k$, $m$ and $k_1$ only.* Therefore, we shall write $t^{n\ell}_{kmj}(g_{n-1}(\theta))$ instead of $t^{n\ell}_{KM}(g_{n-1}(\theta))$ (we replaced $k_1$ by $j$).

The infinite matrices $T^{n\sigma}(g'_{n-1}(\theta))$ and $T^{nR}(g_r)$ have a similar structure. We denote their matrix elements by $t^{n\sigma}_{kmj}(g'_{n-1}(\theta))$ and $t^{nR}_{kmj}(g_r)$ respectively.

It is clear that $t^{n\sigma}_{kmj}(g'_{n-1}(\theta))$ and $t^{nR}_{kmj}(g_r)$ have a meaning for $k \geq j$, $m \geq j$ and $t^{n\ell}_{kmj}(g_{n-1}(\theta))$ has a sense for $\ell \geq k \geq j$, $\ell \geq m \geq j$. In particular,if $m = 0$, then we have $t^{n\ell}_{k00}(g_{n-1}(\theta))$, $t^{n\sigma}_{k00}(g'_{n-1}(\theta))$ and $t^{nR}_{k00}(g_r)$. For simplicity we denote these matrix elements by $t^{n\ell}_{k0}(g_{n-1}(\theta))$, $t^{n\sigma}_{k0}(g'_{n-1}(\theta))$, $t^{nR}_{k0}(g_r)$. For $k = 0$ they coincide with zonal spherical functions.

**9.4.2. Evaluation of associated spherical functions.** The matrix element $t^{n\ell}_{KO}(g)$, $K = (k, k_1, \ldots, k_{n-3})$, is an associated spherical function of the representation $T^{n\ell}$ of $SO(n)$ (see Section 2.3.8). An element $g \in SO(n)$ is representable as

$$g = g_1(\theta_1)g_2(\theta_2)\ldots g_{n-1}(\theta_{n-1})h,\ \ h \in SO(n-1)$$

(see formula (5) of Section 9.1.5). Hence, from (6) and from other results of the previous section we find

$$\begin{aligned} t^{n\ell}_{KO}(g) &= t^{n\ell}_{KO}(g_1(\theta_1)g_2(\theta_2)\ldots g_{n-1}(\theta_{n-1})) \\ &= t^{n\ell}_{k0}(g_{n-1}(\theta_{n-1}))t^{n-1,k}_{K'O}(g_1(\theta_1)g_2(\theta_2)\ldots g_{n-2}(\theta_{n-2})) \end{aligned} \tag{1}$$

where $K' = (k_1, k_2, \ldots, k_{n-3})$. Applying this equality to the matrix element $t^{n-1,k}_{K'O}(g_1(\theta_1)\ldots g_{n-2}(\theta_{n-2}))$ and continuing this procedure of factorization, we conclude that evaluation of associated spherical functions $t^{n\ell}_{KO}(g)$ is reduced to evaluation of the matrix elements $t^{mk}_{r0}(g_{m-1}(\theta_{m-1}))$. It follows from formula (3) of Section 9.4.1 that

$$t^{n\ell}_{k0}(g_{n-1}(\theta)) = \lambda_k \int\limits_{S^{n-2}} (T^{n\ell}(g_{n-1}(\theta))\Xi^{n-1,0}_O(\xi))\overline{\Xi^{n-1,k}_O(\xi)}d\xi. \tag{2}$$

Since $\Xi^{n-1,0}_O(\xi) \equiv 1$ and

$$\Xi^{n-1,k}_O(\xi) = \left[\frac{(2k+n-3)(n+k-4)!}{(n-3)!k!}\right]^{1/2} t^{n-1,k}_{OO}(\xi)$$

(see Section 9.3.1), then, by using formulas (3) of Section 9.2.2 and (4) of Section 9.3.2, we obtain from (2) that

$$t^{n\ell}_{k0}(g_{n-1}(\theta)) = \lambda_k \frac{\Gamma\left(\frac{n-3}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)} \left[\frac{k!(2k+n-3)(n-3)!}{(n+k-4)!}\right]^{1/2}$$

$$\times \int\limits_0^\pi (\cos\theta - i\sin\theta\cos\varphi)^\ell C_k^{(n-3)/2}(\cos\varphi)\sin^{n-3}\varphi d\varphi. \tag{2'}$$

Applying formula (5) of Section 9.3.6 to this integral, we derive

$$t^{n\ell}_{k0}(g_{n-1}(\theta)) = \lambda_k(-i)^k \frac{2^k\Gamma\left(k+\frac{n-2}{2}\right)\ell!}{\Gamma(\frac{n-2}{2})(\ell+k+n-3)!}$$
$$\times\left[\frac{(2k+n-3)(n-3)!(k+n-4)!}{k!}\right]^{1/2}\sin^k\theta\, C^{m+(n-2)/2}_{\ell-m}(\cos\theta). \qquad (3)$$

In order to evaluate $\lambda_k$ we make use of the equality

$$\int\limits_{SO(n)} \left|t^{n\ell}_{KO}(g)\right|^2 dg = \int\limits_{S^{n-1}} \left|t^{n\ell}_{KO}(\xi)\right|^2 d\xi = (\dim T^{n\ell})^{-1}.$$

Taking into account formula (1) for $t^{n\ell}_{K0}(g)$ and formula (4) of Section 9.1.9 for the measure $d\xi$ on $S^{n-1}$, we obtain

$$\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\int\limits_0^\pi \left|t^{n\ell}_{k0}(g_{n-1}(\theta))\right|^2 \sin^{n-2}\theta d\theta = \frac{\dim T^{n-1,k}}{\dim T^{n\ell}}.$$

Let us substitute here expression (3) for $t^{n\ell}_{k0}(g_{n-1}(\theta))$ and take into consideration the orthogonality relation for Gegenbauer polynomials. We find

$$|\lambda_k|^2 = \frac{(\ell+k+n-3)!(\ell-k)!}{\ell!(\ell+n-3)!}.$$

This equality defines $\lambda_k$ up to a constant with the unit absolute value:

$$\lambda_k = (\exp i\alpha_k)\left[\frac{(\ell+k+n-3)!(\ell-k)!}{\ell!(\ell+n-3)!}\right]^{1/2}.$$

As one can see from formulas (1) and (3) of Section 9.4.1, variation of $\exp i\alpha_k$ does not change the scalar product in $\mathfrak{D}^{n-1,\ell}$ and leads to multiplication of basis elements by constants. Let us choose the constants $\exp i\alpha_k$ in such a way that the matrix elements $t^{n\ell}_{k0}(g_{n-1}(\theta))$ are real. In other words, we set

$$\lambda_k = i^k\left[\frac{(\ell+k+n-3)!(\ell-k)!}{\ell!(\ell+n-3)!}\right]^{1/2}. \qquad (4)$$

Then

$$t^{n\ell}_{k0}(g_{n-1}(\theta)) = \frac{1}{\sqrt{\pi}}2^{k+n-3}\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(k+\frac{n-2}{2}\right)$$
$$\times\left[\frac{\ell!(\ell-k)!\dim T^{n-1,k}}{(\ell+k+n-3)!(\ell+n-3)!}\right]^{1/2}\sin^k\theta\, C^{k+(n-2)/2}_{\ell-k}(\cos\theta), \qquad (5)$$

where dim $T^{n-1,k}$ is given by formula (5) of Sectin 9.2.3. Making use of formula (1) we derive

$$t^{n\ell}_{MO}(g) \equiv t^{n\ell}_{MO}(g_1(\theta_1)g_2(\theta_2)\dots g_{n-1}(\theta_{n-1}))$$
$$= \left[\frac{\ell!(n-2)!}{(\ell+n-3)!(2\ell+n-2)}\right]^{1/2} e^{im_{n-2}\theta_1}$$
$$\times \prod_{j=0}^{n-3}\left[\frac{(n-j-2)!(m_j-m_{j+1})!(n-j+2m_j-2)\Gamma^2\left(\frac{n-j-2}{2}+m_{j+1}\right)}{2^{2-2m_{j+1}}\Gamma(m_j+m_{j+1}+n-j-2)\Gamma^2\left(\frac{n-j}{2}\right)}\right]^{1/2}$$
$$\times C^{m_{j+1}+(n-j-2)/2}_{m_j-m_{j+1}}(\cos\theta_{n-j-1})\sin^{m_{j+1}}\theta_{n-j-1}, \tag{6}$$

where $M = (m_1, m_2, \dots, m_{n-2})$.

Substituting this equality into formula (3) of Section 9.3.1 we obtain the expression for $\Xi^{n\ell}_M(\boldsymbol{\xi})$ in terms of spherical coordinates of the point $\boldsymbol{\xi}\in S^{n-1}$.

Analogously, by using the realizations of $T^{n\sigma}$ and $T^{nR}$ in the space $\mathfrak{L}^2(S^{n-2})$ we evaluate associated spherical functions of representations of the groups $SO_0(n-1,1)$ and $ISO(n-1)$. For $SO_0(n-1,1)$ we have

$$t^{n\sigma}_{m0}(g'_{n-1}(\theta)) = \left(T^{n\sigma}(g'_{n-1}(\theta))\Xi^{n-1,0}_O, \Xi^{n-1,m}_O\right)$$
$$= \int_{S^{n-2}} (T^{n\sigma}(g'_{n-1}(\theta))\Xi^{n-1,0}_O(\boldsymbol{\xi}))\overline{\Xi^{n-1,m}_O(\boldsymbol{\xi})}d\boldsymbol{\xi}. \tag{7}$$

Taking into account formulas (3) of Section 9.2.1 and (6) of Section 9.3.6 we obtain

$$t^{n\sigma}_{m0}(g'_{n-1}(\theta)) = \frac{\Gamma\left(p-\frac{1}{2}\right)}{2\sqrt{\pi}\Gamma(p)}\left[\frac{m!\Gamma(2p)(2m+2p-1)}{\Gamma(2p+m-1)}\right]^{1/2}$$
$$\times \int_0^\pi (\cosh\theta - \cos\varphi\sinh\theta)^\sigma C^{p-1/2}_m(\cos\varphi)\sin^{2p-1}\varphi\, d\varphi$$
$$= \frac{(-1)^m 2^{p-3/2}\Gamma\left(p-\frac{1}{2}\right)\Gamma(\sigma+1)}{\Gamma(2p-1)\Gamma(\sigma-m+1)}\left[\frac{(2m+2p-1)\Gamma(2p)\Gamma(2p+m-1)}{m!}\right]^{1/2}$$
$$\times \sinh^{-p+1/2}\theta\,\mathfrak{P}^{-p-m+1/2}_{\sigma+p-1/2}(\cosh\theta), \tag{8}$$

where $p=(n-2)/2$. If $g = hg'_{n-1}(\theta)h'$, $h, h' \in SO(n-1)$, then the associated spherical functions of the representation $T^{n\sigma}$ of $SO_0(n-1,1)$ are of the form

$$t^{n\sigma}_{MO}(g) = t^{n\sigma}_{MO}(hg'_{n-1}(\theta)) = t^{n\sigma}_{m0}(g'_{n-1}(\theta))t^{n-1,m}_{M'0}(h), \tag{9}$$

where $M = (m, M')$ and $t^{n-1,m}_{M'0}(h)$ is an associated spherical function of the representation $T^{n-1,m}$ of $SO(n-1)$.

For $ISO(n-1)$ we have

$$t^{nR}_{m0}(g_r) = (T^{nR}(g_r)\Xi^{n-1,0}_O, \Xi^{n-1,m}_O) = \int\limits_{S^{n-2}} e^{Rr\xi_{n-1}}\overline{\Xi^{n-1,m}_O(\boldsymbol{\xi})}d\boldsymbol{\xi}$$

$$= \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)}\left[\frac{m!(n-4)!(2m+m-3)}{(m+n-4)!(n-3)}\right]^{1/2}$$

$$\times \int\limits_0^\pi e^{Rr\cos\theta}C^{(n-3)/2}_m(\cos\theta)\sin^{n-3}\theta d\theta = i^m\Gamma\left(p+\frac{1}{2}\right)$$

$$\times \left[\frac{\Gamma(2p+m-1)(2m+2p-1)}{m!\Gamma(2p)}\right]^{1/2}\left(\frac{-iRr}{2}\right)^{-p+1/2} J_{m+p-1/2}(-iRr) \quad (10)$$

(see formula (7) of Section 9.3.6), where $p=(n-2)/2$.

If $g=kg_rk'$, $k,k'\in SO(n-1)$, $M=(m,M')$, then for the associated spherical functions of the representation $T^{nR}$ of $ISO(n-1)$ we have

$$t^{nR}_{MO}(g) = t^{nR}_{m0}(g_r)t^{n-1,m}_{M'O}(k). \quad (11)$$

The functions $t^{n\sigma}_{MO}(g)$ and $t^{nR}_{MO}(g)$ are constant on right cosets with respect to $SO(n-1)$. Hence, one can consider $t^{n\sigma}_{MO}(g)$ as a function on the homogeneous space $SO_0(n-1,1)/SO(n-1)\sim H^{n-1}_+$ and $t^{nR}_{MO}(g)$ as a function on $ISO(n-1)/SO(n-1)\sim\mathbb{R}^{n-1}$. In these cases they will be denoted by $\Xi^{n\sigma}_M(\boldsymbol{\xi})$, $\boldsymbol{\xi}\in H^{n-1}_+$, and $\Xi^{nR}_M(\boldsymbol{\xi})$, $\boldsymbol{\xi}\in\mathbb{R}^{n-1}$, respectively. We have

$$\Xi^{n\sigma}_M(\boldsymbol{\xi}) = t^{n\sigma}_{MO}(g) \quad \text{if} \quad g\mathbf{e}_n = \boldsymbol{\xi},$$
$$\Xi^{nR}_M(\boldsymbol{\xi}) = t^{nR}_{MO}(g) \quad \text{if} \quad g\mathbf{0} = \boldsymbol{\xi}$$

where $\mathbf{0}=(0,\ldots,0)$. The function $r^\sigma\Xi^{n\sigma}_M(\mathbf{x}/r)$, where $r^2=[\mathbf{x},\mathbf{x}]>0$, $x_n>0$, will be denoted by $\Xi^{n\sigma}_M(\mathbf{x})$.

It follows from formulas (8) and (9) that for $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_n)\in H^{n-1}_+$ we have

$$\Xi^{n\sigma}_M(\boldsymbol{\xi}) = A\sinh^{-p+1/2}\theta\mathfrak{P}^{-p-m+1/2}_{\sigma+p-1/2}(\cosh\theta)\Xi^{n-1,m}_{M'}(\boldsymbol{\xi}') \quad (12)$$

where $p=(n-2)/2$, $\cos\theta=\xi_n$, $M=(m,M')$, $\boldsymbol{\xi}'=\sinh^{-1}\theta(\xi_1,\ldots,\xi_{n-1})$ and $A$ coincides with the number factor on the right hand side of (8), multiplied by $(\dim T^{n-1,m})^{-1/2}=(m!(n-3)!/(m+n-4)!(2m+n-3))^{1/2}$.

Analogously, formulas (10) and (11) imply that for $\boldsymbol{\xi}\in\mathbb{R}^{n-1}$ we have

$$\Xi^{nR}_M(\boldsymbol{\xi}) = B\left(\frac{-iRr}{2}\right)^{-p+1/2} J_{m+p-1/2}(-iRr)\Xi^{n-1,m}_{M'}(\boldsymbol{\xi}'), \quad (13)$$

where $p = (n-2)/2$, $r^2 = \xi_1^2 + \cdots + \xi_{n-1}^2$, $\boldsymbol{\xi}' = \frac{1}{r}\boldsymbol{\xi}$, $M = (m, M')$ and $B$ coincides with the number factor on the right hand side of (10), multiplied by $(\dim T^{n-1,m})^{-1/2}$.

If $[\mathbf{x},\mathbf{x}] > 0$, $x_n > 0$, $r^2 = [\mathbf{x},\mathbf{x}]$, then we put

$$\Xi_M^{n\sigma}(\mathbf{x}) = r^\sigma \Xi_M^{n\sigma}\left(\frac{\mathbf{x}}{r}\right).$$

**9.4.3. Addition theorems.** Since the rotations $g_{n-1}(\varphi)$ and $g_{n-2}(\theta)$ act in the three-dimensional space with coordinates $x_{n-2}$, $x_{n-1}$, $x_n$, then the relation

$$g_{n-1}(-\theta)g_{n-2}(\psi)g_{n-1}(\varphi) = g_{n-2}(\alpha)g_{n-1}(\beta)g_{n-2}(\gamma) \tag{1}$$

holds, where

$$\left.\begin{aligned} \cos\beta &= \cos\theta\cos\varphi + \sin\theta\sin\varphi\cos\psi, \\ \cos\beta\sin\gamma &= \sin\theta\sin\psi, \\ \sin\beta\sin\alpha &= \sin\psi\sin\varphi. \end{aligned}\right\} \tag{2}$$

This relation was proved for $SO(3)$ in Section 6.1.1. It follows from here that

$$\begin{aligned} T^{n\ell}(g_{n-1}(-\theta))&T^{n\ell}(g_{n-2}(\psi))T^{n\ell}(g_{n-1}(\varphi)) \\ &= T^{n\ell}(g_{n-2}(\alpha))T^{n\ell}(g_{n-1}(\beta))T^{n\ell}(g_{n-2}(\gamma)) \end{aligned}$$

and, therefore,

$$t_{00}^{n\ell}(g_{n-1}(\beta)) = \sum_{m=0}^{\ell} t_{0m}^{n\ell}(g_{n-1}(-\theta))t_{00}^{n-1,m}(g_{n-2}(\psi))t_{m0}^{n\ell}(g_{n-1}(\varphi)), \tag{3}$$

where

$$t_{0m}^{n\ell}(g_{n-1}(-\theta)) = t_{m0}^{n\ell}(g_{n-1}(\theta)) \tag{4}$$

since $t_{m0}^{n\ell}(g_{n-1}(\theta))$ is real. Substituting the expressions for matrix elements in terms of Gegenbauer polynomials into (3), we derive the addition theorem for these polynomials:

$$\begin{aligned} C_\ell^p(\cos\theta\cos\varphi &+ \sin\theta\sin\varphi\cos\psi) = \\ &= \frac{\Gamma(2p-1)}{\Gamma^2(p)} \sum_{m=0}^{\ell} \frac{2^{2m}\Gamma^2(p+m)(\ell-m)!(2p+2m-1)}{\Gamma(2p+\ell+m)} \\ &\times (\sin\varphi\sin\theta)^m C_{\ell-m}^{p+m}(\cos\varphi)C_{\ell-m}^{p+m}(\cos\theta)C_m^{p-1/2}(\cos\psi). \end{aligned} \tag{5}$$

Let us give special cases of (5). If we put $\psi = 0$ and take into account that

$$C_m^{p-1/2}(1) = \frac{\Gamma(2p+m-1)}{m!\Gamma(2p-1)},$$

then we obtain the equality

$$C_\ell^p(\cos(\theta-\varphi)) = \frac{1}{\Gamma^2(p)}\sum_{m=0}^{\ell}\frac{2^{2m}\Gamma^2(p+m)(\ell-m)!\Gamma(2p+m-1)(2m+2p-1)}{m!\Gamma(\ell+m+2p)}$$
$$\times(\sin\varphi\sin\theta)^m C_{\ell-m}^{p+m}(\cos\varphi)C_{\ell-m}^{p+m}(\cos\theta). \tag{6}$$

Similarly, by setting $\psi=\pi/2$ and keeping in mind that $C_{2m+1}^p(0)=0$ and

$$C_{2m}^p(0)=\frac{(-1)^m\Gamma(p+m)}{m!\Gamma(p)}$$

we receive

$$C_\ell^p(\cos\theta\cos\varphi)$$
$$=\frac{\Gamma(2p-1)}{\Gamma^2(p)}\sum_{m=0}^{[\ell/2]}\frac{(-1)^m 2^{4m}\Gamma^2(p+2m)(\ell-2m)!\Gamma\left(p+m-\frac{1}{2}\right)(4m+2p+1)}{m!\Gamma(\ell+2m+2p)\Gamma\left(p-\frac{1}{2}\right)}$$
$$\times(\sin\varphi\sin\theta)^{2m}C_{\ell-2m}^{p+2m}(\cos\varphi)C_{\ell-2m}^{p+2m}(\cos\theta). \tag{7}$$

If we set $\varphi=\theta=\pi/2$ and $\ell-m=2k$ into (5), then we derive the expression for $C_\ell^p(\cos\psi)$ in terms of $C_k^{p-1/2}(\cos\psi)$:

$$C_\ell^p(\cos\psi)=\frac{\Gamma(2p-1)}{\Gamma^2(p)}\sum_{k=0}^{[\ell/2]}\frac{(2k)!(2\ell-4k+2p-1)}{\Gamma(2\ell-2k+2p)}$$
$$\times\left[\frac{2^{\ell-2k}\Gamma(p+\ell-k)}{k!}\right]^2 C_{\ell-2k}^{p-1/2}(\cos\psi). \tag{8}$$

In the same way one obtains addition theorems for Legendre and Bessel functions. Omitting their derivations we give the result. Let $p=\frac{n-2}{2}$ and

$$\cosh\theta=\cosh\theta_1\cosh\theta_2+\sinh\theta_1\sinh\theta_2\cos\varphi. \tag{9}$$

Then

$$\sinh^{-p+1/2}\theta\,\mathfrak{P}_{p+\sigma-1/2}^{-p+1/2}(\cosh\theta)=2^{p-3/2}\Gamma\left(p-\frac{1}{2}\right)\Gamma(\sigma+1)\Gamma(-\sigma-2p+1)$$
$$\times\sum_{k=0}^{\infty}(-1)^k\frac{(2p+2k-1)}{\Gamma(\sigma-k+1)}\Gamma(-\sigma-k-2p+1)(\sinh\theta_1\sinh\theta_2)^{-p+1/2}$$
$$\times\mathfrak{P}_{\sigma+p-1/2}^{-k-p+1/2}(\cosh\theta_1)\mathfrak{P}_{\sigma+p-1/2}^{-k-p+1/2}(\cosh\theta_2)C_k^{p-1/2}(\cos\varphi). \tag{10}$$

Further, let $p = (n-2)/2$ and

$$r = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos\varphi}. \tag{11}$$

Then

$$r^{-p+1/2}J_{p-1/2}(r) = 2^{p-1/2}\Gamma\left(p-\frac{1}{2}\right)\sum_{k=0}^{\infty}(-1)^k\left(k+p-\frac{1}{2}\right) \times (r_1r_2)^{-p+1/2}J_{k+p-1/2}(r_1)J_{k+p-1/2}(r_2)C_k^{p-1/2}(\cos\varphi). \tag{12}$$

We suggest to the reader to write down special cases of this formula, corresponding to $\varphi = 0, \pi/2, \pi$.

For $r_1 = r_2$ formula (12) takes the form

$$\sum_{k=0}^{\infty}(-1)^k\left(k+\frac{s}{2}\right)J_{k+s/2}^2(r)C_k^{s/2}(\cos\varphi) = \frac{r^{s/2}J_{s/2}\left(2r\cos\frac{\varphi}{2}\right)}{2^s\Gamma(\frac{s}{2})\left(\cos\frac{\varphi}{2}\right)^{s/2}}. \tag{12'}$$

**9.4.4. Generalizations of the addition theorems.** The formulas of the preceeding section can be generalized. For example, formula (5) can be extended to non-integral $k$ to give the addition theorem for Gegenbauer functions (see Section 7.4.6):

$$\begin{aligned} &C_\lambda^\alpha(x_1x_2 - z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2}) \\ &= \frac{\Gamma(2\alpha-1)}{\Gamma^2(\alpha)}\sum_{m=0}^{\infty}(-1)^m\frac{2^{2m}\Gamma(\lambda-m+1)\Gamma^2(\alpha+m)(2m+2\alpha-1)}{\Gamma(2\alpha+\lambda+m)} \\ &\times (x_1^2-1)^{m/2}(x_2^2-1)^{m/2}C_{\lambda-m}^{\alpha+m}(x_1)C_{\lambda-m}^{\alpha+m}(x_2)C_m^{\alpha-1/2}(z). \end{aligned} \tag{1}$$

If $\lambda = \ell \in \mathbf{Z}_+ \cup \{0\}$, then the series on the right terminates and we obtain formula (5) of Section 9.4.3. Therefore, it follows from Carleson's theorem[5] that formula (1) with arbitrary $\lambda$ is valid for values of $x_1$, $x_2$, $x$ such that the series converges.

One can show that

$$\begin{aligned} C_\lambda^\alpha(z) \sim \lambda^{\alpha-1}2^{-\alpha}[\Gamma(\alpha)]^{-1}(z^2-1)^{-\alpha/2}\{[z+(z^2-1)^{1/2}]^{\lambda+\alpha} \\ + \exp(\pm i\pi\alpha)[z+(z^2-1)^{1/2}]^{-\lambda-\alpha}\} \end{aligned}$$

if $|\lambda| \to \infty$, $\operatorname{Re}\lambda \geq 0$, $\operatorname{Im} z \gtrless 0$, $|\arg(z\pm 1)| < \pi$ and

$$C_{\lambda-n}^{\alpha+n}(z)\frac{\sin\pi(n-\lambda)}{(n\pi)^{1/2}}2^{-\alpha-n+1}(z+1)^{-\alpha-n+1/2}$$

[5] We mean the following theorem. If for $\operatorname{Re} w \geq 0$ a function $f(w)$ is regular and $f(w) = O(e^{k|w|})$, $k < \pi$, then the condition $f(w) = 0$ for $w = 0, 1, 2, \ldots$ implies that $f(w) \equiv 0$.

if $|\arg(z\pm1)|<\pi$, $\operatorname{Im} z \gtrless 0$ $\operatorname{Re} n\to\infty$. It follows from here that the convergence domain of series (1) is given by the condition

$$|z+(z^2-1)^{1/2}|<|(x_1+1)(x_1+1)/(x_1-1)(x_1-1)|^{1/2}. \tag{2}$$

This condition means that $z$ is inside of the ellipse in the complex $z$-plane with foci at $\pm1$, passing through the point $z=(x_1x_2+1)/[(x_1^2-1)(x_2^2-1)]^{1/2}$. If we set $x_i=\cosh\beta_i$, $|\operatorname{Im}\beta_i|<\pi$, $z=\cosh\varphi$, $|\operatorname{Im}\varphi|<\pi$, then (2) is replaced by

$$|\tanh\frac{\beta_1}{2}\tanh\frac{\beta_2}{2}e^{-\varphi}|<1. \tag{3}$$

The function

$$\begin{aligned}D_\lambda^\alpha(z)&=(\exp i\pi\alpha)\frac{\Gamma(\lambda+2\alpha)}{\Gamma(\alpha)\Gamma(\lambda+\alpha+1)}(2z)^{-\lambda-2\alpha}\\&\quad\times F\left(\alpha+\frac{\lambda}{2},\frac{1}{2}(2\alpha+\lambda+1);\lambda+\alpha+1;z^{-2}\right)\\&=\frac{1}{\sqrt{\pi}}\exp\left(2\pi i\left(\alpha-\frac{1}{4}\right)\right)2^{-\alpha+1/2}\frac{\Gamma(\lambda+2\alpha)}{\Gamma(\alpha)\Gamma(\lambda+1)}\\&\quad\times(z^2-1)^{-\frac{\alpha}{2}+\frac{1}{4}}\mathfrak{Q}_{\lambda+\alpha-1/2}^{-\alpha+1/2}(z)\end{aligned} \tag{4}$$

is called the *Gegenbauer function of the second kind.* With the help of relations for hypergeometric functions we derive that

$$\begin{aligned}&\pm2i\exp(-i\pi\alpha)\exp(\mp i\pi\alpha)(\sin\pi\lambda)D_\lambda^\alpha(z)\\&\qquad=\exp(\pm i\pi\lambda)C_\lambda^\alpha(z)-C_\lambda^\alpha(\exp(\pm i\pi)z).\end{aligned} \tag{5}$$

Making use of this equality we obtain from (1) that

$$\begin{aligned}&D_\lambda^\alpha(x_1x_2-z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2})\\&=\frac{\Gamma(2\alpha-1)}{\Gamma^2(\alpha)}\sum_{m=0}^{\infty}(-1)^m\frac{2^{2m}\Gamma(\lambda-m+1)\Gamma^2(\alpha+m)(2m+2\alpha-1)}{\Gamma(2\alpha+\lambda+m)}\\&\times(x_1^2-1)^{m/2}(x_2^2-1)^{m/2}D_{\lambda-m}^{\alpha+m}(x_1)C_{\lambda-m}^{\alpha+m}(x_2)C_m^{\alpha-1/2}(z),\end{aligned} \tag{6}$$

where

$$|z-(z^2-1)^{1/2}|<|(x_1\mp1)(x_2+1)/(x_1\pm1)(x_2-1)|^{1/2},\ \operatorname{Re}x_1>0. \tag{7}$$

If $x_i=\cosh\beta_i$, $z=\cosh\varphi$, $|\operatorname{Im}\beta_i|<\pi$, $|\operatorname{Im}\varphi|<\pi$, then (7) is rewritten as

$$|\tanh^{-1}\frac{\beta_1}{2}\tanh\frac{\beta_2}{2}e^{-\varphi}|<1,\ |\operatorname{Im}\beta_1|<\frac{\pi}{2},\ |\operatorname{Im}\beta_2|<\pi. \tag{8}$$

One can prove the formula

$$D_\lambda^\alpha(-x_1x_2+z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2})$$
$$=2\pi i\frac{\Gamma(2\alpha-1)}{\Gamma^2(\alpha)}\sum_{\ell=0}^{\infty}4^{-\lambda-2\alpha-\ell}\frac{\ell!\Gamma(2\lambda+2\alpha+\ell+1)(2\lambda+2\alpha+2\ell+1)}{\Gamma^2(\lambda+\alpha+\ell+1)}$$
$$\times[(x_1^2-1)(x_2^2-1)]^{-\frac{1}{2}(\lambda+2\alpha+\ell)}C_\ell^{-\lambda-\alpha-\ell}(x_1)C_\ell^{-\lambda-\alpha-\ell}(x_2)D_{\ell+\lambda}^{\alpha-1/2}(z), \quad (9)$$

where $\lambda$ and $\alpha$ are arbitrary and

$$|z+(z^2-1)^{1/2}|>|(x_1\pm1)(x_2\pm1)/(x_1\mp1)(x_2\mp1)|^{1/2},$$
$$|\arg(x_1\pm1)|<\pi,\ |\arg(x_2\pm1)|<\pi,$$
$$|\arg|z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2}-x_1x_2\pm1|<\pi$$

(all combinations of signs in $(x_1\pm1)$ and $(x_2\pm1)$ must be taken into consideration), and the formula

$$(z^2-1)^{\alpha-1/2}D_\lambda^\alpha(z)=-i2^{2\lambda+1}\left[(x_1^2-1)(x_2^2-1)\right]^{\frac{\lambda+1}{2}}(\xi^2-1)^\alpha$$
$$\times\frac{\Gamma(2\alpha)\Gamma(\lambda+2\alpha)\Gamma^2\left(\lambda+\alpha+\frac{1}{2}\right)}{\Gamma(\lambda+1)\Gamma^2(\alpha)}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell\ell!\Gamma(\lambda+\ell+1)(2\lambda+2\alpha+2\ell+1)}{\Gamma(\lambda+2\alpha+\ell+1)\Gamma(2\lambda+2\alpha+\ell+1)}$$
$$\times C_\ell^{\lambda+\alpha+1/2}(x_1)C_\ell^{\lambda+\alpha+1/2}(x_2)D_{\lambda+\ell}^{\alpha+1/2}(\xi), \quad (10)$$

where $\xi=z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2}-x_1x_2$ and the variables satisfy the conditions

$$|\xi+(\xi^2-1)^{1/2}|>|(x_1+(x_1^2-1)^{1/2})(x_2+(x_2^2-1)^{1/2})|,$$
$$|\arg(\xi\pm1)|\le\pi,\ |\arg(x_1\pm1)|\le\pi,\ |\arg(x_2\pm1)|\le\pi.$$

Relations (1) lead to the addition formula for the Legendre functions $\mathfrak{P}_\nu^\mu(z)$:

$$(\xi^2-1)^{\mu/2}\mathfrak{P}_\nu^\mu(\xi)=2^{-\mu}\frac{\Gamma(-\mu)\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}[(x_1^2-1)(x_2^2-1)]^{\mu/2}$$
$$\times\sum_{m=0}^{\infty}(-1)^m(m-\mu)\frac{\Gamma(\nu-\mu+m+1)}{\Gamma(\nu+\mu-m+1)}\mathfrak{P}_\nu^{\mu-m}(x_1)\mathfrak{P}_\nu^{\mu-m}(x_2)C_m^\mu(z), \quad (11)$$

where

$$\xi=x_1x_2-z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2},$$
$$|z+(z^2-1)^{1/2}|<|(x_1+1)(x_2+1)/(x_1-1)(x_2-1)|^{1/2}.$$

Relation (6) leads to the addition formula

$$(\xi^2-1)^{\mu/2}\mathfrak{Q}_\nu^\mu(\xi) = 2^{-\mu}\frac{\Gamma(-\mu)\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}[(x_1^2-1)(x_2^2-1)]^{\mu/2}$$
$$\times\sum_{m=0}^{\infty}(-1)^m\frac{(m-\mu)\Gamma(\nu-\mu+m+1)}{\Gamma(\nu+\mu-m+1)}\mathfrak{Q}_\nu^{\mu-m}(x_1)\mathfrak{P}_\nu^{\mu-m}(x_2)C_m^\mu(z), \qquad (12)$$

where

$$\xi = x_1x_2 - z(x_1^2-1)^{1/2}(x_2^2-1)^{1/2},$$
$$|z+(z^2-1)^{1/2}| < |(x_1\mp 1)(x_2+1)/(x_1\pm 1)(x_2-1)|^{1/2}.$$

**9.4.5. Product formulas.** Let us multiply both sides of relation (5) of Section 9.4.3 by $\sin^{2p-1}\psi C_k^{p-1/2}(\cos\psi)$ and integrate with respect to $\psi$ from 0 to $\pi$. By virtue of the orthogonality relation for Gegenbauer polynomials we obtain the equality

$$\int_0^\pi C_\ell^p(\cos\theta\cos\varphi+\sin\theta\sin\varphi\cos\psi)C_k^{p-1/2}(\cos\psi)\sin^{2p-1}\psi d\psi$$
$$= \frac{2^{2p+2k-1}\Gamma^2(p+k)\Gamma(2p+k-1)(\ell-k)!}{k!\Gamma(2p-1)\Gamma(\ell+k+2p)}(\sin\theta\sin\varphi)^k C_{\ell-k}^{p+k}(\cos\theta)C_{\ell-k}^{p+k}(\cos\varphi), \qquad (1)$$

called the *product formula* for Gegenbauer polynomials.

Let us make in (1) the substitution

$$\cos\theta\cos\varphi+\sin\theta\sin\varphi\cos\psi = \cos\gamma. \qquad (2)$$

Since

$$\sin\psi = \frac{[\cos(\theta-\varphi)-\cos\gamma]^{1/2}[\cos\gamma-\cos(\theta+\varphi)]^{1/2}}{\sin\theta\sin\varphi},$$
$$d(\cos\psi) = \frac{d(\cos\gamma)}{\sin\theta\sin\varphi}$$

and since the variation of $\psi$ from 0 to $\pi$ leads to the variation of $\gamma$ from $|\theta-\varphi|$ to $\theta+\varphi$, we have

$$\frac{(\ell-k)!}{\Gamma(\ell+k+2p)}C_{\ell-k}^{p+k}(\cos\varphi)C_{\ell-k}^{p+k}(\cos\theta)(\sin\varphi\sin\theta)^k$$
$$= \int_0^\pi C_\ell^p(\cos\gamma)K_k(\varphi,\theta,\gamma)\sin^{2p}\gamma d\gamma, \qquad (3)$$

where $K_k(\varphi,\theta,\gamma)=0$ if $\gamma\bar{\in}[|\theta-\varphi|,\theta+\varphi]$ and

$$K_k(\varphi,\theta,\gamma)=\frac{k!\Gamma(2p-1)}{2^{2k+2p-1}\Gamma^2(p+k)\Gamma(2p+k-1)}C_k^{p-1/2}\left(\frac{\cos\gamma-\cos\theta\cos\varphi}{\sin\theta\sin\varphi}\right)$$
$$\times\frac{\left(\sin\frac{\theta-\varphi+\gamma}{2}\sin\frac{\gamma-\theta+\varphi}{2}\sin\frac{\theta+\varphi-\gamma}{2}\sin\frac{\theta+\varphi+\gamma}{2}\right)^{p-1}}{(\sin\gamma\sin\varphi\sin\theta)^{2p-1}} \tag{4}$$

if $|\theta-\varphi|\le\gamma\le\theta+\varphi$. In particular, for $k=0$ we have

$$C_\ell^p(\cos\varphi)C_\ell^p(\cos\theta)=\int_0^\pi C_\ell^p(\cos\gamma)K(\varphi,\theta,\gamma)\sin^{2p}\gamma d\gamma, \tag{5}$$

where $K(\varphi,\theta,\gamma)=0$ if $\gamma\bar{\in}[|\theta-\varphi|,\theta+\varphi]$ and

$$K(\varphi,\theta,\psi)=$$
$$=\frac{\Gamma(2p+\ell)}{2^{2p-1}\ell!\Gamma^2(p)}\frac{\left(\sin\frac{\theta-\varphi+\gamma}{2}\sin\frac{\gamma-\theta+\varphi}{2}\sin\frac{\theta+\varphi-\gamma}{2}\sin\frac{\theta+\varphi+\gamma}{2}\right)^{p-1}}{(\sin\gamma\sin\varphi\sin\theta)^{2p-1}} \tag{6}$$

if $|\theta-\varphi|\le\gamma<\theta+\varphi$. The kernel $K(\varphi,\theta,\gamma)$ is symmetric with respect to permutations of $\varphi$, $\theta$, $\gamma$.

Making use of the orthogonality relation for Gegenbauer polynomials we derive from (3) that

$$\sum_{\ell=k}^{\infty}\frac{\ell!(\ell+p)(\ell-k)!}{\Gamma(2p+\ell)\Gamma(\ell+k+2p)}C_\ell^p(\cos\gamma)C_{\ell-k}^{p+k}(\cos\varphi)C_{\ell-k}^{p+k}(\cos\theta)$$
$$=\frac{\pi}{2^{2p-1}\Gamma^2(p)}(\sin\varphi\sin\theta)^{-k}K_k(\varphi,\theta,\gamma). \tag{7}$$

In particular, for $k=0$ we have

$$\sum_{\ell=0}^{\infty}\frac{\ell!(\ell+p)}{\Gamma(2p+\ell)}C_\ell^p(\cos\gamma)C_\ell^p(\cos\varphi)C_\ell^p(\cos\theta)=\frac{\pi}{2^{2p-1}\Gamma^2(p)}K(\varphi,\theta,\gamma). \tag{8}$$

The addition formula (10) of Section 9.4.3 implies the product formula for Legendre functions:

$$\int_0^\pi\sinh^{-p+1/2}\theta\mathfrak{P}_{\sigma+p-1/2}^{-p+1/2}(\cosh\theta)C_m^{p-1/2}(\cos\varphi)\sin^{2p-1}\varphi d\varphi$$
$$=\frac{(-1)^m\pi 2^{-p+3/2}\Gamma(\sigma+1)}{m!\Gamma\left(p-\frac{1}{2}\right)\Gamma(\sigma-m+1)}$$
$$\times\frac{\Gamma(-\sigma-2p+1)\Gamma(m+2p-1)}{\Gamma(-\sigma-2-p-m+1)}(\sinh\theta_1\sinh\theta_2)^{-p+1/2}$$
$$\times\mathfrak{P}_{\sigma+p-1/2}^{-m-p+1/2}(\cosh\theta_1)\mathfrak{P}_{\sigma+p-1/2}^{-m-p+1/2}(\cosh\theta_2), \tag{9}$$

where $p=\frac{n-2}{2}$, $\cosh\theta=\cosh\theta_1\cosh\theta_2+\sinh\theta_1\sinh\theta_2\cos\varphi$. For $m=0$ we have

$$\int_0^\pi \sinh^{-p+1/2}\theta\,\mathfrak{P}^{-p+1/2}_{\sigma+p-1/2}(\cosh\theta)\sin^{2p-1}\varphi d\varphi$$
$$=\frac{2^{-p+3/2}\pi\Gamma(2p-1)}{\Gamma\left(p-\frac{1}{2}\right)}(\sinh\theta_1\sinh\theta_2)^{-p+1/2}\mathfrak{P}^{-p+1/2}_{\sigma+p-1/2}(\cosh\theta_1)\mathfrak{P}^{-p+1/2}_{\sigma+p-1/2}(\cosh\theta_2). \quad (10)$$

By replacing the variable $\varphi$ by $\theta$, where

$$\cosh\theta=\cosh\theta_1\cosh\theta_2+\sinh\theta_1\sinh\theta_2\cos\varphi,$$

we rewrite (9) in the form

$$\int_{|\theta_1-\theta_2|}^{\theta_1+\theta_2} K_m(\theta_1,\theta_2,\theta)\mathfrak{P}^{-p+1/2}_{\sigma+p-1/2}(\cosh\theta)\sinh\theta d\theta$$
$$=\mathfrak{P}^{-p-m+1/2}_{\sigma+p-1/2}(\cosh\theta_1)\mathfrak{P}^{-p-m+1/2}_{\sigma+p-1/2}(\cosh\theta_2), \quad (11)$$

where

$$K_m(\theta_1,\theta_2,\theta)=\frac{(-1)^m2^{p-3/2}m!\Gamma(\sigma-m+1)\Gamma(-\sigma-m-2p+1)\Gamma\left(p-\frac{1}{2}\right)m!}{\pi\Gamma(\sigma+1)\Gamma(-\sigma-2p+1)\Gamma(m+2p-1)}$$
$$\times C_m^{p-1/2}\left(\frac{\cosh\theta-\cosh\theta_1\cosh\theta_2}{\sinh\theta_1\sinh\theta_2}\right)$$
$$\times\frac{\{[\cosh(\theta_1+\theta_2)-\cosh\theta][\cosh\theta-\cosh(\theta_1-\theta_2)]\}^{p-1}}{(\sinh\theta_1\sinh\theta_2\sinh\theta)^{p-1/2}}. \quad (12)$$

For $m=0$ the kernel $K_m(\theta_1,\theta_2,\theta)$:

$$K_0(\theta_1,\theta_2,\theta)=$$
$$=\frac{2^{p-3/2}\Gamma(-\sigma-2p+1)\Gamma\left(p-\frac{1}{2}\right)m!}{\pi\Gamma(-\sigma+1)\Gamma(2p-1)}(\sinh\theta_1\sinh\theta_2\sinh\theta)^{-p+1/2}$$
$$\times\left(\sinh\frac{\theta_1+\theta_2-\theta}{2}\sinh\frac{\theta_1-\theta_2+\theta}{2}\sinh\frac{\theta-\theta_1+\theta_2}{2}\sinh\frac{\theta_1+\theta_2+\theta_3}{2}\right)^{p-1} \quad (13)$$

is symmetric with respect to permutations of $\theta_1$, $\theta_2$, $\theta$.

The addition theorem (12) of Section 9.4.3 for Bessel functions leads to the product formula

$$\int_0^\pi r^{-p+1/2} J_{p-1/2}(r) C_m^{p-1/2}(\cos\varphi)\sin^{2p-1}\varphi d\varphi = \frac{(-1)^m \pi\Gamma(m+2p-1)}{2^{p-3/2} m!\Gamma\left(p-\frac{1}{2}\right)} \times (r_1 r_2)^{-p+1/2} J_{m+p-1/2}(r_1) J_{m+p-1/2}(r_2), \tag{14}$$

where $r = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos\varphi}$ and $p = (n-2)/2$. In particular, if $m = 0$, then we have

$$\int_0^\pi r^{-p+1/2} J_{p-1/2}(r)\sin^{2p-1}\varphi d\varphi = \frac{\pi\Gamma(2p-1)}{2^{p-3/2}\Gamma\left(p-\frac{1}{2}\right)} (r_1 r_2)^{-p+1/2} J_{p-1/2}(r_1) J_{p-1/2}(r_2). \tag{15}$$

By replacing the variable $\varphi$ by $r = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2\cos\varphi}$ we derive from (14) that

$$\int_{|r_1-r_2|}^{r_1+r_2} K_m(r_1, r_2, r) J_{p-1/2}(r) r dr = J_{m+p-1/2}(r_1) J_{m+p-1/2}(r_2), \tag{16}$$

where

$$K_m(r_1, r_2, r) = \frac{m!\Gamma\left(p-\frac{1}{2}\right)}{\pi\Gamma(m+2p-1)} (2r_1 r_2 r)^{-p+1/2} C_m^{p-1/2}\left(\frac{r_1^2 + r_2^2 - r^2}{2r_1 r_2}\right) \times [(r_1 + r_2 + r)(r_1 + r_2 - r)(r_1 - r_2 + r)(r - r_1 + r_2)]^{p-1}. \tag{16'}$$

The kernel $K_m(r_1, r_2, r)$ for $m = 0$ is symmetric with respect to permutations of $r_1$, $r_2$, $r$.

Let us replace in (14) $r_1$, $r_2$, $r$ by $Rr_1$, $Rr_2$, $Rr$, respectively, and apply the Fourier-Bessel transform (see Section 4.4.2). For $|r_1 - r_2| < r < r_1 + r_2$, we obtain

$$\int_0^\infty J_{p-1/2}(Rr) J_{m+p-1/2}(Rr_1) J_{m+p-1/2}(Rr_2) R^{-p+3/2} dR = \frac{m!\Gamma\left(p-\frac{1}{2}\right)}{4\pi\Gamma(m+2p-1)S}\left(\frac{8S^2}{r_1 r_2 r}\right)^{p-1/2} C_m^{p-1/2}\left(\frac{r_1^2+r_2^2-r^2}{2r_1 r_2}\right), \tag{17}$$

where $S$ is the area of the triangle with the sides of lengths $r_1, r_2, r$. If $r > r_1 + r_2$ or $r < |r_1 - r_2|$, then the integral in (17) is equal to zero.

We consider the geometric meaning of formula (1) for $k = 0$. It can be rewritten as

$$\frac{\Gamma\left(p+\frac{1}{2}\right)}{\pi\Gamma(p)} \int_0^\pi t_{00}^{n\ell}(\cos\gamma)\sin^{2p-1}\psi d\psi = t_{00}^{n\ell}(\cos\varphi)t_{00}^{n\ell}(\cos\theta), \tag{18}$$

where $\cos\gamma = \cos\theta\cos\varphi + \sin\theta\sin\varphi\cos\psi$. The expression on the left hand side of (18) is the average value of the function $t_{00}^{n\ell}(\cos\gamma)$ on the sphere $S^{n-1}(P,\varphi)$ with spherical radius $\varphi$ and with the center at the point $P$, which is at spherical distance $\theta$ from the pole $N$ of the sphere $S^{n-1}$. Formula (18) means that the average value is equal to $t_{00}^{n\ell}(\cos\theta)t_{00}^{n\ell}(\cos\varphi)$.

Formulas (10) and (15) have a similar meaning, but one has to replace $S^{n-1}$ by $H_+^{n-1}$ and $\mathbb{R}^{n-1}$, respectively.

**9.4.6. Generalized product theorems.** From formula (6) of Section 9.4.4 we derive that

$$(z_1^2-1)^{m/2}(z_2^2-1)^{m/2}D_{\lambda-m}^{\alpha+m}(z_1)C_{\lambda-m}^{\alpha+m}(z_2) = C(\alpha,\lambda,m)$$

$$\times \int_{-1}^{1} D_\lambda^\alpha(z_1z_2 + x(z_1^2-1)^{1/2}(z_2^2-1)^{1/2})C_m^{\alpha-1/2}(x)(1-x^2)^{\alpha-1}dx, \tag{1}$$

where

$$C(\alpha,\lambda,m) = 2^{-2\alpha-2m+1}\frac{\Gamma(2\alpha-1)\Gamma(m+1)\Gamma(\lambda+2\alpha+m)}{\Gamma^2(\alpha+m)\Gamma(m+2\alpha-1)\Gamma(\lambda-m+1)}. \tag{2}$$

From the same formula one can obtain

$$(z_1^2-1)^{m/2}(z_2^2-1)^{m/2}D_{\lambda-m}^{\alpha+m}(z_1)D_{\lambda-m}^{\alpha+m}(z_2) = e^{i\pi(\alpha+2m)}C(\alpha,\lambda,m)$$

$$\times \int_{-1}^{1} D_\lambda^\alpha\left(z_1z_2 + x(z_1^2-1)^{1/2}(z_2^2-1)^{1/2}\right)C_m^{\alpha-1/2}(x)(1-x^2)^{\alpha-1}dx, \tag{3}$$

where $\operatorname{Re}\alpha > 0$, $\operatorname{Re}(\lambda-m+1) > 0$, $\operatorname{Re}\left(m+\alpha-\frac{1}{2}\right) \geq 0$,

$$|\arg(z_1 \pm 1)| < \pi,\ |\arg(z_2 \pm 1)| < \pi,\ |\arg(z_1^2-1)^{1/2}(z_2^2-1)^{1/2}| < \pi.$$

Let $z_2$ tends to the infinity. After changing notations, we have

$$(z^2-1)^{m/2}D_{\lambda-m}^{\alpha+m}(z) =$$

$$= e^{i\pi(\alpha+m)}2^{-m-2\alpha+1}\frac{\Gamma(2\alpha-1)\Gamma(\lambda+2\alpha)\Gamma(m+1)}{\Gamma(\alpha)\Gamma(m+2\alpha-1)\Gamma(\lambda-m+1)\Gamma(\alpha+m)}$$

$$\times \int_1^\infty [z + x(z^2-1)]^{-\lambda-2\alpha}C_m^{\alpha-1/2}(x)(x^2-1)^{\alpha-1}dx. \tag{4}$$

Replacing in (3) $\lambda$ by $\nu$ and setting $m = i\lambda - \alpha + \frac{1}{2}$, $\lambda \in \mathbf{R}$, $\alpha \in \mathbf{R}$, $\alpha > 0$ we find

$$\int_0^\infty D_\nu^\alpha(Z) C_{i\lambda-\alpha+1/2}^{\alpha-1/2}(\cosh t)\left[\sin\pi\left(i\lambda - \alpha + \frac{1}{2}\right)\right]^{-1} \sinh^{2\alpha-1} t dt$$
$$= \frac{C\left(\alpha, \nu, i\lambda - \alpha + \frac{1}{2}\right)}{\sin\pi\left(i\lambda - \alpha + \frac{1}{2}\right)}[(z_1^2 - 1)(z_2^2 - 1)]^{(i\lambda-\alpha+1/2)/2}$$
$$\times D_{\nu-i\lambda+\alpha-1/2}^{i\lambda+1/2}(z_1) D_{\nu-i\lambda+\alpha-1/2}^{i\lambda+1/2}(z_2), \tag{5}$$

where $Z = z_1 z_2 + (z_1^2 - 1)^{1/2}(z_2^2 - 1)^{1/2} \cosh t$.

Since the Gegenbauer function $C_\lambda^\alpha(z)$ is expressed in terms of the associated Legendre function

$$C_\lambda^\alpha(z) = \frac{\sqrt{\pi}\Gamma(\lambda + 2\alpha)}{2^{\alpha-1/2}\Gamma(\alpha)\Gamma(\lambda + 1)}(z^2 - 1)^{-(2\alpha-1)/4}\mathfrak{P}_{\lambda+\alpha-1/2}^{-\alpha+1/2}(z),$$

then the generalized Fock-Mehler transform (6) of Sectin 10.2.8 can be represented in the form

$$\hat{f}(\lambda) = \int_0^\infty \frac{C_{i\lambda-\alpha}^\alpha(\cosh t)}{\sin\pi(i\lambda - \alpha)} f(t) \sinh^{2\alpha} t dt. \tag{6}$$

For $0 \le \lambda < \infty$, $\alpha > 0$ it is a special case of the Jacobi transform (1) of Section 7.8.8. The function $C_{i\lambda-\alpha}^\alpha(\cosh t)/\sin\pi(i\lambda - \alpha)$ is symmetric in $\lambda$. Hence, $\hat{f}(-\lambda) = \hat{f}(\lambda)$ and the inverse transform is written in the form

$$f(t) = \int_0^\infty \frac{C_{i\lambda-\alpha}^\alpha(\cosh t)}{\sin\pi(i\lambda - \alpha)} \hat{f}(\lambda) c(\lambda, \alpha) d\lambda, \tag{7}$$

where

$$c(\lambda, \alpha) = 2^{2\alpha-1} \frac{\lambda\Gamma^2(\alpha)\sinh\pi\lambda}{\Gamma(i\lambda + \alpha)\Gamma(-i\lambda + \alpha)}.$$

Since

$$\frac{C_{i\lambda-\alpha}^\alpha(z)}{\sin\pi(i\lambda - \alpha)} = \frac{e^{-i\pi\alpha}}{i\sinh\pi\lambda}\left[D_{i\lambda-\alpha}^\alpha(z) - D_{-i\lambda-\alpha}^\alpha(z)\right], \tag{8}$$

then one can rewrite inversion formula (7) as

$$f(t) = -ie^{-i\pi\alpha}\int_{-\infty}^\infty \hat{f}(\lambda) D_{i\lambda-\alpha}^\alpha(\cosh t) c(\lambda, \alpha)\frac{d\lambda}{\sinh\pi\lambda}. \tag{9}$$

Applying the formula of the inverse transform to (5) we derive

$$D_\nu^\alpha(Z) = = \frac{\sqrt{\pi}\Gamma\left(\alpha-\frac{1}{2}\right)}{\Gamma(\alpha)} e^{i\pi\alpha} \int_0^\infty \frac{\Gamma\left(i\lambda+\frac{1}{2}\right)\Gamma\left(\nu-i\lambda+\alpha+\frac{1}{2}\right)}{\Gamma\left(-i\lambda+\frac{1}{2}\right)\Gamma\left(\nu+i\lambda+\alpha+\frac{1}{2}\right)} e^{2\pi\lambda}(\tanh\pi\lambda)$$
$$\times 2^{2i\lambda}[(x^2-1)(y^2-1)]^{(i\lambda-\alpha+1/2)/2} D_{\nu-i\lambda+\alpha-1/2}^{i\lambda+1/2}(x) D_{\nu-i\lambda+\alpha-1/2}^{i\lambda+1/2}(y)$$
$$\times \frac{C_{i\lambda-\alpha+1/2}^{\alpha-1/2}(\cosh t)}{\sin\pi\ \left(i\lambda-\alpha+\frac{1}{2}\right)} \lambda d\lambda \tag{10}$$

or

$$D_\nu^\alpha(Z) = = \frac{\sqrt{\pi}\Gamma\left(\alpha-\frac{1}{2}\right)}{\Gamma(\alpha)} \int_{-\infty}^\infty \frac{\Gamma\left(i\nu+\frac{1}{2}\right)\Gamma\left(\nu-i\lambda+\alpha+\frac{1}{2}\right)}{\Gamma\left(-i\nu+\frac{1}{2}\right)\Gamma\left(\nu+i\lambda+\alpha+\frac{1}{2}\right)} \frac{e^{2\pi\lambda}}{\cosh\pi\lambda}$$
$$\times 2^{2i\lambda}[(x^2-1)(y^2-1)]^{(i\lambda-\alpha+1/2)/2} D_{\nu-i\lambda+\alpha-1/2}^{i\lambda+\alpha}(x)$$
$$\times D_{\nu-i\lambda+\alpha-1/2}^{i\lambda+\alpha}(y) D_{i\lambda-\alpha+1/2}^{\alpha-1/2}(\cosh t)\lambda d\lambda, \tag{11}$$

where $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\left(\nu+\alpha+\frac{1}{2}\right) > 0$. The last formula is represented as follows:

$$D_\nu^\alpha(Z) = \frac{\Gamma(2\alpha-1)}{\Gamma^2(\alpha)} \int_{-i\infty-\alpha+1/2}^{i\infty-\alpha+1/2} \frac{4^\ell\Gamma(\nu-\ell+1)\Gamma^2(\alpha+\ell)\Gamma(\ell+2\alpha-1)}{\Gamma(\nu+2\alpha+\ell)}$$
$$\times e^{-2\pi i(\alpha+\ell)}[(x^2-1)(y^2-1)]^{\ell/2} D_{\nu-\ell}^{\alpha+\ell}(x) D_{\nu-\ell}^{\alpha+\ell}(y) D_\ell^{\alpha-1/2}(\cosh t)d\ell. \tag{12}$$

With the help of the relation

$$\lim_{\lambda\to\infty} \lambda^{-2\alpha+1} e^{-i\pi\alpha} D_\lambda^\alpha\left(1+\frac{z^2}{2\lambda^2}\right) = \frac{1}{\sqrt{\pi}\Gamma(\alpha)}(2z)^{-\alpha+\frac{1}{2}} K_{\alpha-1/2}(z) \tag{13}$$

one obtains from (12) the equality

$$\frac{K_\nu(z)}{z^\nu} = \frac{2^\nu\Gamma(\nu)}{i\pi} e^{-i\pi\nu} \int_{-\nu-i\infty}^{-\nu+i\infty} \frac{K_{\nu+m}(x)}{x^\nu} \frac{K_{\nu+m}(y)}{y^\nu} D_m^\nu(\cosh t)(\nu+m)dm, \tag{14}$$
$$z = (x^2+y^2+2xy\cosh t)^{1/2},\ |\arg z| < \frac{\pi}{2},$$

which can be rewritten in the form

$$\frac{K_\nu(z)}{z^\nu} = -\frac{2^\nu\Gamma(\nu)}{\pi}\int_0^\infty \frac{K_{i\lambda}(x)}{x^\nu}\frac{K_{i\lambda}(y)}{y^\nu}\frac{C^\nu_{i\lambda-\nu}(\cosh t)}{\sin\pi(i\lambda-\nu)}(\sinh\pi\lambda)\lambda d\lambda$$

$$= -\frac{2^\nu\Gamma(\nu)}{i\pi}e^{-i\pi\nu}\int_{-\infty}^\infty \frac{K_{i\lambda}(x)}{x^\nu}\frac{K_{i\lambda}(y)}{y^\nu}D^\nu_{i\lambda-\nu}(\cosh t)\lambda d\lambda. \qquad (15)$$

Here $\operatorname{Re}\nu > -\frac{1}{2}$, $|\arg x| < \frac{\pi}{2}$, $|\arg y| < \frac{\pi}{2}$, $|\arg x| + |\arg y| + |\operatorname{Im} t| < \pi$. By means of the Fock-Mehler transform we derive from here the addition formula

$$\int_0^\infty \frac{K_\nu(z)}{z^\nu}\frac{C^\nu_{i\lambda-\nu}(\cosh t)}{\sin\pi(i\lambda-\nu)}\sinh^{2\nu} t dt = -\frac{2^{-\nu+1}}{\pi\Gamma(\nu)}\Gamma(i\lambda+\nu)\Gamma(-i\lambda+\nu)\frac{K_{i\lambda}(x)}{x^\nu}\frac{K_{i\lambda}(y)}{y^\nu}, \qquad (16)$$

where $\lambda \in \mathbb{R}$, $\nu > -\frac{1}{2}$.

The functions $C^\alpha_\lambda(x)$ and $D^\alpha_\lambda(x)$ for real $x$ are connected by the relation

$$\lim_{\varepsilon\to 0+} e^{-i\pi\alpha}D^\alpha_\lambda(x \pm i\varepsilon) = \frac{1}{2}e^{\mp i\pi\lambda}\left[C^\alpha_\lambda(x) \pm iD^\alpha_\lambda(x)\right].$$

Consequently,

$$\left(C^\alpha_\lambda(x)\right)^2 + \left(D^\alpha_\lambda(x)\right)^2 = \lim_{\varepsilon\to 0+} 4e^{-2\pi i\alpha}D^\alpha_\lambda(x+i\varepsilon)D^\alpha_\lambda(x-i\varepsilon).$$

Using this equality and the formula

$$D^\alpha_\lambda(e^{\pm i\pi}z) = e^{\mp i(\lambda+2\alpha)\pi}D^\alpha_\lambda(z)$$

we obtain from (3) that

$$(1-x^2)^m\left\{\left(C^{\alpha+m}_{\lambda-m}(x)\right)^2 + \left(D^{\alpha+m}_{\lambda-m}(x)\right)^2\right\}$$

$$= 2^{-2\alpha-2m+3}e^{-i\pi\alpha}\frac{\Gamma(2\alpha-1)\Gamma(m+1)\Gamma(\lambda+2\alpha+m)}{\Gamma^2(\lambda+m)\Gamma(m+2\alpha-1)\Gamma(\lambda-m+1)}$$

$$\times\int_1^\infty D^\alpha_\lambda(x^2+(1-x^2)z)C^{\alpha-1/2}_m(z)(z^2-1)^{\alpha-1}dz, \qquad (17)$$

where $\operatorname{Re}(\lambda-m+1) > 0$, $\operatorname{Re}\alpha > 0$.

If $x = 1 - y^2/2\lambda^2$, then

$$\lim_{\lambda\to\infty} C^\alpha_\lambda(x) = \frac{2^{2\alpha-1}\sqrt{\pi}}{\Gamma(\alpha)}(2y)^{-\alpha+1/2}J_{\alpha-1/2}(y), \qquad (18)$$

$$\lim_{\lambda\to\infty} D^\alpha_\lambda(x) = -\frac{2^{2\alpha-1}\sqrt{\pi}}{\Gamma(\alpha)}(2y)^{-\alpha+1/2}N_{\alpha-1/2}(y). \qquad (19)$$

Applying (13), (18) and (19) to (17), we obtain

$$J^2_{m+\nu}(x) + N^2_{m+\nu}(x) = \frac{4}{\pi^2}\frac{\Gamma(\nu)\Gamma(m+1)}{\Gamma(m+2\nu)}(4x)^\nu$$
$$\times \int_0^\infty K_\nu(2x\sinh t)C^\nu_m(\cosh 2t)\sinh^\nu t\cosh^{2\nu} t dt. \tag{20}$$

Finally, note that formula (10) of Section 9.4.4 implies the equality

$$(z^2-1)^{\alpha-1/2}D^\alpha_\lambda(z) = \frac{1}{4}(4\sinh\theta_1\sinh\theta_2)^{\lambda+1}\sinh^{2\alpha}\theta$$
$$\times\frac{\Gamma(\lambda+2\alpha)\Gamma(2\alpha)\Gamma^2\left(\alpha+\lambda+\frac{1}{2}\right)}{\Gamma^2(\alpha)\Gamma(\lambda+1)}\int_{-a-i\infty}^{-a+i\infty}\frac{(2\lambda+2\alpha+2\ell+1)\Gamma(\ell+1)\Gamma(\lambda+\ell+1)}{\Gamma(\lambda+2\alpha+\ell+1)\Gamma(2\lambda+2\alpha+\ell+1)}$$
$$\times C_\ell^{\lambda+\alpha+1/2}(\cosh\theta_1)C_\ell^{\lambda+\alpha+1/2}(\cosh\theta_2)D^{\alpha+1/2}_{\lambda+\ell}(\cosh\theta)\frac{d\ell}{\sin\pi\ell}, \tag{21}$$

where $0 < a < 1$ and

$$z = \frac{\cosh\theta + \cosh\theta_1\cosh\theta_2}{\sinh\theta_1\sinh\theta_2}.$$

Indeed, evaluating the right hand side by means of the residue theorem, we obtain the right hand side of formula (10) of Section 9.4.4. The equality

$$\mathfrak{Q}_\lambda(z) = -\frac{i}{2}(4\sinh\theta_1\sinh\theta_2)^{\lambda+1}\sinh\theta\,\Gamma^2(\lambda+1)$$
$$\times\int_{-a-i\infty}^{-a+i\infty}\frac{\Gamma(\ell+1)}{\Gamma(2\lambda+\ell+2)}C_\ell^{\lambda+1}(\cosh\theta_1)C_\ell^{\lambda+1}(\cosh\theta_2)D^1_{\lambda+\ell}(\cosh\theta)\frac{d\ell}{\sin\pi\ell} \tag{22}$$

is a special case of (21).

**9.4.7. The Banach algebras.** The formula

$$\hat{f}_m = \int_{SO(n)} f(g)T^{nm}(g^{-1})dg \tag{1}$$

defines the Fourier transform on the group $SO(n)$. The inverse transform has the form

$$f(g) = \sum_{m=0}^\infty(\dim T^{nm})\mathrm{Tr}(\hat{f}_mT^{nm}(g)). \tag{2}$$

The Fourier transform satisfies the property

$$(f * f')^{\wedge}_m = \hat{f}_m \hat{f}'_m. \tag{3}$$

Let us assume now that a function $f$ on $SO(n)$ is invariant with respect to the left and the right shifts by $SO(n-1)$:

$$f(kgk') = f(g), \quad k, k' \in SO(n-1). \tag{4}$$

Then the operator $\hat{f}_m$ has only one non-zero matrix element in the basis $\{\Xi_M^{nm}\}$, namely, the element

$$\hat{f}(m) \equiv \left(\hat{f}_m \Xi_O^{nm}, \Xi_O^{nm}\right).$$

We have

$$\hat{f}(m) = \int\limits_{SO(n)} f(g) t_{OO}^{nm}(g^{-1}) dg. \tag{5}$$

If functions $f_1$ and $f_2$ satisfy condition (4), then $f_1 * f_2$ has this property too. Moreover, $f_1 * f_2 = f_2 * f_1$. The set $\mathfrak{B}$ of functions $f$, obeying property (4), forms a commutative Banach algebra with respect to the convolution and the norm

$$\|f\| = \int |f(g)| dg.$$

Since elements $g \in SO(n)$ are represented in the form $g = kg_{n-1}(\theta)k'$, $k, k' \in SO(n-1)$, then for functions $f \in \mathfrak{B}$ we can set $f(g) = F(\cos\theta)$. Then

$$(f_1 * f_2)(g) \equiv (F_1 * F_2)(\cos\theta) = c_{n-1} c_n$$
$$\times \int\limits_{-1}^{1} \int\limits_{-1}^{1} F_1(r\cos\theta + (1-r^2)^{1/2} t \sin\theta) F_2(r)(1-r^2)^{\frac{n-3}{2}} (1-t^2)^{\frac{n-4}{2}} dr dt, \tag{6}$$

where $c_n = \Gamma(\frac{n}{2})/\sqrt{\pi}\Gamma(\frac{n-1}{2})$. Repeating reasonings used for derivation of formula (5) of Section 9.4.5, we conclude that the convolution of $F_1$ and $F_2$ can be given as

$$(F_1 * F_2)(\cos\theta) = \int\limits_0^{\pi} \int\limits_0^{\pi} F_1(\cos\theta_1) F_2(\cos\theta_2) K(\theta_1, \theta_2, \theta) d\theta_1 d\theta_2, \tag{7}$$

where $K(\theta_1, \theta_2, \theta)$ is defined by formula (6) of Section 9.4.5.

Thus, we obtain the commutative Banach algebra $L_n^1(-1,1)$ of functions $F(x)$ on the segment $[-1,1]$ with the norm

$$\|F\| = c_n \int\limits_{-1}^{1} |F(t)| (1-t^2)^{\frac{n-3}{2}} dt$$

and with the multiplication (6). Using formula (5) we define the transform

$$\widehat{F}(m) = \frac{n}{2\sqrt{\pi}} \frac{\ell(n-3)!}{(\ell+n-3)!} \int_{-1}^{1} F(t) C_\ell^{\frac{n-2}{2}}(t)(1-t^2)^{\frac{n-3}{2}}\,dt.$$

The space of functions $\widehat{F}(m)$ forms a commutative Banach algebra with respect to the multiplication

$$(F_1 * F_2)^\wedge(m) = \widehat{F}_1(m)\widehat{F}_2(m).$$

**9.4.8. Raising and lowering operators.** Recurrence relations for Gegenbauer polynomials imply those for the matrix elements $t^{n\ell}_{m0}(g_{n-1}(\theta))$. For example, for Gegenbauer polynomials one has the formula

$$4(1-x^2)\alpha(\alpha+1)C_{p-2}^{\alpha+2}(x) - 2\alpha(2\alpha+1)xC_{p-1}^{\alpha+1}(x) + p(2\alpha+p)C_p^\alpha(x) = 0$$

(it follows from differential equation (1) of Section 6.7.6, if one takes into account the equality $\frac{d}{dx}C_p^\alpha(x) = 2\alpha C_{p-1}^{\alpha+1}(x)$). We set here $x = \cos\theta$, $\alpha = m + (n-4)/2$, $p = \ell - m + 1$ and multiply both sides by $\sin^m\theta$. Expressing $\sin^m\theta C_{\ell-m}^{m+(n-2)/2}(\cos\theta)$ in terms of $t^{n\ell}_{m0}(g_{n-1}(\theta))$ we have

$$\begin{aligned}\cos\theta\, t^{n\ell}_{m0}(g_{n-1}(\theta)) &= \\ &= \left[\frac{(\ell+m+n-3)(\ell-m+1)(m+n-4)}{m(2m+n-3)(2m+n-5)}\right]^{1/2} \sin\theta\, t^{n\ell}_{m-1,0}(g_{n-1}(\theta)) \\ &+ \left[\frac{(\ell+m+n-2)(\ell-m)(m+1)}{(2m+n-1)(2m+n-3)(m+n-3)}\right]^{1/2} \sin\theta\, t^{n\ell}_{m+1,0}(g_{n-1}(\theta)).\end{aligned} \quad (1)$$

In the same way we derive the equality

$$\begin{aligned}\frac{d}{d\theta} t^{n\ell}_{m0}(g_{n-1}(\theta)) &= \\ &= \left[\frac{(\ell+m+n-3)(\ell-m+1)(m+n-4)m}{(2m+n-3)(2m+n-5)}\right]^{1/2} t^{n\ell}_{m-1,0}(g_{n-1}(\theta)) \\ &- \left[\frac{(\ell+m+n-2)(\ell-m)(m+n-3)(m+1)}{(2m+n-1)(2m+n-3)}\right]^{1/2} t^{n\ell}_{m+1,0}(g_{n-1}(\theta)).\end{aligned} \quad (2)$$

For the matrix elements $t^{n\sigma}_{m0}(g'_{n-1}(\theta))$ of $T^{n\sigma}(g'_{n-1}(\theta))$ the corresponding equalities are of the form

$$\cosh\theta\, t^{n\sigma}_{m0}(g'_{n-1}(\theta)) =$$
$$= (-\sigma+m-1)\left[\frac{m+n-4}{(2m+n-3)(2m+n-5)m}\right]^{1/2} \times \sinh\theta t^{n\sigma}_{m-1,0}(g'_{n-1}(\theta))$$
$$-(\sigma+m+n-2)\left[\frac{m+1}{(2m+n-1)(2m+n-3)(m+n-3)}\right]^{1/2} \times \sinh\theta\, t^{n\sigma}_{m+1,0}(g'_{n-1}(\theta)), \quad (3)$$

$$\frac{d}{d\theta}t^{n\sigma}_{m0}(g'_{n-1}(\theta)) =$$
$$= (m-\sigma-1)\left[\frac{m(m+n-4)}{(2m+n-3)(2m+n-5)}\right]^{1/2} t^{n\sigma}_{m-1,0}(g'_{n-1}(\theta))$$
$$+(-\sigma-m-n+2)\left[\frac{(m+n-3)(m+1)}{(2m+n-1)(2m+n-3)}\right]^{1/2} t^{n\sigma}_{m+1,0}(g'_{n-1}(\theta)). \quad (4)$$

We suggest to the reader to write down the corresponding equalities for the matrix elements $t^{nR}_{m0}(g_r)$ of the representations of $ISO(n-1)$.

Multiply both sides of (1) by $m\sin^{-1}\theta$ and subtract it from (2):

$$\left[\frac{d}{d\theta} - m\tan^{-1}\theta\right] t^{n\ell}_{m0}(g_{n-1}(\theta))$$
$$= -\left[\frac{(\ell+m+n-2)(\ell-m)(m+1)(2m+n-3)}{(2m+n-1)(m+n-3)}\right]^{1/2} t^{n\ell}_{m+1,0}(g_{n-1}(\theta)). \quad (5)$$

Now we multiply both sides of (1) by $(m+n-3)\sin^{-1}\theta$ and add it to (2):

$$\left[\frac{d}{d\theta} + (m+n-3)\tan^{-1}\theta\right] t^{n\ell}_{m0}(g_{n-1}\theta))$$
$$= \left[\frac{(\ell+m+n-3)(\ell-m+1)(m+n-4)(2m+n-3)}{m(2m+n-5)}\right]^{1/2} t^{n\ell}_{m-1,0}(g_{n-1}(\theta)). \quad (6)$$

The operators on the left hand side of (5) raise the value of the index $m$ in $t^{n\ell}_{m0}(g_{n-1}(\theta))$. It is called the *raising* operator. On the left hand side of (6) we have the *lowering* operator.

From formulas (3) and (4) one finds the raising and the lowering operators for $t^{n\sigma}_{m0}(g'_{n-1}(\theta))$:

$$\left[\frac{d}{d\theta} - m\tanh^{-1}\theta\right] t^{n\sigma}_{m0}(g'_{n-1}(\theta)) = -(\sigma+m+n-2)\left[\frac{(m+1)(2m+n-3)}{(2m+n-1)(m+n-3)}\right]^{1/2} t^{n\sigma}_{m+1,0}(g'_{n-1}(\theta)), \tag{7}$$

$$\left[\frac{d}{d\theta} + (m+n-3)\tanh^{-1}\theta\right] t^{n\sigma}_{m0}(g'_{n-1}(\theta)) = -(\sigma-m+1)\left[\frac{(m+n-4)(2m+n-3)}{m(2m+n-5)}\right]^{1/2} t^{n\sigma}_{m-1,0}(g'_{n-1}(\theta)). \tag{8}$$

For $t^{nR}_{m0}(g_r)$ we have

$$\left[\frac{d}{dr} - \frac{m}{r}\right] t^{nR}_{m0}(g_r) = = R\left[\frac{(m+1)(2m+n-3)}{(2m+n-1)(m+n-3)}\right]^{1/2} t^{nR}_{m+1,0}(g_r), \tag{9}$$

$$\left[\frac{d}{dr} - \frac{m+n-3}{r}\right] t^{nR}_{m0}(g_r) = = R\left[\frac{(m+n-4)(2m+n-3)}{m(2m+n-5)}\right]^{1/2} t^{nR}_{m-1,0}(g_r). \tag{10}$$

The product of the raising and of the lowering operators multiplies the matrix elements $t^{n\ell}_{m0}(g_{n-1}(\theta))$, $t^{n\sigma}_{m0}(g'_{n-1}(\theta))$ and $t^{nR}_{m0}(g_r)$ by a number. This leads to the differential equations for matrix elements:

$$\left[\frac{d^2}{d\theta^2} + (n-2)\tan^{-1}\theta\frac{d}{d\theta} - \frac{m(m+n-3)}{\sin^2\theta} + \ell(\ell+n-2)\right] t^{n\ell}_{m0}(g_{n-1}(\theta)) = 0, \tag{11}$$

$$\left[\frac{d^2}{d\theta^2} + (n-2)\tanh^{-1}\theta\frac{d}{d\theta} - \frac{m(m+n-3)}{\sinh^2\theta} - \sigma(\sigma+n-2)\right] t^{n\sigma}_{m0}(g'_{n-1}(\theta)) = 0, \tag{12}$$

$$\left[\frac{d^2}{dr^2} + \frac{n-2}{r}\frac{d}{dr} - \frac{m(m+n-3)}{r^2} - R^2\right] t^{nR}_{m0}(g_r) = 0. \tag{13}$$

**9.4.9. Relations between spherical functions for the groups of different dimensionalities.** It follows from the explicit expression (8) of Section 9.4.2 for $t^{n\sigma}_{m0}(g'_{n-1}(\theta))$ that

$$t^{n\sigma}_{m0}(g'_{n-1}(\theta)) = -\sigma\left[\frac{n-2}{m(n-1)(n+m-3)}\right]^{1/2} \sinh\theta\, t^{n+2,\sigma-1}_{m-1,0}(g'_{n+1}(\theta)). \qquad (1)$$

Therefore,

$$t^{n\sigma}_{m0}(g'_{n-1}(\theta)) = \frac{(-1)^m\Gamma(\sigma+1)\Gamma(\frac{n-3}{2})}{2^{m+1}\Gamma(n-3)\Gamma\left(m+\frac{n-1}{2}\right)\Gamma(\sigma-m+1)}$$
$$\times\left[\frac{(n+2m-3)\Gamma(n-2)\Gamma(n+m-3)}{m!}\right]^{1/2} \sinh^m\theta\, t^{n+2m,\sigma-m}_{00}(g'_{n+2m-1}(\theta)). \quad (2)$$

For the matrix elements $t^{n\ell}_{m0}(g_{n-1}(\theta))$ we have the equality

$$t^{n\ell}_{m0}(g_{n-1}(\theta)) = \left[\frac{\ell(\ell+n-2)(n-2)}{m(n-1)(m+n-3)}\right]^{1/2} \sin\theta\, t^{n+2,\ell-1}_{m-1,0}(g_{n+1}(\theta)) \qquad (3)$$

which implies that

$$t^{n\ell}_{m0}(g_{n-1}(\theta)) = \frac{\Gamma(\frac{n-3}{2})}{2^m\Gamma\left(m+\frac{n-3}{2}\right)}\left[\frac{(\ell+m+n-3)!(n-3)\ell!(m+n-4)!}{(\ell+n-3)!(n-4)!m!(2m+n-3)}\right]^{1/2}$$
$$\times \sin^m\theta\, t^{n+2m,\ell-m}_{00}(g_{n+2m-1}(\theta)). \qquad (4)$$

For the matrix elements $t^{nR}_{m0}(g_r)$ we obtain

$$t^{nR}_{m0}(g_r) =$$
$$= \frac{i^m\Gamma(\frac{n-1}{2})}{\Gamma\left(m+\frac{n-1}{2}\right)}\left[\frac{(m+n-4)!(2m+n-3)}{m!(n-3)!}\right]^{1/2}\left(\frac{-iRr}{2}\right)^m t^{n+2m,R}_{00}(g_r). \qquad (5)$$

It follows from formula (4) of Section 9.4.8 for $m=0$ that

$$\frac{d}{d\theta}t^{n\sigma}_{00}(g'_{n-1}(\theta)) = \frac{-\sigma-n+2}{\sqrt{n-1}}t^{n\sigma}_{1,0}(g'_{n-1}(\theta)), \qquad (6)$$

where $n \geq 3$. Formula (1) implies that

$$t^{n\sigma}_{1,0}(g_{n-1}(\theta)) = -\frac{\sigma}{\sqrt{n-1}}\sinh\theta\, t^{n+2,\sigma-1}_{00}(g'_{n+1}(\theta)). \qquad (7)$$

From (6) and (7) one derives the differentiation formula

$$\frac{d}{d\theta}t_{00}^{n\sigma}(g'_{n-1}(\theta)) = \frac{\sigma(\sigma+n-2)}{n-1}\sinh\theta\, t_{00}^{n+2,\sigma-1}(g'_{n+1}(\theta)) \tag{8}$$

which can be rewritten in the form

$$\frac{d}{d(\cosh\theta)}t_{00}^{n\sigma}(g'_{n-1}(\theta)) = \frac{\sigma(\sigma+n-2)}{n-1}t_{00}^{n+2,\sigma-1}(g'_{n+1}(\theta)). \tag{9}$$

Consequently,

$$\frac{d^k}{d(\cosh\theta)^k}t_{00}^{n\sigma}(g'_{n-1}(\theta)) = \\ = \frac{\Gamma(\frac{n-1}{2})\Gamma(\sigma+1)\Gamma(\sigma+k+n-2)}{2^k\Gamma\left(k+\frac{n-1}{2}\right)\Gamma(\sigma-k+1)\Gamma(\sigma+n-2)}t_{00}^{n+2k,\sigma-k}(g'_{n+2k-1}(\theta)). \tag{10}$$

The corresponding formulas for matrix elements of representations of the groups $SO(n)$ and $ISO(n-1)$ have the form

$$\frac{d^k}{d(\cos\theta)^k}t_{00}^{n\ell}(g_{n-1}(\theta)) = \\ = \frac{\Gamma(\frac{n-1}{2})\ell!(\ell+k+n-3)!}{2^k\Gamma\left(k+\frac{n-1}{2}\right)(\ell-k)!(\ell+n-3)!}t_{00}^{n+2k,\ell-k}(g_{n+2k-1}(\theta)), \tag{11}$$

$$\frac{d^k}{[d(Rr)^2]^k}t_{00}^{nR}(g_r) = \frac{\Gamma(\frac{n-1}{2})}{2^k\Gamma\left(k+\frac{n-1}{2}\right)}t_{00}^{n+2k,R}(g_r). \tag{12}$$

We derive from (2) and (10) that

$$t_{k0}^{n\sigma}(g'_{n-1}(\theta)) = \frac{(-1)^k\Gamma(\sigma+n-2)}{\Gamma(\sigma+k+n-2)}\left[\frac{(n+2k-3)(k+n-4)!}{k!(n-3)!}\right]^{1/2} \\ \times\sinh^k\theta\frac{d^k}{d(\cosh\theta)^k}t_{00}^{n\sigma}(g'_{n-1}(\theta)) \\ = \frac{(-1)^k\Gamma(\sigma+n-2)}{\Gamma(\sigma+k+n-2)}\left[\dim T^{n-1,k}\right]^{1/2}\frac{d^k}{d\theta^k}t_{00}^{n\sigma}(g'_{n-1}(\theta)). \tag{13}$$

For $t_{m0}^{n\ell}(g_{n-1}(\theta))$ and $t_{m0}^{nR}(g_r)$ we have

$$t_{m0}^{n\ell}(g_{n-1}(\theta)) = \left[\frac{(2m+n-3)(m+n-4)!(\ell-m)!(\ell+n-3)!}{m!\ell!(n-3)!(\ell+m+n-3)!}\right]^{1/2}\sin^m\theta \\ \times\frac{d^m}{d(\cos\theta)^m}t_{00}^{n\ell}(g_{n-1}(\theta)), \tag{13'}$$

$$t_{m0}^{nR}(g_\tau) = 2^m \left[\frac{(m+n-4)!(2m+n-3)}{m!(n-3)!}\right]^{1/2} (Rr)^m \frac{d^m}{[d(Rr)^2]^m} t_{00}^{nR}(g_\tau). \qquad (13'')$$

It was shown in Section 6.5 and in Section 9.3.4 that

$$t_{00}^{3\sigma}(g_2'(\theta)) = \mathfrak{P}_\sigma(\cosh\theta),\ t_{00}^{4\sigma}(g_3'(\theta)) = \frac{\sinh(\sigma+1)\theta}{(\sigma+1)\sinh\theta}.$$

Therefore, setting $n = 3$ and $n = 4$ into (10), we obtain

$$t_{00}^{2k+1,\sigma}(g_{2k}'(\theta)) = \frac{\Gamma(k)\Gamma(\sigma+1)2^{k-1}}{\Gamma(\sigma+2k-1)} \frac{d^{k-1}}{d(\cosh\theta)^{k-1}} \mathfrak{P}_{\sigma+k-1}(\cosh\theta), \qquad (14)$$

$$t_{00}^{2k,\sigma}(g_{2k-1}'(\theta)) = \\ = \frac{\Gamma\left(k-\frac{1}{2}\right)\Gamma(\sigma+1)2^{k-1}}{\sqrt{\pi}\Gamma(\sigma+2k-2)(\sigma+k-1)} \frac{d^{k-1}}{d(\cosh\theta)^{k-1}} (\cosh(\sigma+k-1)\theta). \qquad (15)$$

Replacing $t_{00}^{n\sigma}(g_{n-1}'(\theta))$ by the expressions in terms of Legendre functions, we have

$$\mathfrak{P}_{\tau-1/2}^{-k+1/2}(\cosh\theta) = \frac{\sqrt{2}\Gamma(\tau-k+1)}{\sqrt{\pi}\tau\Gamma(\tau+k)} \sinh^{k-1/2}\theta \frac{d^k}{d(\cosh\theta)^k} \cosh\tau\theta, \qquad (16)$$

$$\mathfrak{P}_\tau^{-k}(\cosh\theta) = \frac{\Gamma(\tau-k+1)}{\Gamma(\tau+k+1)} \sinh^k\theta \frac{d^k}{d(\cosh\theta)^k} \mathfrak{P}_\tau(\cosh\theta), \qquad (17)$$

where $k \in \mathbb{Z}_+$. By virtue of the equality

$$\mathfrak{P}_\tau^m(x) = \frac{\Gamma(\tau+m+1)}{\Gamma(\tau-m+1)} \mathfrak{P}_\tau^{-m}(x), \quad m = 0, 1, 2, \ldots,$$

formula (17) is rewritten as

$$\mathfrak{P}_\tau^m(x) = (x^2-1)^{m/2} \frac{d^m}{dx^m} \mathfrak{P}_\tau(x), \quad m = 0, 1, 2, \ldots. \qquad (18)$$

It follows from (18) and from formula (17) of Seciton 7.4.4 that

$$\mathfrak{Q}_\tau^m(x) = (x^2-1)^{m/2} \frac{d^m}{dx^m} \mathfrak{Q}_\tau(x), \quad m = 0, 1, 2, \ldots. \qquad (19)$$

Analogues of (14) and (15) for the matrix elements $t_{00}^{n\ell}(g_{n-1}(\theta))$ are of the form

$$t_{00}^{2k+1,\ell}(g_{2k}(\theta)) = \frac{\ell!(k-1)!2^{k-1}}{(\ell+2k-2)!} \frac{d^{k-1}}{d(\cos\theta)^{k-1}} P_{\ell+k-1}(\cos\theta) \\ = \frac{(-1)^{\ell+k-1}\ell!(k-1)!}{2^\ell(\ell+k-1)!(\ell+2k-2)!} \frac{d^{\ell+2k-2}}{d(\cos\theta)^{\ell+2k-2}} (\sin^2\theta)^{\ell+k-1}, \qquad (20)$$

$$t_{00}^{2k,\ell}(g_{2k-1}(\theta)) = \frac{2^{k-1}\ell!\Gamma\left(k-\frac{1}{2}\right)}{\sqrt{\pi}(\ell+2k-3)!(\ell+k-1)}\frac{d^{k-1}}{d(\cos\theta)^{k-1}}(\cos(\ell+k-1)\theta), \quad (21)$$

and for $t_{00}^{nR}(g_r)$ of the form

$$t_{00}^{2k+1,R}(g_r) = 4^{k-1}(k-1)!\frac{d^{k-1}}{[d(Rr)^2]^{k-1}}J_0(-iRr), \quad (22)$$

$$t_{00}^{2k,R}(g_r) = \frac{4^{k-1}\Gamma\left(k-\frac{1}{2}\right)}{2\sqrt{\pi}}\frac{d^{k-2}}{[d(Rr)^2]^{k-2}}\frac{\sin iRr}{iRr}. \quad (23)$$

**9.4.10. Asymptotic properties of spherical functions of the group $SO_0(n-1,1)$.** Making use of linear transformation (2) of Section 7.3.5 for hypergeometric function $F(\alpha,\beta;\gamma;x)$ from formula (6) of Section 9.3.2, after simple manipulations we derive the following expression for the zonal spherical function $\varphi^{n\sigma}(\theta)$ of the represenation $T^{n\sigma}$ of $SO_0(n-1,1)$:

$$\begin{aligned}\varphi^{n\sigma}(\theta) &= \\ &= \frac{\Gamma(\frac{n-1}{2})\Gamma\left(-\sigma-\frac{n-2}{2}\right)}{\Gamma\left(-\frac{\sigma}{2}\right)\Gamma\left(-\frac{\sigma-1}{2}\right)}\cosh^{-\sigma-n+2}\theta F\left(\frac{\sigma+n-1}{2},\frac{\sigma+n-2}{2};\sigma+\frac{n}{2};\cosh^{-2}\theta\right) \\ &+ \frac{\Gamma(\frac{n-1}{2})\Gamma\left(\sigma+\frac{n-2}{2}\right)}{\Gamma(\frac{\sigma+n-1}{2})\Gamma(\frac{\sigma+n-2}{2})}\cosh^{\sigma}\theta F\left(-\frac{\sigma}{2},-\frac{\sigma-1}{2};-\sigma-\frac{n}{2}+2;\cosh^{-2}\theta\right) \quad (1)\end{aligned}$$

where $\sigma+\frac{n-2}{2}\bar{\in}\mathbb{Z}$. Since $\cosh^{-2}\theta\to 0$ when $\theta\to\infty$, then for $\theta\to\infty$ we have

$$\varphi^{n\sigma}(\theta)\sim\frac{\Gamma(\frac{n-1}{2})\Gamma\left(-\sigma-\frac{n-2}{2}\right)}{\Gamma\left(-\frac{\sigma}{2}\right)\Gamma\left(-\frac{\sigma-1}{2}\right)}\cosh^{-\sigma-n+2}\theta+\frac{\Gamma(\frac{n-1}{2})\Gamma\left(\sigma+\frac{n-2}{2}\right)}{\Gamma(\frac{\sigma+n-1}{2})\Gamma(\frac{\sigma+n-2}{2})}\cosh^{\sigma}\theta. \quad (2)$$

Let us introduce the notation

$$\mathbf{c}\left(i\left(\sigma+\frac{n-2}{2}\right)\right)=\frac{2^{\sigma+n-2}\Gamma(\frac{n-1}{2})\Gamma\left(-\sigma-\frac{n-2}{2}\right)}{\Gamma\left(-\frac{\sigma}{2}\right)\Gamma\left(-\frac{\sigma-1}{2}\right)}. \quad (3)$$

The function $\mathbf{c}(\lambda)$, $\lambda\in\mathbb{C}$, is called the *Harish-Chandra* **c**-*function* of the group $SO_0(n-1,1)$. By means of this function formula (1) is rewritten in the form

$$\varphi^{n\sigma}(\theta)=\mathbf{c}\left(i\left(\sigma+\frac{n-2}{2}\right)\right)\Phi^{n\sigma}(\theta)+\mathbf{c}\left(-i\left(\sigma+\frac{n-2}{2}\right)\right)\Phi^{n,-\sigma-n+2}(\theta), \quad (4)$$

where

$$\begin{aligned}\Phi^{n\sigma}(\theta) &= (2\cosh\theta)^{-\sigma-n+2}F\left(\frac{\sigma+n-1}{2},\frac{\sigma+n-2}{2};\sigma+\frac{n}{2};\cosh^{-2}\theta\right) \\ &= \frac{e^{i\pi(n-3)/2}\Gamma\left(\sigma+\frac{n}{2}\right)}{2^{(n-1)/2}\sqrt{\pi}\,\Gamma(\sigma+1)}\sinh^{-(n-3)/2}\mathfrak{Q}_{\sigma+(n-3)/2}^{-(n-3)/2}(\cosh\theta) \quad (5)\end{aligned}$$

and $\mathfrak{Q}_\tau^\lambda$ is the associated Legendre function of the second kind (see formula (9) of Seciton 7.4.4).

The function $\Phi^{n\sigma}(\theta)$ satisfies the same differential equation as $\varphi^{n\sigma}(\theta)$ does, that is, equation (3) of Section 9.3.5. The solutions $\Phi^{n\sigma}$ and $\varphi^{n\sigma}$ are linearly independent. The solutions $\Phi^{n\sigma}$ and $\Phi^{n,-\sigma-n+2}$ of equation (3) of Section 9.3.5 are also linearly independent.

It follows from (2) that for $\operatorname{Re}\sigma < -n+2$ we have

$$\varphi^{n\sigma}(\theta) \sim \mathbf{c}\left(i\left(\sigma+\frac{n-2}{2}\right)\right)e^{(-\sigma-n+2)\theta} \quad \text{for} \quad \theta \to +\infty.$$

On the other hand, replacing $\sigma$ by $-\sigma-n+2$ in formula (5) of Section 9.3.2, we obtain for $\theta \to +\infty$ that

$$\varphi^{n\sigma}(\theta) \sim \frac{2^{\sigma+n-2}\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)}e^{(-\sigma-n+2)\theta}\int_0^\pi (1-\cos\varphi)^{-\sigma-n+2}\sin^{n-3}\varphi d\varphi,$$

where $\operatorname{Re}\sigma < -n+2$. Thus, if $\operatorname{Re}\sigma < -n+2$ we have the following integral representation for the Harish-Chandra $\mathbf{c}$-function:

$$\mathbf{c}\left(i\left(\sigma+\frac{n-2}{2}\right)\right) = \frac{2^{\sigma+n-2}\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}\int_0^\pi (1-\cos\theta)^{-\sigma-n+2}\sin^{n-3}\theta d\theta \tag{6}$$

(see formula (3) of Section 9.2.7). In the same way we find from formula (5) of Section 9.3.2 that

$$\mathbf{c}\left(i\left(\sigma+\frac{n-2}{2}\right)\right) = \frac{2^{-\sigma}\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)}\int_0^\pi (1-\cos\varphi)^{\sigma}\sin^{n-3}\varphi d\varphi, \tag{7}$$

where $\operatorname{Re}\sigma > 0$. Since $\varphi^{n\sigma}(\theta) = \varphi^{n\sigma}(-\theta)$ (see Section 9.3.2), then one can replace in formula (5) of Section 9.3.2 $(\cosh\theta - \cos\varphi\sinh\theta)$ by $(\cosh\theta+\cos\varphi\sinh\theta)$. Therefore, $1-\cos\varphi$ in (6) and (7) can be replaced by $1+\cos\varphi$.

In order to find the asymptotic behavior of the function $t_{m0}^{n\sigma}(g'_{n-1}(\theta))$ we note that

$$\begin{aligned} t_{m0}^{n\sigma}(g'_{n-1}(\theta)) &= (T^{n\sigma}(g'_{n-1}(\theta))1, \Xi_O^{n-1,m}) = (1, T^{n,-\bar{\sigma}-n+2}(g'_{n-2}(-\theta))\Xi_O^{n-1,m}) \\ &= A_m^n \int_0^\pi (\cosh\theta+\cos\varphi\sinh\theta)^{-\sigma-n+2} \\ &\qquad \times t_{00}^{n-1,m}\left(\frac{\cosh\theta\cos\varphi+\sinh\theta}{\cosh\theta+\cos\varphi\sinh\theta}\right)\sin^{n-3}\varphi d\varphi, \end{aligned} \tag{8}$$

where

$$A^n_m = [\dim T^{n-1,m}]^{1/2} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)}.$$

Since

$$\frac{\cosh\theta\cos\varphi + \sinh\theta}{\cosh\theta + \cos\varphi\sinh\theta} \underset{\theta\to+\infty}{\sim} 1,$$

then

$$t_{00}^{n-1,m}\left(\frac{\cosh\theta\cos\varphi + \sinh\theta}{\cosh\theta + \cos\varphi\sinh\theta}\right) \underset{\theta\to\infty}{\longrightarrow} 1.$$

Therefore, we find from (8) that for $\theta \to +\infty$

$$t^{n\sigma}_{m0}(g'_{n-1}(\theta)) \sim [\dim T^{n-1,m}]^{1/2} \frac{2^{\sigma+n-2}\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)} e^{(-\sigma-n+2)\theta}$$
$$\times \int_0^\pi (1+\cos\varphi)^{-\sigma-n+2} \sin^{n-3}\varphi d\varphi$$
$$= [\dim T^{n-1,m}]^{1/2} \mathbf{c}\left(i\left(\sigma + \frac{n-2}{2}\right)\right) e^{-(\sigma+n-2)\theta},$$

where $\operatorname{Re}\sigma < -n+2$. Thus, the asymptotics of the function

$$[\dim T^{n-1,m}]^{-1/2} t^{n\sigma}_{m0}(g'_{n-1}(\theta))$$

for $\theta \to +\infty$ does not depend on $m$ and coincides with the asymptotics of the zonal spherical function $\varphi^{n\sigma}(\theta)$.

**9.4.11. Dougall's formula for Gegenbauer polynomials.** Clebsch-Gordan coefficients for the tensor product of representations $T^{n\ell}$ and $T^{nm}$ of the group $SO(n)$ are defined in the same way as in the case of the group $SU(2)$ (see Section 8.1.2). If $\mathbf{e}^\ell_1$ is an orthonormal vector of the carrier space of $T^{n\ell}$, invariant with respect to $SO(n-1)$, then, as in the case of $SU(2)$ (see Section 8.2.4), for the Clebsch-Gordan coefficients

$$(\mathbf{e}^\ell_1 \otimes \mathbf{e}^m_1, \mathbf{e}^s_1) \equiv C^{\ell m s}_{000} \tag{1}$$

we have

$$|C^{\ell m s}_{000}|^2 = (\dim T^{ns}) \int_{SO(n)} t^{n\ell}_{00}(g) t^{nm}_{00}(g) \overline{t^{ns}_{00}(g)} dg.$$

It follows from here that

$$|C^{\ell m s}_{000}|^2 = \frac{\Gamma\left(\frac{n-2}{2}\right)}{2\pi\Gamma\left(\frac{n-1}{2}\right)} \frac{(n-3)!^2\ell!m!(2s+n-2)}{(n+\ell-3)!(n+m-3)!}$$
$$\times \int_0^\pi C^{(n-2)/2}_\ell(\cos\theta) C^{(n-2)/2}_m(\cos\theta) C^{(n-2)/2}_s(\cos\theta) \sin^{n-2}\theta d\theta. \tag{2}$$

Let us denote the integral on the right hand side of (2) by $D\left(\ell, m, s; \frac{n-2}{2}\right)$. We shall prove that if $\ell+m+s=2g$, where $g$ is an integer, and if there exists a triangle with the sides of lengths $\ell$, $m$, $s$, then

$$D(\ell,m,s;p) \equiv \int_{-1}^{1} C_\ell^p(x) C_m^p(x) C_s^p(x)(1-x^2)^{p-1/2}dx$$
$$= \frac{2^{1-2p}\pi\Gamma(g+2p)\Gamma(g-\ell+p)\Gamma(g-m+p)\Gamma(g-s+p)}{[\Gamma(p)]^4\Gamma(g-\ell+1)\Gamma(g-m+1)\Gamma(g-s+1)\Gamma(g+p+1)}. \quad (3)$$

Otherwise $D(\ell,m,s;p)=0$.

Assume that $\ell+m+s$ is not an even integer. Since the polynomial $C_s^p(x)$ and the integer $s$ have the same parity and the integral $\int_{-1}^{1}\varphi(x)dx$ is equal to zero, if $\varphi(x)$ is an odd function, then integral (3) vanishes. If $s>\ell+m$ (in this case there are no triangles with the sides of lengths $\ell$, $m$,$s$), then integral (3) vanishes because of the orthogonality relation for Gegenbauer polynomials. Really, $C_\ell^p(x)C_m^p(x)$ is a polynomial of degree $\ell+m$. This polynomial is expanded in the polynomials $C_s^p(x)$, $s\le \ell+m$.

We now consider the case when $\ell+m+s=2g$, where $g$ is an integer, and there exists a triangle with the sides of lengths $\ell$, $m$, $s$. The formula (3) is proved by induction on $\ell$. For $\ell=0$ we have $m=s=g$ and, therefore, (3) takes the form

$$D(0,n,n;p) = \int_{-1}^{1}[C_n^p(x)]^2(1-x^2)^{p-1/2}dx = \frac{2^{1-2p}\pi\Gamma(n+2p)}{(n+p)\Gamma^2(p)n!}.$$

Taking into account the normalization condition for Gegenbauer polynomials (see Section 6.10.1) we conclude that formula (3) is valid.

Let us now assume that (3) is valid for all positive integers less than $\ell$. We apply to (3) the recurrence relation

$$C_\ell^p(x) = \frac{2(p+\ell-1)}{\ell}xC_{\ell-1}^p(x) - \frac{2p+\ell-2}{\ell}C_{\ell-2}^p(x)$$

(see formula (2) of Section 6.7.6) at first for replacement of $C_\ell^p(x)$ and then for replacement of $xC_{\ell-1}^p(x)$. We obtain

$$D(\ell,m,s;p) = \frac{(p+\ell-1)(s+1)}{\ell(p+s)}D(\ell-1,m,s+1;p)$$
$$+\frac{(p+\ell-1)(2p+s-1)}{\ell(p+s)}D(\ell-1,m,s-1;p) - \frac{2p+\ell-2}{\ell}D(\ell-2,m,s;p).$$

Instead of $D(\dots)$ on the right hand side we substitute their expressions (3) (we can do this by the induction assumption). After simple transformations we obtain expression (3) for $D(\ell, m, s; p)$.

As in the case of the group $SU(2)$, the Clebsch-Gordan coefficients $C_{000}^{\ell m s}$ are defined up to a constant factor with the unit absolute value. We assume that $C_{000}^{\ell m s}$ is positive. Then

$$C_{000}^{\ell m s} = \frac{1}{\Gamma\left(\frac{n-2}{2}\right)}\left[\frac{(n-3)!\ell!m!(2s+n-2)\Gamma(g+n-2)}{2(n+\ell-3)!(n+m-3)!\Gamma(g-\ell+1)} \times \frac{\Gamma\left(g-\ell+\frac{n-2}{2}\right)\Gamma\left(g-m+\frac{n-2}{2}\right)}{\Gamma(g-m+1)\Gamma(g-s+1)\Gamma\left(g+\frac{n}{2}\right)}\right. \quad (4)$$

By virtue of the orthogonality relation for Gegenbauer polynomials we obtain from (3) the expansion

$$C_\ell^p(x)C_m^p(x) = \sum_{s=|\ell-m|}^{\ell+m}{}' \frac{(n+p)\Gamma(s+1)\Gamma(g+2p)}{\Gamma^2(p)(g+p)\Gamma(s+2p)} \times \frac{\Gamma(g-\ell+p)\Gamma(g-m+p)\Gamma(g-s+p)}{\Gamma(g-\ell+1)\Gamma(g-m+1)\Gamma(g-s+1)}C_s^p(x). \quad (5)$$

where $\ell+m+s=2g$ and the summation is over the values of $s$ which are of the same evenness as $\ell+m$ is. It is *Dougall's formula.* By means of formula (4) of Section 9.3.2 it can be written down for zonal spherical functions $t_{00}^{n\ell}(g)$ of $SO(n)$.

One can obtain formulas which generalize Dougall's formula in the following way. From formulas (2′) and (3) of Section 9.4.2 we derive the relation

$$\sin^m\varphi C_{\ell-m}^{p+m}(\cos\varphi) = \frac{m!\Gamma\left(p-\frac{1}{2}\right)\Gamma(2p+m+\ell)i^m}{2^{m+1}\sqrt{\pi}\Gamma(p+m)\Gamma(2p+m-1)} \times \int_0^\pi (\cos\varphi - i\,\sin\varphi\cos\theta)^\ell C_m^{p-1/2}(\cos\theta)\sin^{2p-1}\theta d\theta. \quad (6)$$

Considering (6) as a coefficient in the expansion of the function $(\cos\varphi - i\,\sin\varphi\cos\theta)^\ell$ in Gegenbauer polynomials, we obtain the equality

$$(\cos\varphi - i\,\sin\varphi\cos\theta)^\ell = \frac{\ell!\Gamma(2p-1)}{\Gamma(p)}\sum_{m=0}^{\ell}\frac{(-2i)^m\Gamma(2m+2p-1)\Gamma(p+m)}{\Gamma(2p+\ell+m)} \times \sin^m\varphi C_{\ell-m}^{p+m}(\cos\varphi)C_m^{p-1/2}(\cos\theta). \quad (7)$$

Multiply out these expansions for $\ell = \ell_1$ and $\ell = \ell_2$ and apply to the left hand side the same expansion for $\ell = \ell_1 + \ell_2$. We obtain

$$\sum_j \frac{(-2i)^j(2j+2p-1)\Gamma(p+j)}{\Gamma(2p+\ell_1+\ell_2+j)} \sin^j \varphi C^{p+j}_{\ell_1+\ell_2-j}(\cos\varphi) C^{p-1/2}_j(\cos\theta)$$
$$= \frac{\Gamma(2p-1)\ell_1!\ell_2!}{\Gamma(p)(\ell_1+\ell_2)!} \sum_{k,m} \frac{(-2i)^{k+m}(2k+2p-1)(2m+2p-1)\Gamma(p+k)\Gamma(p+m)}{\Gamma(2p+\ell_1+k)\Gamma(2p+\ell_2+m)}$$
$$\times \sin^{k+m}\varphi C^{p+k}_{\ell_1-k}(\cos\varphi) C^{p+m}_{\ell_2-m}(\cos\varphi) C^{p-1/2}_k(\cos\theta) C^{p-1/2}_m(\cos\theta).$$

Multiply both sides of this equality by $\sin^{2p-1}\theta C^{p-1/2}_j(\cos\theta)$ and integrate with respect to $\theta$ from $0'$ to $\pi$. Taking into consideration the orthogonality relation for Gegenbauer polynomials and formula (3), we obtain the expansion

$$\sin^j \varphi C^{p+j}_{\ell_1+\ell_2-j}(\cos\varphi) = \sum_{k=0}^{\ell_1} \sum_{m=0}^{\ell_2} A^{\ell_1\ell_2 p}_{kmj} \sin^{k+m}\varphi C^{p+k}_{\ell_1-k}(\cos\varphi) C^{p+m}_{\ell_2-m}(\cos\varphi), \tag{8}$$

where the summation is over the values of $k$ and $m$ such that $k+m-j$ is an even integer and there exists a triangle with the sides of lengths $k$, $m$, $j$,

$$A^{\ell_1\ell_2 p}_{kmj} = \frac{2^{2p-3}(-2i)^{k+m-j} j!\ell_1!\ell_2!\Gamma(2p+\ell_1+\ell_2+j)(2k+2p-1)(2m+2p-1)}{\sqrt{\pi}(\ell_1+\ell_2)!\Gamma(p+j)\Gamma(2p+\ell_1+k)\Gamma(2p+\ell_2+m)\Gamma\left(p-\frac{1}{2}\right)}$$
$$\times \frac{\Gamma(p+k)\Gamma(p+m)\Gamma(g+2p-1)\Gamma\left(g-k+p-\frac{1}{2}\right)\Gamma\left(g-m+p-\frac{1}{2}\right)\Gamma\left(g-j+p-\frac{1}{2}\right)}{\Gamma(g-k+1)\Gamma(g-m+1)\Gamma(g-j+1)}$$

and $g = \frac{1}{2}(k+m+j)$. For $j=0$ formula (8) is of the form

$$C^p_{\ell_1+\ell_2}(\cos\varphi) = \sum_{k=0}^{\min(\ell_1,\ell_2)} A^{\ell_1\ell_2}_{kp} \sin^{2k}\varphi C^{p+k}_{\ell_1-k}(\cos\varphi) C^{p+k}_{\ell_2-k}(\cos\varphi), \tag{9}$$

where

$$A^{\ell_1\ell_2}_{kp} = \frac{\ell_1!\ell_2!(-4)^k\Gamma(2p+\ell_1+\ell_2)\Gamma^2(p+k)\Gamma(k+2p-1)(2k+2p-1)}{(\ell_1+\ell_2)!\Gamma^2(p)k!\Gamma(2p+\ell_1+k)\Gamma(2p+\ell_2+k)}.$$

In the same way formula (8) of Section 9.4.2 leads to the relation

$$\sinh^{p-1/2}\theta \mathfrak{P}^{-j-p+1/2}_{\sigma_1+\sigma_2+p-1/2}(\cosh\theta) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} B^{\ell_1\ell_2 p}_{kmj}$$
$$\times \mathfrak{P}^{-k-p+1/2}_{\sigma_1+p-1/2}(\cosh\theta) \mathfrak{P}^{-m-p+1/2}_{\sigma_2+p-1/2}(\cosh\theta), \tag{10}$$

where the summation is over the values of $k$ and $m$ such that $k+m-j$ is an even integer and there exists a triangle with the sides of lengths $k$, $m$, $j$,

$$B_{kmj}^{\ell_1\ell_2p} = \frac{(-1)^{k+m-j}2^{p-1/2}j!\Gamma(\sigma+\sigma_2-j+1)\Gamma(\sigma_1+1)\Gamma(\sigma_2+1)}{\pi\Gamma\left(p-\frac{1}{2}\right)\Gamma(2p+j-1)\Gamma(\sigma_1+\sigma_2+1)\Gamma(\sigma_1-k+1)\Gamma(\sigma_2-m+1)}$$
$$\times\frac{(2p+2k-1)(2p+2m-1)\Gamma(g+2p-1)\Gamma\left(g-k+p-\frac{1}{2}\right)}{\Gamma\left(g+p+\frac{1}{2}\right)\Gamma(g-k+1)}$$
$$\times\frac{\Gamma\left(g-m+p-\frac{1}{2}\right)\Gamma\left(g-j+p-\frac{1}{2}\right)}{\Gamma(g-m+1)\Gamma(g-j+1)}$$

and $g=\frac{1}{2}(k+m+j)$. For $j=0$ we obtain from (10) the relation

$$\sinh^{p-1/2}\theta\mathfrak{P}_{\sigma_1+\sigma_2+p-1/2}^{-p+1/2}(\cosh\theta) = \frac{2^{p-1/2}\Gamma\left(p-\frac{1}{2}\right)\Gamma(\sigma_1+1)\Gamma(\sigma_2+1)}{\Gamma(2p-1)}$$
$$\times\sum_{k=0}^{\infty}\frac{\left(k+p-\frac{1}{2}\right)\Gamma(2p+k-1)}{k!\Gamma(\sigma_1-k+1)\Gamma(\sigma_2-k+1)}\mathfrak{P}_{\sigma_1+p-1/2}^{-p-k+1/2}(\cosh\theta)$$
$$\times\mathfrak{P}_{\sigma_2+p-1/2}^{-p-k+1/2}(\cosh\theta). \tag{11}$$

Formula (10) of Section 9.4.2 implies the expansion

$$e^{itx} = \Gamma(p)\sum_{m=0}^{\infty}i^m(m+p)\frac{J_{p+m}(t)}{(t/2)^p}C_m^p(x). \tag{12}$$

By means of (12) one derives an analog of formula (8) for Bessel functions:

$$\left(\frac{t_1t_2}{2(t_1+t_2)}\right)^p J_{j+p}(t_1+t_2)$$
$$=\frac{j!}{\Gamma(p)\Gamma(j+2p)}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{(m+k-j)/2}}{\Gamma(g+p+1)}(m+p)(k+p)$$
$$\times\frac{\Gamma(g+2p)\Gamma(g+p-j)\Gamma(g+p-m)\Gamma(g+p-k)}{\Gamma(g-j+1)\Gamma(g-m+1)\Gamma(g-k+1)}J_{m+p}(t_1)J_{k+p}(t_2), \tag{13}$$

where the summation and $g$ are the same as in (10). For $j=0$ we obtain from (13) that

$$\left(\frac{t_1t_2}{2(t_1+t_2)}\right)^p J_p(t_1+t_2) = \sum_{m=0}^{\infty}\frac{(-1)^m(m+p)\Gamma(2p+m)}{m!\Gamma(2p)}J_{m+p}(t_1)J_{m+p}(t_2). \tag{14}$$

**9.4.12. Functional relations for Chebyshev polynomials.** Since the Chebyshev polynomials of the second kind $U_\ell(x)$ coincide with $C_\ell^1(x)$ (see Section 6.9.1), then formulas (1) and (5) of Section 9.3.3 imply that

$$U_\ell(\cos\varphi) = \frac{\ell+1}{2}\int_0^\pi (\cos\varphi - i\,\sin\varphi\cos\theta)^\ell \sin\theta d\theta$$
$$= \frac{\ell+1}{2}\int_0^\pi (\cos\varphi - i\,\sin\varphi\cos\theta)^{-\ell-2}\sin\theta d\theta. \quad (1)$$

Since $C_\ell^{1/2}(x) = P_\ell(x)$, then we obtain from formula (8) of Section 9.4.3 the expansion of $U_\ell(x)$ in Legendre polynomials:

$$U_\ell(x) = \sum_{k=0}^{[\ell/2]} \frac{2^{2\ell-4k}(2k)!(2\ell-4k+1)}{(2\ell-2k+1)!}\left[\frac{(\ell-k)!}{k!}\right]^2 P_{\ell-2k}(x). \quad (2)$$

Formula (5) of Section 9.4.5 leads to the product formula

$$\int_0^\pi U_\ell(\cos\theta\cos\varphi - \sin\theta\sin\varphi\cos\psi)\sin\psi d\psi = \frac{2}{\ell+1}U_\ell(\cos\varphi)U_\ell(\cos\theta). \quad (3)$$

It follows from formula (1) of Section 9.4.5 for $\ell \geq k$ that

$$\int_{|\theta-\varphi|}^{\theta+\varphi} U_\ell(\cos\gamma)P_k\left(\frac{\cos\gamma - \cos\theta\cos\varphi}{\sin\theta\sin\varphi}\right)\sin\gamma d\gamma$$
$$= \frac{2^{2k+1}k!^2(\ell-k)!}{(\ell+k+1)!}(\sin\varphi\sin\theta)^{k+1}C_{\ell-k}^{k+1}(\cos\varphi)C_{\ell-k}^{k+1}(\cos\theta). \quad (4)$$

In particular, for $k=0$ we have

$$\int_{|\theta-\varphi|}^{\theta+\varphi} U_\ell(\cos\gamma)\sin\gamma d\gamma = \frac{2}{\ell+1}\sin\varphi\sin\theta\, U_\ell(\cos\varphi)U_\ell(\cos\theta). \quad (5)$$

From formula (8) of Section 9.4.15 we derive the equality

$$\sum_{\ell=0}^\infty \frac{1}{\ell+1}U_\ell(\cos\gamma)U_\ell(\cos\varphi)U_\ell(\cos\theta) = \frac{\pi}{4\sin\gamma\sin\varphi\sin\theta}. \quad (5')$$

Taking into account the relation $\chi_m(u) = U_{2m}\left(\cos\frac{t}{2}\right)$ between characters of the irreducible representations $T_m$, $m = 0, \frac{1}{2}, 1, \ldots$, of $SU(2)$ and Chebyshev polynomials $U_\ell$ (see formula (1) of Section 6.9.3), we derive from (5′) the following relation for characters of representations of $SU(2)$:

$$\sum_{m=0,\frac{1}{2},1,\ldots} \frac{1}{2m+1}\chi_m(u)\chi_m(u')\chi_m(u'') = \frac{\pi}{4\sin t \sin t' \sin t''}, \tag{6}$$

where the angles $t$, $t'$, $t''$ are defined by elements $u$, $u'$, $u''$ of the group $SO(2)$ (see formula (4) of Section 6.9.2).

From formula (5) of Section 9.3.6 for $p = \frac{3}{2}$ we have

$$\int_0^\pi (\cos\varphi - i\sin\varphi\cos\theta)^\ell U_m(\cos\theta)\sin^2\theta d\theta$$

$$= \frac{2^{m+1}\sqrt{\pi}\,\Gamma\left(m+\frac{3}{2}\right)\ell!(m+1)}{i^m(\ell+m+2)!}\sin^m\varphi C_{\ell-m}^{m+3/2}(\cos\varphi). \tag{7}$$

Formula (6) of Section 9.3.8 for $p = \frac{3}{2}$ and $\theta = \frac{\pi}{2}$ gives the expansion of the function $\cos^\ell\theta$ in $U_\ell$:

$$\cos^\ell\theta = \frac{\sqrt{\pi}\,\ell!}{2^{\ell+2}\Gamma\left(\frac{3}{2}\right)}\sum_{k=0}^{[\ell/2]}\frac{(2\ell-4k+2)}{k!(\ell-k+1)!}U_{\ell-2k}(\cos\theta). \tag{8}$$

From formula (2) of Section 9.3.7 we derive the generating function for $U_\ell$:

$$(1-2th+h^2)^{-1} = \sum_{n=0}^{\infty} U_n(t)h^n. \tag{9}$$

The formula (3) of Section 9.3.7 yields the relation

$$\sum_{n=0}^{k} U_n(t)U_{k-n}(t) = C_k^2(t). \tag{10}$$

## 9.5. Matrix Elements of Class 1 Representations and Generalization of Gegenbauer Polynomials, Legendre and Bessel Functions

**9.5.1. Matrix elements of class 1 representations.** Let the representations $T^{n\sigma}$ of $SO_0(n-1,1)$ be realized in the space $\mathfrak{L}^2(S^{n-2})$ (see Section 9.2.1).

Formula (6) of Section 9.4.2 gives an orthonormal basis of this space. For the matrix elements $t^{n\sigma}_{kmj}(g'_{n-1}(\theta))$ of the operators $T^{n\sigma}(g'_{n-1}(\theta))$ in this basis we have

$$t^{n\sigma}_{kmj}(g'_{n-1}(\theta)) = \\ = D^n_{kmj} \int_0^\pi (\cosh\theta - \cos\varphi \sinh\theta)^{\sigma-j} C^{j+(n-3)/2}_{m-j} \left( \frac{\cos\varphi \cosh\theta - \sinh\theta}{\cosh\theta - \cos\varphi \sinh\theta} \right) \\ \times C^{j+(n-3)/2}_{k-j}(\cos\varphi) \sin^{n+2j-3}\varphi d\varphi, \tag{1}$$

where

$$D^n_{kmj} = \\ = \frac{2^{2j+n-5}\Gamma^2\left(j+\frac{n-3}{2}\right)}{\pi} \left[ \frac{(k-j)!(m-j)!(2m+n-3)(2k+n-3)}{(n+k+j-4)!(n+m+j-4)!} \right]^{1/2}. \tag{2}$$

The matrix elements $t^{n\ell}_{KM}(g)$ of the representation $T^{n\ell}$ of $SO(n)$ are given by the formula

$$t^{n\ell}_{KM}(g) = (T^{n\ell}(g)\Xi_M, \Xi_K)_\ell,$$

where $( . , . )$ is the scalar product on $\mathfrak{D}^{n-1,\ell}$ and $\Xi_K$ are defined by formula (6) of Section 9.4.2. As in Section 9.4.2, we obtain for matrix elements $t^{n\ell}_{kmj}(g_{n-1}(\varphi))$ of the operator $T^{n\ell}(g_{n-1}(\varphi))$ the expression

$$t^{n\ell}_{kmj}(g_{n-1}(\varphi)) = \\ = B^{n\ell}_{kmj} \int_0^\pi (\cos\varphi - i\,\cos\theta \sin\varphi)^{\ell-j} C^{j+(n-3)/2}_{m-j} \left( \frac{\cos\theta\cos\varphi - i\,\sin\varphi}{\cos\varphi - i\,\cos\theta\sin\varphi} \right) \\ \times C^{j+(n-3)/2}_{m-j}(\cos\theta) \sin^{n+2j-3}\theta d\theta, \tag{3}$$

where

$$B^{n\ell}_{kmj} = i^{-m+k} \left[ \frac{(\ell+k+n-3)!(\ell-k)!}{(\ell+m+n-3)!(\ell-m)!} \right]^{1/2} D^n_{kmj}, \tag{4}$$

and $D^n_{kmj}$ is given by (2).

For the matrix elements $t^{nR}_{kmj}(g_r)$ of the operators $T^{nR}(g_r)$ of representations of $ISO(n-1)$ one has the integral representation

$$t^{nR}_{kmj}(g_r) = \\ = D^n_{kmj} \int_0^\pi e^{Rr\cos\varphi} C^{j+(n-3)/2}_{m-j}(\cos\varphi) C^{j+(n-3)/2}_{k-j}(\cos\varphi) \sin^{n+2j-3}\varphi d\varphi, \tag{5}$$

where $D^n_{kmj}$ is given by (2).

The function $\mathfrak{P}^{n\sigma}_{kmj}(z)$, defined by the equality

$$\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta) = t^{n\sigma}_{kmj}(g'_{n-1}(\theta)), \tag{6}$$

is a generalization of the associated Legendre function. The formula

$$P^{n\ell}_{kmj}(\cos\varphi) = t^{n\ell}_{kmj}(g_{n-1}(\varphi)) \tag{7}$$

defines a generalization of the Gegenbauer polynomial and the formula

$$J^n_{kmj}(-iRr) = t^{nR}_{kmj}(g_r) \tag{8}$$

gives a generalization of the Bessel function.

It follows from (1) and (5) that $\mathfrak{P}^{n\sigma}_{kmj}(x)$ and $J^n_{kmj}(x)$ are entire analytic functions of parameters $\sigma$ and $x$, respectively.

It was shown in Section 9.1.6 that any element $g \in SO(n)$ is representable in the form

$$g = g^n(\varphi_1,\dots,\varphi_{n-1})h = g^n(\varphi_1,\dots,\varphi_{n-1})g^{n-1}(\theta_1,\dots,\theta_{n-2})h' = \dots,$$
$$h \in SO(n-1),\ h' \in SO(n-2),$$
$$g^n(\varphi_1,\dots,\varphi_{n-1}) = g_1(\varphi_1)g_2(\varphi_2)\cdots g_{n-1}(\varphi_{n-1}).$$

For the matrix elements $t^{n\ell}_{(m,M)(k,K)}(g)$ of the operators $T^{n\ell}(g)$ we have

$$t^{n\ell}_{(m,M)(k,K)}(g) = \sum_P t^{n\ell}_{(m,M)(k,P)}(g^n(\varphi_1,\dots,\varphi_{n-1}))t^{n-1,k}_{PK}(h), \tag{9}$$

where

$$t^{n\ell}_{(m,M)(p,P)}(g^n(\varphi_1,\dots,\varphi_{n-1}))$$
$$= P^{n\ell}_{mpm_1}(\cos\varphi_{n-1})P^{n-1,m}_{m_1p_1m_2}(\cos\varphi_{n-2})P^{n-2,m_1}_{m_2p_2m_3}(\cos\varphi_{n-3})\dots e^{im_{n-3}\varphi_1} \tag{10}$$

and $M = (m_1, m_2, \dots, m_{n-3})$, $P = (p_1, p_2, \dots, p_{n-3})$. Applying formulas (9) and (10) to $t^{n-1,p}_{PK}(h)$ and then to matrix elements of representations of the subgroups $SO(n-2)$, $SO(n-3), \dots$, we find the decomposition of $t^{n\ell}_{(m,M)(k,K)}(g)$ into the sum of products of $P^{ik}_{str}(\cos\varphi)$. Similar decompositions hold for the matrix elements $t^{n\sigma}_{(m,M)(k,K)}(g)$ of representations of $SO_0(n-1,1)$ and for the matrix elements $t^{nR}_{(m,M)(k,K)}(g)$ of representations of $ISO(n-1)$.

**Example 1.** For the representation $T^{4,\ell}$ of the group $SO(4)$ we have

$$t^{4\ell}_{(m,m_1)(p,p_1)}(g) = \sum_j P^{4,\ell}_{mpm_1}(\cos\varphi_3) P^{3,m}_{m_1 j}(\cos\varphi_2) e^{ij\varphi_1} \times P^{3,p}_{jp_1}(\cos\theta_2) e^{ij\theta_1} e^{ip_1\psi_1}, \tag{11}$$

where $g = g^4(\varphi_1,\varphi_2,\varphi_3) g^3(\theta_1,\theta_2) g_2(\psi_1)$.

**Example 2.** For the representation $T^{5,\ell}$ of $SO(5)$ we have

$$t^{5,\ell}_{(m,m_1,m_2)(p,p_1,p_2)}(g) = \sum_{k_1,k_2} P^{5,\ell}_{mpm_1}(\cos\varphi_4) P^{4,m}_{m_1 k_1 m_2}(\cos\varphi_3) \times P^{3,m_1}_{m_2 k_2}(\cos\varphi_2) e^{im_2\varphi_1} t^{4,p}_{(k_1,k_2)(p_1,p_2)}(h),\ h \in SO(4). \tag{12}$$

**9.5.2. Evaluation of the functions $P^{n\ell}_{kmj}(\cos\theta)$.** We make in the integral of formula (3) of Section 9.5.1 the substitution

$$\frac{\cos\theta\cos\varphi - i\,\sin\varphi}{\cos\varphi - i\,\cos\theta\sin\varphi} = -\sin\psi \tag{1}$$

and use the relation $B^{n,-\ell-n+2}_{mkj} = (-1)^{k-m} B^{n\ell}_{kmj}$. Then we obtain the equality

$$P^{n\ell}_{kmj}(\cos\theta) = P^{n,-\ell-n+2}_{mkj}(\cos\theta) \tag{2}$$

which defines the functions $P^{n\ell}_{kmj}$ for negative $\ell$.

Applying the Rodrigues formula for Gegenbauer polynomials to formula (3) of Section 9.5.1 we find that

$$P^{n\ell}_{kmj}(\cos\theta) = \frac{(-1)^{k+m}(m+j+n-4)!\,\Gamma\left(j+\frac{n-2}{2}\right)}{2^{m-j}(m-j)!(2j+n-4)!\,\Gamma\left(m+\frac{n-2}{2}\right)} B^{n\ell}_{kmj} \times \int_{-1}^{1} (1-x^2)^{k+(n-4)/2} \frac{d^{k-j}}{dx^{k-j}} \left[(\cos\theta - ix\sin\theta)^{\ell-j} C^{j+(n-3)/2}_{m-j}\left(\frac{x\,\cos\theta - i\,\sin\theta}{\cos\theta - ix\,\sin\theta}\right)\right] dx. \tag{3}$$

One proves by the induction in $s$ that for $\alpha\delta - \gamma\beta = 1$ the equality

$$\frac{d^s}{dx^s}\left[(\beta x+\delta)^p \varphi\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right)\right] = \sum_{r=0}^{s} \frac{s!(s-p-1)!(-\beta)^r(\beta x+\delta)^{r+p-2s}}{r!(s-r)!(s-p-r-1)!} \frac{d^{s-r}}{dy^{s-r}}\varphi\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right) \tag{4}$$

holds, where $y = \frac{\alpha x+\gamma}{\beta x+\delta}$. Apply (4) to (3) and make the substitution

$$\frac{x\cos\theta - i\sin\theta}{\cos\theta - ix\sin\theta} = -t,$$

remove the parentheses in the expression obtained and integrate term by term. We find that

$$P^{n\ell}_{kmj}(\cos\theta) = \widetilde{G}^{n\ell}_{kmj}\cos^{\ell-j}\theta\sum_{s=0}^{\ell-j}\alpha^{n\ell}_{kmjs}\tan^s\theta, \tag{5}$$

where

$$\widetilde{G}^{n\ell}_{kmj} = \frac{i^{k-m}}{\sqrt{\pi}\,\Gamma\left(m+\frac{n-2}{2}\right)}\left[\frac{(\ell-k)!(m-j)!(k-j)!}{(\ell-m)!(k+j+n-4)!}(m+j+n-4)!(\ell+k+n-3)! \right.$$
$$\left. \times(\ell+m+n-3)!(2m+n-3)(2k+n-3)\right]^{1/2}, \tag{6}$$

$$\alpha^{n\ell}_{kmjs} = \sum_{r=0}^{\min(s,m-j)}\frac{(-i)^s(\ell-r-j)!\Gamma\left(m-r+\frac{n-3}{2}\right)}{2^{r+1}(\ell-j-s)!(s-r)!r!(m-j-r)!(\ell+m+n-r-3)!}$$
$$\times\int_{-1}^{1}t^{s-r}(1-t^2)^{m+(n-4)/2}C^{m-r+(n-3)/2}_{k+r-m}(t)dt. \tag{7}$$

We denote the integral on the right hand side of (7) by $I^n_{kmr}$. In order to evaluate it we make use of the formula

$$C^p_\ell(x) = \frac{1}{\Gamma(p)}\sum_{t=0}^{[\ell/2]}\frac{2^{\ell-2t}(-1)^t\Gamma(p+\ell-t)}{t!(\ell-2t)!}x^{\ell-2t}$$

and integrate term by term. We obtain that $I^n_{kmr} = 0$ if the numbers $m-k$ and $s$ ae of different evenness, and

$$I^n_{kmr} = \frac{2^{k-m+r}}{\Gamma\left(m-r+\frac{n-3}{2}\right)}\sum_{t=0}^{[(k+r-m)/2]}\frac{(-1)^t\Gamma\left(k-t+\frac{n-3}{2}\right)\Gamma\left(\frac{s+k-m+1}{2}-t\right)\Gamma\left(m+\frac{n-2}{2}\right)}{2^{2t}t!\Gamma(k-m+r-2t)\Gamma\left(\frac{s+k+m+n-1}{2}-t\right)}$$
$$= \frac{2^{k+r-m}\Gamma\left(m+\frac{n-2}{2}\right)\Gamma\left(k+\frac{n-3}{2}\right)\Gamma\left(\frac{s+k-m+1}{2}\right)}{\Gamma\left(m-r+\frac{n-3}{2}\right)\Gamma\left(\frac{s+k+m+n-1}{2}\right)\Gamma(k+r-m+1)}$$
$$\times{}_eF_2\left(-\tfrac{k+r-m}{2}, -\tfrac{k+r-m-1}{2}, -\tfrac{s+k+m+n-3}{2}; -k-\tfrac{n-5}{2}, \tfrac{m-s+k+1}{2}; 1\right) \tag{7'}$$

if they have the same evenness. Thus,

$$P^{n\ell}_{kmj}(\cos\theta) = G^{n\ell}_{kmj}\cos^{\ell-j}\theta\sum_{s=0}^{\ell-j}{}'\,\beta^{n\ell}_{kmjs}\tan^s\theta, \tag{8}$$

where the rime at the sum sign means that the summation is over the values $s$ such that $s$ and $m-k$ hae the same evenness,

$$G^{n\ell}_{kmj} = \frac{2^{k-m-1}}{\sqrt{\pi}}\Gamma\left(k+\frac{n-3}{2}\right)\left[\frac{(\ell-k)!(m-j)!(k-j)!}{(\ell-m)!(k+j+n-4)!}(m+j+n-4)!(\ell+k+n-3)!\right.$$
$$\left.\times(\ell+m+n-3)!(2m+n-3)(2k+n-3)\right]^{1/2}, \quad (9)$$

$$\beta^{n\ell}_{kmjs} = \frac{(-1)^{(k-s-m)/2}\Gamma\left(\frac{s+k-m+1}{2}\right)}{(\ell-j-s)!\Gamma\left(\frac{s+k+m+n-1}{2}\right)}\sum_{r=0}^{\min(s,m-j)}\frac{(\ell-r-j)!}{r!(s-r)!(m-j-r)!(\ell+m+n-r-3)!(k-m_r)!}$$
$$\times\,{}_3F_2\left(-\tfrac{k+r-m}{2}, -\tfrac{k+r-m-1}{2}, -\tfrac{s+k+m+n-3}{2}; -k-\tfrac{n-5}{2}, \tfrac{m-s-k+1}{2}; 1\right). \quad (10)$$

Since one of the numbers $\frac{k+r-m}{2}$, $\frac{k+r-m-1}{2}$ is a positive integer, then the hypergeometric series ${}_3F_2(\ldots;1)$ in (10) is finite. For odd $n$ it is expressed in terms of the Hahn polynomials (see formula (5) of Section 8.5.1).

**9.5.3. Evaluation of the functions $\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)$ and $J^n_{kmj}(x)$.** One evaluates the function $\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)$ in the same way as in the case of $P^{n\ell}_{kmj}(\cos\theta)$. Let us apply relation (4) of Section 9.5.2 to formula (1) of Section 9.5.1, make the substitution

$$\frac{x\cosh\theta-\sinh\theta}{\cosh\theta-x\sinh\theta} = -y$$

and expand $(-y\sinh\theta+\cosh\theta)^{\sigma-r-j}$ according to the binomial formula. After simple transformations we obtain

$$\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta) = \widetilde{E}^{n\sigma}_{kmj}\cosh^{-\sigma-j-n+2}\theta\sum_{s=0}^{\infty}\gamma^{n\sigma}_{kmjs}\tanh^s\theta, \quad (1)$$

where

$$\widetilde{E}^{n\sigma}_{kmj} = \frac{\Gamma(k-\sigma)}{2\sqrt{\pi}\Gamma\left(k+\frac{n-2}{2}\right)}\left[\frac{(k-j)!(m-j)!(k+j+n-4)!(2k+n-3)(2m+n-3)}{(m+j+n-4)!}\right]^{1/2}, \quad (2)$$

$$\gamma^{n\sigma}_{kmjs} = \sum_{r=0}^{\min(k-j,s)}\frac{\Gamma\left(k-r+\frac{n-3}{2}\right)\Gamma(-\sigma-r-j-n+3)}{r!(k-j-r)!(s-r)!\Gamma(k-r-\sigma)\Gamma(-\sigma-s-j-n+3)}$$
$$\times\int_{-1}^{1} y^{s-r}(1-y^2)^{k+(n-4)/4}C^{k-r+(n-3)/2}_{m+r-k}(y)\,dy. \quad (3)$$

We have found the integral (3) in Section 9.5.2. In particular, $\gamma^{n\sigma}_{kmjs} = 0$ if $k-m$ and $s$ are of different evenness, that is, the summation in (1) is really carried out over the values of $s$ which have the same evenness as $k-m$.

In order to find $J^n_{kmj}(x)$ we replace in formula (5) of Section 9.5.1 the product of Gegenbauer polynomials by the linear combination of Gegenbauer polynomials (see Section 9.4.11) and utilize formula (7) of Section 9.3.6. We obtain

$$J^n_{kmj}(x) = \frac{1}{2}\left[\frac{(k-j)!(m-j)!(2k+n-3)(2m+n-3)}{(k+j+n-4)!(m+j+n-4)!}\right]^{1/2} \frac{i^{m+k}}{\Gamma\left(j+\frac{n-3}{2}\right)}$$
$$\times \sum_{s=\max(m,k)+j}^{m+k} (-1)^{s-j} \frac{(s+n-4)!\Gamma\left(s-k+\frac{n-3}{2}\right)\Gamma\left(s-m+\frac{n-3}{2}\right)}{\Gamma(2s-m-k-2j+1)\Gamma(s-k-j+1)\Gamma(s-m-j+1)}$$
$$\times \frac{\Gamma\left(m+k+j-s+\frac{n-3}{2}\right)\Gamma(s-2j+1)}{\Gamma(m+k-s+1)\Gamma\left(s-j+\frac{n-3}{2}\right)}\left(\frac{x}{2}\right)^{-j-(n-3)/2} J_{2s-m-k-j+(n-3)/2}(x). \quad (4)$$

If $j \equiv m < k$, then the sum is reduced to one summand and we have

$$J^n_{kmm}(x) = \left[\frac{\sqrt{\pi}(2k+n-3)(k+m+n-4)!\Gamma\left(m+\frac{n-1}{2}\right)}{(k-m)!\Gamma\left(m+\frac{n-2}{2}\right)}\right]^{1/2}$$
$$\times (-i)^{k-m}\frac{J_{k+(n-3)/2}(x)}{x^{k+(n-3)/2}}. \quad (5)$$

**9.5.4. Expansion of $P^{n\ell}_{kmj}(\cos\theta)$ into Fourier series.** It follows from the last expression for $P^{(\alpha,\beta)}_n(t)$ in formula (7) of Section 6.3.8 and from formula (4) of Section 3.5.8 that

$$C^p_m(t) = \frac{\Gamma(2p+m)}{m!\Gamma(2p)}\left(\frac{t+1}{2}\right)^m F\left(-p-m+\frac{1}{2}, -m; p+\frac{1}{2}; \frac{t-1}{t+1}\right).$$

Therefore,

$$(\cos\theta - i\cos\varphi\sin\theta)^{\ell-j} C^{j+(n-3)/2}_{m-j}\left(\frac{\cos\varphi\cos\theta - i\sin\theta}{\cos\theta - i\cos\varphi\sin\theta}\right)$$
$$= \frac{\sqrt{\pi}(m+j+n-4)!\Gamma\left(m+\frac{n-2}{2}\right)}{2^{\ell+j+n-4}\Gamma\left(j+\frac{n-3}{2}\right)}(\ell-m)!(1+\cos\varphi)^{\ell-j}e^{-i(\ell-j)\theta}$$
$$\times \sum_{s=0}^{\ell-j} \beta^{n\ell}_{mjs} e^{2is\theta}\left(\frac{1-\cos\varphi}{1+\cos\varphi}\right)^s, \quad (1)$$

where

$$\beta^{n\ell}_{mjs} = \sum_{q=\max(0,s+m-\ell)}^{\min(m-j,s)} (-1)^q \Bigg[ q!(s-q)!(\ell-m-s+q)!(m-j-q)! \times \Gamma\left(m-q+\frac{n-2}{2}\right)\Gamma\left(q+j+\frac{n-2}{2}\right)\Bigg]^{-1}. \quad (2)$$

Applying (1) to formula (3) of Section 9.5.1, we obtain

$$P^{n\ell}_{kmj}(\cos\theta) = \frac{i^{k-m}}{2^{\ell-j+1}\sqrt{\pi}}\Gamma\left(j+\frac{n-3}{2}\right)\Gamma\left(m+\frac{n-2}{2}\right)\Bigg[(\ell-k)!(\ell-m)! \times \frac{(m-j)!(k-j)!(\ell+k+n-3)!(m+j+n-4)!(2m+n-3)(2k+n-3)}{(k+j+n-4)!(\ell+m+n-3)!}\Bigg]^{1/2} \times \sum_{s=0}^{\ell-j} \beta^{n\ell}_{mjs}\widetilde{I}^{n\ell}_{kjs}e^{-i(\ell-j-2s)\theta}, \quad (3)$$

where

$$\widetilde{I}^{n\ell}_{kjs} = \int_{-1}^{1}(1-t)^{s+j+(n-4)/2}(1+t)^{\ell-s+(n-4)/2}C^{j+(n-3)/2}_{k-j}(t)dt.$$

One can evaluate this integral in the same way as the integral of formula (7) of Section 9.5.2:

$$\widetilde{I}^{n\ell}_{kjs} = \frac{2^{\ell+j+n-3}\Gamma\left(s+j+\frac{n-2}{2}\right)\Gamma\left(\ell-s+\frac{n-2}{2}\right)(k+j+n-4)!}{(k-j)!(2j+n-4)!(\ell+j+n-3)!} \times {}_3F_2\left(-k+j, k+j+n-3, s+j+\frac{n-2}{2}; \ell+j+n-2, j+\frac{n-2}{2}; 1\right). \quad (4)$$

The coefficient $\beta^{n\ell}_{mjs}$ can also be expressed in terms of ${}_3F_2(\dots;1)$. From (2) we find that

$$\beta^{n\ell}_{mjs} = \left[s!(m-j)!(\ell-m-s)!\Gamma\left(m+\frac{n-2}{2}\right)\Gamma\left(j+\frac{n-2}{2}\right)\right]^{-1} \times {}_3F_2\left(-s, -m+j, -m-\frac{n-4}{2}; \ell-m-s+1, j+\frac{n-2}{2}; 1\right) \quad (5)$$

if $\ell-m-s\geq 0$ and

$$\beta^{n\ell}_{mjs} = (-1)^{\ell+m+s}\Bigg[(\ell-m)!(\ell-s-j)!(s+m-\ell)!\Gamma\left(\ell-s+\frac{n-2}{2}\right) \times \Gamma\left(s+m-\ell+j+\frac{n-2}{2}\right)\Bigg]^{-1} {}_3F_2\Bigg(-\ell+m, -\ell+s+j, -\ell+s-\frac{n-4}{2}; -\ell+s+m, s+m+j-\ell+\frac{n-2}{2}; 1\Bigg) \quad (6)$$

if $\ell - m - s \le 0$. The series ${}_3F_2(\dots;1)$ in (4)–(6) are finite.

**9.5.5. Symmetry properties of the functions $\mathfrak{P}^{n\sigma}_{kmj}(x)$, $P^{n\ell}_{kmj}(x)$ and $J^n_{kmj}(x)$.** The equalities

$$\mathfrak{P}^{n\sigma}_{kmj}(x) = \mathfrak{P}^{n+2,\sigma-1}_{k-1,m-1,j-1}(x), \tag{1}$$

$$P^{n\ell}_{kmj}(x) = P^{n+2,\ell-1}_{k-1,m-1,j-1}(x), \tag{2}$$

$$J^n_{kmj}(x) = J^{n+2}_{k-1,m-1,j-1}(x) \tag{3}$$

give remarkable properties of the functions $\mathfrak{P}^{n\sigma}_{kmj}(x)$, $P^{n\ell}_{kmj}(x)$ and $J^n_{kmj}(x)$. They are proved by means of integral representations of these functions. Application of these formulas $j$ times leads to the functions $\mathfrak{P}^{n+2j,\sigma-j}_{k-j,m-j,0}(x)$, $P^{n+2j,\ell-j}_{k-j,m-j,0}(x)$, $J^{n+2j}_{k-j,m-j,0}(x)$.

One can see from (1)–(3) that $\mathfrak{P}^{n\sigma}_{kmj}(x)$, $P^{n\ell}_{kmj}(x)$ and $J^n_{kmj}(x)$ coincide with some of these functions corresponding to the values $n = 5$ and $n = 4$.

We replace $\theta$ by $\pi - \theta$ in formula (8) of Section 9.5.2. Since the summation in this formula is over the values of $s$ which have the same evenness as $m + k$, then

$$P^{n\ell}_{kmj}(-x) = (-1)^{\ell+k+m+j} P^{n\ell}_{kmj}(x). \tag{4}$$

For $J^n_{kmj}(x)$ we have

$$J^n_{kmj}(-x) = (-1)^{k+m} J^n_{kmj}(x). \tag{5}$$

Since $(T^{n\sigma}(g))^* = T^{n,-\bar\sigma-n+2}(g^{-1})$, $g \in SO_0(n-1,1)$, then

$$\overline{t^{n\sigma}_{kmj}(g'_{n-1}(\theta))} = t^{n,-\bar\sigma-n+2}_{mkj}(g'_{n-1}(-\theta)).$$

But

$$t^{n,-\bar\sigma-n+2}_{mkj}(g'_{n-1}(-\theta)) = (-1)^{k+m} t^{n,-\bar\sigma-n+2}_{mkj}(g'_{n-1}(\theta)).$$

Therefore,

$$\overline{\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)} = (-1)^{k+m}\mathfrak{P}^{n,-\bar\sigma-n+2}_{mkj}(\cosh\theta). \tag{6}$$

We derive from integral representation (1) of Section 9.5.1 that

$$\overline{\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)} = \mathfrak{P}^{n\bar\sigma}_{kmj}(\cosh\theta). \tag{7}$$

From (6) and (7) we find that

$$\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta) = (-1)^{k+m}\mathfrak{P}^{n,-\sigma-n+2}_{mkj}(\cosh\theta). \tag{6'}$$

For representations $T^{n\sigma}$ of the principal unitary series one has $\sigma = -\bar\sigma - n + 2 = i\rho - \frac{n-2}{2}$, $\rho \in \mathbb{R}$. Therefore, for these values of $\sigma$ we have

$$\mathfrak{P}^{n\bar\sigma}_{kmj}(\cosh\theta) = \overline{\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)} = (-1)^{k+m}\mathfrak{P}^{n\sigma}_{mkj}(\cosh\theta). \tag{8}$$

If $\sigma$ is not an integer, then for the representations $T^{n\sigma}$ and $T^{n,-\sigma-n+2}$ of $SO_0(n-1,1)$ we have

$$Q(\sigma)T^{n\sigma}(g) = T^{n,-\sigma-n+2}(g)Q(\sigma),$$

where $Q(\sigma) \equiv Q^{\sigma\tau}$ is the intertwining operator. Setting $g = g'_{n-1}(\theta)$ and considering matrix elements of both sides of this relation, we obtain

$$\mathfrak{P}^{n,-\sigma-n+2}_{kmj}(x) = \frac{\Gamma(\sigma+k+n-2)\Gamma(-\sigma+m)}{\Gamma(-\sigma+k)\Gamma(\sigma+m+n-2)}\mathfrak{P}^{n\sigma}_{kmj}(x). \tag{9}$$

This relation can be analytically continued to integral values of the parameter $\sigma$. For representations of the principal unitary series, formula (9) takes the form

$$\mathfrak{P}^{n\bar{\sigma}}_{kmj}(x) = \frac{\Gamma(k-\bar{\sigma})\Gamma(m-\sigma)}{\Gamma(m-\bar{\sigma})\Gamma(k-\sigma)}\mathfrak{P}^{n\sigma}_{kmj}(x). \tag{10}$$

It follows from (9) that

$$\mathfrak{P}^{n,-\sigma-n+2}_{mmj}(x) = \mathfrak{P}^{n\sigma}_{mmj}(x)$$

and (10) implies that $\mathfrak{P}^{n\sigma}_{mmj}(x)$ is real for representations of the principal unitary series.

Formulas (6) and (9) lead to the relation

$$\mathfrak{P}^{n\sigma}_{kmj}(x) = (-1)^{k+m}\frac{\Gamma(\sigma+k+n-2)\Gamma(m-\sigma)}{\Gamma(k-\sigma)\Gamma(\sigma+m+n-2)}\mathfrak{P}^{n\sigma}_{mkj}(x). \tag{11}$$

Similar equality for $P^{n\ell}_{kmj}(x)$ is of the form

$$P^{n\ell}_{kmj}(x) = (-1)^{k+m}P^{n\ell}_{mkj}(x). \tag{12}$$

For $J^{n}_{kmj}(x)$ we have

$$J^{n}_{kmj}(x) = J^{n}_{mkj}(x). \tag{13}$$

Other properties of $J^{n}_{kmj}(z)$ are

$$\begin{aligned}\overline{J^{n}_{kmj}(z)} &= J^{n}_{mkj}(-\bar{z}) = (-1)^{k+m}J^{n}_{mkj}(\bar{z})\\ &= (-1)^{k+m}J^{n}_{kmj}(\bar{z}) = J^{n}_{kmj}(-\bar{z}).\end{aligned} \tag{14}$$

**9.5.6. The functions $\mathfrak{P}^{n\sigma}_{kmm}(x)$, $P^{n\ell}_{kmm}(x)$ and $J^{n}_{kmm}(x)$.** It follows from formula (1) of the preceding section that

$$\mathfrak{P}^{n\sigma}_{kmm}(\cosh\theta) = \mathfrak{P}^{n+2m,\sigma-m}_{k-m,0,0}(\cosh\theta),$$

where $m \le k$. Therefore,

$$\mathfrak{P}^{n\sigma}_{kmm}(\cosh\theta) =$$

$$= (-1)^m \frac{\Gamma(\sigma-m+1}{\Gamma(\sigma+1)} \left[ \frac{k!(k+m+n-3)\Gamma\left(m+\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}{(k-m)!(k+n-4)!\Gamma\left(m+\frac{n-2}{2}\right)\Gamma\left(\frac{n-1}{2}\right)} \right]^{1/2}$$

$$\times \sinh^{-m}\theta \mathfrak{P}^{n\sigma}_{k00}(\cosh\theta)$$

$$= \frac{\sqrt[4]{\pi}\Gamma(\sigma-m+1)}{\Gamma(\sigma-k+1)} \left[ \frac{(k+m+n-4)!(2k+n-3)\Gamma\left(m+\frac{n-1}{2}\right)}{(k-m)!\Gamma\left(m+\frac{n-2}{2}\right)} \right]^{\frac{1}{2}}$$

$$\times \sinh^{-m-\frac{n-3}{2}}\theta \mathfrak{P}^{-k-\frac{n-3}{2}}_{\sigma+\frac{n-3}{2}}(\cosh\theta). \quad (1)$$

In particular,

$$\mathfrak{P}^{n\sigma}_{mmm}(\cosh\theta)$$

$$= \mathfrak{P}^{n+2m,\sigma-m}_{000}(\cosh\theta) = \left(\frac{2}{\sinh\theta}\right)^{m+\frac{n-3}{2}} \Gamma\left(m+\frac{n-1}{2}\right) \mathfrak{P}^{-m-\frac{n-3}{2}}_{\sigma+\frac{n-3}{2}}(\cosh\theta). \quad (2)$$

For $P^{n\ell}_{kmm}(\cosh\varphi)$ we have

$$P^{n\ell}_{kmm}(\cosh\varphi) = P^{n+2m,\ell-m}_{k-m,00}(\cosh\varphi) = \frac{2^{k-m}\Gamma\left(k+\frac{n-2}{2}\right)}{\Gamma\left(m+\frac{n-2}{2}\right)}$$

$$\times \left[ \frac{(\ell-k)!(\ell-m)!(k+m+n-4)!(2m+n-3)!(2k+n-3)}{(k-m)!(\ell+k+n-3)!(\ell+m+n-3)!} \right]^{1/2}$$

$$\times \sin^{k-m}\varphi \sin\varphi C^{k+\frac{n-2}{2}}_{\ell-k}(\cos\varphi). \quad (3)$$

Since $C^p_0(\cos\varphi) = 1$, then for $\ell = k$ we obtain

$$P^{n\ell}_{\ell mm}(\cos\varphi) =$$

$$= \frac{2^{\ell-m}\Gamma\left(\ell+\frac{n-2}{2}\right)}{\Gamma\left(m+\frac{n-2}{2}\right)} \left[ \frac{(2m+n-3)!}{(\ell+m+n-3)!(2\ell+n-4)!} \right]^{1/2} \sin^{\ell-m}\varphi. \quad (4)$$

In particular,

$$P^{n\ell}_{\ell\ell\ell}(\cos\varphi) = 1. \quad (5)$$

We also have

$$P^{n\ell}_{mmm}(\cos\varphi) =$$

$$= P^{n+2m,\ell-m}_{000}(\cos\varphi) = \frac{(2m+n-3)!(\ell-m)!}{(\ell+m+n-3)!} C^{m+(n-2)/2}_{\ell-m}(\cos\varphi). \quad (6)$$

It follows from formula (3) of Section 9.5.5 that

$$J^n_{kmm}(x) = J^{n+2m}_{k-m,0,0}(x), \tag{7}$$

where $k \geq m$. Therefore,

$$\begin{aligned} J^n_{kmm}(x) &= \frac{\Gamma\left(m+\frac{n-1}{2}\right)}{\Gamma\left(k+\frac{n-1}{2}\right)}\left[\frac{k+m+n-4)!(2k+n-3)}{(k-m)!(n+2m-3)!}\right]^{1/2}\left(\frac{-ix}{2}\right)J^{n+2k}_{000}(x) \\ &= \left[\frac{(k+m+n-4)!k!\Gamma\left(m+\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}{(k-m)!(k+n-4)!\Gamma\left(m+\frac{n-2}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}\right]^{1/2} i^m\frac{J^n_{k00}(x)}{x^m} \\ &= \left[\frac{\sqrt{\pi}(2k+n-3)(k+m+n-4)!\Gamma\left(m+\frac{n-1}{2}\right)}{(k-m)!\Gamma\left(m+\frac{n-2}{2}\right)}\right]^{1/2} \\ &\qquad \times (-i)^{k-m}\frac{J_{k+(n-3)/2}(x)}{x^{k+(n-3)/2}}. \end{aligned} \tag{8}$$

In particular,

$$J^n_{mmm}(x) = J^{n+2m}_{000}(x) = \Gamma\left(m+\frac{n-1}{2}\right)\left(\frac{x}{2}\right)^{-m-(n-3)/2} J_{m+(n-3)/2}(x). \tag{9}$$

**9.5.7. The expression for $P^{n\ell}_{kmj}(0)$ in terms of Wilson polynomials.** It follows from formula (3) of Section 9.5.2 that

$$\begin{aligned} P^{n\ell}_{kmj}(0) &= t^{n\ell}_{kmj}\left(g_{n-1}\left(\frac{\pi}{2}\right)\right) \\ &= N\int_{-1}^{1}(1-x^2)^{k+\frac{n-4}{2}}\frac{d^{k-j}}{dx^{k-j}}\left[(-ix)^{\ell-j}C^{j+\frac{n-3}{2}}_{m-j}\left(\frac{1}{x}\right)\right]dx, \end{aligned} \tag{1}$$

where

$$N = \frac{2^{j-m-1}\Gamma\left(j+\frac{n-3}{2}\right)}{i^{k-m}\sqrt{\pi}\Gamma\left(m+\frac{n-3}{2}\right)} \times \left[\frac{(\ell+k+n-3)!(\ell-k)!(m+j+n-4)!(k-j)!(2m+n-3)(2k+n-3)}{(\ell+m+n-3)!(\ell-m)!(m-j)!(k+j+n-4)!}\right]^{1/2}. \tag{2}$$

Formula (4) of Section 9.5.5 shows that for odd values of $\ell+k+m+j$ we have $P^{n\ell}_{kmj}(0) = 0$.

In order to evaluate the integral from (1) we represent the Gegenbauer polynomial as

$$C_{m-j}^{j+\frac{n-3}{2}}\left(\frac{1}{x}\right) = \sum_{r=0}^{[(m-j)/2]} \frac{(-1)^r \Gamma\left(m-r+\frac{n-3}{2}\right)}{r!(m-j-2r)!\Gamma\left(j+\frac{n-3}{2}\right)}\left(\frac{2}{x}\right)^{m-j-2r},$$

carry out term-by-term differentiation and then term-by-term integration. We obtain that $P_{kmj}^{n\ell}(0) = 0$ for odd values of $\ell + j$ and

$$P_{kmj}^{n\ell}(0) = B\ {}_4F_3\left(\begin{matrix} \frac{-m+j}{2}, & \frac{-m+j+1}{2}, \ \frac{\ell-m+1}{2}, & \frac{\ell-m+2}{2} \\ -m-\frac{n-5}{2}, & \frac{\ell+j-m-k+2}{2}, & \frac{\ell+j-m+k+n-1}{2} \end{matrix} \middle|\ 1\right) \tag{3}$$

for even values of $\ell + j$ such that $\ell + j - m - k \geq 0$. Here

$$B = \frac{(-1)^{(\ell-j-k+m)/2}}{2^{\ell-m-k+j+1}} \frac{\Gamma\left(m+\frac{n-3}{2}\right)\Gamma\left(k+\frac{n-2}{2}\right)}{\Gamma\left(\frac{\ell-m-k+j}{2}+1\right)\Gamma\left(\frac{\ell+j-k+m+n-1}{2}\right)\Gamma\left(m+\frac{n-2}{2}\right)}$$
$$\times\left[\frac{(m+j+n-4)!(\ell+k+n-3)!(\ell-k)!(\ell-m)!(k-j)!(2k+n-3)(2m+n-3)}{(m-j)!^2(\ell+m+n-3)!(k+j+n-4)!}\right]^{1/2}.$$

Since one of the numbers $(-m+j)/2$, $(-m+j+1)/2$ is a negative integer, then the hypergeometric series in (3) is finite. Let us replace the sum in the expression for the hypergeometric series in (3) by an inverse one. Then for even values of $m-j$ we obtain

$$P_{kmj}^{n\ell}(0) = A\ {}_4F_3\left(\begin{matrix} \frac{-m+j}{2}, & \frac{m+j+n-3}{2}, \ \frac{-\ell+k}{2}, & -\frac{\ell+k+n-3}{2} \\ \frac{1}{2}, & \frac{-\ell+j}{2}, & \frac{-\ell+j+1}{2} \end{matrix} \middle|\ 1\right) \tag{4}$$

where

$$A = N\frac{(-i)^{\ell-j}(-1)^{(m-j)/2}(\ell-j)!\Gamma\left(k+\frac{n-2}{2}\right)\Gamma\left(\frac{\ell-k+1}{2}\right)\Gamma\left(\frac{m+j+n-3}{2}\right)}{\sqrt{\pi}(\ell-k)!\left(\frac{m-j}{2}\right)!\Gamma\left(j+\frac{n-3}{2}\right)\Gamma\left(\frac{\ell+k+n-1}{2}\right)}$$

and $N$ is given by (2). If $m-j$ is odd then

$$P_{kmj}^{n\ell}(0) = A'{}_4F_3\left(\begin{matrix} \frac{-m+j+1}{2}, & \frac{j+m+n-2}{2}, \ \frac{-\ell+k+1}{2}, & -\frac{\ell+k+n-4}{2} \\ \frac{3}{2}, & \frac{-\ell+j+1}{2}, & \frac{-\ell+j+2}{2} \end{matrix} \middle|\ 1\right) \tag{5}$$

where

$$A' = N\frac{4(-i)^{\ell-j}(-1)^{(m-j-1)/2}(\ell-j-1)!\Gamma\left(k+\frac{n-2}{2}\right)\Gamma\left(\frac{\ell-k}{2}\right)\Gamma\left(\frac{m+j+n-2}{2}\right)}{\sqrt{\pi}(\ell-k-1)!\left(\frac{m-j-1}{2}\right)!\Gamma\left(j+\frac{n-3}{2}\right)\Gamma\left(\frac{\ell+k+n-2}{2}\right)}.$$

By means of recurrence relations for $P^{n\ell}_{kmj}(0)$ and ${}_4F_3(\ldots;1)$ one can show that formulas (4) and (5) are also valid in the case when the condition $\ell+j-m-k<0$ is fulfilled.

Making use of formula (1) of Section 8.5.5, we can express the hypergeometric functions from (4) and (5) in terms of Wilson polynomials. Set $n=(m-j)/2$, $t=(2k+n-3)/4$, $a=-(2\ell+n-3)/4$. Then from (4) we have

$$P^{n\ell}_{kmj}(0) = \\ = Dp_{(m-j)/2}\left(\frac{t^2}{4};-\frac{2\ell+n-3}{4},\frac{2\ell+n-1}{4},\frac{2j+n-3}{4},\frac{2j+n-1}{4}\right), \quad (6)$$

where

$$D = A\frac{\sqrt{\pi}2^{m-j}(\ell-m)!}{(\ell-j)!\Gamma\left(\frac{m-j+1}{2}\right)}.$$

Formula (5) leads to

$$P^{n\ell}_{kmj}(0) = \\ = D'p_{(m-j-1)/2}\left(\frac{t^2}{4};-\frac{2\ell+n-5}{4},\frac{2\ell+n+1}{4},\frac{2j+n-3}{4},\frac{2j+n-1}{4}\right), \quad (7)$$

where

$$t=\frac{2k+n-3}{4}, \qquad D' = A'\frac{\sqrt{\pi}2^{m-j-1}(\ell-m)!}{(\ell-j-1)!\Gamma\left(\frac{m-j+2}{2}\right)}.$$

The unitarity of the matrix $(P^{n\ell}_{kmj}(0))$, that is, the equality

$$\sum_{k=j}^{\ell} P^{n\ell}_{kmj}(0)P^{n\ell}_{krj}(0) = \delta_{mr}$$

gives the orthogonality relation for Wilson polynomials (6) and (7).

The relation

$$g_{n-1}\left(-\frac{\pi}{2}\right)g_{n-2}(\theta)g_{n-1}\left(\frac{\pi}{2}\right) = g_{n-2,n}(\theta) \quad (8)$$

holds in the group $SO(n)$. Since $P^{n\ell}_{kmj}(0)=0$ for odd values of $\ell+k+m+j$, then by virtue of formula (4) of Section 9.5.5 we have

$$t^{n\ell}_{kmj}\left(g_{n-1}\left(-\frac{\pi}{2}\right)\right) = t^{n\ell}_{kmj}\left(g_{n-1}\left(\frac{\pi}{2}\right)\right).$$

Therefore, (8) implies that

$$\sum_{m=\max(j,j')}^{\ell} P^{n\ell}_{kmj}(0)t^{n-1,m}_{jj's}(g_{n-2}(\theta))P^{n\ell}_{mrj'}(0)=t^{n\ell}_{(k,j)(r,j')s}(g_{n-2,n}(\theta)), \qquad (9)$$

where the summation is over the values of $m$ which have the same evenness as $\ell-k+j$,

$$t^{n\ell}_{(k,j)(r,j')s}(g_{n-2,n}(\theta))=t^{n\ell}_{M'M}(g_{n-2,n}(\theta)) \qquad (10)$$

and $M=(r,j',s,N)$, $M'=(k,j,s,N)$.

From the relation

$$g_{n-2}\left(\frac{\pi}{2}\right)g_{n-1}(\theta)g_{n-2}\left(-\frac{\pi}{2}\right)=g_{n-2,n}(\theta)$$

we derive that

$$\sum_{j=s}^{\min(k,m)} P^{n-1,k}_{j'js}(0)t^{n\ell}_{kmj}(g_{n-1}(\theta))P^{n-1,m}_{jj''s}(0)=t^{n\ell}_{(k,j')(m,j'')s}(g_{n-2,n}(\theta)). \qquad (11)$$

Since

$$g_{n-3}\left(\frac{\pi}{2}\right)g_{n-2,n}(\theta)g_{n-3}\left(-\frac{\pi}{2}\right)=g_{n-3,n}(\theta),$$

then

$$\sum_{s=t}^{\min(j,i)} P^{n-2,j}_{s'st}(0)t^{n\ell}_{(k,j)(m,i)s}(g_{n-2,n}(\theta))P^{n-2,i}_{ss''t}(0)=t^{n\ell}_{(k,j,s')(m,i,s'')t}(g_{n-3,n}(\theta)). \quad (12)$$

In the same way one can find matrix elements of the operator $T^{n\ell}(g_{pn}(\theta))$ for arbitrary $p$.

We also have

$$\sum_{j=s}^{\min(k,m)} P^{n-1,k}_{j'js}(0)t^{n\sigma}_{kmj}(g'_{n-1}(\theta))P^{n-1,m}_{jj''s}(0)=t^{n\sigma}_{(k,j')(m,j'')s}(g'_{n-2,n}(\theta))$$

and so on.

**9.5.8. Boundedness properties.** Let $\sigma=i\rho-\frac{n-2}{2}$, $\rho\in\mathbb{R}$. Since $T^{n\sigma}$ is a unitary representation, then

$$\sum_{m=j}^{\infty}\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)\overline{\mathfrak{P}^{n\sigma}_{smj}(\cosh\theta)}=\delta_{ks}. \qquad (1)$$

For $k = s$ this formula takes the form $\sum_{m=j}^{\infty} |\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)|^2 = 1$. Consequently,

$$\left|\mathfrak{P}^{i\rho-(n-2)/2}_{kmj}(\cosh\theta)\right| \le 1 \tag{2}$$

and the equality sign appears when $k = m$ and $\theta = 0$.

In the same way one shows that

$$\left|P^{n\ell}_{kmj}(\cos\theta)\right| \le 1 \tag{3}$$

and the equality is fulfilled when $k = m$ and $\theta = 0$.

We now consider representations $T^{n\sigma}$ with $\sigma \in \mathbb{C}$. It follows from the results of Section 9.2.8 that

$$T^{n,-\bar{\sigma}-n+2}(g_{n-1}(\theta))T^{n\sigma}(g_{n-1}(\theta))^* = I. \tag{4}$$

Writing down this relation for matrix elements of the operators, we obtain

$$\sum_{m=j}^{\infty} \mathfrak{P}^{n,-\sigma-n+2}_{kmj}(\cosh\theta)\mathfrak{P}^{n\sigma}_{smj}(\cosh\theta) = \delta_{ks}. \tag{5}$$

Applying equality (9) of Section 9.5.5 to (5), we have

$$\sum_{m=j}^{\infty} \frac{\Gamma(\sigma+k+n-2)\Gamma(m-\sigma)}{\Gamma(k-\sigma)\Gamma(\sigma+m+n-2)} \mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)\mathfrak{P}^{n\sigma}_{smj}(\cosh\theta) = \delta_{ks}.$$

In particular, for $k = s$ we obtain

$$\sum_{m=j}^{\infty} \frac{\Gamma(\sigma+k+n-2)\Gamma(m-\sigma)}{\Gamma(k-\sigma)\Gamma(\sigma+m+n-2)} \left[\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)\right]^2 = 1. \tag{6}$$

For real values of $\sigma$ the function $\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)$ is real-valued. Since the $\Gamma$-function takes positive values for positive values of the argument, then the coefficients in (6) are positive for $-n+2 < \sigma < 0$. Therefore, for these values of $\sigma$ we have

$$\frac{\Gamma(\sigma+k+n-2)\Gamma(m-\sigma)}{\Gamma(k-\sigma)\Gamma(\sigma+m+n-2)} \left|\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)\right|^2 \le 1,$$

that is,

$$\left|\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)\right| \le \left[\frac{\Gamma(k-\sigma)\Gamma(\sigma+m+n-2)}{\Gamma(\sigma+k+n-2)\Gamma(m-\sigma)}\right]^{1/2}. \tag{7}$$

This property is also fulfilled for real $\sigma$ and $k > \sigma$, $m > \sigma$.

In order to estimate the functions $\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)$, $\sigma\in\mathbf{C}$, we estimate the norms of the operators $T^{n\sigma}(g'_{n-1}(\theta))$. By the definition of a norm we have

$$\|T^{n\sigma}(g'_{n-1}(\theta))\| = \sup_{\|F\|=1} \|T^{n\sigma}(g'_{n-1}(\theta))F\|.$$

Therefore, by virtue of formulas (6) and (7) of Section 9.2.1,

$$\|T^{n\sigma}(g'_{n-1}(\theta))\|^2 \leq \sup_{\|F\|=1} \int_{S^{n-2}} |(\cosh\theta - \cos\varphi_{n-2}\sinh\theta)^{2\sigma}| \cdot |F(\varphi_1,\ldots,\varphi_{n-3},\varphi'_{n-2})|^2 d\xi, \quad (8)$$

where

$$\cos\varphi'_{n-2} = \frac{\cos\varphi_{n-2}\cosh\theta - \sinh\theta}{\cosh\theta - \cos\varphi_{n-2}\sinh\theta}$$

and the integration is carried out with respect to the spherical coordinates $\varphi_1,\varphi_2,\ldots,\varphi_{n-2}$. Going over from the integration with respect to $\varphi_1,\ldots,\varphi_{n-2}$ to the integration with respect to $\varphi_1,\ldots,\varphi_{n-3},\varphi'_{n-2}$ we obtain

$$\|T^{n\sigma}(g'_{n-1}(\theta))\|^2 \leq \sup_{\|F\|=1} \int_{S^{n-2}} |(\cosh\theta - \cos\varphi_{n-2}\sinh\theta)^{-2\sigma-n+2}| \times |F(\varphi_1,\ldots,\varphi_{n-3},\varphi_{n-2})|^2 d\xi \leq [M_\sigma(\theta)]^2,$$

where

$$M_\sigma(\theta) = \sup_\varphi \left|(\cosh\theta - \cos\varphi\sinh\theta)^{-\sigma-(n-2)/2}\right|.$$

Since

$$\sup_\varphi \left|(\cosh\theta - \cos\varphi\sinh\theta)^{-\sigma-(n-2)/2}\right| \leq \max\left(\exp\left|\sigma+\frac{n-2}{2}\right|\theta, \exp\left(-\left|\sigma+\frac{n-2}{2}\right|\theta\right)\right),$$

$$M_\sigma(\theta) = M_{\mathrm{Re}\,\sigma}(\theta),$$

then

$$M_\sigma(\theta) \leq \exp\left(\pm\left|\mathrm{Re}\,\sigma + \frac{n-2}{2}\right|\theta\right). \quad (9)$$

Therefore, $\|T^{n\sigma}(g'_{n-1}(\theta))\| \leq \exp\left(\pm\left|\mathrm{Re}\,\sigma+\frac{n-2}{2}\right|\theta\right)$.

For any functions $F_1$ and $F_2$ from $\mathfrak{L}^2(S^{n-2})$ with the unit norm the inequality

$$|(T^{n\sigma}(g)F_1, F_2)| \leq \|T^{n\sigma}(g)\| \quad (10)$$

holds. It follows from (9) and (10) that

$$\left|\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)\right| \le \exp\left(\pm\left|\operatorname{Re}\sigma + \frac{n-2}{2}\right|\theta\right) \tag{11}$$

if $\sigma \in \mathbf{C}$.

Since the representations $T^{nR}$ of $ISO(n-1)$ are unitary when $R = i\rho$, $\rho \in \mathbf{R}$, then for real $x$ we have

$$\sum_{m=j}^{\infty} J^n_{kmj}(x)\overline{J^n_{smj}(x)} = \delta_{ks}. \tag{12}$$

Hence, $|J^n_{kmj}(x)| \le 1$ for real $x$, and the equality is fulfilled when $k = m$ and $x = 0$.

Let now $\operatorname{Im} R \ne 0$. Then for $r \in \mathbf{R}$ we have $|e^{Rr\cos\varphi}| \le e^{r \operatorname{Re} R}$. Therefore, formula (3) of Section 9.2.4 implies that

$$\|T^{nR}(g_r)\| \le e^{r\operatorname{Re} R}. \tag{13}$$

But then

$$|J^n_{kmj}(x)| \le e^{-\operatorname{Re} ix}.$$

The symmetry relation (5) of Section 9.5.5 allows us to write down the estimate

$$|J^n_{kmj}(x)| \le \exp|\operatorname{Im} x|. \tag{14}$$

**9.5.9. The orthogonality relation for $P^{n\ell}_{kmj}(x)$.** Since the dimensionality of the representation $T^{n\ell}$ of $SO(n)$ is equal to $(\ell+n-3)!(2\ell+n-2)/(\ell!(n-2)!)$, then the orthogonality relation for matrix elements of $T^{n\ell}$ is of the form

$$\int t^{n\ell}_{(k,K)(m,M)}(g)\overline{t^{np}_{(q,Q)(r,R)}(g)}dg = \frac{\ell!(n-2)!\delta_{\ell p}\delta_{kq}\delta_{mr}\delta_{KQ}\delta_{MR}}{(\ell+n-3)!(2\ell+n-2)}. \tag{1}$$

We represent the element $g$ as $g = hg_{n-1}(\theta)h'$, $h, h' \in SO(n-1)$ (see Section 9.1.5) and the measure $dg$ as

$$dg = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})}\sin^{n-2}\theta d\theta dh dh'. \tag{2}$$

For $t^{n\ell}_{(k,K)(m,M)}(g)$ we have

$$t^{n\ell}_{(k,K)(m,M)}(g) = \sum_{j,K''} t^{n-1,k}_{(j',K')(j,K'')}(h)t^{n\ell}_{kmj}(g_{n-1}(\theta))t^{n-1,m}_{(j,K'')(j'',M')}(h'), \tag{3}$$

where $K = (j', K')$, $M = (j'', M')$. Substitute into (1) expresion (2) for $dg$, expression (3) for $t^{n\ell}_{(k,K)(m,M)}(g)$ and analogous expression for $t^{np}_{(q,Q)(r,R)}(g)$ and then integrate with respect to $h$ and $h'$. By virtue of the orthogonality relation for matrix elements of representations of the subgroup $SO(n-1)$ we obtain

$$\sum_{j=0}^{\min(k,m)} \frac{(2j+n-4)(j+n-5)!}{j!} \int_0^\pi P^{n\ell}_{kmj}(\cos\theta) P^{np}_{kmj}(\cos\theta) \sin^{n-2}\theta d\theta$$
$$= \frac{\Gamma(\frac{n-3}{2})\sqrt{\pi}}{\Gamma(\frac{n-2}{2})} \frac{(2m+n-3)(2k+n-3)(m+n-4)!(k+n-4)!\ell!}{m!k!(2\ell+n-2)(\ell+n-3)!}\delta_{\ell p}. \quad (4)$$

This is the orthogonality relation for $P^{n\ell}_{kmj}(t)$.

Note that for $n=4$ the left hand side of (4) has to be replaced by

$$\sum_{j=-\min(k,m)}^{\min(k,m)} \int_0^\pi P^{4,p}_{kmj}(\cos\theta) P^{4,\ell}_{kmj}(\cos\theta)\sin^2\theta d\theta.$$

**9.5.10. Addition theorems and product formulas.** For the one-parameter subgroups $\{g_{n-1}(\varphi)\}$ and$\{g_{n-2}(\psi)\}$ of the group $SO(n)$ the relation

$$g_{n-1}(\alpha)g_{n-2}(\beta)g_{n-1}(\gamma) = g_{n-2}(\varphi)g_{n-1}(\theta)g_{n-2}(\psi). \quad (1)$$

holds, where

$$\left.\begin{aligned} \cos\theta &= \cos\alpha\cos\gamma - \sin\alpha\sin\gamma\cos\beta, \\ \sin\theta\sin\psi &= \sin\alpha\sin\beta, \\ \sin\theta\sin\varphi &= \sin\beta\sin\gamma \end{aligned}\right\} \quad (2)$$

and $\varphi$ and $\psi$ are in the corresponding quadrants. Writing (1) for operators of the representation $T^{n\ell}$ and passing to the matrix elements, we obtain the addition theorem for the functions $P^{n\ell}_{kmj}(x)$:

$$\sum_{s=\max(p,r)}^{\ell} P^{n\ell}_{ksr}(\cos\alpha) P^{n-1,s}_{rpj}(\cos\beta) P^{n\ell}_{smp}(\cos\gamma)$$
$$= \sum_{q=j}^{\min(m,k)} P^{n-1,k}_{rqj}(\cos\varphi) P^{n\ell}_{kmq}(\cos\theta) P^{n-1,m}_{qpj}(\cos\psi), \quad (3)$$

where the angles are connected by relation (2).

Assume that $n \geq 5$. We multiply both sides of (3) by

$$\frac{(2j+n-5)(j+n-6)!}{j!} P^{n-1,s}_{rpj}(\cos\beta)\sin^{n-3}\beta,$$

integrate with respect to $\beta$ from 0 to $\pi$ and sum with respect to $j$ from 0 to $\min(r,p)$. By virtue of the orthogonality relation (4) of Section 9.5.9 we find the product formula for $P^{n\ell}_{kmj}(x)$:

$$AP^{n\ell}_{ksr}(\cos\alpha)P^{n\ell}_{smp}(\cos\gamma) = \sum_{j=0}^{\min(r,p)} \sum_{q=j}^{\min(k,m)} \frac{(2j+n-5)(j+n-6)!}{j!}$$

$$\times \int_0^\pi P^{n\ell}_{kmq}(\cos\theta)P^{n-1,k}_{rqj}(\cos\varphi)P^{n-1,m}_{qpj}(\cos\psi)P^{n-1,s}_{rpj}(\cos\beta)\sin^{n-3}\beta d\beta, \quad (4)$$

where

$$A = \frac{\sqrt{\pi}\Gamma\left(\frac{n-4}{2}\right)(2p+n-4)(2r+n-4)(p+n-5)!(r+n-5)!s!}{\Gamma\left(\frac{n-3}{2}\right)(2s+n-3)(s+n-4)!r!p!} \quad (5)$$

and the angles are connected by relations (2).

For the functions $\mathfrak{P}^{n\sigma}_{kmj}(x)$ the addition formula has the form

$$\sum_{s=\max(r,p)}^{\infty} \mathfrak{P}^{n\sigma}_{ksr}(\cosh\theta_1)P^{n-1,s}_{rpj}(\cos\varphi)\mathfrak{P}^{n\sigma}_{smp}(\cosh\theta_2)$$

$$= \sum_{q=j}^{\min(k,m)} P^{n-1,k}_{rpj}(\cos\alpha)\mathfrak{P}^{n\sigma}_{kmq}(\cosh\theta)P^{n-1,m}_{qpj}(\cos\beta), \quad (6)$$

where the angles are connected by the relations

$$\left.\begin{aligned} \cosh\theta &= \cosh\theta_1\cosh\theta_2 + \sinh\theta_1\sinh\theta_2\cos\varphi, \\ \sinh\theta\sin\beta &= \sinh\theta_1\sin\varphi, \\ \sinh\theta\sin\alpha &= \sinh\theta_2\sin\varphi. \end{aligned}\right\} \quad (7)$$

The addition theorem for $\mathfrak{P}^{n\sigma}_{kmj}(x)$ leads to the product formula

$$A\mathfrak{P}^{n\sigma}_{ksr}(\cosh\theta_1)\mathfrak{P}^{n\sigma}_{smp}(\cosh\theta_2) = \sum_{j=0}^{\min(r,p)} \sum_{q=j}^{\min(k,m)} \frac{(2j+n-5)(j+n-6)!}{j!}$$

$$\times \int_0^\pi \mathfrak{P}^{n\sigma}_{kmq}(\cosh\theta)P^{n-1,k}_{rqj}(\cos\alpha)P^{n-1,m}_{qpj}(\cos\beta)P^{n-1,s}_{rpj}(\cos\varphi)\sin^{n-3}\varphi d\varphi, \quad (8)$$

where $A$ is given by (5) and the angles are connected by (7).

The addition formula for the functions $J^n_{kmj}(x)$ is of the form

$$\sum_{s=\max(t,p)}^{\infty} J^n_{kst}(r_1)P^{n-1,s}_{tpj}(\cos\varphi)J^n_{smp}(r_2) = \sum_{q=j}^{\min(k,m)} P^{n-1,k}_{tqj}(\cos\psi)J^n_{kmq}(r)P^{n-1,m}_{qpj}(\cos\theta), \quad (9)$$

where the parameters $r_1$, $r_2$, $r$, $\varphi$, $\psi$, $\theta$ are connected by the relations

$$\left.\begin{aligned} r\cos\psi &= r_2\cos\varphi + r_1, \\ r\sin\psi &= r_2\sin\varphi, \\ \psi+\theta &= \begin{cases} \varphi & \text{if } \psi\le\varphi, \\ \varphi+2\pi & \text{if } \psi>\varphi. \end{cases} \end{aligned}\right\} \quad (10)$$

The product formula for $J^n_{kmj}(x)$ follows from (9):

$$AJ^n_{kst}(r_1)J^n_{smp}(r_2) = \sum_{j=0}^{\min(t,p)} \sum_{q=j}^{\min(k,m)} \frac{(2j+n-5)(j+n-6)!}{j!}$$
$$\times \int_0^\pi J^n_{kmq}(r)P^{n-1,k}_{tqj}(\cos\psi)P^{n-1,m}_{qpj}(\cos\theta)P^{n-1,s}_{tpj}(\cos\varphi)\sin^{n-3}\varphi d\varphi, \quad (11)$$

where the parameters are connected by (10) and

$$A = (t-5+n)!s!\frac{\sqrt{\pi}\,\Gamma\left(\frac{n-4}{2}\right)(2p+n-4)(2t+n-4)(p+n-5)!}{\Gamma\left(\frac{n-3}{2}\right)(2s+n-3)t!p!(s+n-4)!}. \quad (12)$$

**9.5.11. Generating functions.** Formula (1) of Section 9.5.1 and the orthogonality relation for Gegenbauer polynomials imply the relation

$$\left[\frac{(m-j)!(2m+n-3)}{(m+j+n-4)!}\right]^{1/2}(\cosh\theta - t\sinh\theta)^{\sigma-j}C^{j+(n-3)/2}_{m-j}\left(\frac{t\cosh\theta-\sinh\theta}{\cosh\theta-t\sinh\theta}\right)$$
$$= \sum_{k=j}^{\infty}\left[\frac{(k-j)!(2k+n-3)}{(k+j+n-4)!}\right]^{1/2}\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)C^{j+(n-3)/2}_{k-j}(t). \quad (1)$$

The left hand side of this relation is the generating function for $\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)$ with respect to Gegenbauer polynomials.

For $t = 1, 0, -1$ we derive from (1) the summation formulas

$$\sum_{k=j}^{\infty}\left[\frac{(k+j+n-4)!(2k+n-3)}{(k-j)!}\right]^{1/2}\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)$$
$$=\left[\frac{(m+j+n-4)!(2m+n-3)}{(m-j)!}\right]^{1/2}e^{(j-\sigma)\theta}, \quad (2)$$

$$\sum_{k=j}^{\infty}(-1)^{k-m}\left[\frac{(k+j+n-4)!(2k+n-3)}{(kk-j)!}\right]^{1/2}\mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta)$$
$$=\left[\frac{(m+j+n-4)!(2m+n-3)}{(m-j)!}\right]^{1/2}e^{(\sigma-j)\theta}, \quad (3)$$

$$\sum_{s=0}^{\infty}\frac{(-1)^s\Gamma\left(j+s+\frac{n-3}{2}\right)}{s!\Gamma\left(j+\frac{n-3}{2}\right)}\left[\frac{(2s)!(4s+2j+n-3)}{(2s+2j+n-4)!}\right]^{1/2}\mathfrak{P}^{n\sigma}_{j+2s,m,j}(\cosh\theta)$$
$$=\left[\frac{(m-j)!(2m+n-3)}{(m+j+n-4)!}\right]^{1/2}\cosh^{\sigma-j}\theta C^{j+(n-3)/2}_{m-j}(-\tanh\theta). \quad (4)$$

It follows from formula (3) of Section 9.5.1 and from the orthogonality relation for Gegenbauer polynomials that

$$i^m\left[\frac{(m-j)!(2m+n-3)}{(\ell-m)!(\ell+m+n-3)!(m+j+n-4)!}\right]^{1/2}(\cos\varphi-it\sin\varphi)^{\ell-j}$$
$$\times C^{j+(n-3)/2}_{m-j}\left(\frac{t\cos\varphi-i\sin\varphi}{\cos\varphi-it\sin\varphi}\right)$$
$$=\sum_{k=j}^{\ell}i^k\left[\frac{(k-j)!(2k+n-3)}{(\ell-k)!(\ell+k+n-3)!(k+j+n-4)!}\right]^{1/2}P^{n\ell}_{kmj}(\cos\varphi)C^{j+(n-3)/2}_{k-j}(t). \quad (5)$$

The left hand side of this relation is a generating function for $P^{n\ell}_{kmj}(\cos\varphi)$ with respect to Gegenbauer polynomials.

For $t = 1, 0, -1$ we obtain from (5) the summation formulas

$$i^m\left[\frac{(m+j+n-4)!(2m+n-3)}{(\ell-m)!(m-j)!(\ell+m+n-3)!}\right]^{1/2}e^{i(j-\ell)\varphi}$$
$$=\sum_{k=j}^{\ell}i^k\left[\frac{(k+j+n-4)!(2k+n-3)}{(\ell-k)!(k-j)!(\ell+k+n-3)!}\right]^{1/2}P^{n\ell}_{kmj}(\cos\varphi), \quad (6)$$

$$(-i)^m \left[\frac{(m+j+n-4)!(2m+n-3)}{(\ell-m)!(m-j)!(\ell+m+n-3)!}\right]^{1/2} e^{i(\ell-j)\varphi}$$
$$= \sum_{k=j}^{\ell} (-i)^k \left[\frac{(k+j+n-4)!(2k+n-3)}{(\ell-k)!(k-j)!(\ell+k+n-3)!}\right]^{1/2} P^{n\ell}_{kmj}(\cos\varphi), \quad (7)$$

$$i^m \left[\frac{(m-j)!(2m+n-3)}{(\ell-m)!(\ell+m+n-3)!(m+j+n-4)!}\right]^{1/2} \cos^{\ell-j}\varphi C^{j+(n-3)/2}_{m-j}(-i\tan\varphi)$$
$$= \sum_{k=j}^{\ell}{}' i^k \frac{(-1)^{\frac{k-j}{2}}\Gamma\left(\frac{n+k+j-3)}{2}\right)}{\Gamma\left(\frac{k-j}{2}+1\right)\Gamma\left(\frac{n-3}{2}+j\right)} \left[\frac{(k+j+n-4)!(2k+n-3)}{(\ell-k)!(k-j)!(\ell+k+n-3)!}\right]^{1/2}$$
$$\times P^{n\ell}_{kmj}(\cos\varphi), \quad (8)$$

where the prime means that the summation is over the values of $k$, which have the same evenness as $j$ does.

For $\varphi = \frac{\pi}{2}$ equalities (6)–(8) lead to the summation formulas for Wilson polynomials (6) and (7) of Section 9.5.7.

From formula (5) of Section 9.5.1 and from the orthogonality relation for Gegenbauer polynomials we derive

$$\left[\frac{(2m+n-3)(m-j)!}{(m+j+n-4)!}\right]^{1/2} e^{-ixt} C^{j+(n-3)/2}_{m-j}(t)$$
$$= \sum_{j=1}^{\infty} \left[\frac{(2k+n-3)(k-j)!}{(k+j+n-4)!}\right]^{1/2} J^n_{kmj}(x) C^{j+(n-3)/2}_{k-j}(t). \quad (9)$$

This equality means that the expression on the left hand side is a generating function for $J^n_{kmj}(x)$.

For $t = 1, -1, 0$ we obtain from (9) the relations

$$\left[\frac{(m+j+n-4)!(2m+n-3)}{(m-j)!}\right]^{1/2} e^{-ix}$$
$$= \sum_{k=j}^{\infty} \left[\frac{(k+j+n-4)!(2k+n-3)}{(k-j)!}\right]^{1/2} j^n_{kmj}(x), \quad (10)$$

$$(-1)^m \left[\frac{(m+j+n-4)!(2m+n-3)}{(m-j)!}\right]^{1/2} e^{ix}$$
$$= \sum_{k=j}^{\infty} (-1)^k \left[\frac{(k+j+n-4)!(2k+n-3)}{(k-j)!}\right]^{1/2} J^n_{kmj}(x), \quad (11)$$

$$(-1)^s \frac{(p+s-2)!}{s!} \left[\frac{(2s)!(2s+p-1)}{(2s+2p-3)!}\right]^{1/2} = \sum_{q=0}^{\infty}(-1)^q \frac{(p+q-2)!}{q!} \left[\frac{(2q)!(2q+p-1)}{(2q+2p-3)!}\right]^{1/2} J^{2p+1}_{2s,2q,0}(x), \quad (12)$$

$$(-1)^s \frac{(p+s-1)!}{s!} \left[\frac{(2s+1)!(2s+p)}{(2s+2p-2)!}\right]^{1/2} = \sum_{q=0}^{\infty} \frac{(-1)^q(q+p-1)!}{q!} \left[\frac{(2q+1)!(2q+p)}{(2q+2p-2)!}\right]^{1/2} J^{2p+1}_{2s+1,2q+1,0}(x). \quad (13)$$

**9.5.12. Characters of $T^{n\ell}$and the functions $P^{n\ell}_{kmj}(x)$.** Let us find the character $\chi_\ell(g) \equiv \operatorname{Tr} T^{n\ell}(g)$ of the representation $T^{n\ell}$ of $SO(n)$ at the element $g_{n-1}(\theta)$. We have

$$\chi_\ell(g_{n-1}(\theta)) = \sum_{(m,M)} t^{n\ell}_{(m,M)(m,M)}(g_{n-1}(\theta)), \quad (1)$$

where $M = (m_1, m_2, \ldots, m_{n-3})$, $m \geq m_1 \geq \cdots \geq m_{n-4} \geq |m_{n-3}|$. Since $t^{n\ell}_{(m,M)(m,M)}(g_{n-1}(\theta)) = t^{n\ell}_{mmm_1}(g_{n-1}(\theta))$ and the number of the sets of integers $m_2, m_3, \ldots, m_{n-3}$ such that $m_1 \geq m_2 \geq \cdots \geq |m_{n-3}|$ is equal to $(m_1+n-5)!(2m_1+n-4)/(m_1!(n-4)!)$, then

$$\chi_\ell(g_{n-1}(\theta)) = \sum_{m=0}^{\ell} \sum_{j=0}^{m} \frac{(j+n-5)!(2j+n-4)}{j!(n-4)!} P^{n\ell}_{mmj}(\cos\theta). \quad (2)$$

We can calculate $\chi_\ell(g_{n-1}(\theta))$ in another way. The rotation $g_{n-1}(\theta)$ is conjugate to $g_1(\theta)$. Therefore, $\chi_\ell(g_{n-1}(\theta)) = \chi_\ell(g_1(\theta))$. But the matrix $T^{n\ell}(g_1(\theta))$ is conjugate to the diagonal matrix with the functions $e^{\pm ik\theta}$ on the main diagonal. Every one of these functions is met on the diagonal as many times, as there are sets of integers $(m, m_1, \ldots, m_{n-4}, k)$, with $\ell \geq m \geq m_1 \geq \cdots \geq m_{n-4} \geq |k|$, exist. The number of these sets is equal to $(\ell+n-k-3)!/((\ell-k)!(n-3)!)$. Therefore,

$$\chi_\ell(g_{n-1}(\theta)) = \frac{(\ell+n-3)!}{\ell!(n-3)!} + 2\sum_{k=1}^{\ell} \frac{(\ell+n-k-3)!}{(\ell-k)!(n-3)!} \cos k\theta. \quad (3)$$

It follows from (2) and (3) that

$$\sum_{m=0}^{\ell} \sum_{j=0}^{m} \frac{(j+n-5)!(2j+n-4)}{j!(n-4)!} P^{n\ell}_{mmj}(\cos\theta) = \frac{(\ell+n-3)!}{\ell!(n-3)!} + 2\sum_{k=1}^{\ell} \frac{(\ell+n-k-3)!}{(\ell-k)!(n-3)!} \cos k\theta. \quad (4)$$

**9.5.13. The functions $J^n_{kmj}$ as the limit of $P^{n\ell}_{kmj}$ and of $\mathfrak{P}^{n\sigma}_{kmj}$.** The group $ISO(n-1)$ is obtained by passage to the limit from the groups $SO(n)$ and $SO_0(n-1,1)$. Therefore, the function $J^n_{kmj}$ is the limit of the functions $P^{n\ell}_{kmj}$ and $\mathfrak{P}^{n\sigma}_{kmj}$. In order to prove this we consider integral representation (3) of Section 9.5.1 for $P^{n\ell}_{kmj}$. Assume that in this formula $\varphi \to 0$ and $\ell \to \infty$ such that $\ell\varphi \to -x$. Let us pass to the limit under the integral sign. We have

$$(\cos\varphi - i\cos\theta\sin\varphi)^{\ell-j} = (\cos\varphi)^{\ell-j}\left[\left(1 - i\frac{\cos\theta\sin\varphi}{\cos\varphi}\right)^{-\frac{\cos\varphi}{i\cos\theta\sin\varphi}}\right]^{\frac{i\cos\theta\sin\varphi}{\cos\varphi}(j-\ell)}.$$

Hence,

$$\lim_{\substack{\varphi\to 0\\ \ell\to\infty\\ \varphi\ell\to -x}} (\cos\varphi - i\cos\theta\sin\varphi)^{\ell-j} = e^{ix\cos\theta}.$$

Since $\frac{\cos\theta\cos\varphi - i\sin\varphi}{\cos\varphi - i\cos\theta\sin\varphi} \to \cos\theta$, then

$$i^{m-k}\lim_{\substack{\varphi\to 0\\ \ell\to\infty\\ \ell\varphi\to -x}} P^{n\ell}_{kmj}(\cos\varphi) = J^n_{kmj}(x). \tag{1}$$

In the same way one proves that for $\sigma = i\rho - \frac{n-2}{2}$, $\rho\in\mathbb{R}$, the relation

$$(-1)^{k+m}\lim_{\substack{\theta\to 0\\ \rho\to\infty\\ \theta\rho\to x}} \mathfrak{P}^{n\sigma}_{kmj}(\cosh\theta) = J^n_{kmj}(x) \tag{2}$$

holds.

**9.5.14. Infinitesimal operators of the representations.** Let us differentiate both sides of the relation

$$T^{n\sigma}(g'_{n-1}(\theta_1+\theta_2)) = T^{n\sigma}(g'_{n-1}(\theta_1))T^{n\sigma}(g'_{n-1}(\theta_2))$$

with respect to $\theta_2$ and set $\theta_2 = 0$. We obtain

$$\frac{d}{d\theta}T^{n\sigma}(g'_{n-1}(\theta)) = T^{n\sigma}(g'_{n-1}(\theta))I'^{n\sigma}_{n-1,n},$$

where $I'^{n\sigma}_{n-1,n}$ is the infinitesimal operator corresponding to the one-parameter subgroup $\{g'_{n-1}(\theta)\}$ of hyperbolic rotations. In the matrix form this equality is rewritten as

$$\frac{d}{d\theta}t^{n\sigma}_{kmj}(g'_{n-1}(\theta)) = \sum_s t^{n\sigma}_{ksj}(g'_{n-1}(\theta))(I'^{n\sigma}_{n-1,n})_{smj}. \tag{1}$$

Let us use expression (1) of Section 9.5.1 for $t^{n\sigma}_{kmj}(g'_{n-1}(\theta))$. Differentiating it with respect to $\theta$ under the integration sign, we obtain two summands. The first one contains $C^{j+(n-3)/2}_{m-j}(\cos\varphi')$ and the second one contains $C^{j+(n-3)/2}_{m-j-1}(\cos\varphi')$, where

$$\cos\varphi' = \frac{\cos\varphi\cosh\theta - \sinh\theta}{\cosh\theta - \cos\varphi\sinh\theta}.$$

We apply the recurrence relation

$$2\alpha(1-x^2)C^{\alpha+1}_{n-1}(x) = (n+2\alpha)xC^{\alpha}_{n}(x) - (n+1)C^{\alpha}_{n+1}(x)$$

(see formula (11) of Section 6.7.6) to the second Gegenbauer polynomial, and the formula

$$xC^{\alpha}_{n}(x) = \frac{n+1}{2(n+\alpha)}C^{\alpha}_{n+1}(x) - \frac{2\alpha+n-1}{2(n+\alpha)}C^{\alpha}_{n-1}(x) \qquad (1')$$

(see formula (2) of Section 6.7.6) to $C^{j+(n-3)/2}_{m-j}(\cos\varphi')$. As a result, $\frac{d}{d\theta}t^{n\sigma}_{kmj}(g'_{n-1}(\theta))$ is expressed in terms of two summands, where the integral in the first summand contains $C^{j+(n-3)/2}_{m-j+1}(\cos\varphi')$ and the integral in the second one contains $C^{j+(n-3)/2}_{m-j-1}(\varphi')$. Applying to these summands formula (1) of Section 9.5.1, we find

$$\frac{d}{d\theta}t^{n\sigma}_{kmj}(g'_{n-1}(\theta)) = (-\sigma+m)\left[\frac{(m+j+n-3)(m-j+1)}{(2m+n-3)(2m+n-1)}\right]^{1/2} t^{n\sigma}_{k,m+1,j}(g'_{n-1}(\theta))$$
$$+(-\sigma-m-n+3)\left[\frac{(m+j+n-4)(m-j)}{(2m+n-3)(2m+n-5)}\right]^{1/2} t^{n\sigma}_{k,m-1,j}(g'_{n-1}(\theta)). \quad (2)$$

Comparing this formula with (1), we conclude that the infinitesimal operator $I'^{n\sigma}_{n-1,n}$ of the representation $T^{n\sigma}$ of $SO_0(n-1,1)$ acts upon the basis elements $\Xi^{n-1,m}_{M}$, $M=(m_1,\ldots,m_{n-3})$, of the space $\mathfrak{L}^2(S^{n-2})$ by the formula

$$I'^{n\sigma}_{n-1,n}\Xi^{n-1,m}_{M} = (m-\sigma)\left[\frac{(m+m_1+n-3)(m-m_1+1)}{(2m+n-3)(2m+n-1)}\right]^{1/2}\Xi^{n-1,m+1}_{M}$$
$$+(-\sigma-m-n+3)\left[\frac{(m+m_1+n-4)(m-m_1)}{(2m+n-3)(2m+n-5)}\right]^{1/2}\Xi^{n-1,m-1}_{M}. \quad (3)$$

The infinitesimal operators $I'^{n\sigma}_{jn}$ are obtained by commuting the operator $I'^{n\sigma}_{n-1,n}$ with the infinitesimal operators corresponding to elements of the subalgebra $so(n-1)$. For example,

$$[I^{n\sigma}_{n-2,n-1}, I'^{n\sigma}_{n-1,n}] = I'^{n\sigma}_{n-2,n}. \qquad (4)$$

In the same way one derives formulas for infinitesimal operators of the representations $T^{n\ell}$ and $T^{nR}$ of the groups $SO(n)$ and $ISO(n-1)$. Starting from formula (3) of Section 9.5.1, we obtain the expresion for the infinitesimal operator $I^{n\ell}_{n-1,n}$ of the representation $T^{n\ell}$ of $SO(n)$ (which corresponds to the one-parameter subgroup $\{g_{n-1}(\theta)\}$) in the orthonormal basis

$$\left\{\widetilde{\Xi}^{n-1,k}_K(\xi) \equiv \lambda_k^{-1}\Xi^{n-1,k}_K(\xi)\right\},\ K=(k_1,k_2,\dots,k_{n-3}),$$

of the space $\widetilde{\mathfrak{H}}^{n-1,k}$, where the coefficients $\lambda_k$ are given by formula (4) of Section 9.4.2. It has the form

$$I^{n\ell}_{n-1,n}\widetilde{\Xi}^{n-1,k}_K = \left[\frac{(\ell+k+n-2)(\ell-k)(k+k_1+n-3)(k-k_1+1)}{(2k+n-1)(2k+n-3)}\right]^{1/2}\widetilde{\Xi}^{n-1,k+1}_K$$
$$-\left[\frac{(\ell+k+n-3)(\ell-k+1)(k+k_1+n-4)(k-k_1)}{(2k+n-3)(2k+n-5)}\right]^{1/2}\widetilde{\Xi}^{n-1,k-1}_K. \quad (5)$$

For the infinitesimal operator $J^{nR}_{n-1}$ of the representation $T^{nR}$ of the group $ISO(n-1)$ (which corresponds to the one-parameter subgroup $\{g_r\}$) in the orthonormal basis $\{\Xi^{n-1,k}_K\}$ of the space $\mathfrak{L}^2(S^{n-2})$ we have

$$J^{nR}_{n-1}\Xi^{n-1,k}_K = R\left[\frac{(k+k_1+n-3)(k-k_1+1)}{(2k+n-1)(2k+n-3)}\right]^{1/2}\Xi^{n-1,k+1}_K$$
$$+R\left[\frac{(k+k_1+n-4)(k-k_1)}{(2k+n-3)(2k+n-5)}\right]^{1/2}\Xi^{n-1,k-1}_K, \quad (6)$$

where $K=(k_1,\dots,k_{n-3})$.

We have shown in Section 9.2.7 that the intertwining operator $Q^{\sigma\tau}$ for the representations $T^{n\sigma}$ and $T^{n\tau}$, $\tau=-\sigma-n+2$, of $SO_0(n-1,1)$ is diagonal in the basis $\{\Xi^{n-1,m}_M\}$. Let us find its diagonal elements $q_{\sigma m}$. It follows from the equality

$$Q^{\sigma\tau}T^{n\sigma}(g) = T^{n\tau}(g)Q^{\sigma\tau}, \quad g\in SO_0(n-1,1),$$

that

$$Q^{\sigma\tau}I'^{n\sigma}_{n-1,n} = I'^{n\tau}_{n-1,n}Q^{\sigma\tau}.$$

Let us apply both sides of this equality to the basis function $\Xi^{n-1,m}_M$. As a result, we obtain the recurrence formula

$$(m-\sigma)q_{\sigma,m+1} = (\sigma+m+n-2)q_{\sigma,m}$$

which implies that

$$q_{\sigma,m} = a(\sigma)\frac{\Gamma(\sigma+m+n-2)}{\Gamma(-\sigma+m)}, \quad (7)$$

where $\sigma \bar{\in} \mathbf{Z}$ and $a(\sigma)$ is a constant. By using the fact that $q_{\sigma,0} = (Q^{\sigma\tau}1)(\boldsymbol{\xi})$ is given by formula (3) of Section 9.2.7, we find the expression for $a(\sigma)$:

$$a(\sigma) = \frac{2^{-\sigma-1}\Gamma\left(-\sigma - \frac{n-2}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\Gamma(\sigma+n-2)\Gamma(-\sigma-n+3)}.$$

It follows from (2) that

$$(Q^{\sigma\tau}\Xi_M^{n-1,m})(\boldsymbol{\xi}) = \frac{1}{\Gamma(-\sigma-n+3)}\int_{S^{n-2}} [\boldsymbol{\xi},\boldsymbol{\eta}]^{\tau}\Xi_M^{n-1,m}(\boldsymbol{\eta})d\boldsymbol{\eta} = q_{\sigma,m}\Xi_M^{n-1,m}(\boldsymbol{\xi}). \quad (8)$$

## 9.6. The Groups $O(\infty)$, $IO(\infty)$, the Infinite Dimensional Laplace Operator and Hermite Polynomials

**9.6.1. The Gauss measure. The group $O(\infty)$.** Let $\Phi$ be a real nuclear space. We equip $\Phi$ with the continuous strictly positive scalar product $(.,.)$. Completing $\Phi$ with respect to this scalar product, we obtain the Hilbert space $\mathfrak{H}$. The set of continuous linear functionals on $\Phi$ is denoted by $\Phi^*$. For every $\mathbf{h} \in \mathfrak{H}$ the mapping $\boldsymbol{\xi} \to (\boldsymbol{\xi}, \mathbf{h})$, $\boldsymbol{\xi} \in \Phi$, defines a functional from $\Phi^*$. Vectors $\mathbf{h} \in \mathfrak{H}$ do not exhaust all functionals of $\Phi^*$. We have $\Phi \subset \mathfrak{H} \subset \Phi^*$.

For a fixed positive number $c$ we define the function

$$\chi_c(\boldsymbol{\xi}) = \exp\left(-\frac{c^2}{2}\|\boldsymbol{\xi}\|^2\right) \quad (1)$$

on $\Phi$, where $\|\boldsymbol{\xi}\|^2 = (\boldsymbol{\xi},\boldsymbol{\xi})$. The function $\chi_c(\boldsymbol{\xi})$ is continuous in the topology of the space $\mathfrak{H}$ and *strictly positive definite*, that is, for any complex numbers $\alpha_1,\dots,\alpha_n$ and for any elements $\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_n$ from $\Phi$ we have the inequality

$$\sum_{i,j=1}^{n} \alpha_i\overline{\alpha_j}\chi_c(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j) \geq 0.$$

By Minlos theorem (see, for example, [265]) the formula

$$\chi_c(\boldsymbol{\xi}) = \int_{\Phi^*} \exp\left[i(\boldsymbol{\xi},\mathbf{f})\right] d\mu_c(\mathbf{f}), \quad (2)$$

where $(\boldsymbol{\xi},\mathbf{f})$ is the value of the functional $\mathbf{f}$ at $\boldsymbol{\xi}$, uniquely defines the measure on $\Phi^*$. Sets, generated by cylindrical sets from $\Phi^*$ (that is, sets obtained from them by finite and countable unions and intersections and by taking complements), are $\mu_c$-measurable. Since $\chi_c(\mathbf{0}) = 1$, then $\mu_c(\Phi^*) = 1$. The measure $\mu_c$ is called the *Gauss measure*.

Since $\Phi$ is everywhere dense in $\mathfrak{H}$, then in $\mathfrak{H}$ there exists an orthonormal basis $\xi_1, \xi_2, \ldots$ consisting of elements of $\Phi$. Elements $\xi \in \Phi$ are represented in the form

$$\xi = \sum_{i=1}^{\infty} a_i \xi_i, \quad \text{where} \quad \sum_{i=1}^{\infty} |a_i|^2 < \infty.$$

Elements from $\Phi^*$ can be written as

$$\mathbf{f} = \sum_{i=1}^{\infty} b_i \xi_i, \tag{3}$$

and the series converges in the weak topology of $\Phi^*$, that is, in the topology defined by finite sets of elements from $\Phi$. We have

$$(\xi, \mathbf{f}) = \sum_{i=1}^{\infty} a_i b_i.$$

We set $\xi = \eta \equiv \sum_{i=1}^{n} a_i \xi_i$ into (2) and take into account expression (3) for $\mathbf{f}$. We obtain

$$\chi_c(\eta) = \exp\left[-\frac{c^2}{2}(a_1^2 + \cdots + a_n^2)\right] = \int_{\Phi^*} \exp\left[i(\eta, \mathbf{f})\right] d\mu_c(\mathbf{f})$$

$$= \int_{\Phi^*} \exp\left[i(a_1 b_1 + \cdots + a_n b_n)\right] d\mu_c\left(\sum_{j=1}^{\infty} b_j \xi_j\right)$$

$$= \left(\frac{1}{c\sqrt{2\pi}}\right)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[i(a_1 b_1 + \cdots + a_n b_n)] e^{-b^2/2c^2} db_1 \ldots db_n, \tag{4}$$

where $b^2 = b_1^2 + \cdots + b_n^2$ (we have used the result of Example 1 of Section 3.2.3). This formula allows us to write down the value of the measure $\mu_c$ on the $n$-dimensional subspace

$$\Phi_n^* = \left\{ \mathbf{f}_n = \sum_{i=1}^{n} b_i \xi_i \mid b_i \in \mathbf{R} \right\}$$

of the space $\Phi^*$ in the form

$$d\mu_c(\mathbf{f}_n) = \left(\frac{1}{c\sqrt{2\pi}}\right)^n \exp\left(-\frac{b^2}{2c^2}\right) db_1 \ldots db_n. \tag{5}$$

One can show that this expression depends on the dimensionality $n$ only. Due to (5) we can formally write

$$d\mu_c(\mathbf{f}) = \lim_{n\to\infty}\left[\left(\frac{1}{c\sqrt{2\pi}}\right)^n \exp\left(-\frac{b^2}{2c^2}\right) db_1 \dots db_n\right], \ \mathbf{f}\in\Phi^*.$$

Let $F(\mathbf{f})$ be a function on $\Phi^*$ such that

$$F(\mathbf{f}) = \widetilde{F}(b_1,\dots,b_n), \ \mathbf{f} = \sum_{j=1}^{\infty} b_i\boldsymbol{\xi}_i. \tag{5'}$$

then

$$\int_{\Phi^*} F(\mathbf{f})d\mu_c(\mathbf{f}) = \left(\frac{1}{c\sqrt{2\pi}}\right)^n \int_{\mathbb{R}^n} \widetilde{F}(b_1,\dots,b_n)\exp\left(-\frac{b^2}{2c^2}\right) db_1\dots db_n. \tag{6}$$

Unitary operators in $\mathfrak{H}$, which are one-to-one and mutually continuous on $\Phi$, are called *rotations* of $\Phi$. (Remember that $\Phi$ is a real space.) Rotations form the group, denoted by $O(\infty)$. The equality $(u\boldsymbol{\xi},\mathbf{f}) = (\boldsymbol{\xi}, u^*\mathbf{f})$, $\boldsymbol{\xi}\in\Phi$, $\mathbf{f}\in\Phi^*$, $u\in O(\infty)$, defines the operator $u^*$ on $\Phi^*$. One can easily verify that $(u^{-1})^*$, considered on the subspace $\Phi$ of $\Phi^*$, coincides with $u$. Identifying $(u^{-1})^*$ with $u$, we extend the action of $O(\infty)$ onto $\Phi^*$.

It follows from (1) that $\chi_c(u\boldsymbol{\xi}) = \chi_c(\boldsymbol{\xi})$, $u\in O(\infty)$. Therefore, we have from (2) that

$$\chi_c(u\boldsymbol{\xi}) = \int \exp[i(u\boldsymbol{\xi},\mathbf{f})]d\mu_c(\mathbf{f}) = \int \exp[i(\boldsymbol{\xi},u^{-1}\mathbf{f})]d\mu_c(\mathbf{f})$$
$$= \int \exp[i(\boldsymbol{\xi},\mathbf{f})]d\mu_c(u\mathbf{f}) = \chi_c(\boldsymbol{\xi}) = \int \exp[i(\boldsymbol{\xi},\mathbf{f})]d\mu_c(\mathbf{f}),$$

that is, *$\mu_c$ is invariant with respect to rotations.*

One can show that $\mu_c$ is $O(\infty)$-*ergodic* (see [384]). This means that if a measurable function $F(\mathbf{f})$ on $\Phi^*$ is invariant with respect to $O(\infty)$, i.e. $F(u\mathbf{f}) = F(\mathbf{f})$, $u\in O(\infty)$, then $F(\mathbf{f}) = \text{const}$ almost everywhere.

**9.6.2. The projective limit of spheres.** The space $\Phi^*$ is proved to be identified, up to a set of zero measure, with the limit for $n\to\infty$ (the projective limit) of spheres in $\mathbb{R}^n$. For this identification, we want that the ordinary Euclidean measure on spheres tends to the Gauss measure. Since $\mu_c(\Phi^*) = 1$, then we require that the Euclidean measure of the sphere in $\mathbb{R}^n$ is equal to 1. A simple calculation shows that the radius of this sphere is equal to

$$R_n = \left[\Gamma\left(\frac{n-1}{2}\right)\Big/2\pi^{(n-1)/2}\right]^{\frac{1}{n-1}}.$$

The Stirling formula

$$\Gamma(p) \underset{p\to\infty}{\sim} \sqrt{2\pi} p^{p-1/2} e^{-p} \tag{1}$$

implies that for $n \to \infty$ we have $R_n \sim \sqrt{n/2\pi e}$. In other words, when $n$ is increasing, the radius of the sphere with the unit measure is increasing approximately as $\sqrt{n}$.

Let $\Omega_n$ be the sphere in $\mathbf{R}^{n+1}$ with radius $c\sqrt{n+1}$:

$$\Omega_n = \{\mathbf{x} \in \mathbf{R}^{n+1} \mid x_1^2 + \cdots x_{n+1}^2 = c^2(n+1)\},$$

and $\omega_n$ be the invariant (with respect to $SO(n+1)$) measure on $\Omega_n$ such that $\omega_n(\Omega_n) = 1$ (in order to obtain the measure $\mu_c$ when $n \to \infty$, we have introduced the constant factor $c > 0$). Let $\gamma_{mn}$, $m < n$, be a mapping from $\Omega_n$ into $\Omega_m$ of the form

$$\gamma_{mn}: \Omega_n \ni \mathbf{x}^{(n)} \equiv (x_1^{(n)}, \dots, x_{n+1}^{(n)}) \to \mathbf{x}^{(m)} = (x_1^{(m)}, \dots, x_{m+1}^{(m)}) \in \Omega_m, \tag{2}$$

where

$$x_i^{(m)} = \frac{c\sqrt{m+1}\,x_i^{(n)}}{r_m^{(n)}}, \quad (r_m^{(n)})^2 = (x_1^{(n)})^2 + \cdots + (x_{m+1}^{(n)})^2, \tag{3}$$

($\gamma_{mn}$ is defined at all points from $\Omega_n$ except for points $(x_1^{(n)}, \dots, x_{n+1}^{(n)})$ for which $x_1^{(n)} = \cdots = x_m^{(n)} = 0$). A straightforward calculation shows that the conditions

$$\gamma_{\ell n} = \gamma_{\ell m}\gamma_{mn}, \qquad \ell < m < n, \tag{4}$$

$$\omega_m(A) = \omega_n(\gamma_{mn}^{-1}A), \quad m < n,\ A \subset \Omega_m, \tag{5}$$

hold, where $\gamma_{mn}^{-1}A$ is the set of all points $\mathbf{x}^{(n)} \in \Omega_n$ such that $\gamma_{mn}\mathbf{x}^{(n)} \in A$.

Thus, we have the sequence

$$\Omega_1, \Omega_2, \dots, \Omega_n, \dots \tag{6}$$

of spheres and the collection of mappings $\gamma_{mn}: \Omega_n \to \Omega_m$, $n > m$. They define the limit set $\Omega$ which is called the *projective limit of sets* (6): $\Omega = \operatorname{proj.lim}_{n\to\infty} \Omega_n$. We have $\Omega \subset \mathbf{R}^\infty$. The mappings $\gamma_m = \lim_{n\to\infty} \gamma_{mn}$ from $\Omega$ into $\Omega_m$ are defined. If $\mathbf{x}^\infty = (x_1^\infty, x_2^\infty, \dots) \in \Omega$, then (3) implies that

$$\gamma_m \mathbf{x}^\infty = \mathbf{x}^{(m)} = (x_1^{(m)}, \dots, x_{m+1}^{(m)}) \in \Omega_m, \tag{7}$$

where

$$x_i^{(m)} = c\sqrt{m+1}\,x_i^\infty / r_m^\infty, \quad (r_m^\infty)^2 = (x_1^\infty)^2 + \cdots + (x_{m+1}^\infty)^2. \tag{7'}$$

One can easily prove that $\gamma_m = \gamma_{mn}\gamma_n$, $m < n$.

Let $\mathfrak{F}_n$ be the collection of $\omega_n$-measurable sets in $\Omega_n$. We map these sets into $\Omega$, that is, form the collection of the sets $\gamma_n^{-1}(\mathfrak{F}_n)$. The sets $\gamma_n^{-1}(\mathfrak{F}_n)$, $n = 1, 2, \ldots$, generate the collection $\mathfrak{F}$ of Borel subsets in $\Omega$. By putting

$$\omega(\gamma_n^{-1} A) = \omega_n(A), \quad A \in \mathfrak{F}_n, \tag{8}$$

we obtain the measure $\omega$ on $\Omega$ for which $\mathfrak{F}$ is the collection of measurable subsets.

Starting from $\omega_n$ we shall find an explicit form of the measure $\omega$. Let us evaluate the measure

$$\omega_{nm}(E) = \omega_n(\{\mathbf{x} \in \Omega_n \mid (x_1, \ldots, x_m) \in E\}),$$

where $m < n$ and $E \subset \mathbb{R}^m$. For this we represent $\mathbb{R}^{n+1}$ as the sum $\mathbb{R}^{n+1} = \mathbb{R}^m + \mathbb{R}^{n-m+1}$, where

$$\mathbb{R}^m = \{(x_1, \ldots, x_m, 0, \ldots, 0)\}, \ \mathbb{R}^{n-m+1} = \{(0, \ldots, 0, x_{m+1}, \ldots, x_{n+1})\}.$$

The distance from a point $\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \Omega_n$ to $\mathbb{R}^m$ is

$$r_m(\mathbf{x}) = \sqrt{c^2(n+1) - (x_1^2 + \cdots + x_m^2)}.$$

The angle $\theta$ between the vector, passing from the origin to $\mathbf{x} \in \Omega_n$, and $\mathbb{R}^{n-m+1}$ is given by the formula

$$\cos\theta = r_m(\mathbf{x})/c\sqrt{n+1}.$$

Therefore, for $\omega_{nm}$ we have

$$d\omega_{nm}(x_1, \ldots, x_m) = 0 \quad \text{if} \quad x_1^2 + \cdots + x_m^2 > c^2(n+1) \tag{9}$$

and

$$\begin{aligned} d\omega_{nm}(x_1, \ldots, x_m) &= c_{nm} \frac{r_m(\mathbf{x})}{\cos\theta} dx_1 \ldots dx_m \\ &= c_{nm} c\sqrt{n+1}[c^2(n+1) - (x_1^2 + \cdots + x_m^2)]^{(n-m-1)/2} dx_1 \ldots dx_m \end{aligned} \tag{9'}$$

otherwise, where $c_{nm}$ is the normalizing constant which provides the equality $\omega_{nm}(\mathbb{R}^n) = 1$. Formulas (9) and (9′) can be rewritten as

$$d\omega_{nm}(x_1, \ldots, x_m) = c'_{nm} \left(1 - \frac{x_1^2 + \cdots + x_m^2}{c^2(n+1)}\right)_+^{\frac{n-m-1}{2}} dx_1 \ldots dx_m, \tag{10}$$

where $c'_{nm} = c_{nm}(c^2(n+1))^{(n-m)/2}$.

It follows from (10) that

$$\lim_{n\to\infty} \frac{d\omega_{nm}(x_1,\ldots,x_m)}{c'_{nm}dx_1\ldots dx_m} = \left(\frac{1}{c\sqrt{2\pi}}\right)^m \exp\left(-\frac{x_1^2+\cdots+x_m^2}{2c^2}\right), \qquad (10')$$

where the convergence is uniform in $x_1,\ldots,x_m$. Formulas (10) and (10′) mean that

$$\lim_{n\to\infty} d\omega_{nm}(x_1,\ldots,x_m)$$
$$= d\mu_{c,m}(x_1,\ldots,x_m) \equiv \left(\frac{1}{c\sqrt{2\pi}}\right)^m \exp\left(-\frac{x^2}{2c^2}\right) dx_1\ldots dx_m, \qquad (11)$$

where $x^2 = x_1^2+\cdots+x_m^2$.

Let $\mathbf{x}^\infty \in \Omega$. According to (7) we have $\gamma_n x^\infty \in \Omega_n$. It follows from formulas (8) and (10) that

$$\omega\left(\{\mathbf{x}^\infty\in\Omega \mid ((\gamma_n\mathbf{x}^\infty)_1,\ldots,(\gamma_n x^\infty)_m)\in E\}\right)$$
$$= \lim_{n\to\infty} c'_{nm}\int_E \left(1-\frac{x_1^2+\cdots+x_m^2}{c^2(n+1)}\right)_+^{\frac{n-m-1}{2}} dx_1\ldots dx_m$$
$$= \left(\frac{1}{c\sqrt{2\pi}}\right)^m \int_E \exp\left(-\frac{x_1^2+\cdots x_m^2}{2c^2}\right) dx_1\ldots dx_m, \qquad (12)$$

where $E\subset\mathbf{R}^m$, $m<n$.

We recommend to the reader to prove the equalities

$$\int_\Omega (\gamma_n\mathbf{x}^\infty)_i d\omega(\mathbf{x}^\infty) = 0, \quad 1\le i\le n+1,$$

$$\int_\Omega (\gamma_n\mathbf{x}^\infty)_i(\gamma_m\mathbf{x}^\infty)_j d\omega(\mathbf{x}^\infty) = c^2\sqrt{\frac{n+1}{m+1}}\,\frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{m+2}{2})}{\Gamma(\frac{n+2}{2})\Gamma(\frac{m+1}{2})}\delta_{ij},\ m\le n.$$

They yield

$$\int_\Omega x_i^\infty d\omega(\mathbf{x}^\infty) = 0,\ \int_\Omega x_i^\infty x_j^\infty d\omega(\mathbf{x}^\infty) = c^2\delta_{ij}. \qquad (13)$$

We now imbed $\Omega$ into the space $\Phi^*$, dual to the nuclear space $\Phi$. Let $\xi_1,\xi_2,\ldots$ be an orthonormal basis in $\mathfrak{H}$, consisting of vectors from $\Phi$ (see Section 9.6.1). Since $\Phi$ is a nuclear space, then there exists a sequence of positive numbers $\lambda_1,\lambda_2,\ldots$, such that $\sum_{n=1}^\infty \lambda_n^2<\infty$ and the norm

$$p(\xi) = \left[\sum_{n=1}^\infty \frac{(\xi,\xi_n)^2}{\lambda_n^2}\right]^{1/2}$$

is continuous in the topology of $\Phi$. For any $\mathbf{x} \equiv \mathbf{x}^\infty \in \Omega$ we define the functional $\mathbf{f}_\mathbf{x}$ on $\Phi$ as

$$(\boldsymbol{\xi}, \mathbf{f}_\mathbf{x}) = \sum_{n=1}^{\infty} (\boldsymbol{\xi}, \boldsymbol{\xi}_n) x_n^\infty. \tag{14}$$

Since the norm $p$ is continuous in $\Phi$ and for $\omega$-almost all $\mathbf{x}^\infty \in \Omega$, we have $\sum\limits_{n=1}^{\infty} \lambda_n^2 (x_n^\infty)^2 < \infty$, then

$$\sum_{n=1}^{\infty} |(\boldsymbol{\xi}, \boldsymbol{\xi}_n)| \cdot |x_n^\infty| \leq p(\boldsymbol{\xi}) \left[ \sum_{n=1}^{\infty} \lambda_n^2 (x_n^\infty)^2 \right]^{1/2} < \infty$$

almost everywhere. Thus, $\mathbf{f}_\mathbf{x} \in \Phi^*$ for $\omega$-almost all $\mathbf{x}^\infty \in \Omega$.

We show that $\Omega \ni \mathbf{x}^\infty \to \mathbf{f}_\mathbf{x} \in \Phi^*$ is the desired imbedding of $\Omega$ into $\Phi^*$, defined everywhere except for a set of zero measure. It follows from (14) that to different $\mathbf{x}^\infty$ and $\mathbf{y}^\infty$ there correspond different functionals $\mathbf{f}_\mathbf{x}$ and $\mathbf{f}_\mathbf{y}$ in $\Phi$. It is clear from the construction of measurable sets in $\Omega$ that they are mapped into $\mu_c$-measurable sets in $\Phi^*$. Further, formula (12) implies that if $\boldsymbol{\xi} = \sum\limits_{n=1}^{\infty} a_n \boldsymbol{\xi}_n$, then

$$\int_\Omega \exp[i(\boldsymbol{\xi}, \mathbf{f}_\mathbf{x})] d\omega(\mathbf{x}^\infty) = \exp\left(-\frac{c^2}{2} \|\boldsymbol{\xi}\|^2\right). \tag{15}$$

This equality is extended by continuity onto the space of all elements $\boldsymbol{\xi} \in \Phi$. Since formula (15) uniquely defines the measure, then, by comparing it with formula (2) of Section 9.6.1, we conclude that the measure $\omega$ on $\Omega$ is mapped into the measure $\mu_c$ on $\Phi^*$. Therefore, the mapping $\mathbf{x}^\infty \to \mathbf{f}_\mathbf{x}$ transforms $\Omega$ into a set from $\Phi^*$ of measure 1. Thus, *we have constructed a one-to-one mapping of the set $\widetilde{\Omega}$ from $\Omega$ onto the set $\widetilde{\Phi}^*$ from $\Phi^*$. This mapping preserves the measure and*

$$\omega(\widetilde{\Omega}) = \mu_c(\widetilde{\Phi}^*) = 1.$$

**9.6.3. The infinite dimensional Laplace operator.** In the Euclidean space $E_{n+1}$ the Laplace operator has the form

$$\Delta^{(n+1)} = \sum_{i=1}^{n+1} \frac{\partial^2}{\partial x_i^2}.$$

In spherical coordinates the operator $\Delta^{(n+1)}$ can be written as

$$\Delta^{(n+1)} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0^{(n)},$$

where $\Delta_0^{(n)}$ is the Laplace operator on $S^n$. Let us apply the operator $\Delta_c^{(n+1)} = \Delta_0^{(n)}/c^2(n+1)$ to a function $f(x_1,\dots,x_m)$, $m<n$. Since

$$\Delta_0^{(n)} = r^2\Delta^{(n+1)} - r^2\frac{\partial^2}{\partial r^2} - rn\frac{\partial}{\partial r},$$

then

$$\Delta_c^{(n+1)} f = \frac{r^2}{c^2(n+1)}\left(\sum_{i=1}^{n+1}\frac{\partial}{\partial x_i^2} - \frac{\partial^2}{\partial r^2} - \frac{n}{r}\frac{\partial}{\partial r}\right)f.$$

Substituting the expressions

$$\frac{\partial f}{\partial r} = \sum_{i=1}^{m}\frac{x_i}{r}\frac{\partial f}{\partial x_i},\quad \frac{\partial^2 f}{\partial r^2} = \sum_{i,j=1}^{m}\frac{x_i x_j}{r^2}\frac{\partial^2 f}{\partial x_i \partial x_j},$$

we have

$$\Delta_c^{(n+1)} f\Big|_{r=c\sqrt{n+1}} = \left[\sum_{i=1}^{m}\frac{\partial^2}{\partial x_i^2} - \frac{1}{c^2(n+1)}\sum_{i,j=1}^{m} x_i x_j\frac{\partial^2}{\partial x_i\partial x_j} + \frac{n}{c^2(n+1)}\sum_{i=1}^{m} x_i\frac{\partial}{\partial x_i}\right]f.$$

When $n$ tends to the infinity, we obtain the operator $\Delta_c^\infty = \lim_{n\to\infty}\Delta_c^{(n+1)}$ which acts upon $f(x_1,\dots,x_m)$ according to the formula

$$\Delta_c^\infty f = \sum_{i=1}^{m}\left(\frac{\partial^2}{\partial x_i^2} - \frac{x_i}{c^2}\frac{\partial}{\partial x_i}\right)f. \tag{1}$$

This leads to the following definition of the Laplace operator $\Delta_c$ in the Hilbert space $\mathfrak{L}^2(\Phi^*,\mu_c)$ of functions on $\Phi^*$ with the scalar product

$$(F_1,F_2)_c = \int_{\Phi^*} F_1(\mathbf{f})\overline{F_2(\mathbf{f})}d\mu_c(\mathbf{f}).$$

If $F(\mathbf{f})$ is a function of $\mathfrak{L}^2(\Phi^*,\mu_c)$ such that

$$F(\mathbf{f}) = \widetilde{F}(b_1,\dots,b_n) \quad \text{for} \quad \mathbf{f} = \sum_{i=1}^{\infty} b_i\xi_i \tag{2}$$

(see Section 9.6.1), then

$$\Delta_c F \equiv \Delta_c^\infty F = \sum_{i=1}^{n}\left(\frac{\partial^2}{\partial b_i^2} - \frac{b_i}{c^2}\frac{\partial}{\partial b_i}\right).\widetilde{F} \tag{3}$$

The space spanned by functions of the form (2) for $n = 0, 1, 2, \ldots$ is everywhere dense in $\mathfrak{L}^2(\Phi^*, \mu_c)$. Therefore, (3) defines $\Delta_c$ as an operator (unbounded) on $\mathfrak{L}^2(\Phi^*, \mu_c)$.

It follows from differential equation (13) of Section 5.3.6 for Hermite polynomials $H_n$ that

$$\left[\frac{d^2}{dx^2} - \frac{x}{c^2}\frac{d}{dx}\right] H_n\left(\frac{x}{c\sqrt{2}}\right) = -\frac{n}{c^2} H_n\left(\frac{x}{c\sqrt{2}}\right).$$

Hence, finite products $\prod\limits_i H_{n_i}\left(\frac{x_i}{c\sqrt{2}}\right)$ are eigenfunctions of operator (1) corresponding to the eigenvalues $-\frac{1}{c^2}\sum\limits_i n_i$. Since $(\boldsymbol{\xi}_k, \mathbf{f}) = b_k$, then

$$\widetilde{F}_{(n_1, n_2, \ldots)}(\mathbf{f}) = \prod_{k=1}^{\infty} H_{n_k}\left(\frac{(\boldsymbol{\xi}_k, \mathbf{f})}{c\sqrt{2}}\right), \quad n_1 + n_2 + \ldots < \infty, \tag{4}$$

are eigenfunctions for $\Delta_c$

$$\Delta_c \widetilde{F}_{(n_1, n_2, \ldots)}(\mathbf{f}) = -\frac{1}{c^2}(n_1 + n_2 + \ldots)\widetilde{F}_{(n_1, n_2, \ldots)}(\mathbf{f}). \tag{5}$$

It is easy to see that

$$\|\widetilde{F}_{(n_1, n_2, \ldots)}\|_c^2 = 2^n n_1! n_2! \ldots, \quad n = n_1 + n_2 + \ldots,$$
$$(\widetilde{F}_{(n_1, n_2, \ldots)}(\mathbf{f}),\ \widetilde{F}_{(n'_1, n'_2, \ldots)}(\mathbf{f}))_c = 0$$

if $(n_1, n_2, \ldots) \neq (n'_1, n'_2, \ldots)$. Moreover, the collection of all functions

$$F_{(n_1, n_2, \ldots)}(\mathbf{f}) = (2^{n_1 + n_2 + \cdots} n_1! n_2! \ldots)^{-1/2} \prod_{k=1}^{\infty} H_{n_k}\left(\frac{(\boldsymbol{\xi}_k, \mathbf{f})}{c\sqrt{2}}\right),$$
$$n_1 + n_2 + \ldots < \infty, \tag{6}$$

forms an orthonormal basis in $\mathfrak{L}^2(\Phi^*, \mu_c)$.

Let us denote by $\mathfrak{H}_n$ the closure of the subset in $\mathfrak{L}^2(\Phi^*, \mu_c)$, spanned by the functions

$$F_{(n_1, n_2, \ldots)}(\mathbf{f}), \quad n_2 + n_2 + \ldots = n. \tag{7}$$

Then $\mathfrak{H}_n$ is the eigenspace of $\Delta_c$ corresponding to the eigenvalue $-n/c^2$ and

$$\mathfrak{L}^2(\Phi^*, \mu_c) = \sum_{n=0}^{\infty} \oplus \mathfrak{H}_n \tag{8}$$

is the decomposition of $\mathfrak{L}^2(\Phi^*, \mu_c)$ into the sum of eigenspaces of $\Delta_c$. The subspaces $\mathfrak{H}_n$, $n = 1, 2, \ldots$, are infinite dimensional and $\mathfrak{H}_0$ is one-dimensional.

**9.6.4. The Hilbert space $\mathfrak{L}_c(\Phi)$.** For $F \in \mathfrak{L}^2(\Phi^*, \mu_c)$ we consider the transform $U: F \to \widehat{F}$, where

$$\widehat{F}(\boldsymbol{\xi}) = \int_{\Phi^*} e^{i(\boldsymbol{\xi}, \mathbf{f})} F(\mathbf{f}) d\mu_c(\mathbf{f}), \quad \boldsymbol{\xi} \in \Phi. \tag{1}$$

If $F(\mathbf{f}) = \sum_{j=1}^{n} a_j \exp(i(\boldsymbol{\eta}_j, \mathbf{f}))$, $a_j \in \mathbf{C}$, $\boldsymbol{\eta}_j \in \Phi$, then

$$(UF)(\boldsymbol{\xi}) = \sum_{j=1}^{n} a_j \int_{\Phi^*} \exp[i(\boldsymbol{\xi}, \mathbf{f}) + i(\boldsymbol{\eta}_j, \mathbf{f})] d\mu_c(\mathbf{f}) = \sum_{j=1}^{n} a_j \chi_c(\boldsymbol{\xi} + \boldsymbol{\eta}_j) \tag{2}$$

(see formula (2) of Section 9.6.1). The set

$$\left\{ \sum_{j=1}^{n} a_j \exp[i(\boldsymbol{\eta}_j, \mathbf{f})] \;\middle|\; a_j \in \mathbf{C},\ \boldsymbol{\eta}_j \in \Phi,\ n = 0, 1, 2, \ldots \right\}$$

is dense in $\mathfrak{L}^2(\Phi^*, \mu_c)$. The set of corresponding functions (2) will be denoted by $\mathfrak{L}$. We introduce into $\mathfrak{L}$ the scalar product

$$\left\langle \sum_{j=1}^{n} a_j \chi_c(\boldsymbol{\xi} + \boldsymbol{\eta}_j), \sum_{k=1}^{m} b_k \chi_c(\boldsymbol{\xi} + \boldsymbol{\zeta}_k) \right\rangle_{\mathfrak{L}} = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j \overline{b_k} \chi_c(-\boldsymbol{\eta}_j + \boldsymbol{\zeta}_k). \tag{3}$$

Since $\chi_c$ is a positively definite function, then

$$\left\| \sum_{j=1}^{n} a_j \chi_c(\boldsymbol{\xi} + \boldsymbol{\eta}_j) \right\|_{\mathfrak{L}}^2 = \sum_{j,k=1}^{n} a_j \overline{a_k} \chi_c(\boldsymbol{\eta}_k - \boldsymbol{\eta}_j) \geq 0,$$

where the equality holds if and only if $a_1 = \ldots = a_n = 0$. Hence, (3) gives a scalar product in $\mathfrak{L}$.

Completing $\mathfrak{L}$ with respect to this scalar product, we obtain the Hilbert space denoted by $\mathfrak{L}_c(\Phi)$. It follows from (2) that $U$ is a one-to-one linear isometric transform from $\mathfrak{L}^2(\Phi^*, \mu_c)$ onto $\mathfrak{L}_c(\Phi)$.

We recommend to the reader to prove that for all $\widehat{F}(\boldsymbol{\xi}) \in \mathfrak{L}_c(\Phi)$ we have

$$\left\langle \widehat{F}(\boldsymbol{\xi}), \chi_c(\boldsymbol{\xi} - \boldsymbol{\eta}) \right\rangle_{\mathfrak{L}} = \widehat{F}(\boldsymbol{\eta}), \quad \boldsymbol{\eta} \in \Phi. \tag{4}$$

Hilbert spaces with property (4) are called ***spaces with self-reproducing kernel.***

Let us find the basis of $\mathfrak{L}_c(\Phi)$ which is the image of basis (6) of Section 9.6.3 under transform (1). For this we evaluate the integral

$$I = \int_{\Phi^*} e^{i(\boldsymbol{\xi},\mathbf{f})} H_n\left(\frac{(\boldsymbol{\xi},\mathbf{f})}{c\sqrt{2}\|\boldsymbol{\xi}\|}\right) d\mu_c(\mathbf{f}).$$

Choosing in $\Phi$ the orthonormal basis $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$ such that $\boldsymbol{\xi} = a\boldsymbol{\xi}_1 = \|\boldsymbol{\xi}\|\boldsymbol{\xi}_1$ and making use of the results of Section 9.6.1, we have

$$I = \int_{-\infty}^{\infty} \exp\left(i\|\boldsymbol{\xi}\|b - \frac{b^2}{2c^2}\right) H_n\left(\frac{b}{c\sqrt{2}}\right) db.$$

Substituting the expression

$$H_n\left(\frac{b}{c\sqrt{2}}\right) = (-c)^n \exp\left(\frac{b^2}{2c^2}\right) \frac{d^n}{db^n} \exp\left(-\frac{b^2}{2c^2}\right)$$

(see Section 5.3.6) and then evaluating the integral by parts, we find

$$I = (ic\|\boldsymbol{\xi}\|)^n \exp\left(-\frac{c^2\|\boldsymbol{\xi}\|^2}{2}\right). \tag{5}$$

Using (5), the expansion $\boldsymbol{\xi} = \sum_n (\boldsymbol{\xi}_n, \boldsymbol{\xi})\boldsymbol{\xi}_n$ and the results of Section 9.6.1, we easily obtain that

$$UF_{(n_1,n_2,\ldots)}(\mathbf{f}) = \int_{\Phi^*} e^{i(\boldsymbol{\xi},\mathbf{f})} F_{(n_1,n_2,\ldots)}(\mathbf{f}) d\mu_c(\mathbf{f})$$
$$= \frac{i^n c^n \exp\left(-\frac{c^2\|\boldsymbol{\xi}\|^2}{2}\right)}{\sqrt{n_1!n_2!\ldots}} (\boldsymbol{\xi}_1, \boldsymbol{\xi})^{n_1} (\boldsymbol{\xi}_2, \boldsymbol{\xi})^{n_2} \ldots, \quad n_1 + n_2 + \ldots = n. \tag{6}$$

We denote this function by $\varphi_{(n_1,n_2,\ldots)}(\boldsymbol{\xi})$. It is clear that the collection of these functions $\{\varphi_{(n_1,n_2,\ldots)} \mid n_1 + n_2 + \ldots < \infty\}$ forms an orthonormal basis of $\mathfrak{L}_c(\Phi)$. The closure of the space, spanned by the functions

$$\varphi_{(n_1,n_2,\ldots)}, \quad n_1 + n_2 + \ldots = n, \tag{7}$$

will be denoted by $\mathfrak{L}_c^n$. We have $U\mathfrak{H}_n = \mathfrak{L}_c^n$ and

$$\mathfrak{L}_c(\Phi) = \sum_{n=0}^{\infty} \oplus \mathfrak{L}_c^n. \tag{8}$$

**9.6.5. Irreducible representations of $O(\infty)$.** The formula $(T_g F)(\mathbf{f}) = F(g^{-1}\mathbf{f})$, $g \in O(\infty)$, gives the representation $T$ of the group $O(\infty)$ in $\mathfrak{L}^2(\Phi^*, \mu_c)$. Since the measure $\mu_c$ is invariant with respect to rotations from $O(\infty)$, then $T$ is a unitary representation of $O(\infty)$. By means of the operator $U$ (see Section 9.6.4) we form the representation $R$ of $O(\infty)$: $R_g = UT_gU^{-1}$, $g \in O(\infty)$, in the space $\mathfrak{L}_c(\Phi)$. It follows from formula (1) of Section 9.6.4 that

$$(R_g\widehat{F})(\boldsymbol{\xi}) = \widehat{F}(g^{-1}\boldsymbol{\xi}), \quad \widehat{F} \in \mathfrak{L}_c(\Phi).$$

Formula (6) of the preceding section shows that for any $g \in O(\infty)$ the function $R_g\varphi_{(n_1,n_2,\dots)}, n_1 + n_2 + \dots = n$, is expressed as a series (possibly, infinite) of the functions $\varphi_{(n'_1,n'_2,\dots)}$ with $n'_1 + n'_2 + \dots = n$. In other words, the spaces $\mathfrak{L}_c^n$ from decomposition (8) of Section 9.6.4 are invariant with respect to $O(\infty)$. Consequently, the subspaces $\mathfrak{H}_n$ from decomposition (8) of Section 9.6.3 are also invariant. The restriction of the representation $T$ onto $\mathfrak{H}_n$ is denoted by $T^n$. The representations $T^n$ are irreducible (see [385]). The representation $T^0$ is one-dimensional.

Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots$ be an orthonormal basis in $\mathfrak{H}$ consisting of elements from $\Phi$. By $O(\infty - 1)$ we denote the subgroup of elements from $O(\infty)$, leaving the element $\boldsymbol{\xi}_1$ fixed. In the same way one introduces the subgroups $O(\infty - 2)$, $O(\infty - 3)$ and so on.

Functions (7) of Section 9.6.3 form an orthonormal basis of $\mathfrak{H}_n$. We denote by $\mathfrak{F}_m$ the closure of the subset of $\mathfrak{H}_n$, spanned by the functions $F_{(n_1,n_2,\dots)}$ with $n_1 = m$. We have $\mathfrak{H}_n = \sum_{m=0}^{n} \oplus \mathfrak{F}_m$. The subspaces $\mathfrak{F}_m$ are invariant with respect to $O(\infty - 1)$. Besides, $\mathfrak{F}_m$ is a carrier space of the irreducible representation of $O(\infty - 1)$ which for $O(\infty)$ is denoted by $T^{n-m}$. In order to indicate that this representation relates to $O(\infty - 1)$, we use for it the notation $T_1^{n-m}$. Thus, we have

$$T^n \Big\downarrow^{O(\infty)}_{O(\infty-1)} = \sum_{m_1=0}^{n} \oplus T_1^{m_1}. \tag{1}$$

Similarly, the restriction of the representation $T_1^{m_1}$ of $O(\infty - 1)$ onto the subgroup $O(\infty - 2)$ decomposes into a sum of the irreducible representations $T_2^{m_2}$, $m_2 = 0, 1, \dots, m_1$.

In the same way as in the case of $SO(n)$ (see Section 9.3.1), subsequent decomposition of the space $\mathfrak{H}_n$ into the sum of irreducible subspaces with respect to the subgroups

$$SO(\infty - 1) \supset SO(\infty - 2) \supset SO(\infty - 3) \supset \cdots$$

leads us to the decomposition of $\mathfrak{H}_n$ into the sum of one-dimensional subspaces which are numerated by collections of numbers

$$M = (n, m_1, m_2, m_3, \dots), \quad n \geq m_1 \geq m_2 \geq m_3 \geq \dots \tag{2}$$

These one-dimensional subspaces give the orthonormal basis $\{F(M)\}$ in $\mathfrak{H}_n$, which coincides with basis (7) of Section 9.6.3. We have the correspondence

$$F_{(n_1,n_2,n_3,\dots)} \leftrightarrow F(M) \equiv F(n, n-n_1, n-n_1-n_2, n-n_1-n_2-n_3, \dots). \qquad (3)$$

It is clear from (3) that every collection of numbers $M$, characterizing the basis element, contains finite number of non-zero elements. However, for any $k \in \mathbf{Z}_+$ there exists a collection $M$ for which $m_k \neq 0$.

It follows from (1) that the representations $T^n$ of $O(\infty)$ are of class 1 relative to $O(\infty - 1)$. The basis function

$$F_{(n,0,0,\dots)}(\mathbf{f}) = (2^n n!)^{-1/2} H_n\left(\frac{(\boldsymbol{\xi}_1, \mathbf{f})}{c\sqrt{2}}\right) \qquad (4)$$

is invariant with respect to $O(\infty - 1)$.

**9.6.6. Matrix elements of the representations $T^n$.** Let $g_1(\theta)$ be the rotation

$$g_1(\theta)\boldsymbol{\xi}_1 = \boldsymbol{\xi}_1 \cos\theta + \boldsymbol{\xi}_2 \sin\theta, g_1(\theta)\boldsymbol{\xi}_2 = -\boldsymbol{\xi}_1 \sin\theta + \boldsymbol{\xi}_2\cos\theta, g_1(\theta)\boldsymbol{\xi}_j = \boldsymbol{\xi}_j,\ j \neq 1, 2.$$

Let $N = (n_1, n_2, \dots)$, $N' = (n'_1, n'_2, \dots)$. By integrating we find for matrix elements in basis (7) of Section 9.6.3 that

$$t^n_{NN'}(g_1(\theta)) \equiv (T^n(g_1(\theta))F_{N'}, F_N) = 0 \qquad (1)$$

if $(n_3, n_4, \dots) \neq (n'_3, n'_4, \dots)$. For $(n_3, n_4, \dots) = (n'_3, n'_4, \dots)$ we have

$$t^n_{(\ell-m,m)(\ell-k,k)}(g_1(\theta)) = 2^{-\ell}\pi^{-1}[m!k!(\ell-k)!(\ell-m)!]^{-1/2}$$
$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_{\ell-k}(x\cos\theta + y\sin\theta)H_k(-x\sin\theta + y\cos\theta)H_{\ell-m}(x)H_m(y)e^{-x^2-y^2}\,dx\,dy, \qquad (2)$$

where $n_1 = \ell - m$, $n_2 = m$, $n'_1 = \ell - k$, $n'_2 = k$. We omit the indices $n_3, n_4, \dots$ because matrix element (2) does not depend on them.

The right hand side of (2) is independent on $n$. Therefore, instead of $t^n_{(\ell-m,m)(\ell-k,k)}(g_1(\theta))$ we shall write $t^\ell_{mk}(g_1(\theta))$. Since the basis function, invariant with respect to $O(\infty - 1)$, has the form (4) of Section 9.6.5, then we have

$$t^\ell_{00}(hg_1(\theta)h') = t^\ell_{00}(g_1(\theta)), \quad h, h' \in O(\infty - 1). \qquad (3)$$

Making use of equality (17) of Section 5.3.6 and the integral $\int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{\pi}$, we find from (2) that

$$t^\ell_{00}(g_1(\theta)) = \cos^\ell\theta. \qquad (4)$$

We show that if $t^{n\ell}_{00}(g_1(\theta))$ is the zonal spherical function of the representation $T^{n\ell}$ of $SO(n)$ with respect to the subgroup $\{\mathrm{diag}(1,k) \mid k \in SO(n-1)\}$, then

$$\lim_{n\to\infty} t^{n\ell}_{00}(g_1(\theta)) = t^{\ell}_{00}(g_1(\theta)). \tag{5}$$

The functions $t^{n\ell}_{00}(g_1(\theta))$ are expressed in terms of Gegenbauer polynomials. We start from the integral representation for these polynomials

$$C^p_\ell(x) = \frac{\Gamma(2p+\ell)\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}\ell!\Gamma(2p)\Gamma(p)} \int_{-1}^{1} (x + i\sqrt{1-x^2}\,t)^\ell (1-t^2)^{p-1} dt,\ p = \frac{n-2}{2}. \tag{6}$$

It is clear from the equality

$$\int_{-1}^{1} (1-t^2)^{p-1} dt = \frac{\sqrt{\pi}\Gamma(p)}{\Gamma\left(p+\frac{1}{2}\right)}, \quad \lim_{p\to\infty} (1-t^2)^{p-1} = 0,$$

where $-1 \le t \le 1$, $t \neq 0$, that

$$\lim_{p\to\infty} \frac{\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(p)} (1-t^2)^{p-1} = \delta(t).$$

From here the equality

$$\lim_{p\to\infty} \frac{\ell!\Gamma(2p)}{\Gamma(2p+\ell)} C^p_\ell(x) = \int_{-1}^{1} (x + i\sqrt{1-x^2}\,t)^\ell \delta(t) dt = x^\ell \tag{7}$$

follows, which is equivalent to (5).

By means of formula (2), equality (17) of Section 5.3.6 and orthogonality relation for Hermite polynomials, we find the expressions for matrix elements of the "zero" column:

$$t^\ell_{m0}(g_1(\theta)) = \left(\frac{\ell!}{m!(\ell-m)!}\right)^{1/2} (\sin\theta)^m (\cos\theta)^{\ell-m}, \quad \ell \ge m. \tag{8}$$

We have

$$\lim_{n\to\infty} t^{n\ell}_{m0}(g_1(\theta)) = t^\ell_{m0}(g_1(\theta)), \tag{9}$$

where $t^{n\ell}_{m0}(g_1(\theta))$ is a matrix element of the "zero" column of the operator $T^{n\ell}(g_1(\theta))$ of the representation of $SO(n)$.

We recommend to the reader to prove the following properties of the matrix elements $t^\ell_{mn}(g_1(\theta)) \equiv t^\ell_{mn}(\theta)$:

$$t^\ell_{mn}(\theta) = (-1)^{m+n} t^\ell_{nm}(\theta), \tag{10}$$

$$\sqrt{n}\, t^\ell_{mn}(\theta) = \sqrt{\ell - m}\cos\theta\, t^{\ell-1}_{m,n-1}(\theta) - \sqrt{m}\sin\theta\, t^{\ell-1}_{m-1,n-1}(\theta), \tag{11}$$

$$\frac{d}{d\theta} t^\ell_{mn}(\theta) = \sqrt{n(\ell - n + 1)}\, t^\ell_{m,n-1}(\theta) - \sqrt{(n+1)(\ell - n)}\, t^\ell_{m,n+1}(\theta), \tag{12}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{n!(\ell-n)!}\right)^{1/2} \sin^n\theta_2 \cos^{\ell-n}\theta_2\, t^\ell_{mn}(\theta_1) = \left(\frac{1}{m!(\ell-m)!}\right)^{1/2} \sin^m(\theta_1+\theta_2)\cos^{\ell-m}(\theta_1+\theta_2), \tag{13}$$

$$\sum_{m=0}^{\ell} \left(\frac{n!(\ell-n)!}{m!(\ell-m)!}\right)^{1/2} t^\ell_{mn}(\theta) H_m(x) H_{\ell-m}(y) = H_n(x\cos\theta + y\sin\theta) H_{\ell-n}(-x\sin\theta + y\cos\theta). \tag{14}$$

One can show that $t^\ell_{mk}(\theta)$ are obtained by the limit passage $n \to \infty$ from the matrix elements $t^{n\ell}_{mk0}(g_1(\theta))$ of the operators $T^{n\ell}(g_1(\theta))$ of the representation of $SO(n)$ (see Section 9.5.1). Namely,

$$\lim_{n\to\infty} t^{n\ell}_{mk0}(g_1(\theta)) = t^\ell_{mk}(g_1(\theta)). \tag{15}$$

**9.6.7. Hermite polynomials as the limit of Gegenbauer polynomials.** In Section 9.6.2 we constructed the space $\Omega$ which is the limit of the spheres $\Omega_n$. It is natural to expect that the zonal spherical functions on $S^{n-1}$ for $n \to \infty$ tend to functions on $\Omega$, which are invariant with respect to $O(\infty - 1)$. In order to realize the limit passage we have to pass from zonal spherical functions on $S^{n-1}$ to corresponding functions on $\Omega_{n-1}$.

The zonal spherical function on $S^{n-1}$ is a multiple of the Gegenbauer polynomial $C^p_\ell(x_1)$, $x_1 = \cos\theta$, for which we have

$$C^p_\ell(x_1) = \frac{\Gamma(2p+\ell)\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}\,\ell!\,\Gamma(2p)\Gamma(p)} \int_{-1}^{1} \left(x_1 - i\sqrt{1-x_1^2}\, t\right)^\ell (1-t^2)^{p-1} dt,\ p = \frac{n-2}{2}. \tag{1}$$

We pass to the corresponding homogeneous harmonic polynomial of degree $\ell$ (see formula (3) of Section 9.2.6) and set $r = \sqrt{2p}$. Up to a factor, independent on $x_1$, we obtain the polynomial $C_\ell^p(x/\sqrt{p})$, $x = x_1/\sqrt{2}$. It follows from (1) that

$$p^{-\ell/2} C_\ell^p\left(\frac{x}{\sqrt{p}}\right) = \frac{\Gamma(2p+\ell)\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}\ell!\Gamma(2p)\Gamma(p)p^{\ell+1/2}} \times \int_{-\sqrt{p}}^{\sqrt{p}} \left(x - iu\sqrt{1-\frac{x^2}{p}}\right)^\ell \left(1-\frac{u^2}{p}\right)^{p-1} du \tag{2}$$

(in order to have a function for which the integral over $\Omega_n$ of the square of its module is bounded for any $n$, we have multiplied the Gegenbauer polynomial by $p^{-\ell/2}$; we have also carried out the substitution $t = u/\sqrt{p}$). When $p \to \infty$, then the integral on the right hand side tends to $\int\limits_{-\infty}^{\infty} (x-iu)^\ell e^{-u^2} du$. To find the coefficient at the integral for $p \to \infty$ we use the Stirling formula (1) of Section 9.6.2. As a result, we have

$$\lim_{p\to\infty} p^{-\ell/2} C_\ell^p\left(\frac{x}{\sqrt{p}}\right) = \frac{2^\ell}{\ell!\sqrt{\pi}} \int_{-\infty}^{\infty} (x-iu)^\ell e^{-u^2} du. \tag{3}$$

Expanding $(x-iu)^\ell$ according to the binomial formula, integrating term by term and taking into account formula (6) of Section 5.3.6 for Hermite polynomials, we find

$$\lim_{p\to\infty} p^{-\ell/2} C_\ell^p\left(\frac{x}{\sqrt{p}}\right) = \frac{1}{\ell!} H_\ell(x). \tag{4}$$

This gives the integral representation for Hermite polynomials

$$H_\ell(x) = \frac{2^\ell}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x-it)^\ell e^{-t^2} dt. \tag{5}$$

Passing to the integration over $\Phi^*$ (see Section 9.6.1), we derive

$$H_\ell\left(\frac{x}{c\sqrt{2}}\right) = \frac{2^{\ell/2}}{c^\ell} \int_{\Phi^*} (x - i(\boldsymbol{\xi}_1, \mathbf{f}))^\ell d\mu_c(\mathbf{f}), \tag{6}$$

where $\boldsymbol{\xi}_1$ is the basis element in $\Phi$. Since the measure $\mu_c$ is invariant with respect to $O(\infty)$, then one can replace $\mathbf{f}$ by $T_g\mathbf{f}$. Therefore, taking into consideration the equality $(\boldsymbol{\xi}_1, T_g\mathbf{f}) = (T_g^{-1}\boldsymbol{\xi}_1, \mathbf{f})$, we can assume that $\boldsymbol{\xi}_1$ from (6) is an arbitrary element of $\Phi$ such that $\|\xi_1\| = 1$. It also follows from (6) that for any $\mathbf{f}' \in \Phi^*$ and for any $\xi \in \Phi$, $\|\xi\| = 1$, we have

$$H_\ell\left(\frac{(\xi, \mathbf{f}')}{c\sqrt{2}}\right) = \frac{w^{\ell/2}}{c^\ell} \int_{\Phi^*} (\xi, \mathbf{f}' - i\mathbf{f})^\ell d\mu_c(\mathbf{f}). \tag{7}$$

The function $H_\ell\left(\frac{(\xi_1, \mathbf{f})}{c\sqrt{2}}\right)$ is (up to a constant) one of the basis functions of the space $\mathfrak{H}_\ell$ (see formula (6) of Section 9.6.3). It satisfies the condition

$$\Delta_c H_\ell\left(\frac{(\xi_1, \mathbf{f})}{c\sqrt{2}}\right) = -\frac{\ell}{c^2} H_\ell\left(\frac{(\xi_1, \mathbf{f})}{c\sqrt{2}}\right)$$

and is invariant with respect to the subgroup $O(\infty - 1)$:

$$T_k H_\ell\left(\frac{(\xi_1, \mathbf{f})}{c\sqrt{2}}\right) = H_\ell\left(\frac{(\xi_1, \mathbf{f})}{c\sqrt{2}}\right), \quad k \in O(\infty - 1).$$

That is why sometimes it is called a zonal spherical function of $\Phi^*$ (or of $\Omega$) with respect to the subgroup $O(\infty - 1)$. Unlike the group $SO(n)$, it does not coincide with the matrix element $(T_g F_0, F_0)$ for $O(\infty)$ with respect to the $O(\infty - 1)$-invariant function from $\mathfrak{H}_\ell$.

**9.6.8. Properties of Hermite polynomials.** Relation (4) of the preceding section between Hermite and Gegenbauer polynomials allows us to derive properties of Hermite polynomials by means of the passage to the limit. In particular, in this way one can obtain all properties of these polynomials from Section 5.3.6. For example, the differential equation

$$\left[(1 - x^2)\frac{d^2}{dx^2} - (2p + 1)x\frac{d}{dx} + \ell(\ell + 2p)\right] C_\ell^p(x) = 0$$

for Gegenbauer polynomials (after the replacement of $x$ by $x/\sqrt{p}$, multiplication of both sides by $\ell! p^{-1-\ell/2}$ and the passage to the limit $p \to \infty$) leads to the differential equation for Hermite polynomials

$$\left(\frac{d^2}{dx^2} - 2x\frac{d}{dx} + 2\ell\right) H_\ell(x) = 0.$$

Similarly, recurrence relations for Gegenbauer polynomials yield those for $H_\ell(x)$. From formula (8) of Section 6.3.9 by means of passage to the limit, we find the equality

$$H_\ell(x) = (-1)^\ell e^{x^2} \frac{d^\ell}{dx^\ell} e^{-x^2}.$$

From formula (2) of Section 9.3.7 we obtain

$$e^{-t^2+2tx} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_\ell(x).$$

From the orthogonality relation for $C_\ell^p(x)$ we derive this for Hermite polynomials:

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \begin{cases} 0 & \text{if } m \neq n, \\ 2^n n! \sqrt{\pi} & \text{if } m = n. \end{cases} \tag{1}$$

It follows from the multiplication formula for Gegenbauer polynomials (see Section 9.4.5) that

$$\int_{-\infty}^{\infty} H_\ell(xy + t\sqrt{1-x^2}\,) H_k(t) e^{-t^2} dt = \frac{2^k \ell! \sqrt{\pi}}{(\ell-k)!} x^{\ell-k} (1-x^2)^{k/2} H_{\ell-k}(y), \tag{2}$$

where $\ell \geq k$. For $\ell < k$ this integral vanishes. In particular, if $k = 0$, then

$$\int_{-\infty}^{\infty} H_\ell(xy + t\sqrt{1-x^2}\,) e^{-t^2} dt = \sqrt{\pi}\, x^\ell H_\ell(y). \tag{3}$$

For $k = 0$, $y = 0$, $\ell = 2n$ formula (2) gives

$$\int_{-\infty}^{\infty} H_{2n}(ut) e^{-t^2} dt = \frac{\sqrt{\pi}(2n)!}{n!} (u^2 - 1)^n. \tag{3'}$$

Since

$$\frac{d}{du} H_{2n}(ut) = 4n H_{2n-1}(ut) t, \quad \left(\frac{d}{du}\right)^n (u^2 - 1)^n = 2^n n! P_n(u)$$

(see formula (1) of Section 6.3.10), then (3′) implies

$$\int_{-\infty}^{\infty} t^n H_n(ut) e^{-t^2} dt = n! P_n(n). \tag{4}$$

Further, it follows from formula (5) of Section 9.4.10 that if $\ell + m + n = 2g$, where $g$ is a non-negative integer, and if there exists a triangle with the sides of lengths $\ell$, $m$, $n$, then

$$\int_{-\infty}^{\infty} H_\ell(x) H_m(x) H_n(x) e^{-x^2} dx = \frac{2^g \sqrt{\pi} \ell! m! n!}{(g-\ell)!(g-m)!(g-n)!}. \tag{4'}$$

Otherwise this integral vanishes.

We derive from formula (4) of Section 9.4.5 that

$$\sum_{n=0}^{\infty} \frac{x^n H_n(y) H_n(z)}{2^n n!} = (1-x^2)^{-1/2} \exp\left(\frac{2xyz - (y^2 - z^2)x^2}{1-x^2}\right). \tag{5}$$

We recommend to the reader to carry out details of the corresponding passages to the limit. Let us note that expansion (5) can be obtained from (3) by using the substitution $xy + t\sqrt{1-x^2} = u$ and the orthogonality relation. In the same way formula (2) leads to

$$\sum_{n=0}^{\infty} \frac{x^n H_n(y) H_{n+k}(z)}{2^n n!}$$
$$= (1-x^2)^{-(k+1)/2} \exp\left(\frac{2xyz - (y^2 + z^2)x^2}{1-x^2}\right) H_k\left(\frac{z - xy}{\sqrt{1-x^2}}\right). \tag{6}$$

Some relations for Hermite polynomials can be derived from formula (7) of Section 9.6.7. Setting $\boldsymbol{\xi} = \boldsymbol{\xi}_1 \cos\theta + \boldsymbol{\xi}_2 \sin\theta$, where $\boldsymbol{\xi}_1 \perp \boldsymbol{\xi}_2$, we have

$$H_\ell\left(\frac{(\boldsymbol{\xi}_1, \mathbf{f}')}{c\sqrt{2}} \cos\theta + \frac{(\boldsymbol{\xi}_2, \mathbf{f}')}{c\sqrt{2}} \sin\theta\right)$$
$$= \frac{2^{\ell/2}}{c^\ell} \int_{\Phi^*} [(\boldsymbol{\xi}_1, \mathbf{f}' - i\mathbf{f})\cos\theta + (\boldsymbol{\xi}_2, \mathbf{f}' - i\mathbf{f})\sin\theta]^\ell d\mu_c(\mathbf{f}).$$

The integrand expression can be written as

$$\sum_{n=0}^{\ell} \frac{\ell!}{n!(\ell-n)!} \cos^n\theta \sin^{\ell-n}\theta\, (\boldsymbol{\xi}_1, \mathbf{f}' - i\mathbf{f})^n (\boldsymbol{\xi}_2, \mathbf{f}' - i\mathbf{f})^{\ell-n}.$$

Integrating term by term, we obtain the addition formula

$$H_\ell(x\cos\theta + y\sin\theta) = \sum_{n=0}^{\ell} \frac{\ell!}{n!(\ell-n)!} \cos^n\theta \sin^{\ell-n}\theta H_n(x) H_{\ell-n}(y). \tag{7}$$

If we put

$$H_\ell(x, \alpha) = \frac{\alpha^{\ell/2}}{\ell! 2^{\ell/2}} H_\ell\left(\frac{x}{\sqrt{2\alpha}}\right),$$

then (7) can be written in the form

$$H_\ell(x + y, \alpha + \beta) = \sum_{n=0}^{\ell} H_n(x, \alpha) H_{\ell-n}(y, \beta). \tag{8}$$

In the same way, the formula

$$\prod_{n=0}^{m} H_{\ell_n}\left(\frac{(\xi_n, \mathbf{f}')}{c\sqrt{2}}\right) = \frac{2^{\ell/2}}{c^{\ell}} \int_{\Phi^*} \prod_{n=0}^{m} (\xi_n, \mathbf{f}' - i\mathbf{f})^{\ell_n} d\mu_c(\mathbf{f})$$

leads to the addition formula for products of Hermite polynomials. In particular, for $m = 2$ we have

$$H_\ell(x\cos\theta + y\sin\theta)H_k(-x\sin\theta + y\cos\theta) = \sum_{j=0}^{\ell+k} \alpha_j H_j(x) H_{\ell+k-j}(y), \tag{9}$$

where

$$\alpha_j = \sum_{i=\max(0,j-k)}^{\min(j,\ell)} \frac{\ell! k! (-1)^{i-j}}{i!(j-i)!(\ell-i)!(k-j+i)!} \cos^{k-j+2i}\theta \sin^{j+\ell-2i}\theta.$$

It follows from formula (5) of Section 9.6.4 that

$$\int_{-\infty}^{\infty} e^{ixy} H_n\left(\frac{x}{\sqrt{2}}\right) e^{-x^2/2} dx = \sqrt{2\pi}\,(i\sqrt{2}\,y)^n e^{-y^2/2}. \tag{10}$$

By virtue of the inversion formula for the Fourier transform, we obtain

$$\int_{-\infty}^{\infty} e^{ixy} y^n e^{-y^2/2} dy = \sqrt{2\pi}\left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{x}{\sqrt{2}}\right) e^{-x^2/2}. \tag{11}$$

One can rewrite (10) and (11) in the form

$$\int_{-\infty}^{\infty} e^{ixy} e^{-x^2} H_n(x) dx = \sqrt{\pi}(iy)^n e^{-y^2/4}, \tag{10'}$$

$$\int_{-\infty}^{\infty} e^{ixy} y^n e^{-y^2} dy = \left(\frac{i}{2}\right)^n \sqrt{\pi} e^{-x^2/4} H_n(x). \tag{11'}$$

Putting $t + ix = y$ into (11), we find

$$\int_{-\infty}^{\infty} (t + ix)^n e^{-t^2/2} dt = \sqrt{2\pi}\left(\frac{i}{\sqrt{2}}\right)^n H_n\left(\frac{x}{\sqrt{2}}\right). \tag{12}$$

With the help of the ordinary Fourier transform, we find for the basis elements $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ from $\Phi$ the equality

$$U[(\boldsymbol{\xi}_1,\boldsymbol{\xi})+i(\boldsymbol{\xi}_2,\boldsymbol{\xi})]^n \equiv \int_{\Phi^*} e^{i(\boldsymbol{\xi},\mathbf{f})}[(\boldsymbol{\xi}_1,\mathbf{f})+i(\boldsymbol{\xi}_2,\mathbf{f})]^n d\mu_c(\mathbf{f})$$

$$= i^n c^{2n}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}i^k(\boldsymbol{\xi}_1,\boldsymbol{\xi})^{n-k}(\boldsymbol{\xi}_2,\boldsymbol{\xi})^k e^{-c^2\|\boldsymbol{\xi}\|^2/2}. \tag{13}$$

On the other hand, from formula (6) of Section 9.6.4, we find that

$$U\left[2^{-n/2}c^n\sum_{k=0}^{n}\frac{n!i^k}{k!(n-k)!}H_{n-k}\left(\frac{(\boldsymbol{\xi}_1,\mathbf{f})}{c\sqrt{2}}\right)H_k\left(\frac{(\boldsymbol{\xi}_2,\mathbf{f})}{c\sqrt{2}}\right)\right]$$

coincides with the right hand side of (13). Since $U$ is a one-to-one isometric transform, then

$$[(\boldsymbol{\xi}_1,\mathbf{f})+i(\boldsymbol{\xi}_2,\mathbf{f})]^n = 2^{-n/2}c^n\sum_{k=0}^{n}\frac{n!i^k}{k!(n-k)!}H_{n-k}\left(\frac{(\boldsymbol{\xi}_1,\mathbf{f})}{c\sqrt{2}}\right)H_k\left(\frac{(\boldsymbol{\xi}_2,\mathbf{f})}{c\sqrt{2}}\right).$$

Consequently,

$$2^n(x+iy)^n = \sum_{k=0}^{n}\frac{n!i^k}{k!(n-k)!}H_{n-k}(x)H_k(y). \tag{14}$$

By virtue of the orthogonality relation for Hermite polynomials, we have

$$\int_{-\infty}^{\infty}(x+iy)^n H_m(x)e^{-x^2}dx = \left(\frac{i}{2}\right)^{n-m}\frac{n!}{(n-m)!}H_{n-m}(y),\ n\geq m. \tag{15}$$

One can make use of formulas (4′) and (7) for evaluation of the matrix elements $t^\ell_{mk}(g_1(\theta))$ from Section 9.6.6. Indeed, by means of (7), expression (2) of Section 9.6.6 for $t^\ell_{mk}(g_1(\theta))$ is represented in the form

$$t^\ell_{mk}(g_1(\theta))$$
$$=\frac{1}{2^\ell}\left[\frac{k!(\ell-k)!}{m!(\ell-m)!}\right]^{1/2}\sum_{n=0}^{\ell-k}\sum_{s=0}^{k}\frac{(-1)^s(\sin\theta)^{\ell-k-n+s}(\cos\theta)^{n+k-s}}{n!s!(\ell-k-n)!(k-s)!}$$
$$\times\int_{-\infty}^{\infty}H_n(x)H_s(x)H_{\ell-m}(x)e^{-x^2}dx\int_{-\infty}^{\infty}H_{\ell-k-n}(y)H_{k-s}(y)H_m(y)e^{-y^2}dy.$$

Substituting formula (4′) for the integrals, we find the expression for $t^{\ell}_{mk}(g_1(\theta))$ in the form of a finite sum of trigonometric functions.

**9.6.9. The Wiener transform.** Let $F(\mathbf{f})$ be a function on $\Phi^*$ such that

$$F(\mathbf{f}) = \widetilde{F}((\xi_1, \mathbf{f}), \ldots, (\xi_n, \mathbf{f}))$$

(see formula (5′) of Section 9.6.1) and $\widetilde{F}$ be a polynomial in $n$ variables. For these functions the Wiener transform is defined as

$$(WF)(\mathbf{f}) = \int\limits_{\Phi^*} \widetilde{F}((\xi_1, \sqrt{2}\,\mathbf{f}_1 + i\mathbf{f}), \ldots, (\xi_n, \sqrt{2}\,\mathbf{f}_1 + i\mathbf{f}))d\mu_c(\mathbf{f}_1). \tag{1}$$

If $\widetilde{F}$ is a polynomial, then $F \in \mathfrak{L}^2(\Phi^*, \mu_c)$. By means of formula (6) of Section 9.6.4 and of formula (12) of Section 9.6.8, one can easily show that $WF$ also belongs to $\mathfrak{L}^2(\Phi^*, \mu_c)$ and

$$\|F\|_c = \|WF\|_c. \tag{2}$$

The set of functions $F$ on $\Phi^*$, for which $\widetilde{F}$ are polynomials, is dense in $\mathfrak{L}^2(\Phi^*, \mu_c)$. Therefore, the Wiener transform is continued to be a unitary operator on $\mathfrak{L}^2(\Phi^*, \mu_c)$.

Let $F$ be a function on $\Phi^*$ such that $F(\mathbf{f}) = \widetilde{F}((\xi_1, \mathbf{f}))$, where $\widetilde{F}$ is a polynomial. One has the relation

$$(\xi_1, \mathbf{f})(WF)((\xi_1, \mathbf{f})) = i[W(\mathbf{f}_1 \cdot F(\mathbf{f}_1)]((\xi_1, \mathbf{f})) + 2c^2(D_{\xi_1}F)((\xi_1, \mathbf{f})), \tag{3}$$

where $\mathbf{f}_1 \cdot F(\mathbf{f}_1)$ is a functional, acting by the formula

$$(\mathbf{f}_1 \cdot F(\mathbf{f}_1))(\xi) = (\xi, \mathbf{f}_1)F((\xi, \mathbf{f}_1)),$$

and

$$(D_{\xi_1}F)((\xi_1, \mathbf{f})) = \frac{1}{i}\frac{d}{db}\widetilde{F}(b)\Big|_{b=(\xi_1,\mathbf{f})}.$$

In order to prove (3), we denote $(\xi_1, \mathbf{f})$ by $x$, integrate according to formula (12) of Section 9.6.8 and use the recurrence relation $\frac{d}{dx}H_n(x) = -H_{n+1} + 2xH_n(x)$ (see formulas (10) and (12) of Section 5.3.6). As a result, formula (3) leads to the identity. We recommend to the reader to prove that for any $F \in \mathfrak{L}^2(\Phi^*, \mu_c)$ and for any $\varphi \in \Phi$ the relation

$$(\varphi, \mathbf{f})(WF)((\varphi, \mathbf{f})) = i[W(\mathbf{f}_1 \cdot F(\mathbf{f}_1))]((\varphi, \mathbf{f})) + 2c^2(D_\varphi F)((\varphi, \mathbf{f})) \tag{4}$$

holds, where

$$(D_\varphi F)(\mathbf{f}) = \lim_{\varepsilon\to 0} \frac{F(\mathbf{f}+\varepsilon\varphi) - F(\mathbf{f})}{i\varepsilon}.$$

By setting $F((\xi_1, \mathbf{f})) = H_n\left(\frac{(\xi_1,\mathbf{f})}{c\sqrt{2}}\right)$ into (3), we have

$$-\frac{(\xi_1,\mathbf{f})}{2c^2} W H_n\left(\frac{(\xi_1,\mathbf{f})}{c\sqrt{2}}\right) = W\left[\frac{i(\xi_1,\mathbf{f})}{2c^2} H_n\left(\frac{(\xi_1,\mathbf{f})}{c\sqrt{2}}\right)\right] + W\left[\frac{1}{ic\sqrt{2}} H_n'\left(\frac{(\xi_1,\mathbf{f})}{c\sqrt{2}}\right)\right].$$

Applying recurrence relations (10) and (12) of Section 5.3.6 for Hermite polynomials, we obtain

$$\frac{\sqrt{2}(\xi_1,\mathbf{f})}{c} W H_n\left(\frac{(\xi_1,f)}{c\sqrt{2}}\right) = -iWH_{n+1}\left(\frac{(\xi_1,\mathbf{f})}{c\sqrt{2}}\right) + 2niWH_{n+1}\left(\frac{(\xi_1,\mathbf{f})}{c\sqrt{2}}\right).$$

This equality coincides with the recurrence relation for the polynomials $i^n H_n$. Since $W1 = 1$, then

$$WH_n\left(\frac{(\xi_1,\mathbf{f})}{c\sqrt{2}}\right) = i^n H_n\left(\frac{(\xi_1,\mathbf{f})}{c\sqrt{2}}\right). \tag{5}$$

This means that

$$\int_{-\infty}^{\infty} H_n\left(\frac{\sqrt{2}\,x + iy}{c\sqrt{2}}\right) e^{-x^2/2c^2}\,dx = \sqrt{2\pi}\, i^n c H_n\left(\frac{y}{c\sqrt{2}}\right). \tag{6}$$

It follows from (5) and from the definition of the Wiener transform that, for functions (6) of Section 9.6.3, we have

$$WF_{(n_1,n_2,\dots)} = i^n F_{(n_1,n_2,\dots)}, \quad n = n_1 + n_2 + \dots . \tag{7}$$

In other words, *the basis functions* $F_{(n_1,n_2,\dots)}$ *of the space* $\mathfrak{L}^2(\Phi^*,\mu_c)$ *are eigenfunctions of the Wiener transform.*

We conclude from (7) that the subspaces $\mathfrak{H}_n$ of $\mathfrak{L}^2(\Phi^*,\mu_c)$ are eigenspaces for $W$. Since $\mathfrak{H}_n$ are eigenspaces for the infinite dimensional Laplace operator $\Delta_c$, then

$$W\Delta_c = \Delta_c W. \tag{8}$$

**9.6.10. Representations of the group $IO(\infty)$.** The group of motions of $\Phi$ will be denoted by $IO(\infty)$. It consists of the elements $g = g(u,\varphi)$, $u \in O(\infty)$, $\varphi \in \Phi$, and

$$g(u,\varphi)\mathbf{f} = u\mathbf{f} + \varphi, \quad \mathbf{f} \in \Phi^*. \tag{1}$$

The multiplication in $IO(\infty)$ is given by the formula

$$g(u,\varphi)g(u',\varphi') = g(uu',\varphi + u\varphi'). \tag{2}$$

In particular, $g(u,\varphi) = g(e,\varphi)g(u,\mathbf{0})$, where $e$ is the identity element of the subgroup $O(\infty)$. It is clear that $IO(\infty)$ is the semidirect product of $O(\infty)$ and the additive invariant subgroup $\Phi$: $IO(\infty) = O(\infty) \times \Phi$. Elements $g(u,\mathbf{0})$ and $g(e,\varphi)$ of $IO(\infty)$ will be denoted by $u$ and $\varphi$, respectively.

Let us define the oeprators

$$(T^c_\varphi F)(\mathbf{f}) = e^{-i(\varphi,\mathbf{f})/2c^2} F(\mathbf{f}), \quad \varphi \in \Phi, \tag{3}$$

$$(T^c_u F)(\mathbf{f}) = F(u^{-1}\mathbf{f}), \quad u \in O(\infty), \tag{4}$$

in $\mathfrak{L}^2(\Phi^*,\mu_c)$ and set $T^c_{g(u,\varphi)} = T^c_\varphi T^c_u$. Then the correspondence $g(u,\varphi) \to T^c_{g(u,\varphi)}$ is a unitary representation of $IO(\infty)$. It follows from the results of Section 9.6.5 that

$$T^c \downarrow {IO(\infty) \atop O(\infty)} = \sum_{m=0}^{\infty} \oplus T^m. \tag{5}$$

Therefore, the representation $T^c$ of $IO(\infty)$ is of class 1 with respect to $O(\infty)$. The function $F_0(\mathbf{f}) \equiv 1$, $\mathbf{f} \in \Phi$, of $\mathfrak{L}^2(\Phi^*,\mu_c)$ is invariant for $O(\infty)$.

The Wiener transform (1) of Section 9.6.9 defines the unitary operator $W$ on $\mathfrak{L}^2(\Phi^*,\mu_c)$. By means of $W$ we construct the representation

$$U^c_{g(u,\varphi)} = W^{-1} T^c_{g(u,\varphi)} W.$$

The representation $U^c$ is given by the formulas

$$(U^c_\varphi F)(\mathbf{f}) = \exp\left[-\frac{(\varphi,\mathbf{f})}{2c^2} - \frac{\|\varphi\|^2}{4c^2}\right] F(\mathbf{f}+\varphi), \quad \varphi \in \Phi, \tag{6}$$

$$(U^c_u F)(\mathbf{f}) = F(u^{-1}\mathbf{f}), \quad u \in O(\infty). \tag{7}$$

In order to prove these formulas we use the equality

$$\frac{d\mu_{c,\varphi}(\mathbf{f})}{d\mu_c(\mathbf{f})} = \exp\left[-\frac{(\varphi,\mathbf{f})}{c^2} - \frac{\|\varphi\|^2}{2c^2}\right], \tag{8}$$

where $d\mu_{c,\varphi}(\mathbf{f}) \equiv d\mu_c(\mathbf{f}+\varphi)$. We have

$$(WU^c_\varphi F)(\mathbf{f}) = \exp\left(-\frac{\|\varphi\|^2}{4c^2}\right) \int \exp\left[-\frac{1}{2c^2}(\varphi, \sqrt{2}\,\mathbf{f}_1 + i\mathbf{f})\right] F(\sqrt{2}\,\mathbf{f}_1 + i\mathbf{f} + \varphi) d\mu_c(\mathbf{f}_1)$$

$$= \exp\left(-\frac{\|\varphi\|^2}{4c^2}\right) \int \exp\left[-\frac{1}{2c^2}(\varphi, \sqrt{2}\,\mathbf{f}_1 + i\mathbf{f} - \varphi)\right] F(\sqrt{2}\,\mathbf{f}_1 + i\mathbf{f})$$

$$\times \exp\left[\frac{1}{c\sqrt{2}}(\varphi,\mathbf{f}_1) - \frac{1}{4c^2}\|\varphi\|^2\right] d\mu_c(\mathbf{f}_1)$$

$$= \exp\left(-\frac{i}{2c^2}(\varphi,\mathbf{f})\right) \int F(\sqrt{2}\,\mathbf{f}_1 + i\mathbf{f}) d\mu_c(\mathbf{f}_1) = (T^c_\varphi W F)(\mathbf{f}).$$

In the same way one proves the equality $(WU_{u}^{c}F)(\mathbf{f}) = (T_{u}^{c}WF)(\mathbf{f})$.

We now show that *the representation* $T^c$ *is irreducible*. Let $A$ be a bounded operator in $\mathfrak{L}^2(\Phi^*, \mu_c)$, commuting with all operators $T^c_{g(u,\varphi)}$. It follows from (3) that $A$ is the operator of multiplication by a bounded measurable function $\Psi(\mathbf{f})$, given on $\Phi^*$. We have from (4) that $\Psi(u\mathbf{f}) = \Psi(\mathbf{f})$ for any $u \in O(\infty)$. Since $\mu_c$ is $O(\infty)$-ergodic, then $\Psi$ is constant almost everywhere, that is, $A = aI$, $a \in \mathbf{C}$. Thus, the representation $T^c$ (and consequently, the representation $U^c$) is irreducible.

*The representations* $T^c$, $c > 0$, *are pairwise nonequivalent*. Indeed, let us assume that the representations $T^c$ and $T^{c'}$ are equivalent. Then there exists an isometric mapping $B$ from $\mathfrak{L}^2(\Phi^*, \mu_c)$ onto $\mathfrak{L}^2(\Phi^*, \mu_{c'})$ such that $BU^{c'}_{g(u,\varphi)}B^{-1} = U^c_{g(u,\varphi)}$. But then $B$ commutes with the representations $T^m$ of the subgroup $O(\infty)$ (see decomposition (5)). Since $T^m$ are irreducible, then $B$ is a multiple of the identity operator on their carrier spaces. In particular, $B1 = \alpha 1$, $|\alpha| = 1$. Hence, for all $\varphi \in \Phi$ we have $(U^{c'}_{\varphi}1, 1) = (U^c_{\varphi}1, 1)$. Calculating these scalar products, we obtain

$$\exp(-\|\varphi\|^2/8c^2) = \exp(-\|\varphi\|^2/8c'^2).$$

This means that $c = c'$. Our statement is proved.

It is shown in [297] that, up to a unitary equivalence, the representations $T^c$, $0 < c < \infty$, exhaust all irreducible unitary representations of the group $IO(\infty)$, which are of class 1 with respect to $O(\infty)$.

**9.6.11. Matrix elements of the representations $T^c$.** Let an element $\varphi \in \Phi$ be of the form $\varphi = \sum_{i=1}^{\infty} a_i\varphi_i$. Then for matrix elements of the operator $T^c_{\varphi}$ in basis (6) of Section 9.6.3 (of the space $\mathfrak{L}^2(\Phi^*, \mu_c)$) we have

$$\begin{aligned} t^c_{NN'}(\varphi) \equiv (T^c_{\varphi}F_{N'}, F_N) &= \int_{\Phi^*} \exp\left(-\frac{i(\varphi,\mathbf{f})}{2c^2}\right) F_{N'}(\mathbf{f})\overline{F_N(\mathbf{f})}d\mu_c(\mathbf{f}) \\ &= \prod_{i=1}^{\infty}\left[c\sqrt{2\pi}\,2^{n_i+n'_i}n_i!n'_i!\right]^{-1/2}\int_{-\infty}^{\infty}\exp\left(\frac{\sqrt{-1}\,a_ib_i}{2c^2}\right) \\ &\qquad \times H_{n'_i}\left(\frac{b_i}{c\sqrt{2}}\right)H_{n_i}\left(\frac{b_i}{c\sqrt{2}}\right)e^{-b_i^2/2c^2}db_i, \end{aligned}$$

where $N = (n_1, n_2, \dots)$, $N' = (n'_1, n'_2, \dots)$. The expression under the product sign is denoted by $t^c_{n_in'_i}\left(\frac{a_i}{2c}\right)$. Then we have

$$t^c_{NN'}(\varphi) = \prod_{i=1}^{\infty} t^c_{n_in'_i}\left(\frac{a_i}{2c}\right), \tag{1}$$

where

$$t^c_{km}(x) = (2^{k+m}k!m!\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-\sqrt{2}\,ixy} H_k(y) H_m(y) e^{-y^2} dy. \tag{2}$$

From formula (10′) of Section 9.6.8 we have

$$t^c_{00}(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\sqrt{2}\,ixy} e^{-y^2} dy = e^{-x^2/2}. \tag{3}$$

If $t^{nR}_{00}(g_r)$ is the zonal spherical function of the irreducible representation $T^{nR}$ of $ISO(n-1)$, then

$$\lim_{n\to\infty} t^{ni}_{00}\left(g_{\sqrt{n-3}\,r}\right) = t^c_{00}(r), \quad i = \sqrt{-1}. \tag{4}$$

Indeed, $t^{ni}_{00}(g_r) = \Gamma\left(\frac{n-1}{2}\right)\left(\frac{r}{2}\right)^{-n+3} J_{\frac{n-3}{2}}(r)$. As it is shown in Section 9.3.3, we have

$$\frac{\Gamma(p+1)J_p(x)}{\left(\frac{x}{2}\right)^p} = \frac{\Gamma(p+1)}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)} \int_{-1}^{1} e^{itx}(1-t^2)^{p-1/2} dt,$$

where $p = (n-3)/2$. Let us make the substitution $t = u/\sqrt{p}$ in the integral, replace $x$ by $x\sqrt{2p}$ and take the limit $p \to \infty$. We obtain the equality

$$\lim_{p\to\infty} \frac{\Gamma(p+1)J_p(x)\sqrt{2p}}{\left(\frac{x\sqrt{p}}{\sqrt{2}}\right)^p} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\sqrt{2}\,ixu-u^2} du = e^{-x^2/2}$$

which proves relation (4).

Making use of formula (10′) of Section 9.6.8, we derive from (2) that

$$t^c_{k0}(x) = \frac{(-i)^k}{\sqrt{k!}} x^k e^{-x^2/2}. \tag{5}$$

We suggest to the reader to prove the relations

$$\sqrt{k+1}\, t^c_{k+1,m+1}(x) = \sqrt{m+1}\, t^c_{km}(x) - ix\, t^c_{k,m+1}(x), \tag{6}$$

$$i\frac{d}{dx} t^c_{km}(x) = \sqrt{m+1}\, t^c_{k,m+1}(x) + \sqrt{m}\, t^c_{k,m-1}(x), \tag{7}$$

$$\sum_{s=0}^{\infty} \frac{(-i)^s}{\sqrt{s!}} x_2^s t^c_{ks}(x_1) = \frac{(-i)^k}{\sqrt{k!}} (x_1+x_2)^k, \tag{8}$$

$$\sum_{k=0}^{\infty} \sqrt{\frac{m!}{k!}}\, 2^{(m-k)/2} H_k(y) t^c_{km}(x) = e^{-\sqrt{2}\,ixy} H_m(y). \tag{9}$$

We can find matrix elements of the representations $T^c$ in the polynomial basis. Let $\mathfrak{F}_{n_1,n_2,\dots}$ be the function from $\mathfrak{L}^2(\Phi^*, \mu_c)$ such that

$$\mathfrak{F}_{n_1,n_2\dots}(\mathbf{f}) = (\xi_1, \mathbf{f})^{n_1}(\xi_2, \mathbf{f})^{n_2}\dots, \quad \mathbf{f} \in \Phi^*.$$

By setting $c = 1$ and $\varphi = \sum_{j=1}^{k} a_j\xi_j + a\xi$, where $\|\xi\| = 1$ and $\xi \perp \xi_j$, $j = 1, \dots, k$, we have

$$\begin{aligned}
\hat{t}_{NO}(\varphi) &\equiv (T^1_\varphi 1, \mathfrak{F}_{n_1,n_2,\dots}) = \int e^{-i(\varphi,\mathbf{f})/2}(\xi_1,\mathbf{f})^{n_1}(\xi_2,\mathbf{f})^{n_2}\dots d\mu_1(\mathbf{f}) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(iax+x^2)/2}dx \prod_{j=1}^{k}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{a_jx_j}{2}i - \frac{x_j^2}{2}\right)x_j^{n_j}\,dx_j \\
&= e^{-a^2/8}\prod_{j=1}^{k} e^{-a_j^2/8}\frac{1}{(-i\sqrt{2})^{n_j}}H_{n_j}\left(-\frac{a_j}{2\sqrt{2}}\right) \\
&= \exp\left(-\frac{\|\varphi\|^2}{8}\right)\prod_{j=1}^{k}\frac{1}{(-i\sqrt{2})^{n_j}}H_{n_j}\left(-\frac{(\varphi,\xi_j)}{2\sqrt{2}}\right), \qquad (10)
\end{aligned}$$

where $N = (n_1, \dots, n_k)$.

**9.6.12. Other properties of Hermite polynomials.** Let $\xi_1, \dots, \xi_m$ be vectors from an orthonormal basis of $\Phi$. We set

$$\varphi = -2\sqrt{2}\sum_{k=1}^{m} x_k\xi_k, \quad \psi = \sum_{k=1}^{m} a_k\xi_k, \quad \|\psi\| = 1. \qquad (1)$$

According to formula (10) of the preceding section,

$$(T_\psi 1, \xi^n) = (-i\sqrt{2})^{-n}e^{-x^2}H_n(a_1x_1 + \dots + a_nx_n),\ x^2 = \sum x_j^2.$$

On the other hand,

$$\begin{aligned}
(T_\varphi 1, \psi^n) &= \sum_{i_1+\dots+i_m=n}\frac{n!a_1^{i_1}\dots a_m^{i_m}}{i_1!\dots i_m!}(T_\varphi 1, \xi_1^{i_1}\dots\xi_m^{i_m}) \\
&= \sum_{i_1+\dots+i_m=n}\frac{n!a_1^{i_1}\dots a_m^{i_m}}{i_1!\dots i_m!(-i\sqrt{2})^n}e^{-x^2}\prod_{k=1}^{m}H_{i_k}(x_k).
\end{aligned}$$

Comparing the right hand sides of these relations, we derive the addition formula

$$\begin{aligned}
&H_n(a_1x_1 + \dots + a_mx_m) \\
&\quad = \sum_{i_1+\dots+i_m=m}\frac{n!a_1^{i_1}\dots a_m^{i_m}}{i_1!\dots i_m!}H_{i_1}(x_1)\dots H_{i_m}(x_m), \qquad (2)
\end{aligned}$$

where $a_1^2 + \ldots + a_m^2 = 1$.

Let $x_1 = x_2 = x/\sqrt{2}$ in (1). Then,

$$(T_\varphi 1, \xi_1^n \xi_2^n) = e^{-x^2}(-i\sqrt{2})^{-2n} H_n\left(\frac{x}{\sqrt{2}}\right)^2. \tag{3}$$

Set $\frac{\xi_1 - \xi_2}{\sqrt{2}} = \xi'$, $\frac{\xi + \xi_2}{\sqrt{2}} = \xi$. Then $\xi_1^n \xi_2^n = \left(\frac{\xi^2 - \xi'^2}{2}\right)^n$, $\|\xi\| = \|\xi'\| = 1$, $(\xi, \xi') = 0$. Therefore,

$$\begin{aligned}(T_\varphi 1, \xi_1^n \xi_2^n) &= \frac{1}{2^n} \sum_{k=0}^{n} \frac{n!(-1)^k}{k!(n-k)!}(T_\varphi 1, \xi'^{2k} \xi^{2n-2k}) \\ &= \frac{e^{-x^2}}{2^n(-\sqrt{2}\,i)^{2n}} \sum_{k=0}^{n} \frac{n!(-1)^k}{k!(n-k)!} H_{2k}\left(-\frac{(\varphi, \xi')}{2\sqrt{2}}\right) H_{2n-2k}\left(-\frac{(\varphi, \xi)}{2\sqrt{2}}\right). \end{aligned} \tag{4}$$

Since $(\varphi, \xi) = -2\sqrt{2}\,x$ and

$$H_{2k}\left(-\frac{(\varphi, \xi')}{2\sqrt{2}}\right) = H_{2k}(0) = (-1)^k \frac{(2k)!}{k!},$$

then we obtain from (3) and (4) that

$$\left[H_n\left(\frac{x}{\sqrt{2}}\right)\right]^2 = \frac{n!}{2^n} \sum_{k=0}^{n} \frac{(2k)!}{k!^2(n-k)!} H_{2n-2k}(x). \tag{5}$$

Let $E_m$ be the subspace of $\Phi$ spanned by the basis elements $\xi_1, \ldots, \xi_m$. The space $\Phi$ is representable as $\Phi = E_m \oplus \Phi_m$, where $\Phi_m$ is the closure of the space spanned by $\xi_{m+1}, \xi_{m+2}, \ldots$. The subgroup of $IO(\infty)$, leaving elements from $\Phi_m$ fixed, is isomorphic to the group $IO(m)$ containing the subgroup $ISO(m)$. We denote by $Q^R$ the representation of the group $ISO(m)$ in the Hilbert space $\mathfrak{L}^2(S^{m-1})$ of functions on the sphere $S^{m-1}$ given by the formulas

$$(Q_u^R f)(\eta) = f(u^{-1}\eta), \qquad u \in SO(m), \tag{6}$$

$$(Q_{\mathbf{a}}^R f)(\eta) = e^{-iR(\mathbf{a},\eta)} f(\eta), \qquad \mathbf{a} \in E_m, \tag{7}$$

where $(\mathbf{a}, \eta) = a_1\eta_1 + \ldots + a_m\eta_m$ if $\eta_1, \ldots, \eta_m$ are the Cartesian coordinates of $\eta \in S^{m-1}$. Let $F$ be a function from $\mathfrak{L}^2(\Phi^*, \mu_1)$ such that

$$F(\mathbf{f}) = \widetilde{F}(b_1, \ldots, b_m), \quad \mathbf{f} = \sum_{i=1}^{\infty} b_i \xi_i. \tag{8}$$

Introducing into $E_m$ the spherical coordinates $r, \theta_1, \ldots, \theta_{m-1}$, we represent the function $\widetilde{F}$ in the form $\widetilde{F}(r, \boldsymbol{\eta})$, $\boldsymbol{\eta} \in S^{m-1}$. Comparing formulas (3) and (4) of Section 9.6.10 with formulas (6) and (7), we find that for functions (8) and for $g \in ISO(m) \subset IO(\infty)$ one has the formula

$$(T_g^1 1, F) = \frac{1}{\Gamma(m/2)} \int_0^\infty (Q_g^{r/2} 1, \widetilde{F}(r, \boldsymbol{\eta}))_s r^{m-1} e^{-r^2/2} dr, \tag{9}$$

where $( \, . \, , \, . \, )_s$ denotes the scalar product on $\mathfrak{L}^2(S^{m-1})$.

Let $m = 2$. Then for $\mathbf{a} = (a_1, a_2) \equiv (R\cos\theta, R\sin\theta)$ we have

$$\begin{aligned}(Q_{\mathbf{a}}^r 1, e^{in\psi})_s &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-irR\cos(\psi-\theta)} e^{-in\psi} d\psi \\ &= e^{-in\theta}(-i)^n J_n(rR)\end{aligned} \tag{10}$$

(see Section 4.1.3). For $\boldsymbol{\xi} = (x\cos\theta)\boldsymbol{\xi}_1 + (x\sin\theta)\boldsymbol{\xi}_2$ we obtain

$$\left(T^1_{-\sqrt{8}\boldsymbol{\xi}} 1, \boldsymbol{\xi}_1^n\right) = (-i\sqrt{2})^{-n} e^{-\|\boldsymbol{\xi}\|^2} H_n(x\cos\theta). \tag{11}$$

On the other hand, since

$$\cos^n\theta = \frac{1}{2^n} \sum_{k=0}^n \frac{n!}{k!(n-k)!} e^{i(n-2k)\theta},$$

then according to (9)

$$\begin{aligned}\left(T^1_{-\sqrt{8}\boldsymbol{\xi}} 1, \boldsymbol{\xi}_1^n\right) &= \int_0^\infty (Q_{\mathbf{a}}^{r/2} 1, \cos^n\theta)_s r^{n+1} e^{-r^2/2} dr \\ &= \frac{1}{2^n} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \int_0^\infty (Q_{\mathbf{a}}^{r/2} 1, \, e^{i(n-2k)\theta})_s r^{n+1} e^{-r^2/2} dr,\end{aligned}$$

where $\mathbf{a} = (-\sqrt{8}\,x\cos\theta, -\sqrt{8}\,x\sin\theta)$. Substituting expression (10) for $(Q_{\mathbf{a}}^{r/2} 1, e^{i(n-2k)\theta})_s$ into this relation and comparing its right hand side with the right hand side of (11), we derive the relation

$$H_n(x\cos\theta) = \frac{e^{-x^2}}{2^{n+1}} \sum_{k=0}^n \frac{(-1)^k n!}{k!(n-k)!} e^{i(2n-k)\theta} \int_0^\infty J_{n-2k}(rx) r^{n+1} e^{-r^2/4} dr. \tag{12}$$

The formula

$$\int_0^\infty J_{n-2k}(rx) r^{n+1} e^{-r^2/2} dr = 2^{n+1} k! x^{n-2k} e^{-x^2} L_k^{n-2k}(x^2) \tag{13}$$

leads it to the form

$$H_n(x \cos t) = n! e^{-2x^2} x^n \sum_{k=0}^{n} \frac{(-1)^k x^{-2k}}{(n-k)!} e^{i(2k-n)t} L_k^{n-2k}(x^2). \tag{14}$$

In order to prove (13) we substitute the expression

$$J_{n-2k}(rx) = \frac{(rx)^{n-2k}}{2^{n-2k}(n-2k)!} {}_0F_1\left(n-2k+1; -\frac{r^2x^2}{4}\right)$$

(see formula (1) of Section 3.5.6) instead of $J_{n-2k}(rx)$, pass from the integration with respect to $r$ to the integration with respect to $y = r^2x^2/4$, calculate the obtained integral by formula (12) of Section 3.5.2 and pass from ${}_1F_1$ to Laguerre polynomials in accordance with formula (4) of Section 3.5.7.

It follows from (14) that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} H_n(x \cos t) e^{i(n-2k)t} dt = \frac{n!(-1)^k}{(n-k)!} x^{n-2k} e^{-2x^2} L_k^{n-2k}(x^2) \tag{15}$$

if $k \le n$. Otherwise this integral vanishes.

If $m = 2$ and $\xi$ is the same as in (11), then

$$\left(T^1_{-\sqrt{8}\,\xi} 1, \xi_1^n \xi_2^n\right) = \left(-i\sqrt{2}\right)^{-2n} e^{-\|\varphi\|^2} H_n(x \cos t) H_n(x \sin t). \tag{16}$$

On the other side,

$$\left(T^1_{-\sqrt{8}\,\xi} 1, \xi_1^n \xi_2^n\right) = \frac{1}{2^n} \int_0^\infty (Q_{\mathbf{a}}^{r/2} 1, \sin^n 2\theta)_s r^{2n+1} e^{-r^2/2} dr$$

$$= \frac{1}{(4i)^n} \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} \int_0^\infty (Q_{\mathbf{a}}^{r/2} 1, e^{i(n-2k)2\theta})_s r^{2n+1} e^{-r^2/2} dr. \tag{17}$$

Substituting into (17) expression (10) for $(Q_{\mathbf{a}}^{r/2} 1, e^{i(n-2k)2\theta})_s$ and comparing the right hand side of (16) and (17), we receive the equality

$$H_n(x \cos \theta) H_n(x \sin \theta) = \frac{1}{(2i)^n} \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} e^{(4k-2n)i\theta}$$

$$\times \int_0^\infty J_{2n-4k}\left(\sqrt{2}\, rx\right) r^{2n+1} e^{-r^2/2} dr.$$

Calculating the integral with the help of formula (13), we finally obtain

$$H_n(x\cos\theta)H_n(x\sin\theta) = \frac{2^n n!}{(2i)^n} x^{2n} e^{-x^2} \sum_{k=0}^{n} \frac{(-1)^k (2k)!}{k!(n-2k)!} x^{-4k} e^{i(4k-2n)\theta} L_{2k}^{2n-4k}(x^2). \tag{18}$$

It follows from here that

$$\frac{1}{2\pi}\int_0^{2\pi} H_n(x\cos\theta)H_n(x\sin\theta)e^{i(2n-4k)\theta}d\theta = \frac{(-1)^k(2k)!2^n n!}{k!(n-k)!(2i)^n} e^{-x^2} x^{2n-4k} L_{2k}^{2n-4k}(x^2), \tag{19}$$

where $k \le n$.

Let now $m = 3$. If $\mathbf{a} = (x\cos t, x\sin t, 0)$, then

$$(Q_{\mathbf{a}}^R 1, P_\ell(\cos\theta))_s = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{-iRx(\cos t\cos\theta + \sin t\sin\theta\cos\psi)} \times P_\ell(\cos\theta)\sin\theta d\theta d\psi = \left(\frac{\pi}{2Rx}\right)^{1/2} (-1)^\ell P_\ell(\cos t) J_{\ell-1/2}(Rx). \tag{20}$$

If $\boldsymbol{\xi}$ is the same as in (11), then

$$\left(T^1_{-\sqrt{8}\,\xi} 1, \xi_1^n\right) = (-i\sqrt{2})^{-n} e^{-x^2} H_n(x\cos\theta). \tag{21}$$

On the other hand,

$$\left(T^1_{-\sqrt{8}\,\xi} 1, \xi_1^n\right) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty (Q_{\mathbf{a}}^{r/2} 1, \cos^n\theta)_s r^{n+2} e^{-r^2/2} dr. \tag{22}$$

One has the expansion

$$\cos^n\theta = \frac{\sqrt{\pi}}{2^{n+1}} n! \sum_{k=0}^{[n/2]} \frac{2n-4k+1}{k!\Gamma\left(n-k+\frac{3}{2}\right)} P_{n-2k}(\cos\theta), \tag{23}$$

where $[n/2]$ is the integral part of the number $n/2$. To prove (23) we multiply both sides by $P_{n-2k}(\cos\theta)$, integrate from 0 to $\pi$, use the orthogonality relation for Legendre polynomials and integrate by parts with the help of the formula

$$P_\ell(x) = \frac{(-1)^\ell}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(1-x^2)^\ell.$$

Let us substitute expression (23) for $\cos^n \theta$ into (22), make use of expression (20) for $(Q_{\mathbf{a}}^{r/2}1, P_\ell(\cos\theta))_s$ and compare the right hand sides of (21) and (22). As a result, we obtain

$$e^{-x^2} H_n(x\cos t) = \frac{\sqrt{\pi}\,n!}{2^{1+n/2}} \sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-4k+1)}{k!\Gamma\left(n-k+\frac{3}{2}\right)} P_{n-2k}(\cos t) \times \int_0^\infty J_{n-2k+1/2}(\sqrt{2}\,rx)\frac{r^{n+2}e^{-r^2/2}}{(\sqrt{2}\,rx)^{1/2}}\,dr. \tag{24}$$

This integral can be evaluated in the same way as the integral in (13). It is equal to

$$2^k k!(\sqrt{2}\,x)^{n-2k} e^{-x^2} L_k^{n-2k+1/2}(x^2).$$

Therefore, (24) can be written as

$$\frac{\sqrt{\pi}\,n!}{2} x^n \sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-4k+1)}{\Gamma\left(n-k+\frac{3}{2}\right)} x^{-2k} P_{n-2k}(\cos t) L_k^{n-2k+1/2}(x) = H_n(x\cos t). \tag{25}$$

It follows from (25) and from the orthogonality relation for Legendre polynomials that

$$\int_0^\pi H_n(x\cos t)P_{n-2k}(\cos t)\sin t\,dt = \frac{(-1)^k\sqrt{\pi}\,n!}{\Gamma\left(n-k+\frac{3}{2}\right)} x^{n-2k} L_k^{n-2k+1/2}(x^2), \tag{26}$$

where $k \le [n/2]$.

# Chapter 10.
# Representations of Groups, Related to $SO(n-1)$, in Non-Canonical Bases, Special Functions, and Integral Transforms

In the preceding chapter we have considered spherical functions of irreducible representations of $SO(n)$ and of related groups with respect to the canonical basis $\{\tilde{\Xi}_M^{n-1,m}\}$ in $\mathfrak{L}^2(S^{n-2})$. This basis is connected with subgroup chains of the form

$$SO(n-1) \supset SO(n-2) \supset \ldots \supset SO(2).$$

One can also consider bases related to other subgroup chains. They are obtained by replacement of $SO(k)$ by one of the subgroups $SO_0(k-1,1)$, $ISO(k-1)$, $SO(p) \times SO(q)$, $p+q=k$. These bases and corresponding special functions and integral transforms are considered in this chapter.

## 10.1. Decompositions of Quasi-Regular Representations and Integral Transforms

**10.1.1. Decomposition of the quasi-regular representation of the group $ISO(n-1)$.** The quasi-regular representation $L$ of the group $ISO(n-1)$ is given by shift operators in the space $\mathfrak{L}^2(\mathbb{R}^{n-1})$. Namely, if $g = g(k, \mathbf{a})$, $k \in SO(n-1)$, $\mathbf{a} \in \mathbb{R}^{n-1}$, then

$$(L(g)f)(\mathbf{x}) = f(g^{-1}\mathbf{x}) = f(k^{-1}(\mathbf{x}-\mathbf{a})). \tag{1}$$

In order to decompose $L$ into irreducible representations of $ISO(n-1)$ we make the Fourier transform

$$F(\mathbf{y}) = \int_{\mathbb{R}^{n-1}} f(\mathbf{x}) e^{i(\mathbf{y},\mathbf{x})} d\mathbf{x}. \tag{2}$$

Then the operators $L(g)$ turn into

$$(T(g)F)(\mathbf{y}) = e^{-i(\mathbf{y},\mathbf{a})} F(k^{-1}\mathbf{y}). \tag{3}$$

Since the Fourier transform is invertible, the representations $L$ and $T$ are equivalent.

We set $F(\mathbf{y}) = \Phi(\boldsymbol{\xi}, R)$, where $R = (\mathbf{y},\mathbf{y})^{1/2}$, $\boldsymbol{\xi} = \mathbf{y}/R$. If $F \in \mathfrak{L}^2(\mathbb{R}^{n-1})$, then for a fixed $R$ functions $\Phi(\boldsymbol{\xi}, R)$ form the Hilbert space $\mathfrak{H}_R$, where

$$\|F\|^2 \equiv \int_{\mathbb{R}^{n-1}} |F(\mathbf{y})|^2 d\mathbf{y} = \int_0^\infty R^{n-2} dR \int_{S^{n-2}} |\Phi(\boldsymbol{\xi}, R)|^2 d\boldsymbol{\xi}$$
$$= \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty \|\Phi\|_R^2 R^{n-2} dR.$$

Here $\|\Phi\|_R$ is the norm of $\Phi(\xi, R)$ in $\mathfrak{L}^2(S^{n-2})$. Hence, we have

$$\mathfrak{L}^2(\mathbf{R}^{n-1}) = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty \oplus \mathfrak{H}_R \, R^{n-2} dR. \tag{4}$$

It follows from (3) that

$$(T(g)\Phi)(\xi, R) = e^{-iR(\xi, \mathbf{a})} \Phi(k^{-1}\xi, R).$$

Comparing this formula with formula (3) of Section 9.2.4, we see that the restriction of the representation $T$ onto $\mathfrak{H}_R$ defines the irreducible representation $T^{n,-iR}$ of the group $ISO(n-1)$, equivalent to the representation $T^{n,iR}$.

Thus, the quasi-regular representation $L$ of $ISO(n-1)$ decomposes into the direct sum of unitary irreducible representations $T^{n,iR}$, $0 \leq R < \infty$, of this group. Every one of them appears in the decomposition once.

Making use of the inversion formula for the Fourier transform and the equality

$$d\mathbf{y} = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} R^{n-2} dR d\xi,$$

we obtain

$$f(\mathbf{x}) = c \int_0^\infty R^{n-2} dR \int_{S^{n-2}} \Phi(\xi, R) e^{-iR(\mathbf{x}, \xi)} d\xi, \tag{5}$$

where

$$c = \left[2^{n-2} \pi^{(n-1)/2} \Gamma\left(\frac{n-2}{2}\right)\right]^{-1}. \tag{6}$$

It follows from the Plancherel formula for the Fourier transform that

$$\int_{\mathbb{R}^{n-1}} |f(\mathbf{x})|^2 d\mathbf{x} = c \int_0^\infty R^{n-2} dR \int_{S^{n-2}} |\Phi(\xi, R)|^2 d\xi. \tag{7}$$

**10.1.2. Decomposition of the quasi-regular representation of the group $SO_0(n-1,1)$ in $\mathfrak{L}^2(C_+^{n-1})$.** We denote by $\mathfrak{L}^2(H_+^{n-1})$ the Hilbert space of functions on the hyperboloid $H_+^{n-1}$ (see Section 9.1.1) with the scalar product

$$(f_1, f_2) = \int_{H_+^{n-1}} f_1(\xi) \overline{f_2(\xi)} d\xi,$$

where $d\boldsymbol{\xi}$ is the invariant measure on $H_+^{n-1}$ (see Section 9.1.9). The equality

$$(Q_+^{n-1}(g)f)(\boldsymbol{\xi}) = f(g^{-1}\boldsymbol{\xi})$$

defines a unitary representation of the group $SO_0(n-1,1)$ in $\mathfrak{L}^2(H_+^{n-1})$, called a quasi-regular representation of this group. Another quasi-regular representation $Q_-^{n-1}$ is given by the same formula in the Hilbert space $\mathfrak{L}^2(H_-^{n-1})$ of functions on the one-sheeted hyperboloid $H_-^{n-1} = SO_0(n-1,1)/SO_0(n-2,1)$, and the third quasi-regular representation $Q_0^{n-1}$ is given in the Hilbert space $\mathfrak{L}^2(C_+^{n-1})$ of functions on the cone $C_+^{n-1}$.

Let $\Phi$ be an ordinary or generalized function of one variable. The operator

$$(A_\Phi f)(\boldsymbol{\xi}) = \int\limits_{C_+^{n-1}} f(\boldsymbol{\eta})\Phi([\boldsymbol{\xi},\boldsymbol{\eta}])d\boldsymbol{\eta}, \quad \boldsymbol{\xi} \in H_+^{n-1},$$

intertwines the representations $Q_0^{n-1}$ and $Q_+^{n-1}$. The operator

$$(\widehat{A}_\Phi F)(\boldsymbol{\eta}) = \int\limits_{H_+^{n-1}} F(\boldsymbol{\xi})\Phi([\boldsymbol{\xi},\boldsymbol{\eta}])d\boldsymbol{\xi}, \quad \boldsymbol{\eta} \in C_+^{n-1},$$

intertwines the representations $Q_+^{n-1}$ and $Q_0^{n-1}$. The operators intertwining $Q_0^{n-1}$ and $Q_-^{n-1}$, $Q_+^{n-1}$ and $Q_-^{n-1}$, and so on, are constructed analogously.

In the case when the integral converges, the function

$$\Xi(\mathbf{x},\mathbf{y}) = \int\limits_{C_+^{n-1}} \Phi([\mathbf{x},\boldsymbol{\xi}])\Psi([\mathbf{y},\boldsymbol{\xi}])d\boldsymbol{\xi}, \quad \mathbf{x},\mathbf{y} \in E_{n-1,1},$$

is invariant with respect to shifts by elements $g \in SO_0(n-1,1)$, that is, $\Xi(g\mathbf{x},g\mathbf{y}) = \Xi(\mathbf{x},\mathbf{y})$. Therefore, it is a function (possibly, generalized) of $[\mathbf{x},\mathbf{y}]$, $[\mathbf{x},\mathbf{x}]$, $[\mathbf{y},\mathbf{y}]$.

Let us decompose the quasi-regular representation $Q_0^{n-1}$ into irreducible components. For this with every function $h$ on $C_+^{n-1}$ we associated the Mellin transform for $h(t\boldsymbol{\xi})$:

$$\Phi(\boldsymbol{\xi},\sigma) = \int\limits_0^\infty h(t\boldsymbol{\xi})t^{-\sigma-1}dt. \tag{1}$$

It is obvious that $\Phi(\boldsymbol{\xi},\sigma)$ is a homogeneous function of degree $\sigma$ in $\boldsymbol{\xi} \in C_+^{n-1}$, that is, $\Phi(a\boldsymbol{\xi},\sigma) = a^\sigma\Phi(\boldsymbol{\xi},\sigma)$, $a > 0$. Applying the inversion formula for the Mellin transform and setting $t = 1$, we obtain

$$h(\boldsymbol{\xi}) = \frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty} \Phi(\boldsymbol{\xi},\sigma)d\sigma. \tag{2}$$

We choose on $C_+^{n-1}$ a contour $\Gamma$, intersecting every generatrix of the cone at one point, and denote by $d\gamma$ the measure on $\Gamma$ such that for $\boldsymbol{\xi} = t\gamma$, $\gamma \in \Gamma$, we have $d\boldsymbol{\xi} = t^{n-3}dtd\gamma$ (see formula (14) of Section 9.1.9). Then the equality

$$\int\limits_{C_+^{n-1}} |h(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} = \frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty} d\sigma \int\limits_{\Gamma} |\Phi(\gamma,\sigma)|^2 d\gamma, \ a = -\frac{n-2}{2}, \tag{3}$$

holds. It shows that

$$\mathfrak{L}^2(C_+^{n-1}) = \frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty} \bar{\mathfrak{B}}^{n\sigma} d\sigma, \quad a = -\frac{n-2}{2}, \tag{4}$$

where $\bar{\mathfrak{B}}^{n\sigma}$ is the closure of the space $\mathfrak{B}^{n\sigma}$ from Section 9.2.1 with respect to the corresponding scalar product. Since $Q_0^{n-1}$ defines in $\bar{\mathfrak{B}}^{n\sigma}$ the representation $T^{n\sigma}$ of the principal unitary series (for $a = -\frac{n-2}{2}$), then the quasi-regular representation $Q_0^{n-1}$ of $SO_0(n-1,1)$ is the continuous direct sum of the unitary representations $T^{n\sigma}$, $\sigma = -i\rho - \frac{n-2}{2}$, $-\infty < \rho < \infty$. Since $T^{n\sigma}$ and $T^{n,-\sigma-n+2}$ are equivalent, then every irreducible representation of the principal unitary series appears in the decomposition twice.

**10.1.3. The Gel'fand-Graev transform.** In order to obtain the decomposition of the representation $Q_+^{n-1}$ of $SO_0(n-1,1)$ into irreducible components, we construct the intertwining oeprator for $Q_+^{n-1}$ and $Q_0^{n-1}$. It has the form

$$(Af)((\boldsymbol{\xi}) \equiv h(\boldsymbol{\xi}) = \int\limits_{H_+^{n-1}} f(\boldsymbol{\eta})\delta([\boldsymbol{\eta},\boldsymbol{\xi}]-1)d\boldsymbol{\eta}, \quad \boldsymbol{\xi} \in C_+^{n-1}, \tag{1}$$

where $f$ belongs to $\mathfrak{D}(H_+^{n-1})$ and $\mathfrak{D}(H_+^{n-1})$ is the space of finite infinitely differentiable functions on $H_+^{n-1}$. Let us show that the function $h(\boldsymbol{\xi})$ belongs to $\mathfrak{D}(C_+^{n-1})$, where $\mathfrak{D}(C_+^{n-1})$ is the space of finite infinitely differentiable functions, vanishing in some neighborhood of the point $\boldsymbol{\xi} = 0$.

Infinite differentiability of $h(\boldsymbol{\xi})$ is evident. In order to prove that $h(\boldsymbol{\xi})$ is equal to zero in some neighborhood of the point $\boldsymbol{\xi} = 0$, we utilize the inequality

$$|[\boldsymbol{\eta},\boldsymbol{\xi}]| < \sqrt{\eta_1^2 + \cdots + \eta_{n-1}^2}\ \sqrt{\xi_1^2 + \cdots + \xi_{n-1}^2} + |\eta_n \xi_n|,$$

where $\boldsymbol{\eta} \in H_+^{n-1}$, $\boldsymbol{\xi} \in C_+^{n-1}$. We obtain from here that

$$|[\boldsymbol{\eta},\boldsymbol{\xi}]| < |\xi_n|\left(|\eta_n| + \sqrt{\eta_n^2 - 1}\right). \tag{1'}$$

Since the function $f(\boldsymbol{\eta})$ from (1) is finite, then there is $N > 1$ such that $f(\boldsymbol{\eta}) = 0$ for $|\eta_n| > N$. By virtue of (1′) for $|\xi_n| < \left(N + \sqrt{N^2-1}\right)^{-1}$ and for $|\eta_n| < N$ we have $\|[\boldsymbol{\eta}, \boldsymbol{\xi}]\| < 1$. Therefore, decomposing the integral (1) into the sum of integrals over the domains $|\eta_n| < N$ and $|\eta_n| > N$, we see that it vanishes for $|\xi_n| < \left(N + \sqrt{N^2-1}\right)^{-1}$. In order to show that (1) is a finite function, we use the inequality

$$\|[\boldsymbol{\eta}, \boldsymbol{\xi}]\| > |\xi_n|\left(|\eta_n| - \sqrt{\eta_n^2 - 1}\right)$$

which is proved in the same way as inequality (1′).

Note that the function $h(\boldsymbol{\xi})$ from (1) satisfies the symmetry condition

$$H(\mathbf{a}, t) = H(\mathbf{a}, t^{-1}), \tag{2}$$

where

$$H(\mathbf{a}, t) = \int_{C_+^{n-1}} h(\boldsymbol{\xi})\delta([\mathbf{a}, \boldsymbol{\xi}] - t)d\boldsymbol{\xi}. \tag{3}$$

In fact, the kernel

$$K(\mathbf{a}, \boldsymbol{\eta}, t) = \int_{C_+^{n-1}} \delta([\boldsymbol{\eta}, \boldsymbol{\xi}] - 1)\delta([\mathbf{a}, \boldsymbol{\xi}] - t)d\boldsymbol{\xi} \tag{4}$$

is invariant with respect to simultaneous shifts of the points $\mathbf{a}$ and $\boldsymbol{\eta}$. In particular, it is invariant with respect to the permutation of $\mathbf{a}$ and $\boldsymbol{\eta}$. Replacing in (4) $\boldsymbol{\xi}$ with $t\boldsymbol{\xi}$, we see that this kernel is also invariant with respect to replacement of $t$ by $t^{-1}$. This gives symmetry relation (2).

Let us show that the operator $A$ is invertible on the subspace $\mathfrak{D}_s(C_+^{n-1})$ of functions $h$ from $\mathfrak{D}(C_+^{n-1})$ satisfying condition (2). This will follow from the following statement.

*If $f(\boldsymbol{\eta})$ is a finite infinitely differentiable function on $H_+^{n-1}$ and if $h(\boldsymbol{\xi})$ is defined by (1), then one has the equality*

$$f(\mathbf{a}) = \frac{(-1)^m}{2(2\pi)^{2m}} \int_{C_+^{n-1}} \delta^{(2m)}([\mathbf{a}, \boldsymbol{\xi}] - 1)h(\boldsymbol{\xi})d\boldsymbol{\xi} \tag{5}$$

*if $n = 2m + 2$ and the equality*

$$f(\mathbf{a}) = \frac{(-1)^m\Gamma(2m)}{(2\pi)^{2m}} \int_{C_+^{n-1}} ([\mathbf{a}, \boldsymbol{\xi}] - 1)^{-2m}h(\boldsymbol{\xi})d\boldsymbol{\xi} \tag{6}$$

*if* $n = 2m + 1$. The integral in (6) is understood as the analytic continuation in $\lambda$ into the point $\lambda = -2m$ of the integral

$$\int_{C_+^{n-1}} ([\mathbf{a}, \boldsymbol{\xi}] - 1)^{-\lambda} h(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

The inversion formulas (5) and (6) are proved as follows. At first we note that $\mathbf{a} \in H_+^{n-1}$, $\boldsymbol{\xi} \in C_+^{n-1}$ imply $[\mathbf{a}, \boldsymbol{\xi}] > 0$. The equality

$$M(h) = \int_{C_+^{n-1}} [\mathbf{a}, \boldsymbol{\xi}]^{-\frac{\mu+n-1}{2}} |[\mathbf{a}, \boldsymbol{\xi}] - 1|^{\mu} h(\boldsymbol{\xi}) d\boldsymbol{\xi} \tag{7}$$

defines a linear functional in $\mathfrak{D}_s(C_+^{n-1})$. If $h = Af$, then

$$M(h) = \int_{H_+^{n-1}} \Phi(\mathbf{x}, \mathbf{a}; \mu) f(\mathbf{x}) d\mathbf{x}, \tag{8}$$

where

$$\Phi(\mathbf{x}, \mathbf{a}; \mu) = \int_{C_+^{n-1}} \delta([\mathbf{x}, \boldsymbol{\xi}] - 1)[\mathbf{a}, \boldsymbol{\xi}]^{-\frac{\mu+n-1}{2}} |[\mathbf{a}, \boldsymbol{\xi}] - 1|^{\mu} d\boldsymbol{\xi}. \tag{9}$$

We shall prove that the function $F(\mathbf{x}) \equiv \Phi(\mathbf{x}, \mathbf{a}; \mu)$ at the point $\mu = -n + 1$ has a singularity of the type of the $\delta$-function concentrated at the point $a$. Then equalities (7) and (8) at $\mu = -(n-1)$ will give the expression for $f(\mathbf{a})$ in terms of $h(\boldsymbol{\xi})$.

The integrand expression in (9) is not changed under simultaneous shifts of the points $\mathbf{a}$ and $\mathbf{x}$ by elements from $SO_0(n-1, 1)$, since the expressions $[\mathbf{x}, \boldsymbol{\xi}]$ and $[\mathbf{a}, \boldsymbol{\xi}]$, as well as the measure $d\boldsymbol{\xi}$, are invariant under these shifts. But any pair of points $(\mathbf{a}, \mathbf{x})$, where $\mathbf{a}, \mathbf{x} \in H_+^{n-1}$, can be shifted into the pair $(\mathbf{a}_0, \mathbf{x}_0)$, where $\mathbf{a}_0 = (0, \ldots, 0, \sinh r, \cosh r)$, $\mathbf{x}_0 = (0, \ldots, 0, 1)$ and $\cosh r = [\mathbf{a}, \mathbf{x}]$. Hence,

$$\begin{aligned}
\Phi(\mathbf{x}, \mathbf{a}; \mu)& \\
&= \int_{C_+^{n-1}} \delta(\xi_n - 1)(\xi_n \cosh r - \xi_{n-1} \sinh r)^{-\frac{\mu+n-1}{2}} \\
&\quad \times |\xi_n \cosh r - \xi_{n-1} \sinh r - 1|^{\mu} \frac{d\xi_1 \ldots d\xi_{n-1}}{\xi_n} \\
&= \frac{2\pi^{(n-2)/2}}{\Gamma\left(\frac{n-2}{2}\right)} \int_{S^{n-2}} (\cosh r - \eta_{n-1} \sinh r)^{-\frac{\mu+n-1}{2}} |\cosh r - \eta_{n-1} \sinh r - 1|^{\mu} d\eta.
\end{aligned} \tag{10}$$

We calculate this integral with the help of the spherical coordinates:

$$\Phi(\mathbf{x}, \mathbf{a}; \mu) = \frac{2\pi^{\frac{n-2}{2}}\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{\mu+n-1}{2}\right)} \sinh^{\mu} r F\left(-\frac{\mu}{2}, \frac{1-\mu-n}{2}; \frac{1}{2}; \tanh^2 r\right). \tag{11}$$

Since the function $\Phi(\mathbf{x}, \mathbf{a}; \mu)$ is invariant under simultaneous shifts of $\mathbf{x}$ and $\mathbf{a}$, one can assume that $\mathbf{a} = (0, \ldots, 0, 1)$. In this case $\cosh r = x_n$. Hence, $[\mathbf{x}, \mathbf{x}] = 1$ implies that $x_1^2 + \cdots + x_{n-1}^2 = \sinh^2 r$. Therefore,

$$\Phi(\mathbf{x}, \mathbf{a}; \mu) = \frac{2\pi^{\frac{n-2}{2}}\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{\mu+n-1}{2}\right)} \rho^{\mu} F\left(-\frac{\mu}{2}; \frac{1-\mu-n}{2}; \frac{1}{2}; \frac{\rho^2}{x_n^2}\right), \tag{12}$$

where $\rho^2 = x_1^2 + \cdots + x_{n-1}^2$. In this expression only $\rho^{\mu}$ has a singularity at $\rho = 0$. As we have noted in Section 3.1.6, the function $\rho^{\mu}$ has the simple pole at $\mu = -n+1$ with the residue

$$\operatorname*{Res}_{\mu=-n+1} \rho^{\mu} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \delta(x_1, \ldots, x_{n-1}).$$

If $\mathbf{x} \in H_+^{n-1}$, then for $x_1 = \ldots = x_{n-1} = 0$ we have $\mathbf{x} = (0, \ldots, 0, 1) = \mathbf{a}$. Therefore,

$$\operatorname*{Res}_{\mu=-n+1} \int_{H_+^{n-1}} \rho^{\mu} f(\mathbf{x}) d\mathbf{x} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} f(\mathbf{a}). \tag{13}$$

We now consider the kernel

$$K(\mathbf{a}, \boldsymbol{\xi}) = [\mathbf{a}, \boldsymbol{\xi}]^{-\frac{\mu+n-1}{2}} |[\mathbf{a}, \boldsymbol{\xi}] - 1|^{\mu}$$

from (7). The manifold obtained by intersecting the cone $[\boldsymbol{\xi}, \boldsymbol{\xi}] = 0$ by the plane $[\mathbf{a}, \boldsymbol{\xi}] = 1$ is compact and has no singularities. The generalized function $|t|^{\mu}$ of one variable has a simple pole at $\mu = -n+1 = -2m-1$ with the residue

$$\operatorname*{Res}_{\mu=-2m-1} |t|^{\mu} = \frac{2\delta^{(2m)}(t)}{\Gamma(2m+1)} \tag{14}$$

(see Section 3.1.6). Comparing the right hand sides of (7) and (8) at $\mu = -2m-1$, we obtain (5). But if $\mu = -n+1 = -2m$, then the generalized function $|t|^{\mu}$ is regular at $\mu = -2m$ and takes the value $t^{-2m}$ at this point. Taking into account the expression for $\Phi(\mathbf{x}, \mathbf{a}; \mu)$, we obtain (6).

One can easily show that any function from $\mathfrak{D}(C_+^{n-1})$, satisfying the symmetry condition (2), has the form $h(\boldsymbol{\xi}) = (Af)(\boldsymbol{\xi})$, where $f$ is given by (5) and (6). From here we conclude that the operator $A$ is invertible on $\mathfrak{D}_s(C_+^{n-1})$.

**10.1.4. Decomposition of the quasi-regular representation $Q_+^{n-1}$ of the group $SO_0(n-1,1)$.** Since we have obtained the decomposition of the representation $Q_0^{n-1}$ of $SO_0(n-1,1)$ into irreducible components and the intertwining operator $A$ for the representations $Q_0^{n-1}$ and $Q_+^{n-1}$, then it is easy to find the decomposition[1] of $Q_+^{n-1}$. Since the image of $\mathfrak{L}^2(H_+^{n-1})$ coincides with the space of functions on $C_+^{n-1}$ satisfying symmetry condition (2) of Section 10.1.3, then the representations $T^{n\sigma}$, $\sigma = i\rho - \frac{n-2}{2}$, $0 < \rho < \infty$, appear in the decomposition of $Q_+^{n-1}$ with multiplicities 1.

In order to construct this decomposition in an explicit form, we note that, by virtue of formulas (1) of Section 10.1.2 and (1) of Section 10.1.3, we have

$$\Phi(\boldsymbol{\xi},\sigma) = \int\limits_0^\infty t^\sigma dt \int\limits_{H_+^{n-1}} f(\boldsymbol{\eta})\delta([\boldsymbol{\eta},\boldsymbol{\xi}] - t)d\boldsymbol{\eta}, \quad \boldsymbol{\xi} \in C_+^{n-1} \tag{1}$$

(we have replaced $t$ by $t^{-1}$). Changing the order of integration, we obtain

$$\Phi(\boldsymbol{\xi},\sigma) = \int\limits_{H_+^{n-1}} f(\boldsymbol{\eta})\left[\int\limits_0^\infty t^\sigma \delta([\boldsymbol{\eta},\boldsymbol{\xi}] - t)dt\right]d\boldsymbol{\eta}. \tag{2}$$

Hence,

$$\Phi(\boldsymbol{\xi},\sigma) = \int\limits_{H_+^{n-1}} f(\boldsymbol{\eta})[\boldsymbol{\eta},\boldsymbol{\xi}]^\sigma d\boldsymbol{\eta}. \tag{3}$$

Now by means of inversion formulas (5) and (6) of Section 10.1.3, we derive the expresion for $f(\boldsymbol{\eta})$ in terms of $\Phi(\boldsymbol{\xi},\sigma)$. Using these formulas and equality (1) for $n = 2m+2$, we have

$$f(\boldsymbol{\eta}) = \frac{(-1)^m}{2(2\pi)^{2m+1}i} \int\limits_{a-i\infty}^{a+i\infty} d\sigma \int\limits_{C_+^{n-1}} \Phi(\boldsymbol{\xi},\sigma)\delta^{(2m)}([\boldsymbol{\eta},\boldsymbol{\xi}] - 1)d\boldsymbol{\xi}. \tag{4}$$

Since the function $\Phi(\boldsymbol{\xi},\sigma)$ is homogeneous, it is uniquely defined by its values on any contour intersecting every generatrix of $C_+^{n-1}$ at one point, in particular, on the section of the cone by the plane $\xi_n = 1$ (that is, on the sphere $S^{n-2}$). We denote points of $S^{n-2}$ by $\boldsymbol{\xi}'$. The equality

$$d(t\boldsymbol{\xi}') = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} t^{n-3} dt d\boldsymbol{\xi}'$$

[1] This decomposition can be also obtained by using the inversion formula for the Jacobi transform (see Section 7.8.7). We give the derivation, based on the Gel'fand-Graev transform, since it gives the geometric sense of the problem.

holds. It follows from here that for $n = 2m+2$ we have

$$f(\eta) = \frac{(-1)^m}{2^{2m+1}\pi^{m+1/2}\Gamma\left(m+\frac{1}{2}\right)i} \int_{a-i\infty}^{a+i\infty} d\sigma \int_0^\infty t^{2m-1} \int_{S^{n-2}} \Phi(t\xi', \sigma)$$

$$\times\, \delta^{(2m)}([\eta, t\xi'] - 1)d\xi' dt = \frac{(-1)^m}{2^{2m+1}\pi^{m+1/2}\Gamma\left(m+\frac{1}{2}\right)i}$$

$$\times \int_{a-i\infty}^{a+i\infty} d\sigma \int_{S^{n-2}} \Phi(\xi', \sigma) \int_0^\infty t^{\sigma-2}\delta^{(2m)}([\eta, \xi'] - t^{-1})dt d\xi',$$

where $-2m < a < 0$. However,

$$\int_0^\infty t^{\sigma-2}\delta^{(2m)}([\eta, \xi'] - t^{-1})dt = \int_0^\infty t^{-\sigma}\delta^{(2m)}([\eta, \xi'] - t)dt$$

$$= \frac{\Gamma(\sigma+2m)}{\Gamma(\sigma)}[\eta, \xi']^{-\sigma-2m}$$

and, therefore, for $n = 2m+2$, $-2m < a < 0$ we obtain

$$f(\eta) = \frac{(-1)^m}{2^{2m+1}\pi^{m+1/2}\Gamma\left(m+\frac{1}{2}\right)i}$$

$$\times \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(\sigma+2m)}{\Gamma(\sigma)} d\sigma \int_{S^{n-2}} \Phi(\xi', \sigma)[\eta, \xi']^{-\sigma-2m} d\xi'. \tag{5}$$

Analogously, one derives from formula (6) of Section 10.1.3 that for $n = 2m+1$ we have

$$f(\eta) = \frac{(-1)^{m+1}}{2^{2m}\pi^m\Gamma(m)i} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(\sigma+2m-1)}{\Gamma(\sigma)} \tan^{-1}\pi\sigma\, d\sigma$$

$$\times \int_{S^{n-2}} \Phi(\xi', \sigma)[\eta, \xi']^{-\sigma-2m+1} d\xi', \tag{6}$$

where $-2m+1 < a < 0$. Here we have used the equality

$$\int_0^\infty (t-1)^{-2m} t^{\sigma+2m-2} dt = -\frac{\pi \tan^{-1}\pi\sigma\, \Gamma(\sigma+2m-1)}{\Gamma(2m)\Gamma(\sigma)}.$$

This integral is understood in the sense of regularized value.

Formulas (5) and (6) can be written down as one formula

$$f(\eta) = \frac{(-1)^{(n-2-\varepsilon)/2}}{2(2\pi)^{n-1}i} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(\sigma+n-2)}{\Gamma(\sigma)} \tan^{-\varepsilon} \pi\sigma d\sigma \times \int_{\Gamma} \Phi(\boldsymbol{\xi}',\sigma)[\eta,\boldsymbol{\xi}']^{-\sigma-n+2} d\boldsymbol{\xi}', \tag{7}$$

where $\varepsilon = \frac{1}{2}(1-(-1)^n)$, $-n+2 < a < 0$, $\Gamma$ is a contour on $C_+^{n-1}$ intersecting every generatrix of the cone at one point, and $d\boldsymbol{\xi}'$ is the measure on $\Gamma$ such that $d(t\boldsymbol{\xi}') = t^{n-3}d\boldsymbol{\xi}'$. We have used independence of the integral of the choice of a contour.

We derive from (7) the expression for $\|f\|^2$ in terms of $\Phi(\boldsymbol{\xi}',\sigma)$. For this we replace $f(\eta)$ by expression (7) in the integral

$$\|f\|^2 = \int_{H_+^{n-1}} f(\eta)\overline{f(\eta)}d\eta$$

and make use of formula (3). We find

$$\|f\|^2 = \frac{(-1)^{(n-2-\varepsilon)/2}}{2(2\pi)^{n-1}i} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(\sigma+n-2)}{\Gamma(\sigma)} \tan^{-\varepsilon} \pi\sigma d\sigma \times \int_{\Gamma} \Phi(\boldsymbol{\xi}',\sigma)\overline{\Phi(\boldsymbol{\xi}',-\bar{\sigma}-n+2)}d\boldsymbol{\xi}', \tag{8}$$

where $-n+2 < a < 0$.

For $a = -\frac{n-2}{2}$ this formula has a simpler form. In this case $\sigma = -\bar{\sigma}-n+2$ and, therefore,

$$\|f\|^2 = \frac{(-1)^{(n-2-\varepsilon)/2}}{2(2\pi)^{n-1}} \int_{-\infty}^{+\infty} \frac{\Gamma\left(i\rho+\frac{n-2}{2}\right)}{\Gamma\left(i\rho-\frac{n-2}{2}\right)} \tanh^{\varepsilon} \pi\rho \, d\rho \times \int_{\Gamma} |\psi(\boldsymbol{\xi}',\rho)|^2 d\boldsymbol{\xi}', \tag{9}$$

where $\psi(\boldsymbol{\xi}',\rho) = \Phi\left(\boldsymbol{\xi}', i\rho - \frac{n-2}{2}\right)$.

Making use of symmetry relation (2) of Section 10.1.3, we have

$$\int_\Gamma \left|\Phi\left(\gamma, i\rho - \frac{n-2}{2}\right)\right|^2 d\gamma = \int_\Gamma \left|\Phi\left(\gamma, -i\rho - \frac{n-2}{2}\right)\right|^2 d\gamma. \tag{10}$$

Hence, the integral in (9) can be replaced by the doubled integral over the ray $[0,+\infty)$.

The space of functions $\psi(\boldsymbol{\xi}', \rho)$ from formula (9) with a fixed $\rho$ will be denoted by $\mathfrak{H}_\rho$. It coincides with the Hilbert space $\mathfrak{L}^2(S^{n-2})$. When functions $f(\boldsymbol{\eta})$ are transformed in accordance with the quasi-regular representation $Q_+^{n-1}$, then $\psi(\boldsymbol{\xi}', \rho)$ are transformed in accordance with the unitary representation $T^{n\sigma}$, $\sigma = i\rho - \frac{n-2}{2}$. Therefore, formula (9) implies that

$$\mathfrak{L}^2(H_+^{n-1}) = \tilde{c}_n \int_0^\infty \oplus \mathfrak{H}_\rho d\mu_n(\rho),$$

where $\tilde{c}_n$ coincides with the coefficient before the integral in (9), multiplied by 2,

$$d\mu_n(\rho) = \frac{\Gamma\left(i\rho + \frac{n-2}{2}\right)}{\Gamma\left(i\rho - \frac{n-2}{2}\right)} \tanh^\varepsilon \pi\rho \, d\rho, \tag{11}$$

and that

$$Q_+^{n-1} = \tilde{c}_n \int_0^\infty T^{n, i\rho-(n-2)/2} d\mu_n(\rho). \tag{12}$$

**10.1.5. Restrictions of the representation $T^{n\sigma}$ of $SO_0(n-1,1)$ onto subgroups.** Using the realization of the representation $T^{n\sigma}$ of the group $SO_0(n-1,1)$ in the space $\mathfrak{L}^2(S^{n-2})$ of functions $S^{n-2}$ (see Section 9.1.2), we have found in Section 9.2.6 that

$$T^{n\sigma} \Big\downarrow \begin{matrix} SO_0(n-1,1) \\ SO(n) \end{matrix} = \sum_{\ell=0}^\infty \oplus T^{n-1,\ell}.$$

Let us analyze what representations appear in the decompositions of the restrictions of $T^{n\sigma}$ onto the subgroups $SO_0(n-2,1)$, $ISO(n-1)$ and $SO(x) \times SO_0(n-s-1,1)$. We realize $T^{n\sigma}$ in the spaces of functions on corresponding contours $\Gamma$.

Let $\Gamma_1^\pm$ be the sections of the cone $C_+^{n-1}$ by the planes $\xi_{n-1} = \pm 1$, which coincide with two hyperboloids $H_+^{n-2}$. If $f(\boldsymbol{\xi}) \in \mathfrak{B}^{n\sigma}$ (see Section 9.2.1), then values of $f$ on $\Gamma_1^\pm$ are denoted by

$$F(\boldsymbol{\eta}, \varepsilon) \equiv F(\eta_1, \ldots, \eta_{n-1}, \varepsilon) = f(\eta_1, \ldots, \eta_{n-2}, \varepsilon, \eta_{n-1}). \tag{1}$$

By means of formula (2) of Section 9.2.1 we find that

$$(T^{n\sigma}(h)F)(\eta,\varepsilon)=F(h^{-1}\eta,\varepsilon) \tag{2}$$

for $h\in SO_0(n-2,1)$.

Under the transition from values of functions $f\in\mathfrak{B}^{n\sigma}$ on $S^{n-2}$ to their values on $\Gamma_1^{\pm}$ scalar product (8) of Section 9.2.1 turns into the scalar product

$$(F_1,F_2)=\sum_{\varepsilon=\pm1}\int_{H_+^{n-2}}F_1(\eta,\varepsilon)\overline{F_2(\eta,\varepsilon)}d\eta, \tag{3}$$

where $d\eta$ is the invariant measure on $H_+^{n-2}$ (see Section 9.1.9). Realizing $T^{n\sigma}$ in the Hilbert space of functions $F(\eta,\varepsilon)$ with scalar product (3), we derive from (2) that the restriction of the representation $T^{n\sigma}$ of $SO_0(n-1,1)$ onto $SO_0(n-2,1)$ is equivalent to the sum of two quasi-regular representations of $SO_0(n-2,1)$ in the spaces $\mathfrak{L}^2(H_+^{n-2})$. Applying formula (12) of Section 10.1.4, we conclude that the restriction of $T^{n\sigma}$ onto $SO_0(n-2,1)$ decomposes into the direct integral of the representations $T^{n-1,i\nu-(n-3)/2}$, $0<\nu<\infty$, of this subgroup, and that every representation appears in the decomposition twice.

It follows from decomposition (15) of Section 9.1.6 that the section $\Gamma_2$ of the cone $C_+^{n-1}$ by the plane $x_{n-1}+x_n=1$ intersects every generatrix of the cone at one point. The contour $\Gamma_2$ can be identified with $\mathbb{R}^{n-2}$. We write down values of functions $f\in\mathfrak{B}^{n\sigma}$ on $\Gamma_2$ in the form

$$F(\mathbf{t})\equiv F(t_1,\ldots,t_{n-2})=f\left(t_1,\ldots,t_{n-2},\frac{1}{2}(1-t),\frac{1}{2}(1+t)\right),$$

where

$$t=t_1^2+\ldots+t_{n-2}^2.$$

By means of formula (2) of Section 9.2.1, it is easy to find that

$$(T^{n\sigma}(\bar{n}(\mathbf{s}))F)(\mathbf{t})=F(\mathbf{t}+\mathbf{s}),\quad \bar{n}(\mathbf{s})\in\bar{N}, \tag{4}$$

$$(T^{n\sigma}(h)F)(\mathbf{t})=F(h^{-1}\mathbf{t}),\quad h\in SO(n-2). \tag{5}$$

The correspondence $h\bar{n}(-\mathbf{s})\leftrightarrow g(h,\mathbf{s})\in ISO(n-2)$ gives an isomorphism between $SO(n-2)\bar{N}$ and $ISO(n-2)$. Formulas (4) and (5) show that the restriction of the representation $T^{n\sigma}$ of $SO_0(n-1,1)$ onto the subgroup $ISO(n-2)$ is equivalent to the quasi-regular representation of this subgroup. Making use of the results of Section 10.1.1, we conclude that the restriction of $T^{n\sigma}$ onto $ISO(n-2)$ decomposes into the direct integral of irreducible unitary representations $T^{n-1,iR}$, $0<R<\infty$, of this subgroup and that every one of them appears in the decomposition with multiplicity 1.

Let $\Gamma_3$ be the section of the cone by the cylinder $\xi_1^2+\dots+\xi_p^2=1$. Since $[\boldsymbol{\xi},\boldsymbol{\xi}]=0$, then points $\boldsymbol{\xi}\in C_+^{n-1}$ with fixed $\xi_1,\dots,\xi_p$ form the hyperboloid $H_+^{q-1}$, $p+q=n$. Hence, $\Gamma_3\sim S^{p-1}\times H_+^{q-1}$. One can show that $\Gamma_3$ intersects every generatrix of $C_+^{n-1}$ at one point. We denote values of functions $f\in\mathfrak{B}^{n\sigma}$ on $\Gamma_3$ by $F(\boldsymbol{\eta},\boldsymbol{\zeta})$, $\boldsymbol{\eta}\in S^{p-1}$, $\boldsymbol{\zeta}\in H_+^{q-1}$. With the help of formula (2) of Section 9.2.1, one proves that

$$(T^{n\sigma}(k)F)(\boldsymbol{\eta},\boldsymbol{\zeta})=F(k^{-1}\boldsymbol{\eta},\boldsymbol{\zeta}),\quad k\in SO(p),$$
$$(T^{n\sigma}(h)F)(\boldsymbol{\eta},\boldsymbol{\zeta})=F(\boldsymbol{\eta},h^{-1}\boldsymbol{\zeta}),\quad h\in SO_0(q-1,1).$$

Consequently, the restriction of the representation $T^{n\sigma}$ of the group $SO_0(n-1,1)$ onto the subgroup $K_H\equiv SO(p)\times SO_0(q-1,1)$, $p+q=n$, is equivalent to the tensor product of the quasi-regular representations of $SO(p)$ and of $SO_0(q-1,1)$, realized on the sphere $S^{p-1}$ and on the hyperboloid $H_+^{q-1}$, respectively. Therefore, the restriction of $T^{n\sigma}$ onto $K_H$ decomposes into the direct sum with respect to $m$ and into the direct integral with respect to $\rho$ of the representations

$$T^{pm}\otimes T^{q,i\rho-(q-2)/2},\quad m=0,1,2,\dots,\quad 0<\rho<\infty,$$

of $K_H$. Every one of these representations appears in the decomposition once.

**10.1.6. Decomposition of the quasi-regular representation of $SO_0(p,q)$.** The formula

$$T(g)f(\mathbf{x})=f(g^{-1}\mathbf{x}),\quad \mathbf{x}\in H_+^{p,q},\quad g\in SO_0(p,q), \tag{1}$$

defines the quasi-regular representation of the group $SO_0(p,q)$ in the space $\mathfrak{L}^2(H_+^{pq})$ of functions on the hyperboloid

$$H_+^{pq}=SO_0(p,q)/SO_0(p,q-1).$$

Since the stabilizer subgroup $SO_0(p,q-1)$ of the point $\mathbf{e}_n\in H_+^{pq}$, $p+q=n$, is noncompact, then the decomposition of the representation $T$ of $SO_0(p,q)$ contains both continuous series representations and discrete series representations.

One obtains the decomposition of $T$ into irreducible components by means of the formula

$$\varphi_{\sigma\varepsilon}(\mathbf{y})=\int_{H_+^{pq}}|[\mathbf{x},\mathbf{y}]_{pq}|^{\sigma}\operatorname{sign}^{\varepsilon}[\mathbf{x},\mathbf{y}]_{pq}f(\mathbf{x})d\mathbf{x},\ \ \mathbf{y}\in C^{pq}. \tag{2}$$

A direct verification shows that the functions $\varphi_{\sigma\varepsilon}(\mathbf{y})$ satisfy condition (1) of Section 9.2.9 and that if functions $f(\mathbf{x})$ are transformed in accordance with representation (1), then $\varphi_{\sigma\varepsilon}(\mathbf{y})$ are transformed in accordance with the representation $T_{pq}^{\sigma\varepsilon}$ of

$SO_0(p,q)$ (see formula (2) of Section 9.2.9). The space $\mathfrak{L}^2(H_+^{pq})$ decomposes into irreducible subspaces with respect to $SO_0(p,q)$ as

$$\mathfrak{L}^2(H_+^{pq}) = \frac{1}{2\pi}\sum_{\varepsilon=0,1}\int_0^\infty \mathfrak{L}_\chi^2(C^{pq})|c(\chi)|^{-2}d\rho + \sum_\chi \mathfrak{L}_\chi^2(C^{pq})^0, \tag{3}$$

where $\chi = \left(i\rho - \frac{n-2}{2}, \varepsilon\right)$ in the first summand and the second summation is over all $\chi = \left(\ell - \frac{n-2}{2}, \varepsilon\right)$, for which $\ell + \frac{n-2}{2}$ are positive integers and $\varepsilon \equiv \left(\ell + \frac{n-2}{2} - q\right)$ (mod 2). The space $\mathfrak{L}_\chi^2(C^{pq})$ is the closure of the space $\mathfrak{B}_{pq}^{i\rho-(n-2)/2,\varepsilon}$ (see Section 9.2.9) with respect to scalar product (7) of Section 9.2.9. Representations of the most degenerate principal unitary series are realized in the spaces $\mathfrak{L}_\chi^2(C^{pq})$ from the first summand in (3). The spaces $\mathfrak{L}_\chi^2(C^{pq})^0$ are quotient spaces of the corresponding spaces $\mathfrak{L}_\chi^2(C^{pq})$. In $\mathfrak{L}_\chi^2(C^{pq})^0$ the discrete series representation $T_+^\ell$ (see Section 9.2.10) of $SO_0(p,q)$ is realized. The function $c(\chi)$ is defined by the formula

$$c(\chi) = 2^{n-1}\pi^{(n-2)/2}\frac{\Gamma(i\rho)}{\Gamma\left(i\rho + \frac{n-2}{2}\right)} \times \begin{cases} 1, & \text{if } p \text{ and (or) } q \text{ are odd,} \\ \tan\frac{\pi}{2}\left(i\rho + \frac{n-2}{2} + \varepsilon\right), & \text{if } p \text{ and } q \text{ are even.}\end{cases} \tag{4}$$

The inversion formula for transform (2), taken for a function $f$ from the dense subspace of $\mathfrak{L}^2(H_+^{pq})$ consisting of linear combinations of functions from irreducible subspaces with respect to the action of $K_{pq} = SO(p)\times SO(q)$, has the form

$$\begin{aligned} f(\mathbf{x}) = \frac{2^{-\frac{n-2}{2}}}{2\pi}\sum_{\varepsilon=0,1}\int_0^\infty\int_\Gamma &|[\mathbf{x},\mathbf{y}]_{pq}|^{i\rho-\frac{n-2}{2}}\operatorname{sign}^\varepsilon[\mathbf{x},\mathbf{y}]_{pq} \\ &\times \varphi_\chi(\mathbf{y})d\mathbf{y}|c(\chi)|^{-2}d\rho + \sum_\chi \operatorname{Res}\Big\{c(\chi)^{-1} \\ &\times \int_\Gamma |[\mathbf{x},\mathbf{y}]_{pq}|^{-\ell-\frac{n-2}{2}}\operatorname{sign}^\varepsilon[\mathbf{x},\mathbf{y}]_{pq}\varphi_\chi^0(\mathbf{y})d\mathbf{y}\Big\}, \end{aligned} \tag{5}$$

where $\Gamma = S^{p-1}\times S^{q-1}$, $\varphi_\chi^0(\mathbf{y}) \in \mathfrak{L}_\chi^2(C^{pq})^0$, values of $\chi$ and the summation are the same as in (3). Note that the expression in braces is a meromorphic function in $\ell$ with poles of the first order at the points, over which the summation is carried out.

If $F_\chi(\mathbf{x})$ denotes the integral in the second summand of (5), then for even $q$ the residue of the function $c(\chi)^{-1}F_\chi(\mathbf{x})$ at the point $\chi = \chi_0 \equiv \left(\ell_0 - \frac{n-2}{2}, \varepsilon\right)$ from the summation domain differs from $F_{\chi_0}(\mathbf{x})$ in a constant factor only. And if $q$ is odd, then $c(\chi)^{-1}F_\chi(\mathbf{x})$ differs from

$$\int\limits_{\Gamma} \delta^{(n-1)}([\mathbf{x},\mathbf{y}]_{pq})\varphi_\chi^0(\mathbf{y})d\mathbf{y}$$

in a constant factor only. We leave to the reader calculating this integral for

$$\varphi_\chi^0(\mathbf{y}) = \Xi_M^{pr}(\boldsymbol{\zeta})\Xi_N^{qs}(\boldsymbol{\eta}).$$

## 10.2. The Funk-Hecke Theorem and its Analogs. Continuous Bases and Integral Transforms

**10.2.1. The Funk-Hecke theorem.** The polynomial

$$\check{\Xi}_O^{n\ell}(\mathbf{x}) = \Xi_O^{n\ell}(\mathbf{x})/\Xi_O^{n\ell}(\mathbf{e}_n)$$

(see Section 9.3.1), where $\mathbf{e}_n = (0,\ldots,0,1)$, is the single harmonic polynomial in $\mathbf{x} = (x_1,\ldots,x_n)$ of degree $\ell$, invariant with respect to the action of the group $SO(n-1)$ and such that

$$\check{\Xi}_O^{n\ell}(\mathbf{e}_n) = 1.$$

According to the results of Sections 9.3.1 and 9.3.2 this polynomial can be represented in the form

$$\check{\Xi}_O^{n\ell}(\mathbf{x}) = r^\ell C_\ell^{(n-2)/2}\left(\frac{x_n}{r}\right)\Big/ C_\ell^{(n-2)/2}(1), \quad r^2 = (\mathbf{x},\mathbf{x}),$$

and, since $x_n/r = (\mathbf{x},\mathbf{e}_n)/r$, in the form

$$\check{\Xi}_O^{n\ell}(\mathbf{x}) = r^\ell C_\ell^{(n-2)/2}((\boldsymbol{\xi},\mathbf{e}_n))/C_\ell^{(n-1)/2}(1), \quad \boldsymbol{\xi} = \frac{\mathbf{x}}{r}.$$

If $\mathbf{e}_n$ in this expression is replaced by any vector $\boldsymbol{\eta}\in S^{n-1}$, then we obtain a homogeneous harmonic polynomial, invariant with respect to rotations $h$ such that $h\boldsymbol{\eta} = \boldsymbol{\eta}$. Thus, we have proved the following statement.

**Lemma.** *For any vector $\boldsymbol{\eta}\in S^{n-1}$ there exists one and only one homogeneous harmonic polynomial of degree $\ell$, invariant with respect to rotations $h$, for which $h\boldsymbol{\eta}=\boldsymbol{\eta}$, and equal to the unit for $\mathbf{x}=\boldsymbol{\eta}$. This polynomial coincides with*

$$\check{\Xi}_{\boldsymbol{\eta}}^{n\ell}(\mathbf{x}) = r^\ell C_\ell^{(n-2)/2}((\boldsymbol{\xi},\boldsymbol{\eta}))/C_\ell^{(n-2)/2}(1), \quad \boldsymbol{\xi} = \frac{\mathbf{x}}{r},\ r^2 = (\mathbf{x},\mathbf{x}).$$

It is proved in the theory of invariants that any polynomial of $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, invariant with respect to the action of the group $O(n)$, is a polynomial of $(\mathbf{x},\mathbf{x})$,

$(\mathbf{y},\mathbf{y})$, $(\mathbf{x},\mathbf{y})$. The restriction of this polynomial onto $S^{n-1}$ is a polynomial of $(\boldsymbol{\xi},\boldsymbol{\eta})$. It is obvious that

$$P_\ell(\mathbf{x},\mathbf{y}) = r^\ell R^\ell \int\limits_{S^{n-1}} C_\ell^{(n-2)/2}((\boldsymbol{\xi},\boldsymbol{\eta}))C_\ell^{(n-2)/2}((\boldsymbol{\eta},\boldsymbol{\zeta}))d\boldsymbol{\eta}, \tag{1}$$

where $r^2 = (\mathbf{x},\mathbf{x})$, $R^2 = (\mathbf{y},\mathbf{y})$, $\boldsymbol{\xi} = \frac{\mathbf{x}}{r}$, $\boldsymbol{\zeta} = \frac{\mathbf{y}}{R}$, is a homogeneous polynomial both in $\mathbf{x}$ and in $\mathbf{y}$. It is of degree $\ell$ in $\mathbf{x}$ and of degree $\ell$ in $\mathbf{y}$. Since the measure $d\boldsymbol{\eta}$ is invariant with respect to $SO(n)$, then the equality

$$P_\ell(g\mathbf{x},g\mathbf{y}) = P_\ell(\mathbf{x},\mathbf{y}), \quad g \in SO(n),$$

holds. Hence, for a fixed $\mathbf{y}$ the polynomial $P_\ell(\mathbf{x},\mathbf{y})$ is invariant with respect to the action upon $\mathbf{x}$ of the subgroup of rotations, leaving $\mathbf{y}$ fixed. Applying the lemma proved above, we derive that $P_\ell(\mathbf{x},\mathbf{y}) = A(n,\ell)C_\ell^{(n-2)/2}((\mathbf{x},\mathbf{y}))$. In order to find the coefficient $A(n,\ell)$ we set $\mathbf{x} = \mathbf{y} = \mathbf{e}_n$ into (1):

$$A(n,\ell)C_\ell^{(n-2)/2}(1) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \int\limits_0^\pi \left[C_\ell^{(n-2)/2}(\cos\varphi)\right]^2 \sin^{n-2}\varphi d\varphi.$$

Therefore, $A(n,\ell)$ is evaluated by means of the orthogonality relation and of the values of Gegenbauer polynomials at $\cos\varphi = 1$. Equality (1) leads to the convolution theorem for zonal spherical harmonics:

$$\int\limits_{S^{n-1}} C_\ell^{(n-2)/2}((\boldsymbol{\xi},\boldsymbol{\eta}))C_\ell^{(n-2)/2}((\boldsymbol{\eta},\boldsymbol{\zeta}))d\boldsymbol{\eta} = \frac{n-2}{2\ell+n-2}C_\ell^{(n-2)/2}((\boldsymbol{\xi},\boldsymbol{\zeta})). \tag{2}$$

Since the system of functions $\{\widetilde{\Xi}_M^{n\ell}\}$ is orthogonal in $\widetilde{\mathfrak{H}}^{n\ell}$ and the representations $T^{n\ell}$ of $SO(n)$ are unitary, then

$$Q_\ell(\mathbf{x},\mathbf{y}) = \sum_K \Xi_K^{n\ell}(\mathbf{x})\overline{\Xi_K^{n\ell}(\mathbf{y})}$$

is an $O(n)$-invariant polynomial of degree $\ell$ both in $\mathbf{x}$ and in $\mathbf{y}$. In the same way as in the case of formula (1), we find that for a fixed $\mathbf{y}$ we have

$$\sum_K \Xi_K^{n\ell}(\mathbf{x})\overline{\Xi_K^{n\ell}(\mathbf{y})} = B(n,\ell)r^\ell R^\ell C_\ell^{(n-2)/2}((\boldsymbol{\xi},\boldsymbol{\eta})),$$

where the notations are the same as in (1). By setting $\mathbf{x} = \mathbf{y} = \mathbf{e}_n$ we find that $B(n,\ell) = (2\ell+n-2)/(n-2)$. Consequently, one has

$$\sum_K \Xi_K^{n\ell}(\mathbf{x})\overline{\Xi_K^{n\ell}(\mathbf{y})} = \frac{2\ell+n-2}{n-2}r^\ell R^\ell C_\ell^{(n-2)/2}((\boldsymbol{\xi},\boldsymbol{\eta})). \tag{3}$$

It is the addition theorem for $\Xi_K^{n\ell}$ in general form. The addition theorem for Gegenbauer polynomials from Section 9.4.3 follows from (3).

It follows from (3) that for any $K$ we have

$$\int\limits_{S^{n-1}} C_\ell^{(n-2)/2}((\boldsymbol{\xi},\boldsymbol{\eta}))\Xi_K^{nk}(\boldsymbol{\eta})d\boldsymbol{\eta} = \begin{cases} 0 & \text{if } k \neq \ell, \\ \frac{n-2}{2\ell+n-2}\Xi_K^{n\ell}(\boldsymbol{\xi}) & \text{if } k = \ell. \end{cases} \quad (4)$$

But the functions $\Xi_K^{n\ell}(\boldsymbol{\eta})$ form an orthonormal basis in $\widetilde{\mathfrak{H}}^{n\ell}$ and the functions $C_\ell^{(n-2)/2}(x)$ form a basis in the space of functions on $[-1,1]$ with respect to the weight $(1-x^2)^{(n-3)/2}$. Therefore, the following theorem holds.

**The Funk-Hecke theorem.** *Let $F(x)$ be a function from the space $\mathfrak{L}^2([1,1],\rho(x))$, where $\rho(x)=(1-x^2)^{(n-3)/2}$. Then for every function $\Phi(\boldsymbol{\eta})$ from $\widetilde{\mathfrak{H}}^{n\ell}$ the equality*

$$\int\limits_{S^{n-1}} F((\boldsymbol{\xi},\boldsymbol{\eta}))\Phi(\boldsymbol{\eta})d\boldsymbol{\eta} = \lambda_\ell\Phi(\boldsymbol{\xi}) \quad (5)$$

*holds, where*

$$\lambda_\ell = \frac{2^{n-3}\ell!\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\pi(\ell+n-3)!}\int\limits_{-1}^{1} F(x)C_\ell^{(n-2)/2}(x)(1-x^2)^{(n-3)/2}dx. \quad (6)$$

It follows from formula (7) of Section 9.3.6 that

$$\lambda_\ell = i^\ell 2^{(n-2)/2}\Gamma\left(\frac{n}{2}\right)\int\limits_{-\infty}^{\infty} t^{-(n-2)/2}J_{\ell+(n-2)/2}(t)f(t)dt, \quad (7)$$

where $f(t)$ is the inverse Fourier transform of $F(x)$:

$$f(t) = \frac{1}{2\pi}\int\limits_{-1}^{1} e^{-ixt}F(x)dx. \quad (8)$$

Let us apply the Funk-Hecke theorem for evaluation of some integrals. Let

$$I = \int\limits_{S^{n-2}} (\mathbf{x},\boldsymbol{\zeta})^\ell \Xi_{M'}^{n-1,m}(\boldsymbol{\zeta}')d\boldsymbol{\zeta}', \quad (9)$$

where $\mathbf{x}=(\sin\theta\,\mathbf{x}',\cos\theta)\in S^{n-1}$, $\boldsymbol{\zeta}=(i\boldsymbol{\zeta}',1)$, $\boldsymbol{\zeta}'\in S^{n-2}$, $\mathbf{x}'\in S^{n-2}$, $M'=(m_1,\ldots,\pm m_{n-3})$ and $\Xi_{M'}^{n-1,m}(\boldsymbol{\zeta}')$ is the associated spherical function of the representation $T^{n-1,m}$ of the group $SO(n-1)$ (see Section 9.4.2). Since

$$(\mathbf{x},\boldsymbol{\zeta}) = \cos\theta + i\sin\theta(\mathbf{x}',\boldsymbol{\zeta}'),$$

then according to the Funk-Hecke theorem and formula (5) of Section 9.3.6, we have

$$I = \int_{S^{n-2}} (\cos\theta + i\sin\theta\,(\mathbf{x}',\boldsymbol{\zeta}'))^{\ell}\,\Xi_{M'}^{n-1,m}(\boldsymbol{\zeta}')d\boldsymbol{\zeta}'$$

$$= \frac{2^{m+n-3}\ell!\,\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(m+\frac{n-2}{2}\right)}{\sqrt{\pi}(\ell+m+n-3)!}\sin^m\theta$$

$$\times\, C_{\ell-m}^{m+(n-2)/2}(\cos\theta)\Xi_{M'}^{n-1,m}(\mathbf{x}')$$

$$= \ell!\left(\frac{(n-2)!}{(\ell+m+n-3)!(\ell-m)!(2m+n-2)}\right)^{1/2}\Xi_M^{n\ell}(\mathbf{x}), \qquad (10)$$

where $M=(m,M')$.

Using formulas (6) of Section 9.3.6 and (12′) of Section 9.4.2, we prove analogously that

$$\int_{S^{n-2}} [\mathbf{x},\boldsymbol{\xi}]^{\sigma}\,\Xi_{M'}^{n-1,m}(\boldsymbol{\xi}')d\boldsymbol{\xi}' = \Xi_M^{n\sigma}(\mathbf{x}), \qquad (11)$$

where $\mathbf{x}=(\sinh\theta\,\mathbf{x}',\cosh\theta)\in H_+^{n-1}$, $\boldsymbol{\xi}'=(\boldsymbol{\xi}',1)\in C_+^{n-1}$ and $M$, $\Xi_{M'}^{n-1,m}$ are the same as in formula (10).

With the help of the Funk-Hecke theorem and formula (6′) of Section 9.3.6, we derive that

$$\int_{S^{n-2}} [\mathbf{x},\boldsymbol{\xi}]_{\delta}^{\sigma}\,\Xi_{M'}^{n-1,m}(\boldsymbol{\xi}')d\boldsymbol{\xi}' = \Xi_{M\delta}^{n\sigma}(\mathbf{x}), \qquad (12)$$

where $\mathbf{x}=(\cosh\theta\,\mathbf{x}',\sinh\theta)\in H_-^{n-1}$, $\delta\in\{+,-\}$ and

$$\Xi_{M\delta}^{n\sigma}(\mathbf{x}) = \frac{(-1)^m 2^{(n-4)/2}}{\sqrt{\pi}}\Gamma\left(\frac{n-1}{2}\right)\Gamma(\sigma+1)\cosh^{(2-n)/2}\theta$$

$$\times\, \mathrm{P}_{m+(n-4)/2}^{-\sigma-(n-2)/2}(-\delta\tanh\theta)\Xi_{M'}^{n-1,m}(\mathbf{x}'). \qquad (13)$$

**10.2.2. The analog of the Funk-Hecke theorem for □-harmonic functions.** The statements analogous to those proved in the previous section are valid for spherical functions of representations of the groups $SO_0(n-1,1)$ and $ISO(n-1)$. For example, the analog of relation (3) of Section 10.2.1 has the form

$$\sum_K \Xi_K^{n\sigma}(\mathbf{x})\overline{\Xi_K^{n,-\bar{\sigma}-n+2}(\mathbf{y})} = 2^{(n-3)/2}\Gamma\left(\frac{n-1}{2}\right)(Rr)^{\sigma}$$

$$\times\,([\boldsymbol{\xi},\boldsymbol{\eta}]^2-1)^{-(n-3)/4}\mathfrak{P}_{\sigma+(n-3)/2}^{-(n-3)/2}([\boldsymbol{\xi},\boldsymbol{\eta}]), \qquad (1)$$

where $r^2 = [\mathbf{x},\mathbf{x}] > 0$, $R^2 = [\mathbf{y},\mathbf{y}] > 0$, $\boldsymbol{\xi} = \frac{\mathbf{x}}{r}$, $\boldsymbol{\eta} = \frac{\mathbf{y}}{R}$. In order to prove this relation we take into account that

$$t^{n\sigma}_{KO}(g) = \overline{t^{n,-\bar{\sigma}-n+2}_{OK}(g^{-1})},\ g \in SO_0(n-1,1)$$

(see Section 9.2.7). It follows from here that the left hand side of (1) is invariant with respect to transformations of the form $\mathbf{x} \to g^{-1}\mathbf{x}$, $\mathbf{y} \to g^{-1}\mathbf{y}$, $g \in SO_0(n-1,1)$. This gives the desired result. Let us note that

$$\overline{\Xi^{n,-\bar{\sigma}-n+2}_{K}(\mathbf{y})} = \Xi^{n,-\sigma-n+2}_{\bar{K}}(\mathbf{y}),$$

where $\bar{K} = (k_0, k_1, \ldots, -k_{n-3})$ if $K = (k_0, k_1, \ldots, k_{n-3})$.

The formula

$$\sum_K \Xi^{n,iR}_{K}(\mathbf{x})\overline{\Xi^{n,iR}_{K}(\mathbf{y})} = \Gamma\left(\frac{n-1}{2}\right)(R\|\mathbf{x}-\mathbf{y}\|)^{(3-n)/2} \times J_{(n-3)/2}(R\|\mathbf{x}-\mathbf{y}\|) \qquad (2)$$

is proved analogously. The addition theorems for Legendre and Bessel functions of Section 9.4.3 can be obtained from formulas (1) and (2).

Now we derive an analog of the Funk-Hecke theorem for the group $SO_0(n-1,1)$. A function $\Phi(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in H^{n-1}_+$, is said to be *radial* if $f(h\boldsymbol{\xi}) = f(\boldsymbol{\xi})$, $h \in SO(n-1)$. Every radial function has the form

$$\Phi(\boldsymbol{\xi}) = \varphi([\boldsymbol{\xi}, \mathbf{e}_n]) = \varphi(\cosh\theta), \quad \text{where} \quad \cosh\theta = \xi_n.$$

It follows from formula (12) of Section 9.4.2 and from the orthogonality of functions $\Xi^{n-1,m}_{M'}(\boldsymbol{\xi}')$ on $S^{n-2}$ that if $\Phi(\boldsymbol{\xi})$ is a radial function, then

$$\int_{H^{n-1}_+} \Phi(\boldsymbol{\eta})\Xi^{n\sigma}_{M}(\boldsymbol{\eta})d\boldsymbol{\eta} = \begin{cases} 0 & \text{if } M \neq O, \\ \lambda(\sigma) & \text{if } M = O, \end{cases} \qquad (3)$$

where

$$\lambda(\sigma) = (2\pi)^{(n-1)/2} \times \int_1^\infty \varphi(t)\mathfrak{P}^{-(n-3)/2}_{\sigma+(n-3)/2}(t)(t^2-1)^{(n-3)/4}dt,\ t = [\boldsymbol{\eta}, \mathbf{e}_n]. \qquad (4)$$

Taking into account the equality

$$\Xi^{n\sigma}_{M}(\mathbf{e}_n) = t^{n\sigma}_{MO}(e) = \begin{cases} 0 & \text{if } M \neq O, \\ 1 & \text{if } M = O \end{cases}$$

we obtain

$$\int\limits_{H_+^{n-1}} \varphi([\boldsymbol{\eta},\mathbf{e}_n])\Xi_M^{n\sigma}(\boldsymbol{\eta})d\boldsymbol{\eta} = \lambda(\sigma)\Xi_M^{n\sigma}(\mathbf{e}_n). \tag{5}$$

Since the left hand side of this relation is invariant with respect to the replacements of $\boldsymbol{\eta}$ by $g\boldsymbol{\eta}$ and of $\mathbf{e}_n$ by $g\mathbf{e}_n = \boldsymbol{\xi}$, then (5) implies that

$$\int\limits_{H_+^{n-1}} \varphi([\boldsymbol{\xi},\boldsymbol{\eta}])\Xi_M^{n\sigma}(\boldsymbol{\eta})d\boldsymbol{\eta} = (2\pi)^{(n-1)/2}$$

$$\times\, \Xi_M^{n\sigma}(\boldsymbol{\xi}) \int\limits_1^\infty \varphi(t)\mathfrak{P}_{\sigma+(n-3)/2}^{-(n-3)/2}(t)(t^2-1)^{(n-3)/4}dt. \tag{6}$$

Under the appropriate definition of the scalar product the elements $\Xi_M^{n\sigma}$ form a basis in the space $\mathfrak{H}^{n\sigma}$ of functions on $H_+^{n-1}$, which are eigenfunctions of the operator $\Box_0$ corresponding to the eigenvalue $\sigma(\sigma+n-2)$. We derive from here an analog of the Funk-Hecke theorem:

*Let $\varphi(t)\in\mathfrak{L}_2(I,\rho(t))$, where $I=[1,\infty)$, $\rho(t)=(t^2-1)^{(n-3)/2}$, and $\Psi(\mathbf{x})\in\mathfrak{H}^{n\sigma}$. Then the equality*

$$\int\limits_{H_+^{n-1}} \varphi([\boldsymbol{\xi},\boldsymbol{\eta}])\Psi(\boldsymbol{\eta})d\boldsymbol{\eta} = \lambda(\sigma)\Psi(\boldsymbol{\xi}) \tag{7}$$

*holds, where*

$$\lambda(\sigma) = (2\pi)^{(n-1)/2}\int\limits_1^\infty \varphi(t)\mathfrak{P}_{\sigma+(n-3)/2}^{-(n-3)/2}(t)(t^2-1)^{(n-3)/4}dt. \tag{7'}$$

Formula (10) of Section 9.3.6 implies that

$$\lambda(\sigma) = \frac{2(2\pi)^{(n-4)/2}}{\Gamma\left(\frac{n-1}{2}\right)}\int\limits_{-i\infty}^{i\infty} K_{\sigma-(n-2)/2}(u)u^{-(n-2)/2}f(u)du, \tag{8}$$

where $f(u)$ is the inverse Fourier transform of the function $\varphi$:

$$f(it) = \frac{1}{2\pi}\int\limits_1^\infty e^{-ixt}\varphi(x)dx. \tag{9}$$

As an example we evaluate the integral

$$\overset{*}{\Xi}{}^{n\sigma}_{M\delta}(\mathbf{x}) = \int\limits_{H^{n-2}_{+,\delta}} [\mathbf{x},\boldsymbol{\xi}]^{\sigma}\Xi^{n-1,\tau}_{M'}(\boldsymbol{\xi}')d\boldsymbol{\xi}', \tag{10}$$

where

$$\mathbf{x} = (\cosh\theta\,\mathbf{x}'', \sinh\theta, t\,\cosh\theta) \in H^{n-1}_{+}, \quad \mathbf{x}' = (\mathbf{x}'', t) \in H^{n-2}_{+},$$
$$\boldsymbol{\xi} = (\sinh\varphi\,\boldsymbol{\xi}'', \delta, \cosh\varphi) \in C^{n-1}_{+}, \quad \boldsymbol{\xi}' = (\sinh\varphi\,\boldsymbol{\xi}'', \cosh\varphi) \in H^{n-2}_{+},$$
$$\delta \in \{-1,1\}, \quad M = (\tau, M').$$

According to formula (7) we have

$$\overset{*}{\Xi}{}^{n\sigma}_{M\delta}(\mathbf{x}) = \lambda(\sigma)\Xi^{n-1,\tau}_{M'}(\mathbf{x}'),$$

where

$$\lambda(\sigma) = (2\pi)^{(n-2)/2}\int\limits_{1}^{\infty}(t\,\cosh\theta - \delta\sinh\theta)^{\sigma}\mathfrak{P}^{-(n-4)/2}_{\tau+(n-4)/2}(t) \times (t^2-1)^{(n-4)/4}dt.$$

Evaluating this integral with the help of formula (10′) of Section 9.3.6, we have

$$\overset{*}{\Xi}{}^{n\sigma}_{M\delta}(\mathbf{x}) = (2\pi)^{(n-2)/2}\frac{\Gamma(\tau-\sigma)}{\Gamma(-\sigma)}\Gamma(-\sigma-\tau-n+3)\cosh^{(2-n)/2}\theta \times \mathrm{P}^{\sigma+(n-2)/2}_{\tau+(n-4)/2}(-\delta\tanh\theta)\Xi^{n-1,\tau}_{M'}(\mathbf{x}'). \tag{11}$$

Let now

$$\mathbf{x} = (\mathbf{x}''\sinh\theta, \cosh\theta, t\,\sinh\theta) \in H^{n-1}_{-},\ \theta > 0,$$
$$\mathbf{x}' = (\mathbf{x}'', t) \in H^{n-2}_{+},$$
$$\boldsymbol{\xi} = (\boldsymbol{\xi}''\sinh\varphi, 1, \cosh\varphi) \in H^{n-2}_{+} \subset C^{n-1}_{+},$$
$$\boldsymbol{\xi}' = (\boldsymbol{\xi}''\sinh\varphi, \cosh\varphi),\ \delta \in \{+,-\},\ M = (\tau, M')$$

and

$$\overset{*}{\Xi}{}^{n\sigma}_{M,-}(\mathbf{x}) = \int\limits_{H^{n-2}_{+}} [\mathbf{x},\boldsymbol{\xi}]^{\sigma}_{-}\Xi^{n-1,\tau}_{M'}(\boldsymbol{\xi}')d\boldsymbol{\xi}'. \tag{12}$$

This integral is evaluated with the help of the Funk-Hecke theorem and formula (11) of Section 9.3.6. We have

$$\overset{*}{\Xi}{}^{n\sigma}_{M,-}(\mathbf{x}) = (2\pi)^{(n-2)/2}\Gamma(\sigma+1)\sinh^{(2-n)/2}\theta \times \mathfrak{P}^{-\sigma-(n-2)/2}_{\tau+(n-4)/2}(\tanh^{-1}\theta)\Xi^{n-1,\tau}_{M'}(\mathbf{x}'), \qquad (13)$$

where $M = (\tau, M')$.

**10.2.3. The analog of the Funk-Hecke theorem for the group** $ISO(n-1)$**.** This analog is formulated as follows. *Let* $\varphi(t) \in \mathfrak{L}^2(\mathbb{R}_+, \rho(t))$, *where* $\rho(t) = t^{n-2}dt$, *and let* $\Psi(\mathbf{x}) \in \mathfrak{H}^{n,iR}$, *where* $\mathfrak{H}^{n,iR}$ *is the space of functions on* $\mathbb{R}^{n-1}$ *such that* $\Delta\Psi = -R^2\Psi$. *Then*

$$\int\limits_{\mathbb{R}^{n-1}} \varphi(\|\mathbf{x}-\mathbf{y}\|)\Psi(\mathbf{y})d\mathbf{y} = \lambda(R)\Psi(\mathbf{x}), \qquad (1)$$

*where*

$$\lambda(R) = (2\pi)^{\frac{n-1}{2}} R^{\frac{3-n}{2}} \int\limits_0^\infty \varphi(t)J_{\frac{n-3}{2}}(Rt)t^{\frac{n-2}{2}}dt. \qquad (2)$$

The proof of this statement is similar to that of formula (7) of Section 10.2.2 and we omit it.

As an example we evaluate the integral

$$\overset{**}{\Xi}{}^{n\sigma}_{M}(\mathbf{x}) = \int\limits_{\mathbb{R}^{n-2}} [\mathbf{x},\boldsymbol{\xi}]^\sigma \Xi^{n-1,iR}_{M'}(\boldsymbol{\xi}')d\boldsymbol{\xi}', \qquad (3)$$

where

$$\mathbf{x} = \left(\mathbf{x}', \sinh\theta - \frac{r^2e^{-\theta}}{2}, \cosh\theta + \frac{r^2e^{-\theta}}{2}\right) \in H^{n-1}_+,$$

$$\mathbf{x}' \in \mathbb{R}^{n-2}, r^2 = (\mathbf{x}',\mathbf{x}'),\ \boldsymbol{\xi} = \left(\boldsymbol{\xi}', \frac{1}{2}(1-\rho^2), \frac{1}{2}(1+\rho^2)\right) \in C^{n-1}_+,$$

$$\boldsymbol{\xi}' \in \mathbb{R}^{n-2},\ \rho^2 = (\boldsymbol{\xi}',\boldsymbol{\xi}'),\ M = (iR, M').$$

We have

$$[\mathbf{x},\boldsymbol{\xi}] = \frac{1}{2}(e^{-\theta} + \rho^2e^\theta + r^2e^{-\theta} - (\mathbf{x}',\boldsymbol{\xi}'))$$
$$= \frac{1}{2}e^\theta\left(e^{-2\theta} + \|\mathbf{x}'e^{-\theta} - \boldsymbol{\xi}'\|^2\right).$$

Therefore, according to formula (1), we obtain that

$$\overset{**}{\Xi}{}^{n\sigma}_{M}(\mathbf{x}) = \lambda(R)\Xi^{n-1,iR}_{M'}(e^{-\theta}\mathbf{x}'),$$

where

$$\lambda(R) = (2\pi)^{\frac{n-2}{2}} R^{\frac{4-n}{2}} \left(\frac{e^{\theta}}{2}\right)^{\sigma} \int_0^{\infty} (t^2 + e^{-2\theta})^{\sigma} \times J_{\frac{n-4}{2}}(Rt) t^{(n-2)/2} dt.$$

Applying formula (13) of Section 9.3.6, we conclude that

$$\overset{**}{\Xi}{}^{n\sigma}_{M}(\mathbf{x}) = \frac{2}{\Gamma(-\sigma)} \left(\frac{2\pi}{Re^{\theta}}\right)^{\frac{n-2}{2}} R^{-\sigma} \times K_{\sigma+(n-2)/2}(Re^{-\theta})\Xi^{n-1,iR}_{M'}(e^{-\theta}\mathbf{x}'). \tag{4}$$

Using formula (12) of Section 9.3.6, in the same way we prove that for

$$\mathbf{x} = \left(\mathbf{x}', \cosh\theta - \frac{r^2 e^{-\theta}}{2}, \sinh\theta + \frac{r^2 e^{-\theta}}{2}\right) \in H^{n-1}_{-},$$
$$\mathbf{x}' \in \mathbb{R}^{n-2},\ r^2 = (\mathbf{x}', \mathbf{x}'),$$

we have

$$\overset{**}{\Xi}{}^{n\sigma}_{M,-}(\mathbf{x}) \equiv \int_{\mathbb{R}^{n-2}} [\mathbf{x}, \boldsymbol{\xi}]^{\sigma}_{-} \Xi^{n-1,iR}_{M'}(\boldsymbol{\xi}') d\boldsymbol{\xi}' = \left(\frac{2\pi}{Re^{\theta}}\right)^{\frac{n-2}{2}} \times \Gamma(\sigma+1) R^{-\sigma} J_{\sigma+(n-2)/2}(Re^{-\theta}) \Xi^{n-1,iR}_{M'}(e^{-\theta}\mathbf{x}'), \tag{5}$$

where $\boldsymbol{\xi}$ and $M$ are the same as in formula (3).

**10.2.4. The Bochner theorem and its corollaries.** It follows from the Funk-Hecke theorem and from formula (10) of Section 9.4.2 that

$$\int_{S^{n-1}} e^{i(\mathbf{x},\boldsymbol{\eta})} \Xi^{n\ell}_{M}(\boldsymbol{\eta}) d\boldsymbol{\eta} = \left(\frac{i}{2}\right)^{\ell} \Gamma\left(\frac{n}{2}\right) \left(\frac{r}{2}\right)^{-\ell-(n-2)/2} \times J_{\ell+(n-2)/2}(r) \Xi^{n\ell}_{M}(\mathbf{x}), \tag{1}$$

where $r^2 = (\mathbf{x}, \mathbf{x})$. Hence, if $P(\mathbf{x}) \in \mathfrak{H}^{n\ell}$, then

$$\int_{S^{n-1}} e^{i(\mathbf{x},\boldsymbol{\eta})} P(\boldsymbol{\eta}) d\boldsymbol{\eta} = \left(\frac{i}{2}\right)^{\ell} \Gamma\left(\frac{n}{2}\right) \left(\frac{r}{2}\right)^{-\ell-(n-2)/2} J_{\ell+(n-2)/2}(r) P(\mathbf{x}). \tag{2}$$

Let a function $f(\mathbf{y})$, $\mathbf{y} \in \mathbb{R}^n$, be of the form

$$f(\mathbf{y}) = \varphi(R)\Xi_M^{n\ell}(\boldsymbol{\eta}), \quad R = (\mathbf{y}, \mathbf{y}), \quad \boldsymbol{\eta} = \frac{\mathbf{y}}{R} \in S^{n-1}.$$

Then its Fourier transform is given by the formula

$$\Phi(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{y}) e^{i(\mathbf{x},\mathbf{y})} d\mathbf{y} = \int_0^\infty \varphi(R) R^{n-1} \times \int_{S^{n-1}} \Xi_M^{n\ell}(\boldsymbol{\eta}) e^{iRr(\boldsymbol{\xi},\boldsymbol{\eta})} d\eta dR,$$

where $r^2 = (\mathbf{x}, \mathbf{x})$, $\boldsymbol{\xi} = \mathbf{x}/r$. Applying (2) we obtain that

$$\Phi(\mathbf{x}) = \psi(r)\Xi_M^{n\ell}(\boldsymbol{\xi}), \tag{3}$$

where

$$\psi(r) = \frac{(2\pi)^{\ell+n/2}}{i^\ell r^{\ell+(n-2)/2}} \int_0^\infty \varphi(R) J_{\ell+(n-2)/2}(Rr) R^{\ell+n/2} dR. \tag{4}$$

Thus, we have proved the following statement:

**The Bochner theorem.** *If $f(\mathbf{y}) = \varphi(R)\Xi_M^{n\ell}(\boldsymbol{\eta})$, where $\boldsymbol{\eta} \in \mathbb{R}^n$, $R^2 = (\mathbf{y}, \mathbf{y})$, $\boldsymbol{\eta} = \mathbf{y}/R$, and $\Phi(\mathbf{x})$ is the Fourier transform for $f(\mathbf{y})$, then $\Phi(\mathbf{x}) = \psi(r)\Xi_M^{n\ell}(\boldsymbol{\xi})$, where $\psi(r)$ is given by formula* (4), *and $r^2 = (\mathbf{x}, \mathbf{x})$, $\boldsymbol{\xi} = \mathbf{x}/r$.*

It follows from Example 1 of Section 3.2.3 that

$$\int_{\mathbb{R}^n} e^{-(\mathbf{y},\mathbf{y})/2} e^{i(\mathbf{x},\mathbf{y})} d\mathbf{y} = (2\pi)^{n/2} e^{-(\mathbf{x},\mathbf{x})/2}.$$

Therefore, by applying the Bochner theorem to the function $f(\mathbf{y}) = e^{-(\mathbf{y},\mathbf{y})/2}$ we obtain that

$$\int_0^\infty R^p e^{-R^2/2} J_{p-1}(Rr) dR = r^{p-1} e^{-r^2/2}. \tag{5}$$

This formula is valid for all $p$ such that $\operatorname{Re} p > 1$.

The following statements generalize the Bochner theorem.

**Theorem 1.** *Let $f$ be a rapidly decreasing infinitely differentiable function on $\mathbf{R}$, and let $g$ be a $\Box_{pq}$-harmonic function in $\mathbf{R}^n$ which is analytic in the complement to the cone $C^{pq}$, $p+q=n$, and has the homogeneity degree $(\sigma,\varepsilon)$ (see Section 9.2.9). Set*

$$F(\mathbf{x}) = f\left(|[\mathbf{x},\mathbf{x}]_{pq}|^{1/2}\, sign[\mathbf{x},\mathbf{x}]_{pq}\right) g(\mathbf{x}). \tag{6}$$

*If $\alpha \equiv \sigma + \frac{n-2}{2} \bar{\in} \mathbf{Z}$ for $n \in 2\mathbf{Z}_+$ and if $p+q+\varepsilon$, $q+\sigma-\varepsilon \bar{\in} 2\mathbf{Z}$, then*

$$\int\limits_{\mathbb{R}^n} e^{i[\mathbf{x},\mathbf{y}]_{pq}} F(\mathbf{x})d\mathbf{x} = \mathfrak{F}\left(|[\mathbf{y},\mathbf{y}]_{pq}|^{1/2} sign[\mathbf{y},\mathbf{y}]_{pq}\right) g(\mathbf{y}), \tag{7}$$

*where for $r>0$ the functions $\mathfrak{F}(r)$ and $\mathfrak{F}(-r)$ are given as*

$$\begin{aligned}\mathfrak{F}(r) = i^{\varepsilon}(2\pi)^{n/2} \int\limits_0^\infty (rt)^{-\alpha}\Big[&\left(\cos\frac{\pi}{2}(p+\sigma+\varepsilon)\right) J_\alpha(rt) \\ &- \left(\sin\frac{\pi}{2}(p+\sigma+\varepsilon)\right) Y_\alpha(rt)\Big] f(t)t^{2\sigma+n-1}dt \\ + (-i)^{\varepsilon}(2\pi)^{n/2}\frac{2}{\pi}\int\limits_0^\infty (rt)^{-\alpha} &\left(\sin\frac{\pi}{2}(p+\sigma+\varepsilon)\right) K_\alpha(rt) f(-t) t^{2\sigma+n-1}dt,\end{aligned} \tag{8}$$

$$\begin{aligned}\mathfrak{F}(-r) = (-i)^{\varepsilon}(2\pi)^{n/2}\frac{2}{\pi}\int\limits_0^\infty (rt)^{-\alpha} &\left(\sin\frac{\pi}{2}(q+\sigma-\varepsilon)\right) K_\alpha(rt) \\ &\times f(t)t^{2\sigma+n-1}dt \\ + (-i)^{\varepsilon}(2\pi)^{n/2}\int\limits_0^\infty (rt)^{-\alpha}\Big[&\left(\cos\frac{\pi}{2}(q+\sigma-\varepsilon)\right) J_\alpha(rt) \\ &- \left(\sin\frac{\pi}{2}(q+\sigma-\varepsilon)\right) Y_\alpha(rt)\Big] f(-t)t^{2\sigma+n-1}dt.\end{aligned} \tag{9}$$

Let us note that for some $\sigma$ the integrals in (8) and (9) must be understood in the sense of a regularized value.

The functions $\mathfrak{F}$ from (8) and (9) depend on $p$, $q$, $\sigma$, $\varepsilon$. Therefore, we denote them by $\mathfrak{F}_{pq\sigma\varepsilon}$. It follows from the formulation of Theorem 1 that if $\frac{n}{2}+\sigma = \frac{n'}{2}+\sigma'$ and $p-p'$, $q-q'$, $\varepsilon+\varepsilon'+(p-p'-q+q')/2$ are even numbers, then

$$\begin{aligned}\mathfrak{F}_{p'q'\sigma'\varepsilon'} = (-i)^{\varepsilon-\varepsilon'}&(-1)^{(p-p'+\sigma-\sigma'+\varepsilon-\varepsilon')/2} \\ &\times (2\pi)^{(n'-n)/2}\mathfrak{F}_{pq\sigma\varepsilon}.\end{aligned}$$

**Theorem 2.** *Let $f$ be a rapidly decreasing infinitely differentiable function on $\mathbb{C}$, and let $g(\mathbf{z})$ be a function on $\mathbb{C}^n$ which satisfies the equalities*

$$\sum_{k=1}^{n} \frac{\partial^2 g}{\partial z_k^2} = 0, \quad \sum_{k=1}^{n} \frac{\partial^2 g}{\partial \bar{z}_k^2} = 0,$$
$$g(\lambda \mathbf{z}) = \lambda^a \bar{\lambda}^b g(\mathbf{z}), \quad \lambda \in \mathbb{C},\ \lambda \neq 0,$$

*and is analytic in the complement to the cone $\{\mathbf{z} \in \mathbb{C}^n \mid (\mathbf{z}, \mathbf{z}) = 0\}$, where $(\mathbf{z}, \mathbf{w}) = \sum\limits_{k=1}^{n} z_k w_k$. Then*

$$\left(\frac{i}{2}\right)^n \int\limits_{\mathbb{C}^n} f((\mathbf{z}, \mathbf{z})) g(\mathbf{z}) e^{i\mathrm{Re}(\mathbf{z},\mathbf{w})} d\mathbf{z}\, d\bar{\mathbf{z}}$$
$$= g(\mathbf{w}) i^{b-a} (2\pi)^n \frac{i}{2} \int\limits_{\mathbb{C}} \Phi((\mathbf{w}, \mathbf{w}), t) f(t) dt d\bar{t}, \qquad (10)$$

*where*

$$\Phi(\tau, t) = \frac{1}{2\pi} 4^{n+a+b} \sum_{k=-\infty}^{\infty} \tau^{\frac{k-n}{2}-a} \bar{\tau}^{-\frac{k-n}{2}-b} t^{-\frac{k-2}{2}} \bar{t}^{\frac{k-2}{2}}$$
$$\times G_{04}^{20} \left( \frac{|\tau t|}{16} \,\middle|\, \tfrac{k+2}{2}, \tfrac{k+n+2a}{2}, \tfrac{2-k}{2}, \tfrac{n-k+2b}{2} \right)$$

*($G_{04}^{20}$ is the Maijer G-function, defined in Section 10.6.1). In addition, $\Phi$, as a function of $n$, $a$, $b$, depends on $a + \frac{n}{2}$, $b + \frac{n}{2}$ only.*

The proofs of Theorems 1 and 2 are given in [367].

**10.2.5. The continuous basis in the space $\mathfrak{L}^2(\mathbb{R}^{n-1})$.** The functions $\Xi_M^{n,iR}(\mathbf{x})$ from formula (13) of Section 9.4.2 form a continuous basis of the space $\mathfrak{L}^2(\mathbb{R}^{n-1})$. In fact, the following theorem is fulfilled:

**Theorem.** *Any function $f \in \mathfrak{L}^2(\mathbb{R}^{n-1})$ can be represented in the form*

$$f(\mathbf{x}) = c \sum_M \int\limits_0^\infty a_M(R) \Xi_M^{n,iR}(\mathbf{x}) R^{n-2} dR, \qquad (1)$$

*where $c = \left[2^{n-2}\pi^{(n-1)/2}\Gamma\left(\frac{n-1}{2}\right)\right]^{-1}$ and*

$$a_M(R) = \int\limits_{\mathbb{R}^{n-1}} f(\mathbf{x}) \overline{\Xi_M^{n,iR}(\mathbf{x})} d\mathbf{x}. \qquad (2)$$

*In addition, the analog of the Plancherel formula*

$$\int_{\mathbb{R}^{n-1}} |f(\mathbf{x})|^2 d\mathbf{x} = c\sum_M \int_0^\infty |a_M(R)|^2 R^{n-2} dR \tag{3}$$

*holds.*

*Proof:* We denote by $F(\mathbf{y})$ the Fourier transform of the function $f(\mathbf{x})$. Let $F(\mathbf{y}) = \Phi(\boldsymbol{\xi}, R)$, where $R^2 = (\mathbf{y},\mathbf{y})$, $\boldsymbol{\xi} = \mathbf{y}/R$. For a fixed $R$ the function $\Phi(\boldsymbol{\xi}, R)$ belongs to $\mathfrak{L}^2(S^{n-2})$ and is expanded into the series

$$\Phi(\boldsymbol{\xi}, R) = \sum_M a_M(R)\Xi_{M'}^{n-1,m}(\boldsymbol{\xi}) \tag{4}$$

(see Section 9.3.1), where $M = (m, M')$ and

$$a_M(R) = \int_{S^{n-2}} \Phi(\boldsymbol{\xi}, R)\overline{\Xi_{M'}^{n-1,m}(\boldsymbol{\xi})}d\boldsymbol{\xi}. \tag{5}$$

By the inversion formula for the Fourier transform we have

$$\begin{aligned} f(\mathbf{x}) &= (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} F(\mathbf{y})e^{i(\mathbf{x},\mathbf{y})}d\mathbf{y} \\ &= (2\pi)^{1-n}\frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)}\int_0^\infty R^{n-2}dR \int_{S^{n-2}} \Phi(\boldsymbol{\xi}, R)e^{iR(\mathbf{x},\boldsymbol{\xi})}d\boldsymbol{\xi}. \end{aligned}$$

Replace in this equality the function $\Phi(\boldsymbol{\xi}, R)$ by expansion (4) and then invert the order of integration and summation. Taking into account formula (7) of Section 10.1.6, we obtain expansion (1).

In order to express $a_M(R)$ in terms of $f(\mathbf{x})$, we note that

$$\Phi(\boldsymbol{\xi}, R) = \int_{\mathbb{R}^{n-1}} f(\mathbf{x})e^{-iR(\boldsymbol{\xi},\mathbf{x})}d\mathbf{x}.$$

Therefore, (5) implies that

$$a_M(R) = \int_{S^{n-2}} d\boldsymbol{\xi} \int_{\mathbb{R}^{n-1}} f(\mathbf{x})e^{-iR(\boldsymbol{\xi},\mathbf{x})}\overline{\Xi_{M'}^{n-1,m}(\boldsymbol{\xi})}d\mathbf{x}.$$

Inverting the order of integrations and making use of formula (7) of Section 10.1.6, we obtain equality (2).

Formula (3) follows from the Plancherel formula for the Fourier transform and from the fact that the functions $\Xi_{M'}^{n-1,m}$ are orthogonal in $\mathfrak{L}^2(S^{n-2})$.

**10.2.6. The Fourier-Bessel transform.** Let $f(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^{n-1}$, be a radial function, that is, $f(h\mathbf{x}) = f(\mathbf{x})$ for $h \in SO(n-1)$. Then in expansion (1) of Section 10.2.5 only the summand, for which $M = O$, is different from zero. Since

$$\Xi_0^{n,iR}(\mathbf{x}) = t_{00}^{n,iR}(g_r) = \Gamma\left(\frac{n-1}{2}\right)\left(\frac{Rr}{2}\right)^{-(n-3)/2} J_{(n-3)/2}(Rr)$$

(see Section 10.1.6), then for $f(\mathbf{x}) = F(r)$ we obtain the expansion

$$F(r) = 2^{2-n}\pi^{-(n-1)/2}r^{(3-n)/2}\int_0^\infty a(R)J_{\frac{n-3}{2}}(Rr)R^{(n-1)/2}dR. \tag{1}$$

The inverse transform is given by formula (2) of Section 10.2.5, which in our case is of the form

$$a(R) = 2^{n-2}\pi^{(n-1)/2}R^{\frac{3-n}{2}}\int_0^\infty F(r)J_{\frac{n-3}{2}}(Rr)r^{(n-1)/2}dr. \tag{2}$$

Replacing in (1) and (2) $F(r)$ by $2^{n-2}\pi^{(n-1)/2}r^{\frac{n-3}{2}}F(r)$ and $a(R)$ by $R^{\frac{n-3}{2}} \times a(R)$, we obtain the *direct* and the *inverse Fourier-Bessel transforms*

$$\varphi(r) = \int_0^\infty \Phi(R)J_\nu(Rr)RdR, \tag{3}$$

$$\Phi(R) = \int_0^\infty \varphi(r)J_\nu(Rr)rdr. \tag{4}$$

Here $\nu$ is an integral or half-integral non-negative number. Formula (3) of Section 10.2.5 implies the Plancherel formula for this transform:

$$\int_0^\infty |\Phi(R)|^2 RdR = \int_0^\infty |\varphi(r)|^2 rdr. \tag{5}$$

One can prove that these formulas are valid for any $\nu$.

Transforms (3) and (4) are often written as

$$f(r) = \int_0^\infty F(R)J_\nu(Rr)\sqrt{Rr}\,dR, \tag{6}$$

$$F(R) = \int_0^\infty f(r)J_\nu(Rr)\sqrt{Rr}\,dr \tag{7}$$

and are called the *Hankel transforms*.

Other transforms are connected with the Fourier-Bessel transform. Taking into account expression (11) of Section 3.5.6 for the Neumann function $N_\nu$ and transform (6), we obtain the *Neumann transform*

$$f(r) = \int_0^\infty F(R) N_\nu(Rr)\sqrt{Rr}\, dR. \tag{8}$$

The inverse transform is of the form

$$F(R) = \int_0^\infty f(r)\mathbf{H}_\nu(Rr)\sqrt{Rr}\, dr, \tag{9}$$

where $\mathbf{H}_\nu$ is the *Struve function* which for $\operatorname{Re}\nu > -\frac{1}{2}$ is defined by the formula

$$\begin{aligned} \Gamma\left(\nu+\frac{1}{2}\right)\mathbf{H}_\nu(z) &= \frac{2}{\sqrt{\pi}}\left(\frac{z}{2}\right)^\nu \int_0^{\pi/2} \sin(z\,\cos\varphi)\sin^{2\nu}\varphi d\varphi \\ &= \sum_{m=0}^\infty \frac{(-1)^m (z/2)^{\nu+2m+1}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(\nu+m+\frac{3}{2}\right)} \end{aligned} \tag{10}$$

(compare with formula (26) of Section 3.5.6).

In the same way, from expression (31) of Section 3.5.6 for Macdonald functions and from formula (6), we obtain the transform

$$f(r) = \int_0^\infty F(R) K_\nu(Rr)\sqrt{Rr}\, dR. \tag{11}$$

The inverse transform is given by the formula

$$F(R) = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} f(r) I_\nu(Rr)\sqrt{Rr}\, dr. \tag{12}$$

We now consider examples of Fourier-Bessel transforms. Taking into account in formula (12) of Section 3.5.2 relation (1) of Section 3.5.6 between the function ${}_0F_1$ and the Bessel function $J_\nu$ as well as formula (4) of Section 3.5.7, we obtain

$$n! L_n^\alpha(\lambda) = e^\lambda \lambda^{-\alpha/2}\int_0^\infty e^{-x} x^{n+\alpha/2} J_\alpha\left(2\sqrt{\lambda x}\right) dx. \tag{13}$$

This relation can be written in the form

$$2^n n! e^{-\lambda^2/2} \lambda^{-\alpha} L_n^\alpha\left(\frac{\lambda^2}{2}\right) = \int_0^\infty e^{-x^2/2} x^{2n+\alpha} J_\alpha(\lambda x) x\,dx. \tag{14}$$

Thus, $2^n n! e^{-\lambda^2/2} \lambda^{-\alpha} L_n^\alpha\left(\frac{\lambda^2}{2}\right)$ is the Fourier-Bessel transform of the function $x^{2n+\alpha} e^{-x^2/2}$.

We replace the Laguerre polynomial by its expression in powers of $x^2$ in the integral

$$\int_0^\infty e^{-x^2/2} x^\alpha L_n^\alpha(x^2) J_\alpha(\lambda x) x\,dx,$$

integrate term by term by means of formula (14) and take into account the equality[2]

$$\sum_{m=0}^{n} \frac{(-1)^m \Gamma(n+\alpha+1)}{(n-m)! 2^m \Gamma(m+\alpha+1)} L_m^\alpha\left(\frac{\lambda^2}{2}\right) = L_n^\alpha(\lambda^2). \tag{15}$$

As a result, we obtain the relation

$$\int_0^\infty e^{-x^2/2} x^\alpha L_n^\alpha(x^2) J_\alpha(\lambda x) x\,dx = (-1)^n \lambda^\alpha e^{-\lambda^2/2} L_n^\alpha(\lambda^2) \tag{16}$$

which shows that the functions $e^{-x^2/2} x^\alpha L_n^\alpha(x^2)$ are eigenfunctions for the Fourier-Bessel transform corresponding to the eigenvalues $(-1)^n$.

According to the results of Section 5.5.4 the functions

$$f_n(x) = \left[\frac{n!}{\Gamma(\alpha+n+1)}\right]^{1/2} x^\alpha e^{-x^2/2} L_n^\alpha(x^2), \; n = 0, 1, 2, \ldots,$$

form a complete orthonormal system in the Hilbert space $\mathfrak{L}^2(\mathbf{R}_+, x\,dx)$, where $\mathbf{R}_+$ is the set of positive numbers. Therefore, we derive from (16) that

$$\sum_{n=0}^{\infty} \frac{(-1)^n n!}{\Gamma(\alpha+n+1)} (\lambda x)^\alpha e^{-(x^2+\lambda^2)/2} L_n^\alpha(\lambda^2) L_n^\alpha(x^2) = J_\alpha(\lambda x), \tag{17}$$

$$\alpha > -1.$$

[2] In order to prove this equality it is sufficient to compare coefficients at the same powers of $\lambda^2$ on the left and on the right.

Another example is obtained in the following way. Making the substitution $r = (r_1^2 + r_2^2 + 2r_1r_2\cos\varphi)^{1/2}$ in formula (14) of Section 9.4.5, we find

$$\int\limits_{|r_1-r_2|}^{r_1+r_2} r^{-\nu} J_\nu(r) C_m^\nu\left(\frac{r_1^2+r_2^2-r^2}{2r_1r_2}\right) S^{2\nu-1}(r_1,r_2,r)rdr$$
$$= \frac{\pi\Gamma(m+2\nu)}{m!\Gamma(\nu)}(2r_1r_2)^\nu J_{m+\nu}(r_1)J_{m+\nu}(r_2), \quad (18)$$

where $\nu = (n-3)/2$ and $S(r_1, r_2, r)$ denotes the area of the triangle with sides of the lengths $r_1, r_2, r$. We replace in (18) $r_1, r_2, r$ by $Rr_1, Rr_2, Rr$, respectively, where $R > 0$, and apply the Fourier-Bessel transform. We obtain that

$$\int\limits_0^\infty J_\nu(Rr)J_{\nu+m}(Rr_1)J_{\nu+m}(Rr_2)R^{1-\nu}dR$$
$$= \frac{m!\Gamma(\nu)}{4\pi\Gamma(m+2\nu)S(r_1,r_2,r)}\left(\frac{8S^2(r_1,r_2,r)}{r_1r_2r}\right)^\nu C_m^\nu\left(\frac{r_1^2+r_2^2-r^2}{2r_1r_2}\right) \quad (19)$$

if $|r_1 - r_2| < r < r_1 + r_2$. Otherwise this integral vanishes.

**10.2.7. The continuous basis in the space $\mathfrak{L}^2(H_+^{n-1})$.** The functions $\Xi_{M'}^{n-1,m}(\xi)$ form an orthonormal basis on the section $S^{n-2}$ of the cone $C_+^{n-1}$. The Poisson transform of this basis gives a continuous basis in the space $\mathfrak{L}^2(H_+^{n-1})$. Indeed, one has the following theorem:

**Theorem.** *Any function $f \in \mathfrak{L}^2(H_+^{n-1})$ can be represented in the form*

$$f(\eta) = c_n' \sum_M \int\limits_{a-i\infty}^{a+i\infty} a_M(\sigma)\overline{\Xi_M^{n,-\bar\sigma-n+2}(\eta)}d\mu_n'(\sigma), \quad (1)$$

*where*

$$a_M(\sigma) = \int\limits_{H_+^{n-1}} f(\eta)\Xi_M^{n\sigma}(\eta)d\eta, \quad (2)$$

$$c_n' = \frac{(-1)^{(n-2-\varepsilon)/2}}{2^{n-1}\pi^{(n-1)/2}\Gamma\left(\frac{n-1}{2}\right)i}, d\mu_n'(\sigma) = \frac{\Gamma(\sigma+n-2)}{\Gamma(\sigma)}\cot^\varepsilon \pi\sigma d\sigma. \quad (3)$$

*Proof:* We expand the function $\Phi(\xi', \sigma)$ (for a fixed $\sigma$) from formula (7) of Section 10.1.4 in the basis $\left\{\Xi_{M'}^{n-1,m}\right\}$ of the space $\mathfrak{L}^2(S^{n-2})$:

$$\Phi(\xi',\sigma) = \sum_M a_M(\sigma)\overline{\Xi_{M'}^{n-1,m}(\xi')}, \quad (4)$$

where $M = (m, M')$ and

$$a_M(\sigma) = \int\limits_{S^{n-2}} \Phi(\boldsymbol{\xi}', \sigma) \Xi_{M'}^{n-1,m}(\boldsymbol{\xi}'). \tag{5}$$

Substituting (4) into formula (7) of Section 10.1.4 and taking into consideration formulas (1) and (3) of Section 10.1.6, we obtain (1). In order to prove (2) it is sufficient to substitute expression (3) of Section 10.1.4 for $\Phi(\boldsymbol{\xi}', \sigma)$ into (5) and to take into account formulas (1) and (3) of Section 10.1.6. Theorem is proved.

If $a = -\frac{n-2}{2}$ in formulas (1) and (2), then we obtain the expansion

$$f(\boldsymbol{\eta}) = \frac{1}{2}\tilde{c}_n \sum_M \int\limits_{-\infty}^{\infty} b_M(\rho) \overline{\Xi_M^{n, i\rho-(n-2)/2}(\boldsymbol{\eta})} d\mu_n(\rho), \tag{6}$$

where $\tilde{c}_n$ and $d\mu_n(\rho)$ are the same as in formula (12) of Section 10.1.4 and

$$b_M(\rho) = \int\limits_{H_+^{n-1}} f(\boldsymbol{\eta}) \Xi_M^{n, i\rho-(n-2)/2}(\boldsymbol{\eta}) d\boldsymbol{\eta}. \tag{7}$$

Moreover, the Plancherel formula

$$\tilde{c}_n \sum_M \int\limits_0^{\infty} |b_M(\rho)|^2 d\mu_n(\rho) = \int\limits_{H_+^{n-1}} |f(\boldsymbol{\eta})|^2 d\boldsymbol{\eta} \tag{8}$$

holds. Remember that the functions $\Xi_M^{n\sigma}(\boldsymbol{\eta})$ are associated spherical functions for the representations $T^{n\sigma}$ of $SO_0(n-1, 1)$ (see Section 9.4.2).

**10.2.8. The generalized Fock-Mehler transform.** If the function $f(\boldsymbol{\eta})$ from formula (1) of Section 10.2.7 is invariant with respect to rotations from $SO(n-1)$, that is, if $f(h\boldsymbol{\eta}) = f(\boldsymbol{\eta})$, $h \in SO(n-1)$, then this formula takes the form

$$f(\boldsymbol{\eta}) = c'_{2m+2} \int\limits_{a-i\infty}^{a+i\infty} a(\sigma) t_{OO}^{n,-\sigma-2m}(g_{\boldsymbol{\eta}}) \frac{\Gamma(\sigma+2m)}{\Gamma(\sigma)} d\sigma \tag{1}$$

for $n = 2m+2$ and the form

$$f(\boldsymbol{\eta}) = c'_{2m+1} \int\limits_{a-i\infty}^{a+i\infty} a(\sigma) t_{OO}^{n,-\sigma-2m+1}(g_{\boldsymbol{\eta}}) \frac{\Gamma(\sigma+2m-1)}{\Gamma(\sigma)} \cot \pi\sigma d\sigma \tag{2}$$

for $n = 2m+1$. Here $-2m+1 < a < 1$ and $g_\eta$ is the matrix from $SO_0(n-1,1)$ transferring the point $\eta_0 = (0,\dots,0,1) \in H_+^{n-1}$ into $\eta$. The coefficients $a(\sigma)$ are given by the formula

$$a(\sigma) = \int\limits_{H_+^{n-1}} f(\eta) t_{OO}^{n\sigma}(g_\eta) d\eta. \tag{3}$$

Making use of the expression for the zonal spherical function $t_{OO}^{n\sigma}(g)$ in terms of Legendre functions, we can rewrite (1) and (2) in the form

$$f(\cosh\theta) = \frac{(-1)^m}{2i\sinh^{m-1/2}\theta} \int\limits_{a-i\infty}^{a+i\infty} a(\sigma) \mathfrak{P}_{-\sigma-m-1/2}^{-m+1/2}(\cosh\theta) \frac{\Gamma(\sigma+2m)}{\Gamma(\sigma)} d\sigma \tag{4}$$

if $n = 2m+2$ and in the form

$$f(\cosh\theta) = \frac{(-1)^{m+1}}{2i\sinh^{m-1}\theta} \int\limits_{a-i\infty}^{a+i\infty} a(\sigma) \mathfrak{P}_{-\sigma-m}^{1-m}(\cosh\theta) \frac{\Gamma(\sigma+2m-1)}{\Gamma(\sigma)} \cot\pi\sigma d\sigma \tag{5}$$

if $n = 2m+1$, where $-2m+1 < a < 1$. In addition,

$$a(\sigma) = \int\limits_0^\infty f(\cosh\theta) \sinh^{(n-1)/2}\theta \mathfrak{P}_{\sigma+(n-3)/2}^{(3-n)/2}(\cosh\theta) d\theta. \tag{6}$$

For the principal unitary series representations $T^{n\sigma}$, $\sigma = i\rho - \frac{n-2}{2}$, transforms (4)–(6) can be rewritten as

$$f(\cosh\theta) = \frac{1}{2\sinh^p\theta} \int\limits_{-\infty}^{\infty} b(\rho) \mathfrak{P}_{i\rho-1/2}^{-p}(\cosh\theta) \mu_p(\rho) d\rho, \tag{7}$$

$$b(\rho) = \int\limits_0^\infty f(\cosh\theta) \mathfrak{P}_{-i\rho-1/2}^{-p}(\cosh\theta) \sinh^{p+1}\theta d\theta, \tag{8}$$

where $p$ is an integral or half-integral non-negative number and

$$\mu_p(\rho) = (-1)^{p+1/2} \frac{\Gamma\left(i\rho+p+\frac{1}{2}\right)}{\Gamma\left(i\rho-p-\frac{1}{2}\right)} \quad \text{if } p \text{ is half-integral,} \tag{9}$$

$$\mu_p(\rho) = (-1)^{p+1} \frac{\Gamma\left(i\rho+p+\frac{1}{2}\right)}{\Gamma\left(i\rho-p-\frac{1}{2}\right)} \tanh\pi\rho \quad \text{if } p \text{ is an integer.} \tag{10}$$

Substituting $F(\cosh\theta) = \sinh^p\theta f(\cosh\theta)$ into (7) and (8) and replacing $\cosh\theta$ by $x$, after simple manipulations we obtain the pair of mutually reciprocal transforms

$$F(x) = \int_0^\infty \Phi(\rho)\mathfrak{P}^\mu_{i\rho-1/2}(x)d\rho, \tag{11}$$

$$\Phi(\rho) = \frac{\rho}{\pi}\sinh(\pi\rho)\Gamma\left(i\rho-\mu+\frac{1}{2}\right)\Gamma\left(-i\rho-\mu+\frac{1}{2}\right) \times\int_1^\infty F(x)\mathfrak{P}^\mu_{i\rho-1/2}(x)dx. \tag{12}$$

For $\mu = 0$ they coincide with the direct and the inverse Fock-Mehler transforms.

The function $\mathfrak{P}^{-1/2}_{i\rho-1/2}(\cosh\theta)$ is expressed in terms of $\sin\rho\theta$ (see Section 9.3.4). Therefore, for $\mu = -1/2$ transforms (11) and (12) turn into the Fourier-sine transforms.

Let us note that the following symbolic relations follow from formulas (11) and (12):

$$\frac{1}{\pi}\int_0^\infty \rho\sinh(\pi\rho)\Gamma\left(i\rho-\mu+\frac{1}{2}\right)\Gamma\left(-i\rho-\mu+\frac{1}{2}\right) \times\mathfrak{P}^\mu_{i\rho-1/2}(x)\mathfrak{P}^\mu_{i\rho-1/2}(y)d\rho = \delta(x-y),\ 1\le x,y<\infty, \tag{13}$$

$$\int_1^\infty \mathfrak{P}^\mu_{i\lambda-1/2}(x)\mathfrak{P}^\mu_{i\rho-1/2}(x)dx = \left[\frac{1}{\pi}\rho\sinh(\pi\rho) \times\Gamma\left(i\rho-\mu+\frac{1}{2}\right)\Gamma\left(-i\rho-\mu+\frac{1}{2}\right)\right]^{-1}\delta(\lambda-\rho). \tag{14}$$

Formulas (4) and (5) can be written as

$$f(t) = \frac{(-1)^{(n-\varepsilon-2)/2}}{2i}(t^2-1)^{-(n-3)/4} \times\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\tau+n-2)}{\Gamma(\tau)}\cot^\varepsilon(\pi\tau)\mathfrak{P}^{-(n-3)/2}_{-\tau-(n-3)/2}(t)a(\tau)d\tau, \tag{15}$$

where $\varepsilon = \frac{1}{2}(1-(-1)^n)$, and

$$a(\tau) = \int_1^\infty f(t)(t^2-1)^{(n-3)/4}\mathfrak{P}^{-(n-3)/2}_{\tau+(n-3)/2}(t)dt. \tag{16}$$

Let us apply these formulas to equalities (10), (10′) and (11) of Section 9.3.6. We put $\mu = \frac{4-n}{2}$, $\nu = \tau + \frac{n-4}{2}$ in these equalities and replace $n$ by $n-1$ in (15) and (16). We obtain that if $u > 1$, then

$$(t+u)^\sigma = \frac{(-1)^{(n-\delta-3)/2}}{2i\Gamma(-\sigma)}(t^2-1)^{\frac{4-n}{4}}(u^2-1)^{\frac{\varepsilon}{2}+\frac{n-2}{4}}$$
$$\times\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\tau+n-3)}{\Gamma(\tau)}\cot^\delta(\pi\tau)\Gamma(\tau-\sigma)\Gamma(-\sigma-\tau-n+3)$$
$$\times\mathfrak{P}^{\sigma+(n-2)/2}_{\tau+(n-4)/2}(u)\mathfrak{P}^{(4-n)/2}_{-\tau-(n-2)/2}(t)d\tau, \tag{17}$$

where $\delta = \frac{1}{2}(1-(-1)^{n-1})$. If $|u| < 1$, then we have to replace in (17)

$$(u^2-1)^{\frac{\varepsilon}{2}+\frac{n-4}{4}}\mathfrak{P}^{\sigma+(n-2)/2}_{\tau+(n-4)/2}(u)$$

by

$$(1-u^2)^{\frac{\varepsilon}{2}+\frac{n-4}{4}}\mathrm{P}^{\sigma+(n-2)/2}_{\tau+(n-4)/2}(u).$$

Formula (11) of Section 9.3.6 gives that for $u > 1$ we have

$$(u-t)^\sigma_+ = \frac{(-1)^{(n-2-\delta)/2}}{2i}\Gamma(\sigma+1)(t^2-1)^{-\frac{n-4}{2}}(u^2-1)^{-\sigma-\frac{n-2}{2}}$$
$$\times\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\tau+n-3)}{\Gamma(\tau)}\cot^\delta(\pi\tau)\mathfrak{P}^{(4-n)/2}_{-\tau-(n-2)/2}(t)$$
$$\times\mathfrak{P}^{-\sigma-(n-2)/2}_{\tau+(n-4)/2}(u)d\tau. \tag{18}$$

## 10.3. The Poisson Transforms and Special Functions

**10.3.1. The Poisson transforms.** The classical Poisson transform allows us to find values of a harmonic function inside a contour $\Gamma$, if we know its values on this contour. The Poisson transform in the theory of $\Box_{pq}$-harmonic functions has the same meaning. Let us remember that the representation $T^{n\varepsilon}_{pq}$ of the group $SO_0(p,q)$ is realized in the space $\mathfrak{B}^{\sigma\varepsilon}_{pq}$ of homogeneous infinitely differentiable functions on the cone $C^{pq}$ and is given by the formula $T^{\sigma\varepsilon}_{pq}(g)f(\xi) = f(g^{-1}\xi)$ (see Section 9.2.9).

Let $f \in \mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon}$, $n = p+q$, $\Gamma$ be a contour on $C^{pq}$ intersecting every generatrix of the cone at one point, and $d\boldsymbol{\xi}$ be the measure on $\Gamma$ such that $d(t\boldsymbol{\xi}) = t^{p+q-3}dt d\boldsymbol{\xi}$. Then values of integral

$$(\mathcal{P}_\chi f)(\mathbf{x}) = \int_\Gamma [\mathbf{x},\boldsymbol{\xi}]_{pq}^\sigma \operatorname{sign}^\varepsilon[\mathbf{x},\boldsymbol{\xi}]_{pq} f(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{1}$$

where $\mathbf{x} \in \Omega_\pm^{pq} \equiv \{\mathbf{x} \in E_{pq} \mid [\mathbf{x},\mathbf{x}]_{pq} \gtrless 0\}$, do not depend on the choice of $\Gamma$. The function $\mathcal{P}_\chi f$ is said to be the Poisson transform of $f$ corresponding to $\chi = (\sigma, \varepsilon)$.

It is obvious that $\mathcal{P}_\chi f$ is a homogeneous $\Box_{pq}$-harmonic function of homogeneity degree $\chi = (\sigma,\varepsilon)$. Moreover, by virtue of the equality $[g\mathbf{x}, g\boldsymbol{\xi}]_{pq} = [\mathbf{x},\boldsymbol{\xi}]_{pq}$ and of the invariance of integral (1) with respect to shifts, the operator $f(\boldsymbol{\xi}) \to f(g^{-1}\boldsymbol{\xi})$ turns under the Poisson transform into the operator $(\mathcal{P}_\chi f)(\mathbf{x}) \to (\mathcal{P}_\chi f)(g^{-1}\mathbf{x})$. Thus, the Poisson transform intertwines two realizations of the representation $T_{pq}^{\sigma\varepsilon}$ of $SO_0(p,q)$ (see Section 9.2.9), namely, the realization in the space $\mathfrak{B}_{pq}^{-\sigma-n+2,\varepsilon}$ of homogeneous functions of degree $(-\sigma-n+2,\varepsilon)$ on the cone $C^{pq}$ and the realization in the space of $\Box_{pq}$-harmonic homogeneous functions of degree $(\sigma,\varepsilon)$ in $\Omega_\pm^{pq}$ (let us remember that the representations $T_{pq}^{\sigma\varepsilon}$ and $T_{pq}^{-\sigma-n+2,\varepsilon}$ are equivalent if $\sigma$ is not an integer.)

In order to avoid complications connected with divergence of integrals, we restrict ourselves by the case $p = n-1$, $q = 1$. We shall consider the following contours on the cone $C_+^{n-1}$ (see Section 9.1.1):

(a) spherical contour $\Gamma_0 = \{\boldsymbol{\xi} \in C_+^{n-1} \mid \xi_n = 1\}$;

(b) hyperbolic contour $\Gamma_1 = \Gamma_1^+ \cup \Gamma_1^-$, where

$$\Gamma_1^\delta = \{\boldsymbol{\xi} \in C_+^{n-1} \mid \xi_{n-1} = \delta\};$$

(c) parabolic (or orispherical) contour

$$\Gamma_2 = \{\boldsymbol{\xi} \in C_+^{n-1} \mid \xi_{n-1} + \xi_n = 1\}.$$

In the case $[\mathbf{x},\mathbf{x}] > 0$, $x_n > 0$ we have $[\mathbf{x},\boldsymbol{\xi}] > 0$, and the Poisson transform is of the form

$$(\mathcal{P}_\sigma f)(\mathbf{x}) = \int_\Gamma [\mathbf{x},\boldsymbol{\xi}]^\sigma f(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{1'}$$

Sometimes it is useful to consider the transform

$$(\mathcal{P}_{\sigma\delta} f)(\mathbf{x}) = \int_\Gamma [\mathbf{x},\boldsymbol{\xi}]_\delta^\sigma f(\boldsymbol{\xi}) d\boldsymbol{\xi}, \quad \delta \in \{+,-\}. \tag{2}$$

The kernel of this transform is a linear combination of the kernels of the operators (1) with $p = n-1$, $q = 1$.

The homogeneity degree on $C_+^{n-1}$ is given by one number $\sigma \in \mathbb{C}$. In this case we have the space $\mathfrak{B}^{n\sigma}$ instead of $\mathfrak{B}_{pq}^{\sigma\varepsilon}$ (see Section 9.2.1). Three types of bases of the space $\mathfrak{B}^{n\sigma}$ (and also of the spaces $\mathfrak{L}^2(C_+^{n-1})$ and $\mathfrak{L}^2(H_+^{n-1})$) correspond to three contours on $C_+^{n-1}$, enumerated above. Every one of these bases consists of homogeneous functions on $C_+^{n-1}$ (of homogeneity degree $\sigma$), which coincide, under restrictions onto the corresponding contour $\Gamma$, with basis functions on $\Gamma$. Namely, we denote by

$$\widehat{\Xi}_M^{n\sigma,0}(\boldsymbol{\xi}) \equiv \widehat{\Xi}_{mM'}^{n\sigma,0}(\boldsymbol{\xi})$$

the function from $\mathfrak{B}^{n\sigma}$ which coincides on $\Gamma_0$ with $\Xi_{M'}^{n-1,m}(\boldsymbol{\xi}')$, where $(\boldsymbol{\xi}',1) \in \Gamma_0$ and $M=(m,M')$. By

$$\widehat{\Xi}_{M\delta}^{n\sigma,1}(\boldsymbol{\xi}) \equiv \widehat{\Xi}_{\tau M'\delta}^{n\sigma,1}(\boldsymbol{\xi})$$

we denote the function from $\mathfrak{B}^{n\sigma}$ which coincides with $\Xi_{M'}^{n-1,\tau}(\boldsymbol{\xi}')$ on $\Gamma_1^{\delta}$ and with zero function on $\Gamma_1^{-\delta}$. Here

$$\boldsymbol{\xi} = (\boldsymbol{\xi}'',\delta,\xi_n) \in \Gamma_1^{\delta},\ \boldsymbol{\xi}' = (\boldsymbol{\xi}'',\xi_n),\ \delta \in \{+,-\},\ M=(\tau,M').$$

We also put

$$\widehat{\Xi}_M^{n\sigma,1}(\boldsymbol{\xi}) = \widehat{\Xi}_{M+}^{n\sigma,1}(\boldsymbol{\xi}) + \widehat{\Xi}_{M-}^{n\sigma,1}(\boldsymbol{\xi}).$$

By

$$\widehat{\Xi}_M^{n\sigma,2}(\boldsymbol{\xi}) \equiv \widehat{\Xi}_{RM'}^{n\sigma,2}(\boldsymbol{\xi})$$

we denote the function from $\mathfrak{B}^{n\sigma}$ which takes on the contour $\Gamma_2$ the values $\Xi_{M'}^{n-1,iR}(\boldsymbol{\xi}')$. Here

$$\boldsymbol{\xi} = \left(\boldsymbol{\xi}', \frac{1}{2}(1-\rho^2), \frac{1}{2}(1+\rho^2)\right),\ \boldsymbol{\xi}' \in \mathbb{R}^{n-2},\ \rho^2 = (\boldsymbol{\xi}',\boldsymbol{\xi}'),\ M=(iR,M').$$

It follows from the results of Sections 10.2.5 and 10.2.7 that

(a) the functions

$$\widehat{\Xi}_M^{n\sigma,0}(\boldsymbol{\xi}),\ M=(m,M'),\ M'=(m_1,\ldots,\pm m_{n-3}),$$
$$m \ge m_1 \ge \ldots \ge m_{n-3},$$

constitute a basis of the space $\mathfrak{B}^{n\sigma}$ which is orthogonal with respect to the scalar product in $\mathfrak{L}^2(S^{n-2})$;

(b) the functions

$$\widehat{\Xi}_{M\delta}^{n\sigma,1}(\xi),\ \delta \in \{+,-\},\ M=(\tau,M'),\ \operatorname{Re}\tau = -\frac{n-3}{2},\ \operatorname{Im}\tau \ge 0,$$

constitute a continuous basis of the space $\mathfrak{B}^{n\sigma}$;

(c) the functions

$$\widehat{\Xi}_M^{n\sigma,2}(\boldsymbol{\xi}),\ M=(iR,M'),\ 0\le R<\infty,$$

constitute a continuous basis of $\mathfrak{B}^{n\sigma}$.

The values of a function $f\in\mathfrak{B}^{n\sigma}$ on $C_+^{n-1}$ are connected with its values on $S^{n-2}$ by the formula

$$f(\xi_1,\ldots,\xi_{n-1},\xi_n)=\xi_n^\sigma f\left(\frac{\xi_1}{\xi_n},\ldots,\frac{\xi_{n-1}}{\xi_n},1\right). \tag{3}$$

Hence, the functions $\widehat{\Xi}_M^{n\sigma,0}(\boldsymbol{\xi})\equiv\widehat{\Xi}_{mM'}^{n\sigma,0}(\boldsymbol{\xi})$ are of the form

$$\widehat{\Xi}_{mM'}^{n\sigma,0}(\boldsymbol{\xi})=N_{mm_1}^{n-1}\left(\dim T^{n-1,m}\right)^{1/2}\xi_n^{\sigma-m_1}r_{n-2}^{m_1} \times C_{m-m_1}^{m_1+(n-3)/2}\left(\frac{\xi_{n-1}}{\xi_n}\right)X_{M_1}^{n-2,m_1}(\xi_1,\ldots,\xi_{n-2}), \tag{4}$$

where $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_n)\in C_+^{n-1}$, $r_k^2=\xi_1^2+\ldots+\xi_k^2$, $M'=(m_1,M_1)$,

$$N_{mm_1}^{n-1}= =\frac{2^{m_1}\Gamma\left(m_1+\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-3}{2}\right)}\left[\frac{m!(m-m_1)!(n-4)!(m_1+n-5)!(2m_1+n-4)}{m_1!(m+m_1+n-4)!(m+n-4)!}\right]^{\frac{1}{2}}, \tag{5}$$

$$X_{M_1}^{n-2,m_1}(\xi_1,\ldots,\xi_{n-2})=(\xi_2+\xi_1\operatorname{sign}m_{n-3})^{m_{n-3}} \times\prod_{i=1}^{n-4}N_{m_im_{i+1}}^{n-i-1}\left(\frac{r_{n-i-2}}{r_{n-i-1}}\right)^{m_{i+1}}C_{m_i-m_{i+1}}^{m_{i+1}+(n-i-3)/2}\left(\frac{\xi_{n-i-1}}{r_{n-i-1}}\right). \tag{6}$$

In particular,

$$\widehat{\Xi}_{OO}^{n\sigma,0}(\boldsymbol{\xi})=\xi_n^\sigma,$$

$$\widehat{\Xi}_{mO}^{n\sigma,0}(\boldsymbol{\xi})=\left[\frac{m!(n-4)!(2m+n-3)}{(n-3)(m+n-4)!}\right]^{\frac{1}{2}}\xi_n^\sigma C_m^{(n-3)/2}\left(\frac{\xi_{n-1}}{\xi_n}\right). \tag{4'}$$

With every function $f\in\mathfrak{B}^{n\sigma}$ the formula

$$f(\boldsymbol{\xi})\equiv f(\xi_1,\ldots,\xi_n) =|\xi_{n-1}|^\sigma f\left(\frac{\xi_1}{|\xi_{n-1}|},\ldots,\frac{\xi_{n-2}}{|\xi_{n-1}|},\operatorname{sign}\xi_{n-1},\frac{\xi_n}{|\xi_{n-1}|}\right) \tag{7}$$

associates the function $f(\xi', \ldots, \xi'_{n-2}, \delta, \xi'_n)$ on the contours $\Gamma_1^\delta$. Using formula (7), we write down explicit expressions for the basis functions $\widehat{\Xi}^{n\sigma,1}_{M\delta}(\boldsymbol{\xi}) \equiv \widehat{\Xi}^{n\sigma,1}_{\nu M'\delta}(\boldsymbol{\xi})$, $M = \left(i\nu - \frac{n-3}{2}, M'\right)$. The function $\widehat{\Xi}^{n\sigma,1}_{\nu M'+}(\boldsymbol{\xi})$ for $\xi_{n-1} > 0$ and the function $\widehat{\Xi}^{n\sigma,1}_{\nu M'-}(\boldsymbol{\xi})$ for $\xi_{n-1} < 0$ are given by the formula

$$\widehat{\Xi}^{n\sigma,1}_{\nu M'\delta}(\boldsymbol{\xi}) = N^{n-1}_{\nu m_1} |\xi_{n-1}|^\sigma \left( \frac{\xi_n^2}{|\xi_{n-1}|^2} - 1 \right)^{\frac{4-n}{4}} \times \mathfrak{P}^{-m_1-(n-4)/2}_{i\nu-1/2} \left( \frac{\xi_n}{|\xi_{n-1}|} \right) X^{n-2,m_1}_{M_1}(\xi_1, \ldots, \xi_{n-2}), \quad (8)$$

where $M'$ and $X^{n-2,m_1}_{M_1}$ are the same as in formula (4) and

$$N^{n-1}_{\nu m_1} = \frac{(-1)^{m_1} 2^{(n-4)/2} \Gamma\left(\frac{n-2}{2}\right) \Gamma\left(i\nu - \frac{n-5}{2}\right)}{\Gamma\left(i\nu - m_1 - \frac{n-5}{2}\right)} \left(\dim T^{n-2,m_1}\right)^{\frac{1}{2}}. \quad (9)$$

For other values of $\xi_{n-1}$ we have $\widehat{\Xi}^{n\sigma,1}_{\nu M'\delta}(\boldsymbol{\xi}) \equiv 0$.

The Cartesian coordinates $x_1, \ldots, x_{n-2}$ on the contour $\Gamma_2$ are connected with the Cartesian coordinates $\xi_1, \ldots, \xi_n$ on the cone $C_+^{n-1}$ by the formulas

$$\xi_j = x_j,\ j = 1, 2, \ldots, n-2,\ \xi_{n-1} = \frac{1}{2}(1-\rho^2),\ \xi_n = \frac{1}{2}(1+\rho^2), \quad (10)$$

where $\rho^2 = x_1^2 + \ldots + x_{n-2}^2$. With every function $f \in \mathfrak{B}^{n\sigma}$ the equality

$$f(\boldsymbol{\xi}) \equiv f(\xi_1, \ldots, \xi_n) = (\xi_{n-1} + \xi_n)^\sigma f\left( \frac{\xi_1}{\xi_{n-1}+\xi_n}, \ldots, \frac{\xi_n}{\xi_{n-1}+\xi_n} \right) = (\xi_{n-1} + \xi_n)^\sigma F\left( \frac{\xi_1}{\xi_{n-1}+\xi_n}, \ldots, \frac{\xi_{n-2}}{\xi_{n-1}+\xi_n} \right) \quad (11)$$

associates the function $F$ on $\Gamma_2 \equiv \mathbb{R}^{n-2}$. Hence, the basis functions $\widehat{\Xi}^{n\sigma,2}_{M}(\boldsymbol{\xi}) \equiv \widehat{\Xi}^{n\sigma,2}_{RM'}(\boldsymbol{\xi})$, $M = (iR, M')$, are of the form

$$\widehat{\Xi}^{n\sigma,2}_{RM'}(\boldsymbol{\xi}) = i^{m_1} \Gamma\left(\frac{n-2}{2}\right) (\dim T^{n-2,m_1})^{1/2} (\xi_{n-1}+\xi_n)^\sigma \times \left( \frac{R\sqrt{\xi_n^2 - \xi_{n-1}^2}}{2(\xi_{n-1}+\xi_n)} \right)^{\frac{4-n}{2}} J_{m_1+(n-4)/2} \left( \frac{R\sqrt{\xi_n^2 - \xi_{n-1}^2}}{\xi_{n-1}+\xi_n} \right) \times X^{n-2,m_1}_{M_1}(\xi_1, \ldots, \xi_{n-2}), \quad (12)$$

where $M'$ and $X^{n-2,m_1}_{M_1}$ are the same as in formula (4).

**10.3.2. The Poisson transforms of the bases of the space $\mathfrak{B}^{n\sigma}$ and integral transforms on the hyperboloid.** Formulas (11) and (12) of Section 10.2.1, (10) and (12) of Section 10.2.2, (3) and (4) of Section 10.2.3 provide the Poisson transforms for the functions

$$\hat{\Xi}_M^{n,-\sigma+n_2,0},\quad \hat{\Xi}_{M\delta}^{n,-\sigma-n+2,1},\quad \hat{\Xi}_M^{n,-\sigma-n+2,2},$$

given on the cone $C_+^{n-1}$. If $\mathbf{x} \in H_+^{n-1}$, then

$$\left(\mathcal{P}_\sigma \hat{\Xi}_{M\delta}^{n,-\sigma-n+2,0}\right)(\mathbf{x}) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \Xi_M^{n\sigma}(\mathbf{x}), \tag{1}$$

where $M = (m, M')$,

$$\left(\mathcal{P}_\sigma \hat{\Xi}_{M\delta}^{n,-\sigma-n+2,1}\right)(\mathbf{x}) = \overset{*}{\Xi}{}_{M\delta}^{n\sigma}(\mathbf{x}), \tag{2}$$

where $M = (\tau, M')$, $\delta = \{+, -\}$,

$$\left(\mathcal{P}_\sigma \hat{\Xi}_M^{n,-\sigma-n+2,2}\right)(\mathbf{x}) = \overset{**}{\Xi}{}_M^{n\sigma}(\mathbf{x}), \tag{3}$$

where $M = (iR, M')$.

The form of the factor on the right hand side of (1) is connected with the fact that we choose the normalized measure on $S^{n-2}$ (and not the measure induced by the invariant measure on the cone $C_+^{n-1}$) in formula (11) of Section 10.2.1.

Further, if $\mathbf{x} \in H_-^{n-1}$ and $\delta \in \{+, -\}$, then

$$\left(\mathcal{P}_{\sigma\delta} \hat{\Xi}_M^{n,-\sigma-n+2,0}\right)(\mathbf{x}) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \Xi_{M\delta}^{n\sigma}(\mathbf{x}), \tag{4}$$

where $M = (m, M')$,

$$\left(\mathcal{P}_{\sigma,-} \hat{\Xi}_{M\delta}^{n,-\sigma-n+2,1}\right)(\mathbf{x}) = \overset{*}{\Xi}{}_{M,-,\delta}^{n\sigma}(\mathbf{x}), \tag{5}$$

where $M = (\tau, M')$,

$$\left(\mathcal{P}_{\sigma,-} \hat{\Xi}_M^{n,-\sigma-n+2,2}\right)(\mathbf{x}) = \overset{**}{\Xi}{}_{M,-}^{n\sigma}(\mathbf{x}), \tag{6}$$

where $M = (iR, M')$.

Since the Poisson transform intertwines the representations of the group $SO_0(n-1,1)$ in the spaces of functions on $C_+^{n-1}$ and on $H_+^{n-1}$, and the space $\mathfrak{L}^2(C_+^{n-1})$ is the direct continuous sum of the spaces $\mathfrak{B}^{n\sigma}$, then formulas (1)–(3)

allow us to construct new continuous bases in the space $\mathfrak{L}^2(H_+^{n-1})$. They lead to new integral transforms.

Let $f(\mathbf{x})$ is a smooth finite function on $H_+^{n-1}$ and let

$$\Phi(\boldsymbol{\xi},\sigma) = \int\limits_{H_+^{n-1}} [\mathbf{x},\boldsymbol{\xi}]^\sigma f(\mathbf{x})d\mathbf{x}, \quad \boldsymbol{\xi} \in C_+^{n-1}, \tag{7}$$

be its transform according to Section 10.1.4. Since $\Phi(\boldsymbol{\xi},\sigma) \in \mathfrak{B}^{n\sigma}$, then this function can be expanded in the basis

$$\left\{\widehat{\Xi}_{M\delta}^{n\sigma,1} \mid M = (\tau, M'),\, \delta \in \{+,-\}\right\}.$$

Namely, if $\varphi_\delta(\boldsymbol{\xi}',\sigma)$ is the restriction of $\Phi(\boldsymbol{\xi},\sigma)$ onto $\Gamma_1^\delta$, then according to formulas (1) and (2) of Section 10.2.7, we have

$$\begin{aligned}\varphi_\delta(\boldsymbol{\xi}',\sigma) =&\Phi(\boldsymbol{\xi}',\sigma)\\ &= c'_{n-1}\sum_{M'}\int\limits_{b-i\infty}^{b+i\infty} b_{M'\delta}(\tau,\sigma)\widehat{\Xi}_{\bar{M}\delta}^{n\sigma,1}(\boldsymbol{\xi}')d\mu'_{n-1}(\tau),\end{aligned} \tag{8}$$

where $M = (-\tau - n + 3, \bar{M}')$, $\bar{M}' = (m_1, \ldots, \mp m_{n-3})$ if $M' = (m_1, \ldots, \pm m_{n-3})$, and

$$b_{M'\delta}(\tau,\sigma) = \int\limits_{\Gamma_1^\delta} \Phi(\boldsymbol{\xi},\sigma)\Xi_{M'\delta}^{n-1,\tau}(\boldsymbol{\xi})d\boldsymbol{\xi}, \tag{9}$$

$$c'_{n-1} = \frac{(-1)^{(n-3-\varepsilon)/2}}{2^{n-2}\pi^{(n-2)/2}\Gamma\left(\frac{n-2}{2}\right)i}, \; \varepsilon = \frac{1}{2}(1-(-1)^{n-1}), \tag{10}$$

$$d\mu'_{n-1}(\tau) = \frac{\Gamma(\tau+n-3)}{\Gamma(\tau)}(\cot^\varepsilon \pi\tau)d\tau, \quad 3-n < b < 0. \tag{10'}$$

Let us substitute expansion (8) into formula (7) of Section 10.1.4 and invert the integration order. By using formula (2) we obtain that

$$\begin{aligned}f(\mathbf{x}) = \frac{(-1)^{(n-2-\varepsilon)/2}}{2(2\pi)^{n-1}i}\sum_{M',\delta}\int\limits_{a-i\infty}^{a+i\infty}\int\limits_{b-i\infty}^{b+i\infty} &b_{M'\delta}(\tau,\sigma)\\ &\times \overset{*}{\Xi}{}_{M\delta}^{n\sigma}(\mathbf{x})d\mu'_{n-1}(\tau)d\mu'_n(\sigma),\end{aligned} \tag{11}$$

where $M = (-\tau - n + 3, \bar{M}')$.

In order to express $b_{M'\delta}$ in terms of $f(\mathbf{x})$, we substitute the expression (7) for $\Phi(\boldsymbol{\xi},\sigma)$ into (9) and invert the integration order. By using formula (2) we derive that

$$b_{M'\delta}(\tau,\sigma) = \int\limits_{H_+^{n-1}} f(\mathbf{x})\overset{*}{\Xi}{}^{n\sigma}_{M\delta}(\mathbf{x})d\mathbf{x}. \tag{12}$$

The Plancherel formula

$$\int\limits_{H_+^{n-1}} |f(\mathbf{x})|^2 d\mathbf{x} = \frac{(-1)^{(n-2+\epsilon)/2}c'_{n-1}}{2(2\pi)^{n-1}i}$$

$$\times \sum_{M',\delta} \int\limits_{a-i\infty}^{a+i\infty} \int\limits_{b-i\infty}^{b+i\infty} |b_{M'\delta}(\tau,\sigma)|^2 d\mu'_{n-1}(\tau)d\mu'_n(\sigma) \tag{13}$$

holds. These formulas are of more symmetric form if $\tau = i\lambda - \frac{n-3}{2}$ and $\sigma = \nu - \frac{n-2}{2}$.

If the function $f(\mathbf{x})$ is invariant with respect to the transformations $\mathbf{x} \to h\mathbf{x}$, $h \in SO_0(n-2,1)$, then the obtained formulas can be simplified. In this way, we obtain the transform

$$f(t) = \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} [\sigma_+(\rho)\mathrm{P}^{-i\rho}_{i\nu-1/2}(\tanh t) + a_-(\rho)\mathrm{P}^{-i\rho}_{i\nu-1/2}(-\tanh t)]\mu(\rho)d\rho, \tag{14}$$

where

$$a_\delta(\rho) = \int\limits_{-\infty}^{\infty} f(t)\mathrm{P}^{i\rho}_{i\nu-1/2}(\delta \tanh t)dt, \tag{15}$$

$$\mu(\rho) = \frac{\rho \sinh \pi\rho}{\cosh(\rho+\nu)\pi \cosh(\rho-\nu)\pi}. \tag{16}$$

Another integral transform is related to the contour $\Gamma_2$ on $C_+^{n-1}$. Applying the above reasonings to the basis (12) of Section 10.3.1 of the space $\mathfrak{B}^{n\sigma}$, we obtain that

$$f(\mathbf{x}) = \frac{1}{2}c'_n c \sum -M' \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} a_{M'}(R,\rho)\overset{**}{\Xi}{}^{n\sigma}_{M}(\mathbf{x})$$

$$\times R^{n-3}dRd\mu_n(\rho), \tag{17}$$

where $M = (iR, M')$, $\sigma = i\rho - \frac{n-2}{2}$, $c'_n$ is given by formula (10),

$$c = \frac{\Gamma\left(\frac{n-1}{2}\right)}{2^{n-2}\pi^{n-3/2}\Gamma\left(\frac{n-2}{2}\right)},$$

$$d\mu_n(\rho) = \frac{\Gamma\left(i\rho + \frac{n-2}{2}\right)}{\Gamma\left(i\rho - \frac{n-2}{2}\right)}(\tanh^\varepsilon \pi\rho)\rho d\rho, \ \varepsilon = \frac{1}{2}(1-(-1)^n),$$

and

$$a_{M'}(R,\rho) = \int\limits_{H_+^{n-1}} f(\mathbf{x})\overline{\overset{**}{\Xi}{}_M^{n\sigma}(\mathbf{x})}d\mathbf{x}. \tag{18}$$

The Plancherel formula

$$\int\limits_{H_+^{n-1}} |f(\mathbf{x})|^2 d\mathbf{x} = \frac{1}{2}c'_n c \sum_{M'} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} |a_{M'}(R,\rho)|^2 R^{n-3} dR d\mu_n(\rho) \tag{19}$$

is valid.

If $f(\mathbf{x})$ is one of the form $f(\mathbf{x}) = F(\theta)\Phi(\mathbf{t})$, where $\mathbf{x} = n(\mathbf{t})g'_{n-1}(\theta)\mathbf{e}_n$ (see formula (16) of Section 9.1.5), then it is easy to obtain from (17) and (18) the Kontorovich-Lebedev transforms

$$a(\rho) = \int\limits_0^\infty F(t)K_{i\rho}(t)dt, \tag{20}$$

$$F(t) = \frac{2}{\pi^2 x}\int\limits_0^\infty a(\rho)K_{i\rho}(t)\rho\sinh\pi\rho\, d\rho \tag{21}$$

(see Section 7.8.4). One can rewrite (20) and (21) in more symmetric form

$$A(\rho) = \int\limits_0^\infty f(x)\frac{K_{i\rho}(x)}{\sqrt{x}}dx, \tag{22}$$

$$f(x) = \int\limits_0^\infty A(\rho)\frac{K_{i\rho}(x)}{\sqrt{x}}\ \frac{2\rho\sinh\pi\rho}{\pi^2}d\rho. \tag{23}$$

Formula (19) allows us to derive the Plancherel formula for these transforms:

$$\int\limits_0^\infty |f(x)|^2 dx = \int\limits_0^\infty |A(\rho)|^2\frac{2\rho\sinh\pi\rho}{\pi^2}d\rho.$$

**10.3.3. Expansions of Poisson kernels.** Formulas (10)–(12) of Section 10.2.1 can be considered as expressions for Fourier coefficients of the functions $(\mathbf{x},\boldsymbol{\eta})^\ell$, $[\mathbf{x},\boldsymbol{\xi}]^\sigma$, $[\mathbf{x},\boldsymbol{\xi}[_\delta^\sigma$ under the expansions in orthonormal system of functions

$\Xi_{M'}^{n-1,m}(\boldsymbol{\xi}')$ on $S^{n-2}$. Therefore, we have corresponding expansions into Fourier series. Namely, if $\mathbf{x} \in \mathbb{C}^n$, $\boldsymbol{\zeta} = (-i\boldsymbol{\zeta}', 1)$, $\boldsymbol{\zeta}' \in S^{n-2}$, then

$$(\mathbf{x}, \boldsymbol{\zeta})^{\ell} = \ell! \sum_{M} [(\ell + m + n - 3)!(\ell - m)!(2m + n - 2)]^{-1/2} \times \Xi_{M}^{n\ell}(\mathbf{x}) \Xi_{\bar{M}'}^{n-1,m}(\boldsymbol{\zeta}'), \tag{1}$$

where $M = (m, M')$ and $\bar{M}' = (m_1, \ldots, \mp m_{n-3})$ if $M' = (m_1, \ldots, \pm m_{n-3})$. If $\mathbf{x} \in H_+^{n-1}$, $\boldsymbol{\xi} = (\boldsymbol{\xi}', 1) \in C_+^{n-1}$, then

$$[\mathbf{x}, \boldsymbol{\xi}]^{\sigma} = \sum_{M} \Xi_{M}^{n\sigma}(\mathbf{x}) \Xi_{M'}^{n-1,m}(\boldsymbol{\xi}'), \tag{2}$$

where $M = (m, M')$. If $\mathbf{x} \in H_-^{n-1}$, $\boldsymbol{\xi} = (\boldsymbol{\xi}', 1) \in C_+^{n-1}$, then

$$[\mathbf{x}, \boldsymbol{\xi}]_{\delta}^{\sigma} = \sum_{M} \Xi_{M\delta}^{n\sigma}(\mathbf{x}) \Xi_{\bar{M}'}^{n-1,m}(\boldsymbol{\xi}'), \tag{3}$$

where $M = (m, M')$.

Analogously, applying formulas (1) and (2) of Section 10.2.7 to expansions (10) and (12) of Section 10.2.2, we have for $\mathbf{x} \in H_+^{n-1}$, $\boldsymbol{\xi} \in C_+^{n-1}$ the relations

$$[\mathbf{x}, \boldsymbol{\xi}]^{\sigma} = c'_{n-1} \sum_{M',\varepsilon} \int_{b-i\infty}^{b+i\infty} \overset{*}{\Xi}{}_{M\varepsilon}^{n\sigma}(\mathbf{x}) \Xi_{\bar{M}'}^{n-1,-\tau-n+3}(\boldsymbol{\xi}') d\mu'_{n-1}(\tau), \tag{4}$$

$$[\mathbf{x}, \boldsymbol{\xi}]_{\delta}^{\sigma} = c'_{n-1} \sum_{M',\varepsilon} \int_{b-i\infty}^{b+i\infty} \overset{*}{\Xi}{}_{M\delta\varepsilon}^{n\sigma}(\mathbf{x}) \Xi_{\bar{M}'}^{n-1,-\tau-n+3}(\boldsymbol{\xi}') d\mu'_{n-1}(\tau), \tag{5}$$

where $M = (\tau, M')$, $\varepsilon \in \{+, -\}$.

Applying formulas (1) and (2) of Section 10.2.5 to relations (3) and (4) of Section 10.2.3, we derive the relations

$$[\mathbf{x}, \boldsymbol{\xi}]^{\sigma} = \tilde{c}_{n-1} \sum_{M'} \int_{0}^{\infty} \overset{**}{\Xi}{}_{M}^{n\sigma}(\mathbf{x}) \Xi_{\bar{M}'}^{n-1,iR}(\boldsymbol{\xi}') R^{n-3} dR, \tag{6}$$

$$[\mathbf{x}, \boldsymbol{\xi}]_{\delta}^{\sigma} = \tilde{c}_{n-1} \sum_{M'} \int_{0}^{\infty} \overset{**}{\Xi}{}_{M\delta}^{n\sigma}(\mathbf{x}) \Xi_{\bar{M}'}^{n-1,R}(\boldsymbol{\xi}') R^{n-3} dR, \tag{7}$$

where $M = (iR, M')$, $\tilde{c}_{n-1} = \left[2^{n-3}\pi^{(n-2)/2}\Gamma\left(\frac{n-2}{2}\right)\right]^{-1}$.

The relations of this section are generalizations of formulas (6), (9) and (9′) of Section 9.4.11, (14) and (15) of Section 10.2.6, (17) and (18) of Section 10.2.8.

**10.3.4. Poisson transforms and addition theorems for special functions.** Every one of formulas (2)–(7) of Section 10.3.3 gives addition theorems for special functions. In order to obtain these theorems in general form we have, for example, to multiply both sides of equality (2) of Section 10.3.3 by $[\mathbf{y},\boldsymbol{\xi}]^{-\sigma-n+2}$, $\mathbf{y}\in H_+^{n-1}$, and to integrate with respect to $\boldsymbol{\xi}$ over one of the contours $\Gamma_0$, $\Gamma_1$, $\Gamma_2$. Since the function $[\mathbf{x},\boldsymbol{\xi}]^{\sigma}[\mathbf{y},\boldsymbol{\xi}]^{-\sigma-n+2}$ is of homogeneity degree $-n+2$ in $\boldsymbol{\xi}$, then the integral

$$I_1(\mathbf{x},\mathbf{y})\equiv\int_\Gamma[\mathbf{x},\boldsymbol{\xi}]^{\sigma}[\mathbf{y},\boldsymbol{\xi}]^{-\sigma-n+2}d\boldsymbol{\xi} \tag{1}$$

does not depend on the choice of a contour $\Gamma$. Hence, this integral is invariant with respect to the shifts $\mathbf{x}\to g\mathbf{x}$, $\mathbf{y}\to g\mathbf{y}$, where $g\in SO_0(n-1,1)$. Therefore, we can consider that $\mathbf{x}=(0,\dots,0,1)$ and $\mathbf{y}=(0,\dots,0,\sinh\theta,\cosh\theta)$, where $\cosh\theta=[\mathbf{x},\mathbf{y}]$. If $\Gamma=\Gamma_0$ and $\mathbf{x},\mathbf{y}$ are of these forms, then integral (1) is equal to

$$\frac{2\pi^{\frac{n-2}{2}}}{\Gamma\left(\frac{n-2}{2}\right)}\int_0^\pi(\cosh\theta-\sinh\theta\cos\varphi)^{-\sigma-n+2}\sin^{n-3}\varphi\, d\varphi$$

(the measure on $\Gamma_0=S^{n-2}$ is not normalized). Now by virtue of formula (6) of Section 9.3.6, we have

$$I_1(\mathbf{x},\mathbf{y})=(2\pi)^{\frac{n-1}{2}}\sinh^{\frac{3-n}{2}}\theta\,\mathfrak{P}^{\frac{3-n}{2}}_{\sigma+\frac{n-3}{2}}(\cosh\theta), \tag{1'}$$

where $\cosh\theta=[\mathbf{x},\mathbf{y}]$.

It is analogously proved that for $\mathbf{x}\in H_+^{n-1}$, $\mathbf{y}\in H_-^{n-1}$, $\delta\in\{+,-\}$ we have

$$\begin{aligned}I_2(\mathbf{x},\mathbf{y})&\equiv\int_\Gamma[\mathbf{x},\boldsymbol{\xi}]^{\sigma}[\mathbf{y},\boldsymbol{\xi}]_\delta^{-\sigma-n+2}d\boldsymbol{\xi}\\&=(2\pi)^{\frac{n-2}{2}}\Gamma(-\sigma-n+3)\cosh^{\frac{2-n}{2}}\theta\,\mathrm{P}^{\sigma+\frac{n+2}{2}}_{\frac{n-4}{2}}(-\delta\tanh\theta),\end{aligned} \tag{2}$$

where $\cosh\theta=[\mathbf{x},\mathbf{y}]$.

Now we obtain the addition theorems. By virtue of relation (1) of Section 10.3.2 we have from formula (2) of Section 10.3.3 that

$$\int_\Gamma[\mathbf{x},\boldsymbol{\xi}]^{\sigma}[\mathbf{y},\boldsymbol{\xi}]^{-\sigma-n+2}d\boldsymbol{\xi}=\sum_M\Xi^{n\sigma}_M(\mathbf{x})\Xi^{n,-\sigma-n+2}_{\bar M}(\mathbf{y}),$$

where $\bar{M}$ is obtained from $M$ by changing a sign at $m_{n-3}$. Hence, using formula (1′), we obtain

$$\sinh^{\frac{3-n}{2}}\theta\mathfrak{P}^{\frac{3-n}{2}}_{\sigma+\frac{n-3}{2}}(\cosh\theta) = (2\pi)^{-\frac{n-1}{2}}\sum_{M}\Xi^{n\sigma}_{M}(\mathbf{x})\Xi^{n,-\sigma-n+2}_{\bar{M}}(\mathbf{y}), \tag{3}$$

where $\cosh\theta = [\mathbf{x},\mathbf{y}]$.

Multiplying both sides of relation (2) of Section 10.3.3 by $[\mathbf{y},\boldsymbol{\xi}]^{-\sigma-n+2}$, $\mathbf{y}\in H^{n-1}_{-}$, and integrating over $\Gamma_0 = S^{n-2}$, we derive with the help of formula (4) of Section 10.3.2 that

$$\cosh^{\frac{2-n}{2}}\theta\,\mathrm{P}^{\sigma+\frac{n-2}{2}}_{\frac{n-4}{2}}(-\delta\tanh\theta) = \frac{(2\pi)^{-(n-2)/2}}{\Gamma(-\sigma-n+3)}\sum_{M}\Xi^{n\sigma}_{M}(\mathbf{x})\Xi^{n,-\sigma-n+2}_{\bar{M}\delta}(\mathbf{y}), \tag{4}$$

where $\sinh\theta = [\mathbf{x},\mathbf{y}]$.

Other similar formulas are obtained from formula (4) of Section 10.3.3 and from formulas (2) and (5) of Section 10.3.2. We have

$$\sinh^{\frac{3-n}{2}}\theta\mathfrak{P}^{\frac{3-n}{2}}_{\sigma+\frac{n-3}{2}}(\cosh\theta) = (2\pi)^{-\frac{n-1}{2}}c'_{n-1} \times\sum_{M',\varepsilon,\delta}\int_{b-i\infty}^{b+i\infty}\overset{*}{\Xi}{}^{n\sigma}_{M\varepsilon}(\mathbf{x})\Xi^{n,-\sigma-n+2}_{\bar{M}\delta}(\mathbf{y})d\mu'_{n-1}(\tau), \tag{5}$$

where $M=(\tau,M')$, $\bar{M}=(-\tau-n+3,\bar{M}')$, $\varepsilon,\delta\in\{+,-\}$, $\cosh\theta=[\mathbf{x},\mathbf{y}]$, and

$$\cosh^{\frac{2-n}{2}}\theta\mathrm{P}^{\sigma+\frac{n-2}{2}}_{\frac{n-4}{2}}(-\delta\tanh\theta) = \frac{(2\pi)^{-(n-2)/2}c'_{n-1}}{\Gamma(-\sigma-n+3)} \times\sum_{M',\varepsilon}\int_{b-i\infty}^{b+i\infty}\overset{*}{\Xi}{}^{n\sigma}_{M}(\mathbf{x})\Xi^{n,-\sigma-n+2}_{\bar{M}\varepsilon}(\mathbf{y})d\mu'_{n-1}(\tau), \tag{6}$$

where $\sinh\theta=[\mathbf{x},\mathbf{y}]$ and $M$, $\bar{M}$ are the same as in (5).

We derive from relation (6) of Section 10.3.3 and from formulas (3) and (6) of Section 10.3.2 that

$$\sinh^{\frac{3-n}{2}}\theta\mathfrak{P}^{\frac{3-n}{2}}_{\sigma+\frac{n-3}{2}}(\cosh\theta) = (2\pi)^{-\frac{n-1}{2}}\tilde{c}_{n-1} \times\sum_{M'}\int_{0}^{\infty}\overset{**}{\Xi}{}^{n\sigma}_{M}(\mathbf{x})\overset{**}{\Xi}{}^{n,-\sigma-n+2}_{\bar{M}}(\mathbf{y})R^{n-3}dR, \tag{7}$$

where $\mathbf{x},\mathbf{y}\in H_+^{n-1}$, $\cosh\theta=[\mathbf{x},\mathbf{y}]$, $M=(iR,M')$, $\bar M=(-iR,\bar M')$, and

$$\cosh^{\frac{2-n}{2}}\theta\,\mathrm{P}^{\sigma+\frac{n-2}{2}}_{\frac{n-4}{2}}(-\delta\tanh\theta)=\frac{(2\pi)^{-(n-2)/2}\tilde c_{n-1}}{\Gamma(-\sigma-n+3)}$$
$$\times\sum_{M'}\int_0^\infty \overset{**}{\Xi}{}^{n\sigma}_{M}\overset{**}{\Xi}{}^{n,-\sigma-n+2}_{\bar M\delta}(\mathbf{y})R^{n-3}dR, \tag{8}$$

where $\mathbf{x}\in H_+^{n-1}$, $\mathbf{y}\in H_-^{n-1}$, $\sinh\theta=[\mathbf{x},\mathbf{y}]$.

The obtained addition theorems lead to addition theorems for corresponding special functions, if we consider special values of $\mathbf{x}$ and $\mathbf{y}$. Putting

$$\mathbf{x}=(0,\ldots,0,\sinh\theta_1\sin\varphi,\ \sinh\theta_1\cos\varphi,\ \cosh\theta_1)$$
$$\mathbf{y}=(0,\ldots,0,\cosh\theta_2,\ \sinh\theta_2)$$

into (4), we obtain

$$\cosh^{-p}\theta\,\mathrm{P}^{p+\sigma}_{p-1}(\mp\tanh\theta)=2^{p-3/2}\Gamma\left(p-\frac12\right)\Gamma(\sigma+1)\sinh^{-p+1/2}\theta_1$$
$$\times\cosh^{-p}\theta_2\sum_{m=0}^\infty\frac{(2m+2p-1)}{\Gamma(\sigma-m+1)}\mathfrak{P}^{-p-m-1/2}_{\sigma+p-1/2}(\cosh\theta_1)$$
$$\times\mathrm{P}^{p+\sigma}_{m+p-1}(\mp\tanh\theta_2)C_m^{p-1/2}(\cosh\varphi), \tag{9}$$

where $\sinh\theta=\cosh\theta_1\sinh\theta_2\mp\sinh\theta_1\cosh\theta_2\cos\varphi$.

Let us note that the addition theorems (5) and (6) of Section 9.4.3 can be derived by this method.

Putting

$$\mathbf{x}=(0,\ldots,0,\cosh\theta_1\sinh\varphi,\ \sinh\theta_1,\cosh\theta_1\cosh\varphi),$$
$$\mathbf{y}=(0,\ldots,0,\sinh\theta_2,\ \cosh\theta_2)$$

into relation (5) and using formulas (10) and (11) of Section 10.2.2, we derive the addition theorem

$$\sinh^{-p+1/2}\theta\mathfrak{P}^{-p+1/2}_{\sigma+p-1/2}(\cosh\theta)=\frac{(-1)^{p-\gamma/2}(2^{-1}\pi)^{3/2}}{\Gamma(-\sigma)\Gamma(\sigma+2p)i}$$
$$\times(\cosh\theta_1\cosh\theta_2)^p\sinh^{1-p}\varphi\sum_{\delta,\varepsilon}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(\tau+2p-1)}{\Gamma(\tau)}$$
$$\times\frac{\cot^\gamma\pi\tau}{\sin\pi(\tau-\sigma)\sin\pi(\tau+\sigma)}\mathrm{P}^{\sigma+p}_{\tau+p-1}(-\delta\tanh\theta_1)$$
$$\times\mathrm{P}^{-\sigma-p}_{\tau+p-1}(-\varepsilon\tanh\theta_2)\mathfrak{P}^{1-p}_{\tau+p-1}(\cosh\varphi)d\tau, \tag{10}$$

where $\gamma = \frac{1}{2}(1-(-1)^{n-1})$, $p = \frac{n-3}{2}$

$$\cosh\theta = \cosh\theta_1 \cosh\theta_2 \cosh\varphi - \sinh\theta_1 \sinh\theta_2 .$$

Putting into (6)

$$\mathbf{x} = (0,\dots,0,\cosh\theta_1 \sinh\varphi,\ \sinh\theta_1,\ \cosh\theta_1\cosh\varphi),$$
$$\mathbf{y} = (0,\dots,0,\cosh\theta_2,\ \sinh\theta_2)$$

we derive the addition theorem

$$\cosh^{-p}\theta\, \mathrm{P}^{\sigma+p}_{p-1}(-\tanh\theta) = \frac{(-1)^{p-\gamma/2}}{2i}(\cosh\theta_1\sinh\theta_2)^{-p}\sinh^{1-p}\varphi$$
$$\times \sum_{\delta}\int_{a-i\infty}^{a+i\infty} \frac{\Gamma(\tau+2p-1)}{\Gamma(\tau)}(\cot^{\gamma}\pi\tau)\frac{\Gamma(\tau-\sigma)\Gamma(-\tau-\sigma-2p)}{\Gamma(-\sigma)}$$
$$\times\, \mathrm{P}^{\sigma+p}_{\tau+p-1}(-\delta\tanh\theta_1)\mathfrak{P}^{\sigma+p}_{\tau+p-1}(\tanh^{-1}\theta_2)\mathfrak{P}^{1-p}_{\tau+p-1}(\cosh\varphi)d\tau, \quad (11)$$

where $\gamma = \frac{1}{2}(1-(-1)^{n-1})$, $p = \frac{n-2}{2}$,

$$\cosh\theta = \cosh\theta_1 \sinh\theta_2 \cosh\varphi - \sinh\theta_1 \cosh\theta_2 .$$

Putting into (7)

$$\mathbf{x} = (0,\dots,0,re^{-\theta_1},\sinh\theta_1 + \frac{1}{2}r^2e^{\theta_1},\cosh\theta_1+\frac{1}{2}r^2e^{\theta_1}),$$
$$\mathbf{y} = (0,\dots,0,\sinh\theta_2,\cosh\theta_2)$$

we obtain the addition theorem

$$\sinh^{-p+1/2}\theta\,\mathfrak{P}^{-p+1/2}_{\sigma+p-1/2}(\cosh\theta) = \frac{2^{3/2}e^{\theta_1}e^{p\theta_2}}{\sqrt{\pi}\Gamma(-\sigma)\Gamma(\sigma+2p)r^{p-1}}$$
$$\times \int_0^{\infty} K_{\sigma+p}(Re^{\theta_1})K_{-\sigma-p}(Re^{\theta_2})J_{p-1}(Rre^{\theta_2})R^{1-p}dR, \quad (12)$$

where $\cosh\theta = \cosh(\theta_1+\theta_2) + \frac{r^2}{2}e^{\theta_1-\theta_2}$.

Putting into (8)

$$\mathbf{x} = \left(0,\dots,0,re^{\theta_1},\sinh\theta_1+\frac{r^2e^{\theta_1}}{2},\cosh\theta_1+\frac{r^2e^{\theta_1}}{2}\right),$$
$$\mathbf{y} = (0,\dots,0,\cosh\theta_2,\sinh\theta_2)$$

we derive the addition formula

$$\cosh^{-p}\theta\, \mathrm{P}_{p-1}^{p+\sigma}(\tanh\theta) = \frac{2\Gamma(-\sigma-2p+1)e^{\theta_2}e^{p\theta_1}}{2^{p-1}\Gamma(-\sigma)} \times \int_0^\infty K_{\sigma+p}(Re^{\theta_1})J_{-\sigma-p}(Re^{\theta_2})J_{p-1}(Rr^{\theta_2})R^{1-p}dR, \quad (13)$$

where $\sinh\theta = \sinh(\theta_1+\theta_2) + \frac{r^2}{2}e^{\theta_1-\theta_2}$.

Let us note that the cases $\mathbf{x}, \mathbf{y} \in H_-^{n-1}$ do not lead to addition theorems since the corresponding series and integrals diverge. Probably, formal relations appearing for $\mathbf{x}$ and $\mathbf{y}$ from $H_-^{n-1}$ can be justified by a regularization procedure.

Every addition theorem derived above leads to the corresponding product formula. For example, we derive from (9) the product formula

$$\sinh^{-p+1/2}\theta_1 \cosh^{-p}\theta_2 \mathfrak{P}_{\sigma+p-1/2}^{-k-p+1/2}(\cosh\theta_1)\mathrm{P}_{k+p-1/2}^{\sigma+p}(\mp\tanh\theta_2)$$
$$= \frac{2^{p-3/2}\Gamma(\sigma-k+1)\Gamma\left(p-\frac{1}{2}\right)}{\Gamma(2p+k-1)} \int_0^\pi \cosh^{-p}\theta\, \mathrm{P}_{p-1}^{\sigma+p}(\mp\tanh\theta)$$
$$\times C_k^{p-1/2}(\cos\varphi)\sin^{2p-1}\varphi d\varphi, \quad (14)$$

where $\sinh\theta = \cosh\theta_1 \sinh\theta_2 \mp \sinh\theta_1 \cosh\theta_2 \cos\varphi$.

## 10.4. Spherical Functions in Cylindrical Coordinates and Special Functions

**10.4.1. Harmonic polynomials in bispherical coordinates.** Let $0 < p < n$. We realize the representation $T^{n\ell}$ of the group $SO(n)$ in the space $\mathfrak{H}^{n\ell}$ of homogeneous harmonic polynomials of degree $\ell$ (see Section 9.2.3). Set

$$\mathbf{x} = (\mathbf{y}, \mathbf{t}), \quad \mathbf{y} \in \mathbb{R}^p, \quad \mathbf{t} \in \mathbb{R}^q, \quad p+q=n,$$

and $K_{pq} = SO(p) \times SO(q)$. The action of $K_{pq}$ in $\mathbb{R}^n$ is defined as

$$(b_1, h_2)(\mathbf{y}, \mathbf{t}) = (h_1\mathbf{y}, h_2\mathbf{t}), \; h_1 \in SO(p), \; h_2 \in SO(q).$$

We denote by $\Delta_\mathbf{y}$ the Laplace operator for the variables $y_1, \ldots, y_p$ and by $\Delta_\mathbf{t}$ the Laplace operator for $t_1, \ldots, t_q$. We set

$$\Delta_\mathbf{x} = \Delta_\mathbf{y} + \Delta_\mathbf{t}$$

and

$$(\mathbf{y}, \mathbf{y})_p = y_1^2 + \ldots + y_p^2 = |\mathbf{y}|^2, \; (\mathbf{t}, \mathbf{t})_q = t_1^2 + \ldots + t_q^2 = |\mathbf{t}|^2.$$

**Theorem.** *Let $r, s, m \in \mathbf{Z}_+ \cup \{0\} = \mathbf{N}_0$. Then there exists a non-zero homogeneous polynomial $Y(z, w)$ of degree $m$ such that all polynomials of the form*

$$P(\mathbf{y}, \mathbf{t}) = Y(|\mathbf{y}|^2, |\mathbf{t}|^2) A(\mathbf{y}) B(\mathbf{t}), \tag{1}$$

*where $A(\mathbf{y}) \in \mathfrak{H}^{pr}$, $B(t) \in \mathfrak{H}^{qs}$, are harmonic. The polynomial $Y(z, w)$ is defined uniquely up to a constant factor.*

*Proof:* Let us apply $\Delta_{\mathbf{x}}$ to the function $P(\mathbf{y}, \mathbf{t})$, take into account the fact that $\Delta_{\mathbf{y}} A(\mathbf{y}) = \Delta_{\mathbf{t}} B(\mathbf{t}) = 0$ and make use of the Euler formula. We obtain that $\Delta_{\mathbf{x}} P(\mathbf{y}, \mathbf{t}) = 0$ if and only if

$$\left[2z \frac{\partial^2}{\partial z^2} + (2r + p) \frac{\partial}{\partial z} + 2w \frac{\partial^2}{\partial w^2} + (2s + q) \frac{\partial}{\partial w}\right] Y(z, w) = 0. \tag{2}$$

By setting $Y(z, w) = \sum\limits_{k=0}^{m} a_k z^k w^{m-k}$ we obtain from (2) the recurrence relation for the coefficients $a_k$, which implies that

$$Y(z, w) = A' w^m F\left(-m, -m - s - \frac{q}{2} + 1; r + \frac{p}{2}; -\frac{z}{w}\right). \tag{3}$$

This polynomial can be represented in the form

$$Y(z, w) = A(z + w)^m P_m^{(\alpha, \beta)}\left(\frac{w - z}{w + z}\right), \tag{3'}$$

where

$$\alpha = r + \frac{p-2}{2}, \ \beta = s + \frac{q-2}{2}, \ A' = \frac{\Gamma(m + \alpha + 1)}{2^m m! \Gamma(\alpha + 1)} A.$$

Since every function from $\mathfrak{H}^{pr} \times \mathfrak{H}^{qs}$ can be expanded in functions of the form $\Xi_{M'}^{pr}(\mathbf{y}) \Xi_{N'}^{qs}(\mathbf{t})$, we set $\ell = 2m + r + s$ and

$$\begin{aligned} \Xi_{MN}^{pq\ell}(\mathbf{x}) &= Y(|\mathbf{y}|^2, |\mathbf{t}|^2) \Xi_{M'}^{pr}(\mathbf{y}) \Xi_{N'}^{qs}(\mathbf{t}) \\ &= A |\mathbf{x}|^{\ell - r - s} P_m^{(\alpha, \beta)}\left(\frac{|\mathbf{t}|^2 - |\mathbf{y}|^2}{|\mathbf{x}|^2}\right) \Xi_{M'}^{pr}(\mathbf{y}) \Xi_{N'}^{qs}(\mathbf{t}), \end{aligned} \tag{4}$$

where $M = (r, M')$, $N = (s, N')$, $\alpha = r + \frac{p-2}{2}$, $\beta = s + \frac{q-2}{2}$, $m = \frac{1}{2}(\ell - r - s)$.

We introduce bispherical coordinates on $S^{n-1}$ (see Section 9.1.5) and consider $\Xi_{MN}^{pq\ell}(\mathbf{x})$ on $S^{n-1}$. Since $|\mathbf{y}|^2 = \sin^2\theta$, $|\mathbf{t}|^2 = \cos^2\theta$, then

$$\Xi_{MN}^{pq\ell}(\boldsymbol{\xi}) = Y_{rs}^{pq\ell}(\theta) \Xi_{M'}^{pr}(\boldsymbol{\eta}) \Xi_{N'}^{qs}(\boldsymbol{\tau}), \tag{5}$$

where $M=(r,M')$, $N=(s,N')$, $\boldsymbol{\xi}=\boldsymbol{\eta}\sin\theta+\boldsymbol{\tau}\cos\theta\in S^{n-1}$,

$$\begin{aligned} Y_{rs}^{pq\ell}(\theta) &= A\sin^r\theta\cos^s\theta P_m^{(\alpha,\beta)}(\cos 2\theta) \\ &= A'\tan^r\theta\cos^\ell\theta F\left(-m,-m-s-\frac{q}{2}+1;r+\frac{p}{2};-\tan^2\theta\right). \end{aligned} \tag{6}$$

The functions $\Xi_{M'}^{pr}(\boldsymbol{\eta})$ and $\Xi_{N'}^{qs}(\boldsymbol{\tau})$ are defined by formulas (3) of Section 9.3.1 and (6) of Section 9.4.2. Polynomials of form (5) belong to the space $\widetilde{\mathfrak{H}}^{n\ell}$. The normalizing factor $A$ (see formula (3′)) is chosen such that the polynomials $\Xi_{MN}^{pq\ell}(\boldsymbol{\xi})$ are normalized in $\widetilde{\mathfrak{H}}^{n\ell}$:

$$\int\limits_{S^{n-1}} |\Xi_{MN}^{pq\ell}|^2 d\boldsymbol{\xi}=1, \tag{7}$$

where $d\boldsymbol{\xi}$ is the normalized measure on $S^{n-1}$. Writing down this integral in bi-spherical coordinates and making use of formulas (4) of Section 9.1.9 and (5), we obtain that

$$\frac{2\Gamma(\frac{n}{2})A^2}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{s})}\int\limits_0^{\pi/2}|P_m^{(\alpha,\beta)}(\cos 2\theta)|^2\sin^{2\alpha+1}\theta\cos^{2\beta+1}\theta d\theta=1.$$

Consequently,

$$A=\left(\frac{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})\Gamma(m+\alpha+\beta+1)(2m+\alpha+\beta+1)}{\Gamma(\frac{n}{2})\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}\right)^{1/2}, \tag{8}$$

where $m$, $\alpha$, $\beta$ are the same as above. Therefore,

$$A'=\frac{2^{-m}}{\Gamma(\alpha+1)}\left(\frac{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})\Gamma(m+\alpha+\beta+1)\Gamma(m+\alpha+1)(2m+\alpha+\beta+1)}{m!\Gamma(\frac{n}{2})\Gamma(m+\beta+1)}\right)^{\frac{1}{2}}. \tag{8'}$$

Let us give the special case $p=1$, $q=n-1$. We have

$$\Xi_M^{1,n-1,\ell}(\boldsymbol{\xi})=c\cos^r\theta C_{\ell-r}^{r+(n-2)/2}(\sin\theta)\Xi_{M'}^{n-1,r}(\boldsymbol{\eta}), \tag{9}$$

where $M=(r,M')$ and

$$c=\frac{2^r\Gamma\left(r+\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)}\left(\frac{(\ell-r)!(n-3)!(2\ell+n-2)}{(\ell+r+n-3)!(n-2)}\right)^{1/2}. \tag{10}$$

**10.4.2. Associated $K_{pq}$-spherical functions on the sphere.** If $p$, $q$, $\ell$ are fixed, then the functions $\Xi^{pq\ell}_{MN}(\boldsymbol{\xi})$ are pairwise orthogonal in $\widetilde{\mathfrak{H}}^{n\ell}$, since the functions $\Xi^{pr}_{M'}(\boldsymbol{\eta})$ are pairwise orthogonal in $\mathfrak{L}^2(S^{p-1})$ and the functions $\Xi^{qs}_{N'}(\boldsymbol{\tau})$ are pairwise orthogonal in $\mathfrak{L}^2(S^{q-1})$.

We denote by $\widetilde{\mathfrak{H}}^{pq\ell}_{rs}$ the subspace of $\widetilde{\mathfrak{H}}^{n\ell}$, generated by the functions $\Xi^{pq\ell}_{MN}$ with fixed $\ell$, $r$, $s$, $p$, $q$. It follows from the construction of these functions that

$$\dim \widetilde{\mathfrak{H}}^{pq\ell}_{rs} = (\dim \widetilde{\mathfrak{H}}^{pr})(\dim \widetilde{\mathfrak{H}}^{qs}). \tag{1}$$

Since for $(r,s) \neq (r',s')$ the subspaces $\widetilde{\mathfrak{H}}^{pq\ell}_{rs}$ and $\widetilde{\mathfrak{H}}^{pq\ell}_{r's'}$ are orthogonal, we have

$$\widetilde{\mathfrak{H}}^{n\ell} \supset \sum_{r,s} \oplus \widetilde{\mathfrak{H}}^{pq\ell}_{rs}, \tag{2}$$

where the summation is over all $r$ and $s$ for which $2m = \ell - r - s$ is non-negative and even. Let us prove that the spaces of two sides of formula (2) are equal. For this we note that if $\mathfrak{N}^{n\ell}$ is the space of homogeneous polynomials of degree $\ell$ in $n$ variables, then

$$(1-t)^{-n} = \sum_{\ell=0}^{\infty} (\dim \mathfrak{N}^{n\ell}) t^{\ell}.$$

Therefore, for the dimensionality

$$h(n,\ell) \equiv \dim \mathfrak{H}^{n\ell} = \dim \mathfrak{N}^{n\ell} - \dim \mathfrak{N}^{n,\ell-2}$$

of $\mathfrak{H}^{n\ell}$ (see formula (4) of Section 9.3.2), we obtain

$$(1+t)(1-t)^{1-n} = \sum_{\ell=0}^{\infty} h(n,\ell) t^{\ell}. \tag{3}$$

For the dimensionalities $h(p,r)$ and $h(q,s)$ of spaces on the right hand side of (1), we have

$$(1+t)(1-t)^{1-p} = \sum_{r=0}^{\infty} h(p,r) t^{r},$$

$$(1+t)(1-t)^{1-q} = \sum_{s=0}^{\infty} h(q,s) t^{s}.$$

Multiplying out these equalities and comparing the result with relation (3), we obtain

$$h(n,\ell) - h(n,\ell-2) = \sum_{r+s=\ell} h(p,r) h(q,s).$$

It follows from here that

$$h(n,\ell) = \sum_{r,s} h(p,r)h(q,s)$$

where the summation is the same as in (2). Thus,

$$\widetilde{\mathfrak{H}}^{n\ell} = \sum_{r,s} \oplus \mathfrak{H}^{pq\ell}_{rs}, \tag{4}$$

and the functions $\Xi^{pq\ell}_{MN}$ with fixed $p$, $q$, $\ell$ form an orthonormal basis of $\widetilde{\mathfrak{H}}^{n\ell}$. The left shift operators realize on $\widetilde{\mathfrak{H}}^{n\ell}$ the irreducible representation $T^{n\ell}$ of $SO(n)$. The restriction of $T^{n\ell}$ onto the subgroup $K_{pq}$ and onto the subspace $\mathfrak{H}^{pq\ell}_{rs}$ is equivalent to the representation $T^{pr} \times T^{qs}$ of $K_{pq}$. Hence, we have

$$T^{n\ell} \Big\downarrow \begin{matrix} SO(n) \\ K_{pq} \end{matrix} = \sum_{r,s} \oplus (T^{pr} \times T^{qs}), \tag{5}$$

where the summation is over the same values of $r$ and $s$ as in formula (2).

It follows from formulas (14) and (15′) of Section 9.1.5 that every element $g \in SO(n)$ is representable as

$$g = g^{(p)}(\boldsymbol{\varphi})g^{(q)}(\boldsymbol{\psi})g_{pn}(\theta)k, \tag{6}$$

where $k \in SO(n-1)$, $\boldsymbol{\varphi} \equiv (\varphi_1, \ldots, \varphi_p)$, $\boldsymbol{\psi} \equiv (\psi_1, \ldots, \psi_q)$ and where $g^{(p)}(\boldsymbol{\varphi})g^{(q)}(\boldsymbol{\psi})$ are of the form (7) of Section 9.1.5. If $\widetilde{\Xi}^{n\ell}_{O}(\boldsymbol{\xi})$ is the basis function (3) of Section 9.3.1, then for element (6) we have

$$\begin{aligned}(T^{n\ell}(g)\widetilde{\Xi}^{n\ell}_{O}, \widetilde{\Xi}^{pq\ell}_{MN}) &\equiv t^{np,\ell}_{MN,O}(\boldsymbol{\xi}) \\ &= t^{np,\ell}_{rs,0}(g_{pn}(\theta))t^{pr}_{M'O'}(g^{(p)}(\boldsymbol{\varphi}))t^{qs}_{N'O''}(g^{(q)}(\boldsymbol{\psi})),\ \boldsymbol{\xi} = g\mathbf{e}_n.\end{aligned} \tag{7}$$

Here $M = (r, M')$, $N = (s, N')$ and $t^{pr}_{M'O'}(g^{(p)}(\boldsymbol{\varphi}))$, $t^{qs}_{N'O''}(g^{(q)}(\boldsymbol{\psi}))$ are matrix elements of the form (6) of Section 9.4.2 and $t^{np,\ell}_{rs,0}(g_{pn}(\theta))$ does not depend on $M'$ and $N'$, since $g_{pn}(\theta)$ commutes with elements of the subgroups $SO(p-1)$ and $SO(q-1)$.

Since $t^{np,\ell}_{MN,O}(\boldsymbol{\xi}) \in \widetilde{\mathfrak{H}}^{n\ell}$, $t^{pr}_{M'O'}(\boldsymbol{\eta}) \in \widetilde{\mathfrak{H}}^{pr}$ and $t^{qs}_{N'O''}(\boldsymbol{\tau}) \in \widetilde{\mathfrak{H}}^{qs}$, then $t^{np,\ell}_{rs,0}(g_{pn}(\theta))$ is a multiple of the function $Y^{pq\ell}_{rs}(\theta)$ from Section 10.4.1:

$$t^{np,\ell}_{rs,0}(g_{pn}(\theta)) = c' \sin^r\theta \cos^s\theta P^{(\alpha,\beta)}_m(\cos 2\theta),\ m = \frac{\ell - r - s}{2}. \tag{8}$$

The constant $c'$ is defined by the condition

$$\int \left| t^{np,\ell}_{MN,O}(g) \right|^2 dg = (\dim T^{n\ell})^{-1}.$$

We have

$$c' = \left(\frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)m!\Gamma(m+\alpha+\beta+1)\ell!(p+r-3)!(q+s-3)!}{2\Gamma\left(\frac{n}{2}\right)\Gamma(m+\alpha+1)\Gamma(m+\beta+1)r!s!} \times \frac{(n-2)!(2r+p-2)(2s+q-2)}{(\ell+n-3)!(p-2)!(q-2)!}\right)^{1/2},$$

where $\alpha$ and $\beta$ are the same as above.

The functions $\Xi^{pq\ell}_{MN}(\boldsymbol{\xi})$ (and the functions $t^{np,\ell}_{MN,O}(\boldsymbol{\xi})$ which differ from $\Xi^{pq\ell}_{MN}(\boldsymbol{\xi})$ in a constant factor) are called *associated $K_{pq}$-spherical functions of the symmetric space $S^{n-1} = SO(n)/SO(n-1)$, corresponding to the representation $T^{n\ell}$ of the group $SO(n)$.*

**10.4.3. Associated $K_{pq}$-spherical functions on the hyperboloid $H^{pq}_{+}$.** We denote by $\mathfrak{H}^{pq\sigma}$ the space of homogeneous $\Box_{pq}$-harmonic functions of degree $\sigma$ on $\Omega^{pq} = \{\mathbf{x} \in E_{pq} \mid [\mathbf{x},\mathbf{x}]_{pq} > 0\}$, where $p+q=n$. The following theorem is proved in the same way as that from Section 10.4.1.

**Theorem.** *For every $\lambda \in \mathbb{C}$, $r,s \in \mathbb{N}_0$ there exists in the domain $0 < z < w$ a non-zero homogeneous function $Y(z,w)$ of degree $\lambda$ such that all functions of the form*

$$P(\mathbf{x}) = Y(|\mathbf{y}|^2, |\mathbf{t}|^2)A(\mathbf{y})B(\mathbf{t}), \tag{1}$$

*where $\mathbf{x} = (\mathbf{y},\mathbf{t})$, $\mathbf{y} \in \mathbb{R}^p$, $\mathbf{t} \in \mathbb{R}^q$, $|\mathbf{y}|^2 = (\mathbf{y},\mathbf{y})_p$, $|\mathbf{t}|^2 = (\mathbf{t},\mathbf{t})_q$, $A(\mathbf{y}) \in \mathfrak{H}^{pr}$, $B(\mathbf{t}) \in \mathfrak{H}^{qs}$, are $\Box_{pq}$-harmonic in the domain $\Omega_{pq}$. This function is defined uniquely up to a constant factor and has the form*

$$Y(z,w) = cw^{\lambda}F\left(-\lambda, -\lambda - s - \frac{q}{2} + 1; r + \frac{p}{2}; \frac{z}{w}\right), \tag{2}$$

*where $c \neq 0$.*

In the sequel we shall assume that $c = 1$. Let $A(\mathbf{y}) = \Xi^{pr}_{M'}(\mathbf{y})$, $B(\mathbf{t}) = \Xi^{qs}_{N'}(\mathbf{t})$, $\lambda = (\sigma - r - s)/2$. Then function (1) takes the form

$$\Xi^{pq\sigma}_{MN}(\mathbf{x}) = |\mathbf{t}|^{\sigma-r-s}F\left(\frac{1}{2}(r+s-\sigma), \frac{1}{2}(r-s-\sigma-q)+1; r+\frac{p}{2}; \frac{|\mathbf{y}|^2}{|\mathbf{t}|^2}\right) \times \Xi^{pr}_{M'}(\mathbf{y})\Xi^{qs}_{N'}(\mathbf{t}), \tag{3}$$

where $M = (r, M')$, $N = (s, N')$.

We restrict these functions onto the hyperboloid $H^{pq}_{+}$ and introduce on $H^{pq}_{+}$ the coordinates $\theta$, $\varphi$, $\psi$ (see Section 9.1.7). We obtain the system of functions

$$\Xi^{pq\sigma}_{MN}(\boldsymbol{\xi}) = Y^{pq\sigma}_{rs}(\theta)\Xi^{ps}_{M'}(\boldsymbol{\eta})\Xi^{qs}_{N'}(\boldsymbol{\tau}), \tag{4}$$

where

$$\xi = \eta\sinh\theta + \tau\cosh\theta,\ \eta \in S^{p-1},\ \tau \in S^{q-1},$$

$$Y_{rs}^{pq\sigma}(\theta) = \tag{5}$$

$$= \tanh^r\theta\cosh^\sigma\theta F\left(\frac{1}{2}(r+s-\sigma), \frac{1}{2}(r-s-\sigma-q)+1; r+\frac{p}{2}; \tanh^2\theta\right).$$

The functions $Y_{rs}^{pq\sigma}(\theta)$ are expressed in terms of Jacobi functions.

The system of homogeneous $\Box_{pq}$-harmonic functions in the domain $\Omega_-^{pq} = \{\mathbf{x} \in E_{pq} \mid [\mathbf{x},\mathbf{x}]_{pq} < 0\}$ is constructed analogously. They are defined by their values on the hyperboloid $H_-^{pq} = \{\xi \in E_{pq} \mid [\xi,\xi]_{pq} = -1\}$, which differs from $H_+^{qp}$ in enumeration of coordinates only. The functions $\Xi_{MN}^{pq\sigma}(\xi)$ will be called *associated $K_{pq}$-spherical functions* on $H_+^{pq}$, where, let us recall, $K_{pq} = SO(p) \times SO(q)$.

Let us give the special case of function (4), corresponding to the hyperboloid $H_-^{n-1}$ and to the values $p = 1$, $q = n-1$:

$$\Xi_{MN}^{1,n-1,\sigma}(\xi) = Y_{rs}^{1,n-1,\sigma}(\theta)\Xi^{1r}(\eta)\Xi_N^{n-1,s}(\tau), \tag{6}$$

here $M \equiv r \in \{0,1\}$, $\eta \in \{-1,1\}$, $\tau \in S^{n-2}$, $\Xi^{10}(\eta) = 1$, $\Xi^{11}(\eta) = \operatorname{sign}\eta$.

The formulas

$$\mathrm{P}_\nu^\mu(x) = \frac{1}{2}\left[e^{i\mu\pi/2}\mathfrak{P}_\nu^\mu(x+i0) + e^{-i\mu\pi/2}\mathfrak{P}_\nu^\mu(x-i0)\right], \tag{7}$$

$$\mathrm{Q}_\nu^\mu(x) = \frac{1}{2}e^{-i\mu\pi}\left[e^{-i\mu\pi/2}\mathfrak{Q}_\nu^\mu(x+i0) + e^{i\mu\pi/2}\mathfrak{Q}_\nu^\mu(x-i0)\right] \tag{7'}$$

define Legendre functions of the first and of the second kinds on the cut $-1 < x < 1$. The relations proved for Legendre funtions, in which $z-1$, $z^2-1$, $z+1$ are replaced, respectively, by $(1-x)e^{\pm i\pi}$, $(1-x^2)e^{\pm i\pi}$, $x+1$ $(-1 < x < 1)$ (in accordance with $z = x \pm i0$), are valid for functions (7) and (7′). One can show that

$$\frac{1}{2^\mu\sqrt{\pi}}(1-x^2)^{\mu/2}\mathrm{P}_\nu^\mu(x) = \frac{F\left(-\frac{\mu+\nu}{2}, \frac{1+\nu-\mu}{2}; \frac{1}{2}; x^2\right)}{\Gamma\left(\frac{1-\nu-\mu}{2}\right)\Gamma\left(\frac{1+\nu-\mu}{2}\right)} - \frac{2xF\left(\frac{1-\nu-\mu}{2}, \frac{2+\nu-\mu}{2}; \frac{3}{2}; x^2\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)\Gamma\left(-\frac{\nu+\mu}{2}\right)}, \tag{8}$$

$$\frac{(1-x)^{\mu/2}}{2^\mu\pi^{3/2}}\mathrm{Q}_\nu^\mu(x) = \cot\left[\frac{\pi}{2}(\nu+\mu)\right]\frac{xF\left(\frac{1-\nu-\mu}{2}, \frac{2+\nu-\mu}{2}; \frac{3}{2}; x^2\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)\Gamma\left(-\frac{\nu+\mu}{2}\right)} - \frac{1}{2}\tan\left[\frac{\pi}{2}(\nu+\mu)\right]\frac{F\left(-\frac{\mu+\nu}{2}, \frac{1+\nu-\mu}{2}; \frac{1}{2}; x^2\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)\Gamma\left(\frac{1-\nu-\mu}{2}\right)}. \tag{8'}$$

These equalities and formula (6) for $r = 0, 1$ imply that the functions

$$(\cosh\theta)^{-\sigma-(n-2)/2}\mathrm{P}^{\sigma+(n-2)/2}_{-s-(n-2)/2}(\tanh\theta) \tag{9}$$

and

$$(\cosh\theta)^{-\sigma-(n-2)/2}\mathrm{Q}^{\sigma+(n-2)/2}_{-s-(n-2)/2}(\tanh\theta), \tag{9'}$$

multiplied by $\Xi^{n-1,q}_{N'}(\tau)$, can be chosen as associated $K_{1,n-1}$-spherical functions on $H^{n-1}_{-}$. By virtue of the equality

$$\mathrm{P}^{\mu}_{\nu}(-x) = e^{-(\operatorname{sign} x)\nu\pi i}\mathrm{P}^{\mu}_{\nu}(x) - \frac{2}{\pi}\sin[\pi(\nu+\mu)]e^{-i\mu\pi i}Q^{\mu}_{\nu}(x)$$

one can take the function

$$(\cosh\theta)^{-\sigma-(n-2)/2}\mathrm{P}^{\sigma+(n-2)/2}_{-s-(n-2)/2}(-\tanh\theta) \tag{9''}$$

instead of (9′).

By means of the Whipple formulas (21) and (22) of Section 9.3.5, functions (9) and (9″) are linearly expressed in terms of the functions

$$(\cosh\theta)^{-\sigma-(n-3)/2}\mathfrak{Q}^{s+(n-3)/2}_{-\sigma-(n-1)/2}(i\sinh\theta) \tag{10}$$

and

$$(\cosh\theta)^{-\sigma-(n-3)/2}\mathfrak{Q}^{s+(n-3)/2}_{-\sigma-(n-1)/2}(-i\sinh\theta). \tag{10'}$$

Let us note that for $\sigma = i\rho - \frac{n-2}{2}$ the lower indices of these functions are equal to $-iu\rho - \frac{1}{2}$.

**10.4.4. Differential equations and integral representations for the functions $Y^{pq\ell}_{rs}(\theta)$ and $Y^{pq\sigma}_{rs}(\theta)$.** Since $\Xi^{pq\ell}_{MN}(\boldsymbol{\xi}) \in \tilde{\mathfrak{H}}^{n\ell}$, then

$$\Delta_0\Xi^{pq\ell}_{MN}(\boldsymbol{\xi}) = -\ell(\ell+n-2)\Xi^{pq\ell}_{MN}, \tag{1}$$

where $\Delta_0$ is the Laplace operator on $S^{n-1}$ (see Section 9.1.8). We write down $\Delta_0$ in bispherical coordinates (see formula (9) of Section 9.1.8) and take into account that

$$\Delta^{(p)}_0\tilde{\Xi}^{pr}_{M'}(\boldsymbol{\eta}) = -r(r+p-2)\tilde{\Xi}^{pr}_{M'}(\boldsymbol{\eta}), \quad \boldsymbol{\eta} \in S^{p-1},$$
$$\Delta^{(q)}_0\tilde{\Xi}^{qs}_{N'}(\boldsymbol{\tau}) = -s(s+q-2)\tilde{\Xi}^{qs}_{N'}(\boldsymbol{\tau}), \quad \boldsymbol{\tau} \in S^{q-1}.$$

As a result, we obtain

$$\left[\frac{1}{\sin^{p-1}\theta\cos^{q-1}\theta}\frac{d}{d\theta}\sin^{p-1}\theta\cos^{q-1}\theta\frac{d}{d\theta} - \frac{r(r+p-2)}{\sin^2\theta} - \frac{s(s+q-2)}{\cos^2\theta}\right]Y^{pq\ell}_{rs}(\theta) = -\ell(\ell+n-2)Y^{pq\ell}_{rs}(\theta). \tag{2}$$

The function $Y^{pq\ell}_{rs}(\theta)$ is the single (up to a constant factor) solution of this equation, regular at the point $\theta = 0$. The second linearly independent solution of (2) is

$$\widetilde{Y}^{pq\ell}_{rs}(\theta) = \cot^{p+r-2}\theta\cos^{\ell}\theta F\left(\frac{1}{2}(s-\ell-r-p)+1, 2-\frac{1}{2}(p+q+r+s+\ell);\right.$$
$$\left. -r-\frac{p}{2}+2; -\tan^2\theta\right). \tag{3}$$

Analogously, one proves that the function $Y^{pq\sigma}_{rs}(\theta)$ is the single (up to a constant factor) solution of the equation

$$\left[\frac{1}{\sinh^{p-1}\theta\cosh^{q-1}\theta}\frac{3}{d\theta}\sinh^{p-1}\theta\cosh^{q-1}\theta\frac{d}{d\theta}-\frac{r(r+p-2)}{\sinh^2\theta}\right.$$
$$\left.+\frac{s(s+q-2)}{\cosh^2\theta}\right]Y^{pq\sigma}_{rs}(\theta) = \sigma(\sigma+n-2)Y^{pq\sigma}_{rs}(\theta), \tag{4}$$

which is regular at $\theta = 0$. The second linearly independent solution of (4) is

$$\widetilde{Y}^{pq\sigma}_{rs}(\theta) = \tanh^{-p-r+2}\theta\cosh^{\sigma}\theta F\left(\tfrac{1}{2}(s-\sigma-r-p), 2-\tfrac{1}{2}(p+1+r+s+\sigma);\right.$$
$$\left.-r-\frac{p}{2}+2; \tanh^2\theta\right). \tag{5}$$

Let us derive an integral representation for $\Xi^{pq\ell}_{MN}(\mathbf{x})$. We denote by $\widetilde{C}^{n-1}$ the complex cone $(\boldsymbol{\xi},\boldsymbol{\xi}) = 0$ in $\mathbf{C}^n$, where $(\boldsymbol{\xi},\boldsymbol{\xi}) = \sum_{k=1}^{n}\xi_k^2$, and by $\Gamma$ the set of points from $\widetilde{C}^{n-1}$ of the form $\boldsymbol{\xi} = (\boldsymbol{\eta}, i\boldsymbol{\tau})$, $\boldsymbol{\eta}\in S^{p-1}$, $\boldsymbol{\tau}\in S^{q-1}$. It is evident that $\Gamma$ is homeomorphic to $S^{p-1}\times S^{q-1}$. The function

$$A(\mathbf{x}) = \int\limits_{S^{p-1}}\int\limits_{S^{q-1}}(\mathbf{x},\boldsymbol{\xi})^{\ell}\Xi^{pr}_{M'}(\boldsymbol{\eta})\Xi^{qs}_{N'}(\boldsymbol{\tau})d\boldsymbol{\tau}d\boldsymbol{\eta},\ \boldsymbol{\xi} = (\boldsymbol{\eta}, i\boldsymbol{\tau}), \tag{6}$$

is a homogeneous harmonic polynomial of degree $\ell$ in $x_1,\ldots,x_n$. Using twice the Funk-Hecke theorem and applying the equality $(\mathbf{x},\boldsymbol{\xi}) = (\mathbf{y},\boldsymbol{\eta}) + i(\mathbf{t},\boldsymbol{\tau})$, $\mathbf{y}\in\mathbf{R}^p$, $\mathbf{t}\in\mathbf{R}^q$, we find that $A(\mathbf{x})$ is of the form

$$A(\mathbf{x}) = \lambda(\theta)\Xi^{pr}_{M'}(\mathbf{y})\Xi^{qs}_{N'}(\mathbf{t}), \tag{7}$$

where

$$\lambda(\theta) = \frac{r!s!(p-3)!(q-3)!\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}{\pi(r+p-3)!(s+q-3)!\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{q-1}{2}\right)}\int\limits_{-1}^{1}\int\limits_{-1}^{1}(x\sin\theta+iy\cos\theta)^{\ell}$$
$$\times C^{(p-2)/2}_r(x)C^{(q-2)/2}_s(x)(1-x^2)^{(p-3)/2}(1-y^2)^{(q-3)/2}dxdy. \tag{8}$$

Applying twice formula (4′) of Section 9.3.6, we conclude that

$$\lambda(\theta) = N \sin^r\theta \cos^s\theta \int_{-1}^{1}\int_{-1}^{1} (x\sin\theta + iy\cos\theta)^{\ell-r-s} \times (1-x^2)^{r+(p-3)/2}(1-y^2)^{s+(q-3)/2}\,dx\,dy, \tag{9}$$

where

$$N = \frac{i^s \ell!\,\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}{\pi 2^{r+s}\Gamma\left(r+\frac{p-1}{2}\right)\Gamma\left(s+\frac{q-1}{2}\right)(\ell-r-s)!}.$$

We now note that $A(\mathbf{x})$ is a harmonic polynomial of degree $\ell$ in $x_1,\ldots,x_n$ of the form (4) of Section 10.4.1 and, hence, is a multiple of $\Xi^{pq\ell}_{MN}(\mathbf{x})$. Consequently, by virtue of formula (6) of Section 10.4.1 after cancelling $\sin^r\theta\cos^s\theta$, we obtain

$$P_m^{(\alpha,\beta)}(\cos 2\theta) = C\int_{-1}^{1}\int_{-1}^{1}(x\sin\theta + iy\cos\theta)^{2m}(1-x^2)^{\alpha-1/2}(1-y^2)^{\beta-1/2}\,dx\,dy, \tag{10}$$

where $2m = \ell - r - s$, $\alpha = r + (p-2)/2$, $\beta = s + (q-2)/2$. In order to find $C$ we set $\theta = 0$. Then

$$\frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)} = Ci^{\ell-r-s}\int_{-1}^{1}(1-x^2)^{r+\frac{p-2}{2}}dx\int_{-1}^{1}y^{\ell-r-s}(1-y^2)^{s+\frac{q-2}{2}}dy$$

Applying formula (1) of Section 3.4.6, we derive that

$$C = \frac{(2i)^{\ell-r-s}\Gamma\left(\frac{1}{2}(\ell-r+s+q)\right)\Gamma\left(\frac{1}{2}(\ell+r-s+p)\right)}{\pi\Gamma(\ell-r-s+1)\Gamma\left(r+\frac{p-1}{2}\right)\Gamma\left(s+\frac{q-1}{2}\right)}. \tag{11}$$

The formulas obtained lead to the relation

$$\Xi^{pq\ell}_{MN}(\mathbf{x}) = C^{pq\ell}_{rs}\int_{S^{p-1}}\int_{S^{q-1}}(\mathbf{x},\boldsymbol{\xi})^{\ell}\,\Xi^{pr}_{M'}(\boldsymbol{\eta})\Xi^{qs}_{N'}(\boldsymbol{\tau})\,d\boldsymbol{\eta}\,d\boldsymbol{\tau}, \tag{12}$$

where $M=(r,M')$, $N=(s,N')$, $\boldsymbol{\xi}=(\boldsymbol{\eta},i\boldsymbol{\tau})$, $\boldsymbol{\eta}\in S^{p-1}$, $\boldsymbol{\tau}\in S^{q-1}$,

$$C^{pq\ell}_{rs} = \frac{2^{\ell}i^{r-\ell}}{\ell!}\left[\frac{\Gamma\left(\frac{1}{2}(\ell+r+s+p+q-2)\right)\Gamma\left(\frac{1}{2}(\ell+r-s+p)\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \times \frac{\Gamma\left(\frac{1}{2}(\ell-r+s+q)\right)\left(\frac{\ell-r-s}{2}\right)!(\ell+p+q+1)}{\Gamma\left(\frac{p+q}{2}\right)}\right]^{\frac{1}{2}}.$$

In the same way one proves that

$$\Xi^{pq\sigma}_{MN}(\mathbf{x}) = C^{pq\sigma}_{rs} \int\limits_{S^{p-1}} \int\limits_{S^{q-1}} |(\mathbf{t},\boldsymbol{\tau}) - (\mathbf{y},\boldsymbol{\eta})|^{\sigma} \times \operatorname{sign}^{\varepsilon}((\mathbf{t},\boldsymbol{\tau}) - (\mathbf{y},\boldsymbol{\tau}))\Xi^{pr}_{M'}(\boldsymbol{\eta})\Xi^{qs}_{N'}(\boldsymbol{\tau})d\boldsymbol{\tau}d\boldsymbol{\eta}, \tag{13}$$

where $\mathbf{x} = (\mathbf{y},\mathbf{t})$, $[\mathbf{x},\mathbf{x}]_{pq} > 0$, $r+s \equiv \varepsilon \pmod 2$, $\varepsilon \in \{0,1\}$,

$$C^{pq\sigma}_{rs} = \frac{2^{\sigma}\Gamma\left(\frac{1}{2}(\sigma+r-s+p)\right)\Gamma\left(\frac{1}{2}(\sigma-r+s+q)\right)\Gamma\left(\frac{1}{2}(\sigma-r-s)\right)}{\Gamma(\sigma+1)\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}.$$

Since the system of functions $\Xi^{pr}_{M'}$ and the system of functions $\Xi^{qs}_{N'}$ are orthogonal on $S^{p-1}$ and $S^{q-1}$, respectively, we obtain that for $\mathbf{x} = (\mathbf{y},\mathbf{t})$, $\boldsymbol{\xi} = (\boldsymbol{\eta}, i\boldsymbol{\tau})$ and for $\boldsymbol{\xi} = (\boldsymbol{\eta},\boldsymbol{\tau})$, $\boldsymbol{\eta} \in S^{p-1}$, $\boldsymbol{\tau} \in S^{q-1}$, we have the expansions

$$(\mathbf{x},\boldsymbol{\xi})^{\ell} = \sum_{\substack{M,N \\ r+s\equiv \ell \pmod 2}} (C^{pq\ell}_{rs})^{-1}\Xi^{pq\ell}_{MN}(\mathbf{x})\Xi^{pr}_{M'}(\boldsymbol{\eta})\Xi^{qs}_{N'}(\boldsymbol{\tau}), \tag{14}$$

$$|[\mathbf{x},\boldsymbol{\xi}]_{pq}|^{\sigma}\operatorname{sign}^{\varepsilon}[\mathbf{x},\boldsymbol{\xi}]_{pq} = \sum_{\substack{M,N \\ r+s\equiv \varepsilon \pmod 2}} (C^{pq\sigma}_{rs})^{-1}\Xi^{pq\sigma}_{MN}(\mathbf{x})\Xi^{pr}_{M'}(\boldsymbol{\eta})\Xi^{qs}_{N'}(\boldsymbol{\tau}), \tag{15}$$

where $M = (r, M')$, $N = (s, N')$.

**10.4.5. Addition and product theorems.** The functions $\Xi^{pq\ell}_{MN}(\boldsymbol{\xi})$, as well as the functions $\Xi^{n\ell}_{K}(\boldsymbol{\xi})$, form an orthogonal basis in $\widetilde{\mathfrak{H}}^{n\ell}$. Since the matrix of the transition from one basis to another one is orthogonal, formula (3) of Section 10.2.1 implies the addition theorem for $\Xi^{pq\ell}_{MN}(\boldsymbol{\xi})$:

$$\sum_{M,N} \Xi^{pq\ell}_{MN}(\boldsymbol{\xi}_1)\overline{\Xi^{pq\ell}_{MN}(\boldsymbol{\xi}_2)} = \frac{2\ell+n-2}{n-2}C^{(n-2)/2}_{\ell}((\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)),\ \ \boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in S^{n-1}. \tag{1}$$

By setting $\boldsymbol{\xi}_1 = \sin\theta_1\cos\varphi_2\mathbf{e}_p + \cos\theta_1\cos\varphi_1\mathbf{e}_n$, $\boldsymbol{\xi}_2 = \sin\theta_2\mathbf{e}_p + \cos\theta_2\mathbf{e}_n$ and by writing down formula (1) in bispherical coordinates we obtain the addition theorem for Jacobi polynomials

$$C^{\frac{n-2}{2}}_{\ell}(\cos\theta_1\cos\theta_2\cos\varphi_1 + \sin\theta_1\sin\theta_2\cos\varphi_2) = \frac{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})(n-2)}{\Gamma(\frac{n}{2})(p-2)(q-2)} \times \sum_{r,s} A^{pq\ell}_{rs}(\sin\theta_1\sin\theta_2)^{\frac{p-2}{2}}(\cos\theta_1\cos\theta_2)^{\frac{q-2}{2}}P^{(\alpha,\beta)}_{m}(\cos 2\theta_1) \times P^{(\alpha,\beta)}_{m}(\cos 2\theta_2)C^{\frac{p-2}{2}}_{r}(\cos\varphi_2)C^{\frac{q-2}{2}}_{s}(\cos\varphi_1), \tag{2}$$

where $\alpha$, $\beta$, $m$ have the same sense as in Section 10.4.1 and

$$A_{rs}^{pq\ell} = \frac{(2r+p-2)(2s+q-2)\left(\frac{\ell-r-s}{2}\right)!\Gamma\left(\frac{1}{2}(\ell+r+s+n-2)\right)}{\Gamma\left(\frac{1}{2}(\ell+r-s+p)\right)\Gamma\left(\frac{1}{2}(\ell-r+s+q)\right)}. \tag{3}$$

Formula (2) and the orthogonality of Gegenbauer polynomials lead to the product formula

$$\int_0^\pi\int_0^\pi C_\ell^{\frac{n-2}{2}}(\cos\theta_1\cos\theta_2\cos\varphi_1+\sin\theta_1\sin\theta_2\cos\varphi_2)C_r^{\frac{p-2}{2}}(\cos\varphi_1)C_s^{\frac{q-2}{2}}(\cos\varphi_2)$$
$$\times\sin^{p-2}\varphi_1\sin^{q-2}\varphi_2 d\varphi_1 d\varphi_2 = M_{rs}^{pq\ell}(\sin\theta_1\sin\theta_2)^{\frac{p-2}{2}}(\cos\theta_1\cos\theta_2)^{\frac{q-2}{2}}$$
$$\times P_m^{(\alpha,\beta)}(\cos 2\theta_1)P_m^{(\alpha,\beta)}(\cos 2\theta_2), \tag{4}$$

where $2m=\ell-r-s$, $\alpha=r+(p-2)/2$, $\beta=s+(q-2)/2$,

$$M_{rs}^{pq\ell} = \frac{\pi^2(r+p-3)!(s+q-3)!m!\Gamma\left(\frac{1}{2}(\ell+r+s+n-2)\right)}{2^{n-5}r!s!\Gamma\left(\frac{p-2}{2}\right)\Gamma\left(\frac{q-2}{2}\right)\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{1}{2}(\ell+r-s+p)\right)\Gamma\left(\frac{1}{2}(\ell-r+s+q)\right)}. \tag{5}$$

Let us put $P(\boldsymbol{\eta})=\Xi_{MN}^{pq\ell}(\boldsymbol{\eta})$ into (2) of Section 10.2.4. We obtain that

$$\int_{S^{n-1}} e^{i(\mathbf{x},\boldsymbol{\eta})}\Xi_{MN}^{pq\ell}(\boldsymbol{\eta})d\boldsymbol{\eta} = \left(\frac{i}{2}\right)^\ell\Gamma\left(\frac{n}{2}\right)\left(\frac{r}{2}\right)^{-\ell-\frac{n-2}{2}}J_{\ell+\frac{n-2}{2}}(r)\Xi_{MN}^{pq\ell}(\mathbf{x}). \tag{6}$$

We set here $\mathbf{x}=(\mathbf{x}_1,\mathbf{x}_2)$, $\boldsymbol{\eta}=(\boldsymbol{\eta}_1\sin\theta,\boldsymbol{\eta}_2\cos\theta)$, where $\mathbf{x}_1\in\mathbb{R}^p$, $\mathbf{x}_2\in\mathbb{R}^q$, $\boldsymbol{\eta}_1\in S^{p-1}$, $\boldsymbol{\eta}_2\in S^{q-1}$, and we pass to bispherical coordinates. Taking into account formulas (5), (6) and (8) of Section 10.4.1, we obtain

$$R\int_0^{\pi/2} J_\alpha(R\sin\theta\sin\varphi)J_\beta(R\cos\theta\cos\varphi)P_m^{(\alpha,\beta)}(\cos 2\theta)$$
$$\times\sin^{\alpha+1}\theta\cos^{\beta+1}\theta d\theta$$
$$=(-1)^m J_{2m+\alpha+\beta+1}(R)\sin^\alpha\varphi\cos^\beta\varphi P_m^{(\alpha,\beta)}(\cos 2\varphi), \tag{7}$$

where $\alpha=r+\frac{p-2}{2}$, $\beta=s+\frac{q-2}{2}$. The Bateman formula

$$RJ_\alpha(R\sin\theta\sin\varphi)J_\beta(R\cos\theta\cos\varphi)=(\sin\theta\sin\varphi)^\alpha(\cos\theta\cos\varphi)^\beta$$
$$\sum_{m=0}^\infty(-1)^m\frac{2(\alpha+\beta+2m+1)\Gamma(\alpha+\beta+m+1)m!}{\Gamma(\alpha+m+1)\Gamma(\beta+m+1)}$$
$$\times J_{\alpha+\beta+2m+}(R)P_m^{(\alpha,\beta)}(\cos 2\theta)P_m^{(\alpha,\beta)}(\cos 2\varphi) \tag{8}$$

follows from here. It is valid for all $\alpha$ and $\beta$ except for their integral negative values.

If $m = 0$, then we obtain from (7) Sonin's formula

$$\int_0^{\pi/2} J_\alpha(R\sin\theta\sin\varphi)J_\beta(R\cos\theta\cos\varphi)\sin^{\alpha+1}\theta\cos^{\beta+1}\theta\,d\theta$$
$$= R\sin^\alpha\varphi\cos^\beta\varphi J_{\alpha+\beta+1}(R), \tag{9}$$

which is valid for all $\alpha$ and $\beta$ such that $\operatorname{Re}\alpha > -1$, $\operatorname{Re}\beta > -1$. Let us replace here $R\sin\varphi$ by $a$, $R\cos\varphi$ by $b$, divide both sides by $b^\beta$ and let $b$ tend to zero. We obtain Sonin's integral, which can be written as

$$\int_0^b x^{\nu+1}(b^2-x^2)^\mu J_\nu(ax)dx = 2^\mu\Gamma(\mu+1)a^{-\mu-1}b^{\mu+\nu+1}J_{\mu+\nu+1}(ab). \tag{9'}$$

Let us also give the formulas

$$\int_0^\infty J_\alpha(R\sinh\theta\sinh\varphi)J_\beta(R\cosh\theta\cosh\varphi)\sinh^{\alpha+1}\theta\cosh^{1-\beta}\theta d\theta$$
$$= R^{-1}\sinh^\alpha\varphi\cosh^{-\beta}\varphi J_{\beta-\alpha-1}(R),\ \operatorname{Re}\beta > \operatorname{Re}\alpha > -1, \tag{10}$$

$$\int_0^\infty J_\alpha(R\sinh\theta\sin\varphi)K_\beta(R\cosh\theta\cos\varphi)\sinh^{\alpha+1}\theta\cosh^{1-\beta}\theta d\theta$$
$$= R^{-1}\sin^\alpha\varphi\cos^{-\beta}\varphi K_{\beta-\alpha-1}(R). \tag{11}$$

Applying limit procedure to these formulas, we obtain the formula

$$\int_0^\infty J_\beta(R\cosh\theta\cosh\varphi)\sinh^{2\alpha-1}\theta\cosh^{1-\beta}\theta d\theta$$
$$= 2^\alpha\Gamma(\alpha+1)(R\cosh\varphi)^{-\alpha-1}J_{\beta-\alpha-1}(R\cosh\varphi), \tag{12}$$

where $\operatorname{Re}\left(\frac{\beta}{2}-\frac{1}{4}\right) > \operatorname{Re}\alpha > -1$, and the formula

$$\int_0^\infty K_\beta(R\cosh\theta\cosh\varphi)\sinh^{2\alpha+1}\theta\cosh^{1-\beta}\theta d\theta$$
$$= 2^\alpha\Gamma(\alpha+1)(R\cosh\varphi)^{-\alpha-1}K_{\beta-\alpha-1}(R\cosh\varphi), \tag{13}$$

where $\operatorname{Re}\beta > -1$.

**10.4.6. The Poisson transform of the basis of $\mathfrak{B}^{n\sigma}$, corresponding to the cylindrical section of the cone.** The section $\Gamma_3$ of the cone $C_+^{n-1}$ by the cylinder $x_1^2 + \ldots + x_p^2 = 1$ is the product $S^{p-1} \times H_+^{q-1}$, $p+q=n$, of the sphere and the hyperboloid (see Section 10.1.5). Every function $f \in \mathfrak{B}^{n\sigma}$ is uniquely defined by the function $F(\boldsymbol{\eta}, \boldsymbol{\zeta})$, that is, by its restriction onto $\Gamma_3$, where

$$\boldsymbol{\eta} = \left(\frac{\xi_1}{r}, \ldots, \frac{\xi_p}{r}\right), \quad \boldsymbol{\zeta} = \left(\frac{\xi_{p+1}}{r}, \ldots, \frac{\xi_n}{r}\right),$$
$$\boldsymbol{\xi} \in C_+^{n-1}, \quad r^2 = \xi_1^2 + \ldots + \xi_p^2 = -\xi_{p+1}^2 - \ldots - \xi_{n-1}^2 + \xi_n^2.$$

The functions

$$\Xi_{M'}^{pm}(\boldsymbol{\eta})\Xi_{N'}^{q\tau}(\boldsymbol{\zeta}), \quad \tau = i\nu - \frac{q-2}{2}, \tag{1}$$

form a basis in the space $\mathfrak{L}^2(\Gamma_3)$. The corresponding basis in $\mathfrak{B}^{n\sigma}$ consists of the functions

$$\widehat{\Xi}_{MN}^{pq,\sigma,3}(\boldsymbol{\xi}) = r^\sigma \Xi_{M'}^{pm}(\boldsymbol{\eta})\Xi_{N'}^{q\tau}(\boldsymbol{\zeta}), \tag{2}$$

where $M = (m, M')$, $N = (\tau, N')$. It corresponds to the subgroup chain

$$SO_0(n-1,1) \supset SO(p) \times SO_0(q-1,1) \supset SO(p-1) \times SO(q-1) \supset \ldots .$$

Let us find the Poisson transform of this basis of $\mathfrak{B}^{n,-\sigma-n+2}$, that is, let us evaluate the integral

$$\Xi_{MN}^{pq,\sigma}(\mathbf{x}) = \int_{\Gamma_3} [\mathbf{x}, \boldsymbol{\xi}]^\sigma \widehat{\Xi}_{MN}^{pq,-\sigma-n+2,3}(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{3}$$

Let

$$\boldsymbol{\xi} = (\boldsymbol{\eta}, \boldsymbol{\zeta}), \quad \mathbf{x} = (\sinh\theta\, \mathbf{x}', \cosh\theta\, \mathbf{x}''),$$

where $(\mathbf{x}', \mathbf{x}')_p = 1$, $[\mathbf{x}'', \mathbf{x}'']_q = 1$. Then

$$[\mathbf{x}, \boldsymbol{\xi}] = (\mathbf{x}', \boldsymbol{\eta})\sinh\theta + [\mathbf{x}'', \boldsymbol{\zeta}]\cosh\theta.$$

By applying the Funk-Hecke theorem from Section 10.2.1 and its generalization for the group $SO_0(q-1,1)$ from Section 10.2.2, we obtain that

$$\Xi_{MN}^{pq,\sigma}(\mathbf{x}) = \lambda_{pq}(\theta)\Xi_{M'}^{pm}(\mathbf{x}')\Xi_{N'}^{q\tau}(\mathbf{x}''), \tag{4}$$

where

$$\lambda_{pq}(\theta) = \int_0^\pi \int_0^\infty (\cosh\theta\cosh\psi - \sinh\theta\cos\varphi)^\sigma C_m^{\frac{p-2}{2}}(\cos\varphi)$$
$$\times \mathfrak{P}^{-\frac{q-3}{2}}_{\tau+\frac{q-3}{2}}(\cosh\psi)\sin^{p-2}\varphi\sinh^{\frac{q-1}{2}}\psi\, d\psi\, d\varphi. \tag{4'}$$

Integral (4′) is evaluated in the same way as integral (8) of Section 10.4.4. Omitting this evaluation, we give the result:

$$\lambda_{pq}(\theta) = C_{m\nu}^{pq,\sigma} \tanh^m \theta \cosh^{-\sigma-n+2} \theta \times F\left(\frac{1}{2}(m+\tau+\sigma+n-2), \frac{1}{2}(m-\tau+\sigma+p); m+\frac{p}{2}; \tanh^2\theta\right), \quad (5)$$

where

$$C_{m\nu}^{pq,\sigma} = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}(m+\tau-\sigma-q)\right)\Gamma\left(\frac{1}{2}(m+\tau-\sigma)\right)}{\sqrt{\pi}\, 2^{\sigma+1}\Gamma(-\sigma)\Gamma\left(m+\frac{p}{2}\right)}. \quad (5')$$

Thus,

$$\Xi_{MN}^{pq,\sigma}(\mathbf{x}) = C_{m\nu}^{pq,\sigma} \tanh^m \theta \cosh^{-\sigma-n+2}\theta \times F\left(\frac{1}{2}(m+\tau+\sigma+n-2), \frac{1}{2}(m-\tau+\sigma+p); m+\frac{p}{2}; \tanh^2\theta\right) \times \Xi_{M'}^{pm}(\mathbf{x}')\Xi_{N'}^{q\tau}(\mathbf{x}''). \quad (6)$$

**10.4.7. Integral transforms on $H_+^{n-1}$ related to the subgroup $SO(p)\times SO_0(q-1,1)$.** As in Section 10.3.2, we find that the mutually reciprocal integral transforms

$$f(\mathbf{x}) = \frac{1}{4}\tilde{c}_n\tilde{c}_q \sum_{m,M',N'} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_{mM'N'}(\rho,\nu) \times \Xi_{MN}^{pq,\sigma}(\mathbf{x})d\mu_q(\nu)d\mu_n(\rho), \quad (1)$$

$$a_{mM'N'}(\rho,\nu) = \int_{H_+^{n-1}} f(\mathbf{x})\overline{\Xi_{MN}^{pq,\sigma}(\mathbf{x})}d\mathbf{x}, \quad (2)$$

where $\sigma = i\rho - \frac{n-2}{2}$ and $\mathbf{x}\in H_+^{n-1}$, are connected with functions of Section 10.4.6. The Plancherel formula

$$\int_{H_+^{n-1}} |f(\mathbf{x})|^2 d\mathbf{x} = \frac{1}{4}\tilde{c}_n\tilde{c}_q \sum_{m,M',N'} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |a_{mM'N'}(\rho,\nu)|^2 d\mu_q d\mu_n(\rho) \quad (3)$$

holds. (Note that $\tilde{c}_r$ and $d\mu_r(\alpha)$ are defined in Section 10.1.4.)

Representing $f(\mathbf{x})$ in the form

$$f(\mathbf{x}) = \Phi(\theta)F_1(\mathbf{x}')F_2(\mathbf{x}''),$$

where the variables $\theta$, $\mathbf{x}'$, $\mathbf{x}''$ are the same as in formula (4) of Section 10.4.6, after simplifications we obtain mutually reciprocal transforms of functions of one variable which by means of the function

$$\mathfrak{P}_\tau^{\lambda-\mu,\lambda+\mu}(z) \equiv 2^\mu \mathfrak{P}_{\lambda\mu}^\tau(z) = \frac{1}{\Gamma(\mu-\lambda+1)}(z-1)^{(\mu-\lambda)/2} \times (z+1)^{(\mu+\lambda)/2} F\left(\tau+\mu+1, -\tau+\mu; \mu-\lambda+1; \frac{1-z}{2}\right) \quad (4)$$

(see formula (1) of Section 7.4.1) are written as

$$F(\rho) = \int_1^\infty f(t)\mathfrak{P}_{i\rho-1/2}^{\alpha\beta}(t)dt, \quad (5)$$

$$f(t) = \frac{2^{\alpha-\beta-1}}{\pi}\int_0^\infty F(\rho)\mathfrak{P}_{-i\rho-1/2}^{\alpha\beta}(t)\left|\frac{\Gamma\left(\frac{\beta-\alpha+1}{2}+i\rho\right)\Gamma\left(\frac{1-\alpha-\beta}{2}+i\rho\right)}{\Gamma(2i\rho)}\right|^2 d\rho, \quad (6)$$

where $\alpha$ is a negative half-integer, $\beta \in \mathbb{C}$, and

$$|\mathrm{Re}\,\beta| < 1-\alpha. \quad (7)$$

One can show that these formulas are valid for complex $\alpha$ such that $|\mathrm{Re}\,\beta| < 1 - \mathrm{Re}\,\alpha$. In this case we obtain, in fact, the Jacobi transform from Section 7.8.7.

The function $\mathfrak{P}_\tau^{\alpha\beta}(z)$ is expressed in terms of $\mathfrak{Q}_\tau^{\alpha\beta}(z)$, where

$$\mathfrak{Q}_\tau^{\lambda-\mu,\lambda+\mu}(z) \equiv 2^\mu \mathfrak{Q}_{\lambda\mu}^\tau(z) = e^{i\pi(\lambda-\mu)}\frac{\Gamma(\tau+\lambda+1)\Gamma(\tau-\mu+1)}{2^{-\tau-\mu}\Gamma(2\tau+1)} \times (z-1)^{-\tau-1}\left(\frac{z+1}{z-1}\right)^{\frac{\lambda+\mu}{2}} F\left(\tau+\mu+1, \tau+\lambda+1; 2\tau+2; \frac{2}{1-z}\right) \quad (8)$$

(see formula (6) of Section 7.4.1). We have

$$\frac{\sin 2\tau\pi}{2^{\beta-\alpha+1}\pi}\times \Gamma\left(\tau+\frac{\alpha+\beta}{2}+1\right)\Gamma\left(\tau-\frac{\alpha+\beta}{2}+1\right)\Gamma\left(-\tau-\frac{\alpha+\beta}{2}\right)\Gamma\left(-\tau-\frac{\alpha-\beta}{2}\right)\mathfrak{P}_\tau^{\alpha\beta}(z) = e^{\pi i\alpha}\left[\mathfrak{Q}_\tau^{-\alpha,-\beta}(z) - \mathfrak{Q}_{-\tau-1}^{-\alpha,-\beta}(z)\right]. \quad (9)$$

Substituting this expression for $\mathfrak{P}_\tau^{\alpha,\beta}(z)$ into (6), after simple manipulations we obtain the mutually reciprocal transforms

$$\Phi(\rho) = \int_1^\infty f(t)\mathfrak{P}_{i\rho-1/2}^{\alpha\beta}(t)dt, \quad (10)$$

$$f(t) = \frac{1}{\pi}\int_{-\infty}^\infty \Phi(\rho)e^{\pi i\alpha}\mathfrak{Q}_{i\rho-1/2}^{-\alpha,-\beta}(t)\rho d\rho. \quad (11)$$

Other similar integral transforms are obtained in [48]. If $a > \frac{1}{2}\mathrm{Re}\,\alpha + \frac{1}{2}|\mathrm{Re}\,\beta| - 1$, then the following transforms

$$F(t) = \int_1^\infty f(t)e^{\pi i\alpha}\mathfrak{Q}_\tau^{-\alpha,-\beta}(t)dt, \tag{12}$$

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(\tau)\mathfrak{P}_\tau^{\alpha\beta}(t)(2\tau+1)d\tau \tag{13}$$

are mutually reciprocal. By means of symmetry relations for $\mathfrak{P}_\tau^{\alpha\beta}(z)$ and $\mathfrak{Q}_\tau^{\alpha\beta}(z)$ one can obtain other integral transforms from the above transforms. These symmetry relations are of the form

$$\mathfrak{P}_\tau^{\alpha\beta}(z) = 2^{\tau+1+\beta/2}(1+z)^{-1/2}\mathfrak{P}_{-(\beta+1)/2}^{\alpha,-2\tau-1}\left(\frac{z-3}{-z-1}\right), \tag{14}$$

$$\begin{aligned}-e^{-\pi i\alpha}\mathfrak{Q}_\tau^{\alpha\beta}(z) = {} & 2^{-\tau-1+\beta/2}\Gamma\left(\tau+\frac{\alpha+\beta}{2}+1\right)\Gamma\left(\tau+\frac{\alpha-\beta}{2}+1\right)\\ & \times(1+z)^{-1/2}\mathfrak{P}_{(\beta-1)/2}^{-2\tau-1,-\alpha}\left(\frac{z-3}{z+1}\right),\end{aligned} \tag{15}$$

$$\mathfrak{P}_\tau^{\alpha\beta}(z) = \frac{2^{\tau+2-\alpha/2}(z-1)^{-1/2}e^{-(2\tau+1)\pi i}}{\Gamma\left(\tau-\frac{\alpha-\beta}{2}+1\right)\Gamma\left(\tau-\frac{\alpha+\beta}{2}+1\right)}\mathfrak{Q}_{-(\alpha+1)/2}^{2\tau+1,\beta}\left(\frac{z+3}{z-1}\right), \tag{16}$$

$$\begin{aligned}e^{-\pi i\alpha}\mathfrak{Q}_\tau^{\alpha\beta}(z) = {} & 2^{-\tau-1-\alpha/2}\Gamma\left(\tau+\frac{\alpha+\beta}{2}+1\right)\Gamma\left(\tau+\frac{\alpha-\beta}{2}+1\right)\\ & \times(z-1)^{-1/2}\mathfrak{P}_{(\alpha-1)/2}^{-2\tau-1,\beta}\left(\frac{z+3}{z-1}\right)\end{aligned} \tag{17}$$

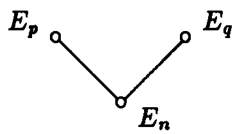
## 10.5. The Tree Method

**10.5.1. The tree method and polyspherical coordinates.** We have constructed the orthonormal bases $\{\Xi_M^{n\ell}\}$ and $\{\Xi_{MN}^{pq\ell}\}$ of the space $\mathfrak{L}^2(S^{n-1})$ (see Section 9.3.1 and Section 10.4.1), corresponding to the spherical and to the bispherical coordinate systems on $S^{n-1}$. In order to construct bispherical coordinates we split $E_n$ into the direct sum $E_n = E_p + E_q$ and represent vectors $\boldsymbol{\xi} \in S^{n-1}$ in the form

$$\boldsymbol{\xi} = \boldsymbol{\eta}\sin\theta_1 + \boldsymbol{\zeta}\cos\theta_1, \quad \boldsymbol{\eta} \in S^{p-1}, \quad \boldsymbol{\zeta} \in S^{q-1} \tag{1}$$

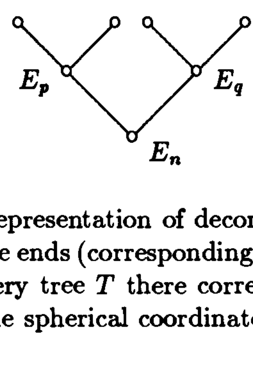
(see Section 9.1.4). One can continue splitting the subspaces $E_p$ and $E_q$ by introducing at every step a new parameter, similar to the parameter $\theta_1$ in decomposition (1). After decomposing the space $E_n$ into the direct sum of one-dimensional subspaces, we obtain the collection of parameters defining polyspherical coordinates on $S^{n-1}$.

To describe decomposing the space $E_n$ into the direct sum of one-dimensional subspaces, we make use of graphs of a special kind, which are called trees. With the decomposition $E_n = E_p + E_q$ we associate the graph



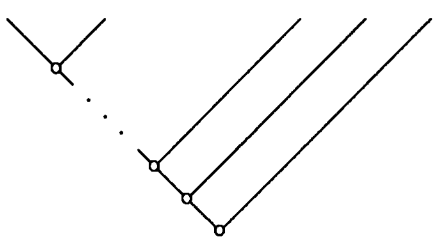

(2)

Here the edge $E_nE_p$ is turned to the left and $E_nE_q$ is turned to the right. This means that the factor $\sin\theta_1$ in (1) corresponds to the vector $\eta \in S^{p-1}$ and the factor $\cos\theta_1$ corresponds to $\zeta \in S^{q-1}$. The similar graphs correspond to the decompositions of the spaces $E_p$ and $E_q$, and we obtain the graph



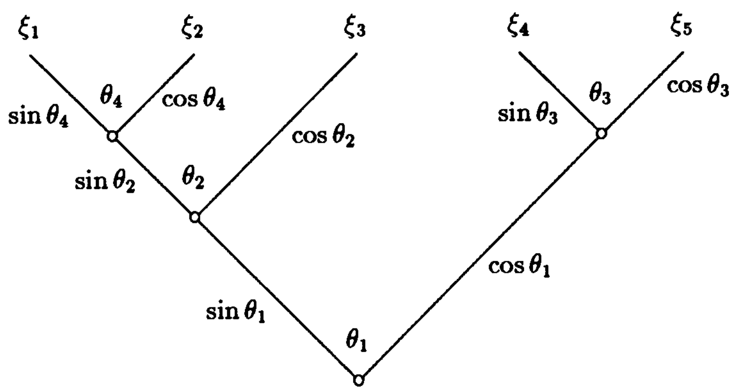

(3)

Continuing graphical representation of decompositions of subspaces, we receive finally a graph with $n$ free ends (corresponding to one-dimensional subspaces), which is called a *tree*. To every tree $T$ there corresponds a polyspherical coordinate system. For example, the spherical coordinate system (2) of Section 9.1.5 is described by the tree



A tree $T$ gives a simple rule for writing the Cartesian coordinates $\xi_1, \ldots, \xi_n$ of the point $\xi \in S^{n-1}$ in terms of the corresponding polyspherical coordinates

$\theta_1, \ldots, \theta_{n-1}$. With every fork of the tree $T$ we associate an angle, and with the edge turned to the left (to the right), we associate sine (cosine) of this angle. To the fork (2) there corresponds the angle $\theta_1$ from formula (1), to the next forks (see graph (3)) there correspond the angles $\theta_2$ and $\theta_3$, respectively, and so on. With free (upper) ends of the tree we associate the corresponding Cartesian coordinates $\xi_1, \ldots, \xi_n$ of the point $\boldsymbol{\xi}$ (instead of one-dimensional subspaces). The tree




(4)

can be considered as an example of a tree for $S^4$.

In order to express the Cartesian coordinates $\xi_i$ in terms of $\theta_1, \theta_2, \ldots, \theta_{n-1}$, we go along the path from the lowest node to the coordinate $\xi_i$. The coordinate $\xi_i$ is equal to the product of trigonometrical functions corresponding to the edges which constitute the path. For example, for (4) we have

$$\xi_1 = \sin\theta_1 \sin\theta_2 \sin\theta_4, \quad \xi_2 = \sin\theta_1 \sin\theta_2 \cos\theta_4,$$
$$\xi_3 = \sin\theta_1 \cos\theta_2, \quad \xi_4 = \cos\theta_1 \sin\theta_3, \quad \xi_5 = \cos\theta_1 \cos\theta_3.$$

In order to find the bounds where the parameters $\theta_1, \theta_2, \ldots, \theta_{n-1}$ vary, we divide graphs with one node and two edges into four types – graphs with two free ends, graphs with one free end and graphs without free ends. To every one of these graphs there corresponds an angle from the parametrization. Depending on the type of a graph, the angle varies in the following bounds:


$0 \le \theta < 2\pi$ $\quad 0 \le \theta_i \le \pi$ $\quad 0 \le \theta_i \le \pi$ $\quad 0 \le \theta_i \le \pi/2$

In particular, for tree (4) we have $0 \le \theta_1 \le \pi/2$, $0 \le \theta_2 \le \pi$, $0 \le \theta_3, \theta_4 < 2\pi$.

To every polyspherical coordinate system on $S^{n-1}$ there corresponds a chain of subgroups in $SO(n)$ (which is the motion group of $S^{n-1}$). To the separated of the angle $\theta_1$ in formula (1) there corresponds the subgroup

$$K_{pq} = SO(p) \times SO(q) \subset SO(n), \quad p+q=n,$$

where $SO(p)$ and $SO(q)$ are the motion groups on the separation spheres $S^{p-1}$ and $S^{q-1}$. In the same way one separates subgroups for further decompositions of the space $E_n$. To the tree (4) there corresponds the chain

$$SO(5) \supset SO(3) \times SO(2) \supset SO(2) \times SO(2) \supset SO(2).$$

As in Section 9.1.5, to every polyspherical coordinate sysem (that is, to every tree) there corresponds a system of representatives of the left cosets from $SO(n)/SO(n-1)$. For example, to the tree (4) there corresponds the representatives

$$g_{12}(\theta_4)g_{23}(\theta_2)g_{45}(\theta_3)g_{34}(\theta_1).$$

Applying them to the basis vector $\mathbf{e}_4 = (0,0,0,1,0)$, we obtain almost all points of the sphere $S^4$.

**10.5.2. The Laplace operator and the invariant measure on $S^{n-1}$.** To every polyspherical coordinate system there corresponds an expression for the Laplace operator $\Delta_0$ on $S^{n-1}$, which is defined by the corresponding tree $T$. To the decomposition $E_n = E_p + E_q$ of the space $E_n$ there corresponds expression (9) of Section 9.1.8 for $\Delta_0$. With further decompositions of $E_n$ we associate the corresponding representations for $\Delta_0^{(p-1)}$ and $\Delta_0^{(q-1)}$. The final expression for $\Delta_0$ is of the form

$$\Delta_0 = \frac{1}{A}\sum_{i=1}^{n-1} \frac{\partial}{\partial\theta_i}\frac{A}{A_i^2}\frac{\partial}{\partial\theta_i}, \quad A = \prod_{i=1}^{n-1} A_i, \; A_1 = 1, \tag{1}$$

where the functions $A_i$ correspond to the angles $\theta_i$.

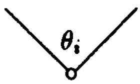
One can define the functions $A_i$ by the tree $T$ as follows. We separate the path along the tree $T$ from the lowest node to the node with angle $\theta_i$. This path is unique. The function $A_i$ is equal to the product of trigonometrical functions corresponding to the edges which constitute the path. For example, for tree (4) of Section 10.5.1 we have

$$A_2 = \sin\theta_1, \quad A_3 = \cos\theta_1, \quad A_4 = \sin\theta_1 \sin\theta_2.$$

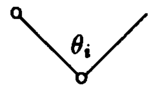
The invariant measure $d\boldsymbol{\xi}$ on $S^{n-1}$ in the polyspherical coordinates $\theta_1, \dots, \theta_{n-1}$ is found by means of formula (4) of Section 9.1.9. We have

$$d\boldsymbol{\xi} = \prod_{i=1}^{n-1} B_i(\theta_i)d\theta_i. \tag{2}$$

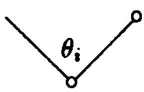
The factors $B_i(\theta_i)d\theta_i$ are defined by the tree $T$. Namely, depending on the fork corresponding to $\theta_i$, for $B_i(\theta_i)d\theta_i$ we have



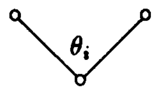

$\frac{1}{2\pi}d\theta_i$




$c_{n_1}\sin^{n_1-1}\theta_i d\theta_i$ (3)




$c_{n_2}\cos^{n_2-1}\theta_i d\theta_i$




$c_{n_1 n_2}\sin^{n_1-1}\theta_i\cos^{n_2-1}\theta_i d\theta_i$ (4)

Here $n_1$ denotes the number of coordinates connected with the upper left node, $n_2$ denotes the number of coordinates connected with the upper right node, and

$$c_{n_1}=\frac{\Gamma\left(\frac{n_1+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n_1}{2}\right)},\quad c_{n_2}=\frac{\Gamma\left(\frac{n_2+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n_2}{2}\right)},\quad c_{n_1n_2}=\frac{2\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}. \tag{5}$$

We have chosen the constants $c_{n_1}$, $c_{n_2}$ and $c_{n_1n_2}$ in such a way that the measure of the whole sphere is equal to 1. In particular, for this choice of the constants we have

$$d\xi = B_1(\theta_1)d\theta_1 d\eta d\zeta$$

if $\xi$ is related with $\eta$ and $\zeta$ by formula (1) of Section 10.5.1.

**10.5.3. Trees and orthonormal bases in $\mathfrak{L}^2(S^{n-1})$.** An orthonormal basis of $\mathfrak{L}^2(S^{n-1})$ corresponds to every tree $T$. It is constructed in the same way as in Section 10.4.3.

To every basis element there corresponds the tree $T$ with labels. Labels are non-negative integers associated with nodes of the tree. They are obtained in the following way. With the lowest node of the tree we associate the index $\ell$ of the irreducible representation $T^{n\ell}$ of the group $SO(n)$. In accordance with the lowest part (2) of Section 10.5.1 for the tree $T$, we restrict $T^{n\ell}$ onto the subgroup $SO(p)\times SO(q)$, $p+q=n$. This restriction decomposes into irreducible components by formula (4′) of Section 10.4.3:

$$T^{n\ell}\downarrow^{SO(n)}_{K_{pq}} = \sum_{r,s}\oplus(T^{pr}\otimes T^{qs}). \tag{1}$$

Therefore, with the upper sides of graph (2) of Section 10.5.1 we associate the indices $r$ and $s$ of the irreducible representation $T^{pr}\otimes T^{qs}$ of the subgroup $SO(p)\times SO(q)$

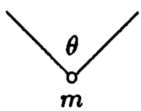
from this formula. Further, we restrict $T^{pr}$ and $T^{qs}$ onto the next subgroups and associate indices with the corresponding nodes of $T$. After a finite number of steps we obtain the tree with labels.

The number of possible trees with labels having index $\ell$ at the lowest node is equal to dim $T^{n\ell}$. To the tree $T$ with different labels $M$ there correspond different basis functions of the space $\mathfrak{L}^2(S^{n-1})$.

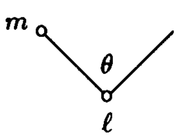
One constructs basis functions by means of formula (6) of Section 10.4.1. The basis function $\Xi_M^T$ corresponding to the tree with labels $T_M$ is represented as the product of functions of one variable:

$$\Xi_M^T(\theta_1,\ldots,\theta_{n-1}) = \prod_{i=1}^{n-1} F_i(\theta_i). \tag{2}$$

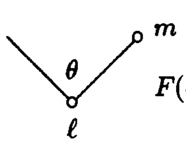
The function $F_i(\theta_i)$ is defined by the fork of $T_M$ corresponding to the angle $\theta_i$. This correspondence is as follows:



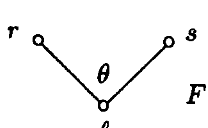

$$F(\theta) \equiv Y_m(\theta) = \frac{1}{\sqrt{2\pi}} e^{im\theta}, \tag{3}$$




$$F(\theta) \equiv Y_\ell^m(\theta) = N_{\ell m} \sin^m \theta C_{\ell-m}^{m+(n_1-1)/2}(\cos\theta), \tag{4}$$




$$F(\theta) \equiv \tilde{Y}_\ell^m(\theta) = N_{\ell m} \cos^m \theta C_{\ell-m}^{m+(n_2-1)/2}(\sin\theta), \tag{5}$$




$$F(\theta) \equiv Y_\ell^{rs}(\theta) = N_{\ell rs} \sin^r \theta \cos^s \theta P_{(\ell-r-s)/2}^{\left(r+\frac{n_1-2}{2},\, s+\frac{n_2-2}{2}\right)}(\cos 2\theta), \tag{6}$$

where $n_1$ and $n_2$ are the same as in formulas (3) and (4) of Section 1.5.2 and

$$N_{\ell m} = \frac{2^m \Gamma\left(m+\frac{n_1-1}{2}\right)}{\Gamma\left(\frac{n_1-1}{2}\right)} \left[\frac{(\ell-m)!(n_1-2)!(2\ell+n_1-1)}{(\ell+m+n_1-2)!(n_1-1)}\right]^{1/2} \tag{7}$$

(one has to replace $n_1$ by $n_2$ for formula (5)),

$$N_{\ell rs} = \left[\frac{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})\Gamma\left(\frac{1}{2}(\ell+r+s+n_1+n_2-2)\right)\Gamma\left(\frac{1}{2}(\ell-r-s+2)\right)(2\ell+n_1+n_2-2)}{2\Gamma\left(\frac{n_1+n_2}{2}\right)\Gamma\left(\frac{1}{2}(\ell+r-s+n_1)\right)\Gamma\left(\frac{1}{2}(\ell-r+s+n_2)\right)}\right]^{1/2}. \quad (8)$$

In the sequel we shall use the notations

$$j_1 = \frac{r}{2}+\frac{n_1}{4}-1, \quad j_2 = \frac{s}{2}+\frac{n_2}{4}-1, \quad j = \frac{\ell}{2}+\frac{n_1+n_2}{4}-1.$$

In these notations function (6) takes the form

$$Y_\ell^{rs}(\theta) = N_{\ell rs}\sin^r\theta\cos^s\theta P^{(2j_1+1,2j_2+1)}_{j-j_1-j_2+1}(\cos 2\theta). \quad (9)$$

Expressions (3)–(5) are reduced to the form (6). By means of formulas (17) and (18) of Section 6.3.9, function (4) is expressed in terms of the Jacobi polynomial of $\cos 2\theta$. We have

$$Y_\ell^m(\theta) = \frac{1}{\sqrt{2}}N_{\ell m0}\sin^m\theta P^{(m+\frac{n_1-2}{2},-\frac{1}{2})}_{(\ell-m)/2}(\cos 2\theta) = \frac{1}{\sqrt{2}}Y_\ell^{m0}(\theta) \quad (10)$$

if $\ell-m$ is even, and

$$Y_\ell^m(\theta) = \frac{1}{\sqrt{2}}N_{\ell m1}\sin^m\theta\cos\theta P^{(m+\frac{n_1-2}{2},\frac{1}{2})}_{(\ell-m-1)/2}(\cos 2\theta) = \frac{1}{\sqrt{2}}Y_\ell^{m1}(\theta) \quad (11)$$

if $\ell-m$ is odd. In the same way we obtain for (5) that

$$\widetilde{Y}_\ell^m(\theta) = \frac{(-1)^{(\ell-m)/2}}{\sqrt{2}}Y_\ell^{m0}(\theta) \quad (12)$$

if $\ell-m$ is even, and

$$\widetilde{Y}_\ell^m(\theta) = \frac{(-1)^{(\ell-m-1)/2}}{\sqrt{2}}Y_\ell^{m1}(\theta) \quad (13)$$

if $\ell-m$ is odd. The functions $Y_\ell^m$ and $\widetilde{Y}_\ell^m$ can be represented in the form (9). For $Y_\ell^m$ we set

$$r=m,\ s=0,\ j_1=\frac{m}{2}+\frac{n_2}{4}-1,\ j_2=-\frac{3}{4},\ j=\frac{\ell}{2}+\frac{n_1+1}{4}-1 \quad (14)$$

if $\ell - m$ is even, and

$$r = m,\ s = 1,\ j_1 = \frac{m}{2} + \frac{n_1}{4} - 1,\ j_2 = -\frac{1}{4},\ j = \frac{\ell}{2} + \frac{n_1+1}{4} - 1 \tag{15}$$

if $\ell - m$ is odd. For $\widetilde{Y}_\ell^m$ we set

$$r = 0,\ s = m,\ j_1 = -\frac{3}{4},\ j_2 = \frac{m}{2} + \frac{n_2}{4} - 1,\ j = \frac{\ell}{2} + \frac{n_2+1}{4} - 1 \tag{16}$$

if $\ell - m$ is even, and

$$r = 1,\ s = m,\ j_1 = -\frac{1}{4},\ j_2 = \frac{m}{2} + \frac{n_2}{4} - 1,\ j = \frac{\ell}{2} + \frac{n_2+1}{4} - 1 \tag{17}$$

if $\ell - m$ is odd.

Instead of (3) it is more convenient to consider the functions

$$Y_M^+(\theta) = \sqrt{2}\cos m\theta, \quad Y_M^-(\theta) = \sqrt{2}\sin m\theta. \tag{18}$$

Since $\cos m\theta = T_n(\cos\theta)$, $\sin m\theta = \sin\theta\, C_{m-1}^1(\cos\theta)$ (see Section 6.9.1), then

$$Y_M^+(\theta) = \frac{1}{2}Y_m^{00}(\theta), \quad Y_m^-(\theta) = \frac{1}{2}Y_m^{11}(\theta). \tag{19}$$

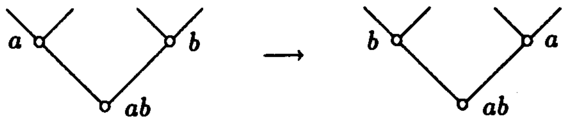
In both cases $n_1 = n_2 = 1$. Therefore, for $Y_M^+(\theta)$ we have $r = s = 0$, $j_1 = j_2 = -\frac{3}{4}$, $j = \frac{m-1}{2}$, and for $Y_m^-(\theta)$ we have $r = s = 1$, $j_1 = j_2 = -\frac{1}{4}$, $j = \frac{m-1}{2}$.

Since all functions from $\widetilde{\mathfrak{H}}^{n\ell}$ satisfy the differential equation $\Delta_0 F = -\ell(\ell + n - 2)F$, then one can say that, to every tree of the form described above, there corresponds a scheme of separation of variables for this equation.

**10.5.4. Relations between orthogonal bases.** Let $T$ and $T'$ be two trees corresponding to different coordinate systems $\{\theta_1, \ldots, \theta_{n-1}\}$ and $\{\theta_1', \ldots, \theta_{n-1}'\}$ on the sphere $S^{n-1}$. To these trees there correspond orthonormal bases $\{\Xi_M^T\}$ and $\{\Xi_{M'}^{T'}\}$ of the space $\mathfrak{L}^2(S^{n-1})$. They are related to each other by the unitary matrix $U^{T'T} \equiv \left(u_{M'M}^{T'T}\right)$:

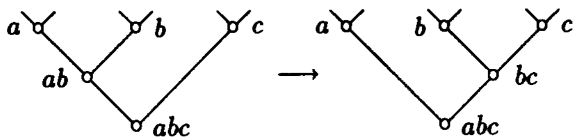
$$\Xi_M^T = \sum_{M'} u_{M'M}^{T'T} \Xi_{M'}^{T'}. \tag{1}$$

In the general case the matrix $U^{T'T}$ is complicated. Therefore, it is useful to represent it as a product of simpler matrices. The possibility of this representation follows from the fact that the transition from $T$ to $T'$ (hence, from $\{\Xi_M^T\}$ to $\{\Xi_{M'}^{T'}\}$) can be realized by successively performing elementary operations. These operations are: 1) the permutation $P \equiv P_{ab}$ of edges at one fork of the tree $T$ together with the subtrees, based on the tops $a$ and $b$:




(2)

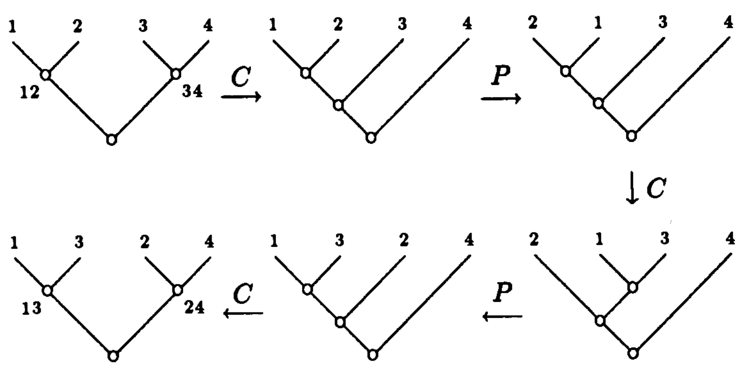
2) the transplantation $C \equiv C_{a(b)c}$ of one of the edges inside the fork:




(3)

For example, the transition between the trees


can be realized by performing the elementary operations:




If we pass from the tree $T$ to the tree $T'$ by means of the elementary operations:

$$T \to T_1 \to T_2 \to \ldots \to T_k \to T',$$

associated with unitary matrices $U_1, U_2, \ldots, U_{k+1}$, then

$$U^{T'T} = U_{k+1} \ldots U_2 U_1.$$

Entries of the matrices $U_P$ and $U_C$ corresponding to the permutation $P$ and to the transplantation $C$ of edges are called *T-coefficients*.

To permutation (2) there corresponds the replacements $\cos\theta \to \sin\theta'$, $\sin\theta \to \cos\theta'$, that is, if with the initial fork we have associated the angle $\theta$, then with the final one we associate the angle $\theta' = \frac{\pi}{2} - \theta$. According to formula (6) of Section 10.5.3, to the initial fork there corresponds the function

$$F(\theta) = N_{\ell rs} \sin^r \theta \cos^s \theta P^{\left(r+\frac{n_1-2}{2}, s+\frac{n_2-2}{2}\right)}_{(\ell-r-s)/2}(\cos 2\theta), \tag{4}$$

and to the final one there corresponds the function

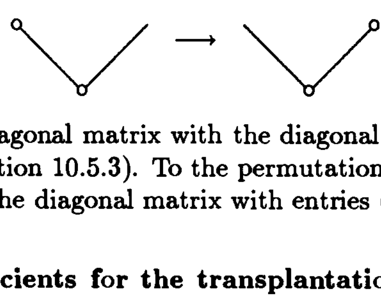
$$F'(\theta') = N_{\ell rs} \sin^r \theta' \cos^s \theta' P^{\left(r+\frac{n_2-2}{2}, s+\frac{n_1-2}{2}\right)}_{(\ell-r-s)/2}(\cos 2\theta'). \tag{5}$$

The remaining factors in the basis function (2) of Section 10.5.3 are not changed by the permutation. In accordance with formula (5) of Section 6.3.8, functions (4) and (5) are connected by the relation

$$F(\theta) = (-1)^{(\ell-r-s)/2} F'(\theta').$$

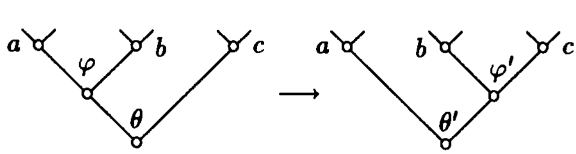
Thus, to permutation (2) there corresponds the diagonal matrix with the diagonal entries $(-1)^{(\ell-r-s)/2}$.

In the same way one proves that to the permutation $P$ of edges in the fork with one free end



there corresponds the diagonal matrix with the diagonal entries $(-1)^{\ell-m}$ (see formulas (4) and (5) of Section 10.5.3). To the permutation in the fork with two free ends there corresponds the diagonal matrix with entries $(-1)^m$ (see formula (3) of Section 10.5.3).

**10.5.5. *T*-coefficients for the transplantation of an edge.** We find $T$-coefficients corresponding to transplantation (3) of Section 10.5.4 of an edge. Let $T$ denote the initial tree and $T'$ denote the tree with transplanted subtree. Then the coordinate system $(\theta_1, \ldots, \theta_{n-1})$ associated with $T$ differs from the coordinate system $(\theta'_1, \ldots, \theta'_{n-1})$ associated with $T'$ only in two angles, which correspond to the lower nodes in formula (3) of Section 10.5.4:

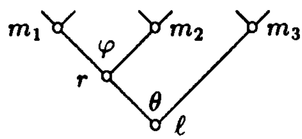



(1)

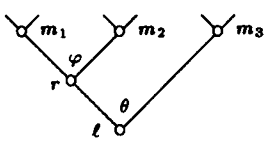
One can easily verify that

$$\cos\theta = \cos\theta' \cos\varphi', \quad \sin\theta \sin\varphi = \sin\theta', \quad \sin\theta\cos\varphi = \cos\theta' \sin\varphi'. \tag{2}$$

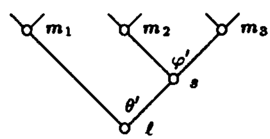
Let us choose fixed labels $M$ of the tree $T$. Assume that the part of $T$ containing the transplanted edge has the labels $M$ of the form




The basis function $\Xi^T_M$ corresponding to $T_M$ is expressed only in terms of basis functions $\Xi^{T'}_{M'}$ for which the labels in $M'$ associated with untransplanted nodes coincide with the labels of corresponding nodes of $T$. We write down this fact in the form




$$= \sum_{s=m_1+m_2}^{\ell} u_{sr} \tag{3}$$




Substitute expressions (2) of Section 10.5.3 for basis functions into (3), cancel equal factors on the left and on the right, and take into account the unitarity of the matrix $(u_{qp})$. We obtain the relation

$$\sum_{j_{12}} u_{j_{12}j_{23}} N_{rm_1m_2} \sin^{m_1}\varphi \cos^{m_2}\varphi P^{(2j_1+1,2j_2+1)}_{j_{12}-j_1-j_2-1}(\cos 2\varphi) N_{\ell r m_3} \sin^r\theta$$
$$\times \cos^{m_3}\theta P^{(2j_{12}+1,2j_3+1)}_{j-j_{12}-j_3-1}(\cos 2\theta) = N_{sm_2m_3} \sin^{m_2}\varphi' \cos^{m_3}\varphi'$$
$$\times P^{(2j_2+1,2j_3+1)}_{j_{23}-j_2-j_3-1}(\cos\varphi') N_{\ell m_1 s} \sin^{m_1}\theta' \cos^s\theta' P^{(2j_1+1,2j_{23}+1)}_{j-j_1-j_{23}-1}(\cos 2\theta'), \tag{4}$$

where $u_{j_{12}j_{23}} \equiv \bar{u}_{sr}$ and

$$j_1 = \frac{m_1}{2} + \frac{n_1}{4} - 1, \qquad j_2 = \frac{m_2}{2} + \frac{n_2}{4} - 1, \qquad j_3 = \frac{m_3}{2} + \frac{n_3}{4} - 1,$$
$$j_{12} = \frac{r}{2} + \frac{n_1+n_2}{4} - 1, \quad j_{23} = \frac{s}{2} + \frac{n_2+n_3}{4} - 1, \quad j = \frac{\ell}{2} + \frac{n_1+n_2+n_3}{4} - 1.$$

Here $n_1$, $n_2$, $n_3$ denote the numbers of coordinates of tree $T$ connected with the nodes $a$, $b$, $c$, respectively (see formula (1)).

Instead of $\cos\theta$ we substitute $\cos\theta'\cos\varphi'$ (see (2)) into (4), multiply both sides by $\cos^{-m_3}\varphi'$ and then set $\theta' = \varphi$, $\theta = \varphi' = \frac{\pi}{2}$. As a result, relation (4) is rewritten as

$$\sum_{j_{12}} \frac{(-1)^{j-j_{12}-j_3-1}\Gamma(j-j_{12}+j_3+1)}{(j-j_{12}-j_3-1)!} N_{\ell r m_3} N_{r m_1 m_2} u_{j_{12} j_{23}} P^{(2j_1+1,2j_2+1)}_{j_{12}-j_1-j_2-1}(\cos 2\varphi)$$

$$= \frac{(-1)^{j_{23}-j_2-j_3-1}\Gamma(j_{23}-j_2+j_3+1)}{(j_{23}-j_2-j_3-1)!}$$

$$\times N_{s m_2 m_3} N_{\ell m_1 s} \cos^{2(j_{23}-j_2-j_3-1)}\varphi P^{(2j_1+1,2j_{23}+1)}_{j-j_1-j_{23}-1}(\cos 2\varphi), \quad (5)$$

where $N_{uvt}$ are the coefficients from formula (6) of Section 10.5.3 and the summation is over the values of $j_{12}$ such that

$$m_1 + m_2 \le r \le \ell.$$

Taking into consideration the orthogonality relation for Jacobi polynomials (see Section 6.10.1), we derive the expression for the $T$-coefficients $u_{j_{12}j_{23}}$:

$$u_{j_{12}j_{23}} =$$

$$N' \int_{-1}^{1} P^{(2j_1+1,2j_2+1)}_{j_{12}-j_1-j_2-1}(x) P^{(2j_1+1,2j_{23}+1)}_{j-j_1-j_{23}-1}(x)(1-x)^{2j_1+1}(1+x)^{j_{23}+j_2-j_3}\,dx, \quad (6)$$

where $N'$ is expressed in terms of $j$, $j_1$, $j_2$, $j_{12}$, $j_{23}$.

The integral (6) is a special case of the integral

$$I \equiv \int_{-1}^{1} P^{(\alpha,\beta)}_n(x) P^{(\alpha,2\gamma+1)}_{r-\gamma-1}(x)(1-x)^{\alpha}(1+x)^{\gamma-a+\beta}\,dx. \quad (7)$$

In order to calculate it we expand $P^{(\alpha,\beta)}_n(x)$ in powers of $(1+x)/2$ (see formula (7) of Section 6.3.8) and invert the order of integration and summation. As a result, $I$ is represented as a linear combination of the integrals

$$\int_{-1}^{1} P^{(\alpha,2\gamma+1)}_{r-\gamma-1}(x)(1-x)^{\alpha}(1+x)^{\gamma-a+\beta+k}\,dx.$$

These integrals are evaluated by means of the equality

$$\int_{-1}^{1} P^{(\alpha,\beta)}_n(x)(1-x)^{\rho}(1+x)^{\sigma}\,dx = \frac{2^{\rho+\sigma+1}\Gamma(\rho+1)\Gamma(\sigma+1)}{\Gamma(\rho+\sigma+2)}$$

$$\times\, {}_3F_2(-n, \alpha+\beta+n+1, \rho+1; \alpha+1, \rho+\sigma+2; 1), \mathrm{Re}\,\rho > -1,\ \mathrm{Re}\,\sigma > -1 \quad (8)$$

(see formula (16) of Section 8.5.1), where one has to set $\alpha=\rho$. Since for $\alpha=\rho$ the right hand side of (8) coincides with ${}_2F_1(\dots;1)$, then, by making use of formula (2) of Section 3.5.1, we derive for (7) the expression

$$I=(-1)^{\gamma-r+n+1}2^{\gamma-a+\alpha+\beta+1}\frac{\Gamma(\gamma-a+\beta+1)\Gamma(r+\alpha-\gamma)\Gamma(r-a)}{\Gamma(r-\gamma)\Gamma(a-\beta+\gamma+1)\Gamma(r-a+\alpha+\beta+n+1)}$$
$$\times\frac{\Gamma(a+r-\beta-n)(\beta+1)_n(a+r+\alpha+1)_n}{n!\Gamma(r-a-n)}$$
$$\times{}_4F_3\left(\begin{matrix}-n,\,\alpha+\beta+n+1,\,a-\gamma,\,a+\gamma+1\\ \beta+1,\,a+\alpha+r+1,\,a-r+1\end{matrix}\middle|\,1\right).$$

We use it to obtain the following expression for $u_{j_{12}j_{23}}$:

$$u_{j_{12}j_{23}}=\frac{\Gamma(j-j_1-j_2-j_3-1)}{\Gamma(2j_2+2)\Gamma(j_1+j_2+j_3+j+3)}A(j,j_1,j_2,j_{12},j_3)A(j,j_3,j_2,j_{23},j_1)$$
$$\times{}_4F_3\left(\begin{matrix}j_1+j_2-j_{12}+1,j_1+j_2+j_{12}+2,j_2+j_3-j_{23}+1,j_2+j_3+j_{23}+2\\ 2j_2+2,j_1+j_2+j_3-j+2,j_1+j_2+j_3+j+j_3\end{matrix}\middle|\,1\right),\quad (9)$$

where

$$A(a,b,c,d,e)=$$
$$\left[\frac{(2d+1)\Gamma(a-d+e+1)\Gamma(a+d+e+2)\Gamma(d-b+c+1)\Gamma(d+b+c+2)}{\Gamma(a+d-e+1)\Gamma(d+b-c+1)(a-d-e-1)!(d-b-c-1)!}\right]^{\frac{1}{2}}.$$

(10)

Formula (1) of Section 8.5.5 allows us to express $u_{j_{12}j_{23}}$ in terms of the Wilson polynomials $p_n(x^2;a,b,c,d)$:

$$u_{j_{12}j_{23}}=(-1)^{j_{12}-j_1-j_2-1}N$$
$$\times\ p_{j_{12}-j_1-j_2-1}\left(\left(j_{23}+\frac{1}{2}\right)^2;j_2+j_3+\frac{3}{2},j_2-j_3+\frac{1}{2},j_1+j_2+\frac{3}{2},j_1-j_2+\frac{1}{2}\right),\quad (11)$$

where

$$N=\left[\frac{(2j_{12}+1)\Gamma(j-j_{12}+j_3+1)\Gamma(j_{12}+j_1+j_2+2)}{\Gamma(j+j_{12}-j_3+1)\Gamma(j_{12}+j_1-j_2+1)\Gamma(j+j_{12}+j_3+2)}\right.$$
$$\times\frac{\Gamma(j-j_{12}-j_3)(2j_{23}+1)\Gamma(j+j_1-j_{23}+1)}{\Gamma(j_{12}-j_1+j_2+1)(j_{12}-j_1-j_2-1)!}$$
$$\left.\times\frac{\Gamma(j+j_1+j_{23}+2)\Gamma(j_{23}+j_2-j_3+1)\Gamma(j_{23}+j_2+j_3+2)}{\Gamma(j-j_1+j_{23}+1)\Gamma(j_{23}-j_2+j_3+1)(j-j_1-j_{23}-1)!(j_{23}-j_2-j_3-1)!}\right]^{1/2}.\quad (12)$$

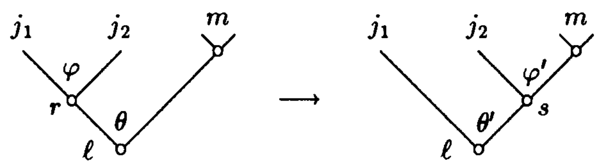
The coefficients $u_{j_{12}j_{23}}$ can be also expressed in terms of the Racah polynomials $r_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ (see Section 8.5.4). The orthogonality relation

$$\sum_{j_{23}} u_{j_{12}j_{23}} u_{j'_{12}j_{23}} = \delta_{j_{12}j'_{12}}$$

for $u_{j_{12}j_{23}}$ is equivalent to that for Racah polynomials.

**10.5.6. *T*-coefficients for the transplantation of an edge (degenerate cases).** We now regard $T$-coefficients for transplantation (1) of Section 10.5.5 of an edge, when one or more of the nodes $a$, $b$, $c$ are absent. In this case one has to substitute into formula (4) of Section 10.5.5 the expressions for functions (3)–(5) of Section 10.5.3 in terms of Jacobi polynomials instead of corresponding functions (6) of Section 10.5.3. As a result, we obtain formulas of Section 10.5.5 for the $T$-coefficients $u_{j_{12}j_{23}}$, in which $j_1$, $j_2$, $j_3$, $j_{12}$, $j_{23}$ take appropriate values (see Section 10.5.3) and the correction factors $\frac{1}{2}$ and $\frac{\sqrt{2}}{2}$ (see formulas (10)–(13) and (19) of Section 10.5.3) are taken into account.

For some cases the expressions for $T$-coefficients are simplified. We consider the edge transplantation



(1)

By virtue of formula (19) of Section 10.5.3 we have $j_1 = j_2 = -\frac{1}{4}$ or $-\frac{3}{4}$. Other $j$ are connected with $m$, $r$, $s$, $\ell$ by the formulas

$$j_3 = \frac{m}{2} + \frac{n_3}{4} - 1,\ j_{12} = \frac{r-1}{2},\ j_{23} = \frac{s}{2} + \frac{n_3+1}{4} - 1,\ j = \frac{\ell}{2} + \frac{n_3}{4} - \frac{1}{2}.$$

Instead of relation (4) of Section 10.5.5 we now have the equality

$$\sum_{j_{12}} u_{j_{12}j_{23}} \frac{1}{\sqrt{2}} \cos r\varphi N_{\ell rm} \sin^r\theta \cos^m\theta\, P^{(2j_{12}+1,2j_3+1)}_{j-j_{12}-j_3-1}(\cos 2\theta)$$
$$= N_{sm} \cos^m\varphi' C^{m+(n_3-1)/2}_{s-m}(\sin\varphi') N_{\ell s} \cos^s\theta' C^{s+n_3/2}_{\ell-s}(\sin\theta') \quad (2)$$

and the equality obtained by replacement of $\cos r\varphi$ by $\sin r\varphi$. As in the case of formula (4) of Section 10.5.5, we replace $\cos\theta$ by $\cos\theta'\cos\varphi'$, multiply both sides by $\cos^{-m}\varphi'$ and then set $\theta = \varphi' = \frac{\pi}{2}$, $\theta' = \varphi$. As a result, we derive for $u_{j_{12}j_{23}}$ the expressions

$$u_{j_{12}j_{23}} = c\int_0^{2\pi} C_{\ell-s}^{s+n_3/2}(\sin\theta)\cos^s\theta\cos r\theta\, d\theta, \tag{3}$$

$$u_{j_{12}j_{23}} = c'\int_0^{2\pi} C_{\ell-s}^{s+n_3/2}(\sin\theta)\cos^s\theta\sin r\theta\, d\theta. \tag{4}$$

The integrals in (3) and (4) are evaluated by means of the formula

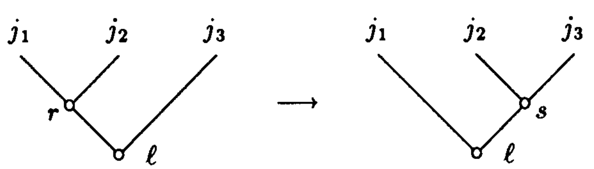
$$\int_0^{2\pi} C_n^\alpha(\cos\varphi)\sin^k\varphi\, e^{-im\varphi}d\varphi$$

$$= \frac{(-1)^m i^{-k}\pi k!\Gamma(\alpha+n)}{n!m!(k-m)!2^{k-1}\Gamma(\alpha)}\, {}_3F_2\left(\begin{matrix}-n,\ -m,\ \alpha\\ -\alpha-n+1, k-m+1\end{matrix}\,\middle|\, 1\right) \tag{5}$$

which can be proved by expanding the Gegenbauer polynomial and the function $\sin^k\varphi$ into Fourier series.

Thus, in this case the coefficients $u_{j_{12}j_{23}}$ are expressed in terms of Hahn polynomials.

We suggest that the reader proves that the transplantation




is fulfilled with the help of coefficients $u_{j_{12}j_{23}}$ which are expressed in terms of Krawtchouk polynomials.

**10.5.7. The tree method and invariant harmonic polynomials.** Let $n = n_1 + n_2$. We fix a tree $T$ having $n_1$ free ends in the left branch and $n_2$ free ends in the right branch. The tree $T$ with different labels $M$ and with fixed lowest label $\ell$ defines the basis of the space $\mathfrak{H}^{n\ell}$ of homogeneous harmonic polynomials of degree $\ell$ in $\mathbf{x} = (x_1, \dots, x_n)$. We set

$$\mathbf{x} = (\mathbf{y}, \mathbf{t}),\ \mathbf{y}\in\mathbb{R}^{n_1},\ \mathbf{t}\in\mathbb{R}^{n_2},\ y = |\mathbf{y}|^2 \equiv y_1^2 + \dots + h_{n_1}^2,\ t = |\mathbf{t}|^2.$$

According to formula (1) of Section 10.5.3 for even $\ell$ there exists in $\mathfrak{H}^{n\ell}$ a unique (up to a constant factor) polynomial, invariant with respect to the subgroup $K_{n_1n_2} \equiv SO(n_1)\times SO(n_2)$, that is, such that

$$p(\mathbf{x}) \equiv p(\mathbf{y}, \mathbf{t}) = p(h_1\mathbf{y}, h_2\mathbf{t}),\quad h_1\in SO(n_1),\ h_2\in SO(n_2). \tag{1}$$

This polynomial corresponds to the tree in which all the labels except for the lowest one are zeros. It is obtained from formula (6) of Section 10.5.3 if we take into account the relation between Cartesian and polyspherical coordinates in $\mathbf{R}^n$. It can be also obtained from formula (5) of Section 10.4.1:

$$\Xi_{OO}^{nn_1,\ell,00}(\mathbf{y},\mathbf{t}) = N(y+t)^{\ell/2} P_{\ell/2}^{(\frac{n_1}{2}-1,\frac{n_2}{2}-1)}\left(\frac{t-y}{t+y}\right). \tag{2}$$

On $S^{n-1}$ (that is, when $y+t=1$) this polynomial is of the form

$$\tilde{\Xi}_{00}^{nn_1,\ell,OO}(\mathbf{x}) = cP_{\ell/2}^{(\frac{n_2}{2}-1,\frac{n_1}{2}-1)}(2y-1) \tag{3}$$

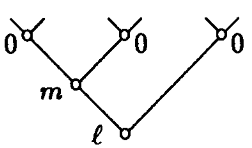
(we have taken into account equality (5) of Section 6.3.8).

Polynomials (2) with $\ell = 0,2,4,\ldots$ form a complete system of polynomials (that is, a basis) in the space of $SO(n_1)\times SO(n_2)$-invariant harmonic polynomials.

Let $\mathfrak{H}^{n\ell}_{n_1n_2n_3}$ be the subspace of $\mathfrak{H}^{n\ell}$, consisting of polynomials invariant with respect to the subgroup

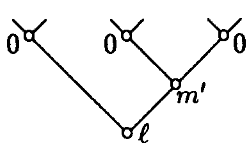
$$SO(n_1)\times SO(n_2)\times SO(n_3), \qquad n_1+n_2+n_3=n.$$

Basis functions of $\mathfrak{H}^{n\ell}_{n_1n_2n_3}$ are also constructed by means of trees with labels. One of these bases corresponds to the tree with labels




$m = 0,2,\ldots,\ell$ ($\ell$ is even) (4)

and another one corresponds to




$m' = 0,2,\ldots,\ell$ ($\ell$ is even) (5)

(to the upper nodes there correspond zero labels and we have omitted them). In order to write down corresponding bases we represent $\mathbf{x}$ in the form

$$\mathbf{x} = (\mathbf{y},\mathbf{t}) = (\mathbf{u},\mathbf{v},\mathbf{t}),\ \mathbf{y}\in\mathbf{R}^{n_1+n_2},\ \mathbf{u}\in\mathbf{R}^{n_1},\ \mathbf{v}\in\mathbf{R}^{n_2},\ \mathbf{t}\in\mathbf{R}^{n_3},$$

and introduce the notations $x = |\mathbf{x}|^2$, $y = |\mathbf{y}|^2$, $u = |\mathbf{u}|^2$ and so on. Making use of the results of Section 10.5.3 and passing to Cartesian coordinates, we find that to the tree with labels (4) there correspond basis elements

$$y^{\frac{\ell}{2}-k} P_{\frac{\ell}{2}-k}^{\left(\frac{n_2}{2}-1,\frac{n_1}{2}-1\right)}\left(2\frac{u}{y}-1\right)(y+t)^k P_k^{\left(\frac{n_3}{2}-1,\frac{n_1+n_2}{2}+\ell-2k-1\right)}\left(\frac{y-t}{y+t}\right), \quad (6)$$

$$k \equiv \frac{\ell-m}{2} = 0,1,\dots,\frac{\ell}{2},$$

and to (5) there correspond the basis elements

$$(y+t-u)^{\frac{\ell}{2}-k} P_{\frac{\ell}{2}-k}^{\left(\frac{n_2}{2}-1,\frac{n_3}{2}-1\right)}\left(1-2\frac{y-u}{y+t-u}\right)(y+t)^k$$

$$\times P_k^{\left(\frac{n_1}{2}-1,\frac{n_2+n_3}{2}+\ell-2k-1\right)}\left(\frac{y-t-2u}{y+t}\right), \quad (7)$$

$$k \equiv \frac{\ell-m'}{2} = 0,1,\dots,\frac{\ell}{2}.$$

On $S^{n-1}$ (that is, for $x=1$) polynomials (6) have the form

$$y^{\frac{\ell}{2}-k} P_{\frac{\ell}{2}-k}^{\left(\frac{n_2}{2}-1,\frac{n_1}{2}-1\right)}\left(2\frac{u}{y}-1\right) P_k^{\left(\frac{n_3}{2}-1,\frac{n_1+n_2}{2}+\ell-2k-1\right)}(2y-1) \quad (8)$$

and polynomials (7) have the form

$$(1-u)^{\frac{\ell}{2}-k} P_{\frac{\ell}{2}-k}^{\left(\frac{n_2}{2}-1,\frac{n_3}{2}-1\right)}\left(1-2\frac{y-u}{1-u}\right) P_k^{\left(\frac{n_1}{2}-1,\frac{n_2+n_3}{2}+\ell-2k-1\right)}(1-2u). \quad (9)$$

We denote by $\mathfrak{L}^2_{n_1n_2n_3}(S^{n-1})$ the subspace of functions $f$ from $\mathfrak{L}^2(S^{n-1})$, invariant with respect to the subgroup $SO(n_1)\times SO(n_2)\times SO(n_3)$:

$$f(h\boldsymbol{\xi}) = f(\boldsymbol{\xi}), \quad h \in SO(n_1)\times SO(n_2)\times SO(n_3).$$

Functions $f \in \mathfrak{L}^2_{n_1n_2n_3}(S^{n-1})$ depend on the variables $y$ and $u$ only: $f(\boldsymbol{\xi}) = F(u,y)$. With the help of the results of Section 10.5.2, one easily shows that

$$\int_{S^{n-1}} |f(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_3}{2}\right)} \iint_{0\le u\le y\le 1} |F(u,y)|^2$$

$$\times (1-y)^{\frac{n_3}{2}-1}(y-u)^{\frac{n_2}{2}-1}u^{\frac{n_1}{2}-1}du\,dy. \quad (10)$$

We obtain the Hilbert space $\mathfrak{L}^2(D)$ of functions $F(u,y)$ on the triangle $D = \{(u,y) \mid 0 \le u \le y \le 1\}$. Polynomials (8) for $\ell = 0,2,4,\dots;\ k = 0,1,\dots,\frac{\ell}{2}$ form an orthogonal basis of $\mathfrak{L}^2(D)$.

Note that the transformations $(u,y) \to (1-y, 1-u)$ and $(n_1,n_2,n_3) \to (n_3,n_2,n_1)$ transfer polynomials (8) into polynomials (9). In addition, the measure on the right hand side of (10) does not change.

If in (8) or in (9) one of the numbers $n_1$, $n_2$, $n_3$ is equal to 1, then one of the Jacobi polynomials becomes a Gegenbauer polynomial. For example, if $n_1 = 1$, then polynomials (8) are the polynomials

$$y^{\frac{\ell}{2}-k} C_{\ell-2k}^{\frac{n_2-1}{2}} \left(\frac{u}{\sqrt{y}}\right) P_k^{\left(\frac{n_3}{2}-1, \frac{n_2}{2}+\ell-2k-\frac{1}{2}\right)}(2y-1). \tag{11}$$

Every one of polynomials (9) is uniquely expanded in polynomials (8) and conversely. For $k = \frac{\ell}{2}$ the polynomial (9) becomes $P_{\ell/2}^{\left(\frac{n_1}{2}-1, \frac{n_1+n_3}{2}-1\right)}(1-2u)$ and we have the expansion

$$P_{\ell/2}^{\left(\frac{n_2+n_3}{2}-1, \frac{n_1}{2}-1\right)}(2u-1) = \sum_{k=0}^{\ell/2} c_{\ell k} y^{\frac{\ell}{2}-k} P_{\frac{\ell}{2}-k}^{\left(\frac{n_2}{2}-1, \frac{n_1}{2}-1\right)}\left(2\frac{u}{y}-1\right) \times P_k^{\left(\frac{n_3}{2}-1, \frac{n_1+n_2}{2}+\ell-2k-1\right)}(2y-1) \tag{12}$$

or, if we set $\alpha = \frac{n_2+n_3}{2}-1$, $\beta = \frac{n_1}{2}-1$, $\gamma = \frac{n_2}{2}-1$, the expansion

$$P_{\ell/2}^{(\alpha,\beta)}(2u-1) = \sum_{k=0}^{\ell/2} c_{\ell k} y^{\frac{\ell}{2}-k} P_{\frac{\ell}{2}-k}^{(\gamma,\beta)}\left(2\frac{u}{y}-1\right) \times P_k^{(\alpha-\gamma-1, \beta+\gamma+\ell-2k+1)}(2y-1). \tag{13}$$

Since the transition from (5) to (4) is a transplantation of the edge, then the coefficient $c_{\ell k}$ from (13) is calculated by means of the results of Section 10.5.5. We have

$$c_{\ell k} = \binom{s}{k}\binom{k+\alpha-\gamma-1}{k}\binom{s-k+\gamma}{s-k}\binom{s+\alpha}{s}^{-1} \times \frac{(\gamma+1)_{s-k}(\alpha+\gamma)_k(s+\alpha+\beta+1)_{s-k}(s-k+\beta+1)_k}{(\alpha+1)_s(s-k+\beta+\gamma+1)_{s-k}(2s-2k+\gamma+\beta+2)_k}, \tag{14}$$

where $s = \ell/2$ and $\binom{\sigma}{n} = \Gamma(\sigma+n+1)/n!\Gamma(\sigma+1)$.

Formula (13) at $y = 1$ leads to the expansion

$$P_s^{(\alpha,\beta)}(x) = \sum_{k=0}^{s} a_{sk} P_{s-k}^{(\gamma,\beta)}(x), \tag{15}$$

where $x = 2u - 1$, $s = \ell/2$ and

$$a_{sk} = \binom{k+\alpha-\gamma-1}{k}^{-1} c_{\ell k},$$

and for $y = t$ we obtain the expansion

$$P_s^{(\alpha,\beta)}(2t-1) = \sum_{k=0}^{s} b_{sk} t^{s-k} P_k^{(\alpha-\gamma-1,\beta+\gamma+\ell-2k+1)}(2t-1), \tag{16}$$

where $s = \ell/2$ and

$$b_{sk} = \binom{s-k+\gamma}{s-k}^{-1} c_{\ell k}.$$

Setting $k = 0$ into (9), we have the polynomial

$$(1-u)^{\ell/2} P_{\ell/2}^{(\frac{n_3}{2}-1,\frac{n_2}{2}-1)}\left(2\frac{y-u}{1-u}-1\right). \tag{9'}$$

It can be expanded in polynomials (8). By setting $\alpha = \frac{n_3}{2}-1$, $\beta = \frac{n_2}{2}-1$, $\gamma = \frac{n_1}{2}-1$, we obtain the expansion

$$(1-u)^s P_s^{(\alpha,\beta)}\left(2\frac{y-u}{1-u}-1\right) = \sum_{k=0}^{s} e_{sk} P_k^{(\alpha,\beta+\gamma+2s-2k+1)}(2y-1) \times y^{s-k} P_{s-k}^{(\beta,\gamma)}\left(2\frac{u}{y}-1\right), \tag{17}$$

where

$$e_{sk} = \binom{s}{k}\binom{\alpha+k}{k}\binom{s-k+\beta}{s-k}\binom{s+\alpha}{s}^{-1} \frac{(-1)^{s-k}(\beta+1)_s(2s-2k+\beta+\gamma+1)}{(s+\beta+\gamma-k+1)_{s+1}}. \tag{18}$$

We now derive some integral formulas for Jacobi polynomials. The polynomial

$$f(\mathbf{x}) = P_s^{(\frac{n_2+n_3}{2}-1,\frac{n_1}{2}-1)}(2u-1), \quad s = \ell/2, \tag{19}$$

where $\mathbf{x} \in S^{n-1}$ and $u = x_1^2 + \ldots + x_{n_1}^2$, is harmonic and invariant with respect to the subgroup $SO(n_1) \times SO(n_2+n_3)$ (see formula (3)). It is easy to verify that if $dh$ is the invariant normalized measure on $SO(n_1+n_2) \times SO(n_3)$, then

$$\varphi(\mathbf{x}) = \int_{SO(n_1+n_2)\times SO(n_3)} f(h\mathbf{x})dh \tag{20}$$

is a polynomial from $\widetilde{\mathfrak{H}}^{n\ell}$, invariant with respect to the subgroup $SO(n_1+n_2) \times SO(n_3)$. Hence,

$$\varphi(\mathbf{x}) = cP_s^{\left(\frac{n_3}{2}-1, \frac{n_1+n_2}{2}-1\right)}(2y-1), \; y = x_1^2 + \ldots + x_{n_1+n_2}^2. \tag{21}$$

Since $u = x_1^2 + \ldots + x_{n_1}^2$, then for integral (20) we have

$$\varphi(\mathbf{x}) = \int\limits_{SO(n_1+n_2)} f(h'\mathbf{x})dh' = \int\limits_{S^{n_1+n_2-1}} f(y\boldsymbol{\xi})d\boldsymbol{\xi},$$

where $y = x_1^2 + \ldots + x_{n_1+n_2}^2$. Making use of expression (4) of Section 9.1.9 for the measure and expression (19) for $f(\mathbf{x})$, we have

$$\varphi(\mathbf{x}) = c' \int\limits_0^1 P_s^{\left(\frac{n_2+n_3}{2}-1, \frac{n_1}{2}-1\right)}(2uy-1)(1-u)^{\frac{n_2}{2}-1}u^{\frac{n_1}{2}-1}du.$$

Setting $y = 0$, we find the relation between $c$ and $c'$, which leads to the equality

$$\frac{P_s^{\left(\frac{n_3}{2}-1, \frac{n_1+n_2}{2}-1\right)}(2y-1)}{P_s^{\left(\frac{n_3}{2}-1, \frac{n_1+n_2}{2}-1\right)}(-1)} = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \times \int\limits_0^1 \frac{P_s^{\left(\frac{n_2+n_3}{2}-1, \frac{n_1}{2}-1\right)}(2uy-1)}{P_s^{\left(\frac{n_2+n_3}{2}-1, \frac{n_1}{2}-1\right)}(-1)}(1-u)^{\frac{n_2}{2}-1}u^{\frac{n_1}{2}-1}du.$$

Putting $\alpha = \frac{n_2+n_3}{2} - 1$, $\beta = \frac{n_1}{2} - 1$, $\gamma = \frac{n_2}{2} - 1$, we have

$$\frac{P_s^{(\alpha-\gamma-1,\beta+\gamma+1)}(2y-1)}{P_s^{(\alpha-\gamma-1,\beta+\gamma+1)}(-1)} = \frac{\Gamma(\beta+\gamma+2)}{\Gamma(\beta+1)\Gamma(\gamma+1)} \int\limits_0^1 \frac{P_s^{(\alpha,\beta)}(2uy-1)}{P_s^{(\alpha,\beta)}(-1)}(1-u)^\gamma u^\beta du. \tag{22}$$

This equality is valid for arbitrary $\alpha$, $\beta$, $\gamma$.

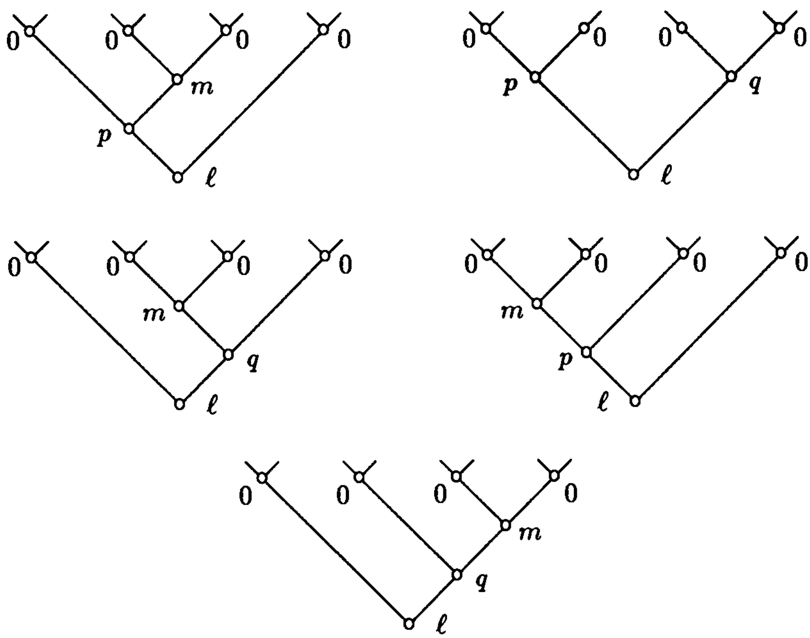
If we take polynomial (9′) instead of polynomial (19), then in the same way we obtain the relation

$$\frac{P_s^{(\alpha,\beta+\gamma)}(2y-1)}{P_s^{(\alpha,\beta+\gamma)}(-1)} = \frac{\Gamma(\beta+\gamma+1)}{\Gamma(\beta+1)\Gamma(\gamma+2)} \times \int\limits_0^1 (1-uy)^s P_s^{(\alpha,\beta)}\left(2\frac{y(1-u)}{1-uy} - 1\right)(1-u)^\beta u^\gamma du. \tag{23}$$

Let $\mathfrak{H}^{n\ell}_{n_1 n_2 n_3 n_4}$ be the subspace of $\mathfrak{H}^{n\ell}$ consisting of polynomials invariant with respect to the subgroup

$$SO(n_1) \times SO(n_2) \times SO(n_3) \times SO(n_4), \quad n_1 + n_2 + n_3 + n_3 = n.$$

Basis functions of this subspace are also constructed by means of trees with labels. There are five types of the corresponding trees:



We represent a point $\mathbf{x} \in \mathbb{R}^n$ in the form $\mathbf{x} = (\mathbf{y}, \mathbf{t})$, $\mathbf{y} \in \mathbb{R}^{n_1+n_2+n_3}$, $\mathbf{t} \in \mathbb{R}^{n_4}$. Let

$$\mathbf{w} = (x_1, \dots, x_{n_1}),\ \mathbf{u} = (x_1, \dots, x_{n_1+n_2}),\ w = |\mathbf{w}|^2,\ u = |\mathbf{u}|^2,\ y = |\mathbf{y}|^2.$$

Basis functions of $\mathfrak{H}^{n\ell}_{n_1 n_2 n_3 n_4}$ corresponding to the first tree are of the form

$$\begin{aligned} &(y+t)^k P_k^{\left(\frac{n_4}{2}-1,\, \frac{n_1+n_2+n_3}{2}+\ell-2k-1\right)} \left(\frac{y-t}{y+t}\right) y^{\frac{\ell}{2}-k} \\ &\times P_r^{\left(\frac{n_1}{2}-1,\, \frac{n_2+n_3}{2}+\ell-2k-2r-1\right)} \left(1-2\frac{w}{y}\right) \left(1-\frac{w}{y}\right)^{\frac{\ell}{2}-k-r} \\ &\times P_{\frac{\ell}{2}-k-r}^{\left(\frac{n_2}{2}-1,\, \frac{n_3}{2}-1\right)} \left(1-2\frac{u-w}{y-w}\right), \end{aligned} \tag{24}$$

$$k \equiv \frac{\ell - p}{2} = 0, 1, \dots, \frac{\ell}{2}, \quad r \equiv \frac{p-m}{2} = 0, 1, \dots, \frac{\ell}{2} - k.$$

On $S^{n-1}$ (that is, when $x \equiv y+t=1$) these polynomials turn into

$$P_k^{\left(\frac{n_4}{2}-1,\frac{n_1+n_2+n_3}{2}+\ell-2k-1\right)}(2y-1)y^{\frac{\ell}{2}-k}\left(1-\frac{w}{y}\right)^{\frac{\ell}{2}-k-r}$$
$$\times P_r^{\left(\frac{n_1}{2}-1,\frac{n_2+n_3}{2}+\ell-2k-2r-1\right)}\left(1-2\frac{w}{y}\right)P_{\frac{\ell}{2}-k-r}^{\left(\frac{n_2}{2}-1,\frac{n_3}{2}-1\right)}\left(1-2\frac{u-w}{y-w}\right). \quad (25)$$

Polynomials (25) for the values

$$\ell=0,2,4,\dots\ , \quad k=0,1,\dots,\frac{\ell}{2}, \quad r=0,1,\dots,\frac{\ell}{2}-k$$

form a complete orthogonal system of functions on the Hilbert space $\mathfrak{L}^2(D,d\mu)$, where $D$ is the simplex

$$\{(w,u,y)\ \big|\ 0\le w\le t\le y\le 1\}$$

and

$$d\mu(w,u,y)=(1-y)^{\frac{n_4}{2}-1}(y-u)^{\frac{n_3}{2}-1}(u-w)^{\frac{n_2}{2}-1}w^{\frac{n_1}{2}-1}dw\,du\,dy.$$

One can obtain analogous complete orthogonal systems of polynomials in the variables $w$, $u$, $y$ for other types of trees with labels. We leave to the reader their constructions.

The polynomial

$$f(\mathbf{x})=P_{\ell/2}^{\left(\frac{n_3+n_4}{2}-1,\frac{n_1+n_2}{2}-1\right)}(2u-1) \quad (26)$$

is a basis polynomial corresponding to the second tree with labels for $p=q=0$. It is expanded in polynomials (25). Calculating (by means of the results of Section 10.5.5) the expansion coefficients and introducing the notations

$$s=\frac{\ell}{2},\quad \alpha=\frac{n_3+n_4}{2}-1,\quad \beta=\frac{n_1+n_2}{2}-1,\quad \gamma=\frac{n_3}{2}-1,\quad \delta=\frac{n_2}{2}-1,$$

we obtain the equality

$$R_s^{(\alpha,\beta)}(2u-1)=\sum_{k=0}^{s}R_k^{(\alpha-\gamma-1,\beta+\gamma+2s-2k+1)}(2y-1)y^{s-k}$$
$$\times\sum_{r=0}^{s-k}b_{skr}R_r^{(\beta+\delta-1,\gamma+\delta+2s-2k-2r+1)}\left(1-2\frac{w}{y}\right)\left(1-\frac{w}{y}\right)^{s-k-r}$$
$$\times R_{s-k-r}^{(\delta,\gamma)}\left(1-2\frac{u-w}{y-w}\right), \quad (27)$$

where

$$R_n^{(a,b)}(x) = \frac{P_n^{(a,b)}(x)}{P_n^{(a,b)}(1)} = \binom{n+a}{n}^{-1} P_n^{(a,b)}(x)$$

and

$$b_{skr} = \frac{(-1)^{s-k}\binom{s}{s} - k - r\binom{k+r}{r}(\gamma+1)_{s-k}(\beta+\delta)_r(\alpha+\gamma)_k(s+\alpha+\beta+1)_{s-k}}{(\alpha+1)_s(\beta+\gamma+2s-2k-r)_r(\beta+\gamma+2s-2k+2)_k(\gamma+\delta+2s-2k-2r+2)_r} \times \frac{1}{(s-k-r+\gamma+\delta+1)_{s-k-r}}. \quad (28)$$

The polynomial

$$f(\mathbf{x}) = (1-u)^{\ell/2} P_{\ell/2}^{\left(\frac{n_4}{2}-1,\frac{n_3}{2}-1\right)}\left(2\frac{y-u}{1-u}-1\right) \quad (29)$$

is another basis function corresponding to the second tree with labels for $p=0$, $q=\ell$. Expanding (29) in polynomials (25), we derive the formula

$$(1-u)^s R_s^{(\alpha,\beta)}\left(2\frac{y-u}{1-u}-1\right) = \sum_{k=0}^{s} R_k^{(\alpha,\beta+\gamma+\delta+2s-2k+2)}(2y-1)y^{s-k} \times \sum_{r=0}^{s-k} g_{skr} R_r^{(\delta,\beta+\gamma+2s-2k-2r+1)}\left(1-2\frac{w}{y}\right)\left(1-\frac{w}{y}\right)^{s-k-r} \times R_{s-k-r}^{(\gamma,\beta)}\left(1-2\frac{u-w}{y-w}\right), \quad (30)$$

where

$$g_{skr} = \frac{(-1)^{s-k}\binom{s}{k}\binom{s-k}{r}(\beta+1)_s(2s-2k+\beta+\gamma+\delta+2)(\gamma+1)_{s-k-r}(\delta+1)_r}{(\beta+\gamma+\delta+s-k)_{s+1}(\gamma+\delta+2)_{s-k}(s-k-r+\beta+\gamma+1)_{s-k-r}} \times \frac{(s-k+\beta+\gamma+\delta+2)_r}{(2s-2k-2r+\beta+\gamma+1)_r}. \quad (31)$$

Equalities (27) and (30) lead to a series of interesting special cases.

If $f(\mathbf{x})$ coincides with (26), then

$$\varphi(\mathbf{x}) = \int_{SO(n_1)\times SO(n_2+n_3)\times SO(n_4)} f(h\mathbf{x})dh$$

is a polynomial from $\mathfrak{H}^{n\ell}$, invariant with respect to the subgroup $SO(n_1)\times SO(n_2+n_3)\times SO(n_4)$. Therefore, it is represented as a linear combination of the polynomials

$$P_{\frac{\ell}{2}-k}^{(\frac{n_2+n_3}{2}-1,\frac{n_1}{2}-1)}\left(2\frac{w}{y}-1\right)P_k^{(\frac{n_4}{2}-1,\frac{n_1+n_2+n_3}{2}+\ell-2k)}(2y-1)y^{\frac{\ell}{2}-k},$$
$$k=0,1,\dots,\ell/2.$$

Let us find the coefficients of this linear combination. Since in (26) $u=x_1^2+\dots+x_{n_1+n_2}^2$, then

$$\varphi(\mathbf{x})=\int\limits_{SO(n_2+n_3)} f(h'\mathbf{x})dh'$$
$$=N\int_0^1 R_{\ell/2}^{(\frac{n_3+n_4}{2}-1,\frac{n_1+n_2}{2}-1)}(2(u(y-w)+w)-1)u^{\frac{n_2}{2}-1}(1-u)^{\frac{n_3}{2}-1}du.$$

Substituting expression (27) for

$$R_{\ell/2}^{(\frac{n_3+n_4}{2}-1,\frac{n_1+n_2}{2}-1)}(2(u(y-w)+w)-1)$$

and integrating with respect to $u$, we obtain the relation, which for

$$\frac{\ell}{2}=s,\ \alpha=\frac{n_3+n_4}{2}-1,\ \beta=\frac{n_1+n_2}{2}-1,\ \gamma=\frac{n_3}{2}-1,\ \delta=\frac{n_2}{2}-1$$

is written down as

$$\int_0^1 R_s^{(\alpha,\beta)}(2(u(y-w)+w)-1)(1-u)^\gamma u^\delta du$$
$$\sum_{k=0}^{s} a_{sk}R_k^{(\alpha-\gamma-1,\beta+\gamma+2s-2k+1)}(2y-1)y^{s-k}$$
$$\times R_{s-k}^{(\gamma+\delta+1,\beta-\delta-1)}\left(2\frac{w}{y}-1\right), \quad (32)$$

where

$$a_{sk}=$$
$$\frac{\Gamma(\gamma+1)\Gamma(\delta+1)}{\Gamma(\gamma+\delta+2)}\binom{s}{k}\frac{(s+\alpha+\beta+1)_{s-k}(\alpha-\gamma)_k(\beta+s-k+1)_k(\gamma+1)_{s-k}}{(\alpha+1)_s(\beta+\gamma+s-k+1)_{s-k}(\beta+\gamma+2s-2k+2)_k}.$$

In the same way, by means of polynomial (29) and expansion (30), we derive the relation

$$\int_0^1 (1-(u(y-w)+w))^s R_s^{(\alpha,\beta)}\left(2\frac{y-(u(y-w)+w)}{1-(u(y-w)+w)}-1\right)(1-u)^\beta u^\gamma du$$
$$= \sum_{k=0}^{s} b_{sk} R_k^{(\alpha,\beta+\gamma+\delta+2k-2k+2)}(2y-1)y^{s-k} R_{s-k}^{(\delta,\beta+\gamma+1)}\left(1-2\frac{w}{y}\right), \quad (33)$$

where

$$b_{sk} = \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+2)} g_{s,k,s-k}$$

and $g_{s,k,s-k}$ is given by (31).

**10.5.8. The tree method and coordinates on the hyperboloid $H_+^{pq}$.** Some coordinate systems on the hyperboloid $H_+^{pq}$, for which the spherical part $\square_0$ of the operator $\square_{pq}$ (see Section 9.1.9) allows separation of variables, are described by means of trees of the same form as trees of Section 10.5.1, whose edges are labelled by "+", "−" and "0". Moreover, the following conditions must be fulfilled:

1) If an edge without free end is labelled by a sign different from "0", then at least one edge, outgoing from its end, is labelled by the same sign.
2) There are no forks, containing two edges, labelled by "0".
3) Any path consisting of edges labelled by "0" ends by a fork, from which edges, labelled by opposite signs, go out.
4) The numbers of edges with free ends is equal to $p+q$, where $p$ of these edges are labelled by "−" and $q$ of these edges are labelled by "+". To these edges there correspond the Cartesian coordinates $x_1, x_2, \ldots, x_{p+q}$ in $\mathbb{R}^{p+q}$.

With every edge we associate a subset from $X \equiv \{x_1, \ldots, x_{p+q}\}$, which consists of coordinates corresponding to free edges which finish all paths starting from the end of this edge. With every fork we associate a number $\theta_j$, where $j$ is the index of the fork. The numbers $\theta_j$ form a coordiante system on $H_+^{pq}$. In addition, the Cartesian coordinates $x_1, \ldots, x_{p+q}$ are expressed in terms of $\theta_1, \ldots, \theta_{p+q-1}$ by the following rules:

(a) If both edges, outgoing from the $j$-th fork, are labelled by the same sign (except for "0"), then all $x_k$ corresponding to the left edge are associated with the factor $\cosh\theta_j$ and those corresponding to the right edge are associated with $\sin\theta_j$.

(b) If these edges are labelled by different signs (except for "0"), then $x_k$ corresponding to "+" are associated with the factor $\cos\theta_j$, and $x_k$ corresponding to "−" are associated with $\sinh\theta_j$.

(c) Let us consider a path consisting of edges labelled by "0" which cannot be extended to a path with this property. Then coordinates corresponding to edges which outgo from this path are multiplied by the numbers $\theta_j$, corresponding to forks from which these edges outgo. With the edge labelled by "+" which outgoes from the end of this path we associate the factor $\frac{1}{2}(\cosh\theta_m + Re^{\theta_m})$ and with the edge, labelled by "−", we associate the factor $\frac{1}{2}(\sinh\theta_m + \mathrm{Re}^{\theta_m})$, where $\theta_m$ is the number corresponding to this final fork, and

$$R = -\sum_{+} x_j^2 + \sum_{-} x_j^2.$$

Here the first summation is over all "+"-edges which outgo from this path, and the second one is over all "−"-edges.

Rules (a)–(c) uniquely define the coordinates in the domain where the expression $-\sum x_i^2 + \sum x_j^2$ with the summation over coordinates, corresponding to the given edge, has the sign which labels this edge. In other words, the hyperboloid $H_+^{pq}$ is divided into domains which are coordinatized in different ways.

As we have noted above, the operator $\Box_{pq}$ written down in these coordinates allows separation of variables. In every case we have a linear combination of two linearly independent solutions of the second order differential equation. The choice of this linear combination depends on boundary conditions. We do not consider these problems and the problems on sewing together eigenfunctions in domains on the hyperboloid with the common boundary. We only note that, as in the case analyzed above, these eigenfunctions are expressed in terms of hypergeometric functions (in particular, in terms of Legendre functions and of cylindrical, trigonometric and exponential functions).

## 10.6. Transition Coefficients for Bases on the Cone and Special Functions

**10.6.1. Maijer G-functions.** *Maijer G-functions* are defined by the formula

$$G_{pq}^{mn}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) \equiv G\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right)$$
$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - t) \prod_{j=1}^{n} \Gamma(1 - a_j + t)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + t) \prod_{j=n+1}^{p} \Gamma(a_j - t)} x^t \, dt, \quad (1)$$

where $0 \le m \le q$, $0 \le n \le q$ and the parameters $a_r$ and $b_s$ are such that the poles of the functions $\Gamma(b_j - t)$, $j = 1, \ldots, m$, do not coincide with any of the poles of the functions $\Gamma(1 - a_k + t)$, $k = 1, \ldots, n$. The contour $L$ is taken in such a way that one of the following possibilities is fulfilled:

1) $L = L_1$, where $L_1$ goes from $= -i\infty$ to $+i\infty$ such that the poles of the functions $\Gamma(b_j - t)$, $j = 1, \ldots, m$, remain on the right and the poles of $\Gamma(1 - a_k + t)$, $k = 1, \ldots, n$, remain on the left. Under this condition the integral (1) converges if $p + q < 2(m + n)$, $|\arg x| < (m + n - \frac{1}{2}(p+q))\pi$;
2) $L = L_2$, where $L_2$ begins and ends at $+\infty$, encircles counterclockwise all poles of the functions $\Gamma(b_j - t)$, $j = 1, \ldots, m$, and does not encircle any pole of the functions $\Gamma(1 - a_k + t)$. Under this condition the integral converges if $q \geq 1$ and either $p < q$ or $p = q$ and $|x| < 1$;
3) $L = L_3$, where $L_3$ begins and ends at $-\infty$, encircles clockwise all poles of the functions $\Gamma(1 - a_k + t)$, $k = 1, \ldots, n$, and does not encircle any pole of the functions $\Gamma(b_j - t)$. Under this condition the integral converges if $p \geq 1$ and either $p > q$ or $q = p$ and $|x| > 1$.

Depending on the conditions satisfied by the parameters and by the variable $x$, the Maijer $G$-function is defined by means of the contour $L_1$, $L_2$ or $L_3$. If it is possible to define the Maijer $G$-function by means of two contours, then the results coincide.

The integral in (1) can be calculated by means of the residue theorem. If no two numbers from $b_j$, $j = 1, 2, \ldots, m$, differ by an integer, then all poles are poles of the first order and, using the contour $L_2$, we have

$$G^{mn}_{pq}\left(x \middle| \begin{matrix} a_r \\ b_s \end{matrix}\right) = \sum_{k=1}^{m} \frac{\sum\limits_{j=1}^{m}{}' \Gamma(b_j - b_k) \prod\limits_{j=1}^{n} \Gamma(a_j + b_k - 1)}{\prod\limits_{j=m+1}^{q} \Gamma(1 + b_k - b_j) \prod\limits_{j=n+1}^{p} \Gamma(a_j - b_k)} x^{b_k}$$
$$\times {}_pF_{q-1}\left(\begin{matrix} 1 + b_k - a_1, \ldots, 1 + b_k - a_p \\ 1 + b_k - b_1, \ldots, *, \ldots, 1 + b_k - b_q \end{matrix} \middle| (-1)^{p-m-n} x\right), \quad (2)$$

where either $p < q$ or $p = q$ and $|x| < 1$. If no two numbers from $a_j$, $j = 1, \ldots, n$, differ by an integer, then, using $L_3$, we obtain

$$G^{mn}_{pq}\left(x \middle| \begin{matrix} a_r \\ b_s \end{matrix}\right) = \sum_{k=1}^{n} \frac{\prod\limits_{j=1}^{n}{}' \Gamma(a_k - a_j) \prod\limits_{j=1}^{m} \Gamma(b_j - a_k + 1)}{\prod\limits_{j=n+1}^{p} \Gamma(a_j - a_k + 1) \prod\limits_{j=m+1}^{q} \Gamma(a_k - b_j)} x^{a_k - 1}$$
$$\times {}_qF_{p-1}\left(\begin{matrix} 1 + b_1 - a_k, \ldots, 1 + b_m - a_k \\ 1 + a_1 - a_k, \ldots, *, \ldots, 1 + a_p - a_k \end{matrix} \middle| (-1)^{q-m-n} x^{-1}\right), \quad (3)$$

where either $q < p$ or $p = q$ and $|x| > 1$. The prime at the product sign in (2) and (3) means that the factor corresponding to $j = k$ is omitted; the star $*$ in $F$ means that the parameter $1 + b_k - b_k$ or $1 + a_k - a_k$, respectively, is omitted.

The Maijer $G$-function is symmetric with respect to permutations of parameters in the collections $\{a_1, \ldots, a_n\}$, $\{a_{n+1}, \ldots, a_p\}$, $\{b_1, \ldots, b_m\}$, $\{b_{m+1}, \ldots, b_q\}$. It is clear that if either one of $a_j$, $j = 1, \ldots, n$, coincides with one of $b_j$, $j = m+1, \ldots, q$, or one of $b_j$, $j = 1, \ldots, m$, coincides with one of $a_j$, $j = n+1, \ldots, p$, then $p$, $q$ and $n$ or $m$ in the $G$-function are decreased by 1.

By means of a change of a variable in integral (1), one shows that

$$x^d G_{pq}^{mn}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) = G_{pq}^{mn}\left(x \,\middle|\, \begin{matrix} a_r + d \\ b_s + d \end{matrix}\right), \tag{4}$$

$$G_{pq}^{mn}\left(x^{-1} \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) = G_{qp}^{nm}\left(x \,\middle|\, \begin{matrix} 1 - b_s \\ 1 - a_s \end{matrix}\right), \tag{5}$$

$$G_{pq}^{mn}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) = (2\pi)^{-m-n+(p+q)/2} 2^{b-a+1+(p-q)/2} \times G_{2p,2q}^{2m,2n}\left(2^{2(p-q)} x^2 \,\middle|\, \begin{matrix} a_r/2, (a_r+1)/2 \\ b_s/2, (b_s+1)/2 \end{matrix}\right), \tag{6}$$

where $a = a_1 + \ldots + a_p$, $b = b_1 + \ldots + b_q$. We easily derive from (1) the relations

$$G_{pq}^{mn}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) = \frac{1}{2\pi i}\left[e^{\pi i b_{m+1}} G_{pq}^{m+1,n}\left(xe^{-\pi i} \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) - e^{-\pi i b_{m+1}} G_{pq}^{m+1,n}\left(xe^{\pi i} \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right)\right], \quad m \le q-1, \tag{7}$$

$$G_{pq}^{mn}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) = \frac{1}{2\pi i}\left[e^{\pi i a_{n+1}} G_{pq}^{m,n+1}\left(xe^{-\pi i} \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) - e^{-\pi i a_{n+1}} G_{pq}^{m,n+1}\left(xe^{\pi i} \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right)\right], \quad n \le p-1. \tag{8}$$

By means of relations (4)–(8), from the properties of Maijer $G$-functions of one type we can derive the properties of Maijer $G$-functions of another type. Majority of known special functions are expressed in terms of Maijer $G$-functions. For example,

$$r^m J_k(rx) = \left(\frac{2}{x}\right)^m G_{02}^{10}\left(\frac{r^2x^2}{4} \,\middle|\, \begin{matrix} \\ \frac{k+m}{2}, \frac{m-k}{2} \end{matrix}\right), \tag{9}$$

$$\begin{aligned} {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) &= \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} G_{p,q+1}^{1p}\left(-x \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_p \\ 0, 1-b_1, \ldots, 1-b_q \end{matrix}\right) \\ &= \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} G_{q+1,p}^{p1}\left(-\frac{1}{x} \,\middle|\, \begin{matrix} 1, b_1, \ldots, b_q \\ a_1, \ldots, a_p \end{matrix}\right). \end{aligned} \tag{10}$$

The following integral relation is valid for Maijer $G$-functions:

$$\int_0^\infty x^{\alpha-1}(x+c)^{-\beta} G_{pq}^{mn}\left(xd \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) dx$$
$$= \frac{c^{\alpha-\beta}}{\Gamma(\beta)} G_{p+1,q+1}^{m+1,n+1}\left(cd \,\middle|\, \begin{matrix} 1-\alpha, a_1,\ldots,a_p \\ \beta-\alpha, b_1,\ldots,b_q \end{matrix}\right). \quad (11)$$

In order to prove it we replace the function $G_{pq}^{mn}\left(xd \,\middle|\, \begin{smallmatrix} a_r \\ b_r \end{smallmatrix}\right)$ on the left hand side by expression (1), invert the integration order, take into account the formula

$$\int_0^\infty x^{\mu-1}(a+x)^{-\nu} dx = a^{\mu-\nu}\mathrm{B}(\mu,\nu-\mu),\ \mathrm{Re}\nu > \mathrm{Re}\mu > 0,\ |\arg a| < \pi$$

(which follows from formula (1) of Section 3.4.6 with the help of the substitution $x = (1-t)/t$), and again apply (1) to the resulting expression. The conditions for which formula (11) is valid depend on the choice of the contour $L$ in (1). To three possible choices of $L$ there correspond three collections of conditions for which (11) is valid. The first collection of conditions is

$$\left.\begin{matrix} p+q < 2(m+n),\ |\arg d| < \left(m+n-\frac{p+q}{2}\right)\pi,\ |\arg c| < \pi, \\ \mathrm{Re}(\alpha+b_j) > 0,\ j=1,\ldots,m;\ \mathrm{Re}(\alpha-\beta+a_j) < 1,\ j=1,\ldots,n; \end{matrix}\right\} \quad (12)$$

the second one is

$$\left.\begin{matrix} p \le q,\ p+q \le 2(m+n),\ |\arg d| < \left(m+n-\frac{p+q}{2}\right)\pi,\ |\arg c| < \pi, \\ \mathrm{Re}(\alpha+b_j) > 0,\ j=1,\ldots,m;\ \mathrm{Re}(\alpha-\beta+a_j) < 1,\ j=1,\ldots,n; \\ \mathrm{Re}\left(a_1+\ldots+a_p-b_1-\ldots-b_q-(q-p)\left(\alpha-\beta-\frac{1}{2}\right)\right) > 1; \end{matrix}\right\} \quad (13)$$

and the third one is

$$\left.\begin{matrix} p \ge q,\ p+q \le 2(m+n),\ |\arg d| \le \left(m+n-\frac{p+q}{2}\right)\pi, |\arg c| < \pi, \\ \mathrm{Re}(\alpha+b_j) > 0,\ j=1,\ldots,m;\ \mathrm{Re}(\alpha-\beta+a_j) < 1,\ j=1,\ldots,n; \\ \mathrm{Re}\left(a_1+\ldots+a_p-b_1-\ldots-b_q+(p-q)\left(\alpha-\frac{1}{2}\right)\right) > 1. \end{matrix}\right\} \quad (14)$$

In the same way, by means of formula (1) of Section 3.4.6, we prove the relations

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} G_{pq}^{mn}\left(xd \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) dx$$
$$= \Gamma(\beta) G_{p+1,q+1}^{m,n+1}\left(d \,\middle|\, \begin{matrix} 1-\alpha, a_1,\ldots,a_p \\ b_1,\ldots,b_q, 1-\alpha-\beta \end{matrix}\right), \quad (15)$$

$$\int_1^\infty x^{-\alpha}(x-1)^{\beta-1} G_{pq}^{mn}\left(xd\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)dx$$

$$= \Gamma(\beta) G_{p+1,q+1}^{m+1,n}\left(d\,\middle|\,\begin{matrix}a_1,\ldots,a_{p,\alpha}\\ \alpha-\beta, b_1,\ldots,b_q\end{matrix}\right). \quad (16)$$

Formula (15) is valid if one of the three collections of conditions

$$\left.\begin{matrix} p+q < 2(m+n),\ |\arg d| < \left(m+n-\frac{p+q}{2}\right)\pi, \operatorname{Re}\beta > 0, \\ \operatorname{Re}(\alpha+b_j) > 0,\ j=1,\ldots,m; \end{matrix}\right\} \quad (17)$$

$$\left.\begin{matrix} p+q \le 2(m+n),\ |\arg d| < \left(m+n-\frac{p+q}{2}\right)\pi,\ \operatorname{Re}\beta > 0, \\ \operatorname{Re}(\alpha+b_j) > 0,\ j=1,\ldots,m, \\ \operatorname{Re}\left(a_1+\ldots+a_p-b_1-\ldots-b_q+(p-q)\left(\alpha-\frac{1}{2}\right)\right) > -\frac{1}{2}; \end{matrix}\right\} \quad (18)$$

$$p<q,\ \operatorname{Re}\beta>0,\ \operatorname{Re}(\alpha+b_j)>0,\ j=1,\ldots,m \quad (19)$$

is satisfied. Formula (16) is valid if one of the three collections of conditions

$$\left.\begin{matrix} p+q < 2(m+n), |\arg d| < \left(m+n-\frac{p+q}{2}\right)\pi, \\ \operatorname{Re}\beta > 0,\ \operatorname{Re}(\alpha-\beta-a_j) > -1,\ j=1,\ldots,m; \end{matrix}\right\} \quad (20)$$

$$\left.\begin{matrix} p+q \le 2(m+n), |\arg d| \le \left(m+n-\frac{p+q}{2}\right)\pi;\ \operatorname{Re}(\alpha-\beta-a_j) > -1, \\ j=1,\ldots,m, \\ \operatorname{Re}\beta>0, \operatorname{Re}\left(a_1+\ldots+a_p-b_1-\ldots-b_q+(q-p)\left(\alpha-\beta+\frac{1}{2}\right)\right) > -\frac{1}{2}, \end{matrix}\right\} \quad (21)$$

$$q<p,\ \operatorname{Re}(\alpha-\beta-a_j) > -1,\ j=1,\ldots,n,\ \operatorname{Re}\beta>0 \quad (22)$$

is fulfilled.

**10.6.2. Transition coefficients for the bases $\{\widehat{\Xi}_{mM}^{n\sigma,0}\}$ and $\{\widehat{\Xi}_{RM}^{n\sigma,2}\}$.** The basis elements $\widehat{\Xi}_{mM}^{n\sigma,0}(\boldsymbol{\xi})$ of the space $\mathfrak{B}^{n\sigma}$ are given by formula (4) of Section 10.3.1 and the basis elements $\widehat{\Xi}_{RM}^{n\sigma,2}(\boldsymbol{\xi})$ are given by formula (12) of Section 10.3.1. Elements of the basis $\{\widetilde{\Xi}_{RM}^{n\sigma,2}\}$, dual to the basis $\{\widehat{\Xi}_{RM}^{n\sigma,2}\}$ with respect to the scalar product

$$(f_1, f_2) = \int_\Gamma f_1(\boldsymbol{\eta})\overline{f_2(\boldsymbol{\eta})}d\boldsymbol{\eta},$$

where $\Gamma$ is a section of the cone $C_+^{n-1}$ and $d\boldsymbol{\eta}$ denotes the invariant measure on $\Gamma$, are of the form

$$\widetilde{\Xi}_{RM}^{n\sigma,2}(\boldsymbol{\xi}) = \widehat{\Xi}_{RM}^{n,-\bar{\sigma}-n+2,2}(\boldsymbol{\xi}). \quad (1)$$

The transition coefficients for the bases $\{\widehat{\Xi}^{n\sigma,0}_{mM}\}$ and $\{\widehat{\Xi}^{n\sigma,2}_{RM}\}$ are defined by the formula

$$C^{n\sigma}(m,M;R,M') \equiv \left(\widehat{\Xi}^{n\sigma,0}_{mM}, \widehat{\Xi}^{n,-\bar{\sigma}-n+2,2}_{RM'}\right)$$
$$= \int_\Gamma \widehat{\Xi}^{n\sigma,0}_{mM}(\eta)\overline{\widehat{\Xi}^{n,-\bar{\sigma}-n+2,2}_{RM'}(\eta)}d\eta. \qquad (2)$$

We choose the contour $\Gamma_2$ from Section 10.3.1 as $\Gamma$. By means of formula (10) of Section 10.3.1 we introduce the coordinates $y_1, \ldots, y_{n-2}$ on $\Gamma_2$. Expressing the measure

$$d\mathbf{y} = \Gamma\left(\frac{n-1}{2}\right)\left(2\pi^{\frac{n-1}{2}}\right)^{-1} dy_1 \ldots dy_{n-2}$$

on $\Gamma_2$ in terms of the spherical coordinates $r, \theta_1, \ldots, \theta_{n-3}$, where $r^2 = y_1^2 + \ldots + y_{n-2}^2$, we have

$$d\mathbf{y} = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)} r^{n-3} dr\, d\boldsymbol{\xi},$$

where $d\boldsymbol{\xi}$ is the invariant measure on $S^{n-3}$. Substitute this expression for the measure $d\mathbf{y}$ and the expressions for the functions $\widehat{\Xi}^{n\sigma,0}_{mM}(\boldsymbol{\xi})$ and $\widehat{\Xi}^{n,\bar{\sigma}-n+2,2}_{RM'}(\mathbf{x})$ on the contour $\Gamma_2$ into (2). After integrating over $S^{n-3}$ we find that

$$C^{n\sigma}(m,M;R,M') = 0 \quad \text{if} \quad M \neq M',$$

and

$$C^{n\sigma}(m,M;R,M) \equiv C^{n\sigma}(m,R,m_1)$$
$$= \frac{(-i)^{m_1} 2^{m_1-1}}{\sqrt{\pi}}\Gamma\left(m_1+\frac{n-3}{2}\right)\left[\frac{(n-3)!(m-m_1)!(2m+n-3)}{(m+m_1+n-4)!}\right]^{1/2}\left(\frac{R}{2}\right)^{(n-4)/2}$$
$$\times \int_0^\infty \left(\frac{1-r^2}{2}\right)^{\sigma-m_1} C^{m_1+\frac{n-3}{2}}_{m-m_1}\left(\frac{1-r^2}{1+r^2}\right) r^{m_1+\frac{n-4}{2}} J_{m_1+\frac{n-4}{2}}(Rr)r\,dr, \qquad (3)$$

where $m_1$ is the first parameter in $M = (m_1, m_2, \ldots, m_{n-3})$.

In order to calculate this integral we utilize the formula

$$C^\alpha_n(x) = \frac{\Gamma(2\alpha+n)}{n!\Gamma(2\alpha)}F\left(-n, 2\alpha+n; \alpha+\frac{1}{2}; \frac{1-x}{2}\right)$$

for expanding the Gegenbauer polynomial into a series, apply formula (9) of Section 10.6.1, invert the order of integration and summation, and apply formula (11) of Section 10.6.1. As a result, we obtain

$$C^{n\sigma}(m,R,m_1) = A^{n\sigma}_{mm_1} R^{-m_1-n+4} G^{n\sigma,21}_{mm_1,13}(R), \qquad (4)$$

where

$$A^{n\sigma}_{mm_1} = (-i)^{m_1} 2^{m_1-\sigma-2} \left[ \frac{(n-3)!(m-m_1)!(2m+n-3)}{(m+m_1+n-4)!} \right]^{1/2}, \tag{5}$$

$$G^{n\sigma,21}_{mm_1,13}(R) = \sum_{k=0}^{m-m_1} \frac{(-1)^k (m+m_1+k+n-4)!}{k!\Gamma\left(m_1+k+\frac{n-2}{2}\right)\Gamma(m_1-\sigma+k)(m-m_1+k)!} \times G^{21}_{13}\left(\frac{R^2}{4} \middle| \begin{matrix} -k \\ m_1-\sigma-1, 0, m_1+\frac{n-4}{2} \end{matrix}\right). \tag{6}$$

**10.6.3. Transition coefficients for the bases $\{\widehat{\Xi}^{n\sigma,2}_{RM}\}$ and $\{\widehat{\Xi}^{n\sigma,1}_{\nu M\epsilon}\}$.** The basis elements $\widehat{\Xi}^{n\sigma,1}_{\nu M\epsilon}(\mathbf{x})$ are given by formula (8) of Section 10.3.1. The transition coefficients for the bases $\{\widehat{\Xi}^{n\sigma,2}_{RM}\}$ and $\{\widehat{\Xi}^{n\sigma,1}_{\nu M\epsilon}\}$ are defined by the formula

$$D^{n\sigma}_{\epsilon}(\nu, M; R, M') \equiv \left(\widehat{\Xi}^{n\sigma,1}_{\nu M\epsilon}, \widehat{\Xi}^{n,-\bar\sigma-n+2,2}_{RM'}\right) = \int_{\Gamma} \widehat{\Xi}^{n\sigma,1}_{\nu M\epsilon}(\eta) \overline{\widehat{\Xi}^{n,-\bar\sigma-n+2,2}_{RM'}(\eta)} d\eta. \tag{1}$$

We put there $\Gamma = \Gamma_2$. As in Section 10.6.2, we find that

$$D^{n\sigma}_{\epsilon}(\nu, M; R, M') = 0 \quad \text{if} \quad M \neq M'.$$

If $M = M'$, then $D^{n\sigma}_{\epsilon}(\nu, M; R, M)$ depends only on $\nu$, $R$ and on the first parameter $m_1$ of $M$. We have

$$D^{n\sigma}_{+}(\nu, M; R, M) \equiv D^{n\sigma}_{+}(\nu, R, m_1) = \frac{i^{m_1}(n-3)!\Gamma\left(i\nu-\frac{n-5}{2}\right) R^{-\frac{n-4}{2}}}{2^{\sigma+\frac{n-2}{2}}\Gamma\left(i\nu-m_1-\frac{n-5}{2}\right)} \times \int_0^1 (1-r^2)^{\sigma+\frac{n-4}{2}} \mathfrak{P}^{-m_1-\frac{n-4}{2}}_{i\nu-1/2}\left(\frac{1+r^2}{1-r^2}\right) J_{m_1+\frac{n-4}{2}}(Rr) r\, dr. \tag{2}$$

To calculate this integral we expand the associated Legendre function in a series by means of the formula

$$\mathfrak{P}^{\mu}_{\nu}(x) = \frac{1}{\Gamma(1-\mu)} \left(\frac{x+1}{x-1}\right)^{\mu/2} F\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right),$$

use formula (9) of Section 10.6.1, invert the order of integration and summation, and apply formula (15) of Section 10.6.1. As a result, we obtain

$$D^{n\sigma}_{+}(\nu, R, m) = A^{n\sigma}_{\nu m+} R^{4-m-n} G^{n\sigma,11}_{\nu m,13}(R), \tag{3}$$

where

$$A^{n\sigma}_{\nu m+} = \frac{i^m(n-3)!\Gamma\left(i\nu - \frac{n-5}{2}\right)}{2^{\sigma-m+2}\Gamma\left(i\nu - m - \frac{n-5}{2}\right)\Gamma\left(\frac{1}{2}-i\nu\right)\Gamma\left(\frac{1}{2}+i\nu\right)}, \tag{4}$$

$$G^{n\sigma,11}_{\nu m,13}(R) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma\left(\frac{1}{2}+k-i\nu\right)\Gamma\left(\frac{1}{2}+k+i\nu\right)\Gamma\left(\sigma-k+\frac{n-4}{2}\right)}{\Gamma\left(m+k+\frac{n-2}{2}\right)} \times G^{11}_{13}\left(\frac{R^2}{4}\middle|\begin{matrix}-k\\ m+\frac{n-4}{2},0,-\sigma-\frac{n-2}{2}\end{matrix}\right). \tag{5}$$

For $D^{n\sigma}_{-}(\nu, M; R, M) \equiv D^{n\sigma}_{-}(\nu, R, m_1)$ we have

$$D^{n\sigma}_{-}(\nu,R,m_1) = \frac{i^{m_1}(n-3)!\Gamma\left(i\nu-\frac{n-5}{2}\right)}{2^{\sigma+\frac{n-2}{2}}\Gamma\left(i\nu-m_1-\frac{n-5}{2}\right)} \times \int_1^{\infty}(r^2-1)^{\sigma+\frac{n-4}{2}}\mathfrak{P}^{-m_1-\frac{n-4}{2}}_{i\nu-1/2}\left(\frac{r^2+1}{r^2-1}\right)J_{m_1+\frac{n-4}{2}}(Rr)r\,dr.$$

The integral from this formula is calculated in the same way as the integral from (2), and we receive

$$D^{n\sigma}_{-}(\nu,R,m) = A^{n\sigma}_{\nu m-}R^{-m}G^{n\sigma,20}_{\nu m,13}(R), \tag{6}$$

where

$$A^{n\sigma}_{\nu m-} = \frac{i^m(n-3)!\Gamma\left(i\nu-\frac{n-5}{2}\right)}{2^{\sigma+m+n-2}\Gamma\left(i\nu-m-\frac{n-5}{2}\right)\Gamma\left(\frac{1}{2}-i\nu\right)\Gamma\left(\frac{1}{2}+i\nu\right)}, \tag{7}$$

$$G^{n\sigma,20}_{\nu m,13}(R) = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(\frac{1}{2}+k-i\nu\right)\Gamma\left(\frac{1}{2}+k+i\nu\right)\Gamma\left(\sigma-k+\frac{n-2}{2}\right)}{k!\Gamma\left(m+k+\frac{n-2}{2}\right)} \times G^{20}_{13}\left(\frac{R^2}{4}\middle|\begin{matrix}0\\ -\sigma+k-\frac{n-2}{2},0,-m_1-\frac{n-4}{2}\end{matrix}\right). \tag{8}$$

**10.6.4. Transition coefficients for the bases $\{\widehat{\Xi}^{n\sigma,0}_{mM}\}$ and $\{\widehat{\Xi}^{n\sigma,1}_{\nu M\varepsilon}\}$.** The transition coefficients for the bases $\{\widehat{\Xi}^{n\sigma,0}_{mM}\}$ and $\{\widehat{\Xi}^{n\sigma,1}_{\nu M\varepsilon}\}$ are defined by the formula

$$E^{n\sigma}_{\varepsilon}(m,M;\nu,M') \equiv \left(\widehat{\Xi}^{n\sigma,0}_{mM},\widehat{\Xi}^{n,-\bar\sigma-n+2,1}_{\nu M'\varepsilon}\right) = \int_{\Gamma}\widehat{\Xi}^{n\sigma,0}_{mM}(\eta)\overline{\widehat{\Xi}^{n,\bar\sigma-n+2,1}_{\nu M'\varepsilon}(\eta)}d\eta. \tag{1}$$

We choose the contour $\Gamma^{\pm}_1$ from Section 10.3.1 as $\Gamma$. In the same way as in Section 10.6.2, one shows that

$$E^{n\sigma}_{\varepsilon}(m,M;\nu,M') = 0 \quad \text{if} \quad M \neq M'.$$

If $M = M'$, then $E_\varepsilon^{n\sigma}(m, M; \nu, M) \equiv E_\varepsilon^{n\sigma}(m, \nu, m_1)$ depends on $m$, $\nu$ and on the first parameter $m_1$ of $M$. We have

$$E_\varepsilon^{n\sigma}(m,\nu,m_1)$$
$$=\frac{(-1)^{m_1}2^{m+\frac{n-4}{2}}\Gamma\left(i\nu-\frac{n-5}{2}\right)\Gamma\left(m_1+\frac{n-3}{2}\right)}{\sqrt{\pi}\Gamma\left(i\nu-m_1-\frac{n-5}{2}\right)}\left[\frac{(n-3)!(m-m_1)!(2m+n-3)}{(m+m_1+n-4)!}\right]^{1/2}$$
$$\times\int_0^\infty \mathfrak{P}_{i\nu-1/2}^{-m_1-\frac{n-4}{2}}(\cosh\theta)C_{m-m_1}^{m_1+\frac{n-3}{2}}\left(\frac{\varepsilon}{\cosh\theta}\right)\tanh^{m_1}\theta\cosh^{\sigma}\theta\sinh^{(n-2)/2}\theta d\theta. \quad (2)$$

Since for Gegenbauer polynomials the expansion

$$C_{m-m_1}^{m_1+p}(x)=\frac{1}{\Gamma(m_1+p)}\sum_{j=0}^{\left[\frac{m-m_1}{2}\right]}\frac{(-1)^j\Gamma(m+p-j)}{j!(m-m_1-2j)!}(2x)^{m-m_1-2j} \quad (3)$$

is valid (see Section 6.3.9), then (2) implies that

$$E_+^{n\sigma}(m,\nu,m_1)=(-1)^{m-m_1}E_-^{n\sigma}(m,\nu,m_1). \quad (4)$$

Using in (2) expansion (3) for $C_{m-m_1}^{m_1+(n-3)/2}\left(\frac{1}{\cosh\theta}\right)$ and taking into account the formula

$$\int_0^\infty \mathfrak{P}_\nu^\mu(\cosh\varphi)\cosh^{-\rho}\varphi\sinh^{1-\mu}\varphi d\varphi$$
$$=\frac{2^{\rho+\mu-2}}{\sqrt{\pi}\,\Gamma(\rho)}\Gamma\left(\frac{1}{2}(\rho+\mu+\nu)\right)\Gamma\left(\frac{1}{2}(\rho+\mu-\nu-1)\right),$$
$$\operatorname{Re}\mu<1,\ \operatorname{Re}(\rho+\mu+\nu)>0,\ \operatorname{Re}(\rho+\mu-\nu)>1,$$

which follows from equalities (8) and (9) of Section 9.3.6, we have

$$E_+^{n\sigma}(m,\nu,m_1)=\frac{\sqrt{\pi}\,c_{\nu m m_1}2^{-m_1-\frac{n-3}{2}}}{\Gamma\left(m_1+\frac{n-3}{2}\right)}\sum_{k=0}^{\left[\frac{m-m_1}{2}\right]}\frac{(-1)^k}{k!} \quad (5)$$
$$\times\frac{\Gamma\left(m-k+\frac{n-3}{2}\right)\Gamma\left(\frac{1}{2}\left(m-m_1-i\nu-\sigma-2k-\frac{n-3}{2}\right)\right)\Gamma\left(\frac{1}{2}\left(m-m_1+i\nu-\sigma-2k-\frac{n-3}{2}\right)\right)}{\Gamma\left(\frac{1}{2}(m-m_1+1)-k\right)\Gamma\left(\frac{1}{2}(m-m_1+2)-k\right)\Gamma\left(\frac{1}{2}(m-\sigma)-k\right)\Gamma\left(\frac{1}{2}(m-\sigma+1)-k\right)},$$

where $c_{\nu m m_1}$ is the coefficient before the integral on the right hand side of (2). Expressing the sum in terms of the hypergeometric series ${}_4F_3(\ldots;1)$ (see Section 3.5.1), we obtain

$$E_+^{n\sigma}(m,\nu,m_1)$$
$$=E^{n\sigma}{}_4F_3\left(\begin{matrix}-\frac{m-m_1-1}{2},\ -\frac{m-m_1}{2},\ \frac{\sigma-m+2}{2},\ \frac{\sigma-m+1}{2}\\ -m-\frac{n+5}{2},\frac{1}{2}\left(\sigma+i\nu-m+m_1+\frac{n-1}{2}\right),\frac{1}{2}\left(\sigma-i\nu-m+m_1+\frac{n+1}{2}\right)\end{matrix}\middle|1\right), \quad (6)$$

where

$$E^{n\sigma} = \frac{(-1)^{m_1}2^{2m-2m_1-\sigma-3/2}\Gamma\left(i\nu-\frac{n-5}{2}\right)\Gamma\left(m+\frac{n-3}{2}\right)\Gamma\left(\frac{1}{2}\left(m-m_1-i\nu-\sigma-\frac{n-3}{2}\right)\right)}{\pi\Gamma(m-\sigma)\Gamma\left(i\nu-m_1-\frac{n-5}{2}\right)}$$
$$\times\Gamma\left(\frac{1}{2}\left(m-m_1+i\nu-\sigma-\frac{n-3}{2}\right)\right)\left[\frac{(n-3)!(2m+n-3)}{(m-m_1)!(m+m_1+n-4)!}\right]^{1/2}.$$

Another expression for $E_+^{n\sigma}(m,\nu,m_1)$ in terms of hypergeometric series is possible. Namely, replacing in (5) the summation index $k$ by $k'=\left[\frac{m-m_1}{2}\right]-k$, we have for even $m-m_1$ that

$$E_+^{n\sigma}(m,\nu,m_1) = \frac{(-1)^{(m-m_1)/2}c_{\nu m m_1}\Gamma\left(\frac{1}{2}(m+m_1+n-2)\right)\Gamma\left(\frac{1}{2}\left(-i\nu-\sigma-\frac{n-3}{2}\right)\right)\Gamma\left(\frac{1}{2}\left(i\nu-\sigma-\frac{n-3}{2}\right)\right)}{\sqrt{\pi}\,2^{-m+\sigma+(n-1)/2}\Gamma\left(m_1+\frac{n-3}{2}\right)\Gamma(m+m_1-\sigma)}$$
$$\times\,{}_4F_3\left(\begin{matrix}-\frac{m-m_1}{2},\ \frac{m+m_1+n-3}{2},\ \frac{1}{2}\left(-i\nu-\sigma-\frac{n-3}{2}\right),\ \frac{1}{2}\left(i\nu-\sigma-\frac{n-3}{2}\right)\\ \frac{1}{2},\ \frac{1}{2}(m+m_1-\sigma),\ \frac{1}{2}(m+m_1-\sigma+1)\end{matrix}\,\middle|\,1\right). \quad (7)$$

Making use of expression (1) of Section 8.5.5 for the Wilson polynomials $p_n(t^2;a,b,c,d)$, we derive from (7) that for even $m-m_1$ the equality

$$E_+^{n\sigma}(m,\nu,m_1) = \frac{(-1)^{(m-m_1)/2}c_{\nu m m_1}\Gamma\left(\frac{1}{2}(m+m_1+n-3)\right)\Gamma\left(\frac{1}{2}\left(i\nu-\sigma-\frac{n-3}{2}\right)\right)\Gamma\left(\frac{1}{2}\left(-i\nu-\sigma-\frac{n-3}{2}\right)\right)}{\sqrt{\pi}\,2^{\sigma+m_1-3m+1}(m-m_1)!\Gamma(2m-\sigma)}$$
$$\times\, p_{\frac{m-m_1}{2}}\left(-\frac{\nu^2}{4};\frac{1}{2}\left(-i\rho+\frac{1}{2}\right),\frac{1}{2}\left(i\rho+\frac{1}{2}\right),\frac{1}{2}\left(m+m_1+\frac{n-3}{2}\right),\frac{1}{2}\left(m+m_1+\frac{n-1}{2}\right)\right) \quad (8)$$

is valid, where $\sigma=i\rho-\frac{n-2}{2}$. In the same way we derive for odd $m-m_1$ the equality

$$E_+^{n\sigma}(m,\nu,m_1) = (-1)^{(m-m_1-1)/2}$$
$$\times\frac{c_{\nu m m_1}\Gamma\left(\frac{1}{2}(m+m_1+n-2)\right)\Gamma\left(\frac{1}{2}\left(-i\nu-\sigma\frac{n-5}{2}\right)\right)\Gamma\left(\frac{1}{2}\left(i\nu-\sigma-\frac{n-5}{2}\right)\right)}{\sqrt{\pi}\,2^{\sigma+m_1-3m}(m-m_1)!\Gamma(2m-\sigma+1)}$$
$$\times\, p_{\frac{m-m_1-1}{2}}\left(-\frac{\nu^2}{4};\frac{1}{2}\left(-i\rho+\frac{3}{2}\right),\frac{1}{2}\left(i\rho+\frac{3}{2}\right),\frac{1}{2}\left(m+m_1+\frac{n-3}{2}\right),\frac{1}{2}\left(m+m_1+\frac{n-1}{2}\right)\right). \quad (9)$$

**10.6.5. The coefficients $C^{n\sigma}(m,R,m_1)$ and Maijer $G$-functions.** It follows from formula (2) of Section 10.6.2 that

$$\overline{\widehat{\Xi}_{RM}^{n,-\bar\sigma-n+2,2}(\xi)} = \sum_{m-m_1}^{\infty} C^{n\sigma}(m,R,m_1)\overline{\widehat{\Xi}_{mM}^{n,-\bar\sigma-n+2,0}(\xi)},\ \xi\in C_+^{n-1}. \quad (1)$$

Multiply both sides of this equality by $[\mathbf{x},\boldsymbol{\xi}]^\sigma$, $x \in H_+^{n-1}$, and integrate over a contour $\Gamma$ on the cone $C_+^{n-1}$:

$$\int_\Gamma [\mathbf{x},\boldsymbol{\xi}]^\sigma \overline{\widehat{\Xi}_{RM}^{n,-\bar\sigma-n+2,2}(\boldsymbol{\xi})} d\boldsymbol{\xi}$$

$$= \sum_{m=m_1}^{\infty} C^{n\sigma}(m,R,m_1) \int_\Gamma [\mathbf{x},\boldsymbol{\xi}]^\sigma \overline{\widehat{\Xi}_{mM}^{n,-\bar\sigma-n+2,0}(\boldsymbol{\xi})} d\boldsymbol{\xi}. \qquad (2)$$

Since the integrals do not depend on the choice of $\Gamma$, then we choose on the right hand side $\Gamma = S^{n-2}$ and on the left hand side $\Gamma = \Gamma_2$, where $\Gamma_2$ is defined in Section 10.3.1. Setting $\mathbf{x} = g'_{n-1}(t)\mathbf{x}_0$, where $\mathbf{x}_0 = (0,\ldots,0,1)$, we represent $\mathbf{x}$ on the right hand side in the form $\mathbf{x} = (0,\ldots,0,\sinh t,\cosh t)$ and on the left hand side in the form

$$\mathbf{x} = \left(0,\ldots,0,\frac{1}{2}\left(\frac{1}{r}-r\right),\frac{1}{2}\left(\frac{1}{r}+r\right)\right), \quad r = \exp(-t).$$

Then for $M = O$ the integral on the right hand side coincides with $t^{n\sigma}_{m0}(g'_{n-1}(t))$ and on the left hand side with

$$\frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}} \int_0^\infty \left[\frac{1}{2r}(r^2+y^2)\right]^\sigma \left(\frac{Ry}{2}\right)^{-\frac{n-4}{2}} J_{\frac{n-4}{2}}(Ry) y^{n-3} dy.$$

This integral is calculated in the same way as the integral from formula (3) of Section 10.6.2 and for the integral on the left hand side of (2) we have the expression

$$\frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,\Gamma(-\sigma)} 2^{-\sigma+n-5} R^{4-n} r^{\sigma+2} G^{21}_{13}\left(\frac{R^2r^2}{4}\middle|\begin{matrix}0\\ -\sigma-1,\frac{n-4}{2},0\end{matrix}\right).$$

Thus, formula (2) leads to the relation

$$\sum_{m=0}^{\infty}\sum_{k=0}^{m} A^\sigma_{mk} \mathfrak{P}^{-m-p}_{\sigma+p}(\cosh t) G^{21}_{13}\left(\frac{R^2}{4}\middle|\begin{matrix}-k\\ -\sigma-1,p-\frac{1}{2},0\end{matrix}\right)$$

$$= \frac{2^p \sin\pi\sigma}{\pi^{3/2}} e^{-(\sigma+2)t} \sinh^p t\, G^{21}_{13}\left(\frac{e^{-2t}R^2}{4}\middle|\begin{matrix}0\\ -\sigma-1,p-\frac{1}{2},0\end{matrix}\right), \qquad (3)$$

where $p = (n-3)/2$ and

$$A^\sigma_{mk} = \frac{(-1)^{m+k}(m+k+2p-1)!(2m+2p)}{k!\Gamma\left(k+p+\frac{1}{2}\right)\Gamma(k-\sigma)\Gamma(\sigma-m+1)(m-k)!}.$$

The transform

$$\widehat{\Xi}^{n\sigma,0}_{mM}(\xi) = \int_0^\infty C^{n\sigma}(m,R,m_1)\widehat{\Xi}^{n\sigma,2}_{RM}(\xi)dR, \quad \xi \in C_+^{n-1}, \tag{4}$$

is reciprocal to relation (2) of Section 10.6.2. Multiply both sides of (4) by $[\mathbf{x},\xi]^{-\sigma-n+2}$, $\mathbf{x} \in H_+^{n-1}$, and integrate over a contour $\Gamma$ on the cone. We set $\Gamma = S^{n-2}$ on the left and $\Gamma = \Gamma_2$ on the right. As in the case of formula (2), after calculating the integrals and substituting value (4) of Section 10.6.2 for $C^{n\sigma}(m,R,m_1)$, we obtain for $m_1 = 0$ the relation

$$\sum_{k=0}^{m} \frac{(-1)^k(m+k+n-4)!}{k!\Gamma\left(k+\frac{n-2}{2}\right)\Gamma(k-\sigma)(m-k)!} \int_0^\infty G^{21}_{13}\left(\frac{R^2}{4}\middle|\begin{matrix}-k\\ -\sigma-1,\frac{n-4}{2},0\end{matrix}\right)$$
$$\times G^{21}_{13}\left(\frac{e^{-2t}R^2}{4}\middle|\begin{matrix}0\\ \sigma+n-3,\frac{n-4}{2},0\end{matrix}\right) R^{8-2n}dR$$
$$= \frac{(-1)^{m+n}\pi^{3/2}(m+n-4)!e^{-(\sigma+n-4)t}}{8^{(n+5)/2}(\sin\sigma\pi)m!(n-3)!\Gamma(-\sigma-m-n+3)} \sinh^{\frac{3-n}{2}} t\,\mathfrak{P}^{-m-\frac{n-3}{2}}_{-\sigma-\frac{n-3}{2}}(\cosh t). \tag{5}$$

By setting $m = 0$ we derive the equality

$$\int_0^\infty G^{21}_{13}\left(\frac{R^2}{4}\middle|\begin{matrix}0\\ -\sigma-1,\frac{p-1}{2},0\end{matrix}\right) G^{21}_{13}\left(\frac{e^{-2t}R^2}{4}\middle|\begin{matrix}0\\ \sigma+p,\frac{p-1}{2},0\end{matrix}\right) R^{2-2p}dR$$
$$= \frac{(-1)^{p+1}\pi^{3/2}\Gamma\left(\frac{p+1}{2}\right)\Gamma(-\sigma)e^{-(\sigma+p-1)t}}{8^{(p+8)/2}(\sin\sigma\pi)p!\Gamma(-\sigma-p)} \sinh^{-p/2} t\,\mathfrak{P}^{-p/2}_{-\sigma-p/2}(\cosh t) \tag{6}$$

(we have replaced $n-3$ by $p$).

In order to obtain other relations for Maijer $G$-functions we consider the operator $T^{n\sigma}(s)$ of the representation $T^{n\sigma}$ of the group $SO_0(n-1,1)$, where $s = \mathrm{diag}(1,\ldots,1,-1,-1,1)$. Function (12) of Section 10.3.1 can be represented in the form

$$\widehat{\Xi}^{n\sigma,2}_{RM}(\xi) = i^{m_1}\Gamma\left(\frac{n-2}{2}\right)(\dim T^{n-2,m_1})^{1/2}(\xi_{n-1}+\xi_n)^\sigma \left(\frac{R}{2}\sqrt{\frac{\xi_n-\xi_{n-1}}{\xi_n+\xi_{n-1}}}\right)^{\frac{4-n}{2}}$$
$$\times J_{m_1+\frac{n-4}{2}}\left(R\sqrt{\frac{\xi_n-\xi_{n-1}}{\xi_n+\xi_{n-1}}}\right) X^{n-2,m_1}_{M_1}(\xi_1,\ldots,\xi_{n-2}),$$

where $M = (m_1, M)$. We have

$$(T^{n\sigma}(s)\widehat{\Xi}^{n\sigma,2}_{RM})(\boldsymbol{\xi})$$

$$= i^{m_1}\Gamma\left(\frac{n-2}{2}\right)(\dim T^{n-2,m_1})^{1/2}(\xi_n - \xi_{n-1})^\sigma \left(\frac{R}{2}\sqrt{\frac{\xi_n+\xi_{n-1}}{\xi_n-\xi_{n-1}}}\right)^{\frac{4-n}{2}}$$

$$\times J_{m_1+\frac{n-4}{2}}\left(R\sqrt{\frac{\xi_n+\xi_{n-1}}{\xi_n-\xi_{n-1}}}\right) X^{n-2,m_1}_{M_1}(\xi_1,\ldots,\xi_{n-3},-\xi_{n-2}).$$

Therefore, for the kernel $K(R,R',m)$ of the operator $T^{n\sigma}(s)$ we derive the expression

$$K(R,R',m) \equiv (T^{n\sigma}(s)\widehat{\Xi}^{n\sigma,2}_{R'M}, \widehat{\Xi}^{n,-\bar{\sigma}-n+2,2}_{RM})$$

$$= \frac{(n-3)!}{2}(RR')^{\frac{4-n}{2}} \int_0^\infty J_{m+\frac{n-4}{2}}(Rr) J_{m+\frac{n-4}{2}}\left(\frac{R'}{r}\right) r^{2\sigma+n-3}dr, \quad (7)$$

where $M = (m, M_1)$. For $M \ne M'$ we have

$$(T^{n\sigma}(s)\widehat{\Xi}^{n\sigma,2}_{R'M'}, \widehat{\Xi}^{n,-\bar{\sigma}-n+2,2}_{RM}) = 0.$$

Let us find the integral on the right hand side of (7). To do this we note that for Bessel function the integral representation

$$J_\nu(x) = \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{2}\right)^{-t}\frac{\Gamma\left(\frac{\nu+t}{2}\right)}{\Gamma\left(1+\frac{\nu-t}{2}\right)}dt, \quad x>0,\ -\mathrm{Re}\,\nu < c < 1, \quad (8)$$

is valid. To prove it one has to calculate the integral by means of the residue theorem and to compare the result with formula (1) of Section 3.5.6. Applying the inversion formula for the Mellin transform (see Section 3.3.4) to (8), we have

$$\int_0^\infty J_\nu(ax)x^{t-1}dx = \frac{2^{t-1}\Gamma\left(\frac{\nu+t}{2}\right)}{a^t\Gamma\left(1+\frac{\nu-t}{2}\right)}, \quad (9)$$

where $a>0$, $-\mathrm{Re}\,\nu < \mathrm{Re}\,t < \frac{3}{2}$. This formula implies that

$$\int_0^\infty x^{t-1}J_\nu(ax)J_\mu\left(\frac{b}{x}\right)dx = \frac{a^{\mu-t}b^\mu\Gamma\left(\frac{1}{2}(\mu+t-\nu)\right)}{2^{2\mu-t+1}\Gamma(\mu+1)\Gamma\left(\frac{1}{2}(\nu+\mu-t+2)\right)}$$

$$\times\, {}_0F_3\left(\mu+1, \frac{1}{2}(\mu-\nu-t+2), \frac{1}{2}(\mu+\nu-t+2); \frac{1}{16}a^2b^2\right)$$

$$+\frac{a^\nu b^{\nu+t}\Gamma\left(\frac{1}{2}(\mu-\nu-t)\right)}{2^{2\nu+t+1}\Gamma(\nu+1)\Gamma\left(\frac{1}{2}(\mu+\nu+t+2)\right)}$$

$$\times\, {}_0F_3\left(\nu+1, \frac{1}{2}(\nu-\mu+t+2), \frac{1}{2}(\nu+\mu+t+2); \frac{1}{16}a^2b^2\right), \quad (10)$$

where $a > 0$, $b > 0$, $-\mathrm{Re}\left(\nu + \frac{3}{2}\right) < \mathrm{Re}\, t < \mathrm{Re}\left(\mu + \frac{3}{2}\right)$. Indeed, after expanding $J_\mu(b/x)$ in powers of $b/x$, integrating term-by-term by means of (9) and taking into consideration expression (2) of Section 3.5.1 for ${}_0F_3(\ldots;x)$, we obtain the right hand side of (10).

Using (10) in formula (7), we find that

$$K(R,R',m) = \frac{(n-3)!}{\Gamma\left(m+\frac{n-2}{2}\right)}\left[\frac{R^{m-2\sigma-n+2}R'^{m}\Gamma\left(\sigma+\frac{n-2}{2}\right)}{2^{2m-2}\Gamma(m-\sigma)}\right.$$
$$\times\, {}_0F_3\left(m+\frac{n}{2}, -\sigma-\frac{n-4}{2}, m-\sigma; \frac{1}{16}(RR')^2\right)$$
$$\left.+\frac{R^m R'^{m+2\sigma+n-2}\Gamma\left(-\sigma-\frac{n-2}{2}\right)}{2^{2m+2\sigma+2n-4}\Gamma(m+\sigma+n-2)}{}_0F_3\left(m+\tfrac{n-2}{2}, \sigma+\tfrac{n}{2}, m+\sigma+n-2; \tfrac{1}{16}(RR')^2\right)\right]. \quad (11)$$

For the basis $\{\widehat{\Xi}^{n\sigma,0}_{pM}\}$, $M = (m, M_1)$, we have

$$T^{n\sigma}(s)\widehat{\Xi}^{n\sigma,0}_{pM} = (-1)^p \Xi^{n\sigma,0}_{pM}. \quad (12)$$

Therefore, for $M = (m, M_1)$, one obtains that

$$\left(T^{n\sigma}(s)\widehat{\Xi}^{n\sigma,0}_{pM}, \widehat{\Xi}^{n,-\bar\sigma-n+2,2}_{RM}\right) = (-1)^p \left(\widehat{\Xi}^{n\sigma,0}_{pM}, \widehat{\Xi}^{n,-\bar\sigma-n+2,2}_{RM}\right) = (-1)^p C^{n\sigma}(p,R,m).$$

Substituting the expression

$$\widehat{\Xi}^{n\sigma,0}_{pM}(\boldsymbol{\xi}) = \int_0^\infty C^{n\sigma}(p,R',m)\Xi^{n\sigma,2}_{R'M}(\boldsymbol{\xi})dR', \quad \boldsymbol{\xi} \in C^{n-1}_+,$$

for $\widehat{\Xi}^{n\sigma,0}_{pM}$ into the left hand side, we have that

$$\int_0^\infty K(R,R',m)C^{n\sigma}(p,R',m)dR' = (-1)^{p-m}C^{n\sigma}(p,R,m).$$

Replacing $C^{n\sigma}(p,R,m)$ and $K(R,R',m)$ by expression (4) of Section 10.6.2 and (11), we obtain the relation

$$\int_0^\infty G^{n\sigma,21}_{pm,13}(R')\left[\frac{R^{-2\sigma-n+2}R'^{4-n}\Gamma\left(\sigma+\frac{n-2}{2}\right)}{2^{2m-2\sigma}\Gamma(m-\sigma)}{}_0F_3\left(m+\tfrac{n}{2}, -\sigma-\tfrac{n-4}{2}, m-\sigma; \tfrac{1}{16}(RR')^2\right)\right.$$
$$\left.+\frac{R'^{2\sigma+2}\Gamma\left(-\sigma-\frac{n-2}{2}\right)}{2^{2\sigma+2m+2n-4}\Gamma(\sigma+m+n-2)}{}_0F_3\left(m+\tfrac{n-2}{2}, \sigma+\tfrac{n}{2}, m+\sigma+n-2; \tfrac{1}{16}(RR')^2\right)\right]dR'$$
$$= \frac{\Gamma\left(m+\frac{n-2}{2}\right)}{(n-3)!}R^{4-2m-n}G^{n\sigma,21}_{pm,13}(R), \quad (13)$$

where $G^{n\sigma,21}_{pm,13}(R)$ is given by formula (6) of Section 10.6.2. For $p=m=0$, it takes the form

$$\int_0^\infty \left[\frac{R^{-2\sigma-2p}R'^{2-2p}\Gamma(\sigma+p)}{2^{-2\sigma}\Gamma(-\sigma)}{}_0F_3\left(p+1,-\sigma-p+1,-\sigma;\frac{1}{16}(RR')^2\right)\right.$$
$$\left.+\frac{R'^{2\sigma+2}\Gamma(-\sigma-p)}{2^{2\sigma+4p}\Gamma(\sigma+2p)}{}_0F_3\left(p,\sigma+p+1,\sigma+2p;\frac{1}{16}(RR')^2\right)\right]$$
$$\times G^{21}_{13}\left(\frac{R'^2}{4}\middle|\begin{matrix}0\\ -\sigma,0,p-1\end{matrix}\right)dR'=\frac{\Gamma(p)}{(2p-1)!}G^{21}_{13}\left(\frac{R^2}{4}\middle|\begin{matrix}0\\ -\sigma,0,p-1\end{matrix}\right). \quad (14)$$

In the same way, the relation

$$\left(T^{n\sigma}(s)\widehat{\Xi}^{n\sigma,2}_{RM},\widehat{\Xi}^{n,-\bar{\sigma}-n+2,2}_{R'M}\right)=\sum_p C^{n\sigma}(p,R,m)C^{n,-\bar{\sigma}-n+2}(p,R',m) \quad (15)$$

(here we have taken into account formula (12)) implies

$$\sum_p(-1)^pA^{n\sigma}_{pm}A^{n,-\sigma-n+2}_{pm}(RR')^{4-2m-n}G^{n\sigma,21}_{pm,13}(R)G^{n,-\sigma-n+2,21}_{pm,13}(R')$$
$$=\frac{(n-3)!}{\Gamma\left(m+\frac{n-2}{2}\right)}\left[\frac{R^{-2\sigma-n+2}\Gamma\left(\sigma+\frac{n-2}{2}\right)}{2^{2m-2\sigma}\Gamma(m-\sigma)}\right.$$
$$\times{}_0F_3\left(m+\tfrac{n}{2},-\sigma-\tfrac{n-4}{2},m-\sigma;\tfrac{1}{16}(RR')^2\right)$$
$$+\frac{R'^{2\sigma+n-2}\Gamma\left(-\sigma-\frac{n-2}{2}\right)}{2^{2\sigma+2m+2n-4}\Gamma(\sigma+m+n-2)}$$
$$\times{}_0F_3\left(m+\tfrac{n-2}{2},\sigma+\tfrac{n-2}{2},m+\sigma+n-2;\tfrac{1}{16}(RR')^2\right), \quad (16)$$

where $A^{n\sigma}_{pm}$ is given by formula (5) of Section 10.6.2.

**10.6.6. The coefficients $D^{n\sigma}_{\pm}(\nu,R,m)$ and Maijer $G$-functions.** It follows from definition (1) of Section 10.6.3 for the coefficients $D^{n\sigma}_{\pm}(\nu,R,m)$ that

$$\widehat{\Xi}^{n\sigma,1}_{\nu M\epsilon}(\boldsymbol{\xi})=\int_0^\infty D^{n\sigma}_{\epsilon}(\nu,R,m)\widehat{\Xi}^{n\sigma,2}_{RM}(\boldsymbol{\xi})dR,\quad \boldsymbol{\xi}\in C^{n-1}_+, \quad (1)$$

where $M=(m,M_1)$. Let us multiply both sides of (1) by $[\mathbf{x},\boldsymbol{\xi}]^{-\sigma-n+2}$, $\mathbf{x}\in H^{n-1}_+$, and let us integrate over a contour $\Gamma$ on the cone. We set $\Gamma=\Gamma^{\epsilon}_1$ on the left hand side and $\Gamma=\Gamma_2$ on the right hand side. By virtue of the results of Sections 10.3.2

and 10.3.3, after some calculations for $\varepsilon=+$, $m=0$ we obtain

$$A^{n\sigma}_{\nu0+}\int\limits_0^\infty G^{n\sigma,11}_{\nu0,13}(R)G^{21}_{13}\left(\frac{e^{-2t}R^2}{4}\,\middle|\,\begin{matrix}0\\ \sigma+n-3,\frac{n-4}{2},0\end{matrix}\right)R^{8-2n}dR \tag{2}$$

$$=\frac{\pi^{\frac{n-1}{2}}\Gamma\left(\sigma-i\nu+\frac{n-1}{2}\right)\Gamma\left(\sigma+i\nu+\frac{n-1}{2}\right)}{2^{\sigma-6+3n/2}\Gamma\left(\frac{n-1}{2}\right)}e^{-(\sigma+n-4)t}\cosh^{\frac{2-n}{2}}t\,\mathrm{P}^{-\sigma-\frac{n-2}{2}}_{i\nu-1/2}(\tanh t),$$

where $A^{n\sigma}_{\nu0+}$ and $G^{n\sigma,11}_{\nu0,13}$ are defined by formulas (4) and (5) of Section 10.6.3. For $\varepsilon=-$, $m=0$ we have the relation

$$A^{n\sigma}_{\nu0-}\int\limits_0^\infty G^{n\sigma,20}_{\nu0,13}(R)G^{21}_{13}\left(\frac{e^{-2t}R^2}{4}\,\middle|\,\begin{matrix}0\\ \sigma+n-3,\frac{n-4}{2},0\end{matrix}\right)R^{4-n}dR \tag{3}$$

$$=\frac{\pi^{\frac{n-1}{2}}\Gamma\left(\sigma-i\nu+\frac{n-1}{2}\right)\Gamma\left(\sigma+i\nu+\frac{n-1}{2}\right)}{2^{\sigma-6+3n/2}\Gamma\left(\frac{n-1}{2}\right)}e^{-(\sigma+n-4)t}\cosh^{\frac{2-n}{2}}t\,\mathrm{P}^{\sigma-\frac{n-2}{2}}_{i\nu-1/2}(-\tanh t),$$

where $A^{n\sigma}_{\nu0-}$ and $G^{n\sigma,20}_{\nu0,13}$ are defined by formulas (7) and (8) of Section 10.6.3.

According to the definition of $D^{n\sigma}_{\pm}(\nu,R,m)$ we have

$$\widehat{\Xi}^{n,-\sigma-n+2,2}_{RM}(\eta)=c_{n-1}a\int\limits_{b-i\infty}^{b+i\infty}D^{n\sigma}_{+}(\nu,R,m)\widehat{\Xi}^{n,-\sigma-n+2,1}_{\nu M+}(\eta)d\mu_{n-1}(\nu)$$
$$+c_{n-1}a\int\limits_{b-i\infty}^{b+i\infty}D^{n\sigma}_{-}(\nu,R,m)\widehat{\Xi}^{n,-\sigma-n+2,1}_{\nu M-}(\eta)d\mu_{n-1}(\nu), \tag{4}$$

where $0<\operatorname{Re}\sigma>-n+2$, $\tau=i\nu-\frac{n-3}{2}$, $a=\Gamma\left(\frac{n-2}{2}\right)/2\pi^{(n-2)/2}$, and $c_{n-1}$ and $d\mu_{n-1}(\nu)$ are the same as in Section 10.3.2. Let us multiply both sides of this equality by $[\mathbf{x},\eta]^\sigma$, $\mathbf{x}\in H^{n-1}_{+}$, and integrate over a contour $\Gamma$ on the cone. We set $\Gamma=\Gamma_2$ on the left hand side and $\Gamma=\Gamma_1^\varepsilon$ on the right hand side. After simple calculations we obtain that

$$c_{n-1}\int\limits_{b-i\infty}^{b+i\infty}\Gamma\left(-i\nu-\sigma-\frac{n-3}{2}\right)\Gamma\left(i\nu-\sigma-\frac{n-3}{2}\right)\Big[A^{n\sigma}_{\nu0+}G^{n\sigma,11}_{\nu0,13}(R)R^{4-n}$$
$$\times\mathrm{P}^{\sigma+\frac{n-2}{2}}_{i\nu-1/2}(\tanh t)+A^{n\sigma}_{\nu0-}G^{n\sigma,20}_{\nu0,13}(R)\mathrm{P}^{\sigma+\frac{n-2}{2}}_{i\nu-1/2}(-\tanh t)\Big]d\mu_{n-1}(\nu) \tag{5}$$
$$=\frac{2^{-\sigma+n-5}\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}R^{m-n+4}\cosh^{\frac{n-2}{2}}t\,e^{-(\sigma+2)t}G^{21}_{13}\left(\frac{e^{-2t}R^2}{4}\,\middle|\,\begin{matrix}0\\ -\sigma-1,\frac{n-4}{2},0\end{matrix}\right).$$

**10.6.7. The coefficients $E_\varepsilon^{n\sigma}(m,\nu,m_1)$ and associated Legendre functions.** According to formula (1) of Section 10.6.4, we have

$$\overline{\widehat{\Xi}_{\nu M\varepsilon}^{n,-\bar\sigma-n+2,1}(\eta)} = \sum_{m=0}^{\infty} E_\varepsilon^{n\sigma}(m,\nu,m_1)\overline{\widehat{\Xi}_{mM}^{n,-\bar\sigma-n+2,0}(\eta)},\ \ \eta\in C_+^{n-1}, \qquad (1)$$

where $M=(m_1,M_1)$. Multiply both sides of (1) by $[\mathbf{x},\eta]^{-\sigma-n+2}$, $\mathbf{x}\in H_+^{n-1}$, and integrate on the left hand side over the contour $\Gamma_1^\varepsilon$ and on the right hand side over $S^{n-2}$. Setting $\varepsilon=+$, $M=0$, after simplifications we obtain

$$\sum_{m=0}^{\infty}\varepsilon^m A_m\ {}_4F_3\left(\begin{matrix}-\frac12(m-1),\ -\frac12 m,\ \frac12(\sigma-m+2),\ \frac12(\sigma-m+1)\\ -m-p-4,\ \frac12(\sigma+i\nu-m+p+2),\ \frac12(\sigma-i\nu-m+p+2)\end{matrix}\middle|\,1\right)$$
$$\times\,\mathfrak{P}_{\sigma+p}^{-m-p}(\cosh t) = \frac{\tanh t}{\sqrt{\cosh t}}\mathrm{P}_{i\nu-1/2}^{\sigma+p+\frac12}(\varepsilon\tanh t), \qquad (2)$$

where $\varepsilon=+$ or $-$, $p=\frac{n-3}{2}$ and

$$A_m=\frac{2^{2m-6}\Gamma(p+1)\Gamma(m+p+1)\Gamma\left(\frac12(m+i\nu-\sigma-p)\right)\Gamma\left(\frac12(m-i\nu-\sigma-p)\right)}{\pi^{m-p+3/2}m!\Gamma(i\nu-\sigma-p)\Gamma(-i\nu-\sigma-p)}.$$

The function ${}_4F_3(\dots;1)$ from (2) can be expressed in terms of a Wilson polynomial.

One more equality connecting associated Legendre function and Wilson polynomials is obtained from the relation

$$\widehat{\Xi}_{mM}^{n\sigma,0}(\eta)=c_{n-1}a\int_{b-i\infty}^{b+i\infty}\left[E_+^{n\sigma}(m,\nu,m_1)\widehat{\Xi}_{\nu M+}^{n\sigma,1}(\eta)+\right.$$
$$\left.+E_-^{n\sigma}(m,\nu,m_1)\widehat{\Xi}_{\nu M-}^{n\sigma,1}(\eta)\right]d\mu_{n-1}(\nu), \qquad (3)$$

where $-n+3<b<0$, $\tau=i\nu-\frac{n-3}{2}$, $a=\Gamma\left(\frac{n-2}{2}\right)/2\pi^{(n-2)/2}$ and $c_{n-1}$, $d\mu_{n-1}(\nu)$ are the same as in Section 10.3.2. Multiply both sides of (3) by $[\mathbf{x},\eta]^{-\sigma-n+2}$, $\mathbf{x}\in H_+^{n-1}$, and integrate over the contour $S^{n-2}$ on the left hand side and over $\Gamma_1^\pm$ on the right hand side. We have

$$c_{n-1}\int_{b-i\infty}^{b+i\infty}E_+^{n\sigma}(m,\nu,0)\left[\mathrm{P}_{i\nu-1/2}^{-\sigma-p-\frac12}(\tanh t)+(-1)^m\mathrm{P}_{i\nu-1/2}^{-\sigma-p-\frac12}(-\tanh t)\right]$$
$$\times\,|\Gamma(i\nu+\sigma+p+1)|^2d\mu_{n-1}(\nu)$$
$$=\frac{(-1)^{m+2p+3}\pi 2^p\Gamma(p+1)(\dim T^{n-1,m})^{1/2}}{(\sin\sigma\pi)\Gamma\left(p+\frac12\right)\Gamma(-\sigma-m-2p)}\frac{\sqrt{\cosh t}}{\tanh^p t}\mathfrak{P}_{-\sigma-p-1}^{-m-p}(\cosh t), \qquad (4)$$

where $p = (n-3)/2$.

## 10.7. Representations of the Group $ISO_0(n-2,1)$ and Special Functions

**10.7.1. Irreducible representations of the group $ISO_0(n-2,1)$.** In Section 9.2.4 we have constructed the representations $T^{nR}$ of the group $ISO_0(n-2,1)$ which act in the space $\mathfrak{T}$ of infinitely differentiable functions $f(\boldsymbol{\xi})$ on the hyperboloid $H_+^{n-2}$ and are given by the formula

$$(T^{nR}(g)f)(\boldsymbol{\xi}) = e^{-R[\mathbf{a},\boldsymbol{\xi}]} f(h^{-1}\boldsymbol{\xi}), \tag{1}$$

where $g = g(h, \mathbf{a}) \in ISO_0(n-2,1)$ (see Section 9.1.3).

The restriction of $T^{nR}$ onto $SO_0(n-2,1)$ coincides with the quasi-regular representation of the group $SO_0(n-2,1)$ in the space $\mathfrak{T}$, and its restriction onto the subgroup $\mathbb{R}^{n-1}$ is given by the formula

$$(T^{nR}(g(e,\mathbf{a}))f)(\boldsymbol{\xi}) = e^{-R[\mathbf{a},\boldsymbol{\xi}]} f(\boldsymbol{\xi}). \tag{2}$$

The infinitesimal operators $A_j$, $1 \le j \le n-1$, corresponding to the one-parameter subgroups of parallel translations along the axes $\xi_j$, act on $\mathfrak{T}$ by the formula

$$(A_j f)(\boldsymbol{\xi}) = -R\xi_j f(\boldsymbol{\xi}).$$

We show that for $R \neq 0$ the representations $T^{nR}$ are operator irreducible. Actually, if for all $g \in ISO_0(n-2,1)$ we have

$$BT^{nR}(g) = T^{nR}(g)B,$$

then the operator $B$ commutes with the infinitesimal operators $A_j$, that is, with all operators of multiplication by $\xi_j$, $1 \le j \le n-1$. Therefore, $B$ is the operator of multiplication by a function: $(Bf)(\boldsymbol{\xi}) = \varphi(\boldsymbol{\xi})f(\boldsymbol{\xi})$. The commutativity of $B$ with all operators $T^{nR}(g(h,0))$ (that is, with shifts by elements $h \in SO_0(n-2,1)$) implies that $\varphi(\boldsymbol{\xi})$ is constant on $H_+^{n-2}$. Hence, $B$ is the operator of multiplication by a scalar. The operator irreducibility of $T^{nR}$ is proved.

Completing the space $\mathfrak{T}$ with respect to the scalar product

$$(f_1, f_2) = \int_{H_+^{n-2}} f_1(\boldsymbol{\xi})\overline{f_2(\boldsymbol{\xi})}d\boldsymbol{\xi},$$

where $d\boldsymbol{\xi}$ is the invariant measure on $H_+^{n-2}$, we obtain the Hilbert space $\mathfrak{L}^2(H_+^{n-2})$. For $R = i\rho$, $\rho \in \mathbb{R}$, the operators $T^{nR}(g)$ can be extended to unitary operators in $\mathfrak{L}^2(H_+^{n-2})$ and we have the unitary representation $T^{n,i\rho}$ of $ISO_0(n-2,1)$.

We introduce on $H_+^{n-2}$ the spherical coordinates $(\beta, \boldsymbol{\theta}) \equiv (\beta, \theta_1, \ldots, \theta_{n-3})$ (see Section 9.1.5) and consider $f \in \mathfrak{T}$ as functions of these coordinates. Then it follows from (2) that for the element $g_r = g(e, \mathbf{a}_r)$, $\mathbf{a}_r = (0, \ldots, 0, r)$, we have

$$(T^{nR}(g_r)F)(\beta, \boldsymbol{\theta}) = e^{-Rr\cosh\beta} F(\beta, \boldsymbol{\theta}). \qquad (3)$$

**10.7.2. Representations of the group $ISO_0(n-2,1)$ by integral operators.** With every function $f \in \mathfrak{T}$ we associate the function

$$a_M(\sigma) = \int\limits_{H_+^{n-2}} f(\boldsymbol{\xi}) \Xi_M^{n-1,\sigma}(\boldsymbol{\xi}) d\boldsymbol{\xi} \qquad (1)$$

(see formula (2) of Section 10.2.7). The space of these functions is denoted by $\mathfrak{D}$. According to formula (1) of Section 10.2.7 the inverse transform is given by the formula

$$f(\boldsymbol{\xi}) = \frac{(-1)^{[(n+1)/2]}}{(2\sqrt{\pi})^{n-2} \Gamma\left(\frac{n-2}{2}\right) i} \times \sum_M \int\limits_{a-i\infty}^{a+i\infty} a_M(\sigma) \overline{\Xi_M^{n-1,-\bar{\sigma}-n+3}(\boldsymbol{\xi})} \frac{\Gamma(\sigma+n-3)}{\Gamma(\sigma)} \tan^{-\varepsilon} \pi\sigma \, d\sigma, \qquad (2)$$

where $[p/2]$ is the integral part of $p/2$.

To $T^{nR}(g)$ there correspond the operators in $\mathfrak{D}$, which will be denoted by $Q^{nR}(g)$. They define another realization of the representation $T^{nR}$. If $h \in SO_0(n-2,1)$, then

$$\begin{aligned} Q^{nR}(g(h,\mathbf{0}))a_M(\sigma) &= \int\limits_{H_+^{n-2}} (T^{nR}(g(h,\mathbf{0}))f)(\boldsymbol{\xi}) \Xi_M^{n-1,\sigma}(\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= \int\limits_{H_+^{n-2}} f(h^{-1}\boldsymbol{\xi}) \Xi_M^{n-1,\sigma}(\boldsymbol{\xi}) d\boldsymbol{\xi} = \int\limits_{H_+^{n-2}} f(\boldsymbol{\xi}) \Xi_M^{n-1,\sigma}(h\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= \sum_K t_{MK}^{n-1,\sigma}(h) a_K(\sigma). \qquad (3) \end{aligned}$$

Hence, the operators $Q^{nR}(g(h,\mathbf{0}))$ act upon the functions $a_M(\sigma)$ with the help of the matrices $(t_{MK}^{n-1,\sigma}(h))$ of the representations $T^{n-1,\sigma}$ of the group $SO_0(n-2,1)$.

For the operator $Q^{nR}(g_r)$ we have

$$Q^{nR}(g_r)a_M(\sigma) = \int\limits_{H_+^{n-2}} e^{-Rr\xi_{n-1}} f(\boldsymbol{\xi}) \Xi_M^{n-1,\sigma}(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

Substituting expression (2) for $f(\boldsymbol{\xi})$, we obtain

$$Q^{nR}(g_r)a_M(\sigma)$$
$$= \frac{(-1)^{[(n+1)/2]}}{(2\sqrt{\pi})^{n-2}\,\Gamma\left(\frac{n-2}{2}\right)i}\int_{H_+^{n-2}}\sum_K\int_{a-i\infty}^{a+i\infty} a_K(\sigma')e^{-Rr\xi_{n-1}}$$
$$\times\, \Xi_M^{n-1,\sigma}(\boldsymbol{\xi})\overline{\Xi_K^{n-1,-\bar{\sigma}'-n+3}(\boldsymbol{\xi})}\,\frac{\Gamma(\sigma'+n-3)}{\Gamma(\sigma')}\tan^{-\epsilon}\pi\sigma'\,d\sigma'\,d\boldsymbol{\xi}. \quad (4)$$

If $\operatorname{Re} R > 0$, then because of the rapid decrease of the function $e^{-Rr\xi_{n-1}}$ for $\xi_{n-1}\to+\infty$ one can invert the order of integrations with respect to $\boldsymbol{\xi}$ and $\rho$. We pass to spherical coordinates on $H_+^{n-2}$ and take into account that

$$d\boldsymbol{\xi} = \frac{2\pi^{(n-2)/2}}{\Gamma\left(\frac{n-2}{2}\right)}\sinh^{n-3}\beta d\beta\, d\boldsymbol{\eta},$$

where $\boldsymbol{\eta}\in S^{n-3}$, and that

$$\Xi_M^{n-1,\sigma}(\boldsymbol{\xi}) = t_{m0}^{n-1,\sigma}(g_{n-2}(\beta))t_{M'O}^{n-2,m}(\boldsymbol{\eta}),$$

where $M=(m,M')$ and $t_{M'O}^{n-2,m}(\boldsymbol{\eta})$ is the associated spherical function of the representation $T^{n-2,m}$ of the group $SO(n-2)$. Integrating over $S^{n-2}$ and taking into consideration the orthogonality relation for $t_{M'O}^{n-2,m}(\boldsymbol{\eta})$, we have

$$Q^{nR}(g_r)a_M(\sigma) = \int_{a-i\infty}^{a+i\infty} K_m(\sigma,\sigma';R,r)a_M(\sigma')d\sigma'. \quad (5)$$

Here the kernel $K_m(\sigma,\sigma';R,r)$ is given by the formula

$$K_m(\sigma,\sigma';R,r) = \frac{(-1)^{[(n+1)/2]}(n-4)!m!\nu(\sigma')\Gamma(\sigma'+n-3)}{2^{n-3}i\Gamma\left(\frac{n-2}{2}\right)^2(2m+n-4)(m+n-5)!\Gamma(\sigma')}\int_0^\infty e^{-Rr\cosh\beta}$$
$$\times\, t_{m0}^{n-1,\sigma}(g_{n-2}(\beta))t_{m0}^{n-1,-\sigma'-n+3}(g_{n-2}(\beta))\sinh^{n-3}\beta d\beta, \quad (6)$$

where

$$\nu(\sigma') = \begin{cases}\tan^{-1}\pi\sigma' & \text{for } n=2m+1,\\ 1 & \text{for } n=2m+2.\end{cases} \quad (6')$$

If we use expression (8) of Section 9.4.2 for $t_{m0}^{n-1,\sigma}(g_{n-2}(\beta))$, then we obtain that

$$K_m(\sigma,\sigma';R,r) = c\int_1^\infty e^{-Rrx}\mathfrak{P}_{\sigma+p}^{-m-p}(x)\mathfrak{P}_{\sigma'+p}^{-m-p}(x)dx, \quad (7)$$

where $p = (n-4)/2$ and

$$c = (-1)^{m+[(n+1)/2]} \frac{\Gamma(\sigma+1)\Gamma(\sigma'+m+n-3)}{2i\Gamma(\sigma-m+1)\Gamma(\sigma')} \nu(\sigma') \tag{8}$$

(we have taken into account that $\mathfrak{P}^{\mu}_{\nu}(x) = \mathfrak{P}^{\mu}_{-\nu-1}(x)$).

Since every element $g \in ISO_0(n-2,1)$ is representable in the form

$$g = g(h, \mathbf{0}) g_r g(h', \mathbf{0}), \quad h, h' \in SO_0(n-2,1),$$

then formulas (3) and (5) completely define the representation $Q^{nR}$ in the space $\mathfrak{D}$.

Since

$$\mathfrak{P}^{-k/2}_{k/2}(x) = \frac{(x^2-1)^{k/4}}{2^{k/2}\Gamma\left(1+\frac{k}{2}\right)},$$

then in the case when $m = \sigma = 0$, formula (7) takes the form

$$K_0(0,\sigma'; R, r) = \frac{(-1)^{[(k+1)/2]}\Gamma(\sigma'+k+1)}{2^{(k+2)/2} i \Gamma\left(1+\frac{k}{2}\right)\Gamma(\sigma')} \nu(\sigma') \times \int_1^{\infty} e^{-Rrx} \mathfrak{P}^{-k/2}_{\sigma'+k/2}(x)(x^2-1)^{k/4} dx, \tag{9}$$

where $k = n-4$. Making use of formula (10) of Section 9.3.6, we have

$$K_0(0,\sigma'; R, r) = \frac{(-1)^{[(k+1)/2]}\Gamma(\sigma'+k+1)\nu(\sigma')}{i\sqrt{\pi}\,\Gamma(\sigma')\Gamma\left(1+\frac{k}{2}\right)(2Rr)^{(k+1)/2}} K_{\sigma'+\frac{k+1}{2}}(Rr), \tag{10}$$

where $K_\mu(x)$ is the Macdonald function.

Since for even $n$ the function $\nu(\sigma)$ has the property

$$\lim_{\sigma\to 0} \frac{\nu(\sigma)}{\Gamma(\sigma)} = \lim_{\sigma\to 0} \frac{\Gamma(1-\sigma)\sin\pi\sigma}{\pi} \frac{\cos\pi\sigma}{\sin\pi\sigma} = \frac{1}{\pi},$$

then for these $n$ we have

$$K_0(\sigma, 0; R, r) = \frac{(-1)^{[(k+1)/2]}\Gamma(k+1)}{2\pi i 2^{k/2}\Gamma\left(1+\frac{k}{2}\right)} \int_0^{\infty} e^{-Rrx} \mathfrak{P}^{-k/2}_{\sigma+k/2}(x)(x^2-1)^{k/4} dx$$
$$= \frac{(-1)^{n/2}(n-4)!}{\pi^{3/2} i \left(\frac{n-4}{2}\right)!(2Rr)^{(n-3)/2}} K_{\sigma+\frac{n-3}{2}}(Rr), \tag{11}$$

where $k=n-4$.

Note that formula (7) can be considered as the Laplace transform of the product $c\mathfrak{P}_{\sigma+p}^{-m-p}(x)\mathfrak{P}_{\sigma'+p}^{-m-p}(x)$, $p=\frac{n-4}{2}$. Applying the inversion formula for this transform, we obtain

$$c\mathfrak{P}_{\sigma+p}^{-m-p}(x)\mathfrak{P}_{\sigma'+p}^{-m-p}(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}K_m(\sigma,\sigma';z)e^{zx}dz, \tag{12}$$

where $K_m(\sigma,\sigma';z)\equiv K_m(\sigma,\sigma';z,1)$, $a>0$. Setting $\sigma=m=0$ into (12) and taking into account formula (10), after some manipulations we obtain

$$\mathfrak{P}_{\sigma+p}^{-p}(x)=\frac{(x^2-1)^{-p}}{\pi i\sqrt{\pi}}\int_{a-i\infty}^{a+i\infty}K_{\sigma+p+1/2}(z)z^{-p-1/2}e^{zx}dz. \tag{13}$$

In particular, for $n=4$ we have

$$\mathfrak{P}_{\sigma}(x)=\frac{1}{\pi\sqrt{2\pi}\,i}\int_{a-i\infty}^{a+i\infty}K_{\sigma+1/2}(z)z^{-1/2}e^{zx}dz. \tag{14}$$

**10.7.3. Addition and product theorems for Macdonald functions.** The element $g_{r_1}g'_{n-2}(t)g_{r_2}$ of the group $ISO_0(n-2,1)$ is represented in the form

$$g_{r_1}g'_{n-2}(t)g_{r_2}=g'_{n-2}(t_1)g_r g'_{n-2}(t_2), \tag{1}$$

where $g(g'_{n-2}(t),\mathbf{0})$ is replaced by $g'_{n-2}(t)$ and where

$$r^2=r_1^2+r_2^2+2r_1r_2\cosh t,\quad t_2=t-t_1,$$
$$\tanh t_1=r_2\sinh t/(r_2\cosh t+r_1).$$

Writing down formula (1) by means of the kernels of the operators, we obtain

$$\sum_{m=0}^{\infty}t_{0m}^{n-1,\sigma}(g'_{n-2}(t_1))K_m(\sigma,\sigma';R,r)t_{m0}^{n-1,\sigma'}(g'_{n-2}(t_2))$$
$$=\int_{a-i\infty}^{a+i\infty}K_0(\sigma,\nu;R,r_1)t_{00}^{n-1,\nu}(g'_{n-2}(t))K_0(\nu,\sigma';R,r_2)d\nu, \tag{2}$$

where $a$ is the same as in formula (2) of Section 10.7.2. Formula (2) is the addition theorem for the kernels $K_m(\sigma,\sigma';R,r)$. Setting $\sigma=\sigma'=0$ and taking into account that

$$t_{0m}^{n-1,0}(g'_{n-2}(t))=0\quad\text{for}\quad m\neq 0$$

and

$$K_0(0,0;R,r) = \frac{(-1)^{n/2}(n-4)!}{\pi^{3/2} i \left(\frac{n-4}{2}\right)!(2Rr)^{(n-3)/2}} K_{\frac{n-3}{2}}(Rr), \tag{3}$$

we derive the following addition theorem for Macdonald functions:

$$\frac{1}{i\sqrt{2\pi}} \int\limits_{a-i\infty}^{a+i\infty} K_{\nu+p+1/2}(r_1)K_{\nu+p+1/2}(r_2)\mathfrak{P}^{-p}_{\nu+p}(\cosh t)$$
$$\times \frac{\Gamma(\nu+2p+1)}{\Gamma(\nu)} \tan^{-1}\pi\nu\, d\nu = (-1)^p \left(\frac{r_1 r_2}{r}\right)^{p+1/2} \sinh^{-p} t\, K_{p+1/2}(r), \tag{4}$$

where $p = \frac{n-4}{2}$, $n$ is even and $r^2 = r_1^2 + r_2^2 + 2r_1r_2\cosh t$. We set $\nu = i\rho - \frac{n-3}{2}$, $\rho \in \mathbb{R}$, into (4). Since $K_{i\rho}(r_1)K_{i\rho}(r_2)\mathfrak{P}^{-p}_{i\rho-1/2}(\cosh t)$ is an even function of $\rho$ and

$$\frac{\Gamma\left(i\rho+\frac{n-3}{2}\right)}{\Gamma\left(-i\rho+\frac{n-3}{2}\right)} \tan^{-1}\pi\left(i\rho - \frac{n-3}{2}\right)$$
$$= \frac{1}{\pi}\left(-i\rho+\frac{n-3}{2}\right)\Gamma\left(i\rho+\frac{n-3}{2}\right)\Gamma\left(-i\rho+\frac{n-3}{2}\right)\cos\pi\left(i\rho-\frac{n-3}{2}\right),$$

then we obtain

$$\int\limits_0^\infty K_{i\rho}(r_1)K_{i\rho}(r_2)\mathfrak{P}^{-p}_{i\rho-1/2}(\cosh t)\left|\Gamma\left(i\rho+p+\frac{1}{2}\right)\right|^2 \rho\sinh\pi\rho\, d\rho$$
$$= \pi\sqrt{\frac{\pi}{2}}\left(\frac{r_1r_2}{2}\right)^{p+1/2} \sinh^p t K_{p+1/2}(r), \tag{5}$$

where $r = (r_1^2 + r_2^2 + 2r_1r_2\cosh t)^{1/2}$, $r_1 > 0$, $r_2 > 0$, and $p = \frac{n-4}{2}$, $n$ is even.

Since

$$\left|\Gamma\left(i\rho+\frac{1}{2}\right)\right|^2 = \frac{\pi}{\cosh\pi\rho}, \quad K_{1/2}(r) = \sqrt{\frac{\pi}{2r}}\, e^{-r},$$

then for $n = 4$ formula (5) can be written as

$$\int\limits_0^\infty K_{i\rho}(r_1)K_{i\rho}(r_2)\mathfrak{P}_{i\rho-1/2}(\cosh t)\rho\tanh\pi\rho\, d\rho = \frac{\pi\sqrt{r_1r_2}}{2re^r}. \tag{6}$$

One can consider formula (5) as the generalized Fock-Mehler transform of the function $K_{i\rho}(r_1)K_{i\rho}(r_2)$ (see Section 10.2.8). Therefore, we obtain the product formula for Macdonald functions

$$K_{i\rho}(r_1)K_{i\rho}(r_2)$$
$$= \sqrt{\frac{\pi}{2}} \int\limits_0^\infty \left(\frac{r_1r_2}{r}\right)^{p+1/2} K_{p+1/2}(r)\mathfrak{P}^{-p}_{i\rho-1/2}(\cosh t)\sinh^{p+1} t\, dt, \tag{7}$$

where $r = (r_1^2 + r_2^2 + 2r_1r_2\cosh t)^{1/2}$, $r_1 > 0$, $r_2 > 0$. In particular,

$$K_{i\rho}(r_1)K_{i\rho}(r_2) = \frac{\pi}{2}\sqrt{r_1r_2}\int_0^\infty (re^r)^{-1}\mathfrak{P}_{i\rho-1/2}(\cosh t)\sinh t\,dt, \tag{8}$$

where $r$ is the same as in (7).

**10.7.4. Evaluation of the kernel $K_m(\sigma,\sigma';z)$ for the general case.** To evaluate the kernel $K_m(\sigma,\sigma';z)$ in the general case we express $\mathfrak{P}_\nu^\mu(x)$ in terms of the hypergeometric function:

$$\mathfrak{P}_\nu^\mu(x) = \left(\frac{x+1}{x-1}\right)^{\mu/2}\left(\frac{x+1}{2}\right)^\nu \frac{1}{\Gamma(1-\mu)}F\left(-\nu,-\nu-\mu;1-\mu;\frac{x-1}{x+1}\right)$$

and substitute this expression into formula (7) of Section 10.7.2.

The formula

$$F(a,b;c;x)F(d,e;f;x) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{n!(c)_n} \times {}_4F_3\left(\begin{matrix}-n,1-c-n,d,e\\ 1-a-n,1-b-n,f\end{matrix}\middle|\,1\right)x^n \tag{1}$$

holds. In order to prove (1) it is sufficient to note that the coefficient at $x^n$ in the product $F(a,b;c;x)F(d,e;f;x)$ is equal to

$$I = \sum_{s=0}^n \frac{(a)_{n-s}(b)_{n-s}}{(n-s)!(c)_{n-s}}\cdot\frac{(d)_s(e)_s}{s!(f)_s}, \tag{2}$$

and

$$(a)_{n-s} = \frac{(-1)^s(a)_n}{(1-a-n)_s},\quad \frac{1}{(n-s)!} = \frac{(-1)^s(s-n)_s}{s!}.$$

Taking into account formula (1), we have for $K_m(\sigma,\sigma';z)$ the expression

$$K_m(\sigma,\sigma';z) = \frac{c2^{-\sigma-\sigma'-k}}{\Gamma\left(m+\frac{k}{2}+1\right)}\sum_{s=0}^\infty \frac{\Gamma\left(-\sigma-\frac{k}{2}+s\right)\Gamma(m-\sigma+s)}{s!\gamma\left(-\sigma-\frac{k}{2}\right)\Gamma(m-\sigma)\Gamma\left(m+\frac{k}{2}+s+1\right)}$$
$$\times {}_4F_3\left(\begin{matrix}-s,-s-m-\frac{k}{2},m-\sigma',-\sigma'-\frac{k}{2}\\ \sigma+\frac{k}{2}-s+1,\sigma-m-s+1,m+\frac{k}{2}+1\end{matrix}\middle|\,1\right)$$
$$\times\int_1^\infty e^{-zx}(x-1)^{m+s+k/2}(x+1)^{\sigma+\sigma'-m-s+k/2}dx, \tag{3}$$

where $k = n - 4$ and $c$ is given by formula (8) of Section 10.7.2.

By the substitution $u = \frac{x-1}{2}$ we reduce the integral from (3) to integral (2) of Section 3.5.7. Therefore, this integral is equal to

$$\frac{1}{2}\left(\frac{z}{2}\right)^{-1-(\sigma+\sigma'+k)/2}\Gamma\left(m+s+\frac{k}{2}+1\right)W_{\frac{\sigma+\sigma'}{2}-m-s,\frac{\sigma+\sigma'+k+1}{2}}(2z).$$

Thus, for $\operatorname{Re} z > 0$ we obtain

$$K_m(\sigma,\sigma';z) = \frac{(-1)^{[(k+1)/2]}\Gamma(\sigma'+m+k+1)\nu(\sigma')}{i\Gamma\left(m+\frac{k}{2}+1\right)\Gamma\left(-\sigma-\frac{k}{2}\right)\Gamma(-\sigma)\Gamma(\sigma')}(2z)^{-1-(\sigma+\sigma'+k)/2}$$
$$\times\sum_{s=0}^{\infty}\frac{1}{s!}\left(-\sigma-\frac{k}{2}+s\right)\Gamma(m-\sigma+s)\,{}_4F_3\left(\begin{matrix}-s,-s-m-\frac{k}{2},m-\sigma',-\sigma'-\frac{k}{2}\\ \sigma+\frac{k}{2}-s+1,\sigma-m-s+1,m+\frac{k}{2}+1\end{matrix}\,\middle|\,1\right)$$
$$\times W_{\frac{\sigma+\sigma'}{2}-m-s,\frac{\sigma+\sigma'+k+1}{2}}(2z), \quad (4)$$

where $k = n - 4$. In particular,

$$K_0(\sigma,0;z) = \frac{(-1)^{[(k+1)/2]}k!(2z)^{-1-(\sigma+k)/2}}{i\Gamma\left(\frac{k}{2}+1\right)\Gamma(-\sigma)\Gamma\left(-\sigma-\frac{k}{2}\right)}$$
$$\times\sum_{s=0}^{\infty}\frac{1}{s!}\Gamma(s-\sigma)\Gamma\left(s-\sigma-\frac{k}{2}\right)W_{\frac{\sigma}{2}-s,\frac{\sigma+k+1}{2}}(2z). \quad (5)$$

Comparing the right hand side of formula (11) of Section 10.7.2 with (5), we derive the identity

$$\sum_{s=0}^{\infty}\frac{1}{s!}(-2\tau)_s\left(-2\tau-p+\frac{1}{2}\right)_s W_{\tau-s,\tau+p}(x) = \frac{x^{\tau+1/2}}{\sqrt{\pi}}K_{2\tau+p}\left(\frac{x}{2}\right), \quad (6)$$

where $p$ is the half of a positive odd integer and $2\tau = \sigma \in \mathbb{C}$.

**10.7.5. Class 1 representations of the group $HSO(n)$ and the function ${}_1F_1$.** The group $HSO(n)$ consisting of matrices

$$g(\omega,\tau,\mathbf{x},r) = \begin{pmatrix} e^{\tau}\omega & 0 & \mathbf{x}\\ 0 & e^{2\tau} & r\\ 0 & 0 & 1\end{pmatrix}, \quad (1)$$

where $\tau, r \in \mathbb{R}$, $\omega \in SO(n)$, $\mathbf{x} \in \mathbb{R}^n$, is a multivariate generalization of the group $G_2$ considered in Section 5.3.5. It is obvious that

$$g(\omega_1,\tau_1,\mathbf{x}_1,r_1)g(\omega_2,\tau_2,\mathbf{x}_2,r_2)$$
$$= g(\omega_1\omega_2,\tau_1+\tau_2,e^{\tau_1}\omega_1\mathbf{x}_2+\mathbf{x}_1,e^{2\tau_1}r_2+r_1). \quad (2)$$

The subgroup $H$ of matrices of the form $g(\omega, \tau, \mathbf{0}, 0)$ is isomorphic to $SO(n) \times \mathbf{R}$, and the subgroup of matrices of the form $g(I, 0, \mathbf{x}, r)$, where $I$ denotes the identity rotation, is isomorphic to $\mathbf{R}^{n+1}$. Every element $g = g(\omega, \tau, \mathbf{x}, r) \in HSO(n)$ can be represented as

$$g = g(\omega_1, \tau_1, \mathbf{0}, 0) g(I, 0, \mathbf{e}_n, r_1) g(\omega_2, \tau_2, \mathbf{0}, 0), \tag{3}$$

where $\mathbf{e}_n = (0, \ldots, 0, 1)$,

$$e^{\tau_1} = (\mathbf{x}, \mathbf{x})^{1/2}, \quad \omega_1 \mathbf{e}_n = (\mathbf{x}, \mathbf{x})^{-1/2} \mathbf{x}, \quad \omega_2 = \omega_1^{-1} \omega,$$
$$\tau_2 = \tau - \tau_1, \quad r_1 = (\mathbf{x}, \mathbf{x})^{-1} r.$$

In this factorization $\omega$ is defined up to the right multiplication by an element $k \in SO(n-1)$ and, hence, can be considered as a point from $S^{n-1}$.

The invariant measure on $HSO(n)$ is given by the formula

$$dg = (\mathbf{x}, \mathbf{x})^{-(n+2)/2} d\omega\, d\tau\, d\mathbf{x}\, dr.$$

The natural action of the group $HSO(n)$ on the plane $x_{n+2} = 1$ of $\mathbf{R}^{n+2}$ is reduced to homothetic transformations in $\mathbf{R}^n$ with the coefficient $e^{\tau}$ and to analogous transformations in $\mathbf{R}$ with the coefficient $e^{2\tau}$. Restricting this action onto the subgroup $H$, we conclude that this plane splits into the paraboloids

$$\mathcal{P}_\alpha = \{(\mathbf{y}, s) \mid 2\alpha s = (\mathbf{y}, \mathbf{y})\}, \qquad -\infty < \alpha < \infty.$$

To every complex-valued function $f$ on $\mathbf{R}^n$ there corresponds the function $\hat{f}$ on $\mathcal{P}_\alpha$, given by the formula $\hat{f}(\mathbf{x}, r) = f(\mathbf{x})$. Using this fact and applying the Fourier transform, we see that the quasi-regular representation of the group $HSO(n)$ in $\mathbf{R}^{n+1}$ decomposes into the continuous direct sum of the representations $T_\alpha$, defined by the formula

$$(T_\alpha(g(\omega, \tau, \mathbf{x}, r))f)(\mathbf{y}) = e^{-\alpha(r(\mathbf{y},\mathbf{y}) + (\mathbf{x},\mathbf{y}))} f(e^{\tau} \omega^{-1} \mathbf{y}). \tag{4}$$

A simple verification shows that the representation $T_\alpha$ is irreducible for $\alpha \neq 0$ and is unitary with respect to the scalar product

$$(f_1, f_2) = \int_{\mathbf{R}^n} f_1(\mathbf{x}) \overline{f_2(\mathbf{x})} (\mathbf{x}, \mathbf{x})^{-n/2} d\mathbf{x}$$

for $\alpha \in i\mathbf{R}$.

We set $\mathbf{y} = \rho \boldsymbol{\eta}$, $\rho \geq 0$, $\boldsymbol{\eta} \in S^{n-1}$ and

$$F(\lambda, \boldsymbol{\eta}) = \int_0^\infty \rho^{\lambda - 1} f(\rho \boldsymbol{\eta}) d\rho. \tag{5}$$

Making use of the inversion formula for the Mellin transform, we derive that to $T_\alpha$ there corresponds the representation $Q_\alpha$ in the space of functions $F(\lambda, \boldsymbol{\eta})$, given by the formulas

$$(Q_\alpha(g(\omega, \tau, \mathbf{0}, 0))F)(\lambda, \boldsymbol{\eta}) = e^{-\lambda\tau} F(\lambda, \omega^{-1}\boldsymbol{\eta}), \tag{6}$$

$$(Q(g(I, 0, \mathbf{e}_n, r))F)(\lambda, \boldsymbol{\eta}) = \int_{a-i\infty}^{a+i\infty} K_\alpha(\lambda - \mu, r, \boldsymbol{\eta}) F(\mu, \boldsymbol{\eta}) d\mu, \tag{7}$$

where

$$K_\alpha(\lambda, r, \boldsymbol{\eta}) = \frac{1}{2\pi i} \int_0^\infty \rho^{\lambda-1} e^{-\alpha(r\rho^2 + \rho(\mathbf{e}_n, \boldsymbol{\eta}))} d\rho \tag{8}$$

(because of (3) these equalities define $Q_\alpha$ completely).

We expand functions $F(\lambda, \boldsymbol{\eta})$ in spherical functions:

$$F(\lambda, \boldsymbol{\eta}) = \sum_{m,M} a_M^{nm}(\lambda) \overline{\Xi_M^{nm}(\boldsymbol{\eta})},$$

where

$$a_M^{nm}(\lambda) = \int_{S^{n-1}} F(\lambda, \boldsymbol{\eta}) \Xi_M^{nm}(\boldsymbol{\eta}) d\boldsymbol{\eta}.$$

Then to the operators $Q_\alpha(g)$ there correspond the operators $\widetilde{Q}_\alpha(g)$ in the space of the functions $a_M^{nm}(\lambda)$, and we have

$$\widetilde{Q}_\alpha(g(\omega, \tau, \mathbf{0}, 0)) a_M^{nm}(\lambda) = e^{-\lambda\tau} \sum_L t_{LM}^{nm}(\omega) a_L^{nm}(\lambda),$$

$$\widetilde{Q}_\alpha(g(I, 0, \mathbf{e}_n, r)) a_M^{nm}(\lambda) = \sum_{\ell, L} \int_{a-i\infty}^{a+i\infty} K_{LM}^{\ell m,\alpha}(\lambda - \mu, r) a_L^{n\ell}(\mu) d\mu, \tag{9}$$

where $t_{LM}^{nm}(\omega)$ are matrix elements of the representation $T^{nm}$ of the group $SO(n)$ (see Section 9.4.1) and

$$K_{LM}^{\ell m,\alpha}(\lambda, r) = \frac{1}{2\pi i} \int_0^\infty \int_{S^{n-1}} \rho^{\lambda-1} \exp[-\alpha(\rho^2 r + \rho(\mathbf{e}_n, \boldsymbol{\eta}))] \overline{\Xi_L^{n\ell}(\eta)} \Xi_M^{nm}(\boldsymbol{\eta}) d\boldsymbol{\eta}\, d\rho.$$

Since the function $\exp[-\alpha r(\mathbf{e}_n, \boldsymbol{\eta})]$ is invariant with respect to rotations from $SO(n-1)$, the kernel $K_{LM}^{\ell m,\alpha}(\lambda, r)$ differs from zero for $M = L$ only. We evaluate it for $m = 0$. In this case $M = L = O$ and therefore,

$$K_{OO}^{\ell 0,\alpha}(\lambda, r) = \frac{1}{2\pi i}\int_0^\infty \int_{S^{n-1}} \rho^{\lambda-1} \exp[-\alpha(\rho^2 r + \rho(\mathbf{e}_n, \boldsymbol{\eta}))]$$

$$\times\, \Xi_O^{n\ell}(\boldsymbol{\eta})d\boldsymbol{\eta}\, d\rho = \frac{1}{2\pi i}\frac{\ell!(n-3)!\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}(\ell+n-3)!\Gamma\left(\frac{n-1}{2}\right)}\left[\frac{(\ell+n-3)!(2\ell+n-2)}{\ell!(n-2)!}\right]^{1/2}$$

$$\times \int_0^\infty \int_0^\pi \rho^{\lambda-1}\exp[-\alpha(\rho^2 r + \rho\cos\theta)] C_\ell^{(n-2)/2}(\cos\theta)\sin^{n-2}\theta d\theta\, d\rho. \quad (10)$$

We evaluate this integral for $\alpha r > 0$ by integrating at first with respect to $\theta$ and then with respect to $\rho$. Making use of formula (7) of Section 9.3.6, we obtain

$$K_{OO}^{\ell 0,\alpha}(\lambda, r) = \frac{2^{(n-2)/2}\Gamma\left(\frac{n}{2}\right) i^{\ell-(n-2)/2}}{2\pi i \Gamma\left(\frac{n-1}{2}\right)}\left[\frac{(\ell+n-3)!(2\ell+n-2)}{\ell!(n-2)!}\right]^{1/2}$$

$$\times\, \alpha^{-(n-2)/2}\int_0^\infty \rho^{\lambda-n/2} e^{-\alpha\rho^2 r} J_{\ell+(n-2)/2}(i\alpha\rho)d\rho.$$

Further, we expand the Bessel function into a power series and integrate term by term. We have

$$K_{OO}^{\ell 0,\alpha}(\lambda, r) = A(n,\ell,\lambda)\alpha^{\frac{\ell-\lambda}{2}} r^{-\frac{\lambda+\ell}{2}}\, {}_1F_1\left(\frac{\lambda+\ell}{2}; \ell+\frac{n}{2}; \frac{\alpha}{4r}\right), \quad (11)$$

where

$$A(n,\ell,\lambda) = \frac{(-1)^\ell \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{\lambda+\ell}{2}\right)}{2^{\ell+2}\pi i\Gamma\left(\ell+\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}\left[\frac{(n+\ell-3)!(2\ell+n-2)}{\ell!(n-2)!}\right]^{1/2}. \quad (12)$$

One can evaluate integral (10) by integrating first with respect to $\rho$ and then with respect to $\theta$. Using formula (1) of Section 5.3.5, we obtain that

$$K_{OO}^{\ell 0,\alpha}(\lambda, r) = B(n,\ell,\lambda) r^{-\lambda/2}\int_{-1}^{1}\left(\exp\frac{\alpha x^2}{8r}\right) D_{-\lambda}\left(\frac{x\sqrt{\alpha}}{\sqrt{2r}}\right)$$

$$\times\, C_\ell^{(n-2)/2}(x)(1-x^2)^{(n-3)/2}dx, \quad (13)$$

where

$$B(n,\ell,\lambda)=\frac{\ell!(n-3)!\Gamma(\frac{n}{2})\Gamma(\lambda)}{2\pi i(\ell+n-3)!\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}(2\alpha)^{-\lambda/2}\left[\frac{(\ell+n-3)!(2\ell+n-2)}{\ell!(n-2)!}\right]^{1/2}. \tag{14}$$

Comparing formulas (11) and (13), we have

$$\int_{-1}^{1}\left(\exp\frac{x^2\beta^2}{8}\right)D_{-\lambda}\left(\frac{\beta x}{\sqrt{2}}\right)C_\ell^{(n-2)/2}(x)(1-x^2)^{(n-3)/2}dx$$
$$=\frac{(-1)^\ell\sqrt{\pi}\,2^{-\ell-1+\lambda/2}(\ell+n-3)!\Gamma\left(\frac{\lambda+\ell}{2}\right)\beta^\ell}{\ell!(n-3)!\Gamma(\lambda)\Gamma\left(\ell+\frac{n}{2}\right)}{}_1F_1\left(\frac{\lambda+\ell}{2};\ell+\frac{n}{2};\frac{\beta^2}{4}\right) \tag{15}$$

(we have replaced $\alpha/r$ by $\beta$).

Formulas (3), (6) and (9) lead to the equality

$$\widetilde{Q}_\alpha(g(I,0,R\mathbf{e}_n,r))a_O^{n0}(\lambda)=\sum_{\ell=0}^{\infty}\int_{a-i\infty}^{a+i\infty}R^{\mu-\lambda}K_{OO}^{\ell 0,\alpha}(\lambda-\mu,R^{-2}r)a_O^{n\ell}(\mu)d\mu. \tag{16}$$

Now we derive the addition theorem for the functions ${}_1F_1$. It follows from (3) that

$$g(I,0,R_1\mathbf{e}_n,r_1)g(\omega(\varphi),0,0,0)g(I,0,R_2\mathbf{e}_n,r_2)$$
$$=g(\omega(\varphi_1),\tau_1,\mathbf{0},0)g(I,0,\mathbf{e}_n,r)g(\omega(\varphi_2),\tau_2,\mathbf{0},0), \tag{17}$$

where $\omega(\varphi)$ denotes the rotation by the angle $\varphi$ in the plane $(x_{n-1},x_n)$ and

$$\varphi=\varphi_1+\varphi_2,\quad \tau_1=-\tau_2,\quad e^{2\tau_1}=R_1^2+2R_1R_2\cos\varphi+R_2^2, \tag{18}$$
$$r=e^{-2\tau_1}(r_1+r_2).$$

Acting by corresponding operators of the representtion $\widetilde{Q}_\alpha$ upon the element $a_O^{n0}(\lambda)$ and comparing in the resulting relation the coefficients of the same element, we obtain

$$K_{OO}^{00,\alpha}(\lambda-\mu,r)e^{-\mu\tau_2-\lambda\tau_1}=\sum_{\ell=0}^{\infty}t_{OO}^{n\ell}(\varphi)$$
$$\times\int_{a-i\infty}^{a+i\infty}R_1^{\nu-\lambda}R_2^{\mu-\nu}K_{OO}^{\ell 0,\alpha}(\lambda-\nu,R_1^{-2}r_1)K_{OO}^{\ell 0,\alpha}(\nu-\mu,R_2^{-2}r_2)d\nu, \tag{19}$$

where $r$, $r_1$, $r_2$, $\tau_1$, $\tau_2$, $\varphi$, $R_1$, $R_2$ are connected by relations (18). Taking into account formula (11), we derive the addition theorem

$$A(n,0,\lambda-\mu)e^{-\lambda\tau_1-\mu\tau_2}r^{\frac{\mu-\lambda}{2}}{}_1F_1\left(\frac{\lambda-\mu}{2};\frac{n}{2};\frac{\alpha}{4r}\right)$$

$$=\sum_{\ell=0}^{\infty}\frac{\ell!\Gamma(n-2)}{\Gamma(\ell+n-2)}\alpha^{\ell}r_1^{-\frac{\ell+\lambda}{2}}r_2^{-\frac{\ell-\mu}{2}}R_1^{-\lambda}R_2^{\mu}C_\ell^{\frac{n-2}{2}}(\cos\varphi)$$

$$\times\int_{a-i\infty}^{a+i\infty}A(n,\ell,\lambda-\nu)A(n,\ell,\nu-\mu)\left(\frac{R_1r_1}{R_2r_2}\right)^{\nu}$$

$$\times{}_1F_1\left(\frac{\lambda-\nu+\ell}{2};\ell+\frac{n}{2};\frac{\alpha R_1^2}{4r_1}\right){}_1F_1\left(\frac{\nu-\mu+\ell}{2};\ell+\frac{n}{2};\frac{\alpha R_2^2}{4r_2}\right)d\nu, \quad (20)$$

where the variables are connected by formulas (18) and the coefficients $A(n,\ell,\lambda)$ are given by (12).

Making use of the orthogonality relations for Gegenbauer polynomials and the inversion formula for the Mellin transform in $\nu$, one can derive from (20) a product formula.

# Chapter 11.
# Special Functions Connected with the Groups $U(n)$, $U(n-1,1)$ and $IU(n-1)$

## 11.1. The Groups $U(n)$, $U(n-1,1)$, $IU(n-1)$ and Related Homogeneous Spaces

**11.1.1. The groups $U(n)$, $U(n-1,1)$ and related homogeneous spaces.** Let $E_n^C$ be the $n$-dimensional unitary (complex) space with the scalar product $(\mathbf{z},\mathbf{w}) = z_1\bar{w}_1 + \ldots + z_n\bar{w}_n$. The number $r = \|\mathbf{z}\| \equiv (\mathbf{z},\mathbf{z})^{1/2}$ is called the length of $\mathbf{z} \in E_n^C$ and the number $\|\mathbf{z}-\mathbf{w}\|$ is called the distance between $\mathbf{z}$ and $\mathbf{w}$. The set of points $\mathbf{z} \in E_n^C$ such that $\|\mathbf{z}\| = a$, $a > 0$, forms the sphere $S_C^{n-1}(a)$ of radius $a$. The sphere $S_C^{n-1}(1)$ will be denoted by $S_C^{n-1}$.

The group $U_n$ consists of unitary transformations in $E_n^C$, i.e. of linear transformations preserving $(\mathbf{z},\mathbf{w})$: $(u\mathbf{z}, u\mathbf{w}) = (\mathbf{z},\mathbf{w})$. If $u \in U(n)$, then $u^*u = I_n$ and, hence, $|\det u| = 1$. Unimodular transformations from $U(n)$ form the subgroup $SU(n)$. The groups $U(n)$ and $SU(n)$ are connected and compact. The group $SU(n)$ is simply connected and simple. The scalar matrices $e^{i\varphi}I_n$ form an invariant subgroup of $U(n)$, isomorphic to $U(1)$. We have $SU(n) \sim U(n)/U(1)$.

The action of the group $U(n)$ splits $E_n^C$ into the spheres $S_C^{n-1}(\tau)$, $0 \le \tau < \infty$. The stabilizer of the basis element $\mathbf{e}_n = (0,\ldots,0,1) \in E_n^C$ is isomorphic to $U(n-1)$ and, therefore, $S_C^{n-1} \sim U(n)/U(n-1)$. The action of $U(n)$ on $U(n)/U(n-1)$ by left shifts defines rotations of $S_C^{n-1}$.

One can consider $E_n^C$ as a real space with doubled number of coordinates (real and imaginary parts of the coordinates $z_1,\ldots,z_n$). Then $E_n^C$ is identified with the Euclidean space $E_{2n}$, the sphere $S_C^{n-1} \subset E_n^C$ with the sphere $S^{2n-1} \subset E_{2n}$, the group $U(n)$ with the subgroup in $SO(2n)$.

Let us identify the points $e^{i\varphi}\mathbf{z}$, $0 \le \varphi < 2\pi$, in $S_C^{n-1}$. These identifications lead us to the space denoted by $P_C^{n-1}$. Since $U(n)/U(1) \sim SU(n)$, then rotations of $S_C^{n-1}$ define rotations of $P_C^{n-1}$ and $SU(n)$ acts transitively upon $P_C^{n-1}$. In addition,

$$P_C^{n-1} \sim SU(n)/S(U(n-1)\times U(1)). \tag{1}$$

Thus, $P_C^{n-1}$ is a compact symmetric Riemannian space (see Section 1.2.3). It is clear that

$$SU(n)/S(U(n-1)\times U(1)) \sim U(n)/U(n-1)\times U(1).$$

The complex $n$-dimensional vector space with the Hermitian form

$$[\mathbf{z},\mathbf{w}] = -z_1\bar{w}_1 - \ldots - z_{n-1}\bar{w}_{n-1} + z_n\bar{w}_n \tag{2}$$

is said to be the pseudo-unitary space of signature $(n-1,1)$. It is denoted by $E_{n-1,1}^C$. The formula

$$r^2(\mathbf{z},\mathbf{w}) = [\mathbf{z}-\mathbf{w}, \mathbf{z}-\mathbf{w}]$$

gives the distance between $\mathbf{z}$ and $\mathbf{w}$. The value of $r^2(\mathbf{z},\mathbf{w})$ can be positive, negative and zero. Points $\mathbf{z} \in E^C_{n-1,1}$, for which $[\mathbf{z},\mathbf{z}] = 0$, $\mathbf{z} \neq 0$, form the cone $C^{n-1}_C$ in $E^C_{n-1,1}$.

The set of linear transformations of $E^C_{n-1,1}$ preserving $[\mathbf{z},\mathbf{w}]$ forms the group $U(n-1,1)$ of pseudo-unitary transformations. The condition det $g = 1$ separates in $U(n-1,1)$ the subgroup $SU(n-1,1)$. The scalar matrices $e^{i\varphi}I_n$, $0 \le \varphi < 2\pi$, form an invariant subgroup in $U(n-1,1)$, isomorphic to $U(1)$. We have $SU(n-1,1) \sim U(n-1,1)/U(1)$. The groups $U(n-1,1)$ and $SU(n-1,1)$ are connected and locally compact. (They are not compact.) The group $SU(n-1,1)$ is simple.

The action of $U(n-1,1)$ splits $E^C_{n-1,1}$ into orbits consisting of points $\mathbf{z}$ for which $[\mathbf{z},\mathbf{z}] = r$, $-\infty < r < \infty$. The origin is one of the orbits.

Points $\mathbf{z}$, such that $[\mathbf{z},\mathbf{z}] = 1$, form the hyperboloid $H^{n-1}_C$ in $E^C_{n-1,1}$. The stabilizer of $\mathbf{e}_n = (0,\dots,0,1) \in H^{n-1}_C$ coincides with the subgroup isomorphic to $U(n-1)$. Therefore,

$$H^{n-1}_C \sim U(n-1,1)/U(n-1).$$

The space $E^C_{n-1,1}$, considered as a real space with doubled number of parameters (real and imaginary parts of $z_1,\dots,z_n$), is identified with the pseudo-Euclidean space $E_{2n-2,2}$. Then to the hyperboloid $H^{n-1}_C$ in $E^C_{n-1,1}$ there corresponds the hyperboloid $[\mathbf{x},\mathbf{x}]_{2,2n-2} = 1$ in $E_{2n-2,2}$ with $SO_0(2n-2,2)$ as a transitive motion group. The stabilizer of the point $\mathbf{z}_0 = (0,\dots,0,1)$ of this hyperboloid coincides with $SO_0(2n-2,1)$. Hence,

$$U(n-1,1)/U(n-1) \sim SO_0(2n-2,2)/SO_0(2n-2,1).$$

The second quotient space is a symmetric pseudo-Riemannian space (see Section 1.2.4). The group $U(n-1,1)$ is imbedded into $SO_0(2n-2,2)$.

By identifying the points $e^{i\varphi}\mathbf{z}$, $0 \le \varphi < 2\pi$, of $H^{n-1}_C$ we obtain the space denoted by $\mathcal{P}^{n-1}_C$. The transitive motion group of $\mathcal{P}^{n-1}_C$ coincides with $SU(n-1,1)$. The subgroup $S(U(n-1)\times U(1))$ is the stabilizer of the element

$$p_n = \{(0,\dots,0,e^{i\varphi}) \mid 0 \le \varphi < 2\pi\} \in \mathcal{P}^{n-1}_C.$$

Consequently,

$$\mathcal{P}^{n-1}_C \sim SU(n-1,1)/S(U(n-1)\times U(1)) \sim U(n-1,1)/U(n-1)\times U(1).$$

Since $S(U(n-1)\times U(1))$ is the maximal compact subgroup in $SU(n-1,1)$, then $\mathcal{P}^{n-1}_C$ is a symmetric Riemannian space of noncompact type. It is dual by Cartan to the space $P^{n-1}_C$.

As in the case of the group $SO_0(n-1,1)$ (see Section 9.1.1), one shows that $\mathcal{P}^{n-1}_C$ is diffeomorphic to the ball

$$D^{n-1}_C = \{\mathbf{z} \in \mathbb{C}^{n-1} \mid |z_1|^2 + \dots + |z_{n-1}|^2 < 1\}. \tag{3}$$

This diffeomorphism is given by the formula

$$Q: (z_1, \dots, z_n) \longrightarrow \left( \frac{z_1}{z_n}, \dots, \frac{z_{n-1}}{z_n} \right). \tag{4}$$

In order to define the action of $U(n-1,1)$ on $D_C^{n-1}$ we associate with $\mathbf{z} \in D_C^{n-1}$ the point $\mathbf{w} \in (z_1, \dots, z_{n-1}, 1) \in E_{n-1,1}^C$. If $g\mathbf{w} = (w_1, \dots, w_n) \in E_{n-1,}^C$, $g \in U(n-1,1)$, then we define

$$g\mathbf{z} = \left( \frac{w_1}{w_n}, \dots, \frac{w_{n-1}}{w_n} \right). \tag{5}$$

One can easily verify that the diffeomorphism $Q$ transforms the action of $U(n-1,1)$ on $H_C^{n-1}$ into the action (5) on $D_C^{n-1}$.

Let us choose the point $\mathbf{a} = (0, \dots, 0, 1, 1)$ on the cone $C_C^{n-1}$. The stabilizer of this point in $U(n-1,1)$ coincides with the subgroup $\widetilde{M}N$, where $\widetilde{M}$ is isomorphic to $U(n-2)$ and $N$ consists of the matrices

$$n \equiv n(\mathbf{z}, \alpha) = \begin{pmatrix} I_{n-2} & -\mathbf{z}^t & \mathbf{z}^t \\ \mathbf{z} & 1 - \alpha i - \frac{z^2}{2} & \alpha i + \frac{z^2}{2} \\ \mathbf{z} & -\alpha i - \frac{z^2}{2} & 1 + i\alpha + \frac{z^2}{2} \end{pmatrix}, \quad \begin{matrix} \mathbf{z} = (z_1, \dots, z_{n-2}) \in \mathbb{C}^{n-2}, \\ z^2 = |z_1|^2 + \dots + |z_{n-2}|^2, \\ \alpha \in \mathbb{R}. \end{matrix} \tag{6}$$

Therefore, $C_C^{n-1} \sim U(n-1,1)/U(n-2)N$. Replacing $\mathbf{a}$ by the point $\mathbf{b} = (0, \dots, 0, -1, 1) \in C_C^{n-1}$, we obtain that $C_C^{n-1} \sim U(n-1,1)/U(n-2)\bar{N}$, where the subgroup $\bar{N}$ consists of the matrices

$$\bar{n} \equiv \bar{n}(\mathbf{z}, \alpha) = \begin{pmatrix} I_{n-2} & -\mathbf{z}^t & -\mathbf{z}^t \\ \mathbf{z} & 1 - \alpha i - \frac{z^2}{2} & -\alpha i - \frac{z^2}{2} \\ \mathbf{z} & \alpha i + \frac{z^2}{2} & 1 + \alpha i + \frac{z^2}{2} \end{pmatrix}. \tag{7}$$

**11.1.2. The Lie algebras of the groups $U(n)$ and $U(n-1,1)$.** The group $U(n)$ contains the subgroup $SO(n)$. We select in $U(n)$ the one-parameter subgroups $\{g_{ij}(\theta)\}$, $i < j$, contained in $SO(n)$ (see Section 9.1.1), and the one-parameter subgroups

$$\{d_k(\varphi) = \mathrm{diag}(1, \dots, 1, e^{i\varphi}, 1, \dots, 1)\},$$

where $e^{i\varphi}$ is on the $k$-th position. Since

$$\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}, \tag{1}$$

then the subgroups $\{g_{ij}(\theta)\}$ and $\{d_k(\varphi)\}$ generate a sequence of one-parameter subgroups $\{g^C_{rs}(\theta)\}$, $r < s$, consisting of the transformations $\mathbf{z} \to \mathbf{z}'$ of $E^C_n$, where $z'_j = z_j$, $j \neq r$, $j \neq s$,

$$z'_r = z_r \cos\theta + iz_s \sin\theta, \quad z'_s = iz_r \sin\theta + z_s \cos\theta. \tag{2}$$

The tangent matrices to the subgroups $\{g_{ij}(\theta)\}$, $\{g^C_{ij}(\theta)\}$, $\{d_k(\theta)\}$ at the point $\theta = 0$ have, respectively, the form

$$I_{ij} = E_{ij} - E_{ji}, \quad J_{ij} = \sqrt{-1}(E_{ij} + E_{ji}), \quad \sqrt{-1}\, E_{kk}, \tag{3}$$

where $E_{ij}$ is the matrix with entries $(E_{ij})_{st} = \delta_{is}\delta_{jt}$. The matrices

$$I_{ij}, \quad J_{ij}, \quad i < j, \quad \sqrt{-1}\, E_{kk}, \quad k = 1, 2, \ldots, n, \tag{4}$$

form a basis of the Lie algebra $\mathfrak{u}(n)$ of the group $U(n)$. Therefore, $\mathfrak{u}(n)$ consists of all skew-symmetric matrices of order $n$.

Since $\det u = 1$ for elements $u$ from $SU(n)$, then the Lie algebra $\mathfrak{su}(n)$ of the group $SU(n)$ consists of matrices from $\mathfrak{u}(n)$ with zero trace. Therefore,

$$I_{ij}, \quad J_{ij}, \quad i < j, \quad \sqrt{-1}(E_{kk} - E_{k+1,k+1}), \quad k = 1, 2, \ldots, n-1, \tag{5}$$

is a basis of $\mathfrak{su}(n)$.

As it was shown in Section 11.1.1, $U(n)$ is imbedded into $SO(2n)$. Let us construct this embedding. If $(z_1, \ldots, z_n)$ are Cartesian coordinates in $E^C_n$ and $z_j = x_j + iy_j$, then $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in E_{2n}$ and elements from $SO(n) \subset U(n)$ act simultaneously in the space of coordinates $x_1, \ldots, x_n$ and in the space of coordinates $y_1, \ldots, y_n$. Hence, elements $g_{ij}(\theta) \in SO(n) \subset U(n)$ are imbedded into $SO(2n)$ as

$$g_{ij}(\theta) \to \operatorname{diag}(g_{ij}(\theta), g_{ij}(\theta)) \in SO(2n). \tag{6}$$

Since

$$(\cos\varphi + i\,\sin\varphi)(x + iy) = (\cos\varphi\, x - \sin\varphi\, y) + i(\sin\varphi\, x + \cos\varphi\, y),$$

then to the elements $d_k \in U(n)$ there correspond the rotations $g_{k,n+k}(-\varphi)$ from $SO(2n)$. One can easily find (with the help of formula (1)) the matrices $g$ of $SO(2n)$ corresponding to the elements $g^C_{ij}(\theta) \in U(n)$. On the intersections of rows and columns of $g$, indexed by $i$, $j$, $i+n$, $j+n$, one has the matrix

$$\begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{pmatrix}. \tag{7}$$

Other entries of $g$, except for the main diagonal whose elements are equal to 1, are zeroes.

Since $SO_0(n-1,1) \subset U(n-1,1)$, then $U(n-1,1)$ contains the one-parameter subgroups $\{g_{ij}(\theta)\}$, $i<j<n$, and $\{g'_{in}(\theta)\}$, $i=1,2,\dots,n-1$, of $SO_0(n-1,1)$ (see Section 9.1.3). The one-parameter subgroups $\{d_k(\theta)\}$, $k=1,\dots,n$, and $\{g^C_{ij}(\theta)\}$, $i<j<n$, of $U(n)$ also belong to $U(n-1,1)$. We select one more class of one-parameter subgroups, namely, the subgroups $\{g'^C_{jn}(\theta)\}$, $i=1,2,\dots,n-1$, consisting of transformations $\mathbf{z} \to \mathbf{z}'$ of $E^C_{n-1,1}$, where $z_k = z'_k$, $k \neq j$, $k \neq n$,

$$z'_j = z_j \cosh\theta + iz_n \sinh\theta, \quad z'_n = -iz_j \sinh\theta + z_n \cosh\theta. \tag{8}$$

The basis of $\mathfrak{u}(n-1,1)$ consists of the tangent matrices to these one-parameter subgroups at the point $\theta = 0$. They are

$$\begin{aligned} I_{ij} &= E_{ij} - E_{ji}, \quad I'_{in} = E_{in} + E_{ni}, \quad \sqrt{-1}\,E_{kk}, \\ J_{ij} &= \sqrt{-1}(E_{ij} + E_{ji}), \quad J'_{in} = \sqrt{-1}(E_{in} - E_{ni}). \end{aligned} \tag{9}$$

In order to find the basis of the Lie algebra $\mathfrak{su}(n-1,1)$ one has to replace the matrices $\sqrt{-1}\,E_{kk}$, $1 \le k \le n$, by $\sqrt{-1}(E_{kk} - E_{k+1,k+1})$, $1 \le k \le n-1$.

For studying representations of $U(n-1,1)$ it is convenient to consider, along with matrices (9) from $\mathfrak{u}(n-1,1)$, the matrices

$$E_{jn} = \frac{1}{2}(I'_{jn} - iJ'_{jn}), \quad E_{nj} = \frac{1}{2}(I'_{jn} + iJ'_{jn}), \qquad 1 \le j \le n-1, \tag{10}$$

$$E_{jk} = \frac{1}{2}(I_{jk} - iJ_{jk}), \quad E_{kj} = \frac{1}{2}(I_{jk} + iJ_{jk}), \qquad 1 \le j < k \le n-1. \tag{11}$$

The mapping $\theta\colon X \to JXJ$, where $J = \mathrm{diag}(-1,\dots,-1,1)$, is an involutive automorphism of $\mathfrak{u}(n-1,1)$. Therefore, the space of the Lie algebra $\mathfrak{u}(n-1,1)$ decomposes into the direct sum of eigenspaces corresponding to the eigenvalues 1 and $-1$ of $\theta$:

$$\mathfrak{u}(n-1,1) = (\mathfrak{u}(n-1) + \mathfrak{u}(1)) + \mathfrak{p},$$

where

$$\mathfrak{p} = \sum_{i=1}^{n-1} \mathbb{R}I'_{in} + \sum_{i=1}^{n-1} \mathbb{R}J'_{in}$$

corresponds to the eigenvalue $-1$.

For the element

$$X = \sum \alpha_{ij} I_{ij} + \sum \beta_{in} I'_{in} + \sum \gamma_{ij} J_{ij} + \sum \delta_{in} J'_{in} + \sum \sqrt{-1}\,\varepsilon_k E_{kk} \in u(n-1,1)$$

the Killing form (see Section 1.1.6) of $\mathfrak{u}(n-1,1)$ is of the form

$$\begin{aligned} B(X,X) &\equiv \mathrm{Tr}(\mathrm{ad}\, X\, \mathrm{ad}\, X) \\ &= -2n\left[\sum \alpha_{ij}^2 + \sum \gamma_{ij}^2 + \sum \varepsilon_k^2 - \sum \beta_{in}^2 - \sum \delta_{in}^2\right]. \end{aligned}$$

It is positive on $\mathfrak{p}$ and negative on $\mathfrak{u}(n-1)+\mathfrak{u}(1)$. The equality

$$\langle X, Y\rangle = -B(X, \theta Y), \qquad X, Y \in u(n-1,1), \tag{12}$$

defines a positive scalar product on $\mathfrak{u}(n-1,1)$.

The operator $\mathrm{ad}\, I'_{n-1,n}$ is symmetric with respect to the scalar product (12). Therefore, $\mathfrak{u}(n-1,1)$ decomposes into the orthogonal sum of eigenspaces:

$$\mathfrak{u}(n-1,1) = \mathfrak{n}_1 + \mathfrak{n}_2 + (\mathfrak{m}+\mathfrak{a}) + \bar{\mathfrak{n}}_1 + \bar{\mathfrak{n}}_2.$$

Here $\mathfrak{a} = \mathbb{R} I'_{n-1,n}$, $\mathfrak{m}$ is isomorphic to the Lie algebra $\mathfrak{u}(n-2)+\mathfrak{u}(1)$ and consists of matrices $\mathrm{diag}(a, i\alpha, i\alpha)$, $a \in \mathfrak{u}(n-2)$, $\alpha \in \mathbb{R}$, $\mathfrak{n}_1$ consists of matrices

$$\begin{pmatrix} 0_{n-2} & -\mathbf{w}^T & \mathbf{w}^T \\ \bar{\mathbf{w}} & 0 & 0 \\ \bar{\mathbf{w}} & 0 & 0 \end{pmatrix}, \quad \mathbf{w} \in \mathbb{C}^{n-2}, \tag{13}$$

$\bar{\mathfrak{n}}_1$ consists of matrices

$$\begin{pmatrix} 0_{n-2} & -\mathbf{w}^T & -\mathbf{w}^T \\ \bar{\mathbf{w}} & 0 & 0 \\ \bar{\mathbf{w}} & 0 & 0 \end{pmatrix}, \quad \mathbf{w} \in \mathbb{C}^{n-2}, \tag{14}$$

$\mathfrak{n}_2$ consists of matrices

$$\begin{pmatrix} 0_{n-2} & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & -i\alpha & i\alpha \\ \mathbf{0} & -i\alpha & i\alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}, \tag{15}$$

$\bar{\mathfrak{n}}_2$ consists of matrices

$$\begin{pmatrix} 0_{n-2} & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & -i\alpha & -i\alpha \\ \mathbf{0} & i\alpha & i\alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}. \tag{16}$$

The subspaces $(\mathfrak{m}+\mathfrak{a})$, $\mathfrak{n}_1$, $\mathfrak{n}_2$, $\bar{\mathfrak{n}}_1$, $\bar{\mathfrak{n}}_2$ correspond to the eigenvalues 0, 1, 2, −1, −2, respectively.

The subspace $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$ is a maximal nilpotent subalgebra of $\mathfrak{u}(n-1,1)$. The subspace $\bar{\mathfrak{n}} = \bar{\mathfrak{n}}_1 + \bar{\mathfrak{n}}_2$ has the same property.

**11.1.3. Subgroups of $U(n-1,1)$.** The subgroup $A' = \exp \mathfrak{a}$ coincides with the one-parameter subgroup of matrices $g'_{n-1,n}(\theta) \equiv g'_{n-1}(\theta)$, $\theta \in \mathbf{R}$. The maximal compact subgroup of $U(n-1,1)$ coincides with $K = U(n-1) \times U(1)$. The subgroup $M = \exp \mathfrak{m}$ consists of the matrices $\mathrm{diag}(h, e^{i\varphi}, e^{i\varphi})$, $h \in U(n-2)$, $0 \le \varphi < 2\pi$, and is isomorphic to $U(n-2) \times U(1)$. It is the centralizer of $A'$ in $K = U(n-1) \times U(1)$. The normalizer of $A'$ in $K$ coincides with $M' = M \cup g_w M$, where $g_w = \mathrm{diag}(I_{n-3}, s, 1)$, $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We also have $N = \exp(\mathfrak{n}_1 + \mathfrak{n}_2)$, $\bar{N} = \exp(\bar{\mathfrak{n}}_1 + \bar{\mathfrak{n}}_2)$, where $N$ and $\bar{N}$ are maximal nilpotent subgroups in $U(n-1,1)$. The multiplication in $N$ is given by the formula

$$n(\mathbf{z}, \alpha) n(\mathbf{z}', \alpha') = n(\mathbf{z} + \mathbf{z}', \alpha + \alpha' + \mathrm{Im}(\mathbf{z}', \mathbf{z})), \tag{1}$$

where $(\mathbf{z}', \mathbf{z}) = z'_1 \bar{z}_1 + \ldots + z'_{n-2} \bar{z}_{n-2}$. The $N$ is isomorphic to the group of matrices

$$n'(\mathbf{z}, \alpha) = \begin{pmatrix} 1 & \bar{\mathbf{z}} & i\alpha + \frac{(\mathbf{z},\mathbf{z})}{2} \\ 0 & I_{n-2} & \mathbf{z}^T \\ 0 & \mathbf{0} & 1 \end{pmatrix}, \tag{2}$$

which is called the Heisenberg group (see Section 1.1.8).

It is easy to verify that

$$g'_{n-1}(\theta) n(\mathbf{z}, \alpha) g'_{n-1}(-\theta) = n(e^{\theta}\mathbf{z}, e^{2\theta}\alpha). \tag{3}$$

Therefore, $A'N = NA'$ is a subgroup in $U(n-1,1)$ which is the semidirect product of $A'$ and $N$, where $N$ is an invariant subgroup.

For the matrix $m = \mathrm{diag}(g, e^{i\varphi}, e^{i\varphi})$, $g \in U(n-2)$, of the subgroup $M$ one has the relation

$$mn(\mathbf{z}, \alpha) m^{-1} = n(e^{-i\varphi} g\mathbf{z}, \alpha). \tag{4}$$

Consequently, $MN = NM$ is a subgroup of $U(n-1,1)$. It is the semidirect product of its subgroups $M$ and $N$, where $N$ is an invariant subgroup.

Let us consider the subgroup $U(n-2)$ of $M$. Then $U(n-2)N$ is the semidirect product of $U(n-2)$ and $N$. The elements $un(\mathbf{z}, \alpha) \in U(n-2)N$ will be denoted by $g(u; \mathbf{z}, \alpha)$. The multiplication in $U(n-2)N$ is given by the formula

$$\begin{aligned} &g(u_1; \mathbf{z}_1, \alpha_1) g(u_2; \mathbf{z}_2, \alpha_2) \\ &\qquad = g(u_1 u_2; u_2^{-1}\mathbf{z}_1 + \mathbf{z}_2, \alpha_1 + \alpha_2 + \mathrm{Im}(\mathbf{z}_2, u_2^{-1}\mathbf{z}_1)). \end{aligned} \tag{5}$$

Elements of $U(n-2)N$ can be written in the matrix form

$$\begin{pmatrix} 1 & \bar{\mathbf{z}} & i\alpha + \frac{(\mathbf{z},\mathbf{z})}{2} \\ 0 & u & u\mathbf{z}^T \\ 0 & 0 & 1 \end{pmatrix}, \quad u \in U(n-2). \tag{6}$$

**11.1.4. The group $IU(n-1)$.** The set of $n \times n$-matrices

$$g(u;\varphi,\mathbf{z}) = \begin{pmatrix} u & \mathbf{z} \\ 0 & e^{i\varphi} \end{pmatrix}, \tag{1}$$

where $u \in U(n-1)$, $0 \le \varphi < 2\pi$, $\mathbf{z}$ is a column with $n-1$ complex elements, forms the group[1] $JU(n-1)$. The product of matrices of the form (1) is given by the formula

$$g(u_1;\varphi_1,\mathbf{z}_1)g(u_2;\varphi_2,\mathbf{z}_2) = g(u_1u_2;\varphi_1+\varphi_2, u_1\mathbf{z}_2 + e^{i\varphi_2}\mathbf{z}_1). \tag{2}$$

The matrices $g(u;\varphi,\mathbf{0})$ form the subgroup $K$ isomorphic to $U(n-1)\times U(1)$, and the matrices $g(e;0,\mathbf{z})$ form a commutative subgroup denoted by $T$. The group $JU(n-1)$ is the semidirect product of $K$ and $T$, where $T$ is an invariant subgroup. The group $JU(n-1)$ is triple to the groups $U(n)$ and $U(n-1,1)$ (see Section 1.2.3). It is obtained by "straightening" $U(n)$ and $U(n-1,1)$.

It is clear that the space $JU(n-1)/U(n-1)\times U(1) \sim T$ is diffeomorphic to $\mathbb{C}^{n-1}$. This symmetric Riemannian space is triple to the symmetric spaces $P_C^{n-1}$ and $\mathcal{P}_C^{n-1}$.

The subgroup $B$ of the matrices $g(e;\varphi,\mathbf{0})$ is an invariant subgroup of $JU(n-1)$. The quotient group $JU(n-1)/B$ is isomorphic to the group $IU(n-1)$ of the matrices $g(u;\mathbf{z}) \equiv g(u;0,\mathbf{z})$. The multiplication in $IU(n-1)$ is given by the formula

$$g(u_1,\mathbf{z}_1)g(u_2,\mathbf{z}_2) = g(u_1u_2, u_1\mathbf{z}_2 + \mathbf{z}_1). \tag{3}$$

The group $IU(n-1)$ is the semidirect product of the subgroups $U(n-1)$ and $T$.

**11.1.5. Coordinate systems on $S_C^{n-1}$ and $H_C^{n-1}$.** Every point $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n) \in S_C^{n-1}$ is representable in the form

$$\boldsymbol{\xi} = \left(e^{i\varphi_1}\eta_1,\ldots,e^{i\varphi_n}\eta_n\right), \tag{1}$$

where $0 \le \varphi_j < 2\pi$ (we set $\varphi_j = 0$ if $\xi_j = 0$), $\eta_j \ge 0$ and $(\eta_1,\ldots,\eta_n) \in S^{n-1}$. Hence, the spherical and the polyspherical coordinate systems on the real sphere $S^{n-1}$ lead to the spherical and to the polyspherical coordinate systems on $S_C^{n-1}$, respectively. If $\theta_1,\ldots,\theta_{n-1}$ are spherical coordinates on $S^{n-1}$, then spherical coordinates $\varphi_1,\ldots,\varphi_n$, $\theta_1,\ldots,\theta_{n-1}$ on $S_C^{n-1}$ are connected with Cartesian coordinates $\xi_1,\ldots,\xi_n$ by the formulas

$$\left.\begin{array}{l} \xi_1 = e^{i\varphi_1}\sin\theta_{n-1}\ldots\sin\theta_2\sin\theta_1, \\ \xi_2 = e^{i\varphi_2}\sin\theta_{n-1}\ldots\sin\theta_2\cos\theta_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \xi_{n-1} = e^{i\varphi_{n-1}}\sin\theta_{n-1}\cos\theta_{n-2}, \\ \xi_n = e^{i\varphi_n}\cos\theta_{n-1}, \end{array}\right\} \tag{2}$$

[1] In accordance with the notations introduced in Section 1.1.1, one has to denote this group by $I(U(n-1)\times U(1))$. For brevity we use the notation $JU(n-1)$.

where

$$\varphi_j = \arg \xi_j, \quad \sin\theta_j = \frac{|\xi_{j+1}|}{r_{j+1}}, \quad \cos\theta_j = \frac{r_j}{r_{j+1}},$$

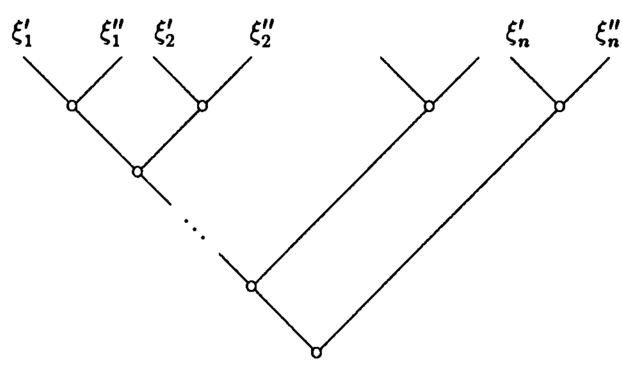
$$r_j = \left(\sum_{s=1}^{j} |\xi_j|^2\right)^{1/2}. \tag{3}$$

When the parameters $\varphi_j$, $\theta_k$ vary in the domain

$$0 \le \varphi_j < 2\pi, \quad 0 \le \theta_k \le \frac{\pi}{2}, \tag{4}$$

then the point $\boldsymbol{\xi} = (\xi_1 \ldots, \xi_n)$ runs all over the sphere $S_C^{n-1}$. In the same way one can write down the formulas connecting Cartesian coordinates on $S_C^{n-1}$ with polyspherical coordinates.

As we noted in Section 11.1.1, the sphere $S_C^{n-1}$ in $E_n^C$ can be considered as the sphere $S^{2n-1}$ in the real space $E_{2n}$. Hence, to every coordinate system on $S_C^{n-1}$ there corresponds a coordinate system on $S^{2n-1}$. If $\xi_1, \ldots, \xi_n$ are Cartesian coordinates on $S_C^{n-1}$ and $\xi_j = \xi_j' + i\xi_j''$, $\xi_j', \xi_j'' \in \mathbb{R}$, then $\xi_1', \xi_1'', \ldots, \xi_n', \xi_n''$ are Cartesian coordinates on $S^{2n-1}$. Spherical and polyspherical coordinates on $S_C^{n-1}$ turn into polyspherical coordinates on $S^{2n-1}$, corresponding to the appropriate trees. For example, to spherical coordinates (2) on $S_C^{n-1}$ there correspond coordinates on $S^{2n-1}$ related to the tree




(5)

Every point $\boldsymbol{\xi} \in S_C^{n-1}$ with coordinates (2) is obtained from $\mathbf{e}_n = (0, \ldots, 0, 1) \in S_C^{n-1}$ by means of the transformation

$$\begin{aligned} g^n(\boldsymbol{\varphi}, \boldsymbol{\theta}) &\equiv g^n(\varphi_1, \ldots, \varphi_n, \theta_1 \ldots, \theta_{n-1}) \\ &= d_1(\varphi_1) \cdot d_2(\varphi_2) g_1(\theta_1) \cdot \ldots \cdot d_n(\varphi_n) g_{n-1}(\theta_{n-1}) \end{aligned} \tag{6}$$

from $U(n)$. Since $S_C^{n-1} = U(n)/U(n-1)$ then every matrix $g \in U(n)$ is representable as a product

$$g = g^n(\boldsymbol{\varphi}, \boldsymbol{\theta})k, \qquad k \in U(n-1), \tag{7}$$

or as the product

$$g = g^{n-1}(\boldsymbol{\varphi}', \boldsymbol{\theta}')d_n(\varphi_n)g_{n-1}(\theta_{n-1})k, \tag{8}$$

where $\boldsymbol{\varphi}' = (\varphi_1, \ldots, \varphi_{n-1})$, $\boldsymbol{\theta}' = (\theta_1, \ldots, \theta_{n-2})$. Thus

$$U(n) = KU_n(1)AK, \tag{9}$$

where $K = U(n-1)$, $A = \{g_{n-1}(\theta)\}$ and $U_n(1) = \{d_n(\varphi)\}$ is a subgroup, isomorphic to $U(1)$.

Polyspherical coordinate systems on $S_C^{n-1}$ lead to factorizations of elements $g \in U(n)$:

$$g = k'k''g_{sn}(\theta)k, \tag{10}$$

where $k \in U(n-1)$, $k' \in U(s)$, $k'' \in U(n-s)$.

As in the case of $S_C^{n-1}$, every point $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in H_C^{n-1}$ is representable in the form

$$\boldsymbol{\xi} = \left(e^{i\varphi_1}\eta_1, \ldots, e^{i\varphi_n}\eta_n\right),$$

where $0 \le \varphi_j < 2\pi$, $\eta_j \ge 0$, $(\eta_1, \ldots, \eta_n) \in H_+^{n-1}$. Hence, to every coordinate system on the upper sheet $H_+^{n-1}$ of the real hyperboloid there corresponds a coordinate system on $H_C^{n-1}$. For example, the coordinate system (11) of Section 9.1.4 on $H_+^{n-1}$ leads to the coordinates $\varphi_1, \ldots, \varphi_n$, $\theta_1, \ldots, \theta_{n-1}$ on $H_C^{n-1}$ such that

$$\left.\begin{array}{l} \xi_1 = e^{i\varphi_1} \sinh\theta_{n-1} \sin\theta_{n-2} \ldots \sin\theta_2 \sin\theta_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \xi_{n-1} = e^{i\varphi_{n-1}} \sinh\theta_{n-1} \cos\theta_{n-2}, \\ \xi_n = e^{i\varphi_n} \cosh\theta_{n-1}, \end{array}\right\} \tag{11}$$

where $0 \le \varphi_j < 2\pi$, $0 \le \theta_{n-1} < \infty$, $0 \le \theta_j \le \frac{\pi}{2}$, $1 \le j \le n-2$.

The transformation

$$g^n(\boldsymbol{\varphi}, \boldsymbol{\theta}) = g^{n-1}(\boldsymbol{\varphi}', \boldsymbol{\theta}')d_n(\varphi_n)g'_{n-1}(\theta_{n-1}) \tag{12}$$

of the group $U(n-1,1)$, where $g^{n-1}(\boldsymbol{\varphi}', \boldsymbol{\theta}')$ is an element of the form (6) from $U(n-1)$, transforms the point $\mathbf{e}_n \in H_C^{n-1}$ into the point with Cartesian coordinates (11). It follows from here that any element $g \in U(n-1,1)$ can be represented as

$$g = g^{n-1}(\boldsymbol{\varphi}', \boldsymbol{\theta}')d_n(\varphi_n)g'_{n-1}(\theta_{n-1})k, \tag{13}$$

where $g^{n-1}(\varphi', \theta') \in U(n-1)$ and $k \in U(n-1)$. Thus,

$$U(n-1,1) = KU_n(1)A'K, \quad K = U(n-1), \quad A' = \{g'_{n-1}(\theta)\}. \tag{14}$$

One can construct for the group $JU(n-1)$ the factorization analogous to (13):

$$g(u;\varphi,\mathbf{z}) = g(e;\varphi,\mathbf{z})g(u,0,\mathbf{0}) = g(u',0,\mathbf{0})g(e,\varphi,\mathbf{0})g_r g(u,0,\mathbf{0}), \tag{15}$$

where $u' = g^{n-1}(\varphi', \theta')$, $u \in U(n-1)$, $g_r = g(0,0,\mathbf{r})$, $\mathbf{r} = (0,\ldots,0,r)$, $r > 0$. Thus,

$$JU(n-1) = U(n-1)U_n(1)\widetilde{A}U(n-1), \quad \widetilde{A} \equiv \{g_r\}. \tag{16}$$

For $IU(n-1)$ we have

$$IU(n-1) = U(n-1)\widetilde{A}U(n-1). \tag{17}$$

**11.1.6. Coordinate systems on $C_C^{n-1}$.** Let us represent points $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n) \in C_C^{n-1}$ in the form

$$\boldsymbol{\xi} = \left(e^{i\varphi_1}\eta_1,\ldots,e^{i\varphi_n}\eta_n\right), \qquad 0 \le \varphi_j < 2\pi,\ \eta_j \ge 0, \tag{1}$$

where $(\eta_1,\ldots,\eta_n)$ is a point of the upper sheet $C_+^{n-1}$ of the real cone. Consequently, coordinate systems on $C_C^{n-1}$ are constructed with the help of those for $C_+^{n-1}$ (see Section 9.1.6). For example, coordinates (4) of Section 9.1.6 lead to the spherical coordinates $\varphi_1,\ldots,\varphi_n$, $\theta_1,\ldots,\theta_{n-2}$, $r$ on $C_C^{n-1}$:

$$\left.\begin{array}{l} \xi_1 = re^{i\varphi_1}\sin\theta_{n-2}\ldots\sin\theta_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \xi_{n-1} = re^{i\varphi_{n-1}}\cos\theta_{n-2}, \\ \xi_n = re^{i\varphi_n} \end{array}\right\}. \tag{2}$$

The point $\boldsymbol{\xi}$ with these coordinates is obtained from $\mathbf{a} = (0,\ldots,0,1,1) \in C_C^{n-1}$ with the help of the transformation

$$g^{n-1}(\varphi',\theta')d_n(\varphi_n)g'_{n-1}(\theta), \quad g^{n-1}(\varphi',\theta') \in U(n-1), \quad r = e^{\theta}. \tag{3}$$

Since $C_C^{n-1} = U(n-1,1)/MN$, then every element $g \in U(n-1,1)$ can be represented as

$$g = g^{n-1}(\varphi',\theta')d_n(\varphi_n)g'_{n-1}(\theta)mn, \qquad n \in N.$$

Hence, we have the decomposition

$$U(n-1,1) = U(n-1)U_n(1)A'N \tag{4}$$

of the group $U(n-1,1)$. It is called the ***Iwasawa decomposition.*** it is uniquely defined for every element $g \in U(n-1,1)$. Considering the point $\mathbf{b} = (0,\dots,0,-1,1)$ instead of $\mathbf{a} = (0,\dots,1,1)$, we obtain the Iwasawa decomposition in the form

$$U(n-1,1) = U(n-1)U_n(1)A'\bar{N}. \tag{4'}$$

**11.1.7. Laplace operators.** Let $\mathbf{z} = (z_1,\dots,z_n) \in E_n^C$. We represent the coordinates $z_j$ of $\mathbf{z}$ in the form $z_j = x_j + iy_j$, where $x_j, y_j \in \mathbf{R}$. Let $\mathfrak{L}$ be the space of infinitely differentiable functions on $E_n^C$. The operator

$$\Delta = \sum_{j=1}^{n} \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right) \tag{1}$$

is defined on $\mathfrak{L}$. It is called the *Laplace operator* on $E_n^C$.

The operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \tag{2}$$

are often used in complex analysis instead of $\partial/\partial x_j$ and $\partial/\partial y_j$. In these notations the operator $\Delta$ is of the form

$$\Delta = 4\sum_{j=1}^{n} \frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z_j}} \tag{3}$$

One can consider $E_n^C$ as the real space $E_{2n}$. Then $\Delta$ becomes the Laplace operator on $E_{2n}$. Therefore, we have

$$\Delta = \frac{1}{r^{2n-1}}\frac{\partial}{\partial r}r^{2n-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0 \tag{4}$$

(see Section 9.1.8), where $\Delta_0$ is the Laplace operator on $S^{2n-1}$ and, consequently, on $S_C^{n-1}$. In order to write down $\Delta_0$ in spherical coordinates (2) of Section 11.1.5 we take into account that $\Delta_0$ coincides with the Laplace operator on $S^{2n-1}$ corresponding to tree (5) of Section 11.1.5. Keeping in mind the results of Section 10.5.2, we have

$$\Delta_0 \equiv \Delta_0^{(n-1)} = \frac{1}{\sin^{2n-3}\theta\cos\theta}\frac{\partial}{\partial\theta}\sin^{2n-3}\theta\cos\theta\frac{\partial}{\partial\theta} + \frac{\Delta_0^{(n-2)}}{\sin^2\theta} + \frac{1}{\cos^2\theta}\frac{\partial^2}{\partial\varphi^2} \tag{5}$$

(we have replaced $\theta_{n-1}$ by $\theta$ and $\varphi_n$ by $\varphi$), where $\Delta_0^{(n-2)}$ is the Laplace operator on $S_C^{n-2}$ (and, consequently, on the real sphere $S^{2n-3}$).

The left quasi-regular representation $L$ of the group $U(n)$ is defined in $\mathfrak{L}$ by operators $L(g)$:

$$(L(g)f)(\mathbf{z}) = f(g^{-1}\mathbf{z}), \qquad g \in U(n). \tag{6}$$

One can consider $\mathfrak{L}$ as the space $\mathfrak{L}_R$ of the infinitely differentiable functions on $E_{2n}$. The left quasi-regular representation of the group $SO(2n)$ in $\mathfrak{L}_R$ is defined. Since $U(n) \subset SO(2n)$, then the representation $L$ of $U(n)$ coincides with the restriction of the left quasi-regular representation of $SO(2n)$ onto $U(n)$. Hence, it follows from the results of Section 9.1.8 that the operators $L(g)$, $g \in U(n)$, commute with the Laplace operator $\Delta$ and, consequently, with $\Delta_0$.

In polyspherical coordinates the operator $\Delta_0$ has the form

$$\Delta_0 = \frac{1}{\sin^{2s-1}\theta\cos^{2n-2s-1}\theta}\frac{\partial}{\partial\theta}\sin^{2s-1}\theta\cos^{2n-2s-1}\theta\frac{\partial}{\partial\theta} + \\ + \frac{1}{\sin^2\theta}\Delta_0^{(s-1)} + \frac{1}{\cos^2\theta}\Delta_0^{(n-s-1)}, \tag{7}$$

where $\Delta_0^{(s-1)}$ and $\Delta_0^{(n-s-1)}$ are the Laplace operators on $S_C^{s-1}$ and on $S_C^{(n-s-1)}$, respectively.

Instead of the Laplace operator, on the space $E_{n-1,1}^C$ we have the operator

$$\Box = 4\left(\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} + \ldots + \frac{\partial^2}{\partial z_{n-1}\partial \overline{z_{n-1}}} - \frac{\partial^2}{\partial z_n \partial \overline{z_n}}\right) = \\ = \sum_{j=1}^{n-1}\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right) - \frac{\partial^2}{\partial x_n^2} - \frac{\partial^2}{\partial y_n^2}. \tag{8}$$

Multiplying the left and the right hand sides of formula (1) of Section 11.1.5 by $r$, $0 \le r < \infty$, we obtain the coordinate system on $E_{n-1,1}^C$ in which

$$\Box = -\frac{1}{r^{2n-1}}\frac{\partial}{\partial r}r^{2n-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\Box_0, \tag{9}$$

where $\Box_0$ is the Laplace operator on the hyperboloid $H_C^{n-1}$. In coordinates (11) of Section 11.1.5 we have

$$\Box_0 = \frac{1}{\sinh^{2n-3}\theta\cosh\theta}\frac{\partial}{\partial\theta}\sinh^{2n-3}\theta\cosh\theta\frac{\partial}{\partial\theta} + \frac{\Delta_0^{(n-2)}}{\sinh^2\theta} - \frac{1}{\cosh^2\theta}\frac{\partial^2}{\partial\varphi^2}, \tag{10}$$

where $\Delta_0^{(n-2)}$ is the Laplace operator on $S_C^{n-2}$. The operator $\Box_0$ commutes with the operators of the left quasi-regular representation of the group $U(n-1,1)$ on $H_C^{n-1}$.

**11.1.8. Invariant measures.** The invariant measure on $S_C^{n-1}$ in spherical coordinates (2) of Section 11.1.5 coincides with the invariant measure on the real sphere $S^{2n-1}$ in coordinates corresponding to tree (5) of Section 11.1.5:

$$d\xi \equiv d\xi(\varphi,\theta) = \frac{(n-1)!}{2\pi^n} \prod_{r=1}^{n} d\varphi_r \prod_{k=1}^{n-1} \sin^{2k-1}\theta_k \cos\theta_k d\theta_k. \tag{1}$$

The factor is chosen such that the measure of the whole sphere is equal to 1.

The elements $g^n(\varphi,\theta) \in U(n)$ correspond to points $\xi(\varphi,\theta) \in S_C^{n-1}$ (see formula (6) of Section 11.1.5). Therefore, measure (1) on $S_C^{n-1}$ defines the measure $dg^n(\varphi,\theta)$ on the elements $g^n(\varphi,\theta)$ which parametrize points of the quotient space $U(n)/U(n-1)$. If $g \in U(n)$ is represented as product (7) of Section 11.1.5, then the invariant measure $dg$ on $U(n)$ is given by the formula

$$dg = dg^n(\varphi,\theta)dk, \qquad k \in U(n-1), \tag{2}$$

where $dk$ is the normalized invariant measure on $U(n-1)$.

Note that (1) can be written as

$$d\xi(\varphi,\theta) = \frac{n-1}{\pi} \sin^{2n-3}\theta_{n-1} \cos\theta_{n-1} d\theta_{n-1} d\varphi_n d\xi'(\varphi',\theta'), \tag{3}$$

where $\xi'(\varphi',\theta') \in S_C^{n-2}$, $\varphi' = (\varphi_1,\dots,\varphi_{n-1})$, $\theta' = (\theta_1,\dots,\theta_{n-2})$.

In polyspherical coordinates on $S_C^{n-1}$ corresponding to factorization (10) of Section 11.1.5 of elements from $U(n)$, the normalized invariant measure is given by the formula

$$d\xi = \frac{2(n-1)!}{(s-1)!(n-s-1)!} \sin^{2s-1}\theta \cos^{2n-2s-1}\theta d\theta\, d\eta\, d\zeta, \tag{4}$$

where $d\eta$ and $d\zeta$ are the normalized invariant measures on $S_C^{s-1}$ and $S_C^{n-s-1}$, respectively.

The measure $d\xi$ on $H_C^{n-1}$, invariant with respect to $U(n-1,1)$, in coordinates (11) of Section 11.1.5 is given by the formula

$$d\xi = \sinh^{2n-3}\theta_{n-1} \cosh\theta_{n-1} d\theta_{n-1} \prod_{r=1}^{n} d\varphi_r \prod_{k=1}^{n-2} \sin^{2k-1}\theta_k \cos\theta_k d\theta_k =$$

$$= \frac{2\pi^{n-1}}{(n-2)!} \sinh^{2n-3}\theta_{n-1} \cosh\theta_{n-1} d\varphi_n d\theta_{n-1} d\eta, \tag{5}$$

where $d\eta$ is the normalized invariant measure on $S_C^{n-2}$. By means of measure (5) and formula (13) of Section 11.1.5, one defines the measure on $U(n-1,1)$:

$$dg = d\xi dk, \qquad k \in U(n-1). \tag{6}$$

## 11.2. Class 1 Representations of the Groups $U(n)$, $U(n-1,1)$ and $IU(n-1)$

**11.2.1. Harmonic polynomials on $\mathbf{C}^n$.** We denote by $\mathfrak{R}_C^{n\ell\ell'}$ the linear space of homogeneous polynomials $p(\mathbf{z},\bar{\mathbf{z}}) \equiv p(z_1,\ldots,z_n,\bar{z}_1,\ldots,\bar{z}_n)$ of homogeneity degree $\ell$ in $z_1,\ldots,z_n$ and of homogeneity degree $\ell'$ in $\bar{z}_1,\ldots,\bar{z}_n$:

$$p(a\mathbf{z},b\bar{\mathbf{z}}) = a^{\ell}b^{\ell'}p(\mathbf{z},\bar{\mathbf{z}}), \quad a,b \in \mathbf{R}. \tag{1}$$

It is clear that

$$\mathfrak{R}_C^{n\ell\ell'} = \mathfrak{R}_C^{n\ell 0}\mathfrak{R}_C^{n0\ell'}. \tag{2}$$

The dimensionalities of the spaces $\mathfrak{R}_C^{n\ell 0}$ and $\mathfrak{R}_C^{n0\ell}$ coincide and are equal to

$$r(n,\ell) = \frac{(n+\ell-1)!}{(n-1)!\ell!}.$$

Consequently,

$$\dim \mathfrak{R}_C^{n\ell\ell'} = \frac{(n+\ell-1)!(n+\ell'-1)!}{(n-1)!^2\ell!\ell'!}. \tag{3}$$

It is easy to verify that

$$\frac{1}{(1-t)^n(1-s)^n} = \sum_{\ell,\ell'=0}^{\infty}(\dim \mathfrak{R}_C^{n\ell\ell'})t^{\ell}s^{\ell'}. \tag{4}$$

Polynomials $p \in \mathfrak{R}_C^{n\ell\ell'}$ are uniquely defined by their values on $S_C^{n-1}$. The space of polynomials from $\mathfrak{R}_C^{n\ell\ell'}$, considered on $S_C^{n-1}$, is denoted by $\widetilde{\mathfrak{R}}_C^{n\ell\ell'}$. It follows from (1) that for $\varphi \in \widetilde{\mathfrak{R}}_C^{n\ell\ell'}$ we have

$$\varphi(e^{i\theta}\boldsymbol{\xi}) = e^{i(\ell-\ell')\theta}\varphi(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in S_C^{n-1}. \tag{5}$$

One can regard polynomials $p \in \mathfrak{R}_C^{n\ell\ell'}$ as polynomials $p(\mathbf{x}+i\mathbf{y},\mathbf{x}-i\mathbf{y})$ of real and imaginary parts $x_j$, $y_j$ of the variables $z_j$. It follows from (1) that these polynomials are homogeneous of degree $\ell+\ell'$ in $x_1,\ldots,x_n$, $y_1,\ldots,y_n$. Thus, if $\mathfrak{R}^{2n,\ell+\ell'}$ is the space of homogeneous polynomials of degree $\ell+\ell'$ in $x_1,\ldots,x_n$, $y_1,\ldots,y_n$, then

$$\mathfrak{R}_C^{n\ell\ell'} \subset \mathfrak{R}^{2n,\ell+\ell'}. \tag{6}$$

If $p \in \mathfrak{R}_C^{n,\ell-1,\ell'-1}$ and $(\mathbf{z},\mathbf{z}) = |z_1|^2+\ldots+|z_n|^2$, then $(\mathbf{z},\mathbf{z})p(\mathbf{z},\bar{\mathbf{z}}) \in \mathfrak{R}_C^{n\ell\ell'}$. Therefore,

$$\widetilde{\mathfrak{R}}_C^{n,\ell-1,\ell'-1} \subset \widetilde{\mathfrak{R}}_C^{n\ell\ell'}. \tag{6'}$$

A polynomial $p$ of $z_1,\ldots,z_n$, $\bar{z}_1,\ldots,\bar{z}_n$ is said to be harmonic if

$$\Delta p \equiv \frac{1}{4}\left(\frac{\partial^2}{\partial z_1\partial\overline{z_1}}+\ldots+\frac{\partial^2}{\partial z_n\partial\overline{z_n}}\right)p(\mathbf{z},\bar{\mathbf{z}}) = 0. \tag{7}$$

The set of harmonic polynomials from $\mathfrak{R}_C^{n\ell\ell'}$ will be denoted by $\mathfrak{H}_C^{n\ell\ell'}$. The set of functions on $S_C^{n-1}$ obtained by restricting polynomials from $\mathfrak{H}_C^{n\ell\ell'}$ onto $S_C^{n-1}$ will be denoted by $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$. It follows from (7) that

$$\mathfrak{H}_C^{n\ell 0} = \mathfrak{R}_C^{n\ell 0}, \quad \mathfrak{H}_C^{n0\ell} = \mathfrak{R}_C^{n0\ell}. \tag{8}$$

As an example of a polynomial from $\mathfrak{H}_C^{n\ell\ell'}$, one can regard the polynomial

$$(a_1 z_1 + \ldots + a_n z_n)^\ell (b_1 \overline{z_1} + \ldots + b_n \overline{z_n})^{\ell'}, \tag{8'}$$

where $a_1 b_1 + \ldots + a_n b_n = 0$.

Considering $p \in \mathfrak{H}_C^{n\ell\ell'}$ as polynomials from $\mathfrak{R}^{2n,\ell+\ell'}$, we obtain the imbedding

$$\mathfrak{H}_C^{n\ell\ell'} \subset \mathfrak{H}^{2n,\ell+\ell'}, \tag{9}$$

where $\mathfrak{H}^{2n,\ell+\ell'}$ is the space of harmonic polynomials from $\mathfrak{R}^{2n,\ell+\ell'}$.

Let $\mathfrak{L}^2(S_C^{n-1})$ be the Hilbert space of functions on $S_C^{n-1}$ with the scalar product

$$(f_1, f_2) = \int_{S_C^{n-1}} f_1(\xi)\overline{f_2(\xi)}d\xi, \tag{10}$$

where $d\xi$ is the measure invariant with respect to $U(n)$. By identifying $S_C^{n-1}$ with $S^{2n-1}$, we identify $\mathfrak{L}^2(S_C^{n-1})$ with $\mathfrak{L}^2(S^{2n-1})$. It is clear that $\widetilde{\mathfrak{H}}_C^{n\ell\ell'} \subset \mathfrak{L}^2(S_C^{n-1})$.

If $(\ell, \ell') \neq (m, m')$, then the spaces $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$ and $\widetilde{\mathfrak{H}}_C^{nmm'}$ are orthogonal in $\mathfrak{L}^2(S_C^{n-1})$. Actually, if $\ell + \ell' \neq m + m'$, then $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$ and $\widetilde{\mathfrak{H}}_C^{nmm'}$ belong to the orthogonal spaces $\widetilde{\mathfrak{H}}^{2n,\ell+\ell'}$ and $\widetilde{\mathfrak{H}}^{2n,m+m'}$ and, therefore, are orthogonal. If $\ell + \ell' = m + m'$, then $\ell - \ell' \neq m - m'$. Then for $f_1 \in \widetilde{\mathfrak{H}}_C^{n\ell\ell'}$, $f_2 \in \widetilde{\mathfrak{H}}_C^{nmm'}$ we have

$$(f_1, f_2) = \int_{S_C^{n-1}} f_1(e^{i\theta}\xi)\overline{f_2(e^{i\theta}\xi)}d\xi = e^{i(\ell-\ell'-m+m')\theta}(f_1, f_2) \tag{11}$$

(see formula (5)). Hence, $(f_1, f_2) = 0$ and $\widetilde{\mathfrak{H}}_C^{n\ell\ell'} \perp \widetilde{\mathfrak{H}}_C^{nmm'}$.

The space $\mathfrak{L}^2(S_C^{n-1})$ also contains $\widetilde{\mathfrak{R}}_C^{n\ell\ell'}$. Setting $f_1 \in \widetilde{\mathfrak{R}}_C^{n\ell\ell'}$, $f_2 \in \widetilde{\mathfrak{R}}_C^{nmm'}$ into (11), we obtain

$$\widetilde{\mathfrak{R}}_C^{n\ell\ell'} \perp \widetilde{\mathfrak{R}}_C^{nmm'} \quad \text{if} \quad \ell - \ell' \neq m - m'. \tag{12}$$

It follows from (9) that $\mathfrak{H}_C^{n\ell\ell'} \subset \mathfrak{H}^{2n,m}$, if $\ell + \ell' = m$. Let us show that

$$\mathfrak{H}^{2n,m} = \sum_{\ell+\ell'=m} \oplus \mathfrak{H}_C^{n\ell\ell'}. \tag{13}$$

For this we fix $f \in \mathfrak{H}^{2n,m}$. Then $f = f_0 + f_1 + \ldots + f_m$, where $f_k \in \mathfrak{R}_C^{n,k,m-k}$. We have $\Delta f_k \in \mathfrak{R}_C^{n,k-1,m-k-1}$. It follows from here that $\Delta f = \sum_k \Delta f_k = 0$. Since $\Delta f_k$, $k = 0, 1, \ldots, m$, belong to different spaces $\mathfrak{R}_C^{nss'}$, then $\Delta f_k = 0$, that is, $f_k \in \mathfrak{H}_C^{n,k,m-k}$. Formula (13) is proved.

We have from formula (2′) of Section 9.2.6 and (13) that

$$\mathfrak{L}^2(S_C^{n-1}) = \sum_{\ell,\ell'=0}^{\infty} \oplus \widetilde{\mathfrak{H}}_C^{n\ell\ell'}. \tag{14}$$

Now we show that

$$\widetilde{\mathfrak{R}}_C^{n\ell\ell'} = \sum_{s=0}^{p} \oplus \widetilde{\mathfrak{H}}_C^{n,\ell-s,\ell'-s}, \tag{15}$$

where $p = \min(\ell, \ell')$. For this we note that (6), (13) and the formula

$$\widetilde{\mathfrak{R}}^{nk} = \sum_{r=0}^{[k/2]} \oplus \widetilde{\mathfrak{H}}^{n,k-r}$$

(it follows from equality (1) of Section 9.2.3) imply that

$$\widetilde{\mathfrak{R}}_C^{n\ell\ell'} \subset \sum_{m,m'} \oplus \widetilde{\mathfrak{H}}_C^{nmm'}, \qquad m + m' \le \ell + \ell'. \tag{16}$$

From (6′) and from the fact that $\widetilde{\mathfrak{H}}_C^{nkk'} \subset \widetilde{\mathfrak{R}}_C^{nkk'}$ we derive

$$\widetilde{\mathfrak{R}}_C^{n\ell\ell'} \supset \sum_{s=0}^{p} \oplus \widetilde{\mathfrak{H}}_C^{n,\ell-s,\ell'-s}. \tag{17}$$

Let $q$ and $q'$ be non-negative integers such that $q - q' \ne \ell - \ell'$. Then according to (12), we have $\widetilde{\mathfrak{R}}_C^{n\ell\ell'} \perp \widetilde{\mathfrak{R}}_C^{nqq'}$. Since $\widetilde{\mathfrak{H}}_C^{nqq'} \subset \widetilde{\mathfrak{R}}_C^{nqq'}$, then $\widetilde{\mathfrak{R}}_C^{n\ell\ell'} \perp \widetilde{\mathfrak{H}}_C^{nqq'}$. Therefore, (16) and (17) imply equality (15).

By considering polynomials on the whole space $\mathbf{C}^n$, we can rewrite (15) in the form

$$\mathfrak{R}_C^{n\ell\ell'} = \sum_{s=0}^{p} r^{2s} \mathfrak{H}_C^{n,\ell-s,\ell'-s}, \tag{15'}$$

where the sum is direct and $r^2 = |z_1|^2 + \ldots + |z_n|^2$. Therefore,

$$\mathfrak{R}_C^{n\ell\ell'} = \mathfrak{H}_C^{n\ell\ell'} + r^2 \mathfrak{R}_C^{n,\ell-1,\ell'-1}. \tag{15''}$$

We derive from (3) and (15″) that

$$\dim \mathfrak{H}_C^{n\ell\ell'} = \dim \mathfrak{R}_C^{n\ell\ell'} - \dim \mathfrak{R}_C^{n,\ell-1,\ell'-1} = \frac{(\ell+n-2)!(\ell'+n-2)!(\ell+\ell'+n-1)}{(n-1)!(n-2)!\ell!\ell'!}. \tag{18}$$

The first part of this formula yields

$$\sum_{\ell,\ell'=0}^{\infty} (\dim \mathfrak{H}_C^{n\ell\ell'}) t^\ell s^{\ell'} = (1-ts) \sum_{\ell,\ell'=0}^{\infty} (\dim \mathfrak{R}_C^{n\ell\ell'}) t^\ell s^{\ell'}.$$

It follows from this equality and from (4) that

$$\frac{1-ts}{(1-t)^n(1-s)^n} = \sum_{\ell,\ell'=0}^{\infty} (\dim \mathfrak{H}_C^{n\ell\ell'}) t^\ell s^{\ell'}. \tag{19}$$

The Laplace operator $\Delta_0$ on $S_C^{n-1}$ coincides with the Laplace operator on the real sphere $S^{2n-1}$ (see Section 11.1.7). Therefore, it follows from formula (4) of Section 9.2.6 and from (9) that

$$\Delta_0 f = -(\ell+\ell')(\ell+\ell'+2n-2)f, \tag{20}$$

where $f \in \widetilde{\mathfrak{H}}_C^{n\ell\ell'}$. Thus, (15) is the decomposition of $\widetilde{\mathfrak{R}}_C^{n\ell\ell'}$ into the sum of eigenspaces of the Laplace operator $\Delta_0$.

**11.2.2. The representations $T^{n\ell\ell'}$ of $U(n)$.** The space $\mathfrak{R}_C^{n\ell\ell'}$ is invariant with respect to the operators

$$(L^{n\ell\ell'}(g)p)(\mathbf{z}, \bar{\mathbf{z}}) = p(g^{-1}\mathbf{z}, g^{-1}\bar{\mathbf{z}}), \quad g \in U(n). \tag{1}$$

Hence, the correspondence $g \to L^{n\ell\ell'}(g)$ is a representation of the group $U(n)$ in $\mathfrak{R}_C^{n\ell\ell'}$.

The Laplace operator $\Delta$ on $E_n^C$ commutes with the left shift operators (see Section 11.1.7). Consequently,

$$\Delta L^{n\ell\ell'}(g) = L^{n\ell\ell'}(g)\Delta, \quad g \in U(n).$$

It means that eigenspaces of the operator $\Delta$, in particular $\mathfrak{H}_C^{n\ell\ell'}$, are invariant with respect to the operators $L^{n\ell\ell'}(g)$, $g \in U(n)$. The obtained representation of the group $U(n)$ in the space $\mathfrak{H}_C^{n\ell\ell'}$ of homogeneous harmonic polynomials of $\mathbf{z}$ and $\bar{\mathbf{z}}$ is denoted by $T^{n\ell\ell'}$:

$$(T^{n\ell\ell'}(g)f)(\mathbf{z}, \bar{\mathbf{z}}) = f(g^{-1}\mathbf{z}, g^{-1}\bar{\mathbf{z}}), \quad g \in U(n), \ f \in \mathfrak{H}_C^{n\ell\ell'}. \tag{2}$$

Functions $f(\mathbf{z},\bar{\mathbf{z}})$ from $\mathfrak{R}_C^{n\ell\ell'}$ are uniquely determined by their values on $S_C^{n-1}$. Therefore, one can consider the representations $L^{n\ell\ell'}$ and $T^{n\ell\ell'}$ on $\widetilde{\mathfrak{R}}_C^{n\ell\ell'}$ and $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$, respectively:

$$(L^{n\ell\ell'}(g)f)(\boldsymbol{\xi}) = f(g^{-1}\boldsymbol{\xi}), \quad (T^{n\ell\ell'}(g)f)(\boldsymbol{\xi}) = f(g^{-1}\boldsymbol{\xi}), \quad g \in U(n). \tag{3}$$

Formula (10) of Section 11.2.1 gives a scalar product on $\widetilde{\mathfrak{R}}_C^{n\ell\ell'}$. It is easy to see that the operators $L^{n\ell\ell'}(g)$, $g \in U(n)$, conserve this scalar product. Hence, the representations $L^{n\ell\ell'}$ and $T^{n\ell\ell'}$ are unitary in $\widetilde{\mathfrak{R}}_C^{n\ell\ell'}$ and $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$.

Irreducibility and pairwise nonequivalence of the representations $T^{n\ell\ell'}$, $\ell,\ell' \in \mathbb{Z}_+ \cup \{0\}$, can be proved in the same way as it was done in Section 9.2.6 for the representations $T^{n\ell}$ of the group $SO(n)$. We use another method. Let $P_{\ell\ell'}$ is the projection operator from $\mathfrak{L}^2(S_C^{n-1})$ onto $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$. Since for a fixed $\boldsymbol{\xi}_0 \in S_C^{n-1}$ the correspondence $f \to (P_{\ell\ell'}f)(\boldsymbol{\xi}_0)$ is a bounded linear functional on $\mathfrak{L}^2(S_C^{n-1})$, then according to the Riesz lemma, there exists the unique function $K_{\boldsymbol{\xi}_0} \in \mathfrak{L}^2(S_C^{n-1})$ such that

$$(P_{\ell\ell'}f)(\boldsymbol{\xi}_0) = \int_{S_C^{n-1}} f(\boldsymbol{\xi})\overline{K_{\boldsymbol{\xi}_0}(\boldsymbol{\xi})}d\boldsymbol{\xi} \equiv (f, K_{\boldsymbol{\xi}_0}). \tag{4}$$

Since $P_{\ell\ell'}f = 0$ for $f \perp \widetilde{\mathfrak{H}}_C^{n\ell\ell'}$, then $K_{\boldsymbol{\xi}_0} \in \widetilde{\mathfrak{H}}_C^{n\ell\ell'}$. If $f \in \widetilde{\mathfrak{H}}_C^{n\ell\ell'}$, then we have from (4) that $f(\boldsymbol{\xi}) = (f, K_{\boldsymbol{\xi}})$. In particular,

$$K_{\boldsymbol{\eta}}(\boldsymbol{\xi}) = (K_{\boldsymbol{\eta}}, K_{\boldsymbol{\xi}}), \tag{5}$$

that is,

$$K_{\boldsymbol{\eta}}(\boldsymbol{\xi}) = \overline{K_{\boldsymbol{\xi}}(\boldsymbol{\eta})}. \tag{6}$$

Let $L(g)$, $g \in U(n)$, be an operator of the left quasi-regular representation of the group $U(n)$ in the space $\mathfrak{L}^2(S_C^{n-1})$:

$$(L(g)f)(\boldsymbol{\xi}) = f(g^{-1}\boldsymbol{\xi}), \quad f \in \mathfrak{L}^2(S_C^{n-1}).$$

Since the projection operator $P_{\ell\ell'}$ commutes with $L(g)$, then

$$(f, K_{g\boldsymbol{\xi}}) = (P_{\ell\ell'}f)(g\boldsymbol{\xi}) = (L(g^{-1})(P_{\ell\ell'}f))(\boldsymbol{\xi}) = (L(g^{-1})f, K_{\boldsymbol{\xi}}) = (f, L(g)K_{\boldsymbol{\xi}}).$$

Since $f$ is arbitrary, then

$$K_{g\boldsymbol{\xi}}(\boldsymbol{\eta}) = (L(g)K_{\boldsymbol{\xi}})(\boldsymbol{\eta}). \tag{7}$$

We derive from here that

$$K_{g\boldsymbol{\xi}}(g\boldsymbol{\xi}) = K_{\boldsymbol{\xi}}(\boldsymbol{\xi}), \qquad g \in U(n), \tag{8}$$

$$K_{\boldsymbol{\xi}}(\boldsymbol{\eta}) = (L(h)K_{\boldsymbol{\xi}})(\boldsymbol{\eta}), \qquad h \in H_{\boldsymbol{\xi}}, \tag{9}$$

where $H_{\xi}$ is the stabilizer of $\xi$. It follows from (8) that for any $\xi$ and $\eta$ from $S_C^{n-1}$, we have

$$K_{\xi}(\xi) = K_{\eta}(\eta). \tag{10}$$

Now we show that *for every* $\xi_0 \in S_C^{n-1}$ *there exists the unique function* $f$ *in* $\mathfrak{H}_C^{n\ell\ell'}$ *such that* $f(\xi_0) = 1$ *and* $(L(h)f)(\xi) = f(\xi)$, $h \in H_{\xi_0}$. Existence of this function follows from (9). Since the representation $T^{n\ell\ell'}$ of $U(n)$ acts in $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$, and $S_C^{n-1}$ is transitive with respect to $U(n)$, then it is sufficient to prove uniqueness for the point $\mathbf{e}_n \in S_C^{n-1}$. In this case $H_{\xi_0} = U(n-1)$. We write down points $\mathbf{z} \in \mathbb{C}^n$ in the form $\mathbf{z} = (\mathbf{z}', z_n)$, where $\mathbf{z}' = (z_1, \ldots, z_{n-1})$. Then the invariance $(L(h)f)(\xi) = f(\xi)$, $h \in U(n-1)$, means that for every fixed $z_n$ the function $f$ is a polynomial of $r^2 \equiv |\mathbf{z}'|^2 = |z_1|^2 + \ldots + |z_{n-1}|^2$. Since $f \in \mathfrak{H}_C^{n\ell\ell'}$, then

$$f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{i=0}^{p} c_i r^{2i} z_n^{\ell-i}\, \overline{z_n}^{\ell'-i}, \tag{11}$$

where $p = \min(\ell, \ell')$, $c_i \in \mathbb{C}$, $c_0 = 1$. Differentiating (11), we have

$$4\Delta f \equiv \sum_{k=1}^{n} \frac{\partial^2 f}{\partial z_k \partial \overline{z_k}} = \sum_{i=0}^{p-1} b_i r^{2i} z_n^{\ell-i-1}\, \overline{z_n}^{\ell'-i-1}, \tag{12}$$

where

$$b_i = (\ell - i)(\ell' - i)c_i + (i+1)(n+i-1)c_{i+1}, \quad 0 \le i < p. \tag{13}$$

Since $f$ is a harmonic polynomial, then $\Delta f = 0$. Hence, the coefficients $b_i$ in (12) are equal to 0 and the coefficients $c_i$ are successively defined from (13). Therefore, uniqueness of the function $f$ with required properties is proved.

Let $A$ be an operator from $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$ into $\widetilde{\mathfrak{H}}_C^{nmm'}$ such that for all $g \in U(n)$ we have

$$AT^{n\ell\ell'}(g) = T^{nmm'}(g)A. \tag{14}$$

Let us show that $A = 0$ if $(\ell, \ell') \neq (m, m')$, and $A$ is a multiple of the identity operator if $(\ell, \ell') = (m, m')$.

Let $K_{\xi_0}$ be a function from formula (4), belonging to the space $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$, and let $K'_{\xi_0}$ be a similar function from the space $\widetilde{\mathfrak{H}}_C^{nmm'}$. For every element $h \in H_{\xi_0}$, we have

$$L(h)(AK_{\xi_0}) = A(L(h)K_{\xi_0}) = AK_{\xi_0}.$$

Then, by virtue of uniqueness of the function invariant with respect to $H_{\xi_0}$, there exists $c(\xi) \in \widetilde{\mathfrak{H}}_C^{nmm'}$ such that $AK_{\xi_0} = c(\xi_0)K'_{\xi_0}$. In particular,

$$(AK_{\xi_0})(\xi_0) = c(\xi_0)K'_{\xi_0}(\xi_0). \tag{15}$$

In accordance with (10), $K'_{\xi_0}(\xi_0)$ does not depend on $\xi_0$. It follows from (7) and (8) that, for $\xi = g\xi_0$, $g \in U(n)$, we have

$$(AK_\xi)(\xi) = (AL(g)K_{\xi_0})(g\xi_0) = (AK_{\xi_0})(\xi_0).$$

Thus, $(AK_{\xi_0})(\xi_0)$ is also independent of $\xi_0$. Therefore, $c(\xi_0)$ does not depend on $\xi_0$: $c(\xi_0) = c$. We have $AK_{\xi_0} = cK'_{\xi_0}$, $\xi_0 \in S_C^{n-1}$.

If $f \in \widetilde{\mathfrak{H}}_C^{n\ell\ell'}$, then we derive from (4) and (6) that

$$f(\xi) = \int f(\eta)K_\eta(\xi)d\eta.$$

Hence, taking into account equality (4) for $P_{mm'}$, we have

$$Af = \int f(\eta)(AK_\eta)d\eta = c\int f(\eta)K'_\eta d\eta = cP_{mm'}f, \quad f \in \widetilde{\mathfrak{H}}_C^{n\ell\ell'}.$$

If $(\ell,\ell') \neq (m,m')$, then $P_{mm'}f = 0$ and $A \equiv 0$. If $(\ell,\ell') = (m,m')$, then $A = cI$, where $I$ is the identity operator in $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$.

It was proved that if an operator $A$ intertwines representations $T^{n\ell\ell'}$ and $T^{nmm'}$ of the group $U(n)$, then $A = 0$ if $(\ell,\ell') \neq (m,m')$ and $A = cI$ if $(\ell,\ell') = (m,m')$. From here and from the results of Section 2.2.8, we derive irreducibility and pairwise nonequivalence of representations $T^{n\ell\ell'}$.

Formula (18) of Section 11.2.1 implies the formula for dimensionality of the representation $T^{n\ell\ell'}$:

$$\dim T^{n\ell\ell'} = \frac{(\ell+n-2)!(\ell'+n-2)!(\ell+\ell'+n-1)}{(n-1)!(n-2)!\ell!\ell'!}. \tag{16}$$

The diagonal matrices $d(\varphi) = \mathrm{diag}(e^{i\varphi},\ldots,e^{i\varphi})$ commute with elements of the group $U(n)$. Therefore, the operators $T^{n\ell\ell'}(d(\varphi))$ are multiples of the identity operator. It follows from (2) that

$$T^{n\ell\ell'}(d(\varphi)) = e^{i(\ell'-\ell)\varphi}I. \tag{17}$$

The subspace of functions from $\mathfrak{L}^2(_C^{n-1})$, satisfying differential equation (20) of Section 11.2.1 and the equality

$$(L(d(\varphi))f)(\xi) \equiv f(d(-\varphi)\xi) = e^{i(\ell'-\ell)\varphi}f(\xi), \tag{18}$$

coincides with $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$.

**11.2.3. Spectral decompositions of the representations $T^{n\ell\ell'}$.** Formula (14) of Section 11.2.1 and the definition of $T^{n\ell\ell'}$ imply the decomposition of the left quasi-regular representation $L$ of the group $U(n)$ in the space $\mathfrak{L}^2(S_C^{n-1})$:

$$L = \sum_{\ell,\ell'=0}^{\infty} \oplus T^{n\ell\ell'}. \tag{1}$$

Formula (15) of Section 11.2.1 leads to the decomposition of the representation $L^{n\ell\ell'}$ of $U(n)$:

$$L^{n\ell\ell'} = \sum_{s=0}^{p} \oplus T^{n,\ell-s,\ell'-s}, \quad p = \min(\ell,\ell'). \tag{2}$$

The irreducible representation $T^{2n,m}$ of $SO(2n)$ is realized in the space $\mathfrak{H}^{2n,m}$ (see Section 9.2.3). Therefore, we obtain from formula (13) of Section 11.2.1 the decomposition of the restriction of $T^{2n,m}$ onto the subgroup $U(n)$:

$$T^{2n,m} \downarrow {SO(2n) \atop U(n)} = \sum_{\ell+\ell'=m} \oplus T^{n\ell\ell'}. \tag{3}$$

The representation $T^{n\ell 0}$ of $U(n)$ is realized in the space $\mathfrak{H}_C^{n\ell 0} = \mathfrak{R}_C^{n\ell 0}$ (see formula (8) of Section 11.2.1). The restriction of $T^{n\ell 0}$ onto $SO(n)$ is equivalent to the representation

$$(L^{n\ell}(g)p)(x) = p(g^{-1}x), \qquad g \in SO(n),$$

in the space $\mathfrak{R}^{n\ell}$ of homogeneous polynomials of degree $\ell$ in $x_1, \ldots, x_n$. Since $\mathfrak{R}^{n\ell} = \sum_{k=0}^{[\ell/2]} \oplus \mathfrak{H}^{n,\ell-2k}$, then $L^{n\ell}$ decomposes into the sum of the irreducible representations $T^{n,\ell-2k}$, $k = 0, 1, \ldots, [\ell/2]$, of the group $SO(n)$. Thus,

$$T^{n\ell 0} \downarrow {U(n) \atop SO(n)} = \sum_{k=0}^{[\ell/2]} \oplus T^{n,\ell-2k}. \tag{4}$$

In the same way we derive from the equality $\mathfrak{H}_C^{n0\ell} = \mathfrak{R}_C^{n0\ell}$ that

$$T^{n0\ell} \downarrow {U(n) \atop SO(n)} = \sum_{k=0}^{[\ell/2]} \oplus T^{n,\ell-2k}. \tag{5}$$

Since $\mathfrak{R}_C^{n\ell 0}\mathfrak{R}_C^{n0\ell'} = \mathfrak{R}_C^{n\ell\ell'}$ and the representation $T^{n\ell 0} \otimes T^{n0\ell'}$ of $U(n)$ is realized in $\mathfrak{R}_C^{n\ell 0}\mathfrak{R}_C^{n0\ell'}$, then we find from (2) that

$$T^{n\ell 0} \otimes T^{n0\ell'} = \sum_{s=0}^{p} \oplus T^{n,\ell-s,\ell'-s}, \quad p = \min(\ell,\ell'). \tag{6}$$

Let us restrict the representation $T^{n\ell\ell'}$ of the group $U(n)$, $n \geq 3$, onto $U(n-1)$. The obtained representation of $U(n-1)$ decomposes into irreducible components as follows:

$$T^{n\ell\ell'} \downarrow^{U(n)}_{U(n-1)} = \sum_{m=0}^{\ell} \sum_{m'=0}^{\ell'} \oplus T^{n-1,mm'}. \tag{7}$$

One can prove this formula in the same way as formula (1) of Section 9.2.6. We give the proof without details. The representation $T^{n\ell\ell'}$ of $U(n)$ is realized in the quotient space $\mathfrak{R}_C^{n\ell\ell'}/r^2\mathfrak{R}_C^{n,\ell-1,\ell'-1}$, where $r^2 = |z_1|^2 + \ldots + |z_n|^2$. The space $\mathfrak{R}_C^{n\ell\ell'}$ contains the space $\mathfrak{Z}^{mm'} \equiv z_n^{\ell-m}\overline{z_n}^{\ell'-m'}\mathfrak{H}_C^{n-1,mm'}$, $0 \leq m \leq \ell$, $0 \leq m' \leq \ell'$, where $\mathfrak{H}_C^{n-1,mm'}$ is the space of homogeneous harmonic polynomials of $z_1, \ldots, z_{n-1}$, $\overline{z_1}, \ldots, \overline{z_{n-1}}$ which are of degree $m$ in $z_1, \ldots, z_{n-1}$ and of degree $m'$ in $\overline{z_1}, \ldots, \overline{z_{n-1}}$. In the spaces $\mathfrak{Z}^{mm'}$ the operators$T^{n\ell\ell'}(h)$, $h \in U(n-1)$, realize the irreducible representations $T^{n-1,mm'}$ of $U(n-1)$. Let us show that

$$\mathfrak{R}_C^{n\ell\ell'} = r^2\mathfrak{R}_C^{n,\ell-1,\ell'-1} + \sum_{m=0}^{\ell} \sum_{m'=0}^{\ell'} \mathfrak{Z}^{mm'}, \tag{8}$$

where the sum is direct. At first we prove that the dimensionalities of the left and of the right hand sides coincide, that is

$$\dim \mathfrak{R}_C^{n\ell\ell'} - \dim \mathfrak{R}_C^{n,\ell-1,\ell-1} = \dim \mathfrak{H}_C^{n\ell\ell'} = \sum_{m=0}^{\ell} \sum_{m'=0}^{\ell'} \dim \mathfrak{H}_C^{n-1,mm'}.$$

By making use of formula (19) of Section 11.2.1 we obtain

$$\sum_{\ell,\ell'=0}^{\infty} (\dim \mathfrak{H}_C^{n\ell\ell'})t^\ell s^{\ell'} = \frac{1-ts}{(1-t)^{n-1}(1-s)^{n-1}} \frac{1}{(1-t)(1-s)} =$$

$$= \sum_{p,q=0}^{\infty} (\dim \mathfrak{H}_C^{n-1,pq})t^p s^q \sum_{i,j=0}^{\infty} t^i s^j = \sum_{\ell,\ell'=0}^{\infty} \left[ \sum_{m=0}^{\ell} \sum_{m'=0}^{\ell'} (\dim \mathfrak{H}_C^{n-1,mm'}) \right] t^\ell s^{\ell'}.$$

It proves the statement on dimensionalities.

Now we prove that the spaces on the right hand side of (8) generate $\mathfrak{R}_C^{n\ell\ell'}$. For this we represent a polynomial $f(\mathbf{z}) \in \mathfrak{R}_C^{n\ell\ell'}$ in the form

$$f(\mathbf{z}) = r^2 F(\mathbf{z}) + \sum_{m=0}^{\ell} z_n^m f_m(\mathbf{z}') + \sum_{m'=0}^{\ell'} \bar{z}_n^{m'} \tilde{f}_{m'}(\mathbf{z}'), \tag{9}$$

where $\mathbf{z}' = (z_1, \ldots, z_{n-1})$, $f_m \in \mathfrak{R}_C^{n-1,\ell-m,\ell'}$, $f_{m'} \in \mathfrak{R}_C^{n-1,\ell,\ell'-m'}$. Then we expand the functions $f_m$ and $f_{m'}$ according to formula (15′) of Section 11.2.1, written down for $\mathfrak{R}_C^{n-1,kk'}$. Replace $r_{n-1}^2 \equiv |z_1|^2 + \ldots + |z_{n-1}|^2$ by $r^2 - |z_n|^2$ and carry over all terms containing $r^2$ to the first summand on the right hand side of (9). As a result, (9) becomes the expansion of $f$ in polynomials from the spaces on the right hand side of (8). This proves decomposition (8) and, hence, formula (7).

It follows from (7) that *the representations $T^{n\ell\ell'}$ of $U(n)$ are of class* 1 *relative to the subgroup $U(n-1)$*, i.e. in the carrier spaces of $T^{n\ell\ell'}$ there are vectors, invariant with respect to $U(n-1)$.

As in the case of the representations $T^{n\ell}$ of $SO(n)$ (see Section 9.2.6), one proves that *every irreducible unitary representation $T$ of $U(n)$, which is of class 1 relative to $U(n-1)$, is unitarily equivalent to one of the representations $T^{n\ell\ell'}$.* From this statement we conclude that *the subgroup $U(n-1)$ is massive in $U(n)$.*

For restriction of the representation $T^{n\ell\ell'}$ of $U(n)$ onto the subgroup $K' = U(n-1) \times U(1)$ consisting of matrices diag$(g, e^{i\varphi})$, $g \in U(n-1)$, we have

$$T^{n\ell\ell'} \downarrow^{U(n)}_{K'} = \sum_{m=0}^{\ell} \sum_{m'=0}^{\ell'} \oplus (T^{n-1,mm'} \otimes T^{1,(\ell-\ell')-(m-m')}), \tag{10}$$

where $T^{1,k}$ is the one-dimensional representation $e^{i\varphi} \to e^{ik\varphi}$ of $U(1)$.

It follows from (10) that the representation $T^{n\ell\ell'}$ of $U(n)$ is of class 1 relative to the subgroup $K'$ if and only if $\ell = \ell'$.

If $C$ is the complex conjugation of polynomials, then $C\mathfrak{H}_C^{n\ell\ell'} = \mathfrak{H}_C^{n\ell'\ell}$. Therefore, the space $\mathfrak{H}_C^{n\ell'\ell}$ is dual to $\mathfrak{H}_C^{n\ell\ell'}$. For $f \in \mathfrak{H}_C^{n\ell\ell'}$, $\varphi \in \mathfrak{H}_C^{n\ell'\ell}$ we have

$$\varphi(f) = \int_{S_C^{n-1}} f(\xi)\varphi(\xi)d\xi = \int_{S_C^{n-1}} f(\xi)\overline{\overline{\varphi(\xi)}}d\xi = (f, \bar{\varphi}).$$

Hence, the representation $T^{n\ell'\ell}$ is contragradient to $T^{n\ell\ell'}$ (see Section 2.2.2).

**11.2.4. The representations $T^{n\sigma k}$ of $U(n-1,1)$.** Let $\sigma$ be a complex number and $k$ be an integer. We denote by $\mathfrak{B}^{n\sigma k}$ the linear space of smooth functions on $C_C^{n-1}$ satisfying the homogeneity conditions

$$f(a\zeta) = a^\sigma f(\zeta),\ a > 0,\ f(u\zeta) = u^k f(\zeta),\ u \in \mathbb{C},\ |u| = 1. \tag{1}$$

One can rewrite (1) as

$$f(b\zeta) = b^{(\sigma+k)/2}\bar{b}^{(\sigma-k)/2} f(\zeta), \quad b \in \mathbb{C}. \tag{2}$$

It is evident that $\mathfrak{B}^{n\sigma k}$ is invariant with respect to the shift operators

$$(T(g)f)(\zeta) = f(g^{-1}\zeta), \quad g \in U(n-1,1). \tag{3}$$

Restricting these operators onto the space $\mathfrak{B}^{n\sigma k}$ with fixed $\sigma \in \mathbf{C}$, $k \in \mathbf{Z}$, we obtain the representation $T^{n\sigma k}$ of the group $U(n-1,1)$.

Since homogeneous functions on the cone $C_C^{n-1}$ are uniquely determined by their values on the contour $\Gamma$ intersecting every complex generatrix at one point, then $T^{n\sigma k}$ can be realized in the space of functions on these contours. As in the case of representations of $SO_0(n-1,1)$ (see Section 9.2.1), one shows that the operators $T^{n\sigma k}(g)$, $g \in U(n-1,1)$, in these spaces are given by the formula

$$(T^{n\sigma k}(g)F)(\boldsymbol{\eta}) = (\alpha(\boldsymbol{\eta}, g))^{(\sigma+k)/2}(\overline{\alpha(\boldsymbol{\eta}, g)})^{(\sigma-k)/2} F(\widehat{\boldsymbol{\eta}}), \tag{4}$$

where $\boldsymbol{\eta} \in \Gamma$, $\widehat{\boldsymbol{\eta}} \in \Gamma$ and $\alpha(\boldsymbol{\eta}, g)$ is found from the equality

$$\widehat{\boldsymbol{\eta}} = \alpha^{-1}(\boldsymbol{\eta}, g)(g^{-1}\boldsymbol{\eta}).$$

In particular, if $\Gamma = \Gamma_0$ is the section of $C_C^{n-1}$ by the plane $\xi_n = 1$, then $\Gamma_0 = S_C^{n-2}$. For $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_{n-1}, 1)$ and $g^{-1}\boldsymbol{\eta} = (\eta_1', \ldots, \eta_n')$, we have $\alpha(\boldsymbol{\eta}, g) = \eta_n'$ and $\widehat{\boldsymbol{\eta}} = (\eta_1'/\eta_n', \ldots, \eta_{n-1}'/\eta_n', 1)$. Thus, we obtain the realization of $T^{n\sigma k}$ in the space $\mathfrak{D}$ of smooth functions on $\Gamma_0 = S_C^{n-2}$, and to elements $k \in U(n-1)$ there corresond the operators

$$(T^{n\sigma k}(k)F)(\boldsymbol{\eta}) = F(k^{-1}\boldsymbol{\eta}) \tag{5}$$

and to the element $g_{n-1}'(t)$ there corresponds the operator

$$(T^{n\sigma k}(g_{n-1}'(t))F)(\boldsymbol{\eta}) = (\eta_n')^{(\sigma+k)/2}(\overline{\eta_n'})^{(\sigma-k)/2} F\left(\frac{\boldsymbol{\eta}'}{\eta_n'}\right). \tag{6}$$

Here $\eta_i' = \eta_i$, $1 \le i \le n-2$, and

$$\eta_{n-1}' = \eta_{n-1}\cosh t - \sinh t, \quad \eta_n' = \cosh t - \eta_{n-1}\sinh t. \tag{7}$$

Since Cartesian coordinates of a point $(\eta_1., \ldots, \eta_{n-1}) \in S_C^{n-2}$ are connected with its spherical coordinates by formulas of the form (2) of Section 11.1.5, then spherical coordinates $\varphi_1', \ldots, \varphi_{n-1}'$, $\theta_1', \ldots, \theta_{n-2}'$ of the transformed point are expressed in terms of spherical coordinates $\varphi_1, \ldots, \varphi_{n-1}$, $\theta_1, \ldots, \theta_{n-2}$ of the point $(\eta_1, \ldots, \eta_{n-1})$ by the formulas

$$e^{2i\varphi_{n-1}'} = \frac{(e^{i\varphi_{n-1}}\cos\theta_{n-2}\cosh t - \sinh t)(\cosh t - e^{-i\varphi_{n-1}}\cos\theta_{n-2}\sinh t)}{(e^{-i\varphi_{n-1}}\cos\theta_{n-2}\cosh t - \sinh t)(\cosh t - e^{i\varphi_{n-1}}\cos\theta_{n-2}\sinh t)}, \tag{8}$$

$$\cos^2\theta_{n-2}' = \frac{(e^{i\varphi_{n-1}}\cos\theta_{n-2}\cosh t - \sinh t)(e^{-i\varphi_{n-1}}\cos\theta_{n-2}\cosh t - \sinh t)}{(\cosh t - e^{i\varphi_{n-1}}\cos\theta_{n-2}\sinh t)(\cosh t - e^{-i\varphi_{n-1}}\cos\theta_{n-2}\sinh t)}, \tag{9}$$

$$\sin^2\theta'_{n-2} = \frac{\sin^2\theta_{n-2}}{(\cosh t - e^{i\varphi_{n-1}}\cos\theta_{n-2}\sinh t)(\cosh t - e^{-i\varphi_{n-1}}\cos\theta_{n-2}\sinh t)}, \quad (10)$$

$$\cos\theta'_i = \cos\theta_i, \quad \sin\theta'_i = \sin\theta_i, \quad i = 1, 2, \ldots, n-3, \quad (11)$$

$$e^{2i\varphi'_j} = e^{2i\varphi_j}\frac{\cosh t - e^{-i\varphi_{n-1}}\cos\theta_{n-2}\sinh t}{\cosh t - e^{i\varphi_{n-1}}\cos\theta_{n-2}\sinh t}, \ j = 1, 2, \ldots, n-2. \quad (12)$$

Hence, in spherical coordinates formula (6) takes the form

$$(T^{n\sigma k}(g'_{n-1}(t))F)(e^{i\varphi_1}, \ldots, e^{i\varphi_{n-1}}, \cos\theta_1, \ldots, \cos\theta_{n-2}) =$$
$$= (\cosh t - e^{i\varphi_{n-1}}\cos\theta_{n-2}\sinh t)^{(\sigma+k)/2}(\cosh t - e^{-i\varphi_{n-1}}\cos\theta_{n-2}\sinh t)^{(\sigma-k)/2}$$
$$\times F(e^{i\varphi'_1}, \ldots, e^{i\varphi'_{n-1}}, \cos\theta_1, \ldots, \cos\theta_{n-3}, \cos\theta'_{n-2}), \quad (13)$$

where $\varphi'_1, \ldots, \varphi'_{n-1}, \theta'_{n-2}$ are connected with $\varphi_1, \ldots, \varphi_{n-1}, \theta_{n-2}$ by formulas (8)–(12). For the element $\mathrm{diag}(1, \ldots, 1, e^{i\varphi}) \equiv d_n(e^{i\varphi}) \in U(n-1, 1)$ we have

$$(T^{n\sigma k}(d_n(\varphi))F)(e^{i\varphi_1}, \ldots, e^{i\varphi_{n-1}}, \cos\theta_1, \ldots, \cos\theta_{n-2}) =$$
$$= e^{-ki\varphi}F(e^{i(\varphi_1+\varphi)}, \ldots, e^{i(\varphi_{n-1}+\varphi)}, \cos\theta_1, \ldots, \cos\theta_{n-2}). \quad (14)$$

By virtue of formulas (1) of Section 11.2.3 and (5) we obtain

$$T^{n\sigma k}\downarrow^{U(n-1,1)}_{U(n-1)} = \sum_{\ell,\ell'=0}^{\infty} \oplus T^{n-1,\ell\ell'}. \quad (15)$$

Therefore, the representations $T^{n\sigma k}$ are of class 1 relative to the subgroup $U(n-1)$. The function $F_0(\eta) \equiv 1$ is invariant with respect to the operators $T^{n\sigma k}(k)$, $k \in U(n-1)$. By virtue of (14) the representation $T^{n\sigma k}$ is of class 1 relative to the maximal compact subgroup $U(n-1) \times U(1)$ if and only if $k = 0$.

We equip $\mathfrak{D}$ with the scalar product

$$(F_1, F_2) = \int_{S_C^{n-2}} F_1(\eta)\overline{F_2(\eta)}d\eta, \quad (16)$$

where $d\eta$ is the invariant measure on $S_C^{n-2}$, given by formula (1) of Section 11.1.8. The space $\mathfrak{D}$ is completed to the Hilbert space $\mathfrak{L}^2(S_C^{n-2})$ with the scalar product (16), and the operators $T^{n\sigma k}(g)$, $g \in U(n-1, 1)$, are extended to bounded operators in $\mathfrak{L}^2(S_C^{n-2})$. As a result, we obtain representations of the group $U(n-1, 1)$ in $\mathfrak{L}^2(S_C^{n-2})$ which are also denoted by $T^{n\sigma k}$.

One directly verifies that the representations $T^{n\sigma k}$ and $T^{n,-\bar{\sigma}-2n+2,k}$ of $U(n-1, 1)$ are Hermitian-adjoint, that is, for any $g \in U(n-1, 1)$ and for any $F_1, F_2 \in \mathfrak{L}^2(S_C^{n-2})$ we have

$$(T^{n\sigma k}(g)F_1, T^{n,-\bar{\sigma}-2n+2,k}(g)F_2) = (F_1, F_2). \quad (16')$$

Consequently, the representations $T^{n\sigma k}$, $\sigma = i\rho - n + 1$, $\rho \in \mathbf{R}$, are unitary. They form so-called spherical (with respect to $U(n-1)$) principal unitary series of representations.

We shall show in Section 11.6.3 that the representation $T^{n\sigma k}$ of $U(n-1,1)$ is irreducible if and only if either $\sigma$ is not an integer or if $\sigma$ is an integer such that

$$-n+2 < \frac{\sigma+k}{2} \leq 0 \quad \text{and} \quad 0 \leq \frac{k-\sigma}{2} < n-2. \tag{17}$$

For the representations $T^{n\sigma k}$ and $T^{n,-\sigma-2n+2,k}$ there exists an intertwining operator:

$$Q^{\sigma k} T^{n\sigma k}(g) = T^{n,-\sigma-2n+2,k}(g) Q^{\sigma k}, \quad g \in U(n-1,1). \tag{18}$$

If $T^{n\sigma k}$ and $T^{n\tau k}$, $\tau = -\sigma - 2n + 2$, are realized in the spaces $\mathfrak{B}^{n\sigma k}$ and $\mathfrak{B}^{n\tau k}$, respectively, then the operator $Q^{\sigma k}$ is given by the formula

$$(Q^{\sigma k} f)(\boldsymbol{\xi}) = \int_{\Gamma} [\boldsymbol{\xi}, \boldsymbol{\eta}]^{-\sigma-2n+2} f(\boldsymbol{\eta}) d\boldsymbol{\eta}, \quad \boldsymbol{\xi}, \boldsymbol{\eta} \in \Gamma, \tag{19}$$

where $\Gamma$ is a contour intersecting every complex generatrix of $C_C^{n-1}$ at one point. The proof of the intertwining property of $Q^{\sigma k}$ is the same as in the case of the group $SO_0(n-1,1)$ (see Section 9.2.7). The matrix elements of $Q^{\sigma k}$ will be found in Section 11.6.3.

**11.2.5. Other realizations of $T^{n\ell\ell'}$ and $T^{n\sigma k}$.** If $\frac{\sigma+k}{2}$ and $\frac{\sigma-k}{2}$ are non-negative integers, then $\mathfrak{B}^{n\sigma k}$ contains the subspace $\widetilde{\mathfrak{B}}^{\ell\ell'}$, $\ell = \frac{\sigma+k}{2}$, $\ell' = \frac{\sigma-k}{2}$, invariant under $T^{n\sigma k}(g)$, $g \in U(n-1,1)$. It consists of restrictions onto $C_C^{n-1}$ of homogeneous polynomials of $z_1, \ldots, z_n, \overline{z_1}, \ldots, \overline{z_n}$ of degree $\ell$ in $z_1, \ldots, z_n$ and of degree $\ell'$ in $\overline{z_1}, \ldots, \overline{z_n}$. We denote by $\widehat{T}^{n\ell\ell'}$ the restriction of $T^{n\sigma k}$ onto $\widetilde{\mathfrak{B}}^{\ell\ell'}$. The representation $\widehat{T}^{n\ell\ell'}$ is finite dimensional. As in the case of the representations of the groups $SO_0(n-1,1)$ and $SO(n)$ (see Section 9.2.2), we now go over from the representations $\widehat{T}^{n\ell\ell'}$ of $U(n-1,1)$ to the representations $\check{T}^{n\ell\ell'}$ of $U(n)$. The representations $\check{T}^{n\ell\ell'}$ act in the spaces $\mathfrak{D}^{n-1,\ell\ell'}$ consisting of restrictions of functions of the spaces $\widetilde{\mathfrak{B}}^{\ell\ell'}$ onto the section of the cone by the plane $z_n = 1$. These representations are given by the formulas

$$(\check{T}^{n\ell\ell'}(k)f)(\boldsymbol{\zeta}) = f(k^{-1}\boldsymbol{\zeta}), \; k \in U(n-1), \; \boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_{n-1}) \in S_C^{n-2}, \tag{1}$$

$$\begin{aligned}(\check{T}^{n\ell\ell'}(g_{n-1}(\theta))f)(\boldsymbol{\zeta}) &= (\cos\theta - i\zeta_{n-1}\sin\theta)^{\ell}\overline{(\cos\theta - i\zeta_{n-1}\sin\theta)}^{\ell'} \\ &\times f\left(\frac{\zeta_1}{\cos\theta - i\zeta_{n-1}\sin\theta}, \ldots, \frac{\zeta_{n-2}}{\cos\theta - i\zeta_{n-1}\sin\theta}, \frac{\zeta_{n-1}\cos\theta - i\sin\theta}{\cos\theta - i\zeta_{n-1}\sin\theta}\right),\end{aligned} \tag{2}$$

$$(\check{T}^{n\ell\ell'}(d_n(e^{i\varphi}))f)(\zeta) = e^{-i(\ell-\ell')\varphi}f(e^{i\varphi}\zeta). \tag{3}$$

The representation $\check{T}^{n\ell\ell'}$ is equivalent to $T^{n\ell\ell'}$. We suggest that the reader carries out a detailed proof of these statements.

It is clear from (1) that $\check{T}^{n\ell\ell'}\downarrow^{U(n)}_{U(n-1)}$ is a subrepresentation of the left quasi-regular representation of the group $U(n-1)$ in $\mathfrak{L}^2(S_C^{n-2})$. Consequently, we obtain from formula (1) of Section 11.2.3 that

$$\mathfrak{D}^{n-1,\ell\ell'} = \sum_{m=0}^{\ell}\sum_{m'=0}^{\ell'} \oplus\widetilde{\mathfrak{H}}_C^{n-1,mm'}, \tag{4}$$

where the spaces $\widetilde{\mathfrak{H}}_C^{n-1,mm'}$ are defined in Section 11.2.1.

Going over in (2) to spherical coordinates on $S_C^{n-2}$, we find

$$\begin{aligned}(\check{T}^{n\ell\ell'}(g_{n-1}(\varphi))F)&(e^{i\varphi_1},\ldots,e^{i\varphi_{n-1}},\cos\theta_1,\ldots,\cos\theta_{n-2}) =\\ &= (\cos\varphi - ie^{i\varphi_{n-1}}\cos\theta_{n-2}\sin\varphi)^{\ell}(\cos\varphi - ie^{-i\varphi_{n-1}}\cos\theta_{n-2}\sin\varphi)^{\ell'}\\ &\qquad\times F(e^{i\varphi'_1},\ldots,e^{i\varphi'_{n-1}},\cos\theta_1,\ldots,\cos\theta_{n-3},\cos\theta'_{n-2}).\end{aligned} \tag{5}$$

The parameters $\varphi'_1,\ldots,\varphi'_{n-1},\theta'_{n-2}$ are connected with $\varphi_1,\ldots,\varphi_{n-1},\theta_{n-2}$ by formulas (8)–(12) of Section 11.2.4, where $\sinh t$ and $\cosh t$ are replaced by $i\sin\varphi$ and $\cos\varphi$, respectively. For the operator $\check{T}^{n\ell\ell'}(d_n(e^{i\varphi}))$ we have

$$\begin{aligned}(\check{T}^{n\ell\ell'}(d_n(e^{i\varphi}))F)&(e^{i\varphi_1},\ldots,e^{i\varphi_{n-1}},\cos\theta_1,\ldots,\cos\theta_{n-2}) =\\ &= e^{-i(\ell-\ell')\varphi}F(e^{i(\varphi_1+\varphi)},\ldots,e^{i(\varphi_{n-1}+\varphi)},\cos\theta_1,\ldots,\cos\theta_{n-2}).\end{aligned} \tag{6}$$

A function $f(\mathbf{z})$, defined either inside or outside of the cone $C_C^{n-1}$, is said to be $\Box$-harmonic if $\Box f = 0$, where the operator $\Box$ is given by formula (8) of Section 11.1.7. If $[\boldsymbol{\xi},\boldsymbol{\xi}] = 0$, then the function

$$f(\mathbf{z}) = [\mathbf{z},\boldsymbol{\xi}]^{\tau}\overline{[\mathbf{z},\boldsymbol{\xi}]}^{\tau+k}, \qquad \tau\in\mathbb{C},\ k\in\mathbb{Z},$$

is $\Box$-harmonic.

Let us denote by $\mathfrak{H}^+(n,\sigma,k)$ and $\mathfrak{H}^-(n,\sigma,k)$ the spaces of $\Box$-harmonic functions which are defined, respectively, inside and outside of the cone $C_C^{n-1}$ and satisfy the homogeneity conditions

$$f(a\mathbf{z}) = a^{\sigma}f(\mathbf{z}),\quad a>0,\quad f(u\mathbf{z}) = u^k f(\mathbf{z}),\quad u\in\mathbb{C},\quad |u|=1. \tag{7}$$

The equality $(R_{\pm}^{n\sigma k}(g)f)(\mathbf{z}) = f(g^{-1}\mathbf{z})$ gives representations of $U(n-1,1)$ in $\mathfrak{H}^+(n,\sigma,k)$ and $\mathfrak{H}^-(n,\sigma,k)$. These representations are equivalent to $T^{n,-\sigma-2n+2,k}$. The formula

$$(QF)(\mathbf{z}) = \frac{1}{\Gamma(-\sigma-2n+3)}\int_{\Gamma}[\mathbf{z},\boldsymbol{\zeta}]^{\frac{\sigma+k}{2}}\overline{[\mathbf{z},\boldsymbol{\zeta}]}^{\frac{\sigma-k}{2}}F(\boldsymbol{\zeta})d\boldsymbol{\zeta},\ [\mathbf{z},\mathbf{z}]\lessgtr 0, \tag{8}$$

gives an intertwining operator for $T^{n,-\sigma-2n+2,k}$ and $R_{\pm}^{n\sigma k}$. Here $\Gamma$ is the same as in formula (19) of Section 11.2.4.

The homogeneity condition (7) allows us to consider functions $f \in \mathfrak{H}^-(n,\sigma,k)$ (respectively, $f \in \mathfrak{H}^+(n,\sigma,k)$) only on the hyperboloid $H_C^{n-1}$ (respectively, on the hyperboloid $[\mathbf{z},\mathbf{z}] = -1$). It follows from formulas (9) of Section 11.1.7 and (8) that these functions on $H_C^{n-1}$ satisfy the equation

$$\Box_0 f(\boldsymbol{\xi}) = \sigma(\sigma+2n-2)f(\boldsymbol{\xi}). \tag{9}$$

**11.2.6. The representations $T^{nkR}$ of the group $JU(n-1)$.** Let $\mathbf{z} = (z_1,\ldots,z_{n-1}) \in \mathbb{C}^{n-1}$ and $\boldsymbol{\zeta} = (\zeta_1,\ldots,\zeta_{n-1}) \in S_C^{n-2}$. By $(\mathbf{z},\boldsymbol{\zeta})$ we denote the number

$$(\mathbf{z},\boldsymbol{\zeta}) = \sum_{j=1}^{n-1}(x_j\xi_j + y_j\eta_j), \tag{1}$$

where $z_j = x_j + iy_j$, $\zeta_j = \xi_j + i\eta_j$. Let us fix $k \in \mathbb{Z}$ and $R \in \mathbb{C}$. With every element $g = g(u;\varphi,\mathbf{z}) \in JU(n-1)$ we associate the operator

$$(T^{nkR}(g)f)(\boldsymbol{\zeta}) = \exp[R(\mathbf{z}, e^{i\varphi}\boldsymbol{\zeta}) - ik\varphi]f(e^{i\varphi}u^{-1}\boldsymbol{\zeta}) \tag{2}$$

in the space $\mathfrak{L}^2(S_C^{n-2})$. The correspondence $g \to T^{nkR}(g)$ is a representation of the group $JU(n-1)$. It follows from (2) that

$$(T^{nkR}(g(e;\varphi,\mathbf{0}))f)(\boldsymbol{\zeta}) = e^{-ik\varphi}f(e^{i\varphi}\boldsymbol{\zeta}), \tag{3}$$

$$(T^{nkR}(g(u;0,\mathbf{0}))f)(\boldsymbol{\zeta}) = f(u^{-1}\boldsymbol{\zeta}). \tag{4}$$

Hence, on $U(n-1)\times U(1)$ the representation $T^{nkR}$ of $JU(n-1)$ coincides with the representation $T^{n\sigma k}$ of $U(n-1,1)$. It is clear that $T^{nkR}$ is of *class 1 relative to* $U(n-1)$.

We write down a function $f \in \mathfrak{L}^2(S_C^{n-2})$ in spherical coordinates:

$$f(\boldsymbol{\zeta}) = F(\varphi_1,\ldots,\varphi_{n-1},\theta_1,\ldots,\theta_{n-2}) \equiv F(\boldsymbol{\varphi},\boldsymbol{\theta}).$$

Then for the shift $g_r$ by the value $r \in \mathbb{R}$ along the $(n-1)$th axis we have

$$(T^{nkR}(g_r)F)(\boldsymbol{\varphi},\boldsymbol{\theta}) = e^{Rr\cos\theta\cos\varphi}F(\boldsymbol{\varphi},\boldsymbol{\theta}), \tag{5}$$

where $\theta \equiv \theta_{n-2}$, $\varphi \equiv \varphi_{n-1}$. For the shift $g_{ir}$ by the imaginary value $ir$ along the same axis we obtain

$$(T^{nkR}(g_{ir})F)(\boldsymbol{\varphi},\boldsymbol{\theta}) = e^{Rr\cos\theta\sin\varphi}F(\boldsymbol{\varphi},\boldsymbol{\theta}). \tag{6}$$

Note that one obtains (5) from formula (13) of Section 11.2.4 by replacing $\sigma$ by $\alpha + i\tau$, $\tau \in \mathbb{R}$, and by passing to the limit $t \to 0$, $\tau \to \infty$ such that $\tau t \to Rr/2$.

A direct evaluation shows that *the representation* $T^{nkR}$ *is unitary if and only if* $R+\bar{R}=0$, *i.e. if* $R$ *is a purely imaginary number.* The representations $T^{nkR}$ and $T^{n,k,-\bar{R}}$ are Hermitian-adjoint, i.e. for all $g \in JU(n-1)$ we have

$$(T^{nkR}(g)f_1, T^{nk,-\bar{R}}(g)f_2) = (f_1, f_2), \quad f_1, f_2 \in \mathfrak{L}^2(S_C^{n-2}).$$

The representations $T^{nkR}$ and $T^{nk,-R}$ are equivalent. The equivalence is given by the operator $A: (Af)(\zeta) = f(-\zeta)$. For $R \neq 0$ the representations $T^{nkR}$ are irreducible.

## 11.3. Zonal and Associated Spherical Functions

**11.3.1. The orthonormal basis in $\mathfrak{L}^2(S_C^{n-1})$.** According to formula (14) of Section 11.2.1 we have

$$\mathfrak{L}^2(S_C^{n-1}) = \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} \oplus \widetilde{\mathfrak{H}}_C^{n\ell\ell'},$$

where $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$ is the carrier space of the irreducible representation $T^{n\ell\ell'}$ of $U(n)$. We decompose every $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$ into the orthogonal sum of subspaces invariant with respect to the subgroup $U(n-1)$ (see formula (7) of Section 11.2.3). Decomposing the spaces obtained into irreducible components with respect to the subgroups $U(n-2)$, $U(n-3), \ldots, U(1)$ and taking into account the fact that invariant irreducible spaces for $U(1)$ are one-dimensional, we obtain the decomposition

$$\mathfrak{L}^2(S_C^{n-1}) = \sum_{\ell,\ell'} \sum_{M} \oplus \widetilde{\mathfrak{H}}_M^{n\ell\ell'} \tag{1}$$

of the space $\mathfrak{L}^2(S_C^{n-1})$ into the sum of one-dimensional subspaces $\widetilde{\mathfrak{H}}_M^{n\ell\ell'}$, where $M = (m, m'; m_1, m_1'; \ldots; m_{n-3}, m_{n-3}'; m_{n-2})$ and

$$\ell \geq m \geq m_1 \geq \ldots \geq m_{n-3} \geq m_{n-2}, \; \ell' \geq m' \geq m_1' \geq \ldots \geq m_{n-3}' \geq m_{n-2}. \tag{2}$$

Choosing the unit vectors $\Xi_M^{n\ell\ell'}$ in the spaces $\widetilde{\mathfrak{H}}_M^{n\ell\ell'}$ we obtain an orthonormal basis in $\mathfrak{L}^2(S_C^{n-1})$. It is clear that the vectors of this basis are uniquely defined (up to factors with unit moduli) by the chain of subgroups $U(n-1) \supset U(n-2) \supset \ldots \supset U(1)$.

The functions $\Xi_M^{n\ell\ell'}$ with fixed $\ell$ and $\ell'$ form an orthonormal basis of the space $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$. We denote the matrix elements of $T^{n\ell\ell'}$ in the basis $\{\Xi_M^{n\ell\ell'}\}$ by $t_{MK}^{n\ell\ell'}(g)$:

$$t_{MK}^{n\ell\ell'}(g) = (T^{n\ell\ell'}(g)\Xi_K^{n\ell\ell'}, \Xi_M^{n\ell\ell'}). \tag{3}$$

For $K=O$ these matrix elements are spherical functions of $T^{n\ell\ell'}$. They are constant on right cosets with respect to $U(n-1)$ and, hence, can be considered as functions $t_{MO}^{n\ell\ell'}(\zeta)$ on $S_C^{n-1}$. By repeating reasonings of Section 9.3.1, we conclude that

$$\Xi_M^{n\ell\ell'}(\zeta) = (\dim T^{n\ell\ell'})^{1/2}\, \overline{t_{MO}^{n\ell\ell'}(\zeta)}, \tag{4}$$

where $\dim T^{n\ell\ell'}$ is given by formula (16) of Section 11.2.2.

Let $M = (m, m'; m_1, m_1'; \dots)$, $K = (k, k'; k_1, k_1'; \dots)$. As in the case of the group $SO(n)$ (see Section 9.4.1), one shows that $t^{n\ell\ell'}_{MK}(g_{n-1}(\theta)) = 0$ if $K' \equiv (k_1, k_1'; \dots) \neq M' = (m_1, m_1'; \dots)$ and $t^{n\ell\ell'}_{MK}(g_{n-1}(\theta))$ depends on $m_1, m_1'$ and does not depend on other indices of $M'$ if $K' = M'$. Therefore, instead of $t^{n\ell\ell'}_{MK}(g_{n-1}(\theta))$, $M' = K'$, we shall write $t^{n\ell\ell'}_{(mm')(k,k')(jj')}(g_{n-1}(\theta))$ (we have replaced $m_1$ and $m_1'$ by $j$ and $j'$, respectively). If $k = k' = 0$, then $j = j' = 0$. The corresponding matrix element is denoted by $t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))$. Similar statements are valid for matrix elements of the operators $T^{n\sigma k}(g'_{n-1}(t))$ and $T^{nkR}(g_r)$ of representations of the groups $U(n-1, 1)$ and $JU(n)$. Analogous notations are introduced for them.

**11.3.2. Evaluation of zonal spherical functions.** The zonal spherical functions

$$t^{n\ell\ell'}_{OO}(g) = (T^{n\ell\ell'}(g)\Xi^{n\ell\ell'}_O, \Xi^{n\ell\ell'}_O) \tag{1}$$

of the representation $T^{n\ell\ell'}$ of $U(n)$ with respect to the subgroup $U(n-1)$ are constant on two-sided cosets with respect to $U(n-1)$. It follows from factorization (8) of Section 11.1.5 that if $g = k_1 d_n(\varphi) g_{n-1}(\theta) k_2$, $k_1, k_2 \in U(n-1)$, then

$$t^{n\ell\ell'}_{OO}(g) = t^{n\ell\ell'}_{OO}(d_n(\varphi) g_{n-1}(\theta)) = e^{i(\ell'-\ell)\varphi} t^{n\ell\ell'}_{00}(g_{n-1}(\theta)) \tag{2}$$

(see formula (5) of Section 11.2.1). The matrix element $t^{n\ell\ell'}_{00}(g_{n-1}(\theta))$ will be denoted by $\varphi^{n\ell\ell'}(\theta)$.

Evaluation of $\varphi^{n\ell\ell'}(\theta)$ is similar to that for zonal spherical functions of representations of $SO(n)$. We obtain the following integral representation of $\varphi^{n\ell\ell'}(\theta)$:

$$\varphi^{n\ell\ell'}(\theta) = \frac{n-2}{\pi} \int_0^{2\pi} \int_0^{\pi/2} (\cos\theta - ie^{i\varphi}\cos\psi\sin\theta)^{\ell} \times \\ \times (\cos\theta - ie^{-i\varphi}\cos\psi\sin\theta)^{\ell'} \sin^{2n-5}\psi\cos\psi\, d\psi\, d\varphi. \tag{3}$$

In particular, for the representations $T^{n\ell\ell'}$ of class 1 relative to the subgroup $U(n-1) \times U(1)$, we find

$$t^{n\ell\ell}_{OO}(g_{n-1}(\theta)) \equiv \varphi^{n\ell\ell}(\theta) \\ = \frac{n-2}{\pi} \int_0^{2\pi} \int_0^{\pi/2} (\cos^2\theta - \cos^2\psi\sin^2\theta - i\sin 2\theta\cos\psi\cos\varphi)^{\ell} \sin^{2n-5}\psi\cos\psi\, d\psi\, d\varphi. \tag{3'}$$

We take $\cos\theta$ out from the binomial expressions on the right hand side of (3), expand $(1 - ie^{i\varphi}\cos\psi\tan\theta)^{\ell}$ and $(1 - ie^{-i\varphi}\cos\psi\tan\theta)^{\ell'}$ by the binomial formula and integrate term by term at first with respect to $\varphi$ and then with respect to $\psi$. We obtain

$$\begin{aligned}\varphi^{n\ell\ell'}(\theta) &= (\cos\theta)^{\ell+\ell'}F(-\ell,-\ell';n-1;-\tan^2\theta)\\ &= (\cos\theta)^{\ell-\ell'}F(-\ell',\ell+n-1;n-1;\sin^2\theta)\\ &= (\cos\theta)^{\ell'-\ell}F(-\ell,\ell'+n-1;n-1;\sin^2\theta) \qquad (4)\end{aligned}$$

(see formula (3) of Section 3.5.3). The function $\varphi^{n\ell\ell'}(\theta)$ is expressed in terms of Jacobi polynomials:

$$\begin{aligned}\varphi^{n\ell\ell'}(\theta) &= \frac{\ell!(n-2)!}{(\ell+n-2)!}\cos^{\ell'-\ell}\theta P_{\ell}^{(n-2,\ell'-\ell)}(\cos 2\theta)\\ &= \frac{\ell'!(n-2)!}{(\ell'+n-2)!}\cos^{\ell-\ell'}\theta P_{\ell'}^{(n-2,\ell-\ell')}(\cos 2\theta). \qquad (5)\end{aligned}$$

These formulas imply that

$$\varphi^{n\ell\ell'}(\theta) = \frac{q!(n-2)!}{(q+n-2)!}(\cos\theta)^{|\ell-\ell'|}P_q^{(n-2,|\ell-\ell'|)}(\cos 2\theta), \qquad (5')$$

where $q=\min(\ell,\ell')$.

For the representations $T^{n\ell 0}$ and $T^{n0\ell}$ of $U(n)$ we have

$$\varphi^{n\ell 0}(\theta) = \varphi^{n0\ell}(\theta) = \cos^\ell\theta. \qquad (6)$$

For zonal spherical functions of the representations $T^{n\sigma k}$ of $U(n-1,1)$ we find

$$t_{OO}^{n\sigma k}(g) = t_{OO}^{n\sigma k}(kd_n(\varphi)g'_{n-1}(t)k') = e^{-ik\varphi}t_{OO}^{n\sigma k}(g'_{n-1}(t)), \qquad (7)$$

where $k,k'\in U(n-1)$. By making use of formula (13) of Section 11.2.4 we obtain

$$\begin{aligned}t_{00}^{n\sigma k}(g'_{n-1}(t)) \equiv \varphi^{n\sigma k}(t) = \frac{n-2}{\pi}\int_0^{2\pi}\int_0^{\pi/2}&(\cosh t - e^{i\varphi}\cos\theta\sinh t)^{(\sigma+k)/2}\\ &\times(\cosh t - e^{-i\varphi}\cos\theta\sinh t)^{(\sigma-k)/2}\sin^{2n-5}\theta\cos\theta\,d\theta\,d\varphi. \qquad (8)\end{aligned}$$

In particular, for the representations $T^{n\sigma 0}$ of class 1 relative to $U(n-1)\times U(1)$ we have

$$\begin{aligned}\varphi^{n\sigma 0}(t) = \frac{n-2}{\pi}\int_0^{2\pi}\int_0^{\pi/2}&(\cosh^2 t + \sinh^2 t\cos^2\theta - \sinh 2t\cos\theta\cos\varphi)^{\sigma/2}\\ &\times\sin^{2n-5}\theta\cos\theta\,d\theta\,d\varphi. \qquad (8')\end{aligned}$$

We take $\cosh t$ out from the binomial expressions on the right hand side of (8), apply the binomial formula and integrate with respect to $\varphi$:

$$\varphi^{n\sigma k}(t) = 2(n-2)\cosh^\sigma t \sum_{p=0}^{\infty} \frac{\Gamma\left(\frac{\sigma+k+2}{2}\right)\Gamma\left(\frac{\sigma-k+2}{2}\right)\tanh^{2p} t}{p!p!\Gamma\left(\frac{\sigma+k+2}{2}-p\right)\Gamma\left(\frac{\sigma-k+2}{2}-p\right)} \times \int_0^{\pi/2} \sin^{2n-5}\theta \cos^{2p+1}\theta\, d\theta. \quad (8'')$$

The summation with respect to $p$ coincides with the hypergeometric series. Thus, we have the integral representation

$$\varphi^{n\sigma k}(t) = 2(n-2)\cosh^\sigma t \int_0^{\pi/2} F\left(\frac{-\sigma-k}{2}, \frac{k-\sigma}{2}; 1; \cos^2\theta \tanh^2 t\right) \sin^{2n-5}\theta \cos\theta d\theta. \quad (9)$$

Calculating the integrals in (8″) with the help of formula (1) of Section 3.4.6, we find

$$\begin{aligned}\varphi^{n\sigma k}(t) &= \cosh^\sigma t\ F\left(\frac{-\sigma-k}{2}, \frac{k-\sigma}{2}; n-1; \tanh^2 t\right)\\ &= \cosh^{-\sigma-2n+2} t\ F\left(\frac{\sigma-k}{2}+n-1, \frac{\sigma+k}{2}+n-1; n-1; \tanh^2 t\right)\\ &= \cosh^k t\ F\left(\frac{\sigma+k}{2}+n-1, \frac{k-\sigma}{2}; n-1; -\sinh^2 t\right)\\ &= \cosh^{-k} t F\left(\frac{-\sigma-k}{2}, \frac{\sigma-k}{2}+n-1; n-1; -\sinh^2 t\right) \quad (10)\end{aligned}$$

(see formulas (3) and (5) of Section 3.5.3). The first and the second expressions for $\varphi^{n\sigma k}(t)$ give the equality

$$\varphi^{n\sigma k}(t) = \varphi^{n,-\sigma-2n+2,k}(t) \quad (10')$$

which is a consequence of the equivalence of the representations $T^{n\sigma k}$ and $T^{n,-\sigma-2n+2,k}$ (see Section 11.2.4).

If $\sigma$ is an integer, then for some $k$ the hypergeometric series terminate and are expressed in terms of Jacobi polynomials. We separate two cases:

1) $\nu_1 = \dfrac{\sigma+k}{2} \geq 0$, $\nu_2 = \dfrac{k-\sigma}{2} \geq n-2$, $\nu_1 + n - 2 \leq \nu_2$,

2) $\nu_1 \equiv \frac{\sigma+k}{2} \leq -n+2,\ \nu_2 \equiv \frac{k-\sigma}{2} \leq 0,\ \nu_1+n-2 \leq \nu_2.$

In the first case

$$\varphi^{n\sigma k}(t) \equiv \psi_-^{\nu_1\nu_2}(t) = \frac{\nu_1!(n-2)!}{(\nu_1+n-2)!} \cosh^{-\nu_1-\nu_2} t\ P_{\nu_1}^{(n-2,-\nu_1-\nu_2)}(\cosh 2t), \tag{11}$$

and in the second case

$$\varphi^{n\sigma k}(t) \equiv \psi_+^{\nu_1\nu_2}(t) = \frac{(-\nu_2)!(n-2)!}{(-\nu_2+n-2)!} \cosh^{\nu_1+\nu_2} t\ P_{-\nu_2}^{(n-2,\nu_1+\nu_2)}(\cosh 2t). \tag{11'}$$

For zonal spherical functions of the representations $T^{nkR}$ of the group $JU(n-1)$ we have

$$t_{OO}^{nkR}(g) = t_{OO}^{nkR}(kd_n(\varphi)g_rk') = e^{-ik\varphi}t_{00}^{nR}(g_r), \tag{12}$$

where $k, k' \in U(n-1)$. By means of formula (5) of Section 11.2.6 we find

$$t_{00}^{nR}(g_r) = \frac{n-2}{\pi} \int_0^{2\pi}\int_0^{\pi/2} e^{Rr\cos\theta\cos\varphi} \sin^{2n-5}\theta\cos\theta\, d\theta\, d\varphi. \tag{13}$$

As above, we obtain

$$t_{00}^{nR}(g_r) = 2(n-2)\int_0^{\pi/2} J_0(-iRr\cos\theta)\sin^{2n-5}\theta\cos\theta\, d\theta, \tag{14}$$

$$t_{00}^{nR}(g_r) = (n-2)!\left(\frac{-iRr}{2}\right)^{-n+2} J_{n-2}(-iRr). \tag{15}$$

These formulas imply

$$\int_0^1 J_0\left(x\sqrt{1-t^2}\right)t^{2n-1}dt = \frac{(n-1)!}{2}\left(\frac{x}{2}\right)^{-n} J_n(x). \tag{16}$$

This relation is a special case of Sonin's formula

$$z^{\mu+\nu+1}J_{\mu+\nu+1}(z) = \frac{1}{2^\mu\Gamma(\mu+1)}\int_0^z r^{\nu+1}J_\nu(r)(z^2-r^2)^\mu dr. \tag{17}$$

**11.3.3. Asymptotic properties of zonal spherical functions.** By using equality (2) of Section 7.3.5 we derive from formula (10) of the preceding section the following expression for $\varphi^{n\sigma k}$:

$$\varphi^{n\sigma k}(t) = \frac{(n-2)!\Gamma(\sigma+n-1)\cosh^{\sigma} t}{\Gamma\left(\frac{\sigma+k}{2}+n-1\right)\Gamma\left(\frac{\sigma-k}{2}+n-1\right)} \times F\left(\frac{-\sigma-k}{2}, \frac{k-\sigma}{2}; -\sigma-n+2; \cosh^{-2} t\right)$$
$$+ \frac{(n-2)!\Gamma(-\sigma-n+1)}{\Gamma\left(\frac{-\sigma-k}{2}\right)\Gamma\left(\frac{k-\sigma}{2}\right)} \cosh^{-\sigma-2n+2} t \times F\left(\frac{\sigma+k}{2}+n-1, \frac{\sigma-k}{2}+n-1; \sigma+n; \cosh^{-2} t\right), \tag{1}$$

where $\sigma\bar{\in}\mathbb{Z}$. The function

$$\mathbf{c}_k(i(\sigma+n-1)) = \frac{2^{\sigma+2n-2}(n-2)!\Gamma(-\sigma-n+1)}{\Gamma\left(\frac{-\sigma-k}{2}\right)\Gamma\left(\frac{k-\sigma}{2}\right)} \tag{2}$$

is called the *Harish-Chandra* **c**-*function* of the group $U(n-1,1)$. By means of this function formula (1) is rewritten as

$$\varphi^{n\sigma k}(t) = \mathbf{c}_k(i(\sigma+n-1))\Phi^{n\sigma k}(t) + \mathbf{c}_k(-i(\sigma+n-1))\Phi^{n,-\sigma-2n+2,k}(t), \tag{3}$$

where

$$\Phi^{n\sigma k}(t) = (2\cosh t)^{-\sigma-2n+2} F\left(\frac{\sigma+k}{2}+n-1, \frac{\sigma-k}{2}+n-1; \sigma+n; \cosh^{-2} t\right). \tag{4}$$

Since $\cosh^{-2} t \to 0$ when $t\to\infty$, then we find from (3) that

$$\varphi^{n\sigma k}(t) \underset{t\to\infty}{\sim} \mathbf{c}_k(i(\sigma+n-1))(2\cosh t)^{-\sigma-2n+2} + \mathbf{c}_k(-i(\sigma+n-1))(2\cosh t)^{\sigma}. \tag{5}$$

Consequently,

$$\varphi^{n\sigma k}(t) \underset{t\to\infty}{\sim} \mathbf{c}_k(i(\sigma+n-1))e^{(-\sigma-2n+2)t}$$

where $\operatorname{Re}\sigma < -n+1$.

On the other hand, it follows from formulas (8) and (11) of Section 11.3.2 that

$$\varphi^{n\sigma k}(t) \underset{t\to\infty}{\sim} \frac{2^{\sigma+2n-2}(n-2)}{\pi} e^{(-\sigma-2n+2)t}$$
$$\times \int_0^{2\pi}\int_0^{\pi/2} (1-e^{i\varphi}\cos\theta)^{\frac{k-\sigma}{2}-n+1}(1-e^{-i\varphi}\cos\theta)^{-\frac{k+\sigma}{2}-n+1}\sin^{2n-5}\theta\cos\theta\, d\theta\, d\varphi,$$

where $\operatorname{Re}\sigma < -n+1$. Thus, for $\operatorname{Re}\sigma < -n+1$ we have the integral representation

$$\mathrm{c}_k(i(\sigma+n-1)) = \frac{2^{\sigma+2n-2}(n-2)}{\pi}\int_0^{2\pi}\int_0^{\pi/2}(1-e^{i\varphi}\cos\theta)^{\frac{k-\sigma}{2}-n+1}$$
$$\times(1-e^{-i\varphi}\cos\theta)^{-\frac{\sigma+k}{2}-n+1}\sin^{2n-5}\theta\cos\theta\, d\theta\, d\varphi. \quad (6)$$

Since $\varphi^{n\sigma k}(t) = \varphi^{n\sigma k}(-t)$, then in formula (8) of Section 11.3.2 one can replace $(\cosh t - e^{\pm i\varphi}\cos\theta\sinh t)$ by $(\cosh t + e^{\pm i\varphi}\cos\theta\sinh t)$. Therefore, $(1-e^{\pm i\varphi}\cos\theta)$ in (6) can be replaced by $(1+e^{\pm i\varphi}\cos\theta)$, respectively. Making use of the equality

$$(T^{n\sigma k}(g'_{n-1}(t))F_0, F_0) = (F_0, T^{n,-\bar{\sigma}-2n+2,k}(g'_{n-1}(-t))F_0),$$

writing down the integral representation for the right hand side and taking the limit $t\to\infty$, we derive the integral representation for $\mathrm{c}_k(\lambda)$, which coincides with (6) if we replace $e^{\pm i\varphi}$ by $e^{\mp i\varphi}$.

**11.3.4. Associated spherical functions.** We realize the representation $T^{n\ell\ell'}$ of $U(n)$ in the space $\mathfrak{D}^{n-1,\ell\ell'}$. As in the case of the group $SO(n)$ (see Section 9.4.1), the restriction of the invariant scalar product $(\,.\,,\,.\,)_{\ell\ell'}$ of $\mathfrak{D}^{n-1,\ell\ell'}$ onto $\widetilde{\mathfrak{H}}_C^{n-1,mm'}$ differs from the scalar product in $\mathfrak{L}^2(S_C^{n-2})$ in a constant factor only. That is, if

$$\varphi(\eta) = \sum_{m=0}^{\ell}\sum_{m'=0}^{\ell'}\varphi_{mm'}(\eta),\ \psi(\eta) = \sum_{m=0}^{\ell}\sum_{m'=0}^{\ell'}\psi_{mm'}(\eta),\ \eta\in S_C^{n-2},$$

where $\varphi_{mm'},\psi_{mm'}\in\widetilde{\mathfrak{H}}_C^{n-1,mm'}$, then

$$(\varphi,\psi)_{\ell\ell'} = \sum_{m=0}^{\ell}\sum_{m'=0}^{\ell'}\lambda_{mm'}\int_{S_C^{n-2}}\varphi_{mm'}(\eta)\overline{\psi_{mm'}(\eta)}d\eta. \quad (1)$$

The analog of formula (3) of Section 9.4.1 for representations of the group $U(n)$ coincides with

$$t^{n\ell\ell'}_{MK}(g) = \frac{\lambda_{mm'}}{\lambda_{kk'}}\int_{S_C^{n-2}}\left(T^{n\ell\ell'}(g)\Xi_{K'}^{n-1,kk'}(\eta)\right)\overline{\Xi_{M'}^{n-1,mm'}(\eta)}d\eta, \quad (2)$$

where $M=(m,m';M')$, $K=(k,k';K')$. In particular,

$$t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta)) = \lambda_{mm'}\int_{S_C^{n-2}}\left(T^{n\ell\ell'}(g_{n-1}(\theta)\Xi_O^{n-1,00}(\eta)\right)\overline{\Xi_O^{n-1,mm'}(\eta)}d\eta, \quad (3)$$

where $\Xi_O^{n-1,00}(\boldsymbol{\eta}) \equiv 1$. By virtue of formulas (4) of Section 11.3.1 and (5) of Section 11.2.5, we obtain

$$t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))$$
$$= \frac{n-2}{\pi}\lambda_{mm'}\left[\dim T^{n-1,mm'}\right]^{1/2}\int_0^{2\pi}\int_0^{\pi/2}(\cos\theta - ie^{i\varphi}\cos\psi\sin\theta)^{\ell}$$
$$\times(\cos\theta - ie^{-i\varphi}\cos\psi\sin\theta)^{\ell'}e^{-i(m-m')\varphi}t_{00}^{n-1,mm'}(g_{n-2}(\psi))$$
$$\times\sin^{2n-5}\psi\cos\psi\,d\psi\,d\varphi, \quad (4)$$

where $t_{00}^{n-1,mm'}(g_{n-2}(\psi))$ is defined by formula (5) of Section 11.3.2. The explicit expression for $\lambda_{mm'}$ will be found in Section 11.3.5.

Every element $g \in U(n)$ is represented in the form (7) of Section 11.1.5. Therefore,

$$t^{n\ell\ell'}_{MO}(g) = t^{n\ell\ell'}_{MO}(g^n(\boldsymbol{\varphi},\boldsymbol{\theta})). \quad (5)$$

Repeating the proof of formula (1) of Section 9.4.2, we find

$$t^{n\ell\ell'}_{MO}(g^n(\boldsymbol{\varphi},\boldsymbol{\theta}))$$
$$= e^{-i(\ell-\ell'-m+m')\varphi_n}t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta_{n-1}))t^{n-1,mm'}_{M'O}(g^{n-1}(\boldsymbol{\varphi}',\boldsymbol{\theta}')), \quad (6)$$

where $M = (m, m'; M')$, $\boldsymbol{\varphi}' = (\varphi_1, \ldots, \varphi_{n-1})$, $\boldsymbol{\theta}' = (\theta_1, \ldots, \theta_{n-2})$ and

$$t^{n-1,mm'}_{M'O}(g^{n-1}(\boldsymbol{\varphi}',\boldsymbol{\theta}')) =$$
$$= e^{im_{n-2}\varphi_1}\prod_{r=0}^{n-3}e^{i(m_{r+1}-m'_{r+1}-m_r+m'_r)\varphi_{n-r-1}}t^{n-r-1,m_rm'_r}_{(m_{r-1},m'_{r+1})0}(g_{n-r-2}(\theta_{n-r-2})) \quad . \quad (7)$$

Here $m_0 = m$, $m'_0 = m'$.

We derive from formulas (20) of Section 11.2.1 and (4) of Section 11.3.1 that

$$\Delta_0 t^{n\ell\ell'}_{MO}(g^n(\boldsymbol{\varphi},\boldsymbol{\theta})) = -(\ell+\ell')(\ell+\ell'+2n-2)t^{n\ell\ell'}_{MO}(g^n(\boldsymbol{\varphi},\boldsymbol{\theta})). \quad (8)$$

Representing the Laplace operator $\Delta_0$ in the form (5) of Section 11.1.7 and $t^{n\ell\ell'}_{MO}(g^n(\boldsymbol{\varphi},\boldsymbol{\theta}))$ in the form (6), we find the differential equation for $t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))$:

$$\left[\frac{1}{\sin^{2n-3}\theta\cos\theta}\frac{d}{d\theta}\sin^{2n-3}\theta\cos\theta\frac{d}{d\theta} - \frac{(m+m')(m+m'+2n-4)}{\sin^2\theta} -\right.$$
$$\left. - \frac{(\ell-\ell'-m+m')^2}{\cos^2\theta} + (\ell+\ell')(\ell+\ell'+2n-2)\right]t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta)) = 0. \quad (9)$$

For the associated spherical functions $t^{n\sigma k}_{MO}(g)$ of the representation $T^{n\sigma k}$ of $U(n-1,1)$ we have

$$t^{n\sigma k}_{MO}(g) = e^{-i(k-m+m')\varphi_n} t^{n\sigma k}_{(mm')0}(g'_{n-1}(t)) t^{n-1,mm'}_{M'O}(g^{n-1}(\varphi', \theta')), \quad (10)$$

where $M = (m, m'; M')$ and $t^{n-1,mm'}_{M'O}(g^{n-1}(\varphi', \theta'))$ is the same as in (6). For the matrix element $t^{n\sigma k}_{(mm')0}(g'_{n-1}(t))$ one has the integral representation

$$t^{n\sigma k}_{(mm')0}(g'_{n-1}(t)) = \frac{n-2}{\pi}\left[\dim T^{n-1,mm'}\right]^{1/2} \int_0^{2\pi}\int_0^{\pi/2} (\cosh t - e^{i\varphi}\cos\theta \sinh t)^{\frac{\sigma+k}{2}}$$
$$\times (\cosh t - e^{-i\varphi}\cos\theta\sinh t)^{\frac{\sigma-k}{2}} e^{-i(m-m')\varphi} t^{n-1,mm'}_{00}(g_{n-2}(\theta)) \times \sin^{2n-5}\theta\cos\theta\, d\theta\, d\varphi, \quad (11)$$

where $t^{n-1,mm'}_{00}(g_{n-2}(\theta))$ is given by formula (5) of Section 11.3.2.

In order to find the differential equation for $t^{n\sigma k}_{(mm')0}(g'_{n-1}(t))$ we note that if $\Gamma = S^{n-2}_C$, then $\alpha(\eta, g)$ from formula (4) of Section 11.2.4 is representable in the form $[g\mathbf{e}_n, \eta]$ where $\mathbf{e}_n = (0, \ldots, 0, 1) \in H^{n-1}_C$. But then formula (8) of Section 11.2.5 implies that associated spherical functions (10) belong to the space $\mathfrak{H}^+(n, \sigma, k)$ whose elements satisfy equation (9) of Section 11.2.5. Now by making use of expression (10) of Section 11.1.7 for $\Box_0$ we find

$$\left[\frac{1}{\sinh^{2n-3} t\cosh t}\frac{d}{dt}\sinh^{2n-3} t\cosh t\frac{d}{dt} - \frac{(m+m')(m+m'+2n-4)}{\sinh^2 t} + \frac{(k-m+m')^2}{\cosh^2 t} - \sigma(\sigma+2n-2)\right] t^{n\sigma k}_{(mm')0}(g'_{n-1}(t)) = 0. \quad (12)$$

For associated spherical functions of the representations $T^{nkR}$ of $JU(n-1)$ we have

$$t^{nkR}_{MO}(g) = e^{-i(k-m+m')\varphi_n} t^{nR}_{(mm')0}(g_r) t^{n-1,mm'}_{MO}(g^{n-1}(\varphi', \theta')), \quad (13)$$

where $t^{n-1,mm'}_{M'O}(g^{n-1}(\varphi', \theta'))$ is the same as in (6) and

$$t^{nR}_{(mm')0}(g_r) = \frac{n-2}{\pi}\left[\dim T^{n-1,mm'}\right]^{1/2} \int_0^{2\pi}\int_0^{\pi/2} e^{Rr\cos\theta\cos\varphi}$$
$$\times e^{-i(m-m')\varphi} t^{n-1,mm'}_{00}(g_{n-2}(\theta)) \sin^{2n-5}\theta\cos\theta\, d\theta\, d\varphi. \quad (14)$$

The differential equation for $t^{nR}_{(mm')0}(g_r)$ is obtained from (12) by passage to the limit (see Section 11.2.6):

$$\left[\frac{d^2}{dr^2}+\frac{2n-2}{r}\frac{d}{dr}-\frac{(m+m')(m+m'+2n-4)}{r^2}-R^2\right]t^{nR}_{(mm')0}(g_r)=0. \qquad (15)$$

**11.3.5. Evaluation of associated spherical functions.** In order to evaluate associated spherical functions of representations of the groups $U(n)$, $U(n-1,1)$, $JU(n-1)$ it is sufficient to find the matrix elements $t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))$, $t^{n\sigma k}_{(mm')0}(g'_{n-1}(t))$, $t^{nR}_{(mm')0}(g_r)$. To find $t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))$ we solve differential equation (9) of the preceding section. Its solution, regular at $\theta=0$, has the form

$$u(\theta)=\cos^{\ell+\ell'}\theta\tan^{m+m'}\theta F(-\ell+m,-\ell'+m';m+m'+n-1;-\tan^2\theta). \qquad (1)$$

Since

$$\int\limits_{S_C^{n-2}}\left|t^{n\ell\ell'}_{MO}(g^n(\varphi,\theta))\right|^2 d\eta=\left[\dim T^{n\ell\ell'}\right]^{-1},$$

then

$$2(n-1)\int\limits_0^{\pi/2}\left|t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))\right|^2\sin^{2n-3}\theta\cos\theta\,d\theta=\frac{\dim T^{n-1,mm'}}{\dim T^{n\ell\ell'}}. \qquad (2)$$

Formulas (1) and (2) define $t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))$ up to a constant factor with unit module. We choose this factor such that $t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))$ is real. We have

$$t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))=d^{n\ell\ell'}_{mm'}\cos^{\ell+\ell'}\theta\tan^{m+m'}\theta$$
$$\times F(-\ell+m,-\ell'+m';m+m'+n-1;-\tan^2\theta), \qquad (3)$$

where

$$d^{n\ell\ell'}_{mm'}=\frac{(-1)^m}{(m+m'+n-3)!}$$
$$\times\left[\frac{\ell!\ell'!(m+n-3)!(m'+n-3)!(\ell'+m+n-2)!(\ell+m'+n-2)!(n-2)}{(m+m'+n-2)m!m'!(\ell-m)!(\ell'-m')!(\ell+n-2)!(\ell'+n-2)!}\right]^{1/2}. \qquad (4)$$

For $t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))$ one has also the expressions

$$t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))=$$
$$=d^{n\ell\ell'}_{mm'}\cos^{\ell-\ell'+2m'}\theta\tan^{m+m'}\theta F(-\ell'+m',\ell+m'+n-1;m+m'+n-1;\sin^2\theta)$$
$$=d^{n\ell\ell'}_{mm'}\cos^{\ell'-\ell+2m}\theta\tan^{m+m'}\theta F(-\ell+m,\ell'+m+n-1;m+m'+n-1;\sin^2\theta). \qquad (5)$$

One can express $t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))$ in terms of $P^{\ell}_{mn}(x)$ (see Section 6.3.4):

$$t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta)) = = (-1)^{m'}\left[\frac{(n-2)(m+m'+n-2)\ell!\ell'!(m+n-3)!(m'+n-3)!}{m!m'!(\ell+n-2)!(\ell'+n-2)!}\right]\frac{P^{a}_{bc}(\cos 2\theta)}{\sin^{n-2}\theta}, \quad (6)$$

where

$$a = \frac{\ell+\ell'+n-2}{2}, \quad b = m - \frac{\ell-\ell'-n+2}{2}, \quad c = -m' - \frac{\ell-\ell'+n-2}{2},$$

and in terms of Jacobi polynomials:

$$\begin{aligned} t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta)) &= \\ &= (-1)^{m} b^{n\ell\ell'}_{mm'} \cos^{\ell-\ell'+2m'}\theta \tan^{m+m'}\theta P^{(m+m'+n-2,\ell-\ell'+m'-m)}_{\ell'-m'}(\cos 2\theta) \\ &= (-1)^{m} b^{n\ell'\ell}_{m'm} \cos^{\ell'-\ell+2m}\theta \tan^{m+m'}\theta P^{(m+m'+n-2,\ell'-\ell-m'+m)}_{\ell-m}(\cos 2\theta), \quad (7) \end{aligned}$$

where

$$b^{n\ell\ell'}_{mm'} = = \left[\frac{\ell!\ell'!(\ell'-m')!(m+n-3)!(m'+n-3)!(\ell+m'+n-2)!(n-2)(m+m'+n-2)}{m!m'!(\ell-m)!(\ell+n-2)!(\ell'+n-2)!(\ell'+m+n-2)!}\right]^{1/2}. \quad (7')$$

In order to evaluate $t^{n\sigma k}_{(mm')0}(g'_{n-1}(t))$ we find the asymptotics of this function for $t \to \infty$. Since $T^{n\sigma k}$ and $T^{n,-\bar\sigma-2n+2,k}$ are Hermitian-adjoint, then

$$\begin{aligned} t^{n\sigma k}_{(mm')0}(g'_{n-1}(t)) &= \left(\Xi^{n-1,00}_{O}, T^{n,-\bar\sigma-2n+2,k}(g'_{n-1}(-t))\Xi^{n-1,mm'}_{O}\right) \\ &= \left(\dim T^{n-1,mm'}\right)^{1/2}\frac{n-2}{\pi}\int_0^{2\pi}\int_0^{\pi/2}(\cosh t + e^{-i\varphi}\cos\theta\sinh t)^{\frac{k-\sigma}{2}-n+1} \\ &\quad \times (\cosh t + e^{i\varphi}\cos\theta\sinh t)^{-\frac{k-\sigma}{2}-n+1} e^{-i(m-m')\varphi'} t^{n-1,mm'}_{00}(g_{n-1}(\theta')) \\ &\quad \times \sin^{2n-5}\theta\cos\theta\, d\theta\, d\varphi, \quad (8) \end{aligned}$$

where

$$e^{2i\varphi'} = \frac{(e^{i\varphi}\cos\theta\cosh t + \sinh t)(\cosh t + e^{-i\varphi}\cos\theta\sinh t)}{(e^{-i\varphi}\cos\theta\cosh t + \sinh t)(\cosh t + e^{i\varphi}\cos\theta\sinh t)}, \quad (9)$$

$$\cos^2\theta' = \frac{(e^{i\varphi}\cos\theta\cosh t + \sinh t)(e^{-i\varphi}\cos\theta\cosh t + \sinh t)}{(\cosh t + e^{i\varphi}\cos\theta\sinh t)(\cosh t + e^{-i\varphi}\cos\theta\sinh t)}. \quad (9')$$

Dividing the numerators and the denominators in (9) and (9′) by $\cosh^2 t$ and turning $t$ to the infinity, we obtain that $e^{2i\varphi'} \to 1$, $\cos^2\theta' \to 1$ when $t \to +\infty$. From here, from the integral representation for the Harish-Chandra function and from (8), we derive the asymptotics for $t^{n\sigma k}_{(mm')0}(g'_{n-1}(t))$ when $t \to +\infty$:

$$t^{n\sigma k}_{(mm')0}(g'_{n-1}(t)) \sim (\dim T^{n-1,mm'})^{1/2} \mathbf{c}_k(i(\sigma+n-1)) e^{(-\sigma-2n+2)t}, \tag{10}$$

where $\operatorname{Re}\sigma < -n+1$.

The function $t^{n\sigma k}_{(mm')0}(g'_{n-1}(t))$ is a solution of equation (12) of Section 11.3.4, which is regular at the point $t = 0$. Hence,

$$t^{n\sigma k}_{(mm')0}(g'_{n-1}(t)) = \\ = c^{n\sigma k}_{mm'} \cosh^\sigma t \tanh^{m+m'} t F\left(m - \frac{\sigma+k}{2}, m' - \frac{\sigma-k}{2}; m+m'+n-1; \tanh^2 t\right). \tag{11}$$

By means of (10), we find

$$c^{n\sigma k}_{mm'} = \left[\frac{(m+n-3)!(m'+n-3)!(n-2)}{m!m'!(m+m'+n-2)}\right]^{1/2} \\ \times \frac{\Gamma\left(\frac{\sigma-k}{2}+1\right)\Gamma\left(\frac{\sigma+k}{2}+1\right)}{(m+m'+n-3)!\Gamma\left(\frac{\sigma+k}{2}-m+1\right)\Gamma\left(\frac{\sigma-k}{2}-m'+1\right)}. \tag{12}$$

Formulas (11) and (12) define $t^{n\sigma k}_{(mm')0}(g'_{n-1}(t))$ for $\operatorname{Re}\sigma < -n+1$. But the right hand side of formula (11) of Section 11.3.4 for every fixed $t$ is an entire analytic function in $\sigma$. Consequently, $t^{n\sigma k}_{(mm')0}(t'_{n-1}(t))$ is an entire analytic function in $\sigma$. Therefore, formulas (11) and (12) are valid for all $\sigma \in \mathbb{C}$.

The matrix elements $t^{nR}_{(mm')0}(g_r)$ are obtained from (11) by passage to the limit (see Section 11.2.7):

$$t^{nR}_{(mm')0}(g_r) = i^{m+m'} \left[\frac{(n-2)(m+m'+n-2)(m+n-3)!(m'+n-3)!}{m!m'!}\right]^{1/2} \\ \times \left(\frac{-iRr}{2}\right)^{-n+2} J_{m+m'+n-2}(-iRr). \tag{13}$$

Let us invert the order of integration in formula (11) of Section 11.3.4. The intrinsic integral is of the form

$$I \equiv \frac{1}{2\pi}\int_0^{2\pi} (\cosh t - e^{i\varphi}\cos\theta\sinh t)^{\frac{\sigma+k}{2}} (\cosh t - e^{-i\varphi}\cos\theta\sinh t)^{\frac{\sigma-k}{2}} e^{iq\varphi}\, d\varphi. \tag{14}$$

In order to calculate it we take cosh $t$ out from the binomial expression, apply the binomial formula, multiply out the obtained series and integrate term by term. We obtain that

$$I = \frac{\Gamma\left(\frac{\sigma-k}{2}+1\right)\cosh^{\sigma} t\,\tanh^{q} t}{q!\,\Gamma\left(\frac{\sigma-k}{2}-q+1\right)}\cos^{q}\theta F\left(-\frac{\sigma+k}{2},\frac{k-\sigma}{2}+q;q+1;\cos^2\theta\tanh^2 t\right) \quad (15)$$

if $q \geq 0$ and

$$I = \frac{\Gamma\left(\frac{\sigma+k}{2}+1\right)\cosh^{\sigma} t\,\tanh^{-q} t}{(-q)!\,\Gamma\left(\frac{\sigma+k}{2}+q+1\right)}\cos^{-q}\theta F\left(-q-\frac{\sigma+k}{2},\frac{k-\sigma}{2};-q+1;\cos^2\theta\tanh^2 t\right), \quad (15')$$

if $q \leq 0$. Hence,

$$t^{n\sigma k}_{(mm')0}(g'_{n-1}(t)) = \frac{2(n-2)\Gamma\left(\frac{\sigma-k}{2}+1\right)}{(m'-m)!\,\Gamma\left(\frac{\sigma-k}{2}+m-m'+1\right)}\left(\dim T^{n-1,mm'}\right)^{1/2}\cosh^{\sigma} t\,\tanh^{m'-m} t$$
$$\times\int_0^{\pi/2} F\left(\frac{k-\sigma}{2}+m'-m,-\frac{k+\sigma}{2};m'-m+1;\cos^2\theta\tanh^2 t\right)\varphi^{n-1,mm'}(\theta)$$
$$\times\sin^{2n-5}\theta\cos^{m'-m+1}\theta\,d\theta \quad (16)$$

if $m'-m \geq 0$, and

$$t^{n\sigma k}_{(mm')0}(g'_{n-1}(t)) = \frac{2(n-2)\Gamma\left(\frac{\sigma+k}{2}+1\right)}{(m-m')!\,\Gamma\left(\frac{\sigma+k}{2}-m+m'+1\right)}\left(\dim T^{n-1,mm'}\right)^{1/2}\cosh^{\sigma} t$$
$$\times\tanh^{m-m'} t\int_0^{\pi/2} F\left(-\frac{\sigma+k}{2}+m-m',\frac{k-\sigma}{2};m-m'+1;\cos^2\theta\tanh^2 t\right)\varphi^{n-1,mm'}(\theta)$$
$$\times\sin^{2n-5}\theta\cos^{m-m'+1}\theta\,d\theta \quad (16')$$

if $m - m' \geq 0$. In the same way we obtain for $t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))$ the expression

$$t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta)) = \\ = \frac{2(n-2)(-1)^{m+m'}}{(m'-m)!(\ell+m-m')!}\left(\dim\, T^{n-1,mm'}\right)^{1/2}\left[\frac{\ell'!(\ell-m)!(\ell'-m')!(\ell+m'+n-2)!}{\ell!(\ell+n-2)!(\ell'+n-2)!}\right. \\ \left.\times(\ell'+m+n-2)!\right]^{1/2}\cos^{\ell+\ell'}\theta\tan^{m'-m}\theta \\ \times\int_0^{\pi/2}F(-\ell,-\ell'+m'-m;m'-m+1;-\cos^2\psi\tan^2\theta) \\ \times\varphi^{n-1,mm'}(\psi)\sin^{2n-5}\psi\cos^{m'-m+1}\psi\,d\psi, \quad (17)$$

where $m' \geq m$. If $m' < m$, then one has to permute $m$ and $m'$, $\ell$ and $\ell'$ and to remove $(-1)^{m+m'}$ from the right hand side of (17).

Expressions (11) and (12) for $t^{n\sigma k}_{(mm')0}(g'_{n-1}(t))$ can be derived from (16) and (16$'$). For this we represent the hypergeometric function as a series, replace $\varphi^{n-1,mm'}(\theta)$ by one of expressions (5) of Section 11.3.2, invert the order of integration and summation and integrate term by term by means of the formula

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{p}P_m^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+p+1}p!\Gamma(\alpha+m+1)\Gamma(p-\beta+1)}{m!\Gamma(p-m-\beta+1)\Gamma(p+\alpha+m+2)},$$

where $\operatorname{Re}\alpha > -1$, $p \geq 0$.

In the same way one can evaluate $t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))$ . Comparing the expression obtained with formula (3), we find the expression for the coefficients $\lambda_{mm'}$ from formulas (1)–(4) of Section 11.3.4:

$$\lambda_{mm'} = (-1)^m i^{-m-m'}\left[\dim\, T^{n-1,mm'}\right]^{1/2} \\ \times\left[\frac{(\ell-m)!(\ell'-m')!(\ell+m'+n-2)!(\ell'+m+n-2)!}{\ell!\ell'!(\ell+n-2)!(\ell'+n-2)!}\right]^{1/2}. \quad (18)$$

For the matrix elements $t^{nR}_{(mm')0}(g_r)$ we derive from formula (14) of Section 11.3.4 the expression

$$t^{nR}_{(mm')0}(g_r) = 2(n-2)\left[\frac{(m+n-3)!(m'+n-3)!(m+m'+n-2)}{m!m'!(n-3)!(n-2)!}\right]^{1/2} i^{m'-m} \\ \times\int_0^{\pi/2}J_{m'-m}(-iRr\cos\theta)\varphi^{n-1,mm'}(\theta)\sin^{2n-5}\theta\cos\theta\,d\theta. \quad (19)$$

Substituting into (16′) expressions for the matrix elements, we obtain

$$\int_0^{\pi/2} F(\tau,\tau+2k+m-m';m-m'+1;r^2\cos^2\theta)P_{m'}^{(n-3,m-m')}(\cos 2\theta) \times \sin^{2n-5}\theta\cos^{m-m'+1}\theta\,d\theta$$
$$= \frac{(m-m')!(n-3)!}{2(m+m'+n-2)!}\frac{\Gamma(-\tau+1)\Gamma(-\tau+2k+1)r^{2m'}}{\Gamma(-\tau-m+1)\Gamma(-\tau+2k-m'+1)} \times F(m+\tau,\tau+2k+m';m-m'+n-1;r^2), \quad (20)$$

where $\tau\in\mathbf{C}$, $k\in\mathbf{Z}$, $m,m'\in\mathbf{Z}_+\cup\{0\}$. A similar relation follows from (16). In the same way formula (19) yields

$$\int_0^{\pi/2} J_\beta(x\cos\theta)P_n^{(\alpha,\beta)}(\cos 2\theta)\sin^{2\alpha+1}\theta\cos^{\beta+1}\theta\,d\theta$$
$$= (-1)^n\frac{2^\alpha\Gamma(\alpha+n+1)}{x^{\alpha+1}}J_{\alpha+\beta+2n-1}(x), \quad (21)$$

where $\alpha$ and $\beta$ are non-negative integers.

**11.3.6. Special cases.** For $\ell=m$ and $\ell'=m'$ the hypergeometric function in formula (3) of Section 11.3.5 is equal to 1. Hence, we have

$$t^{n\ell\ell'}_{(m\ell')0}(g_{n-1}(\theta)) = (-1)^{\ell-m}t^{n\ell'\ell}_{(\ell' m)0}(g_{n-1}(\theta))$$
$$= (-1)^m\left[\frac{(n-2)\ell!(m+n-3)!(\ell+\ell'+n-2)!}{m!(\ell-m)!(\ell+n-2)!(\ell'+m+n-3)!(\ell'+n-2)}\right]^{1/2}\sin^{\ell'+m}\theta\cos^{\ell-m}\theta. \quad (1)$$

In particular,

$$t^{n\ell 0}_{(m0)0}(g_{n-1}(\theta)) = (-1)^m t^{n0\ell}_{(0m)0}(g_{n-1}(\theta))$$
$$= (-1)^m\left[\frac{\ell!}{m!(\ell-m)!}\right]^{1/2}\sin^m\theta\cos^{\ell-m}\theta, \quad (2)$$

$$t^{n\ell\ell'}_{(\ell\ell')0}(g_{n-1}(\theta)) = (-1)^\ell\left[\frac{(n-2)(\ell+\ell'+n-2)}{(\ell+n-2)(\ell'+n-2)}\right]^{1/2}\sin^{\ell+\ell'}\theta. \quad (3)$$

**11.3.7. Symmetry relations.** Integral representation (4) of Section 11.3.4 and expressions (3)–(5) of Section 11.3.5 for $t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta)) \equiv t^{n\ell\ell'}_{(mm')0}(\theta)$ imply the symmetry relations

$$t^{n\ell\ell'}_{(mm')0}(\theta) = (-1)^{m+m'}t^{n\ell'\ell}_{(m'm)0}(\theta) = (-1)^{m+m'}t^{n\ell\ell'}_{(mm')0}(-\theta). \quad (1)$$

For $t^{nR}_{(mm')0}(g_r)$ we have

$$t^{nR}_{(mm')0}(g_r) = t^{nR}_{(m'm)0}(g_r) = (-1)^{m+m'} t^{nR}_{(mm')0}(g_{-r}), \tag{2}$$

$$\overline{t^{nR}_{(mm')0}(g_r)} = t^{n\bar{R}}_{(mm')0}(g_r), \tag{3}$$

and for $t^{n\sigma k}_{(mm')0}(g'_{n-1}(t)) \equiv t^{n\sigma k}_{(mm')0}(t)$ we have

$$t^{n\sigma k}_{(mm')0}(t) = t^{n\sigma,-k}_{(m'm)0}(t) = (-1)^{m+m'} t^{n\sigma k}_{(mm')0}(-t) = \overline{t^{n\bar{\sigma}k}_{(mm')0}(t)}$$
$$= \left[\frac{\Gamma\left(\frac{\sigma-k}{2}-m'+1\right)\Gamma\left(\frac{\sigma+k}{2}-m+1\right)\Gamma\left(\frac{-\sigma-k}{2}-n+2\right)\Gamma\left(\frac{k-\sigma}{2}-n+2\right)}{\Gamma\left(\frac{\sigma-k}{2}+1\right)\Gamma\left(\frac{\sigma+k}{2}+1\right)\Gamma\left(\frac{-\sigma-k}{2}-m'-n+2\right)\Gamma\left(\frac{k-\sigma}{2}-m-n+2\right)}\right]^{-1}$$
$$\times\, t^{n,-\sigma-2n+2,k}_{(mm')0}(t). \tag{4}$$

**11.3.8. Raising and lowering operators.** Raising and lowering operators for $t^{n\ell\ell'}_{(mm')0}(\theta)$, $t^{n\sigma k}_{(mm')0}(t)$ and $t^{nR}_{(mm')0}(g_r)$ are derived in the same way as for the corresponding matrix elements of representations of the groups $SO(n)$, $SO_0(n-1,1)$ and $ISO(n-1)$. They can be also found by means of recurrence relations for the special functions which appear in the expressions for the matrix elements. The raising and lowering operators for $t^{n\ell\ell'}_{(mm')0}(\theta)$ are of the form

$$\left[\frac{d}{d\theta} - (\ell-\ell')\tan\theta - 2m'\tan^{-1}\theta - 2(m-m')\tan^{-1}2\theta\right] t^{n\ell\ell'}_{(mm')0}(\theta)$$
$$= 2\left[\frac{(\ell'+m+n-1)(\ell-m)(m+1)(m+m'+n-2)}{(m+n-2)(m+m'+n-1)}\right]^{1/2} t^{n\ell\ell'}_{(m+1,m')0}(\theta), \tag{1}$$

$$\left[\frac{d}{d\theta} + (\ell-\ell')\tan\theta - 2m\,\tan^{-1}\theta + 2(m-m')\tan^{-1}2\theta\right] t^{n\ell\ell'}_{(mm')0}(\theta)$$
$$= -2\left[\frac{(\ell+m'+n-1)(\ell'-m')(m'+1)(m+m'+n-2)}{(m'+n-2)(m+m'+n-1)}\right]^{1/2}$$
$$\times\, t^{n\ell\ell'}_{(m,m'+1)0}(\theta), \tag{2}$$

$$\left[\frac{d}{d\theta} - (\ell-\ell')\tan\theta + 2(m+n-2)\tan^{-1}\theta - 2(m-m')\tan^{-1}2\theta\right] t^{n\ell\ell'}_{(mm')0}(\theta)$$
$$= 2\left[\frac{(\ell+m'+n-2)(\ell'-m'+1)(m'+n-3)(m+m'+n-2)}{m'(m+m'+n-3)}\right]^{1/2}$$
$$\times\, t^{n\ell\ell'}_{(m,m'-1)0}(\theta), \tag{3}$$

$$\left[\frac{d}{d\theta}+(\ell-\ell')\tan\theta+2(m'+n-2)\tan^{-1}\theta+2(m-m')\tan^{-1}2\theta\right]t^{n\ell\ell'}_{(mm')0}(\theta)$$
$$=-2\left[\frac{(\ell-m+1)(\ell'+m+n-2)(m+n-2)(m+m'+n-2)}{m(m+m'+n-3)}\right]^{1/2}$$
$$\times\, t^{n\ell\ell'}_{(m-1,m')0}(\theta). \quad (4)$$

For $t^{n\sigma k}_{(mm')0}(t)$ we have

$$\left[\frac{d}{dt}+k\tanh t-2m'\tanh^{-1}t-2(m-m')\tanh^{-1}2t\right]t^{n\sigma k}_{(mm')0}(t)$$
$$=2\left(\frac{k-\sigma}{2}-m-n+1\right)\left[\frac{(m+1)(m+m'+n-2)}{(m+n-2)(m+m'+n-1)}\right]^{1/2}t^{n\sigma k}_{(m+1,m')0}(t), \quad (5)$$

$$\left[\frac{d}{dt}+k\tanh t+2(m+n-2)\tanh^{-1}t-2(m-m')\tanh^{-1}2t\right]t^{n\sigma k}_{(mm')0}(t)$$
$$=2\left(\frac{k-\sigma}{2}+m'-1\right)\left[\frac{(m'+n-3)(m+m'+n-2)}{m'(m+m'+n-3)}\right]^{1/2}t^{n\sigma k}_{(m,m'-1)0}(t), \quad (6)$$

$$\left[\frac{d}{dt}-k\tanh t-2m\tanh^{-1}t+2(m-m')\tanh^{-1}2t\right]t^{n\sigma k}_{(mm')0}(t)$$
$$=-2\left(\frac{\sigma+k}{2}+m'+n-1\right)\left[\frac{(m'+1)(m+m'+n-2)}{(m'+n-2)(m+m'+n-1)}\right]^{1/2}$$
$$\times\, t^{n\sigma k}_{(m,m'+1)0}(t), \quad (7)$$

$$\left[\frac{d}{dt}-k\tanh t+2(m'+n-2)\tanh^{-1}t+2(m-m')\tanh^{-1}2t\right]t^{n\sigma k}_{(mm')0}(t)$$
$$=-2\left(\frac{\sigma+k}{2}-m+1\right)\left[\frac{(m+n-3)(m+m'+n-2)}{m(m+m'+n-3)}\right]^{1/2}t^{n\sigma k}_{(m-1,m')0}(t). \quad (8)$$

We suggest that the reader derives corresponding formulas for $t^{nR}_{(mm')0}(g_r)$.

**11.3.9. Relations between spherical functions for groups of different dimensionalities.** We derive from formula (5) of Section 11.3.5 that

$$t^{n\ell\ell'}_{(mm')0}(\theta)=\left[\frac{(n-2)(\ell+n-1)\ell'}{m'(n-1)(m+n-2)}\right]^{1/2}\sin\theta\, t^{n+1,\ell,\ell'-1}_{(m,m'-1)0}(\theta)$$
$$=-\left[\frac{(n-2)\ell(\ell'+n-1)}{m(n-1)(m'+n-2)}\right]^{1/2}\sin\theta\, t^{n+1,\ell-1,\ell'}_{(m-1,m')0}(\theta). \quad (1)$$

Similarly, it follows from formula (3) of Section 11.3.5, that

$$t^{n\ell\ell'}_{(mm')0}(\theta) = -\left[\frac{(m+1)(\ell+n-1)(m'+n-3)\ell'}{(m+n-2)(\ell+1)m'(\ell'+n-2)}\right]^{1/2} t^{n,\ell+1,\ell'-1}_{(m+1,m'-1)0}(\theta). \quad (2)$$

Formula (1) implies

$$t^{n\ell\ell'}_{(mm')0}(\theta) =$$
$$= (-1)^m \left[\frac{(n-2)\ell!(\ell'+m+n-2)!(m'+n-3)!}{(m+n-2)m!(\ell-m)!(\ell'+n-2)!(m+m'+n-3)!}\right]^{1/2} \sin^m\theta t^{n+m,\ell-m,\ell'}_{(0,m')0}(\theta)$$
$$= \left[\frac{(n-2)\ell'!(\ell+m'+n-2)!(m+n-3)!}{(m'+n-2)m'!(\ell'-m')!(\ell+n-2)!(m+m'+n-3)!}\right]^{1/2} \sin^{m'}\theta t^{n-m',\ell,\ell'-m'}_{(m0)0}(\theta). \quad (3)$$

Consequently, $t^{n\ell\ell'}_{(mm')0}(\theta)$ can be expressed in terms of $\varphi^{n'pp'}(\theta)$:

$$t^{n\ell\ell'}_{(mm')0}(\theta) = \frac{(-1)^m}{(m+m'+n-3)!}$$
$$\times \left[\frac{\ell!\ell'!(\ell+m'+n-2)!(\ell'+m+n-2)!(m+n-3)!(m'+n-3)!(n-2)}{m!m'!(\ell-m)!(\ell'-m')!(\ell'+n-2)!(\ell+n-2)(m+m'+n-2)}\right]^{1/2}$$
$$\times \sin^{m+m'}\theta \varphi^{n+m+m',\ell-m,\ell'-m}(\theta). \quad (4)$$

From the expressions of Section 11.3.5, we find for $t^{n\sigma k}_{(mm')0}(t)$ the relations

$$t^{n\sigma k}_{(mm')0}(t) = \left[\frac{n-2}{(n-1)m'(m+n-2)}\right]^{1/2} \frac{\sigma-k}{2} \sinh t\, t^{n+1,\sigma-1,k+1}_{(m,m'-1)0}(t)$$
$$= \left[\frac{n-2}{(n-1)m(m'+n-2)}\right]^{1/2} \frac{\sigma+k}{2} \sinh t\, t^{n+1,\sigma-1,k-1}_{(m-1,m')0}(t)$$
$$= \left[\frac{(m+1)(m'+n-3)}{m'(m+n-2)}\right]^{1/2} \frac{\sigma+k-2}{\sigma-k+2} t^{n,\sigma,k-2}_{(m+1,m'-1)0}(t). \quad (5)$$

Hence,

$$t^{n\sigma k}_{(mm')0}(t) =$$
$$= \left[\frac{(m+n-3)!(m'+n-3)!(n-2)}{m!m'!(m+m'+n-2)}\right]^{1/2} \frac{\Gamma\left(\frac{\sigma-k}{2}+1\right)\Gamma\left(\frac{\sigma+k}{2}+1\right)}{\Gamma\left(\frac{\sigma-k}{2}-m'+1\right)\Gamma\left(\frac{\sigma+k}{2}-m+1\right)}$$
$$\times \frac{\sinh^{m+m'} t}{(m+m'+n-3)!} \varphi^{n+m+m',\sigma-m-m',k+m-m'}(t). \quad (6)$$

For the functions $t^{nR}_{(mm')0}(g_r)$ we have

$$t^{nR}_{(mm')0}(g_r) = \frac{Rr}{2}\left[\frac{n-2}{(n-1)m'(m+n-2)}\right]^{1/2} t^{n+1,R}_{(m,m'-1)0}(g_r)$$
$$= \frac{Rr}{2}\left[\frac{n-2}{(n-1)m(m'+n-2)}\right]^{1/2} t^{n+1,R}_{(m-1,m')0}(g_r)$$
$$= \left[\frac{(m'+n-3)(m+1)}{m'(m+n-2)}\right]^{1/2} t^{nR}_{(m+1,m'-1)0}(g_r). \tag{7}$$

**11.3.10. Relations for zonal spherical functions.** Let us set $m = m' = 0$ into formula (1) of Section 11.3.8 and take into account that

$$t^{n\ell\ell'}_{(1,0)0}(\theta) = -\left[\frac{\ell(\ell'+n-1)}{n-1}\right]^{1/2} \sin\theta\, t^{n+1,\ell-1,\ell'}_{00}(\theta) \tag{1}$$

(see formula (1) of Section 11.3.9). We obtain

$$\left[\frac{d}{d\theta} - (\ell-\ell')\tan\theta\right]\varphi^{n\ell\ell'}(\theta) = -2\frac{\ell(\ell'+n-1)}{n-1}\sin\theta\,\varphi^{n+1,\ell-1,\ell'}(\theta). \tag{2}$$

In the same way from formulas (3) of Section 11.3.8 and (1) of Section 11.3.9 we have

$$\left[\frac{d}{d\theta} + (\ell-\ell')\tan\theta\right]\varphi^{n\ell\ell'}(\theta) = -2\frac{\ell'(\ell+n-1)}{n-1}\sin\theta\,\varphi^{n+1,\ell,\ell'-1}(\theta). \tag{3}$$

The sum of relations (2) and (3) gives the differentiation formula

$$\frac{d}{d\theta}\varphi^{n\ell\ell'}(\theta) = -\frac{\ell(\ell'+n-1)}{n-1}\sin\theta\,\varphi^{n+1,\ell-1,\ell'}(\theta) - \frac{\ell'(\ell+n-1)}{n-1}\sin\theta\,\varphi^{n+1,\ell,\ell'-1}(\theta). \tag{4}$$

Multiplying (2) by $-1$ and summing it with (3), we find

$$\varphi^{n\ell\ell'}(\theta) = = \frac{\ell(\ell'+n-1)}{(\ell-\ell')(n-1)}\cos\theta\,\varphi^{n+1,\ell-1,\ell'}(\theta) - \frac{\ell'(\ell+n-1)}{(\ell-\ell')(n-1)}\cos\theta\,\varphi^{n+1,\ell,\ell'-1}(\theta). \tag{5}$$

It follows from formulas (5) of Section 11.3.8 and (5) of Section 11.3.9 that

$$\left[\frac{d}{dt} + k\tanh t\right]\varphi^{n\sigma k}(t) = -2\left(\frac{\sigma-k}{2}+n-1\right)\frac{\sigma+k}{2(n-1)} \times \sinh t\,\varphi^{n+1,\sigma-1,k-1}(t). \tag{6}$$

In the same way formulas (7) of Section 11.3.8 and (5) of Section 11.3.9 yield

$$\left[\frac{d}{dt} - k\tanh t\right]\varphi^{n\sigma k}(t) = -2\left(\frac{\sigma+k}{2}+n-1\right)\frac{\sigma-k}{2(n-1)} \times \sinh t\,\varphi^{n+1,\sigma-1,k+1}(t). \tag{7}$$

From (6) and (7) we derive that

$$\frac{d}{dt}\varphi^{n\sigma k}(t) = \frac{(\sigma+k)(k-\sigma-2n+2)}{4(n-1)}\sinh t\,\varphi^{n+1,\sigma-1,k-1}(t) - \frac{(\sigma-k)(\sigma+k+2n-2)}{4(n-1)}\sinh t\,\varphi^{n=1,\sigma-1,k+1}(t), \tag{8}$$

$$\varphi^{n\sigma k}(t) = -\frac{(\sigma+k)(\sigma-k=2n-2)}{4k(n-1)}\cosh t\,\varphi^{n+1,\sigma-1,k-1}(t) + \frac{(\sigma-k)(\sigma+k+2n-2)}{4k(n-1)}\cosh t\,\varphi^{n+1,\sigma-1,k+1}(t). \tag{9}$$

**11.3.11. The connection of spherical functions of the groups $U(n)$ and $IU(n-1)$ with spherical functions of the groups $SO(2n)$ and $ISO(2n-1)$.** The differential equation for matrix elements $t^{U(n),\ell\ell'}_{(mm')0}(g_{n-1}(\theta))$ of the operators $T^{n\ell\ell'}(g_{n-1}(\theta))$ of the representations of $U(n)$ has the form (9) of Section 11.3.4. This equation looks like the differential equation

$$\left[\frac{1}{\sin^{2n-3}\theta\cos\theta}\frac{d}{d\theta}\sin^{2n-3}\theta\cos\theta\frac{d}{d\theta} - \frac{p(p+2n-4)}{\sin^2\theta} - \frac{q^2}{\cos^2\theta} + r(r+2n-2)\right]u(\theta) = 0$$

for the function $t^{SO(2n),r}_{pq,0}(g_{2n-2,2n}(\theta))$ which appears in the associated $SO(2n-2)\times SO(2)$-spherical function of the symmetric space $SO(2n)/SO(2n-1)$ corresponding to the representation $T^{2n,r}$ of the group $SO(2n)$ (see Section 10.4.2). Comparing these differential equations, we obtain

$$t^{U(n),\ell\ell'}_{(mm')0}(g_{n-1}(\theta)) = a^{\ell\ell'}_{mm'}t^{SO(2n),\ell+\ell'}_{(m+m',\ell-\ell'-m+m')0}(g_{2n-2,2n}(\theta)). \tag{1}$$

In order to find the constant $a^{\ell\ell'}_{mm'}$ we note that the invariant measures on $S^{n-1}_C$ and $S^{2n-1}$ coincide. And since

$$2(n-1)\int_0^{\pi/2}\left|t^{U(n),\ell\ell'}_{(mm')0}(g_{n-1}(\theta))\right|^2\sin^{2n-3}\theta\cos\theta\,d\theta = \frac{\dim T^{U(n-1),mm'}}{\dim T^{U(n),\ell\ell'}},$$

$$2(n-1)\int_0^{\pi/2}\left|t^{SO(2n),r}_{pq,0}(g_{2n-2,2n}(\theta))\right|^2\sin^{2n-3}\theta\cos\theta\,d\theta = \frac{\dim T^{SO(2n-2),p}}{\dim T^{SO(2n),r}},$$

then

$$a^{\ell\ell'}_{mm'} = \left[\frac{\dim T^{SO(2n),\ell+\ell'} \dim T^{U(n-1),mm'}}{\dim T^{SO(2n-2),m+m'} \dim T^{U(n),\ell\ell'}}\right]^{1/2}. \tag{2}$$

Substituting expressions for the dimensionalities of representations, after simplifications we derive

$$a^{\ell\ell'}_{mm'} = \\ = \left[\frac{(\ell+\ell'+2n-3)!\ell!\ell'!(m+m')!(m+n-3)!(m'+n-3)!(n-2)}{2(\ell+\ell')!(\ell+n-2)!(\ell'+n-2)!(m+m'+2n-5)!m!m'!(2n-3)}\right]^{1/2}. \tag{3}$$

For the zonal spherical function of the group $U(n)$ we have

$$t^{U(n),\ell\ell'}_{00}(g_{n-1}(\theta)) = \\ = \left[\frac{\dim T^{SO(2n),\ell+\ell'}}{\dim T^{U(n),\ell\ell'}}\right]^{1/2} t^{SO(2n),\ell+\ell'}_{(0,\ell-\ell')0}(g_{2n-2,2n}(\theta)). \tag{4}$$

Comparing expression (13) of Section 11.3.5 for the matrix element $t^{IU(n-1),R}_{(mm')0}(g_r)$ of the operator $T^{nkR}(g_r)$ of the representation of $JU(n-1)$ with expression (10) of Section 9.4.2 for the matrix element of the operator $T^{2n-1,R}(g_r)$ of the representation of $ISO(2n-2)$, we obtain

$$t^{IU(n-1),R}_{(mm')0}(g_r) = a_{mm'} t^{ISO(2n-2),R}_{m+m',0}(g_r), \tag{5}$$

where

$$a_{mm'} = \left[\left(\dim T^{U(n-1),mm'}\right)\left(\dim T^{SO(2n-2),m+m'}\right)^{-1}\right]^{1/2}. \tag{6}$$

In particular,

$$t^{IU(n-1),R}_{00}(g_r) = t^{ISO(2n-2),R}_{00}(g_r). \tag{7}$$

**11.3.12. $K_{pq}$-spherical functions on the complex sphere.** The results analogous to those of Section 10.4 for harmonic polynomials on $\mathbb{R}^n$ are valid for harmonic polynomials on $\mathbb{C}^n$. Let $1 < p < n$. Set $\mathbf{z} = (\mathbf{w}, \mathbf{t})$, $\mathbf{w} \in \mathbb{C}^p$, $\mathbf{t} \in \mathbb{C}^q$, $p+q=n$, and denote by $K_{pq}$ the subgroup $U(p) \times U(q)$ of $U(n)$. Matrices from $K_{pq}$ act in $\mathbb{C}^n$ according to the formula

$$(h_1, h_2)(\mathbf{w}, \mathbf{t}) = (h_1\mathbf{w}, h_2\mathbf{t}), \qquad U_1 \in U(p),\ h_2 \in U(q).$$

For $\mathbf{z} \in \mathbb{C}^r$, we set $|\mathbf{z}|^2 = z_1\bar{z}_1 + \ldots + z_r\bar{z}_r$. For harmonic polynomials on $\mathbb{C}^n$, the analog of the theorem from Section 10.4.1 is formulated as follows.

*Let $r, r'$, $s$, $s'$, $m$ be non-negative integers. Then there exists the non-zero homogeneous polynomial $Y(x,y)$ of degree $m$ in two real variables, such that all polynomials of the form*

$$P(\mathbf{w},\mathbf{t}) = Y(|\mathbf{w}|^2, |\mathbf{t}|^2)A(\mathbf{w})B(\mathbf{t}), \tag{1}$$

*where $A(\mathbf{w}) \in \mathfrak{H}_C^{prr'}$, $B(\mathbf{t}) \in \mathfrak{H}_C^{qss'}$, are harmonic. This polynomial is of the form*

$$Y(x,y) = \mathrm{c}'(x+y)^m P_m^{(r+r'+p-1,s+s'+q-1)}\left(\frac{x-y}{x+y}\right), \tag{2}$$

*where $\mathrm{c}'$ is a constant factor.*

The functions

$$\Xi_{M'}^{prr'}(\mathbf{w}) = |\mathbf{w}|^{r+r'}\Xi_{M'}^{prr'}(\boldsymbol{\zeta}), \tag{3}$$

$$\xi_{M''}^{qss'}(\mathbf{t}) = |\mathbf{t}|^{s+s'}\Xi_{M''}^{qss'}(\boldsymbol{\eta}), \tag{3'}$$

where $\boldsymbol{\zeta} = \frac{\mathbf{w}}{|\mathbf{w}|} \in S_C^{p-1}$, $\boldsymbol{\eta} = \frac{\mathbf{t}}{|\mathbf{t}|} \in S_C^{q-1}$ and $\Xi_{M'}^{prr'}(\boldsymbol{\zeta})$, $\xi_{M''}^{qss'}(\boldsymbol{\eta})$ are given by formula (4) of Section 11.3.1, form bases in the spaces $\mathfrak{H}_C^{prr'}$ and $\mathfrak{H}_C^{qss'}$, respectively. Let us set

$$\ell = m + r + s, \qquad \ell' = m + r' + s' \tag{4}$$

and introduce the functions

$$\Xi_{M'M''}^{pq,\ell\ell',rr',ss'}(\mathbf{z}) = \mathrm{c}''Y(|\mathbf{w}|^2, |\mathbf{t}|^2)\Xi_{M'}^{prr'}(\mathbf{w})\Xi_{M''}^{qss'}(\mathbf{t}). \tag{5}$$

It is clear that the parameters $\ell$, $\ell'$, $r$, $r'$, $s$, $s'$ satisfy the conditions

$$r - r' + s - s' = \ell - \ell', \qquad r + r' + s + s' \le \ell + \ell'. \tag{6}$$

In bispherical coordinates on $S_C^{n-1}$ functions (5) take the form

$$\Xi_{M'M''}^{pq,\ell\ell',rr',ss'}(\boldsymbol{\xi}) = \mathrm{c}\,\sin^{r+r'}\theta\cos^{s+s'}\theta\, P_{(\ell+\ell'-r-r'-s-s')/2}^{(r+r'+p-1,s+s'+q-1)}(\cos 2\theta) \times \Xi_{M'}^{prr'}(\boldsymbol{\zeta})\Xi_{M''}^{qss'}(\boldsymbol{\eta}), \tag{7}$$

where $\boldsymbol{\xi} = \sin\theta\boldsymbol{\zeta} + \cos\theta\boldsymbol{\eta}$. We choose the constant c such that

$$\int_{S_C^{n-1}} \left|\Xi_{M'M''}^{pq,\ell\ell',rr',ss'}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} = 1.$$

As in the case of formula (7) of Section 10.4.1, we find that

$$c = \left[\frac{s!p!q!\Gamma\left(\frac{1}{2}(\ell+\ell'-r-r'-s-s'+2)\right)\Gamma\left(\frac{1}{2}(\ell+\ell'+r+r'+s+s'-2n-2)\right)(\ell+\ell'+n-1)}{n!\Gamma\left(\frac{1}{2}(\ell+\ell'+r+r'-s-s'+2p)\right)\Gamma\left(\frac{1}{2}(\ell+\ell'+s+s'-r-r'+2q)\right)!}\right]^{1/2}. \quad (8)$$

Repeating the reasonings of Section 10.4.2, we show that functions (7) are orthogonal. Therefore,

$$\widetilde{\mathfrak{H}}_C^{n\ell\ell'} \supset \sum_{r,r',s,s'} \oplus \mathfrak{H}_{rr',ss'}^{pq,\ell\ell'}, \quad (9)$$

where $\mathfrak{H}_{rr',ss'}^{pq,\ell\ell'}$ denotes the subspace of $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$, spanned by functions (7) with fixed $r$, $r'$, $s$, $s'$, and the summation is over all integral non-negative values of $r$, $r'$, $s$, $s'$ satisfying conditions (6).

We prove that, in fact, spaces (9) coincide. It is sufficient to show that their dimensionalities coincide. The dimensionality of $\mathfrak{H}_{rr',ss'}^{pq,\ell\ell'}$ is equal to the product of the dimensionalities of the spaces $\mathfrak{H}_C^{prr'}$ and $\mathfrak{H}_C^{qss'}$. For the dimensionalities $h(p;r,r')$ and $h(q;s,s')$ of these spaces we have

$$\frac{1-xy}{(1-x)^p(1-y)^p} = \sum_{r,r'=0}^{\infty} h(p;r,r')x^r y^{r'},$$

$$\frac{1-xy}{(1-x)^q(1-y)^q} = \sum_{s,s'=0}^{\infty} h(q;s,s')x^s y^{s'}$$

(see Section 11.2.1). Multiplying these equations, we find

$$(1-xy)\frac{1-xy}{(1-x)^n(1-y)^n} = \sum_{\ell,\ell'=0}^{\infty}\left[\sum_{r+s=\ell}\ \sum_{r'+s'=\ell'} h(p;r,r')h(q;s,s')\right]x^\ell y^{\ell'}.$$

Comparing this formula with formula (19) of Section 11.2.1, for the dimensionality $h(n;\ell,\ell')$ of $\mathfrak{H}_C^{n\ell\ell'}$ we derive the relation

$$h(n;\ell,\ell') - h(n;\ell-1,\ell'-1) = \sum_{r+s=\ell}\ \sum_{r'+s'=\ell'} h(p;r,r')h(q;s,s')$$

which implies that

$$h(n;\ell,\ell') = \sum_{r,r',s,s'} h(p;r,r')h(q;s,s'),$$

where the summation is the same as in (9). This proves coincidence of spaces (9):

$$\widetilde{\mathfrak{H}}_C^{n\ell\ell'} = \sum_{r,r',s,s'} \oplus \mathfrak{H}_{rr',ss'}^{pq,\ell\ell'}. \tag{10}$$

Left shift operators realize in $\widetilde{\mathfrak{H}}_C^{n\ell\ell'}$ the irreducible representation $T^{n\ell\ell'}$ of the group $U(n)$ (see Section 11.2.2). In addition, the irreducible representation $T^{prr'} \otimes T^{qss'}$ of the group $K_{pq} = U(p) \times U(q)$ is realized in $\mathfrak{H}_{rr',ss'}^{pq,\ell\ell'}$. Hence,

$$T^{n\ell\ell'} \downarrow^{U(n)}_{I_{pq}} = \sum_{r,r',s,s'} \oplus \left(T^{prr'} \otimes T^{qss'}\right), \tag{11}$$

where the summation is the same as in (9).

Every element $g \in U(n)$ is represented in the form

$$g = g^p(\varphi,\theta) g^q(\psi,\tau) g_{pn}(\theta) k, \tag{12}$$

where $k \in U(n-1)$, $g^p(\varphi,\theta)$ and $g^q(\psi,\tau)$ are elements of type (7) of Section 11.1.5 from the subgroups $U(p)$ and $U(q)$, respectively. If $\Xi_O^{n\ell\ell'}(\boldsymbol{\xi})$ is the basis function (4) of Section 11.3.1 with $M = O$, then for (12) we have

$$\begin{aligned}\left(T^{n\ell\ell'}(g)\Xi_O^{n\ell\ell'}, \Xi_{M'M''}^{pq,\ell\ell',rr',ss'}\right) &\equiv t^{pq,\ell\ell'}_{(rr'M',ss'M'')O}(g) \\ &= t^{pq,\ell\ell'}_{(rr',ss')0}(g_{pn}(\theta)) t^{prr'}_{M'O}(g^p(\varphi,\theta)) t^{qss'}_{M''O}(g^q(\psi,\tau)),\end{aligned} \tag{13}$$

where $t^{prr'}_{M'O}(g^p)$ and $g^{qss'}_{M''O}(g^q)$ are associated spherical functions of the representations $T^{prr'}$ and $T^{qss'}$ of the subgroups $U(p)$ and $U(q)$, respectively, and $t^{pq,\ell\ell'}_{(rr',ss')0}(g_{pn}(\theta))$ does not depend on $M'$ and $M''$.

The matrix element

$$t^{pq,\ell\ell'}_{(rr',ss')0}(g_{pn}(\theta)) = \left(T^{n\ell\ell'}(g_{pn}(\theta))\Xi_O^{n\ell\ell'}, \Xi_{OO}^{pq,\ell\ell',rr',ss'}\right)$$

is evaluated by means of (7) in the same way as in the case of the group $SO(n)$ (see Section 10.4.2). We have

$$t^{pq,\ell\ell'}_{(rr',ss')0}(g_{pn}(\theta)) = c \sin^{r+r'}\theta \cos^{s+s'}\theta P^{(r+r'+p-1,s+s'_q-1)}_{(\ell+\ell'-r-r'-s-s')/2}(\cos 2\theta), \tag{14}$$

where

$$c = \left[\frac{\Gamma\left(\frac{1}{2}(\ell+\ell'+r+r'+s+s'+2n-2)\right)\left(\frac{1}{2}(\ell+\ell'-r-r'-s-s')\right)!(r+p-2)!(r'+p-2)!}{\Gamma\left(\frac{1}{2}(\ell+\ell'+r+r'-s-s'+2p)\right)\Gamma\left(\frac{1}{2}(\ell+\ell'-r-r'+s+s'+2q)\right)!(q-2)!(p-2)!}\right.$$
$$\left.\times \frac{(s+q-2)!(s'+q-2)!(s+s'+q-1)!(r+r'+p-1)!\ell!\ell'!(n-2)!}{(\ell'+n-2)!(\ell+n-2)!r!r'!s!s'!}\right]^{1/2}. \tag{15}$$

Functions (13) are said to be associated $K_{pq}$-spherical functions of the space $S_C^{n-1} = U(n)/U(n-1)$. For $r = r' = s = s' = 0$ (in accordance with (6) in this case $\ell = \ell'$), we obtain zonal $K_{pq}$-spherical functions on $S_C^{n-1}$:

$$\varphi_\ell^{K_{pq}K}(\theta) \equiv t^{pq,\ell\ell}_{(00,00)0}(g_{pn}(\theta))$$
$$= \ell! \left[\frac{(p-1)!(n-2)!(q-1)!\ell!}{(\ell+p-1)!(\ell+q-1)!(\ell+n-2)!}\right]^{1/2} P_\ell^{(p-1,q-1)}(\cos 2\theta). \quad (16)$$

In formulas (13)–(16) we have $1 < p < n$. For $p = n-1$ we obtain ordinary associated and zonal spherical functions of the space $U(n)/U(n-1)$ from Section 11.3. Let $p = 1$. In this case

$$T^{n\ell\ell'} \downarrow^{U(n)}_{U(1)\times U(n-1)} = \sum_{s=0}^{\ell} \sum_{s'=0}^{\ell'} \oplus \left(T^{1,\ell-\ell'-s+s'} \otimes T^{n-1,ss'}\right)$$

and instead of functions (14), we have

$$t^{1,n-1,\ell\ell'}_{(k,ss')0}(g_{1n}(\theta)) = c \sin^k\theta \cos^{s+s'}\theta P^{(k,s+s'+n-2)}_{\ell'-s'}(\cos 2\theta), \quad (17)$$

where $k = \ell - \ell' - s + s'$ and $c$ is a constant factor.

The differential equation for functions (14) is derived from the equation

$$[\Delta_0 + (\ell+\ell')(\ell+\ell'+2n-2)]t^{pq,\ell\ell'}_{(rr'M',ss'M'')O}(g) = 0$$

(see formula (13)) and from the differential equations for $t^{prr'}_{M'O}(g^p)$ and $t^{qss'}_{M''O}(g^q)$. It has the form

$$\left[\frac{1}{\sin^{2p-1}\theta\cos^{2q-1}\theta}\frac{d}{d\theta}\sin^{2p-1}\theta\cos^{2q-1}\theta\frac{d}{d\theta} - \frac{(r+r')(r+r'+2p-2)}{\sin^2\theta}\right.$$
$$\left. - \frac{(s+s')(s+s'+2q-2)}{\cos^2\theta}\right]F(\theta) = -(\ell+\ell')(\ell+\ell'+2n-2)F(\theta). \quad (18)$$

This equation coincides with differential equation (2) of Section 10.1.5 for the matrix elements

$$t^{SO(2n)2p,\ell+\ell'}_{(r+r',s+s')0}(g_{2p,2n}(\theta)) \equiv t^{2n,2p,\ell+\ell'}_{r+r',s+s';0}(g_{2p,2n}(\theta))$$

(see formula (8) of Section 10.4.2) of the operators $T^{2n,\ell+\ell'}(g_{2p,2n}(\theta))$ of representations of $SO(2n)$ corresponding to the subgroup $SO(2p) \times SO(2q)$. Therefore,

$$t^{U(n)p,\ell\ell'}_{(rr',ss')0}(g_{pn}(\theta)) = N t^{SO(2n)2p,\ell+\ell'}_{(r+r',s+s')0}(g_{2p,2n}(\theta)). \quad (19)$$

In order to calculate $N$ we make use of formula (7′) of Section 10.4.2 and the analogous formula for $t^{U(n)p,\ell\ell'}_{(rr',ss')0}(g_{pn}(\theta))$. We have

$$N = \left[\frac{(\dim T^{SO(2n),\ell+\ell'})(\dim T^{U(p),rr'})(\dim T^{U(q),ss'})}{(\dim T^{SO(2p),r+r'})(\dim T^{SO(2q),s+s'})(\dim T^{U(n),\ell\ell'})}\right]^{1/2}. \tag{20}$$

The analogous formula for matrix element (17) is of the form

$$t^{U(n)1,\ell\ell'}_{(k,ss')0}(g_{1n}(\theta)) = \left[\frac{(\dim T^{SO(2n),\ell+\ell'})(\dim T^{U(n-1),ss'})}{(\dim T^{SO(2n-2),s+s'})(\dim T^{U(n),\ell\ell'})}\right]^{1/2} \times t^{SO(2n)2,\ell+\ell'}_{(k,s+s')0}(g_{2,2n}(\theta)). \tag{21}$$

## 11.4. Functional Relations for Jacobi Polynomials and Functions and for Bessel Functions

**11.4.1. Integral representations.** From formulas (3) and (5′) of Section 11.3.1 we obtain the representation for Jacobi polynomials in the form of a double integral:

$$P^{(\alpha,\beta)}_\ell(\cos 2\theta) = \frac{\Gamma(\ell+\alpha+1)}{\pi \ell! \Gamma(\alpha)\cos^\beta\theta}\int_0^{2\pi}\int_0^1 (\cos\theta - ie^{i\varphi} r\sin\theta)^\ell \times (\cos\theta - ie^{-i\varphi} r\sin\theta)^{\ell+\beta} r(1-r^2)^{\alpha-1}\,dr\,d\varphi. \tag{1}$$

In particular, for $\ell = \ell'$ we have

$$P^{(\alpha,0)}_\ell(\cos 2\theta) = \frac{2\Gamma(\ell+\alpha+1)}{\pi \ell! \Gamma(\alpha)}\int_0^{\pi}\int_0^1 (\cos^2\theta - r^2\sin^2\theta - ir\cos\varphi\sin 2\theta)^\ell r(1-r^2)^{\alpha-1}\,dr\,d\varphi. \tag{2}$$

In order to derive similar formulas for the Jacobi function

$$\mathfrak{P}^{(\alpha,\beta)}_\mu(z) = F\left(-\mu, \mu+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}\right) \tag{3}$$

we take into account that

$$\begin{aligned} t^{n\sigma k}_{(mm')0}(g_{n-1}(t)) &= \\ &= c^{n\sigma k}_{mm'} \sinh^{m+m'} t \cosh^{k+m'-m} t\, \mathfrak{P}^{(m+m'+n-2,k+m'-m)}_{\frac{\sigma-k}{2}-m'}(\cosh 2t) \\ &= c^{n\sigma k}_{mm'} \sinh^{m+m'} t \cosh^{m-m'-k} t\, \mathfrak{P}^{(m+m'+n-2,m-m'-k)}_{\frac{\sigma+k}{2}-m}(\cosh 2t) \end{aligned} \tag{4}$$

(it follows from the results of Section 11.3.5), where $c^{n\sigma k}_{mm'}$ is given by formula (12) of Section 11.3.5. In particular,

$$t^{n\sigma k}_{00}(g'_{n_1}(t)) = \cosh^k t\, \mathfrak{P}^{(n-2,k)}_{\frac{\sigma-k}{2}}(\cosh 2t) = \cosh^{-k} t\, \mathfrak{P}^{(n-2,-k)}_{\frac{\sigma+k}{2}}(\cosh 2t). \qquad (5)$$

The last formula implies that

$$\mathfrak{P}^{(\alpha,\beta)}_{\mu}(\cosh 2t) = \cosh^{-2\beta} t \mathfrak{P}^{(\alpha,-\beta)}_{\mu+\beta}(\cosh 2t). \qquad (5')$$

It follows from the results of Sections 11.3.4 and 11.3.5 that

$$\mathfrak{P}^{(\alpha,k)}_{\mu}(\cosh 2t) = \frac{\alpha}{\pi}\cosh^{-k} t \int_0^{2\pi}\int_0^1 (\cosh t - e^{i\varphi} r \sinh t)^{\mu+k}(\cosh t - e^{-i\varphi} r\sinh t)^{\mu} \times r(1-r^2)^{\alpha-1} dr\, d\varphi, \qquad (6)$$

$$\mathfrak{P}^{(\alpha,0)}_{\mu}(\cosh 2t) = \frac{\alpha}{\pi}\int_0^{2\pi}\int_0^1 (\cosh^2 t + r^2\sinh^2 t - r\sinh 2t\cos\varphi)^{\mu} \times r(1-r^2)^{\alpha-1} dr\, d\varphi. \qquad (7)$$

**11.4.2. Addition theorems for Jacobi polynomials.** For elements of the group $U(n)$ we have

$$g_{n-1}(\theta_1)d_{n-1}(\tau)g_{n-2}(\varphi)g_{n-1}(-\theta_2) = g_{n-2}(\psi_1)d_n(\nu)g_{n-1}(\theta)d_{n-1}(\omega)g_{n-2}(\psi_2), \qquad (1)$$

where the matrices $g_j(\psi)$ and $d_k(\varepsilon)$ are defined in Section 11.1.2 and the angles are connected by the formulas

$$\left.\begin{aligned} e^{i\nu}\cos\theta &= \cos\theta_1\cos\theta_2 + e^{i\tau}\sin\theta_1\sin\theta_2\cos\varphi,\\ \sin\theta\sin\psi_1 &= -\sin\varphi\sin\theta_2,\\ e^{i(\nu+\omega)}\sin\theta\sin\psi_2 &= e^{i\tau}\sin\theta_1\sin\varphi. \end{aligned}\right\} \qquad (2)$$

Since

$$t^{n\ell\ell'}_{0(mm')}(g_{n-1}(\theta)) = \overline{t^{n\ell\ell'}_{(mm')0}(g_{n_1}(-\theta))} = (-1)^{m+m'} t^{n\ell\ell'}_{(mm')0}(g-n-1(\theta)), \qquad (2')$$

then (1) implies the addition theorem for $t^{n\ell\ell'}_{(mm')0}(g-n_1(\theta)) \equiv t^{n\ell\ell'}_{(mm')0}(\theta)$:

$$\sum_{m=0}^{\ell}\sum_{m'=0}^{\ell'} e^{-i(m-m')\tau} t^{n-1,mm'}_{00}(\varphi) t^{n\ell\ell'}_{(mm')0}(\theta_1) t^{n\ell\ell'}_{(mm')0}(\theta_2) = e^{-i(\ell-\ell')\nu} t^{n\ell\ell'}_{00}(\theta), \quad (3)$$

where the angles are connected by formulas (2). Replacing $t^{n\ell\ell'}_{(mm')0}(\psi)$ by the second expression of formula (7) of Section 11.3.5, we derive

$$(e^{i\nu}\cos\theta)^{\beta}P_\ell^{(\alpha,\beta)}(\cos 2\theta)$$
$$=\sum_{m=0}^{\ell}\sum_{m'=0}^{\ell+\beta} b^{\alpha\beta\ell}_{mm'}(\sin\theta_1\sin\theta_2)^{m+m'}(\cos\theta_1\cos\theta_2)^{\beta+m-m'}(e^{i\tau}\cos\varphi)^{m'-m}$$
$$\times P_m^{(\alpha-1,m'-m)}(\cos 2\varphi)P_{\ell-m}^{(\alpha+m+m',\beta+m-m')}(\cos 2\theta_1)$$
$$\times P_{\ell-m}^{(\alpha+m+m',\beta+m-m')}(\cos 2\theta_2), \quad (4)$$

where $\alpha=n-2$, $\beta=\ell-\ell'$ and

$$b^{\alpha\beta\ell}_{mm'}=\frac{(\ell-m)!(\alpha+\beta+\ell+1)_m(\beta+\ell-m'+1)_{m'}(\alpha+m+m')}{m'!(\alpha+m')_{\ell+1}}. \quad (5)$$

Set $\ell=\ell'$ into (3) and replace $t^{n\ell\ell'}_{(mm')0}(\psi)$ by the first expression of formula (7) of Section 11.3.5 if $m'\geq m$ and by the second one if $m>m'$. After simple transformations we obtain the addition theorem for Jacobi polynomials:

$$P_\ell^{(\alpha,0)}\left(2|\cos\theta_1\cos\theta_2+re^{i\tau}\sin\theta_1\sin\theta_2|^2-1\right)$$
$$=\sum_{m=0}^{\ell}\sum_{m'=0}^{m} a^{\alpha\ell}_{mm'}(\sin\theta_1\sin\theta_2)^{m+m'}(\cos\theta_1\cos\theta_2)^{m-m'}r^{m-m'}\cos(m-m')\tau$$
$$\times P_{m'}^{(\alpha-1,m-m')}(2r^2-1)P_{\ell-m}^{(\alpha+m+m',m-m')}(\cos 2\theta_1)P_{\ell-m}^{(\alpha+m+m',m-m')}(\cos 2\theta_2), \quad (6)$$

where $\alpha=n-2$ and

$$a^{\alpha\ell}_{mm'}=\delta\frac{(\alpha+\ell+m)!(\alpha+m-1)!\ell!(\ell-m)!(\alpha+m+m')}{(\alpha+\ell+m')!(\alpha+\ell)!(\ell-m')!m!}. \quad (7)$$

Here $\delta=1$ if $m=m'$ and $\delta=2$ if $m\neq m'$.

Taking into account the formulas

$$2|\cos\theta_1\cos\theta_2+re^{i\tau}\sin\theta_1\sin\theta_2|^2$$
$$=2\cos^2\theta_1\cos^2\theta_2+2r^2\sin^2\theta_1\sin^2\theta_2+r\cos\tau\sin 2\theta_1\sin 2\theta_2,$$
$$\cos(m-m')\tau=T_{m-m'}(\cos\tau),$$

we differentiate both sides of (6) $k$ times with respect to $\cos\tau$. Keeping in mind the relations

$$\left(\frac{d}{dx}\right)^k P_n^{(\alpha,0)}(x)=2^{-k}\frac{(\alpha+n+k)!}{(\alpha+n)!}P_{n-k}^{(\alpha+k,k)}(x),$$
$$\left(\frac{d}{dx}\right)^k T_n(x)=2^{k-1}(k-1)!nC^k_{n-k}(x)$$

(see formulas (2) of Section 6.3.8, (1) of Section 6.3.9 and (5) of Section 6.9.1), after simplifications and renotations we have

$$P_\ell^{(\alpha,\beta)}\left(2|\cos\theta_1\cos\theta_2+re^{i\tau}\sin\theta_1\sin\theta_2|^2-1\right)=\sum_{m=0}^{\ell}\sum_{m'=0}^{m}a_{mm'}^{\alpha\beta\ell}$$
$$\times(\sin\theta_1\sin\theta_2)^{m+m'}(\cos\theta_1\cos\theta_2)^{m-m'}r^{m-m'}\frac{\beta+m-m'}{\beta}C_{m-m'}^{\beta}(\cos\tau)$$
$$\times P_{m'}^{(\alpha-\beta-1,\beta+m-m')}(2r^2-1)P_{\ell-m}^{(\alpha+m+m',\beta+m-m')}(\cos 2\theta_1)$$
$$\times P_{\ell-m}^{(\alpha+m+m',\beta+m-m')}(\cos 2\theta_2), \quad (8)$$

where $\alpha,\beta\in\mathbf{Z}_+\cup\{0\}$, $\alpha>\beta$ and

$$a_{mm'}^{\alpha\beta\ell}=$$
$$=\frac{(\alpha+\beta+\ell+1)_m(\ell+\beta-m'+1)_{m'}\Gamma(\alpha+m)\Gamma(\ell-m+1)(\alpha+m+m')}{(\beta+1)_m\Gamma(\alpha+\ell+m'+1)}. \quad (9)$$

Both sides of (8) are rational functions of $\alpha$ and $\beta$. Therefore, (8) can be analytically continued in $\alpha$ and $\beta$. Thus, formulas (8) and (9) are valid for all $\alpha$ and $\beta$ such that expressions do not become infinite.

For $r=0$ we have from (8) that

$$P_\ell^{(\alpha,\beta)}(2\cos^2\theta_1\cos^2\theta_2-1)=\sum_{m=0}^{\ell}d_{\ell m}^{\alpha\beta}(\sin\theta_1\sin\theta_1)^{2m}$$
$$\times P_{\ell-m}^{(\alpha+2m,\beta)}(\cos 2\theta_1)P_{\ell-m}^{(\alpha+2m,\beta)}(\cos 2\theta_2), \quad (10)$$

where

$$d_{\ell m}^{\alpha\beta}=(-1)^m\frac{(\alpha+\beta+\ell+1)_m(\ell+\beta-m+1)_m\Gamma(\ell-m+1)(\alpha+2m)}{m!(\alpha+m)_{\ell+1}}. \quad (10')$$

We leave to the reader to write down special cases of (8) for $r=1$ and $\tau=0$.

In the group $U(n)$ we have the equality

$$g_{n-1}(\theta_1)d_n(\omega)d_{n-1}(-\omega)g_{n-1}(\theta_2)=d_{n-1}(-\delta)g_{n-1}(\theta)d_{n-1}(-\gamma), \quad (11)$$

where the angles satisfy the conditions

$$\left.\begin{array}{l}\cos 2\theta=\cos 2\theta_1\cos 2\theta_2-\sin 2\theta_1\sin 2\theta_2\cos 2\omega,\\ e^{i(\delta+\gamma)}\cos\theta=e^{i\omega}\cos\theta_1\cos\theta_2-e^{-i\omega}\sin\theta_1\sin\theta_2,\\ e^{i(\delta-\gamma)}\sin\theta=e^{i\omega}\cos\theta_1\sin\theta_2+e^{-i\omega}\sin\theta_1\cos\theta_2.\end{array}\right\} \quad (12)$$

From (11) we have

$$\sum_{m=0}^{\ell}\sum_{m'=0}^{\ell'} e^{i(\ell'-\ell+2m-2m')\omega} t^{n\ell\ell'}_{(mm')0}(\theta_1)(-1)^{m+m'} t^{n\ell\ell'}_{(mm')0}(\theta_2) = e^{i(\ell-\ell')(\delta+\gamma)} t^{n\ell\ell'}_{00}(\theta).$$

Substituting expressions for the matrix elements $t^{n\ell\ell'}_{(mm')0}(\varphi)$, after simplifications we obtain

$$e^{-i\beta(\omega+\delta+\gamma)}\cos^\beta\theta P^{(\alpha,\beta)}_\ell(\cos 2\theta) = \sum_{m=0}^{\ell}\sum_{m'=0}^{\ell-\beta} e^{2i(m-m')\omega} a^{mm'}_{\ell\alpha\beta}(\sin\theta_1\sin\theta_2)^{m+m'}(\cos\theta_1\cos\theta_2)^{\beta+m-m'} \times P^{(\alpha+m+m',\beta+m-m')}_{\ell-m}(\cos 2\theta_1) P^{(\alpha+m+m',\beta+m-m')}_{\ell-m}(\cos 2\theta_2), \quad (13)$$

where $\alpha = n-2$, $\beta = \ell-\ell'$ and the angles satisfy conditions (12) and

$$a^{mm'}_{\ell\alpha\beta} = (-1)^{m+m'}\frac{(\ell+\beta-m'+1)_{m'}(\ell+\alpha+\beta+1)_m(\alpha)_{m'}(\ell-m)!\Gamma(\alpha+m)(\alpha+m+m')}{m!m'!\Gamma(\ell+m'+\alpha+1)}.$$

**11.4.3. Addition theorems for Jacobi and Bessel functions.** In the group $U(n-1,1)$ we have the relation

$$g'_{n-1}(t_1)d_{n-1}(\tau)g_{n-2}(\varphi)g'_{n-1}(t_2) = g_{n-2}(\psi_1)d_n(\nu)g'_{n-1}(t)d_{n-1}(\omega)g-n-2(\psi_2), \quad (1)$$

where the angles are connected by the equalities

$$\left.\begin{aligned} &e^{i\nu}\cosh t = \cosh t_1\cosh t_2 + e^{i\tau}\sinh t_1\sinh t_2\cos\varphi,\\ &\sinh t\sin\psi_1 = \sin\varphi\sinh t_2,\\ &e^{i(\nu+\omega)}\sinh t\sin\psi_2 = e^{i\tau}\sinh t_1\sin\varphi. \end{aligned}\right\} \quad (2)$$

Making use of the relation

$$t^{n\sigma k}_{0(mm')}(g'_{n-1}(t)) = (-1)^{m-m'} t^{n,-\sigma-2n+2,k}_{(mm')0}(g'_{n-1}(t)) \quad (3)$$

in the same way as in the case of Jacobi polynomials, we obtain the addition theorem for Jacobi functions:

$$e^{-ik\nu}\cosh^{-k}t\,\mathfrak{P}^{(\alpha,-k)}_\mu(\cosh 2t) = \sum_{m=0}^{\infty}\sum_{m'=0}^{\infty}(-1)^{m+m'}a^{\mu k}_{mm'}e^{i(m'-m)\tau}\cos^{m'-m}\varphi \times(\sinh t_1\sinh t_2)^{m+m'}(\cosh t_1\cosh t_2)^{m-m'-k}\mathfrak{P}^{(\alpha+m+m',m-m'-k)}_{\mu-m}(\cosh 2t_1) \times P^{(\alpha-1,m'-m)}_m(\cos 2\varphi)\mathfrak{P}^{(\alpha+m+m',m-m'-k)}_{\mu-m}(\cosh 2t_2), \quad (4)$$

where $\alpha = n-2$ and the angles satisfy formulas (2) and

$$a^{\mu k}_{mm'} = \frac{(\mu-k-m'+1)_{m'}(\mu-m+1)_m(-\mu-\alpha-m')_{m'}(k-\mu-\alpha-m)_m}{m!(\alpha+1)_{m+m'}(\alpha+m')_m}. \tag{5}$$

In particular, for $k=0$ we have

$$\mathfrak{P}^{(\alpha,0)}_\mu(\cosh 2t) = \sum_{m=0}^{\infty}\sum_{m'=0}^{\infty}(-1)^{m+m'}a^{\mu 0}_{mm'}e^{i(m'-m)\tau}(\sinh t_1 \sinh t_2)^{m+m'} \times (\cosh t_1 \cosh t_2)^{m-m'}\cos^{m'-m}\varphi P^{(\alpha-1,m'-m)}_m(\cos 2\varphi) \times \mathfrak{P}^{(\alpha+m+m',m-m')}_{\mu-m}(\cosh 2t_1)\mathfrak{P}^{(\alpha+m+m',m-m')}_{\mu-m}(\cosh 2t_2), \tag{6}$$

where

$$\cosh 2t = 2|\cosh t_1 \cosh t_2 + e^{i\tau}\sinh t_1 \sinh t_2 \cos\varphi|^2 - 1. \tag{7}$$

This relation can be written as

$$\mathfrak{P}^{(\alpha,0)}_\mu(\cosh 2t) = \sum_{m=0}^{\infty}\sum_{m'=0}^{\infty}(-1)^{m+m'}b^{\mu 0}_{mm'}e^{i(m'-m)\tau}(\sinh t_1 \sinh t_2)^{m+m'} \times (\cosh t_1 \cosh t_2)^{m-m'}r^{m-m'}P^{(\alpha-1,m-m')}_{m'}(2r^2-1) \times \mathfrak{P}^{(\alpha+m+m',m-m')}_{\mu-m}(\cosh 2t_1)\mathfrak{P}^{(\alpha+m+m',m-m')}_{\mu-m}(\cosh 2t_2), \tag{8}$$

where

$$\cosh 2t = 2|\cosh t_1 \cosh t_2 + re^{i\tau}\sinh t_1 \sinh t_2|^2 - 1, \tag{9}$$

$$b^{\mu 0}_{mm'} = \frac{\varepsilon(\mu-k-m'+1)_{m'}(\mu-m+1)_m(-\mu-\alpha-m')_{m'}(k-\mu-\alpha-m)_m}{m!(\alpha+1)_{m+m'}(\alpha+m)_{m'}},$$

$\varepsilon = 1$ if $m = m'$ and $\varepsilon = 2$ if $m \neq m'$.

If $\mu$ is real, then all functions in (8) are real and $e^{i(m'-m)\tau}$ can be replaced by $\cos(m-m')\tau$. Let us differentiate both sides of (8) with respect to $\cos\tau$. Taking into account the relation

$$\frac{d}{dz}\mathfrak{P}^{(\alpha,\beta)}_\mu(z) = \frac{\mu(\mu+\alpha+\beta+1)}{2(\alpha+1)}\mathfrak{P}^{(\alpha+1,\beta+1)}_{\mu-1}(z),$$

after simple transformations we obtain

$$\mathfrak{P}^{(\alpha,\beta)}_\mu(2|\cosh t_1 \cosh t_2 + re^{i\tau}\sinh t_1 \sinh t_2|^2 - 1) = \sum_{m=0}^{\infty}\sum_{m'=0}^{m} a^{\mu\alpha\beta}_{mm'}r^{\beta+m-m'}(\cosh 2t_1) \times (\sinh t_1 \sinh t_2)^{m+m'}(\cosh t_1 \cosh t_2)^{m-m'}\mathfrak{P}^{(\alpha+m+m',m-m')}_{\mu-m} \times C^{\beta}_{m-m'}(\cos\tau)P^{(\alpha-\beta-1,\beta+m-m')}_{m'}(2r^2-1)\mathfrak{P}^{(\alpha+m+m',\beta+m-m')}_{\mu-m}(\cosh 2t_2), \tag{10}$$

where $\alpha, \beta \in \mathbf{Z}_+$, $\alpha > \beta$, $\beta \neq 0$ and

$$a^{\mu\alpha\beta}_{mm'} = \frac{(\alpha+\beta+\mu+1)_m(\alpha+\mu+1)_{m'}(\mu-m+1)_m(\beta+\mu-m'+1)_{m'}\Gamma(\alpha+1)(\beta+m-m')}{\beta(\beta+1)_m(\alpha+m)_{m'+1}\Gamma(\alpha+m+m')}.$$

For $r=0$ we derive from (10) that

$$\mathfrak{P}^{(\alpha,\beta)}_{\mu}(2\cosh^2 t_1 \cosh^2 t_2 - 1) = \sum_{m=0}^{\infty} b^{\mu\alpha\beta}_m (\sinh t_1 \sinh t_2)^{2m} \mathfrak{P}^{(\alpha+2m,\beta)}_{\mu-m}(\cosh 2t_1)\mathfrak{P}^{(\alpha+2m,\beta)}_{\mu-m}(\cosh 2t_2), \quad (11)$$

where

$$b^{(\mu\alpha\beta)}_m = (-1)^m \frac{(\alpha+\beta+\mu+1)_m(\beta+\mu-m+1)_m(\alpha+\mu+1)_m(\mu-m+1)_m\Gamma(\alpha+1)}{m!(\alpha+m)_{m+1}\Gamma(\alpha+2m)}.$$

In the group $JU(n-1)$ the relation

$$g_{r_1} g_{n-1}(\theta) d_n(\varphi) g_{r_2} = k d_n(\varphi) g_r k'$$

holds, where $k, k' \in U(n-1)$ and

$$r = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos\theta \cos\varphi}. \quad (12)$$

Since $t^{nR}_{0(mm')}(g_r) = t^{nR}_{(mm')0}(g_r)$, then we derive from here the addition theorem for Bessel functions:

$$\left(\frac{r_1 r_2}{2r}\right)^n J_n(r) = \sum_{m=0}^{\infty}\sum_{m'=0}^{\infty} \frac{(-1)^{m+m'}}{m'!}(m'+n-1)!(m+m'+n)e^{i(m-m')\varphi} \times \cos^{m'-m}\theta\, P^{(n-1,m'-m)}_m(\cos 2\theta) J_{m+m'+n}(r_1) J_{m+m'+n}(r_w), \quad (13)$$

where $r$ is given by (12). This relation is reduced to the form

$$\left(\frac{r_1 r_2}{2r}\right)^n J_n(r) = \sum_{m=0}^{\infty}\sum_{m'=0}^{m} \frac{\varepsilon(-1)^{m+m'}}{m'}(m+n-1)!(m+m'+n)\cos(m-m')\varphi \times \cos^{m-m'}\theta P^{(n-1,m-m')}_{m'}(\cos 2\theta) J_{m+m'+n}(r_1) J_{m+m'+n}(r_2), \quad (14)$$

where $\varepsilon = 1$ for $m = m'$ and $\varepsilon = 2$ for $m \neq m'$. If $\theta = \frac{\pi}{2}$, then we have

$$\sum_{m=0}^{\infty}(-1)^m \frac{(m+n-1)!(2m+n)}{m!} J_{2m+n}(r_1)J_{2m+n}(r_2) = \left(\frac{r_1 r_2}{2\sqrt{r_1^2+r_2^2}}\right)^n J_n\left(\sqrt{r_1^2+r_2^2}\right). \tag{15}$$

Let us replace $\cos(m-m')\varphi$ in formula (14) by $T_{m-m'}(\cos\varphi)$ and differentiate both sides $k$ times with respect to $x = \cos\varphi$. Since for

$$r = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos\theta\cos\varphi}$$

we have

$$\left(\frac{d}{d\cos\varphi}\right)^k [r^{-n}J_n(r)] = (-1)^k r^{-n-k}(r_1 r_2\cos\theta)^k J_{n+k}(r),$$
$$\left(\frac{d}{dx}\right)^k T_n(x) = 2^{k-1}(k-1)!nC_{n-k}^k(x), \quad k \geq 1,$$

then we obtain the addition theorem

$$\frac{2}{(k-1)!}\left(\frac{r_1 r_2}{2r}\right)^{n+k} J_{n+k}(r) = \sum_{m=0}^{\infty}\sum_{m'=0}^{m-k} \varepsilon(-1)^{m+m'}\frac{(m-m')}{m!}(m+n-1)!(m+m'+n)\cos^{m-m'-k}\theta \times C_{m-m'-k}^k(\cos\varphi)P_{m'}^{(n-1,m-m')}(\cos 2\theta)J_{m+m'+n}(r_1)J_{m+m'+n}(r_2), \tag{16}$$

where $\varepsilon = 1$ if $m' = m-k$, $\varepsilon = 2$ if $m' \neq m-k$, $k \geq 0$ and $r$ is given by (12). For $\theta = \frac{\pi}{2}$ we obtain

$$\sum_{m=0}^{\infty}(-1)^m \frac{(m+n-1)!(2m-k+n)}{(m-k)!} J_{2m-k+n}(r_1)J_{2m-k+n}(r_2) = \frac{2}{(k-1)!}\left(\frac{r_1 r_2}{2r}\right)^{n+k} J_{n+k}(r), \tag{17}$$

where $n \geq k \geq 1$ and $r = \sqrt{r_1^2+r_2^2}$.

**11.4.4. Expansions in zonal spherical functions of the group $U(n)$.** Let $\mathfrak{L}_0^2(U(n))$ be the space of functions $f \in \mathfrak{L}^2(U(n))$ such that for any $k, k' \in$

$U(n-1)$ we have $f(kgk') = f(g)$. By virtue of equality (8) of Section 11.1.5 functions $f$ from $\mathfrak{L}_0^2(U(n))$ can be considered as functions $F(\varphi, \theta)$ of angles $\varphi$, $\theta$, where $0 \le \varphi < 2\pi$, $0 \le \theta \le \frac{\pi}{2}$. It follows from formulas (2) and (4) of Section 11.1.8 that

$$\int\limits_{U(n)} |f(g)|^2 dg = \frac{n-1}{\pi} \int\limits_0^{2\pi} \int\limits_0^{\pi/2} |F(\varphi,\theta)|^2 \sin^{2n-3}\theta \cos\theta \, d\theta \, d\varphi. \quad (1)$$

The functions

$$t_{00}^{n\ell\ell'}(d_n(\varphi) g_{n-1}(\theta)) = e^{-i(\ell-\ell')\varphi} t_{00}^{n\ell\ell'}(g_{n-1}(\theta)) \quad (2)$$

are zonal spherical functions of the group $U(n)$ with respect to the subgroup $U(n-1)$. They belong to $\mathfrak{L}_0^2(U(n))$. According to the results of Section 2.3.9, one can expand $F(\varphi, \theta)$ in functions (2):

$$F(\varphi,\theta) = \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} \left(\dim T^{n\ell\ell'}\right) a_{\ell\ell'} e^{-i(\ell-\ell')\varphi} t_{00}^{n\ell\ell'}(g_{n-1}(\theta)). \quad (3)$$

The coefficients $a_{\ell\ell'}$ are given by the formula

$$a_{\ell\ell'} - \frac{n-1}{\pi} \int\limits_0^{2\pi} \int\limits_0^{\pi/2} F(\varphi,\theta) e^{i(\ell-\ell')\varphi} t_{00}^{n\ell\ell'}(g_{n-1}(\theta)) \sin^{2n-3}\theta \cos\theta \, d\theta \, d\varphi. \quad (4)$$

In addition, the Parseval formula

$$\frac{n-1}{\pi} \int\limits_0^{2\pi} \int\limits_0^{\pi/2} |F(\varphi,\theta)|^2 \sin^{2n-3}\theta \cos\theta \, d\theta \, d\varphi = \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} \left(\dim T^{n\ell\ell'}\right) |a_{\ell\ell'}|^2 \quad (5)$$

holds.

**11.4.5. Product formulas for Jacobi and Laguerre polynomials.** Considering formula (3) of Section 11.4.2 as expansion (3) of Section 11.4.4 for the function $F(\nu, \theta) = e^{-i(\ell-\ell')\nu} \varphi^{n\ell\ell'}(\theta)$, we have

$$t_{(mm')0}^{n\ell\ell'}(\theta_1) t_{(mm')0}^{n\ell\ell'}(\theta_2) = \frac{n-2}{\pi} \left(\dim T^{n-1,mm'}\right) \int\limits_0^{2\pi} \int\limits_0^{\pi/2} e^{-i(\ell-\ell')\nu}$$
$$\times \varphi^{n\ell\ell'}(\theta) e^{i(m-m')\tau} \varphi^{n-1,mm'}(\varphi) \sin^{2n-5}\varphi \cos\varphi \, d\varphi \, d\tau, \quad (1)$$

where the angles are connected by relations (2) of Section 11.4.2. Substituting the expression for $t^{n\ell\ell'}_{(mm')0}(\psi)$, we derive the product formula

$$(\sin\theta_1\sin\theta_2)^{m+m'}(\cos\theta_1\cos\theta_2)^{\beta+m-m'}P_{\ell-m}^{(\alpha+m+m',\beta+m-m')}(\cos 2\theta_1)$$
$$\times P_{\ell-m}^{(\alpha+m+m',\beta+m-m')}(\cos 2\theta_2) = a^{\alpha\beta}_{\ell mm'}\int_0^{2\pi}\int_0^1 e^{i\beta\nu}e^{i(m-m')\tau}\cos^\beta\theta$$
$$\times P_\ell^{(\alpha,\beta)}(\cos 2\theta)P_m^{(\alpha-1,m'-m)}(2r^2-1)r^{m'-m+1}(1-r^2)^{\alpha-1}dr\,d\tau, \quad (2)$$

where

$$a^{\alpha\beta}_{\ell mm'} = \frac{m!\Gamma(\ell+m'+\alpha+1)}{\pi(\ell+\beta-m'+1)_{m'}(\alpha+\beta+\ell+1)_m(\ell-m)!\Gamma(\alpha+m)}.$$

In particular, for $m = m' = 0$, we have

$$(\cos\theta_1\cos\theta_2)^\beta P_\ell^{(\alpha,\beta)}(\cos 2\theta_1)P_\ell^{(\alpha,\beta)}(\cos 2\theta_2)$$
$$= \frac{\Gamma(\ell+\alpha+1)}{\pi\ell!\Gamma(\alpha)}\int_0^{2\pi}\int_0^1 e^{i\beta\nu}\cos^\beta\theta P_\ell^{(\alpha,\beta)}(\cos 2\theta)(1-r^2)^{\alpha-1}r\,dr\,d\tau, \quad (3)$$

where the angles satisfy relations (2) of Section 11.4.2.

Let us regard equality (8) of Section 11.4.2 as the expansion of the function $P_\ell^{(\alpha,\beta)}(\dots)$ in Gegenbauer polynomials $C_p^\beta(\cos\tau)$. Multiply both sides of this equality by $\sin^{2\beta}\tau C_p^\beta(\cos\tau)$ and integrate with respect to $\tau$ from 0 to $\pi$. Now, considering the obtained equality as the expansion in Jacobi polynomials of $(2r^2-1)$, we multiply both sides by $(1-r^2)^{\alpha-\beta-1}r^{2\beta+p+1}P_{m'}^{(\alpha-\beta-1,\beta+p)}(2r^2-1)$ and integrate with respect to $r$ from 0 to 1. Setting $m = m' = 0$, we obtain the product formula for Jacobi polynomials

$$P_\ell^{(\alpha,\beta)}(\cos 2\theta_1)P_\ell^{(\alpha,\beta)}(\cos 2\theta_2) = \frac{2\Gamma(\ell+\alpha+1)}{\sqrt{\pi}\,\ell!\Gamma(\alpha-\beta)\Gamma\left(\beta+\frac{1}{2}\right)}$$
$$\times\int_0^\pi\int_0^1 P_\ell^{(\alpha,\beta)}\left(2|\cos\theta_1\cos\theta_2+re^{i\tau}\sin\theta_1\sin\theta_2|^2-1\right)r^{2\beta+1}(1-r^2)^{\alpha-\beta-1}$$
$$\times\sin^{2\beta}\tau\,dr\,d\tau, \quad (4)$$

where $\alpha > \beta > -\frac{1}{2}$.

This formula allows us to derive a generalization of equality (2) of Section 11.4.1. For this we have to divide both sides of (4) by $P_\ell^{(\alpha,\beta)}(\cos 2\theta_2)$, to replace

$\cos 2\theta_2$ by $t$ and to shift $t$ to the infinity. Replacing $\theta_1$ by $\theta$, we have

$$P_\ell^{(\alpha,\beta)}(\cos 2\theta) = \frac{2\Gamma(\ell+\alpha+1)}{\sqrt{\pi}\ell!\Gamma(\alpha-\beta)\Gamma\left(\beta+\frac{1}{2}\right)} \int_0^\pi \int_0^1 (\cos\theta - ire^{i\tau}\sin\theta)^\ell \times (\cos\theta - ire^{-i\tau}\sin\theta)^\ell r^{2\beta+1}(1-r^2)^{\alpha-\beta-1}\sin^{2\beta}\tau\, dr\, d\tau, \quad (5)$$

where $\alpha > \beta > -\frac{1}{2}$. This formula can be written as

$$\frac{P_\ell^{(\alpha,\beta)}(x)}{P_\ell^{(\alpha,\beta)}(1)} = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma\left(\beta+\frac{1}{2}\right)} \int_0^1 \int_0^\pi \left[\frac{1+x-(1-x)u^2}{2} + i\sqrt{1-x^2}\, u\cos\theta\right]^\ell \times (1-u^2)^{\alpha-\beta-1}u^{2\beta+1}\sin^{2\beta}\theta\, d\theta\, du. \quad (5')$$

The substitution $u^2 = \frac{(1-y)(1+x)}{(1+y)(1-x)}$ transforms this equality into

$$\frac{P_\ell^{(\alpha,\beta)}(x)}{P_\ell^{(\alpha,\beta)}(1)} = \frac{2^{\alpha-\beta}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma\left(\beta+\frac{1}{2}\right)} \int_x^1 \int_0^\pi \frac{(1-y)^\beta(1+x)^{\ell+\beta+1}}{(1-x)^\alpha(1+y)^{\ell+\alpha+1}}(y-x)^{\alpha-\beta-1} \times \left(y + i\sqrt{1-y^2}\cos\theta\right)^\ell \sin^{2\beta}\theta\, d\theta\, dy. \quad (5'')$$

According to formula (1) of Section 9.3.3, the integral with respect to $\theta$ coincides with a Gegenbauer polynomial which can be expressed in terms of $P_\ell^{(\beta,\beta)}(y)$. Hence, we have

$$\frac{(1-x)^\alpha}{(1+x)^{n+\beta+1}} \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} = \frac{2^{\alpha-\beta}\Gamma(\alpha+1)}{\Gamma(\alpha-\beta)\Gamma(\beta+1)} \times \int_x^1 \frac{(1-y)^\beta}{(1+y)^{n+\alpha+1}} \frac{P_n^{(\beta,\beta)}(y)}{P_n^{(\beta,\beta)}(1)}(y-x)^{\alpha-\beta-1}dy. \quad (6)$$

Formula (4) can be represented in another form. For this, by means of the formula

$$e^{i\varphi}\cos\theta = \cos\theta_1\cos\theta_2 + re^{i\tau}\sin\theta_1\sin\theta_2, \quad (6')$$

we introduce new variables $\theta$ and $\varphi$ instead of $\tau$ and $r$. Equality (6′) is equivalent to two relations

$$\cos\theta\cos\varphi - \cos\theta_1\cos\theta_2 = r\sin\theta_1\sin\theta_2\cos\tau, \quad (7)$$

$$\cos\theta\sin\varphi = r\sin\theta_1\sin\theta_2\sin\tau. \quad (8)$$

Excluding $\cos\tau$ from (7) and (8), after simple transformations we find that

$$(1-r^2)\sin^2\theta_1\sin^2\theta_2 = 1-\cos^2\theta_1-\cos^2\theta_2-\cos^2\theta+2\cos\theta\cos\theta_1\cos\theta_2\cos\varphi. \quad (9)$$

From (8) we obtain

$$r\sin\tau = \frac{\cos\theta\sin\varphi}{\sin\theta_1\sin\theta_2}. \quad (10)$$

The Jacobian of transition from $r$, $\tau$ to $\theta$, $\varphi$ is equal to

$$\frac{\partial(r,\tau)}{\partial(\theta,\varphi)} = -\frac{\sin\theta\cos\theta}{2\sin^2\theta_1\sin^2\theta_2}. \quad (11)$$

The argument of the integrand Jacobi polynomial in (4) is represented in terms of $\cos 2\theta$:

$$2|\cos\theta_1\cos\theta_2 + re^{i\tau}\sin\theta_1\sin\theta_2|^2 - 1 = \cos 2\theta. \quad (12)$$

As it is obvious from (4), the parameters $\theta_1$ and $\theta_2$ lie in the interval $\left[0, \frac{\pi}{2}\right]$. It follows from (6′) that $\theta$ can be considered to lie in this interval. Then, either

$$|\theta_1-\theta_2| < \theta < \theta_1+\theta_2 \quad (13)$$

or

$$\theta_1+\theta_2 < \theta < \frac{\pi}{2}. \quad (14)$$

In the first case, for any $r>0$, $0<\tau<\pi$ and $\theta$, there exists $\varphi\in[0,\gamma]$ satisfying (10), where

$$\cos\gamma = \frac{\cos^2\theta_1+\cos^2\theta_2+\cos^2\theta-1}{2\cos\theta_1\cos\theta_2\cos\theta}. \quad (15)$$

In the second case, $\varphi\in[0,\pi]$. Substituting expressions (9)–(11) for $(1-r^2)$, $r\sin\tau$ and $\partial(r,\tau)/\partial(\theta,\varphi)$ into (4), we obtain

$$P_\ell^{(\alpha,\beta)}(\cos 2\theta_1)P_\ell^{(\alpha,\beta)}(\cos 2\theta_2) = \int_0^{\pi/2} K(\theta_1,\theta_2,\theta)P_\ell^{(\alpha,\beta)}(\cos 2\theta)\sin^{2\alpha+1}\theta\cos^{2\beta+1}\theta\, d\theta, \quad (16)$$

where for any $\ell$ the kernel $K(\theta_1,\theta_2,\theta)$ is given by the formula

$$K(\theta_1,\theta_2,\theta) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\left(\beta+\frac{1}{2}\right)}(\sin\theta_1\sin\theta_2\sin\theta)^{-2\alpha}$$
$$\times\int_0^\delta(1-\cos^2\theta_1-\cos^2\theta_2-\cos^2\theta+2\cos\theta_1\cos\theta_2\cos\theta\cos\varphi)^{\alpha-\beta-1}\sin^{2\beta}\varphi\, d\varphi. \quad (17)$$

Here, $\delta = \gamma$ (see formula (15)) if $\theta$ belongs to interval (13), $\delta = \pi$ if $\theta$ belongs to interval (14), and

$$K(\theta_1, \theta_2, \theta) = 0 \tag{18}$$

if $\theta$ does not belong to any of these intervals.

The integral in (17) is expressed in terms of the hypergeometric function. If $\delta = \gamma$, then, replacing $\cos\varphi$ by $1 - y$ and then $y/(1-B)$ by $t$, where

$$B = \frac{\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta - 1}{2\cos\theta_1\cos\theta_2\cos\theta}, \tag{19}$$

we reduce this integral to

$$\int_0^1 (1-t)^{\alpha-\beta-1} t^{\beta-1/2}\left(1 - \frac{1-B}{2}t\right)^{\beta-1/2} dt.$$

The last integral can be computed by means of formula (13) of Section 3.5.2. Thus, if $\theta$ lies in interval (13), then

$$\begin{aligned} K(\theta_1, \theta_2, \theta) &= \frac{2^{\alpha-3/2}\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma\left(\beta+\frac{1}{2}\right)} \frac{(\cos\theta_1\cos\theta_2\cos\theta)^{\alpha-\beta-1}}{(\sin\theta_1\sin\theta_2\sin\theta)^{2\alpha}}(1-B^2)^{\alpha-1/2} \\ &\times F\left(\beta+\frac{1}{2}, -\beta+\frac{1}{2}; \alpha+\frac{1}{2}; \frac{1-B}{2}\right) = \frac{\Gamma(2\alpha+1)}{2^{\alpha+3/2}\Gamma(\alpha+\beta)\Gamma\left(\beta+\frac{1}{2}\right)} \\ &\times \frac{(\cos\theta_1\cos\theta_2\cos\theta)^{\alpha-\beta-1}}{(\sin\theta_1\sin\theta_2\sin\theta)^{2\alpha}}(1-B^2)^{\frac{\alpha}{2}-\frac{1}{4}} \mathrm{P}_{\beta-1/2}^{-\alpha+\frac{1}{2}}(B), \end{aligned} \tag{20}$$

where $B$ is given by formula (19) and $\mathrm{P}_\nu^\mu(x)$ is an associated Legendre function on the cut $-1 < x < 1$. If $\delta = \pi$ in (17), i.e. $\theta$ lies in interval (14), then, similarly, we find

$$\begin{aligned} K(\theta_1, \theta_2, \theta) &= \frac{2^{\alpha-\beta-1}\Gamma(\alpha+1)}{\Gamma(\alpha-\beta)\Gamma(\beta+1)} \frac{(\cos\theta_1\cos\theta_2\cos\theta)^{\alpha-\beta-1}}{(\sin\theta_1\sin\theta_2\sin\theta)^{2\alpha}} B^{\alpha-\beta-1} \\ &\times F\left(\frac{1}{2}(\beta-\alpha+1), \frac{1}{2}(\beta-\alpha+2); \beta+1; B^{-2}\right). \end{aligned} \tag{21}$$

One can rewrite (16) in the form

$$P_\ell^{(\alpha,\beta)}(x) P_\ell^{(\alpha,\beta)}(y) = \int_{-1}^{1} \mathcal{K}(x,y,t) P_\ell^{(\alpha,\beta)}(t)(1-t)^\alpha(1+t)^\beta dt, \tag{22}$$

where

$$\mathcal{K}(\cos 2\theta_1, \cos 2\theta_2, \cos 2\theta) = 2^{-\alpha-\beta-1} K(\theta_1, \theta_2, \theta). \tag{23}$$

Setting $\ell = 0$ into (22), we have

$$\int_{-1}^{1} \mathcal{K}(x, y, t)(1-t)^{\alpha}(1+t)^{\beta} dt = 1. \tag{24}$$

Considering formula (22) as the coefficient in the expansion of $\mathcal{K}(x, y, t)$ in Jacobi polynomials, we derive the equality

$$\mathcal{K}(x, y, t) = \sum_{\ell=0}^{\infty} a_{\ell}^{(\alpha,\beta)} P_{\ell}^{(\alpha,\beta)}(x) P_{\ell}^{(\alpha,\beta)}(y) P_{\ell}^{(\alpha,\beta)}(t), \tag{25}$$

where

$$a_{\ell}^{(\alpha,\beta)} = \frac{\ell!\Gamma(\alpha+\beta+\ell+1)(\alpha+\beta+2\ell+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+\ell+1)\Gamma(\beta+\ell+1)} \tag{26}$$

(see Section 6.10.2).

As in the case of Gegenbauer polynomials (see Section 9.4.5), one can construct the Banach algebra of functions of one variable with multiplication given with the help of the kernel $K(x, y, t)$. We suggest that the reader carry out corresponding constructions.

One can obtain Laguerre polynomials from Jacobi polynomials by passage to the limit

$$L_n^{\alpha}(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right). \tag{27}$$

Therefore, addition and product formulas for Jacobi polynomials obtained above give those for Laguerre polynomials. By setting

$$R_{mn}^{\alpha}(z) = \frac{n!\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} P_n^{(\alpha,m-n)}(2z\bar{z}-1) z^{m-n}$$

for $m \geq n$ and

$$R_{mn}^{\alpha}(z) = \frac{m!\Gamma(\alpha+1)}{\Gamma(\alpha+m+1)} P_m^{(\alpha,n-m)}(2z\bar{z}-1) \bar{z}^{n-m}$$

for $m \leq n$, we rewrite formula (4) of Section 11.4.2 in the form

$$\begin{aligned}
&[(1-x^2)^{1/2}(1-y^2)^{1/2} + xyre^{i\psi}]^{m-n} \\
&\quad \times P_n^{(\alpha,m-n)}(2(1-x^2)(1-y^2) + 2x^2y^2r^2 + 4xy(1-x^2)^{1/2}(1-y^2)^{1/2} r\cos\psi - 1) \\
&\quad = \sum_{k=0}^{m}\sum_{p=0}^{n} \frac{\alpha m! n!^2 (\alpha+n+1)(\alpha+m+1)_p [(\alpha+k+p+1)_{q(k,p)}]^2}{(\alpha+k+p)k!p!(m-k)!(n-p)!(\alpha+p)_k(\alpha+k)_p(\alpha+1)_n(q(k,p))^2} \\
&\qquad \times (xy)^{k+p}[(1-x^2)(1-y^2)]^{|m-n-k+p|/2} P_{q(k,p)}^{(\alpha+k+p,|m-n-k+p|)}(1-2x^2) \\
&\qquad\qquad \times P_{q(k,p)}^{(\alpha+k+p,|m-n-k+p|)}(1-2y^2) R_{kp}^{\alpha-1}(re^{i\psi}),
\end{aligned} \tag{28}$$

where $q(k,p) = \min(m-k, n-p)$. We replace here $x$ by $m^{-1/2}x$, $y$ by $m^{-1/2}y$ and shift $m$ to infinity. Taking into account formula (27), we obtain the addition formula

$$\exp(xye^{i\psi})L_n^\alpha(x^2+y^2-2xyr\cos\psi)$$
$$= \frac{\Gamma(\alpha+n+2)}{\Gamma(\alpha)} \sum_{k=0}^{\infty}\sum_{p=0}^{n} \frac{(n-p)!\Gamma(\alpha+k+p+1)\Gamma(\alpha+k+p)}{p!k!(\alpha+p)_k(\alpha+k)_p\Gamma^2(\alpha+n+k+1)}$$
$$\times (xy)^{k+p} L_{n-p}^{\alpha+k+p}(x^2) L_{n-p}^{\alpha+k+p}(y^2) R_{kp}^{\alpha-1}(re^{i\psi}), \quad (29)$$

where $x \geq 0$, $y \geq 0$, $0 \leq r \leq 1$, $0 \leq \varphi < 2\pi$, $\alpha > 0$.

By integration with respect to $r$ and $\psi$, we derive from (29) the product formula

$$L_n^\alpha(x^2)L_n^\alpha(y^2) = \frac{2\Gamma(\alpha+n+1)}{\pi n!\Gamma(\alpha)} \int_0^\pi \int_0^1 e^{-xyr\cos\psi}$$
$$\times L_n^\alpha(x^2+y^2+2xyr\cos\psi)\cos(xyr\sin\psi) r(1-r^2)^{\alpha-1} dr\, d\psi, \quad (30)$$

where $x \geq 0$, $y \geq 0$, $\alpha \geq 0$. Setting $r\cos\psi = \cos\theta$, $r\sin\psi = \sin\theta\cos\psi$ into (30), we have

$$(xy)^{\alpha-1/2} L_n^\alpha(x^2)L_n^\alpha(y^2) = \frac{2^{\alpha-1/2}\Gamma(\alpha+n+1)}{\sqrt{\pi}\, n!}$$
$$\times \int_0^\pi e^{-xy\cos\theta} L_n^\alpha(x^2+y^2+2xy\cos\theta) J_{\alpha-1/2}(xy\sin\theta) \sin^{\alpha+1/2}\theta\, d\theta, \quad (31)$$

where $x \geq 0$, $y \geq 0$, $\alpha > -\frac{1}{2}$.

If we introduce the notation

$$\mathcal{L}_{n\mu}^\alpha(x,t) = \frac{n!\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} L_n^\alpha(|\mu|x) e^{i\mu t - |\mu| x/2},$$

then formula (30) can be written as

$$\mathcal{L}_{n\mu}^\alpha(x^2,s)\mathcal{L}_{n\mu}^\alpha(y^2,t)$$
$$= \frac{\alpha}{\pi} \int_0^{2\pi} \int_0^1 \mathcal{L}_{n\mu}^\alpha(x^2+y^2+2xyr\cos\varphi, s+t+xyr\sin\varphi) r(1-r^2)^{\alpha-1} dr\, d\varphi. \quad (32)$$

The formula

$$L_n^\alpha(z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \Phi(-n;\alpha+1;z)$$

defines Laguerre polynomials. Replacing $n$ by non-integral parameter $\nu$, we obtain the Laguerre function

$$L_\nu^\alpha(z) = \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\alpha+1)}\Phi(-\nu;\alpha+1;z) = \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\alpha+1)}e^z\Phi(\nu+\alpha+1;\alpha+1;-z).$$

The formula

$$N_\nu^\alpha(z) = \frac{1}{2}\Gamma(\nu+\alpha+1)e^z\Psi(\nu+\alpha+1;\alpha+1;-z), \tag{33}$$

where $0 < \arg z < 2\pi$, defines Laguerre functions of the second kind. Durand [96] proved the analog of formula (30) for Laguerre functions of the second kind:

$$(xy)^{(k+\ell)/2}N_{\nu-\ell}^{\alpha+k+\ell}(x)N_{\nu-\ell}^{\alpha+k+\ell}(y)$$
$$= c_{\nu\alpha k\ell}\int_0^\infty\int_{-\infty}^\infty \exp\left[-\sqrt{xy}\cosh\varphi e^\psi - (k-\ell)\psi\right]$$
$$\times N_\nu^\alpha(x+y+2\sqrt{xy}\cosh\varphi\cosh\psi)P_\ell^{(\alpha-1,k-\ell}(\cosh 2\varphi)\sinh^{2\alpha-1}\varphi$$
$$\times \cosh^{k-\ell+1}\varphi\, d\psi\, d\varphi, \tag{34}$$

where

$$c_{\nu\alpha k\ell} = \frac{\Gamma(\ell+1)\Gamma(\nu+\alpha+k+1)}{\Gamma(\nu-\ell+1)\Gamma(\alpha+\ell)}e^{i\pi(k+\ell)},$$
$$\mathrm{Re}\,\alpha > 0,\ \mathrm{Re}(\alpha+k+\ell)\geq 0,\ \mathrm{Re}(\nu-\alpha-k-\ell+1)\geq 0,$$
$$\mathrm{Re}(\nu+\alpha+\beta+k-\ell+1) > 0,$$
$$0 < \arg x < 2\pi,\ 0 < \arg y < 2\pi,\ \frac{\pi}{2} < \arg\sqrt{xy} < \frac{3}{2}\pi,$$

and the analog of formula (29):

$$\exp\left(-\sqrt{xy}\,e^\psi\cosh\varphi\right)N_\nu^\alpha\left(x+y+2\sqrt{xy}\cosh\varphi\cosh\psi\right)$$
$$= \int_{-\infty}^\infty\int_{-\infty}^\infty q(\lambda,\mu)(xy)^{(i\lambda-\alpha)/2}N_{\nu-(i\lambda-i\mu-\alpha)/2}^{i\lambda}(x)N_{\nu-(i\lambda-i\mu-\alpha)/2}^{i\lambda}(y)$$
$$\times \Omega_{(i\lambda-i\mu-\alpha)/2}^{(\alpha-1,i\mu)}(\cosh 2\varphi)\cosh^{i\mu}\varphi\, e^{i\mu\psi}d\lambda\, d\mu, \tag{35}$$

where

$$\alpha > -\frac{1}{2},\ \mathrm{Re}(\nu+1) > 0,\ 0 < \arg x < 2\pi,\ 0 < \arg y < 2\pi,$$
$$\frac{\pi}{2} < \arg\sqrt{xy} < \frac{3\pi}{2},$$
$$q(\lambda,\mu) = \frac{i\lambda}{2\pi^2}\frac{\Gamma\left(\nu+1-\frac{1}{2}(i\lambda-i\mu-\alpha)\right)\Gamma\left(\frac{1}{2}(i\lambda+i\mu+\alpha)\right)}{\Gamma\left(\nu+1+\frac{1}{2}(i\lambda+i\mu+\alpha)\right)\Gamma\left(\frac{1}{2}(i\lambda+i\mu-\alpha)+1\right)}e^{\pi\lambda-i\pi\alpha}.$$

**11.4.6. Product formulas for Jacobi and Bessel functions.** From formula (4) of Section 11.4.3 we derive the product formula for Jacobi functions:

$$(\sinh t_1 \sinh t_2)^{m+m'}(\cosh t_1 \cosh t_2)^{m-m'-k}\mathfrak{P}_{\mu-m}^{(\alpha+m+m',m-m'-k)}(\cosh 2t_1)$$
$$\times \mathfrak{P}_{\mu-m}^{(\alpha+m+m',m-m'-k)}(\cosh 2t_2) = d_{mm'}^{\mu k}\int_0^{2\pi}\int_0^1 e^{-ik\nu}e^{i(m-m')\tau}\cosh^{-k} t$$
$$\times \mathfrak{P}_{\mu}^{(\alpha,-k)}(\cosh 2t)P_m^{(\alpha-1,m'-m)}(2r^2-1)^{m'-m+1}(1-r^2)^{\alpha-1}\,dr\,d\tau, \quad (1)$$

where the angles are connected by formulas (2) of Section 11.4.3 and

$$d_{mm'}^{\mu k} = \frac{(-1)^{m+m'}m!(\alpha+m)_{m'+1}(\alpha+1)_{m+m'}}{\pi(\mu-m+1)_m(\mu-k-m'+1)_{m'}(k-\mu-\alpha-m)_m(-\mu-\alpha-m')_{m'}}. \quad (2)$$

In particular, for $m = m' = 0$, we have

$$(\cosh t_1 \cosh t_2)^{-k}\mathfrak{P}_{\mu}^{(\alpha,-k)}(\cosh 2t_1)\mathfrak{P}_{\mu}^{(\alpha,-k)}(\cosh 2t_2)$$
$$= \frac{\alpha}{\pi}\int_0^{2\pi}\int_0^1 e^{-ik\nu}\cosh^{-k} t\,\mathfrak{P}_{\mu}^{(\alpha,-k)}(\cosh 2t)(1-r^2)^{\alpha-1}r\,dr\,d\tau, \quad (3)$$

where $e^{i\nu}\cosh t = \cosh t_1 \cosh t_2 + re^{i\tau}\sinh t_1 \sinh t_2$.

Let us now consider equality (10) of Section 11.4.3 as an expansion in Gegenbauer polynomials of $\cos\tau$ and in Jacobi polynomials of $(2r^2-1)$. Taking inverse transforms, we obtain the product formula

$$\mathfrak{P}_{\mu}^{(\alpha,\beta)}(\cosh 2t_1)\mathfrak{P}_{\mu}^{(\alpha,\beta)}(\cosh 2t_2) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma\left(\beta+\frac{1}{2}\right)}$$
$$\times \int_0^{\pi}\int_0^1 \mathfrak{P}_{\mu}^{(\alpha,\beta)}(2|\cosh t_1 \cosh t_2 - re^{i\tau}\sinh t_1 \sinh t_2|^2 - 1)r^{2\beta+1}$$
$$\times (1-r^2)^{\alpha-\beta-1}\sin^{2\beta}\tau\,dr\,d\tau, \quad (4)$$

where $\alpha, \beta \in \mathbb{Z}_+$, $\alpha > \beta$, $\mu \in \mathbb{R}$. By analytic continuation in $\mu$ we prove formula (4) for complex $\mu$. Formula (4) also allows analytic continuation in $\alpha$ and $\beta$ to their real values such that $\alpha > \beta > -\frac{1}{2}$.

We divide both sides of (4) by $\mathfrak{P}_{\mu}^{(\alpha,\beta)}(\cosh 2t_2)$ and shift $t_2$ to the infinity. Replacing $t_1$ by $t$, we obtain the generalization of integral representation (5) of

Section 11.4.1:

$$\mathfrak{P}_{\mu}^{(\alpha,\beta)}(\cosh 2t) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma\left(\beta+\frac{1}{2}\right)} \int_0^{\pi}\int_0^1 (\cosh^2 t + r^2\sinh^2 t + r\sinh 2t\cos\varphi)^{\mu} \times r^{2\beta+1}(1-r^2)^{\alpha-\beta-1}\sin^{2\beta}\varphi\,dr\,d\varphi. \quad (5)$$

One can also obtain this formula from integral representation (5) of Section 11.4.5 for Jacobi polynomials with the help of the formula

$$\mathfrak{P}_{\ell}^{(\alpha,\beta)}(\cosh 2t) = \frac{\ell!\Gamma(\alpha+1)}{\Gamma(\ell+\alpha+1)} P_{\ell}^{(\alpha,\beta)}(\cosh 2t), \quad \ell \in \mathbb{Z}_+, \quad (6)$$

and of Carleson's theorem[2] on analytic continuation.

Formula (4) can be represented in the form of formula (16) of Section 11.4.5. For this we go over in (4) from the variables $r$ and $\tau$ to $t$ and $\varphi$ by means of the substitution

$$e^{i\varphi}\cosh t = \cosh t_1\cosh t_2 - re^{i\tau}\sinh t_1\sinh t_2.$$

The variable $t$ varies in the interval $(|t_1-t_2|, t_1+t_2)$, and for every fixed $t$ the variable $\varphi$ varies from 0 to $\gamma$, where

$$\cos\gamma = \frac{\cosh^2 t_1 + \cosh^2 t_2 + \cosh^2 t - 1}{2\cosh t_1\cosh t_2\cosh t}. \quad (7)$$

Repeating the proof of formula (16) of Section 11.4.5, we obtain

$$\mathfrak{P}_{\mu}^{(\alpha,\beta)}(\cosh 2t_1)\mathfrak{P}_{\mu}^{\alpha,\beta}(\cosh 2t_2) = \frac{1}{2\pi}\int_0^{\infty} \mathfrak{P}_{\mu}^{(\alpha,\beta)}(\cosh 2t)K(t_1,t_2,t) \times \sinh^{2\alpha+1} t\cosh^{2\beta+1} t\,dt, \quad (8)$$

where $K(t_1,t_2,t) = 0$ if $t\bar{\in}(|t_1-t_2|, t_1+t_2)$ and

$$K(t_1,t_2,t) = \frac{\Gamma(\alpha+1)}{2^{\alpha+\beta-1/2}\Gamma(\alpha-\beta)\Gamma\left(\beta+\frac{1}{2}\right)} \frac{(2\cosh t_1\cosh t_2\cosh t)^{\alpha-\beta-1}}{(\sinh t_1\sinh t_2\sinh t)^{2\alpha}} \times \int_0^{\gamma} (\cos\tau - \cos\gamma)^{\alpha-\beta-1}\sin^{2\beta}\tau\,d\tau \quad (9)$$

[2]We have in mind the following theorem. If for $\operatorname{Re} z \geq 0$ a function $f(z)$ is regular and $f(z) = O(e^{k|z|})$, $k < \pi$, then the condition $f(z) = 0$ for $z = 0, 1, 2, \ldots$ implies that $f(z) \equiv 0$.

if $t \in (|t_1 - t_2|, t_1 + t_2)$. Here $\cos\gamma$ and $\gamma$ are given by (7).

The integral in (9) is calculated in the same way as the integral (17) of Section 11.4.5, and we have

$$K(t_1, t_2, t) = \frac{\Gamma(\alpha+1)}{2^{\alpha+\beta+1/2}\Gamma(\alpha-\beta)\Gamma\left(\alpha+\frac{1}{2}\right)} \frac{(\cosh t_1 \cosh t_2 \cosh t)\alpha - \beta - 1}{(\sinh t_1 \sinh t_2 \sinh t)^{2\alpha}}$$
$$\times (1-B^2)^{\alpha-1/2} F\left(\alpha+\beta, \alpha-\beta; \alpha+\frac{1}{2}; \frac{1-B}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(\alpha+1)}{2^{2\alpha+\beta-1}\Gamma(\alpha+\beta)\Gamma\left(\beta+\frac{1}{2}\right)}$$
$$\times \frac{(\cosh t_1 \cosh t_2 \cosh t)^{\alpha-\beta-1}}{(\sinh t_1 \sinh t_2 \sinh t)^{2\alpha}} (B^2-1)^{\frac{\alpha}{2}-\frac{1}{4}} \mathrm{P}^{-\alpha+1/2}_{\beta-1/2}(B), \quad (10)$$

where $|t_1 - t_2| < t < t_1 + t_2$ and $B$ denotes expression (7).

We derive from formula (14) of Section 11.4.3 the product formula for Bessel functions:

$$\left(\frac{r_1 r_2}{2}\right)^{-n} J_{m+m'+n}(r_1) J_{m+m'+n}(r_2) = \frac{(-1)^{m+m'} m!}{\pi(m+n-1)!} \int_0^{2\pi} \int_0^{\pi/2} r^{-n} J_n(r)$$
$$\times e^{i(m'-m)\varphi} P_m^{(n-1, m'-m)}(\cos\theta) \sin^{2n-1}\theta \cos^{m'-m+1}\theta \, d\theta \, d\varphi, \quad (11)$$

where $r = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos\theta \cos\varphi}$. In particular, for $m = m' = 0$, we have

$$\left(\frac{r_1 r_2}{2}\right)^{-n} J_n(r_1) J_n(r_2) = \frac{1}{\pi(n-1)!} \int_0^{2\pi} \int_0^{\pi/2} r^{-n} J_n(r) \sin^{2n-1}\theta \cos\theta \, d\theta \, d\varphi. \quad (12)$$

Considering formula (13) of Section 11.4.3 as the Fourier series of the function $(r_1 r_2/2r)^n J_n(r)$, for $q \geq 0$ we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{J_n\left(\sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos\theta \cos\varphi}\right)}{(r_1^2 + r_2^2 + 2r_1 r_2 \cos\theta \cos\varphi)^{n/2}} e^{-iq\varphi} d\varphi = \left(\frac{r_1 r_2}{2}\right)^{-n} (-\cos\theta)^q$$
$$\times \sum_{m=q}^{\infty} \frac{(m+n-1)!(2m-q+n)}{m!} P_{m-q}^{(n-1, q)}(\cos 2\theta) J_{2m-q+n}(r_1) J_{2m-q+n}(r_2). \quad (13)$$

It follows from (13) that, for $q > 0$ and for any $\theta$, we have

$$\int_0^{2\pi} \frac{J_n\left(\sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos\theta \cos\varphi}\right)}{(r_1^2 + r_2^2 + 2r_1 r_2 \cos\theta \cos\varphi)^{n/2}} \sin q\varphi \, d\varphi = 0. \quad (14)$$

Durand [96] proved the product formula for Jacobi functions of the second kind

$$\mathfrak{Q}_\mu^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+1)\Gamma(2\mu+\alpha+\beta+1)}{\Gamma(\mu+\alpha+1)\Gamma(\mu+\alpha+\beta+1)}\left(\frac{z+1}{2}\right)^\mu \times F\left(-\mu, -\mu-\beta; -2\mu-\alpha-\beta;' \frac{1+z}{2}\right).$$

It has the form

$$[(x-1)(y-1)]^{\ell+m/2}[(x+1)(y+1)]^{m/2}\mathfrak{Q}_{\nu-\ell-m}^{(\alpha+2\ell+m,\beta+m)}(x)\mathfrak{Q}_{\nu-\ell-m}^{(\alpha+2\ell+m,\beta+m)}(y)$$
$$= N_{\nu\ell m}^{\alpha\beta}\int_0^\infty\int_0^\infty \mathfrak{Q}_\nu^{(\alpha,\beta)}(Z)P_\ell^{(\alpha-\beta-1,\beta+m)}(\cosh 2u)C_m^\beta(\cosh t)$$
$$\times \sinh^{2\alpha-2\beta-1} u \cosh^{2\beta+m+1} u \sinh^{2\beta} t\, dt\, du, \quad (15)$$

where

$$\mathrm{Re}\alpha > \mathrm{Re}\beta > -\frac{1}{2}, \mathrm{Re}(m+\beta) \geq 0, \mathrm{Re}\left(\ell+\frac{\alpha}{2}+\frac{m}{2}\right) \geq 0, \mathrm{Re}(\nu-\ell-m+1) > 0,$$
$$\mathrm{Re}(\nu+\alpha-\beta-m+1) > 0, |\arg(x\pm 1)| < \pi, |\arg(y\pm 1)| < \pi,$$
$$|\arg(x-1)(y-1)| < \pi, |\arg(x^2-1)^{1/2}(y^2-1)^{1/2}| < \pi,$$

and $Z$ and $N_{\nu\ell m}^{\alpha\beta}$ are given by the formulas

$$Z = xy + (x^2-1)^{1/2}(y^2-1)^{1/2}\cosh t \cosh u + \frac{1}{2}(x-1)(y-1)\sinh^2 u,$$

$$N_{\nu\ell m}^{\alpha\beta} = 2^{2(\beta+\ell+m)}\frac{\Gamma(\beta)\Gamma(\nu+\beta-\ell+1)\Gamma(\nu+\alpha+\ell+1)\Gamma(\ell+1)}{\Gamma(\nu-\ell-m+1)\Gamma(\nu+\beta+1)\Gamma(\nu+\alpha+\beta+\ell+m+1)} \times \frac{\Gamma(m+1)\Gamma(\nu+\alpha+\beta+1)}{\Gamma(m+2\beta)\Gamma(\ell+\alpha-\beta)}.$$

By means of the Jacobi transform, one derives from (15) the continual addition formula

$$\mathfrak{Q}_\nu^{(\alpha,\beta)}(Z) = \int_{-\infty}^\infty\int_{-\infty}^\infty W(\lambda,\mu)[(x-1)(y-1)]^{(i\lambda-\alpha)/2}[(x+1)(y+1)]^{(i\mu-\beta)/2}$$
$$\times \mathfrak{Q}_{\nu-(i\lambda+i\mu-\alpha-\beta)/2}^{(i\lambda,i\mu)}(x)\mathfrak{Q}_{\nu-(i\lambda+i\mu-\alpha-\beta)/2}^{(i\lambda,i\mu)}(y)$$
$$\times \mathfrak{Q}_{(i\lambda-i\mu-\alpha+\beta)/2}^{(\alpha-\beta-1,i\mu)}(\cosh 2u)D_{i\mu-\beta}^{\beta}(\cosh t)\cosh^{i\mu-\beta} u\, d\lambda\, d\mu, \quad (16)$$

where the function $D^\tau_\sigma(z)$ is defined by formula (4) of Section 7.4.6,

$$|\arg(x \pm 1)| < \pi, |\arg(y \pm 1)| < \pi, |\arg(x^2-1)^{\frac{1}{2}}(y^2-1)^{\frac{1}{2}}| < \pi,$$
$$|\arg(x-1)(y-1)| < \pi,$$
$$\mathrm{Re}(\nu+\alpha+1) > 0, \mathrm{Re}\left(\nu + \frac{\alpha+\beta}{2}+1\right) > 0, \mathrm{Re}\alpha > \mathrm{Re}\beta > -\frac{1}{2}$$

and $W(\lambda,\mu)$ is given as

$$W(\lambda,\mu) = -\frac{1}{\pi^2}e^{-i\pi\beta}2^{-i\lambda-i\mu+\alpha+\beta+1}\frac{\lambda\mu\Gamma(\beta)\Gamma(\nu+\beta+1)}{\Gamma(\nu+\alpha+\beta+1)}\times$$
$$\frac{\Gamma\left(\frac{1}{2}(i\lambda+i\mu+\alpha-\beta)\right)\Gamma\left(\nu+1+\frac{1}{2}(\alpha+\beta-i\lambda-i\mu)\right)\Gamma\left(\nu+1+\frac{1}{2}(i\lambda+i\mu+\alpha+\beta)\right)}{\Gamma\left(\frac{1}{2}(i\lambda+i\mu-\alpha+\beta)+1\right)\Gamma\left(\nu+1+\frac{1}{2}(i\mu-i\lambda+\alpha+\beta)\right)\Gamma\left(\nu+1+\frac{1}{2}(i\lambda-i\mu+\alpha+\beta)\right)}.$$

If we set

$$k = \frac{1}{2}(i\lambda+i\mu-\alpha-\beta), \quad \ell = \frac{1}{2}(i\lambda-i\mu-\alpha+\beta),$$
$$x = \cosh 2\theta_1, \quad y = \cosh 2\theta_2, \quad t = \varphi, \quad u = \psi,$$

then we obtain an analog of formula (10) of Section 11.4.3:

$$\mathfrak{Q}^{(\alpha,\beta)}_\nu(\cosh 2\theta_1\cosh 2\theta_2 + \sinh 2\theta_1 \sinh 2\theta_2 \cosh\varphi\cosh\psi + 2\sinh^2\theta_1\sinh^2\theta_2\sinh^2\psi)$$
$$= \frac{1}{\pi^2}e^{-i\pi\beta}\int\limits_{-(\alpha+\beta)/2-i\infty}^{-(\alpha+\beta)/2+i\infty}\int\limits_{-(\alpha-\beta)/2-i\infty}^{-(\alpha-\beta)/2+i\infty} c_{k\ell}(\nu,\alpha,\beta)(\sinh\theta_1\sinh\theta_2)^{k+\ell}$$
$$\times(\cosh\theta_1\cosh\theta_2)^{k-\ell}\mathfrak{Q}^{(\alpha+k+\ell,\beta+k-\ell)}_{\nu-k}(\cosh 2\theta_1)\mathfrak{Q}^{(\alpha+k+\ell,\beta+k-\ell)}_{\nu-k}(\cosh\theta_2)$$
$$\times\mathfrak{Q}^{(\alpha-\beta-1,\beta+k-\ell)}_\ell(\cosh 2\psi)D^\beta_{k-\ell}(\cosh\varphi)d\ell\, dk, \quad (17)$$

where

$$c_{k\ell}(\nu,\alpha,\beta) = (\alpha+k+\ell)(\beta+k-\ell)\Gamma(\beta)$$
$$\times\frac{\Gamma(\nu-k+1)\Gamma(\nu+\alpha+\beta+k+1)\Gamma(\alpha+k)\Gamma(\nu+\beta+1)}{\Gamma(\nu+\alpha+\beta+1)\Gamma(\nu+\alpha+\ell+1)\Gamma(\beta+k+1)\Gamma(\nu+\beta-\ell+1)}.$$

Flensted-Jensen and Koornwinder [107] proved the product formula

$$\mathfrak{Q}^{(\alpha,\beta)}_\mu(x)\mathfrak{P}^{(\alpha,\beta)}_\mu(y) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma\left(\beta+\frac{1}{2}\right)}\int\limits_0^\pi\int\limits_0^1\mathfrak{Q}^{(\alpha,\beta)}_\mu\left(\frac{1}{2}(x+1)(y+1)\right.$$
$$\left.+\frac{1}{2}(x-1)(y-1)r^2 + r^2(x^2-1)^{1/2}(y^2-1)^{1/2}\cos\tau - 1\right)$$
$$\times r^{2\beta+1}(1-r)^{\alpha-\beta-1}\sin^{2\beta}\tau\, dr\, d\tau, \quad (18)$$

where $x > y \geq 1$.

**11.4.7. Generating functions.** Considering formula (4) of Section 11.3.4 as the coefficient $a_{mm'}$ in expansion of the function

$$F(\varphi,\theta) \equiv (\cos\theta - ie^{i\varphi}\cos\psi\sin\theta)^{\ell}(\cos\theta - ie^{-i\varphi}\cos\psi\sin\theta)^{\ell'} \tag{1}$$

in the basis functions $e^{i(m-m')\varphi}t_{00}^{n-1,mm'}(\psi)$, and taking into account expression (18) of Section 11.3.5 for $\lambda_{mm'}$, we derive

$$(\cos\theta - ie^{i\varphi}\cos\psi\sin\theta)^{\ell}(\cos\theta - ie^{-i\varphi}\cos\psi\sin\theta)^{\ell'} = \sum_{m=0}^{\ell}\sum_{m'=0}^{\ell'} a_{mm'}^{\ell\ell'} t(\theta)e^{i(m-m')\varphi}t_{00}^{n-1,mm'}(\psi), \tag{2}$$

where

$$a_{mm'}^{\ell\ell'} = (-1)^m i^{m+m'}\left[\frac{\ell!\ell'!(n-2)_m(n-2)_{m'}(m+m'+n-2)}{m!m'!(\ell-m)!(\ell'-m')!(\ell+n-1)_{m'}(\ell'+n-1)_m(n-2)}\right]^{1/2}. \tag{3}$$

Thus, (1) is the generating function for $t^{n\ell\ell'}_{(mm')0}(g_{n-1}(\theta))$ with respect to zonal spherical functions of the group $U(n-1)$.

Substituting into (2) values of spherical functions, we have

$$\begin{aligned}(\cos\theta = ie^{i\varphi}\cos\psi\sin\theta)^{\ell}&(\cos\theta = ie^{-i\varphi}\cos\psi\sin\theta)^{\ell+\beta}\\ &= \sum_{m=0}^{\ell}\sum_{m'=0}^{\ell+\beta} i^{m+m'}\frac{\ell!(\alpha)_m(\ell+\beta-m'+1)_{m'}(\alpha+m+m')}{m!m'!(m'+\alpha)_{\ell+1}}\sin^{m-m'}\theta\cos^{\beta+m-m'}\theta\\ &\quad\times P_{\ell-m}^{(\alpha+m+m',\beta+m-m')}(\cos 2\theta)e^{i(m-m')\varphi}t_{00}^{n-1,mm'}(g_{n-2}(\psi)),\end{aligned} \tag{4}$$

where $\beta = \ell' - \ell$, $\alpha = n-2$. For $\varphi = 0$ and $\psi = \pi/2$ we obtain from (4) the equality

$$\sum_{m=0}^{\ell}\frac{\ell!(\ell+\beta)!(\alpha+m-1)!(\alpha+2m)}{m!(\ell+\beta-m)!(\ell+\alpha+m)}\sin^{2m}\theta\, P_{\ell-m}^{(\alpha+2m,\beta)}(\cos 2\theta) = \cos^{2\ell}\theta. \tag{5}$$

Formulas (11) and (14) of Section 11.3.4 give the relation

$$\begin{aligned}(\cosh t - e^{i\varphi}\sinh t\cos\theta)^{\mu}&(\cosh t - e^{-i\varphi}\sinh t\cos\theta)^{\mu-k}\\ &= \sum_{m=0}^{\infty}\sum_{m'=0}^{\infty} a_{mm'}^{\alpha\mu k}\sinh^{m+m'}t\cosh^{m-m'-k}t\,\mathfrak{P}_{\mu-m}^{(m+m'+\alpha,m-m'-k)}(\cosh 2t)\\ &\quad\times e^{i(m-m')\varphi}t_{00}^{n-1,mm'}(g_{n-2}(\theta)),\end{aligned} \tag{6}$$

where $\alpha = n-2$,

$$a^{\alpha\mu k}_{mm'} = \frac{(\alpha)_m(\mu-m+1)_m(\mu-k-m'+1)_{m'}}{m!m'!(\alpha+m')_m}, \tag{6'}$$

and the relation

$$e^{ix\cos\theta\cos\varphi} = \sum_{m=0}^{\infty}\sum_{m'=0}^{\infty} i^{m+m'}\frac{(m+n-1)!(m+m'+n)}{m!}e^{i(m-m')\varphi} \times \cos^{m-m'}\theta P^{(n-1,m-m')}_{m'}(\cos 2\theta)\left(\frac{x}{2}\right)^{-n} J_{m+m'+n}(x). \tag{7}$$

For $\varphi = 0$ and $\theta = 0, \frac{\pi}{2}$, we obtain from (6) that

$$\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty} a^{\alpha\mu k}_{mm'}\sinh^{m+m'}t\cosh^{m-m'-k}t \times \mathfrak{P}^{(\alpha+m+m',m-m'-k)}_{\mu-m}(\cosh 2t) = e^{(2\mu-k)t}, \tag{8}$$

$$\sum_{m=0}^{\infty}(-1)^m\frac{(\mu-m+1)_m(\mu-k-m+1)_m}{m!(\alpha+m)_m}\sinh^{2m}t\,\mathfrak{P}^{(2m+\alpha,-k)}_{\mu-k}(\cosh 2t) = \cosh^{2\mu}t. \tag{8'}$$

Multiply both sides of (7) by $e^{iq\varphi}/2\pi$ and integrate with respect to $\varphi$ from 0 to $2\pi$. By virtue of formula (10) of Section 3.5.6, we have

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}(m+n-1)!(2m+n+q)P^{(n-1,-q)}_{m+q}(\cos 2\theta)J_{2m+n+q}(x) = \cos^q\theta\left(\frac{x}{2}\right)^n J_q(x\cos\theta). \tag{9}$$

Consequently, the function on the right hand side is the generating function for Bessel functions $J_{2m+n+q}$, $m = 0,1,2,\dots$, with respect to Jacobi polynomials. For $\theta = 0$ the relation (9) yields

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{(m+n-1)!(m+q+n-1)!(2m+n+q)}{(m+q)!}J_{2m+n+q}(x) = \left(\frac{x}{2}\right)^n J_q(x). \tag{10}$$

Let us consider (6) as the Fourier series of the function on the left hand side. Taking into account values (15) and (15′) of Section 11.3.5 of the integral

$$\frac{1}{2\pi}\int_0^{2\pi}(\cosh t - e^{i\varphi}\cos\theta\sinh t)^{\frac{\sigma+k}{2}}(\cosh t - e^{-i\varphi}\cos\theta\sinh t)^{\frac{\sigma-k}{2}}e^{iq\varphi}d\varphi,$$

we have the equality

$$\sum_{m'=0}^{\infty}\frac{(\mu-q-m'+1)_{m'}(\mu-k-m'+1)_{m'}}{m'!(\alpha+m')_{m'+q}}\sinh^{2m'}t\mathfrak{P}_{\mu-q-m'}^{(\alpha+2m'+q,q-k)}(\cosh 2t)$$
$$\times P_{m'+q}^{(\alpha-1,-q)}(\cos 2\theta)$$
$$=\frac{1}{q!}\cosh^{2\mu-2q}t\cos^{2q}\theta F(k-\mu,q-\mu;q+1;\cos^2\theta\tanh^2 t)\quad(11)$$

if $m-m'\equiv q>0$ and the equality

$$\sum_{m=0}^{\infty}\frac{(\mu-m+1)_m(\mu-k-q-m+1)_m}{(m+q)!(\alpha+q+m+2)_{m-2}}\sinh^{2m}t\mathfrak{P}_{\mu-m}^{(\alpha+2m+q,-k-q)}(\cosh 2t)$$
$$\times P_m^{(\alpha-1,q)}(\cos 2\theta)$$
$$=\frac{1}{q!}\cosh^{2\mu}tF(-\mu,q+k-\mu;q+1;\cos^2\theta\tanh^2 t)\quad(12)$$

if $m-m'\equiv -q<0$. In particular, for $\theta=0$, we obtain

$$\sum_{m=0}^{\infty}\frac{(\alpha)_m(\mu-q-m+1)_m(\mu-k-m+1)_m}{m!m+q)!(\alpha+m+q)_m}\sinh^{2m}t\mathfrak{P}_{\mu-q-m}^{(\alpha+q+2m,q-k)}(\cosh 2t)$$
$$=\frac{1}{q!}\cosh^{2\mu-2q}tF(k-\mu,q-k;q+1;\tanh^2 t),\quad(13)$$

$$\sum_{m=0}^{\infty}\frac{(\alpha)_m(\mu-m+1)_m(\mu-k-q-m+1)_m}{m!(m+q)!(\alpha+m+q+2)_{m-2}}\sinh^{2m}t\mathfrak{P}_{\mu-m}^{(\alpha+2m+q,-k-q)}(\cosh 2t)$$
$$=\frac{1}{q!}\cosh^{2\mu}tF(-\mu,q+k-\mu;q+1;\tanh^2 t),\quad(14)$$

where $q\geq 0$.

If we fix $q=m-m'$ on the left hand side of (8), then the remaining sum coincides with the sum in (13) or in (14). Therefore, we obtain

$$\sum_{q=1}^{\infty}\frac{1}{q!(\mu-k-q+1)_q}\tanh^q tF(-\mu,q_k-\mu;q_1;\tanh^2 t)$$
$$+\sum_{q=0}^{\infty}\frac{1}{q!(\mu-q+1)_q}\tanh^q tF(k-\mu,q-\mu;q+1;\tanh^2 t)$$
$$=\cosh^{k-2\mu}t\,e^{(2\mu-k)t}.\quad(15)$$

A more general formula is derived from equalities (6) with $\varphi = 0$, and from (11) and (12):

$$\sum_{q=0}^{\infty} \frac{\tanh^q t \cos^q \theta}{q!(\mu - q + 1)_q} F(k-\mu, q-\mu; q+1; \cos^2\theta \tanh^2 t)$$
$$+ \sum_{q=1}^{\infty} \frac{\tanh^q t \cos^q \theta}{q!(\mu - k - q + 1)_q} F(-\mu, q+k-\mu; q+1; \cos^2\theta \tanh^2 t)$$
$$= \cosh^{k-2\mu} t\, (\cosh t - \sinh t \cos\theta)^{2\mu - k}. \quad (16)$$

Formula (4) gives the analogs of relations (11) and (12) for Jacobi polynomials:

$$\sum_{r=0}^{\min(\ell+\beta, \ell-q)} \frac{(-1)^r (\alpha + q + 2r) \sin^{2r}\theta}{r!(r+\alpha)_{\ell+1}(\ell+\beta-r)!} P_{\ell-q-r}^{(\alpha+q+2r, \beta+q)}(\cos 2\theta) P_{r+q}^{(\alpha-1,-1)}(\cos 2\psi)$$
$$= \frac{(-1)^q \sin^{2q-2r}\theta \cos^{-q-2\ell}\theta}{q!(\ell-q)!(\ell+\beta)!} F(q-\ell, -\ell; q+1; -\cos^2\psi \tan^2\theta), \quad (17)$$

where $q = m - m' \geq 0$,

$$\sum_{m=0}^{\min(\ell, \ell+\beta-q)} \frac{(-1)^m (\ell+\beta-q-m+1)_m (\alpha+2m+q)}{(m+q)!(\alpha+m+q)_{\ell+1}} \sin^{2m}\theta P_{\ell-m}^{(\alpha+2m+q, \beta-q)}(\cos 2\theta)$$
$$\times P_m^{(\alpha-1,q)}(\cos 2\psi) = \frac{\cos^{2\ell}\theta}{\ell! q!} F(-\ell, q-\ell-\beta; q+1; -\cos^2\psi \tan^2\theta), \quad (18)$$

where $q = m' - m \geq 0$.

**11.4.8. Expansion in zonal spherical functions of the group $U(n-1,1)$.** Let $\mathfrak{L}_0^2(U(n-1,1))$ be the subspace in $\mathfrak{L}^2(U(n-1,1))$ consisting of functions invariant with respect to right and left shifts by elements from $U(n-1)$. The space $\mathfrak{L}_0^2(U(n-1,1))$ is identified with the space of functions $f(\varphi, t)$, $0 \leq \varphi < 2\pi$, $0 \leq t < \infty$, such that

$$\|f\| \equiv \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} |f(\varphi, t)|^2 \sinh^{2n-3} t \cosh t \, dt \, d\varphi < \infty. \quad (1)$$

We denote the Hilbert space of these functions by $\mathfrak{L}_0^2(H_C^{n-1})$. Carrying out the Fourier transform

$$F_k(\cosh t) = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi, t) e^{ik\varphi} d\varphi, \quad (2)$$

we represent $\mathfrak{L}_0^2(H_C^{n-1})$ as the orthogonal sum of the spaces $\mathfrak{H}_k$, $k = 0, \pm 1, \pm 2, \dots$, of functions $F_k(\cosh t)$ with the scalar product

$$(F_k, G_k) = \int_0^\infty F_k(\cosh t) G_k(\cosh t) \sinh^{2n-3} t \cosh t \, dt. \tag{3}$$

It it clear that the spaces $\mathfrak{H}_k$ coincide.

We expand functions $F_k$ of the spaces $\mathfrak{H}_k$ in the functions $\varphi^{n\sigma k}(t)$ (see Section 11.3.2). For this we note that, for the principal unitary series representations $T^{n\sigma k}$, $\sigma = i\rho - n + 1$, $\rho \in \mathbb{R}$, the functions $\varphi^{n\sigma k}(t)$ are expressed in terms of $\mathfrak{P}^\tau_{mn}(\cosh 2t)$:

$$\varphi^{n,i\rho-n+1,k}(t) = \frac{(n-2)!\Gamma\left(\frac{1}{2}(k - i\rho - n + 3)\right)}{\Gamma\left(\frac{1}{2}(k - i\rho + n - 1)\right) \sinh^{n-2} t} \mathfrak{P}^{-(i\rho+1)/2}_{\frac{n-k}{2}-1, -\frac{n+k}{2}+1}(\cosh 2t). \tag{4}$$

For the functions $\psi_-^{\nu_1\nu_2}(t)$ and $\psi_+^{\nu_1\nu_2}(t)$ (see formulas (11) and (11′) of Section 11.3.2), we have

$$\psi_-^{\nu_1\nu_2}(t) = \frac{(\nu_2 - n + 1)!(n-2)!}{(\nu_2 - 1)! \sinh^{n-2} t} \mathcal{P}^{\frac{1}{2}(\nu_1 - \nu_2 + n) - 1}_{\frac{1}{2}(-\nu_1 - \nu_2 + n) - 1, \frac{1}{2}(-\nu_1 - \nu_2 - n) + 1}(\cosh 2t), \tag{4′}$$

$$\psi_+^{\nu_1\nu_2}(t) = \frac{(-\nu_1 - n + 1)(n-2)!}{(-\nu_2 - 1)! \sinh^{n-2} t} \mathcal{P}^{\frac{1}{2}(\nu_1 - \nu_2 + n) - 1}_{\frac{1}{2}(\nu_1 + \nu_2 + n) - 1, \frac{1}{2}(\nu_1 + \nu_2 - n) + 1}(\cosh 2t), \tag{4″}$$

where the functions $\mathcal{P}^\ell_{mn}(x)$ are defined by formula (9) of Section 6.5.6 and are related to the discrete series representations of the group $SU(1,1)$. Remember that in (4″) and (4′) $\nu_1, \nu_2 \in \mathbb{Z}$. Besides, in (4′) $0 \le \nu_1 \le \nu_2 - n + 2$ and in (4″) $0 \ge \nu_2 \ge \nu_1 + n - 2$.

The functions

$$\varphi^{n,-i\rho-n+1,k}(\varphi, t) \equiv e^{-ik\varphi} \varphi^{n,-i\rho-n+1,k}(t)$$
$$\psi_\pm^{\nu_1\nu_2}(\varphi, t) \equiv e^{-i(\nu_1+\nu_2)\varphi} \psi_\pm^{\nu_1\nu_2}(t)$$

are zonal spherical functions of the corresponding representations $T^{n\sigma k}$ of the group $U(n-1,1)$ (see Section 11.3.2).

Now we consider expansions (8)–(11) of Section 7.8.4 for $f(x)$ in $\mathfrak{P}^\tau_{mm'}(x)$ and $\mathcal{P}^\ell_{mm'}(x)$ and write them down in the form

$$f(\cosh 2t) = \frac{1}{2} \int_{-\infty}^\infty a(\rho) \mathfrak{P}^{i\rho - 1/2}_{mm'}(\cosh 2t) \rho \tanh \pi(\rho + i\varepsilon) d\rho$$
$$+ \frac{1}{2} \sum_{\ell=-1-\varepsilon}^{N} (-2\ell - 1) b(\ell) \mathcal{P}^\ell_{mm'}(\cosh 2t), \tag{5}$$

$$a(\rho) = 4\int_0^\infty f(\cosh 2t)\overline{\mathfrak{P}^{i\rho-1/2}_{mm'}(\cosh 2t)}\sinh t\cosh t\,dt, \tag{6}$$

$$b(\ell) = 4\int_0^\infty f(\cosh 2t)\mathfrak{P}^{\ell}_{mm'}(\cosh 2t)\sinh t\cosh t\,dt, \tag{7}$$

$$4\int_0^\infty |f(\cosh 2t)|^2\sinh t\cosh t\,dt = \int_0^\infty |a(\rho)|^2\rho\tanh\pi(\rho+i\varepsilon)d\rho$$
$$+\frac{1}{2}\sum_{\ell=-1-\varepsilon}^{N}(-2\ell-1)|b(\ell)|^2, \tag{8}$$

where $N = -\min(|m|,|m'|)$ for $mm' > 0$ and $N = 0$ for $mm' \leq 0$.

Carry out in (5)–(8) the substitution

$$F_k(t) = \frac{f(\cosh 2t)}{\sinh^{n-2} t},\quad b_k(\rho) = \frac{\Gamma\left(\frac{k+i\rho}{2} - \frac{n-3}{2}\right)(n-2)!}{\Gamma\left(\frac{k+i\rho}{2} + \frac{n-1}{2}\right)}a\left(\frac{\rho}{2}\right)$$

and pass from $F_k(t)$ to

$$f(\varphi,t) = \sum_{k=-\infty}^{\infty} F_k(t)e^{-ik\varphi}.$$

We obtain the expansion

$$f(\varphi,t) = \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} b_k(\rho)\varphi^{n,-i\rho-n+1,k}(\varphi,t)d\mu_k(\rho)$$
$$+\sum_{\nu_1,\nu_2}\frac{(\nu_2-1)!^2(\nu_2-\nu_1-n+1)}{2(\nu_2-n+1)!^2(n-2)!^2}b(\nu_1,\nu_2)\psi_-^{\nu_1\nu_2}(\varphi,t)$$
$$+\sum_{\nu_1,\nu_2}\frac{(-\nu_1-1)!^2(\nu_2-\nu_1-n+1)}{2(-\nu_1-n+1)!^2(n-2)!^2}b'(\nu_1,\nu_2)\psi_+^{\nu_1\nu_2}(\varphi,t), \tag{9}$$

where the summation in the second term is over integral values of $\nu_1$ and $\nu_2$ for which $0 \leq \nu_1 < \nu_2 - n + 1$ and the summation in the third term is over integral values $\nu_1$ and $\nu_2$ for which $0 \geq \nu_2 > \nu_1 + n - 1$. Besides,

$$b_k(\rho) = \frac{2}{\pi}\int_0^{2\pi}\int_0^\infty f(\varphi,t)\overline{\varphi^{n,-i\rho-n+1,k}(\varphi,t)}\sinh^{2n-3} t\cosh t\,dt\,d\varphi, \tag{10}$$

$$b(\nu_1,\nu_2) = \frac{2}{\pi}\int_0^{2\pi}\int_0^{\infty} f(\varphi,t)\overline{\psi_-^{\nu_1\nu_2}(\varphi,t)}\sinh^{2n-3} t\cosh t\,dt\,d\varphi, \tag{11}$$

$$b'(\nu_1,\nu_2) = \frac{2}{\pi}\int_0^{2\pi}\int_0^{\infty} f(\varphi,t)\overline{\psi_+^{\nu_1\nu_2}(\varphi,t)}\sinh^{2n-3} t\cosh t\,dt\,d\varphi, \tag{12}$$

$$d\mu_k(\rho) = \frac{1}{4}\left|\frac{\Gamma\left(\frac{1}{2}(k+i\rho+n-1)\right)}{\Gamma\left(\frac{1}{2}(k+i\rho-n+3)\right)(n-2)!}\right|^2 \rho\tanh\left(\frac{\pi\rho}{2}+i\varepsilon\right)d\rho. \tag{13}$$

Here $\varepsilon = 0$ if $n-k$ is even and $\varepsilon = \frac{1}{2}$ if $n-k$ is odd. The Plancherel formula for transformations (9)–(12) has the form

$$\begin{aligned}\frac{2}{\pi}\int_0^{2\pi}\int_0^{\infty} |f(\varphi,t)|^2\sinh^{2n-3} t\cosh t\,dt\,d\varphi &= \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} |b_k(\rho)|^2 d\mu_k(\rho)\\ &+ \sum_{\nu_1,\nu_2}\frac{(\nu_2-1)!^2(\nu_2-\nu_1-n+1)}{2(\nu_2-n+1)!^2(n-2)!^2}|b(\nu_1,\nu_2)|^2\\ &+ \sum_{\nu_1,\nu_2}\frac{(-\nu_1-1)!^2(\nu_2-\nu_1-n+1)}{2(-\nu_1-n+1)!^2(n-2)!^2}|b'(\nu_1,\nu_2)|^2,\end{aligned} \tag{14}$$

where the summations with respect to $\nu_1$ and $\nu_2$ are the same as in (9).

## 11.5. Orthogonal Polynomials on the Disk

**11.5.1. The definition.** Zonal spherical functions of the representations $T^{n\ell\ell'}$ of the group $U(n)$ can be written as

$$\begin{aligned}t_{00}^{n\ell\ell'}(d_n(\varphi)g_{n-1}(\theta)) &= \frac{q!(n-2)!}{(q+n-2)!}e^{i(\ell'-\ell)\varphi}\cos^{|\ell-\ell'|}\theta\, P_q^{(n-2,|\ell-\ell'|)}(\cos 2\theta)\\ &= e^{i(\ell'-\ell)\varphi}\cos^{|\ell-\ell'|}\theta F(-q,|\ell-\ell'|+q+n-1;n-1;\sin^2\theta),\end{aligned} \tag{1}$$

where $q = \min(\ell,\ell')$. Let us consider this function in another coordinate system. Let

$$x = \cos\theta\cos\varphi,\quad y = \cos\theta\sin\varphi. \tag{2}$$

The mapping $(\theta,\varphi)\to(x,y)$ transforms the north hemisphere $S$ of the unit sphere $S^2$, i.e. the domain $\{(\theta,\varphi)\,|\,0\le\theta\le\frac{\pi}{2}, 0\le\varphi<2\pi\}$ into the unit disk $\bar{B}^2 \equiv \{(x,y)\,|\,x^2+y^2\le 1\}$. The measure

$$ds = \frac{n-1}{\pi}\sin^{2n-3}\theta\cos\theta\,d\theta\,d\varphi$$

on $S$ turns into the measure

$$\frac{n-1}{\pi}(1-x^2-y^2)^{n-2}dx\,dy \tag{3}$$

on $\bar{B}^2$. The space $\mathfrak{L}^2(S) \equiv \mathfrak{L}^2(U(n-1)\backslash U(n)/U(n-1))$ becomes the Hilbert space $\mathfrak{L}^2(\bar{B}^2)$ with the scalar product

$$(f_1, f_2) = \frac{n-1}{\pi}\int_{\bar{B}^2} f_1(x,y)\overline{f_2(x,y)}(1-x^2-y^2)^{n-2}dx\,dy. \tag{4}$$

Under the mapping $(\theta,\varphi) \to (x,y)$ functions (1) turn into the polynomials

$$\begin{aligned} p^n_{\ell\ell'}(x,y) &= (x+iy\,\mathrm{sign}(\ell'-\ell))^{|\ell-\ell'|}F(-q,|\ell-\ell'|+q+n-1;n-1;1-x^2-y^2) \\ &= \frac{q!(n-2)!}{(q+n-2)!}(x+iy\,\mathrm{sign}(\ell'-\ell))^{|\ell-\ell'|}P_q^{(n-2,|\ell-\ell'|)}(2(x^2+y^2)-1). \end{aligned} \tag{5}$$

Since zonal spherical functions (1) are orthogonal (see Section 11.4.4), then the polynomials $p^n_{\ell\ell'}(x,y)$ are orthogonal:

$$\frac{n-1}{\pi}\int_{\bar{B}^2} p^n_{\ell\ell'}(x,y)\overline{p^n_{mm'}(x,y)}(1-x^2-y^2)^{n-2}dx\,dy = \left(\dim T^{n\ell\ell'}\right)^{-1}\delta_{\ell m}\delta_{\ell' m'}. \tag{6}$$

It follows from the results of Section 11.4.4 that, for a fixed $n$, the polynomials $p^n_{\ell\ell'}(x,y)$, $0 \le \ell,\ell' < \infty$, form an orthogonal basis in $\mathfrak{L}^2(\bar{B}^2)$.

The equality $\varphi^{n\ell\ell'}(\theta) = \varphi^{n\ell'\ell}(\theta)$ implies that

$$p^n_{\ell\ell'}(x,y) = \left(\frac{x+iy}{\sqrt{x^2+y^2}}\right)^{2(\ell'-\ell)} p^n_{\ell'\ell}(x,y). \tag{7}$$

**11.5.2. Integral representation and differential equations.** It follows from integral representation (3) of Section 11.3.2 for $\varphi^{n\ell\ell'}(\theta)$ that

$$\begin{aligned} t^{n\ell\ell'}_{00}(d_n(\varphi)g_{n-1}(\theta)) = \frac{n-2}{\pi}\int_0^{2\pi}\int_0^{\pi/2} &(e^{-i\varphi}\cos\theta - ie^{i\tau}\cos\psi\sin\theta)^{\ell} \\ &\times (e^{i\varphi}\cos\theta - ie^{-i\tau}\cos\psi\sin\theta)^{\ell'}\sin^{2n-5}\psi\cos\psi\,d\psi\,d\tau. \end{aligned} \tag{1}$$

It gives the integral representation for $p^n_{\ell\ell'}(x,y)$:

$$\begin{aligned} p^n_{\ell\ell'}(x,y) = \frac{n-2}{\pi}\int_{\bar{B}^2} &\left[(x-iy) - i(s+it)\sqrt{1-x^2-y^2}\right]^{\ell} \\ &\times \left[(x+iy) - i(s-it)\sqrt{1-x^2-y^2}\right]^{\ell'}(1-s^2-t^2)^{n-3}ds\,dt. \end{aligned} \tag{2}$$

Differential equation (9) of Section 11.3.4 for $\varphi^{n\ell\ell'}(\theta)$ implies the differential equation for $t_{00}^{n\ell\ell'}(d_n(\varphi)g_{n-1}(\theta)) \equiv u(\varphi,\theta)$:

$$\left[\frac{1}{\sin^{2n-3}\theta\cos\theta}\frac{\partial}{\partial\theta}\sin^{2n-3}\theta\cos\theta\frac{\partial}{\partial\theta}+\frac{1}{\cos^2\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$
$$\left.+(\ell+\ell')(\ell+\ell'+2n-2)\right]u(\varphi,\theta)=0. \quad (3)$$

Besides, it is clear that

$$\frac{1}{i}\frac{\partial}{\partial\varphi}u(\varphi,\theta)=(\ell'-\ell)u(\varphi,\theta). \quad (4)$$

From (3) and (4) we derive the differential equations for $p_{\ell\ell'}^n(x,y)$:

$$\left[\frac{y^2+x^2(1-x^2-y^2)}{x^2+y^2}\frac{\partial^2}{\partial x^2}+\frac{x^2+y^2(1-x^2-y^2)}{x^2+y^2}\frac{\partial^2}{\partial y^2}-2xy\frac{\partial^2}{\partial x\partial y}\right.$$
$$\left.-(2n-1)x\frac{\partial}{\partial x}-(2n-1)y\frac{\partial}{\partial y}+(\ell+\ell')(\ell+\ell'+2n-2)\right]p_{\ell\ell'}^n(x,y)=0, \quad (5)$$

$$\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)p_{\ell\ell'}^n(x,y)=(\ell'-\ell)p_{\ell\ell'}^n(x,y). \quad (6)$$

These equations define $p_{\ell\ell'}^n(x,y)$ uniquely up to a constant.

One can derive recurrence relations for $p_{\ell\ell'}^n(x,y)$ from those for Jacobi polynomials. For example, if $\ell<\ell'$, then we have

$$2(q+n-1)(q+n+|\ell-\ell'|-1)(2q+n+|\ell-\ell'|-2)p_{\ell+1,\ell'+1}^n(x,y)-(2q+n+|\ell-\ell'|-1)$$
$$\times[(2q+n+|\ell-\ell'|-2)(2q+n+|\ell-\ell'|)(2x^2+2y^2-1)+(n-2)^2-(\ell-\ell')^2]p_{\ell\ell'}^n(x,y)$$
$$+2q(q+|\ell-\ell'|)(2q+n+|\ell-\ell'|)p_{\ell-1,\ell'-1}^n(x,y)=0. \quad (7)$$

And if $\ell>\ell'$, then we have to permute $p_{\ell+1,\ell'+1}^n(x,y)$ and $p_{\ell-1,\ell'-1}^n(x,y)$.

**11.5.3. The addition theorem and the product formula.** In formula (1) of Section 11.4.2 we replace $g_{n-1}(\theta_1)d_{n-1}(\tau)g_{n-2}(\varphi)g_{n-1}(-\theta_2)$ by

$$g=d_n(\varphi_1)g_{n-1}(\theta_1)d_{n-1}(\tau)g_{n-2}(\psi)d_n(-\varphi_2)g_{n-1}(-\theta_2).$$

Then

$$g=kd_n(\nu)g_{n-1}(\theta)k', \qquad k,k'\in U(n-1),$$

where

$$e^{i\nu}\cos\theta=e^{i\varphi_1}\cos\theta_1e^{i\varphi_2}\cos\theta_2+e^{i\tau}\sin\theta_1\sin\theta_2\cos\psi. \quad (1)$$

The new factor $\exp(-i(\ell-\ell')(\varphi_1+\varphi_2))$ appears on the left hand sides of formulas (3) and (4) of Section 11.2.4. We pass in these formulas from $\varphi_1$, $\theta_1$, $\varphi_2$, $\theta_2$, $\tau$, $\psi$, $\nu$, $\theta$ to

$$\begin{aligned} x_1 &= \cos\theta_1\cos\varphi_1, & y_1 &= \cos\theta_1\sin\varphi_1,\\ x_2 &= \cos\theta_2\cos\varphi_2, & y_2 &= \cos\theta_2\sin\varphi_2\\ x_3 &= \cos\psi\cos\tau, & y_3 &= \cos\psi\sin\tau,\\ x_4 &= \cos\theta\cos\nu, & y_4 &= \cos\theta\sin\nu. \end{aligned}$$

Then (1) yields that

$$x = x_3\sqrt{1-x_2^2-y_2^2}\,\sqrt{1-x_1^2-y_1^2}+x_1x_2+y_1y_2, \tag{2}$$

$$y = y_3\sqrt{1-x_2^2-y_2^2}\,\sqrt{1-x_1^2-y_1^2}+x_1y_2-x_2y_1, \tag{3}$$

and we have

$$p^n_{\ell\ell'}(x,y) = \sum_{m=0}^{\ell}\sum_{m'=0}^{\ell'} d^n_{\ell\ell'mm'}p^{n-1}_{mm'}(x_3,y_3)[(1-x_1^2-y_1^2)(1-x_2^2-y_2^2)]^{\frac{m+m'}{2}} \times \overline{p^{n+m+m'}_{\ell-m,\ell'-m'}(x_1,y_1)}p^{n+m+m'}_{\ell-m,\ell'-m'}(x_2,y_2), \tag{4}$$

where

$$d^n_{\ell\ell'mm'} = \frac{q!p!(n-2)(p_1+n-3)!(q_1+n-3)!(m+m'+n-1)_{q_0}(m+m'+n-2)_{p_0+1}}{p_0!q_0!p_1!q_1!}$$

and $q=\min(\ell,\ell')$, $p=|\ell-\ell'|-q$, $q_0=\min(m,m')$, $p_0=|m-m'|-q_0$, $q_1=\min(\ell-m,\ell'-m')$, $p_1=|\ell-\ell'-m+m'|-q_1$. Formula (4) is *the addition theorem for the polynomials* $p^n_{\ell\ell'}(x,y)$.

Making use of the orthogonality relation for $p^{n-1}_{mm'}(x_3,y_3)$, we derive from (4) the *product formula*

$$\frac{n-2}{\pi}\int_{\bar{B}^2} p^n_{\ell\ell'}(x,y)\overline{p^{n-1}_{mm'}(x_3,y_3)}(1-x_3^2-y_3^2)^{n-3}dx_3dy_3 = c^n_{\ell\ell'mm'}[(1-x_1^2-y_1^2)(1-x_2^2-y_2^2)]^{(m+m')/2}p^{n+m+m'}_{\ell-m,\ell'-m'}(x_2,y_2) \times \overline{p^{n+m+m'}_{\ell-m,\ell'-m'}(x_1,y_1)}, \tag{5}$$

where $x$ and $y$ are given by formulas (2) and (3),

$$c^n_{\ell\ell'mm'} = \frac{p!q!(p_0+m+m'+n-2)!(q_0+m+m'+n-2)!(n-2)!^2}{p_0!q_0!(p+n-2)!(q+n-2)!(m+m'+n-2)!^2}.$$

In particular, for $m = m' = 0$ we have

$$\frac{n-2}{\pi}\int_{\tilde{B}^2} p^n_{\ell\ell'}(x,y)(1-x_3^2-y_3^2)^{n-3}dx_3dy_3 = \overline{p^n_{\ell\ell'}(x_1,y_1)}p^n_{\ell\ell'}(x_2,y_2), \qquad (6)$$

where, as in (5), $x$ and $y$ are given by (2) and (3).

**11.5.4. Special functions on the exterior of the disk.** Let us now regard the zonal spherical functions

$$t^{n\sigma k}_{00}(d_n(\varphi)g'_{n-1}(\theta)) = e^{-ik\varphi}\cosh^k t\, F\left(\frac{\sigma+k}{2}+n-1, \frac{k-\sigma}{2}; n-1; -\sinh^2 t\right) \qquad (1)$$

of the representations $T^{n\sigma k}$ of the group $U(n-1,1)$ in other coordinates. Let

$$x = \cosh t\cos\varphi, \quad y = \cosh t \sin\varphi, \quad 0 \le t < \infty, \quad 0 \le \varphi \le \frac{\pi}{2}. \qquad (2)$$

Points $(x,y)$ run all over the exterior $F^2$:

$$F^2 = \{(x,y) \mid x^2+y^2 \ge 1\}$$

of the disk. The measure $\sinh^{2n-3} t\cosh t dt\, d\varphi$ turns into the measure

$$(x^2+y^2-1)^{n-2}dx\, dy \qquad (3)$$

on $F^2$. The space $\mathfrak{L}^2_0(U(n-1,1))$ (see Section 11.4.8) becomes the Hilbert space $\mathfrak{L}^2(F^2)$ of functions on $F^2$ with the scalar product

$$(f_1,f_2) = \int_{F^2} f_1(x,y)\overline{f_2(x,y)}(x^2+y^2-1)^{n-2}dx\, dy. \qquad (4)$$

The transformation $(t,\varphi) \to (x,y)$ transforms functions (1) into the functions

$$f^n_{\sigma k}(x,y) = (x - iy\,\mathrm{sign}\, k)^k F\left(\frac{\sigma+k}{2}+n-1, \frac{k-\sigma}{2}; n-1; 1-x^2-y^2\right). \qquad (5)$$

Zonal spherical functions (11) and (11′) of Section 11.3.2 in the coordinates $x$ and $y$ turn into the polynomials

$$f^n_{\nu_1\nu_2}(x,y) = \frac{\nu_2!(n-2)!}{(\nu_2+n-2)!}(x-iy)^{\nu_1+\nu_2}P^{(n-2,\nu_1+\nu_2)}_{\nu_2}(2(x^2+y^2)-1), \qquad (6)$$

$$0 \le \nu_1 < \nu_2 - n + 1,$$

$$f^n_{\nu_1\nu_2}(x,y)=\frac{(-\nu_1)!(n-2)!}{(-\nu_1+n-2)!}(x+iy)^{-\nu_1-\nu_2}P^{(n-2,-\nu_1-\nu_2)}_{-\nu_1}(2(x^2+y^2)-1), \qquad (7)$$
$$0\ge\nu_2>\nu_1+n-1$$

(we have made use of the relation between Jacobi polynomials following from equality (5) of Section 11.3.2).

It follows from the results of Section 11.4.8 that the functions $f^n_{i\rho-n+1,k}(x,y)$, $0\le\rho<\infty$, and polynomials (6) and (7) form a complete system of functions in $\mathfrak{L}^2(F^2)$. Namely, for every function $F\in\mathfrak{L}^2(F^2)$, one has the expansion

$$\begin{aligned}F(x,y)=&\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}b_k(\rho)f^n_{-i\rho-n+1,k}(x,y)d\mu_k(\rho)\\&+\sum_{0\le\nu_1<\nu_2-n+1}b_{\nu_1\nu_2}b(\nu_1,\nu_2)f^n_{\nu_1\nu_2}(x,y)\\&+\sum_{0\ge\nu_2>\nu_1+n-1}b'_{\nu_1\nu_2}b(\nu_1,\nu_2)f^n_{\nu_1\nu_2}(x,y),\end{aligned}\qquad (8)$$

where $d\mu_k(\rho)$ is given by formula (13) of Section 11.4.8,

$$b_{\nu_1\nu_2}=\frac{(\nu_2-1)!^2(\nu_2-\nu_1-n+1)}{2(\nu_2-n+1)!^2(n-2)!^2},\qquad b'_{\nu_1\nu_2}=\frac{(-\nu_1-1)!^2(\nu_2-\nu_1-n+1)}{2(-\nu_1-n+1)!^2(n-2)!^2},$$

$$b_k(\rho)=\frac{2}{\pi}\int_{F^2}F(x,y)\overline{f^n_{-i\rho-n+1,k}(x,y)}(x^2+y^2-1)^{n-2}dx\,dy, \qquad (9)$$

$$b(\nu_1,\nu_2)=\frac{2}{\pi}\int_{F^2}F(x,y)\overline{f^n_{\nu_1\nu_2}(x,y)}(x^2+y^2-1)^{n-2}dx\,dy. \qquad (10)$$

For expansion (8) the Plancherel formula

$$\begin{aligned}\frac{2}{\pi}\int_{F^2}|F(x,y)|^2(x^2+y^2-1)^{n-2}dx\,dy=&\sum_{k=-\infty}^{\infty}\int_0^{\infty}|b_k(\rho)|^2d\mu_k(\rho)\\&+\sum_{0\le\nu_1<\nu_2-n+1}b_{\nu_1\nu_2}|b(\nu_1,\nu_2)|^2\\&+\sum_{0\ge\nu_2>\nu_1+n-1}b'_{\nu_1\nu_2}|b(\nu_1,\nu_2)|^2\end{aligned}\qquad (11)$$

holds.

**11.5.5. Integral representation and differential equations for $f^n_{\sigma k}$.** Formula (8) of Section 11.3.2 implies the integral representation for zonal spherical functions (1) of the preceding section:

$$t^{n\sigma k}_{00}(d_n(\varphi)g'_{n-1}(t)) = \frac{n-2}{\pi}\int_0^{2\pi}\int_0^{\pi/2}(e^{-i\varphi}\cosh t - e^{i\tau}\sinh t\cos\theta)^{\frac{\sigma+k}{2}}$$
$$\times (e^{i\varphi}\cosh t - e^{-i\tau}\sinh t\cos\theta)^{\frac{\sigma-k}{2}}\sin^{2n-5}\theta\cos\theta\, d\theta\, d\tau. \quad (1)$$

It gives the integral representation for $f^n_{\sigma k}(x,y)$:

$$f^n_{\sigma k}(x,y) = \frac{n-2}{\pi}\int_{B^2}\left[(x-iy)-(s+it)\sqrt{x^2+y^2-1}\right]^{\frac{\sigma+k}{2}}$$
$$\times \left[(x+iy)-(s-it)\sqrt{x^2+y^2-1}\right]^{\frac{\sigma-k}{2}}(1-s^2-t^2)^{n-3}ds\, dt. \quad (2)$$

The differential equations for $f^n_{\sigma k}(x,y)$ are derived from equation (12) of Section 11.3.4 and from the equation of the form (4) of Section 11.5.2:

$$\left[\frac{y^2+x^2(x^2+y^2-1)}{x^2+y^2}\frac{\partial^2}{\partial x^2} + \frac{x^2+y^2(x^2+y^2-1)}{x^2+y^2}\frac{\partial^2}{\partial y^2} - 2xy\frac{\partial^2}{\partial x\partial y}\right.$$
$$\left. -(2n-1)x\frac{\partial}{\partial x} - (2n-1)y\frac{\partial}{\partial y} - \sigma(\sigma+2n-2)\right] f^n_{\sigma k}(x,y) = 0, \quad (3)$$

$$\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) f^n_{\sigma k}(x,y) = -k f^n_{\sigma k}(x,y). \quad (4)$$

The function $f^n_{\sigma k}(x,y)$ is defined by equations (3) and (4) uniquely up to a constant.

In the same way as in the case of polynomials $p^n_{\ell\ell'}(x,y)$, we can derive the addition theorem and the product formula for $f^n_{\sigma k}(x,y)$.

Properties of the polynomials $f^n_{\nu_1\nu_2}(x,y)$ follow from those of the functions $f^n_{\sigma k}(x,y)$ if one takes into account that $f^n_{\nu_1\nu_2}(x,y)$ coincide with $f^n_{\sigma k}(x,y)$ if $\nu_1 = (\sigma+k)/2$, $\nu_2 = (k-\sigma)/2$.

## 11.6. Matrix Elements of Class 1 Representations

**11.6.1. Integral representation for matrix elements of the operators $T^{n\sigma k}(g'_{n-1}(t))$.** The following matrix elements of the operators $T^{n\sigma k}(g'_{n-1}(t))$ of representations of the group $U(n-1,1)$ differ from zero:

$$t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t)) = \left(T^{n\sigma k}(g'_{n-1}(t))\Xi_P^{n-1,pp'}, \Xi_P^{n-1,mm'}\right), \quad (1)$$

where $P=(jj',P')$ (see Section 11.3.1). These matrix elements do not depend on $P'$. Therefore, we can assume that $P'=O$. Substitute into (1) the expressions

$$\Xi_P^{n-1,mm'}(\varphi,\theta)=\left[\dim T^{n-1,mm'}\right]^{1/2} e^{-i(m'-m+j-j')\varphi} \times t^{n-1,mm'}_{(jj')0}(g_{n-1}(\theta))t^{n-2,jj'}_{00}(g^{n-2}(\varphi',\theta')) \quad (1')$$

for the basis functions $\Xi_P^{n-1,mm'}$ (see Section 11.3.1), take into account expression (1) of Section 11.1.8 for the measure $d\xi$ on $S_C^{n-2}$ and formula (13) of Section 11.2.4 for $T^{n\sigma k}(g'_{n-1}(t))$. After integration with respect to $\varphi'=(\varphi_1,\ldots,\varphi_{n-2})$ and $\theta'=(\theta_1,\ldots,\theta_{n-3})$, we obtain the integral representation for matrix elements (1):

$$\begin{aligned} t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t)) &= D^n_{mm'pp'jj'}\int_0^{2\pi}\int_0^{\pi/2}(\cosh t-e^{i\varphi}\cos\theta\sinh t)^{\frac{\sigma+k-p+p'}{2}} \\ &\times(\cosh t-e^{-i\varphi}\cos\theta\sinh t)^{\frac{\sigma-k-p'+p}{2}}(e^{i\varphi}\cos\theta\cosh t-\sinh t)^{\frac{p-p'-j+j'}{2}} \\ &\times(e^{-i\varphi}\cos\theta\cosh t-\sinh t)^{\frac{p'-p-j'+j}{2}}e^{i(m'-m-j'+j)\varphi} \\ &\times t^{n-1,pp'}_{(jj')0}(g_{n-2}(\theta'))t^{n-1,mm'}_{(jj')0}(g_{n-2}(\theta))\sin^{2n-5}\theta\cosh\theta\,d\theta\,d\varphi, \end{aligned} \quad (2)$$

where $\cos\theta'$ and $\sin\theta'$ are defined by formulas (9) and (10) of Section 11.2.4 and

$$D^n_{mm'pp'jj'} = \frac{[(m+1)_{n-3}(m'+1)_{n-3}(p+1)_{n-3}(p'+1)_{n-3}(m+m'+n-2)(p+p'+n-2)]^{1/2}}{\pi(j+1)_{n-4}(j'+1)_{n-4}(n-3)(j+j'+n-3)} \quad (3)$$

**11.6.2. Infinitesimal operators of the representations $T^{n\sigma k}$.** An analog of relation (1) of Section 9.5.14 for the representations $T^{n\sigma k}$ of the group $U(n-1,1)$ has the form

$$\frac{d}{dt}t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t))=\sum_{s,s'}t^{n\sigma k}_{(mm')(ss')(jj')}(g'_{n-1}(t))(I'_{n-1,n})_{(ss')(pp')(jj')}, \quad (1)$$

where the element $I'_{n-1,n}$ of the Lie algebra $\mathfrak{u}(n-1,1)$ is defined by formula (9) of Section 11.1.2. As $t^{n\sigma k}_{\dots}(g'_{n-1}(t))$ we take integral representation (2) of Section 11.6.1. Replace $t^{n-1,pp'}_{(jj')0}(g_{n-2}(\theta'))$ in the integrand function by the second expression of formula (7) of Section 11.3.5 and differentiate with respect to $t$ under the integral sign. As a result, we obtain two summands under the integral sign. The first one contains $P^{(\alpha,\beta)}_{p-j}(\cos 2\theta')$ and the second one contains $\frac{d}{d\theta'}P^{(\alpha,\beta)}_{p-j}(\cos 2\theta')$, where

$\alpha = j + j' + n - 3$, $\beta = p' - p - j' + 1$ and $\theta'$ is given by formulas (9) and (10) of Section 11.2.4. Applying recurrence relation (5) of Section 6.7.4 to the polynomial $P_{p-j}^{(\alpha,\beta)}(\cos 2\theta')$, we express it in terms of $P_{p-j}^{(\alpha,\beta-1)}(\cos 2\theta')$ and $P_{p-j+1}^{(\alpha,\beta-1)}(\cos 2\theta')$. By means of formula (2) of Section 6.3.8, we express $\frac{d}{d\theta'} P_{p-j}^{(\alpha,\beta)}(\cos 2\theta')$ in terms of $P_{p-j-1}^{(\alpha+1,\beta+1)}(\cos 2\theta')$. Using recurrence formula (4) of Section 6.7.4, we express $P_{p-j-1}^{(\alpha+1,\beta+1)}(\cos 2\theta')$ in terms of $P_{p-j-1}^{(\alpha,\beta+1)}(\cos 2\theta')$ and $P_{p-j}^{(\alpha,\beta+1)}(\cos 2\theta')$. As a result, the integrand function contains four summands. Now with the help of formula (7) of Section 11.3.5, we express the Jacobi polynomials in terms of

$$t_{(jj')0}^{n-1,p,p'-1}(\dots), t_{(jj')0}^{n-1,p+1,p'}(\dots), t_{(jj')0}^{n-1,p-1,p'}(\dots), t_{(jj')0}^{n-1,p,p'+1}(\dots),$$

respectively. Finally, we obtain the explicit form of formula (1) which gives expressions for matrix elements of the infinitesimal operator $I'_{n-1,n}$ in the representation $T^{n\sigma k}$. In the basis $\{\Xi_M^{n-1,mm'}\}$ of the space $\mathfrak{L}^2(S_C^{n-2})$ the operator $I'_{n-1,n} \equiv I_{n-1,n}^{n\sigma k}$ has the form

$$\begin{aligned} I_{n-1,n}^{n\sigma k} \Xi_M^{mm'} &= -\left(\frac{\sigma+k}{2} - m\right) \left[\frac{(m-j+1)(m+j'+n-2)}{(m+m'+n-2)(m+m'+n-1)}\right]^{1/2} \Xi_M^{m+1,m'} \\ &- \left(\frac{\sigma+k}{2} + m' + n - 2\right) \left[\frac{(m'+j+n-3)(m'-j')}{(m+m'+n-2)(m+m'+n-3)}\right]^{1/2} \Xi_M^{m,m'-1} \\ &+ \left(\frac{\sigma-k}{2} + m + n - 2\right) \left[\frac{(m-j)(m+j'+n-3)}{(m+m'+n-2)(m+m'+n-3)}\right]^{1/2} \Xi_M^{m-1,m'} \\ &+ \left(\frac{\sigma-k}{2} - m'\right) \left[\frac{(m'+j+n-2)(m'-j'+1)}{(m+m'+n-2)(m+m'+n-1)}\right]^{1/2} \Xi_M^{m,m'+1} \end{aligned} \tag{2}$$

(we have omitted the index $n-1$ in $\Xi_M^{n-1,mm'}$).

It follows from formula (14) of Section 11.2.4 that to the element $iE_{nn}$ of the Lie algebra $\mathfrak{u}(n-1,1)$ there corresponds the operator $iE_{nn}^{n\sigma k}$ acting upon the basis functions $\Xi_M^{n-1,mm'}$ in accordance to the formula

$$iE_{nn}^{n\sigma k} \Xi_M^{mm'} = i(m - m' - k)\Xi_M^{mm'}. \tag{3}$$

Since for the elements $iE_{nn}$, $I'_{n-1,n} = E_{n-1,n} + E_{n,n-1}$, $J'_{n-1,n} = i(E_{n-1,n} - E_{n,n-1})$ of $\mathfrak{u}(n-1,1)$ we have

$$[I'_{n-1,n}, iE_{nn}] = J'_{n-1,n},$$

then in the representation $T^{n\sigma k}$ to the element $J'_{n-1,1} \in \mathfrak{u}(n-1,1)$ there corresponds the operator $J_{n-1,n}^{n\sigma k} = [I_{n-1,n}^{n\sigma k}, iE_{nn}^{n\sigma k}]$. Therefore, $J_{n-1,n}^{n\sigma k}$ acts upon $\Xi_M^{mm'}$ according to the formula which is obtained from (2) by multiplication of the first and the second summands by $-i$, and the third and the fourth summands by $i$. For the operators

$$E^{n\sigma k}_{n-1,n} = \frac{1}{2}\left(I^{n\sigma k}_{n-1,n} - iJ^{n\sigma k}_{n-1,n}\right), \quad E^{n\sigma k}_{n,n-1} = \frac{1}{2}\left(I^{n\sigma k}_{n-1,n} + iJ^{n\sigma k}_{n-1,n}\right),$$

corresponding to the matrices $E_{n-1,n}$ and $E_{n,n-1}$ (see Section 11.1.2), we have

$$E^{n\sigma k}_{n,n-1}\Xi^{mm'}_{M} = -\left(\frac{\sigma+k}{2} - m\right)\left[\frac{(m-j+1)(m+j'+n-2)}{(m+m'+n-2)(m+m'+n-1)}\right]^{1/2}\Xi^{m+1,m'}_{M}$$
$$-\left(\frac{\sigma+k}{2} + m' + n - 2\right)\left[\frac{(m'+j+n-3)(m'-j')}{(m+m'+n-2)(m+m'+n-3)}\right]^{1/2}\Xi^{m,m'-1}_{M}, \quad (4)$$

$$E^{n\sigma k}_{n-1,n}\Xi^{mm'}_{M}$$
$$= -\left(\frac{\sigma-k}{2} + m + n - 2\right)\left[\frac{(m-j)(m+j'+n-3)}{(m+m'+n-2)(m+m'+n-3)}\right]^{1/2}\Xi^{m-1,m'}_{M}$$
$$-\left(\frac{\sigma-k}{2} - m'\right)\left[\frac{(m'+j+n-2)(m'-j'+1)}{(m+m'+n-2)(m+m'+n-1)}\right]^{1/2}\Xi^{m,m'+1}_{M}. \quad (5)$$

The operators

$$E^{n\sigma k}_{jn} = \frac{1}{2}\left(I^{n\sigma k}_{jn} - iJ^{n\sigma k}_{jn}\right), \quad E^{n\sigma k}_{nj} = \frac{1}{2}\left(I^{n\sigma k}_{jn} + iJ^{n\sigma k}_{jn}\right), \quad (6)$$
$$j = 1, 2, \ldots, n-2,$$

are obtained from $E^{n\sigma k}_{n-1,n}$ and $E^{n\sigma k}_{n,n-1}$ by commutation with the operators $E^{n\sigma k}_{ij}$, $1 \le i, j \le n-1$, corresponding to basis elements (11) of Section 11.1.2:

$$\left[E^{n\sigma k}_{j,n-1}, E^{n\sigma k}_{n-1,n}\right] = E^{n\sigma k}_{jn}, \quad \left[E^{n\sigma k}_{n,n-1}, E^{n\sigma k}_{n-1,j}\right] = E^{n\sigma k}_{nj}. \quad (7)$$

Therefore, the action of the operators $E^{n\sigma k}_{jn}$ upon $\Xi^{mm'}_{M}$ increases $m$ by the unit or decreases $m'$ by the unit. The operators $E^{n\sigma k}_{nj}$ decrease $m$ or increase $m'$.

**11.6.3. Irreducibility of the representations $T^{n\sigma k}$ for non-integral $\sigma$.** The restriction of the representation $T^{n\sigma k}$ of $U(n-1,1)$ onto the subgroup $U(n-1)$ coincides with the left quasi-regular representation of $U(n-1)$ in $\mathfrak{L}^2(S^{n-2}_{C})$. It decomposes into the orthogonal sum of all irreducible representations $T^{n-1,mm'}$ of $U(n-1)$ and each of these representations appears in the decomposition only once. The representation $T^{n-1,mm'}$ acts in the subspace $\widetilde{\mathfrak{H}}^{n-1,mm'}_{C}$, spanned by the basis functions $\Xi^{n-1,mm'}_{M}$ with given $m$ and $m'$. By using these facts, as in the case of the representations $T^{n\sigma}$ of the group $SO_0(n-1,1)$ (see Section 9.2.6), one proves the following. If for any $m$ and $m'$, $m \geq 0$, $m' \geq 1$, the coefficients $\left(\frac{-\sigma-k}{2}+m\right)$ and $\left(-\frac{\sigma+k}{2}-m'-n+2\right)$ from formula (4) of Section 11.6.2 are non-zero and if for any $m$ and $m'$, $m \geq 1$, $m' \geq 0$, the coefficients $\left(\frac{\sigma-k}{2}+m+n-2\right)$ and $\left(\frac{\sigma-k}{2}-m'\right)$ from formula (5) of Section 11.6.2 are non-zero, then the representation $T^{n\sigma k}$ is irreducible.

It follows from this statement that the representation $T^{n\sigma k}$ of the group $U(n-1,1)$ is irreducible if $\sigma$ is either non-integral or if $\sigma$ is an integer such that

$$-n+2 < \frac{\sigma+k}{2} \leq 0 \quad \text{and} \quad 0 \leq \frac{-\sigma+k}{2} < n-2.$$

By making use of the formulas for infinitesimal operators of Section 11.6.2, one can easily show that, for other values of $\sigma$, the representations are reducible.

As it was shown in Section 11.2.6, for the irreducible representations $T^{n\sigma k}$ and $T^{n,-\sigma-2n+2,k}$ of the group $U(n-1,1)$, there exists the non-zero intertwining operator $Q^{\sigma k}$:

$$Q^{\sigma k}T^{n\sigma k}(g) = T^{n,-\sigma-2n+2,k}(g)Q^{\sigma k}, \quad g \in U(n-1,1). \tag{1}$$

As in the case of representations of $SO_0(n-1,1)$ (see Section 9.2.7), one proves that, on every subspace $\widetilde{\mathfrak{H}}_C^{n-1,mm'}$ of the space $\mathfrak{L}^2(S_C^{n-2})$, the operator $Q^{\sigma k}$ is a multiple of the identity operator:

$$Q^{\sigma k}\Xi_M^{n-1,mm'} = q^{\sigma k}(m,m')\Xi_M^{n-1,mm'}. \tag{2}$$

It follows from (1) that

$$Q^{\sigma k}E_{n,n-1}^{n\sigma k} = E_{n,n-1}^{n,-\sigma-2n+2,k}Q^{\sigma k}. \tag{3}$$

We apply both sides of this equality to the basis function $\Xi_M^{n-1,mm'}$ and equate coefficients at the functions $\Xi_M^{n-1,m,m'-1}$ and $\Xi_M^{n-1,m+1,m'}$. We obtain the recurrence relations

$$-\left(\frac{\sigma+k}{2}-m\right)q^{\sigma k}(m+1,m') = \left(\frac{\sigma-k}{2}+m+n-1\right)q^{\sigma k}(m,m'),$$

$$-\left(\frac{\sigma+k}{2}+m'+n-2\right)q^{\sigma k}(m,m'-1) = \left(\frac{\sigma-k}{2}-m'+1\right)q^{\sigma k}(m,m')$$

which give the expression for $q^{\sigma k}(m,m')$ in terms of $q^{\sigma k}(0,0)$. For fixed $\sigma$ and $k$, the intertwining operator $Q^{\sigma k}$ is defined by equality (1) uniquely up to a constant factor. Up to a value independent of $m$ and $m'$, we have

$$q^{\sigma k}(m,m') = \frac{\Gamma\left(\frac{\sigma-k}{2}+m+n-1\right)\Gamma\left(\frac{\sigma-k}{2}-m'+1\right)}{\Gamma\left(\frac{-\sigma-k}{2}+m\right)\Gamma\left(\frac{-\sigma-k}{2}-m'-n+2\right)}. \tag{4}$$

**11.6.4. Infinitesimal operators of the representations $T^{n\ell\ell'}$ of the group $U(n)$.** Let us realize the representation $T^{n\ell\ell'}$ of $U(n)$ in the space $\mathfrak{D}^{n-1,\ell\ell'}$ (see Section 11.2.5) and equip this space with scalar product (1) of Section 11.3.4. As in the case of Section 11.6.1, by means of this scalar product, the integral representation for the matrix elements $t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta))$ of the representation $T^{n\ell\ell'}$ of the group $U(n)$ is derived. Formulas for the infinitesimal operators $E^{n\ell\ell'}_{n-1,n}$ and $E^{n\ell\ell'}_{n,n-1}$, corresponding to the elements

$$E_{n-1,n} = \frac{1}{2}\left(I_{n-1,n} - iJ_{n-1,n}\right), \quad E_{n,n-1} = \frac{1}{2}\left(I_{n-1,n} + iJ_{n-1,n}\right)$$

(see formula (3) of Section 11.1.2), are derived by means of the integral representation for $t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta))$ in the same way as in the case of the group $U(n-1,1)$. We obtain the following result. In the orthonormal basis

$$\tilde{\Xi}^{mm'}_M \equiv (-1)^{m+m'}[(\ell-m)!(\ell+m'+n-2)!(\ell'+m+n-2)!(\ell'-m')!]^{1/2}\Xi^{n-1,mm'}_M$$

of the space $\mathfrak{D}^{n-1,\ell\ell'}$, the operators $E^{n\ell\ell'}_{n-1,n}$ and $E^{n\ell\ell'}_{n,n-1}$ are given by the formulas

$$\begin{aligned} E^{n\ell\ell'}_{n,n-1}\tilde{\Xi}^{mm'}_M &= \left[\frac{(\ell-m)(\ell'+m+n-1)(m-j+1)(m+j'+n-2)}{(m+m'+n-2)(m+m'+n-1)}\right]^{1/2}\tilde{\Xi}^{m+1,m'}_M \\ &+ \left[\frac{(\ell+m'+n-2)(\ell'-m'+1)(j+m'+n-3)(m'-j')}{(m+m'+n-2)(m+m'+n-3)}\right]^{1/2}\tilde{\Xi}^{m,m'-1}_M, \end{aligned} \qquad (1)$$

$$\begin{aligned} E^{n\ell\ell'}_{n-1,n}\tilde{\Xi}^{mm'}_M &= \left[\frac{(\ell-m+1)(\ell'+m+n-2)(m-j)(m+j'+n-3)}{(m+m'+n-2)(m+m'+n-3)}\right]^{1/2}\tilde{\Xi}^{m-1,m'}_M \\ &+ \left[\frac{(\ell'-m')(\ell+m'+n-1)(m'+j+n-2)(m'-j'+1)}{(m+m'+n-2)(m+m'+n-1)}\right]^{1/2}\tilde{\Xi}^{m,m'+1}_M, \end{aligned} \qquad (2)$$

wehre $M = (jj', \dots)$. For the infinitesimal operator $E^{n\ell\ell'}_{nn}$ we have

$$E^{n\ell\ell'}_{nn}\tilde{\Xi}^{mm'}_M = (\ell'-\ell+m-m')\tilde{\Xi}^{mm'}_M. \qquad (2')$$

**11.6.5. Infinitesimal operators of the representations $T^{nkR}$ of the group $JU(n-1)$.** The tangent vector to the one-parameter subgroup of elements $g(e,0,\mathbf{x}_j)$, where $\mathbf{x}_j = (0,\dots,0,a,0,\dots,0)$ ($a$ is on $j$-th position), will be denoted by $E_j$ if $a \in \mathbb{R}$ and by $E^j$ if $a = ib$, $b \in \mathbb{R}$. To the elements $E_j$ and $E^j$ of the Lie algebra $\mathrm{iu}(n-1)$ of the group $JU(n-1)$ in the representation $T^{nkR}$ there correspond the operators $E_j^R$ and $\bar{E}_j^R$ which are given as

$$(E_j^R f)(\boldsymbol{\xi}) = \frac{d}{da}\left[(T^{nkR}(g(e,0,\mathbf{x}_j))f)(\boldsymbol{\xi})\right]_{a=0}, \quad a \in \mathbf{R},$$
$$(\bar{E}_j^R f)(\boldsymbol{\xi}) = \frac{d}{db}\left[(T^{nkR}(g(e,0,\mathbf{x}_j))f)(\boldsymbol{\xi})\right]_{b=0}, \quad a = ib.$$

It follows from formulas (5) and (6) of Section 11.2.6 that

$$(E_{n-1}^R F)(\varphi,\boldsymbol{\theta}) = R\cos\theta\cos\varphi F(\varphi,\boldsymbol{\theta}), \tag{1}$$
$$(\bar{E}_{n-1}^R F)(\varphi,\boldsymbol{\theta}) = R\cos\theta\sin\varphi F(\varphi,\boldsymbol{\theta}). \tag{2}$$

Instead of $E_j^R$ and $\bar{E}_j^R$ it is convenient to consider the operators

$$F_j^R \equiv E_j^R - i\bar{E}_j^R, \qquad F^{j,R} \equiv E_j^R + i\bar{E}_j^R. \tag{3}$$

Then

$$(F_{n-1}^R)(\varphi,\boldsymbol{\theta}) = R\cos\theta e^{-i\varphi}F(\varphi,\boldsymbol{\theta}), \tag{4}$$
$$(F^{n-1,R}F)(\varphi,\boldsymbol{\theta}) = R\cos\theta e^{i\varphi}(\varphi,\boldsymbol{\theta}). \tag{5}$$

We represent the basis function $\Xi_M^{n-1,mm'}$ of the space $\mathfrak{L}^2(S_C^{n-2})$ in the form (1′) of Section 11.6.1. Taking into account the first expression of formula (7) of Section 11.3.5 for $t_{(jj')0}^{n-1,mm'}(g_{n-2}(\theta))$ and recurrence relation (8) of Section 6.7.4 for Jacobi polynomials, we find that

$$\begin{aligned} &e^{-i\varphi}\cos\theta\left[\dim T^{n-1,mm'}\right]^{1/2} t_{(pp')0}^{n-1,mm'}(d_{n-1}(\varphi)g_{n-2}(\theta)) \\ &= \left[\frac{(m-p+1)(m+p'+n-2)}{(m+m'+n-2)(m+m'+n-2)}\right]^{1/2}\left[\dim T^{n-1,m+1,m'}\right]^{1/2} \\ &\qquad\times t_{(pp')0}^{n-1,m+1,m'}(d_{n-1}(\varphi)g_{n-2}(\theta)) \\ &+ \left[\frac{(p+m'+n-3)(m'-p')}{(m+m'+n-2)(m+m'+n-3)}\right]^{1/2}\left[\dim T^{n-1,m,m'-1}\right]^{1/2} \\ &\qquad\times t_{(pp')0}^{n-1,m,m'-1}(d_{n-1}(\varphi)g_{n-2}(\theta)) \quad . \end{aligned} \tag{6}$$

It follows from (5) and (6) that in the basis $\left\{\Xi_M^{n-1,mm'} \equiv \Xi_{(pp',M')}^{n-1,mm'}\right\}$ the operator $F^{n-1,R}$ is of the form

$$\begin{aligned} F^{n-1,R}\Xi_M^{n-1,mm'} &= R\left[\frac{(m-p+1)(m+p'+n-2)}{(m+m'+n-2)(m+m'+n-1)}\right]^{1/2}\Xi_M^{n-1,m+1,m'} \\ &+ R\left[\frac{(m'-p')(p+m'+n-3)}{(m+m'+n-2)(m+m'+n-3)}\right]^{1/2}\Xi_M^{n-1,m,m'-1}. \end{aligned} \tag{7}$$

In the same way, we find that

$$F^R_{n-1}\Xi_M^{n-1,mm'} = R\left[\frac{(m-p)(m+p'+n-3)}{(m+m'+n-2)(m+m'+n-3)}\right]^{1/2}\Xi_M^{n-1,m-1,m'} + R\left[\frac{(m'-p'+1)(m'+p+n-2)}{(m+m'+n-2)(m+m'+n-1)}\right]^{1/2}\Xi_M^{n-1,m,m'+1}. \quad (8)$$

**11.6.6. Matrix elements of finite dimensional representations of the group $GL(n,\mathbb{C})$.** Let $\mathfrak{gl}(n,\mathbb{C})$ be the Lie algebra of the group $GL(n,\mathbb{C})$. The matrices $E_{ij}$, $1 \le i,j \le n$, form a basis of $\mathfrak{gl}(n,\mathbb{C})$. To the element $E_{ij} \in \mathfrak{gl}(n,\mathbb{C})$ there corresponds the one-parameter subgroup $\exp tE_{ij}$ in $GL(n,\mathbb{C})$. It is obvious that

$$\exp tE_{ii} = (\exp t)E_{ii} + (I - E_{ii}), \quad (1)$$

$$\exp tE_{ij} = I + tE_{ij}, \quad i \ne j, \quad (2)$$

where $I$ is the identity matrix.

Let us consider the finite dimensional irreducible representations $T^{n\ell\ell'}$ of the group $U(n)$. Since $GL(n,\mathbb{C})$ is the complexification of $U(n)$ (see Section 2.2.6), then to $T^{n\ell\ell'}$ there corresponds a finite dimensional irreducible representation of $GL(n,\mathbb{C})$. It will be also denoted by $T^{n\ell\ell'}$.

To the element $E_{ij}$ of $\mathfrak{gl}(n,\mathbb{C})$ there corresponds in the representation $T^{n\ell\ell'}$ the operator $T^{n\ell\ell'}(E_{ij}) = E_{ij}^{n\ell\ell'}$. The operators $E_{ij}^{n\ell\ell'}$ are connected with the representation operators for $GL(n,\mathbb{C})$ as

$$T^{n\ell\ell'}(\exp tE_{ij}) = \exp tE_{ij}^{n\ell\ell'} \quad (3)$$

(see Section 2.1.5). It follows from formula (2′) of Section 11.6.4 that

$$T^{n\ell\ell'}(\exp tE_{nn})\tilde{\Xi}_M^{mm'} = e^{(\ell'-\ell+m-m')t}\tilde{\Xi}_M^{mm'}. \quad (4)$$

In Section 11.6.4 we have found the operators $E_{n-1,n}^{n\ell\ell'}$ and $E_{n,n-1}^{n\ell\ell'}$. We use these operators to evaluate the matrices of the operators $T^{n\ell\ell'}(\exp tE_{n-1,n})$ and $T^{n\ell\ell'}(\exp tE_{n,n-1})$.

According to (3), we have

$$T^{n\ell\ell'}(\exp tE_{n-1,n}) = I + tE_{n-1,n}^{n\ell\ell'} + \frac{t^2}{2!}(E_{n-1,n}^{n\ell\ell'})^2 + \dots\ . \quad (5)$$

It follows from formula (2) of Section 11.6.4 that $E_{n-1,n}^{n\ell\ell'}$ transforms the basis function $\tilde{\Xi}_M^{mm'}$ into a linear combination of the basis functions $\tilde{\Xi}_M^{pp'}$ for which

$(m-m')-(p-p')=1$. Therefore, the operator $(E^{n\ell\ell'}_{n-1,n})^k$ transfers $\widetilde{\Xi}^{mm'}_M$ into a linear combination of the basis functions $\widetilde{\Xi}^{pp'}_M$ for which $(m-m')-(p-p')=k$. Consequently, we conclude from (5) that if $p\le m$, $p'\ge m'$ and $(m-m')-(p-p')=k$, then

$$\left(T^{n\ell\ell'}(\exp tE_{n-1,n})\widetilde{\Xi}^{mm'}_M,\widetilde{\Xi}^{pp'}_M\right)=\frac{t^k}{k!}\left(\left(E^{n\ell\ell'}_{n-1,n}\right)^k\widetilde{\Xi}^{mm'}_M,\widetilde{\Xi}^{pp'}_M\right). \tag{6}$$

Let us show that if $p\le m$, $p'\ge m'$, $(m-m')-(p-p')=k$, then

$$\begin{aligned}&\left(T^{n\ell\ell'}(\exp tE_{n-1,n})\,\widetilde{\Xi}^{mm'}_M,\widetilde{\Xi}^{pp'}_M\right)\\&\quad=\frac{t^k}{k!}\left[\frac{(p+m'+n-3)!^2(p'-j')!(m-j)!(p'+j+n-3)!(\ell-p)!}{((m-p)!(p'-m')!(m+p'+n-2)!)^2(m'-j')!(p-j)!}\right.\\&\times\frac{(\ell'-m')!(m+j'+n-3)!(\ell+p'+n-2)!}{(m'+j+n-3)!(p-j'+n-3)!(\ell+m'+n-2)!}\\&\qquad\left.\times\frac{(\ell'+m+n-2)!(m+m'+n-2)(p+p'+n-2)}{(\ell'+p+n-2)!(\ell-m)!(\ell'-p')!}\right],\end{aligned} \tag{7}$$

where $M=(jj',\dots)$. We use the induction method. By means of (6), one can easily verify that (7) is valid for $k=1$. Assume that (7) is valid for all $k$ less than $r$. We have

$$\begin{aligned}\frac{t^r}{r!}\left(\left(E^{n\ell\ell'}_{n-1,n}\right)^r\widetilde{\Xi}^{mm'}_M,\widetilde{\Xi}^{pp'}_M\right)&=\frac{t^r}{r!}\left(\left(E^{n\ell\ell'}_{n-1,n}\right)^{r-1}E^{n\ell\ell'}_{n-1,n}\widetilde{\Xi}^{mm'}_M,\widetilde{\Xi}^{pp'}_M\right)\\&=\frac{a_1t^r}{r!}\left(\left(E^{n\ell\ell'}_{n-1,n}\right)^{r-1}\widetilde{\Xi}^{m-1,m}_M,\widetilde{\Xi}^{pp'}_M\right)+\frac{a_2t^r}{r!}\left(\left(E^{n\ell\ell'}_{n-1,n}\right)^{r-1}\widetilde{\Xi}^{m,m'+1}_M,\widetilde{\Xi}^{pp'}_M\right),\end{aligned}$$

where $a_1$ and $a_2$ are the coefficients at $\widetilde{\Xi}^{m-1,m'}_M$ and $\widetilde{\Xi}^{m,m'+1}_M$ in formula (2) of Section 11.6.4. Substituting expressions for matrix elements of the operators $(E^{n\ell\ell'}_{n-1,n})^{r-1}$, following from (7), after simplifications we obtain formula (7) for $k=r$. Therefore, (7) is proved.

In the same way, by means of formula (1) of Section 11.6.4 we prove that if $p\ge m$, $p'\le m'$, $(p-p')-(m-m')=k$, then

$$\begin{aligned}&\left(T^{n\ell\ell'}(\exp tE_{n,n-1})\widetilde{\Xi}^{mm'}_M,\widetilde{\Xi}^{pp'}_M\right)\\&\quad=\frac{t^k}{k!}\left[\frac{(m+p'+n-3)!^2(p-j)!(m'-j')!(\ell-m)!(p+j'+n-3)!}{((p-m)!(m'-p')!(p+m'+n-2)!)^2(m+j'+n-3)!}\right.\\&\times\frac{(\ell'-p')!(m'+j+n-3)!(\ell+m'+n-2)!}{(\ell'-m')!(p'+j+n-3)!(\ell+p'+n-2)!}\\&\qquad\left.\times\frac{(\ell'-p+n-2)!(m+m'+n-2)(p+p'+n-2)}{(\ell'+m+n-2)!(m-j)!(p'-j')!(\ell-p)!}\right]^{1/2},\end{aligned} \tag{8}$$

where $M = (jj', \dots)$.

**11.6.7. Matrix elements of the representations $T^{n\ell\ell'}$ of the group $U(n)$.** Let us evaluate matrix elements of the operator $T^{n\ell\ell'}(g_{n-1}(\theta))$. Considering the matrix $g_{n-1}(\theta)$ as an element of the group $GL(n, \mathbf{C})$, we can represent it in the form

$$g_{n-1}(\theta) = [\exp((-\tan\theta)E_{n-1,n})]d_{n-1}(\cos^{-1}\theta)d_n(\cos\theta)[\exp((\tan\theta)E_{n,n-1})], \quad (1)$$

and in the form

$$g_{n-1}(\theta) = [\exp((\tan\theta)E_{n,n-1})]d_{n-1}(\cos\theta)d_n(\cos^{-1}\theta)[\exp((-\tan\theta)E_{n-1,n})], \quad (2)$$

where we recall that $d_i(a)$ is the diagonal matrix which differs from the identity matrix only in one element, namely, the $i$-th entry of the main diagonal of $d_i(a)$ is equal to $a$. From (1) we have

$$t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta)) = \sum_{r'=j'}^{\min(m',p')} \sum_{r=\max(m,p)}^{\ell} (\cos\theta)^{\ell'-\ell-j+j'+24-2r'}$$
$$\times \left(T^{n\ell\ell'}(\exp(-\tan\theta)E_{n-1,n})\tilde{\Xi}_M^{rr'}, \tilde{\Xi}_M^{mm'}\right)\left(T^{n\ell\ell'}(\exp(\tan\theta)E_{n,n-1})\tilde{\Xi}_M^{pp'}, \tilde{\Xi}_M^{rr'}\right).$$

By making use of formulas (7) and (8) of Section 11.6.6, we obtain

$$t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta)) = A_{mm'}A_{pp'}(\tan\theta)^a(\cos\theta)^{a'} \sum_{r=\max(m,p)}^{\ell} \sum_{r'=j'}^{\min(m',p')} (-1)^{p+p'}$$
$$\times \frac{(r+r'+n-2)(p+r'+n-3)!(m+r'+n-3)!(\ell'-r')!}{(r+p'+n-2)!(r+m'+n-2)!(r-p)!(r-m)!(p'-r')!}$$
$$\times \frac{(r-j')!(\ell'+r+n-2)!(r+j'+n-3)!(-\sin^2\theta)^{r-r'}}{(m'-r')!(\ell-r)!(r'-j')!(\ell+r'+n-2)!(j+r'+n-3)!}, \quad (3)$$

where $a = m' - m + p' - p$, $a' = j' - j + \ell' - \ell$,

$$A_{kk'} = \left[\frac{(k+k'+n-2)(\ell-k)!(\ell+k'+n-2)!(j+k'+n-3)!(k'-j')!}{(\ell'-k')!(k+\ell'+n-2)!(k+j'+n-3)!(k-j)!}\right]^{1/2}. \quad (4)$$

If formula (2) is used instead of (1), then in the same way we obtain

$$t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta)) = B_{mm'}B_{pp'}(\tan\theta)^b(\cos\theta)^{b'} \sum_{r=j}^{\min(m,p)} \sum_{r'=\max(m',p')}^{\ell} (-1)^{m+m'}$$
$$\times \frac{(r+r'+n-2)(r+p'+n-3)!(r+m'+n-3)!(\ell-r)!}{(p+r'+n-2)!(m+r'+n-2)!(p-r)!(m-r)!(r'-p')!}$$
$$\times \frac{(\ell+r'+n-2)!(j+r'+n-3)!(r'-j')!(-\sin^2\theta)^{r-r'}}{(r'-m')!(\ell'-r')!(r-j)!(\ell'+r+n-2)!(r+j'+n-3)!}, \quad (5)$$

where $b = m - m' + p - p'$, $b' = j - j' + \ell - \ell'$,

$$B_{kk'} = \left[\frac{(k+k'+n-2)(\ell'-k')!(k+\ell'+n-2)!(k+j'+n-3)!(k-j)!}{(\ell-k)!(\ell+k'+n-2)!(j+k'+n-3)!(k'-j')!}\right]^{1/2}. \quad (6)$$

The matrix elements $t^{n\ell\ell'}_{\cdots}(g_{n-1}(\theta))$ can be represented in terms of the generalized hypergeometric function ${}_5F_4(\dots)$. Replace in (3) the summation with respect to $r$ by the summation with respect to $s = r - \max(m,p)$. Then the sum with respect to $s$ is expressed in terms of ${}_5F_4(\dots)$. For $p \geq m$ we have

$$t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta))$$
$$= A\sin^d\theta\cos^{d'}\theta \sum_{r'=j'}^{N} \frac{(-1)^{p'+r'}(\ell'-r')!(m+r'+n-3)!(p+r'+n-2)!\sin^{-2r'}\theta}{(p'-r')!(m'-r')!(r'-j')!(\ell+r'+n-2)!(j+r'+n-3)!}$$
$$\times {}_5F_4\left(\begin{matrix} p-\ell, p+r'+n-1, p-j+1, \ell'+p+n-1, p+j'+n-2 \\ p+r'+n-2, p-m+1, p+p'+n-1, p+m'+n-1 \end{matrix}\middle| \sin^2\theta\right), \quad (7)$$

where $d = p + p' - m + m'$, $d' = m - m' + p - p' - \ell + \ell' - j + j'$, $N = \min(m', p')$,

$$A = [(m+m'+n-2)(p+p'+n-1)_{\ell-p}(m+\ell'+n-1)_{p-m}(m+j'+n-2)_{p-m}$$
$$\times (\ell-p+1)_{p-m}(m-j+1)_{p-m}(\ell+m'+n-2)!$$
$$\times (j+m'+n-3)!(p'-j')!(m'-j')!]^{1/2}$$
$$[(j+p'+n-2)_{p-j}(\ell'-p')!(\ell'-m')!(p'-m)!^2(p+m'+n-2)!^2]^{-1/2}. \quad (8)$$

For $m > p$ one has to permute $m$ and $p$ and to replace $(-1)^{r'+p'}$ by $(-1)^{r'+m+p+p'}$.

If in (3) one replaces the summation with respect to $r'$ by that with respect to $s' = r' - j'$, then the sum with respect to $s'$ is represented in terms of ${}_5F_4(\dots)$.

We obtain

$$t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta)) = B_{mm'}B_{pp'}(-1)^{j'+p+p'}\sin^t\theta\cos^{t'}\theta$$
$$\times\sum_{r=\max(p,m)}^{\ell}\frac{(-1)^r(r-j)!(r+\ell'+n-2)!(r+j'+n-2)!}{(r-p)!(r-m)!(r+p'+n-2)!(\ell-r)!(r+m'+n-2)!}\sin^{2r}\theta$$
$$\times {}_5F_4\left(\begin{matrix}-p'+j',-m'+j',p+j'+n-2,m+j'+n-2,r+j'+n-1\\ j'-\ell',\ell+j'+n-1,j+j'+n-2,r+j'+n-2\end{matrix}\,\middle|\,\sin^{-2}\theta\right) \quad (9)$$

where $t = m'-m+p'-p-2j$, $t' = m-m'+p-p'-j+j'-\ell+\ell'$,

$$B_{kk'} = \left[\frac{(k+k'+n-2)(\ell-k)!(\ell+k'+n-2)!(j+k'+n-3)!(k+j'+n-3)!(\ell'-j')!}{(\ell'-k')!(k+\ell'+n-2)!(k'-j')!(k-j)!(\ell+j'+n-2)!(j+j'+n-3)!}\right]^{1/2}$$

Similar expressions for $t^{n\ell\ell'}_{\dots}(g_{n-1}(\theta))$ can be obtained from (5).

**11.6.8. Matrix elements of the representations $T^{n\sigma k}$ of the group $U(n-1,1)$.** Matrix elements of the representations $T^{n\sigma k}$ of $U(n-1,1)$ are evaluated in the same way as for those for the representations $T^{n\ell\ell'}$ of the group $U(n)$. For this we need to carry out an analytic continuation of $T^{n\sigma k}$ to a representation of the complexification of the group $U(n-1,1)$, that is, to a representation of $GL(n,\mathbb{C})$. It is impossible to continue $T^{n\sigma k}$ to a global representation of $GL(n,\mathbb{C})$ because in this case we obtain a singular operator function. But it follows from the results of paper [138] that the representations $T^{n\sigma k}$ of $U(n-1,1)$ can be analytically continued to a local representation of $GL(n,\mathbb{C})$, that is, to the representation $\widehat{T}^{n\sigma k}$ given on a neighborhood $W$ of the identity element of $GL(n,\mathbb{C})$. This representation acts in some everywhere dense invariant linear subspace $\mathfrak{H}$ of $\mathfrak{L}^2(S_C^{n-2})$ and the basis functions $\Xi_M^{n-1,mm'}$ belong to $\mathfrak{H}$.[3] In addition, if $X$ is an element of the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ of the group $GL(n,\mathbb{C})$ such that $\exp X\in W$, then

$$\exp\widehat{T}^{n\sigma k}(X) = \widehat{T}^{n\sigma k}(\exp X).$$

Making use of formula (5) of Section 11.6.2, in the same way as in Section 11.6.6 we find that for $|t|<\varepsilon$, where $\varepsilon$ is a positive number, we have

$$\left(\left(\exp tE^{n\sigma k}_{n-1,n}\right)\Xi_M^{mm'},\Xi_M^{pp'}\right) = \frac{t^r}{r!}\left(\left(E^{n\sigma k}_{n-1,n}\right)^r\Xi_M^{mm'},\Xi_M^{pp'}\right)$$
$$= \frac{(-1)^{m+m'+p+p'}t^r(p+m'+n-3)!}{(m-p)!(p'-m')!(m+p'+n-2)!}\,\frac{\Gamma\left(\frac{\sigma-k}{2}-m'+1\right)\Gamma\left(\frac{\sigma-k}{2}+m+n-1\right)}{\Gamma\left(\frac{\sigma-k}{2}-p'+1\right)\Gamma\left(\frac{\sigma-k}{2}+p+n-1\right)}$$
$$\times\left[\frac{(j+p'+n-3)!(p'-j')!(m+j'+n-3)!(m-j)!(m+m'+n-2)(p+p'+n-2)}{(p+j'+n-3)!(j+m'+n-3)!(p-j)!(m'-j')!}\right]^{1/2}, \quad (1)$$

[3] Concerning the detailed description of applying the results of [138] to the representations of the group $U(n-1,1)$, see paper [189] or reference [24] of the first volume.

where $m \geq p$, $m' \leq p'$ and $(m-m')-(p-p') = r$. By means of formula (4) of Section 11.6.1, we derive that, for $m \leq p$, $m' \geq p'$, $(p-p')-(m-m') = r$ and for $|t| < \varepsilon$, where $\varepsilon$ is a positive number,

$$\left(\left(\exp tE^{n\sigma k}_{n,n-1}\right)\Xi^{mm'}_{M},\Xi^{pp'}_{M}\right) = \frac{t^r}{r!}\left(\left(E^{n\sigma k}_{n,n-1}\right)^r\Xi^{mm'}_{M},\Xi^{pp'}_{M}\right)$$
$$= \frac{(-1)^{m+m'+p+p'}t^r(p+m'+n-3)!}{(p-m)!(m'-p')!(p+m'+n-2)!}\frac{\Gamma\left(\frac{\sigma+k}{2}-m+1\right)\Gamma\left(\frac{\sigma+k}{2}+m'+n-1\right)}{\Gamma\left(\frac{\sigma+k}{2}-p+1\right)\Gamma\left(\frac{\sigma+k}{2}+p'+n-1\right)}$$
$$\times\left[\frac{(p+j'+n-3)!(p-j)!(j+m'+n-3)!(m'-j')!(m+m'+n-2)(p+p'+n-2)}{(j+p'+n-3)!(m+j'+n-3)!(p'-j')!(p-j)!}\right]^{1/2}. \quad (2)$$

The matrix $g'_{n-1}(t) \in U(n-1,1)$, considered as an element of $GL(n,\mathbf{C})$, is representable in the form

$$g'_{n-1}(t) = [\exp((\tanh t)E_{n,n-1})]d_{n-1}(\cosh t)d_n(\cosh^{-1}t)[\exp((\tanh t)E_{n-1,n})] \quad (3)$$

and in the form

$$g'_{n-1}(t)$$
$$= [\exp((\tanh t)E_{n-1,n})]d_{n-1}(\cosh^{-1}t)d_n(\cosh t)[\exp((\tanh t)E_{n,n-1})]. \quad (4)$$

Let us write down relation (4) for the matrices of the local representation $\widehat{T}^{n\sigma k}$ of the group $GL(n,\mathbf{C})$. By means of formulas (1) and (2), we find for small $t$ that

$$t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t)) = \frac{\Gamma\left(\frac{\sigma+k}{2}-p+1\right)\Gamma\left(\frac{\sigma+k}{2}+p'+n-1\right)}{\Gamma\left(\frac{\sigma-k}{2}+m+n-1\right)\Gamma\left(\frac{\sigma-k}{2}-m'+1\right)}$$
$$\times\left[\frac{(j+p'+n-3)!(j+m'+n-3)!(p'-j')!(m'-j')!(m+m'+n-2)(p-p'+n-2)}{(p+j'+n-3)!(m+j'+n-3)!(p-j)!(m-j)!}\right]^{1/2}$$
$$\times(\tanh t)^{p'-p+m-m'}(\cosh t)^{j'-j-k}\sum_{r=\max(m,p)}^{\infty}\sum_{r'=j'}^{\min(m'p')}\frac{\Gamma\left(\frac{\sigma-k}{2}-r'+1\right)}{\Gamma\left(\frac{\sigma+k}{2}+r'+n-1\right)}$$
$$\times\frac{\Gamma\left(\frac{\sigma-k}{2}+r+n-1\right)}{\Gamma\left(\frac{\sigma+k}{2}-r+1\right)}\frac{(r+j+n-3)!(r-j)!(m+r'+n-3)!}{(j+r'+n-3)!(r'-j')!(r-m)!(r+m'+n-2)!}$$
$$\times\frac{(p+r'+n-3)!(r+r'+n-2)(\sin^2 t)^{r-r'}}{(m'-r')!(r-p)!(r+p'+n-2)!(p'-r')!}. \quad (5)$$

After replacing the summation with respect to $r$ by the summation with respect to $s = r-\max(m,p)$, the sum is expressed in terms of ${}_5F_4(\dots)$, and for $m \geq p$ we have

$$t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t)) = A \sinh^{a} t \cosh^{a'} t$$
$$\times \sum_{r'=j'}^{\min(m',p')} \frac{\Gamma\left(\frac{\sigma-k}{2}-r'+1\right)(p+r'+n-3)!(m+r'+n-2)!}{\Gamma\left(\frac{\sigma+k}{2}+r'+n-1\right)(j+r'+n-3)!(r'-j')!(m'-r')!}\frac{\sinh^{-2r'} t}{(p'-r')!}$$
$$\times {}_5F_4\left(\begin{matrix} m-\frac{\sigma+k}{2}, \frac{\sigma-k}{2}+m+n-1, m+j'+n-2, m-j+1, m+r'+n-1 \\ m+m'+n-1, m-p+1, m+p'+n-1, m+r'+n-2 \end{matrix}\middle| -\sinh^2 t\right), \tag{6}$$

where $a = p' - p + m' + m$, $a' = p - p' + m - m' - j + j' - k$ and

$$A = \frac{\Gamma\left(\frac{\sigma+k}{2} - p + 1\right)\Gamma\left(\frac{\sigma+k}{2} + p' + n - 1\right)}{\Gamma\left(\frac{\sigma-k}{2} - m' + 1\right)\Gamma\left(\frac{\sigma+k}{2} - m + 1\right)}$$
$$\times \left[\frac{(p+p'+n-2)(m+j'+n-3)!(m-j)!(j+p'+n-3)!}{(m+m'+n-2)(m+m'+n-3)!^2(m-p)!^2}\right.$$
$$\left.\times \frac{(j+m'+n-3)!(p'-j')!(m'-j')!}{(m+p'+n-2)!^2(p-j)!(p+j'+n-3)!}\right]^{1/2}. \tag{7}$$

If $m < p$, one has to permute $m$ and $p$ and to replace

$$\frac{\Gamma\left(\frac{\sigma+k}{2} - p + 1\right)}{\Gamma\left(\frac{\sigma+k}{2} - m + 1\right)} \quad \text{by} \quad \frac{\Gamma\left(\frac{\sigma-k}{2} + p + n - 1\right)}{\Gamma\left(\frac{\sigma-k}{2} + m + n - 1\right)}. \tag{8}$$

Using relation (3) instead of (4), in the same way we obtain for $p' \geq m'$ that

$$t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t)) = B \sinh^{b} t \cosh^{b'} t$$
$$\times \sum_{r=j}^{\min(m,p)} \frac{\Gamma\left(\frac{\sigma+k}{2}-r+1\right)(r+p'+n-2)!(r+m'+n-3)!}{\Gamma\left(\frac{\sigma-k}{2}+r+n-1\right)(r+j'+n-3)!(r-j)!(p-r)!}\frac{\sinh^{-2r} t}{(m-r)!}$$
$$\times {}_5F_4\left(\begin{matrix} \frac{\sigma+k}{2}+p'+n-1, \frac{k-\sigma}{2}+p', j+p'+n-2, p'-j'+1, p'+r+n-1 \\ p+p'+n-1, m+p'+n-1, p'-m'+1, r+p'+n-2 \end{matrix}\middle| -\sinh^2 t\right) \tag{9}$$

where $b = p + p' + m - m'$, $b' = p' - p + m' - m + j - j' - k$ and

$$B = \frac{\Gamma\left(\frac{\sigma-k}{2} + p + n - 1\right)\Gamma\left(\frac{\sigma+k}{2} + p' + n - 1\right)}{\Gamma\left(\frac{\sigma+k}{2} - m + 1\right)\Gamma\left(\frac{\sigma+k}{2} + m' + n - 1\right)}$$
$$\times \left[\frac{(m+m'+n-2)(j+p'+n-3)!(p'-j')!(j'+p+n-3)!}{(p+p'+n-2)(p+p'+n-3)!^2(m+p'+n-2)!^2}\right.$$
$$\left.\times \frac{(m+j'+n-3)!(p-j)!(m-j)!}{(p'-m')!^2(j+m'+n-3)!(m'-j')!}\right]^{1/2}. \tag{10}$$

If $p' < m'$, then one has to permute $p'$ and $m'$ on the right hand side and to replace

$$\frac{\Gamma\left(\frac{\sigma+k}{2}+p'+n-1\right)}{\Gamma\left(\frac{\sigma+k}{2}+m'+n-1\right)} \quad \text{by} \quad \frac{\Gamma\left(\frac{\sigma-k}{2}-p'+1\right)}{\Gamma\left(\frac{\sigma-k}{2}-m'+1\right)}. \tag{11}$$

The formulas for $t^{n\sigma k}_{\dots}(g'_{n-1}(t))$ are derived for small values of $t$. But, as we see from integral representation (2) of Section 11.6.1, $t^{n\sigma k}_{\dots}(g'_{n-1}(t))$ as a function of $t$ is analytic in some neighborhood of the real line of values of $t$. Consequently, if we have the values of $t^{n\sigma k}_{\dots}(g'_{n-1}(t))$ in a neighborhood of some real point $t = t_0$, then by analytic continuation we obtain its values on the real line. Therefore, the obtained expressions for $t^{n\sigma k}_{\dots}(g'_{n-1}(t))$ are valid on the whole real line.

**11.6.9. Matrix elements of the representations $T^{nkR}$ of the group $JU(n-1)$.** The integral representation for matrix elements of the operators $T^{nkR}(g_r)$ is derived in the same way as formula (2) of Section 11.6.1. By using formula (5) of Section 11.2.6, we find

$$t^{nkR}_{(mm')(pp')(jj')}(g_r) = \frac{B}{\pi}\int_0^{2\pi}\int_0^{\pi/2} e^{Rr\cos\theta\cos\varphi} e^{i(p-p'-m+m')\varphi} t^{n-1,pp'}_{(jj')0}(g_{n-2}(\theta)) \\ \times t^{n-1,mm'}_{(jj')0}(g_{n-2}(\theta)) \sin^{2n-5}\theta\cos\theta\, d\theta\, d\varphi, \tag{1}$$

where

$$B = \frac{[(m+n-3)!(m'+n-3)!(m+m'+n-2)(p+n-3)!}{[m!m'!p!p'!]^{1/2}(j+n-4)!(j'+n-4)!} \\ \times \frac{(p'+n-3)!(p+p'+n-2)]^{1/2} j!j'!}{(j+j'+n-3)(n-3).} \tag{2}$$

Carrying out integration with respect to $\varphi$ in (1), we obtain

$$t^{nR}_{(mm')(pp')(jj')}(g_r) = 2Bi^{m'-m-p'+p}\int_0^{\pi/2} J_{m'-m+p-p'}(-iRr\cos\theta) \\ \times t^{n-1,pp'}_{(jj')0}(g_{n-2}(\theta)) t^{n-1,mm'}_{(jj')0}(g_{n-2}(\theta)) \sin^{2n-5}\theta\cos\theta\, d\theta, \tag{3}$$

where $B$ is given by (2).

In order to obtain the matrix elements $t^{nkR}_{(mm')(pp')(jj')}(g_r)$ in the form of series, one uses matrix elements of the operators $\exp \frac{r}{2} F^{n-1,R}$ and $\exp \frac{r}{2} F^{R}_{n-1}$. They are found in the same way as matrix elements of the operators $\exp tE^{n\ell\ell'}_{n-1,n}$ and

$\exp tE^{n\ell\ell'}_{n,n-1}$ (see Section 11.6.6). By means of formulas (7) and (8) of Section 11.6.5, we find that

$$\left(\left(\exp \frac{r}{2} F^{R}_{n-1}\right) \Xi^{mm'}_{M}, \Xi^{pp'}_{M}\right) = \frac{(Rr/2)^k (p+m'+n-3)!}{(m+p'+n-2)!(m-p)!(p'-m')!} \times \left[\frac{(m+m'+n-2)(p+p'+n-2)(j+p'+n-3)!}{(p+j'+n-3)!(p-j)!(j+m'+n-3)!(m'-j')!} \times (p'-j')!(m-j)!(m+j'+n-3)!\right]^{1/2} \quad (4)$$

if $m \geq p$, $m' \leq p'$, $(m-p)+(p'-m') = k$ and

$$\left(\left(\exp \frac{r}{2} F^{n-1,R}\right) \Xi^{mm'}_{M}, \Xi^{pp'}_{M}\right) = \frac{(Rr/2)^k (m+p'+n-3)!}{(p+m'+n-2)!(p-m)!(m'-p')!} \times \left[\frac{(m+m'+n-2)(p+p'+n-2)(p+j'+n-3)!}{(j+p'+n-3)!(m+j'+n-3)!(p'-j')!(m-j)!} \times (p-j)!(m'-j')!(j+m'+n-3)!\right]^{1/2} \quad (5)$$

if $m \leq p$, $m' \geq p'$, $(m'-p')+(p-m) = k$. Here $M = (jj', \dots)$.

Since $E^{R}_{n-1} = \frac{1}{2}(F^{R}_{n-1} + F^{n-1,R})$ (see formula (3) of Section 11.6.5), then

$$T^{nR}(g_r) = \exp rE^{R}_{n-1} = \left(\exp \frac{r}{2} F^{n-1,R}\right)\left(\exp \frac{r}{2} F^{R}_{n-1}\right) = \left(\exp \frac{r}{2} F^{R}_{n-1}\right)\left(\exp \frac{r}{2} F^{n-1,R}\right). \quad (6)$$

In the same way as in the case of the group $U(n)$ (see Section 11.6.7), using formulas (4) and (5) in the first of relations (6), we obtain

$$t^{nR}_{(mm')(pp')(jj')}(g_r) = N \sum_{s=j}^{\min(p,m)} \sum_{s'=\max(p',m')}^{\infty} \frac{(j+s'+n-3)!(s'-j')!(s+p'+n-3)!}{(s+j'+n-3)!(s-j)!(p-s)!(p+s'+n-2)!} \times \frac{(s+m'+n-3)!(s+s'+n-2)}{(s'-p')!(m-s)!(s'-m')!(m+s'+n-2)!} \left(\frac{Rr}{2}\right)^{p-p'+m-m'-2s+2s'}, \quad (7)$$

where

$$N = \left[\frac{(p+j'+n-3)!(m+j'+n-3)!(p-j)!(m-j)!}{(j+p'+n-3)!(j+m'+n-3)!(p'-j')!(m'-j')!} \times (m+m'+n-2)(p+p'+n-2)\right]^{1/2}.$$

Replace the summation with respect to $s'$ by that with respect to $r = s' - \max(p', m')$. Then the sum with respect to $r$ is expressed in terms of the hypergometric function ${}_3F_4$, and for $m' \leq p'$ we have

$$t^{nR}_{(mm')(pp')(jj')}(g_r) = N' \left(\frac{Rr}{2}\right)^a \sum_{s=j}^{\min(p,m)} \frac{(s+p'+n-2)!(s+m'+n-3)!(Rr/2)^{-2s}}{(s+j'+n-3)!(s-j)!(p-s)!(m-s)!}$$

$$\times {}_3F_4\left(\begin{matrix} j+p'+n-2, p'-j'+1, s+p'+n-1 \\ p+p'+n-1, m+p'+n-1, p'-m'+1, s+p'+n-2 \end{matrix} \middle| \left(\frac{Rr}{2}\right)^2\right), \quad (8)$$

where $a = p + p' + m - m'$,

$$N' = \left[\frac{(m+m'+n-2)(j+p'+n-3)!(p'-j')!(p+j'+n-3)!}{(p+p'+n-2)(p+p'+n-3)!^2(m+p'+n-2)!^2} \times \frac{(p-j)!(m-j)!(m+j'+n-3)!}{(p'-m')!^2(j+m'+n-3)!(m'-j')!}\right]^{1/2}.$$

If $p' < m'$, one has to permute $m'$ and $p'$.

Using the second expression of formula (6) for $\exp rE^R_{n-1}$ instead of the first one, we obtain for $m \geq p$ that

$$t^{nR}_{(mm')(pp')(jj')}(g_r) = N'' \left(\frac{Rr}{2}\right)^b \sum_{s'=j'}^{\min(m',p')} \frac{(m+s'+n-2)!(p+s'+n-3)!(Rr/2)^{-2s'}}{(s'+j+n-3)!(s'-j')!(m'-s')!(p'-s')!}$$

$$\times {}_3F_4\left(\begin{matrix} m+j'+n-2, m-j+1, m+s'+n-1 \\ m+m'+n-1, m-p+1, m+p'+n-1, m+s'+n-2 \end{matrix} \middle| \left(\frac{Rr}{2}\right)^2\right), \quad (9)$$

where $b = m + m' - p + p'$,

$$N'' = \left[\frac{(p+p'+n-2)(m+j'+n-3)!(m-j)!(j+p'+n-3)!}{(m+m'+n-2)(m+p'+n-2)!^2(m+m'+n-3)!^2} \times \frac{(j+m'+n-3)!(p'-j')!(m'-j')!}{(m-p)!^2(p+j'+n-3)!(m-j)!}\right]^{1/2}.$$

If $m < p$, one has to permut $m$ and $p$.

**11.6.10. Symmetry relations.** It follows from integral representation (2) of Section 11.6.1 for matrix elements of the operators $T^{n\sigma k}(g'_{n-1}(t))$ that

$$\overline{t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t))} = t^{n,\bar{\sigma},-k}_{(m'm)(p'p)(j'j)}(g'_{n-1}(t)). \quad (1)$$

From formula (5) of Section 11.6.8 we have

$$\overline{t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t))} = t^{n\bar{\sigma}k}_{(mm')(pp')(jj')}(g'_{n-1}(t)). \tag{2}$$

Consequently,

$$t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t)) = t^{n\sigma,-k}_{(m'm)(p'p)(j'j)}(g'_{n-1}(t)). \tag{3}$$

From integral representation (1) of Section 11.6.9 for matrix elements of the operators $T^{nR}(g_r)$ we obtain

$$\overline{t^{nR}_{(mm')(pp')(jj')}(g_r)} = t^{n\bar{R}}_{(m'm)(p'p)(j'j)}(g_r). \tag{4}$$

By making use of formula (7) of Section 11.6.9 we derive

$$\overline{t^{nR}_{(mm')(pp')(jj')}(g_r)} = t^{n\bar{R}}_{(mm')(pp')(jj')}(g_r). \tag{5}$$

Therefore,

$$t^{nR}_{(mm')(pp')(jj')}(g_r) = t^{nR}_{(m'm)(p'p)(j'j)}(g_r). \tag{6}$$

For matrix elements of the operators $T^{n\ell\ell'}(g_{n-1}(\theta))$ we have

$$t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta)) = t^{n\ell'\ell}_{(m'm)(p'p)(j'j)}(g_{n-1}(\theta)). \tag{7}$$

Since the representations $T^{n\sigma k}$ and $T^{n,-\bar{\sigma}-2n+2,k}$ of the group $U(n-1,1)$ are Hermitian-adjoint, then

$$\overline{t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t))} = t^{n,-\bar{\sigma}-2n+2,k}_{(pp')(mm')(jj')}(g'_{n-1}(-t)). \tag{8}$$

It follows from formula (5) of Section 11.6.8 that

$$t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(-t)) = (-1)^{p-p'+m-m'}\, t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t)). \tag{9}$$

From (2), (8) and (9), we derive the relation

$$t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t)) = (-1)^{p-p'+m-m'}\, t^{n,-\sigma-2n+2,k}_{(pp')(mm')(jj')}(g'_{n-1}(t)). \tag{10}$$

The results of Section 11.6.3 lead to the equality

$$t^{n,-\sigma-2n+2,k}_{(mm')(pp')(jj')}(g'_{n-1}(t)) \\ = \frac{\Gamma\left(\frac{\sigma-k}{2}+m+n-1\right)\Gamma\left(\frac{\sigma-k}{2}-m'+1\right)\Gamma\left(p-\frac{\sigma+k}{2}\right)\Gamma\left(-\frac{\sigma+k}{2}-p'-n+2\right)}{\Gamma\left(m-\frac{\sigma+k}{2}\right)\Gamma\left(-\frac{\sigma+k}{2}-m'-n+2\right)\Gamma\left(\frac{\sigma-k}{2}+p+n-1\right)\Gamma\left(\frac{\sigma-k}{2}-p'+1\right)} \\ \times t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t)). \tag{11}$$

From (10) and (11), we obtain

$$t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t)) = (-1)^{p-p'+m-m'} \frac{\Gamma\left(\frac{\sigma-k}{2}+p+n-1\right)\Gamma\left(\frac{\sigma-k}{2}-p'+1\right)}{\Gamma\left(p-\frac{\sigma+k}{2}\right)\Gamma\left(-\frac{\sigma+k}{2}-p'-n+2\right)}$$
$$\times \frac{\Gamma\left(m-\frac{\sigma+k}{2}\right)\Gamma\left(-\frac{\sigma+k}{2}-m'-n+2\right)}{\Gamma\left(\frac{\sigma-k}{2}+m+n-1\right)\Gamma\left(\frac{\sigma-k}{2}-m'+1\right)} t^{n\sigma k}_{(pp')(mm')(jj')}(g'_{n-1}(t)). \quad (12)$$

For $(p,p') = (m,m')$, we have from (11) that

$$t^{n\sigma k}_{(mm')(mm')(jj')}(g'_{n-1}(t)) = t^{n,-\sigma-2n+2,k}_{(mm')(mm')(jj')}(g'_{n-1}(t)). \quad (13)$$

For the principal unitary series representations this equality is of the form

$$\overline{t^{n\sigma k}_{(mm')(mm')(jj')}(g'_{n-1}(t))} = t^{n\sigma k}_{(mm')(mm')(jj')}(g'_{n-1}(t)). \quad (14)$$

Writing down relation (11) for the principal unitary series representations, we derive that for $\sigma = i\rho - n + 1$, $\rho \in \mathbb{R}$, the function

$$\frac{\Gamma\left(\frac{\sigma-k}{2}-m'+1\right)\Gamma\left(p-\frac{\sigma+k}{2}\right)}{\Gamma\left(\frac{\sigma-k}{2}-p'+1\right)\Gamma\left(m-\frac{\sigma+k}{2}\right)} t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t))$$

is real.

For the matrix elements $t^{n\ell\ell'}_{\cdots}(g_{n-1}(\theta))$ the relation

$$t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta)) = (-1)^{m-m'+p-p'} t^{n\ell\ell'}_{(pp')(mm')(jj')}(g_{n-1}(\theta)) \quad (15)$$

holds. It follows from formula (5) of Section 11.6.7 that

$$t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\pi-\theta)) = (-1)^a t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta)), \quad (16)$$

where $a = \ell - \ell' + m - m' + p - p' + j - j'$.

Since the representations $T^{nkR}$ and $T^{nk,-\bar{R}}$ of the group $JU(n-1)$ are Hermitian-adjoint, then

$$\overline{t^{nR}_{(mm')(pp')(jj')}(g_r)} = t^{n\bar{R}}_{(pp')(mm')(jj')}(g_r). \quad (17)$$

From (4) and (17) we derive the equality

$$t^{nR}_{(mm')(pp')(jj')}(g_r) = t^{nR}_{(pp')(mm')(jj')}(g_r). \quad (18)$$

The explicit expressions for matrix elements of the operators $T^{nkR}(g_r)$ imply the symmetry relations

$$t^{nR}_{(mm')(pp')(jj')}(g_r) = \left[\frac{(m+m'+n-2)(p+p'+n-2)}{(p+m'+n-2)(m+p'+n-2)}\right]^{1/2} t^{nR}_{(pm')(mp')(jj')}(g_r)$$
$$= \left[\frac{(m+m'+n-2)(p+p'+n-2)}{(m+p'+n-2)(p+m'+n-2)}\right]^{1/2} t^{nR}_{(mp')(pm')(jj')}(g_r). \quad (19)$$

We derive from formula (5) of Section 11.6.7 that

$$t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta))$$
$$= (-1)^{p+m}\left[\frac{(p+p'+n-2)(m+m'+n-2)}{(p+m'+n-2)(m+p'+n-2)}\right]^{\frac{1}{2}} t^{n\ell\ell'}_{(pm')(mp')(jj')}(g_{n-1}(\theta))$$
$$= (-1)^{p'+m'}\left[\frac{(p+p'+n-2)(m+m'+n-2)}{(p+m'+n-2)(m+p'+n-2)}\right]^{1/2} t^{n\ell\ell'}_{(mp')(pm')(jj'}(g_{n-1}(\theta)). \quad (20)$$

Similar relations for $t^{n\sigma k}_{\dots}(g'_{n-1}(t))$ follow from formula (5) of Section 11.6.8.

**11.6.11. Relations between matrix elements of representations for groups of different dimensionalities.** Let us consider integral representation (2) of Section 11.6.1 for matrix elements of the operators $T^{n\sigma k}(g'_{n-1}(t))$. We replace the function $t^{n-1,pp'}_{(jj')0}(g_{n-2}(\theta'))$ by the function

$$-\left[\frac{(j+1)(p+n-2)(j'+n-4)p'}{(p+1)(j+n-3)(p'+n-3)j'}\right]^{1/2} t^{n-1,p+1,p'-1}_{(j+1,j'-1)0}(g'_{n-2}(\theta'))$$

(see formula (2) of Section 11.3.9) in this integral representation and carry out the same replacement for $t^{n-1,mm'}_{(jj')0}(g_{n-2}(\theta))$. As a result, we obtain

$$t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t)) = t^{n\sigma,k+2}_{(m+1,m'-1)(p+1,p'-1)(j+1,j'-1)}(g'_{n-1}(t)). \quad (1)$$

An analogous relation for matrix elements of the operators $T^{n\ell\ell'}(g_{n-1}(\theta))$ is of the form

$$t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta)) = t^{n,\ell+1,\ell'-1}_{(m+1,m'-1)(p+1,p'-1)(j+1,j'-1)}(g_{n-1}(\theta)). \quad (2)$$

For matrix elements of the operators $T^{nkR}(g_r)$ we have

$$t^{nR}_{(mm')(pp')(jj')}(g_r) = t^{nR}_{(m+1,m'-1)(p+1,p'-1)(j+1,j'-1)}(g_r). \quad (3)$$

Applying relation (1) of Section 11.3.9 to integral representation (2) of Section 11.6.1, we have

$$t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t)) = t^{n+1,\sigma-1,k-1}_{(m-1,m')(p-1,p')(j-1,j')}(g'_n(t))$$
$$= t^{n+1,\sigma-1,k+1}_{(m,m'-1)(p,p'-1)(j,j'-1)}(g'_n(t)). \quad (4)$$

In the same way, we obtain

$$t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta)) = t^{n+1,\ell-1,\ell'}_{(m-1,m')(p-1,p')(j-1,j')}(g_n(\theta))$$
$$= t^{n+1,\ell,\ell'-1}_{(m,m'-1)(p,p'-1)(j,j'-1)}(g_n(\theta)), \quad (5)$$

$$t^{nR}_{(mm')(pp')(jj')}(g_r) = t^{n+1,R}_{(m-1,m')(p-1,p')(j-1,j')}(g_r)$$
$$= t^{n+1,R}_{(m,m'-1)(p,p'-1)(j,j'-1)}(g_r). \quad (6)$$

It follows from relations (4) that

$$t^{n\sigma k}_{(mm')(pp')(jj')}(g'_{n-1}(t)) = t^{n+j+j',\sigma-j-j',k-j+j'}_{(m-j,m'-j')(p-j,p'-j')(00)}(g'_{n+j+j'-1}(t)). \quad (7)$$

Analogous relations are valid for $t^{n\ell\ell'}_{\dots}(g_{n-1}(\theta))$ and $t^{nR}_{\dots}(g_r)$.

We derive from (5) that

$$t^{n\ell\ell'}_{(mm')(pp')(jj')}(g_{n-1}(\theta)) = t^{4,\ell+n-4,\ell'}_{(m+n-4,m')(p+n-4,p')(j+n-4,j')}(g_3(\theta))$$
$$= t^{4,\ell,\ell'+n-4}_{(m,m'+n-4)(p,p'+n-4)(j,j'+n-4)}(g_3(\theta)). \quad (8)$$

For $t^{n\sigma k}_{\dots}(g'_{n-1}(t))$ and $t^{nR}_{\dots}(g_r)$ similar equalities are obtained from (4) and (6).

It follows from formula (7) with $p=j$, $p'=j'$ that

$$t^{n\sigma k}_{(mm')(pp')(pp')}(g'_{n-1}(t)) = t^{n+p+p',\sigma-p-p',k-p+p'}_{(m-p,m'-p')0}(g'_{n+p+p'-1}(t)). \quad (9)$$

Applying formula (5) of Section 11.3.9 to the right hand side, we obtain

$$t^{n\sigma k}_{(mm')(pp')(pp')}(g'_{n-1}(t)) = A(\sinh t)^{-p-p'}t^{n\sigma k}_{(mm')0}(g'_{n-1}(t)), \quad (10)$$

where

$$A = \frac{\Gamma\left(\frac{\sigma-k}{2}-p'+1\right)\Gamma\left(\frac{\sigma+k}{2}-p+1\right)}{\Gamma\left(\frac{\sigma-k}{2}+1\right)\Gamma\left(\frac{\sigma+k}{2}+1\right)}$$
$$\times\left[\frac{(p+p'+n-2)(m'+p+n-2)!(m+p'+n-3)!m!m'!}{(n-2)(m+n-3)!(m'+n-3)!(m-p)!(m'-p')!}\right]^{1/2}.$$

In just the same way one derives the relation

$$t^{n\ell\ell'}_{(mm')(pp')(pp')}(g_{n-1}(\theta)) = B(\sin\theta)^{-p-p'} t^{n\ell\ell'}_{(mm')0}(g'_{n-1}(\theta)), \tag{11}$$

where

$$B = \left[\frac{(p+p'+n-2)(m'+p+n-3)!(m+p'+n-3)!m!m'!}{(m+n-3)!(m'+n-3)!(n-2)(m-p)!(m'-p')!\ell!\ell'!} \times \frac{(\ell-p)!(\ell'-p')!(\ell+n-2)!(\ell'+n-2)!}{(\ell+p'+n-2)!(\ell'+p+n-2)!}\right]^{1/2},$$

and the relation

$$t^{nR}_{(mm')(pp')(pp')}(g_r) = \left[\frac{(p+p'+n-2)(m+m'+n-2)(m+p'+n-3)!(m'+p+n-3)!}{(m-p)!(m'-p')!}\right]^{1/2} \times i^{m+m'+p+p'}(-iRr/2)^{p+p'-n+2} J_{m+m'+n-2}(-iRr). \tag{12}$$

**11.6.12. The functions $t^{n\ell 0}(g_{n-1}(\theta))$.** Let us set $\ell' = m' = p' = j'$ into formula (5) of Section 11.6.7. For $p \geq m$ we have

$$t^{n\ell\ell'}_{(m\ell')(p\ell')(j\ell')}(g_{n-1}(\theta)) = \left[\frac{(\ell-m)!(p-j)!}{(\ell-p)!(m-j)!}\right]^{1/2} \frac{\sin^{p-m}\theta\cos^{p+m-\ell-j}\theta}{(p-m)!} \times F(-\ell+p, p-j+1; p-m+1; \sin^2\theta). \tag{1}$$

For $p < m$ one has to permute $m$ and $p$ and multiply the right hand side by $(-1)^{m+p}$.

It follows from (1) that the matrix elements $t^{n\ell\ell'}_{(m\ell')(p\ell')(j\ell')}(g_{N-1}(\theta))$ do not depend on $\ell'$ and $n$. In particular, the matrix elements of the operator $T^{n\ell 0}(g_{n-1}(\theta))$ do not depend on $n$. Formula (7) of Section 11.6.10 implies that the matrix elements $t^{n\ell'\ell}_{(\ell' m)(\ell' p)(\ell' j)}(g_{n-1}(\theta))$ are also given by formula (1) and do not depend on $\ell'$ and $n$. The matrix elements of the operator $T^{n0\ell}(g_{n-1}(\theta))$ do not depend on $n$.

Matrix elements of the operator $T^{n\ell 0}(g_{n-1}(\theta))$ can be expressed in terms of $P^r_{ms}(\cos 2\theta)$:

$$t^{n\ell}_{mpj}(\theta) \equiv t^{n\ell 0}_{(m0)(p0)(j0)}(g_{n-1}(\theta)) = P^{(\ell-j)/2}_{p-(\ell+j)/2,\, m-(\ell+j)/2}(\cos 2\theta) \tag{2}$$

and in terms of Jacobi polynomials:

$$t^{n\ell}_{mpj}(\theta) = \left[\frac{(\ell-p)!(p-j)!}{(\ell-m)!(m-j)!}\right]^{1/2} \sin^{p-m}\theta\cos^{a}\theta P^{(p-m,a)}_{\ell-p}(\cos 2\theta), \tag{3}$$

where $a = p + m - \ell - j$. In (2) and (3) $p \geq m$. If $p < m$, one has to permute $m$ and $p$ and multiply the right hand side by $(-1)^{m+p}$.

**11.6.13. Special cases of matrix elements of the operators $T^{n\ell\ell'}(g_{n-1}(\theta))$.** Let us set $p' = j'$ into formula (7) of Section 11.6.7. Then for $p \geq m$ we have

$$t^{n\ell\ell'}_{(mm')(pj')(jj')}(g_{n-1}(\theta)) = N \sin^a \theta \cos^b \theta \times {}_3F_2\left( \begin{matrix} -\ell + p, p - j + 1, p + \ell' + n - 1 \\ p - m + 1, p + m' + n - 1 \end{matrix} \middle| \sin^2 \theta \right), \qquad (1)$$

where $a = p - m + m' - j$, $b = m - m' - \ell + \ell' + p - j$ and

$$N = \left[ \frac{(m+m'+n-2)(\ell-m)!(p+\ell'+n-2)!(\ell+m'+n-2)!(p+j'+n-2)!}{(\ell-p)!(\ell'-m')!(m+\ell'+n-2)!(m-j)!(m'-j')!(\ell+j'+n-2)!} \times \frac{(j+m'+n-3)!(m+j'+n-3)!(p-j)!(\ell'-j')!}{(j+j'+n-3)!(p-m)!^2(p+m'+n-2)!^2} \right]^{1/2}.$$

If $p < m$, one has to permute $m$ and $p$ and multiply the right hand side of (1) by $(-1)^{p+m}$.

Formula (9) of Section 11.6.7 for $p = \ell$ yields

$$t^{n\ell\ell'}_{(mm')(\ell p')(jj')}(g_{n-1}(\theta)) = (-1)^{p'-j'} M \sin^r \theta \cos^s \theta \times {}_3F_2\left( \begin{matrix} -p' + j', -m' + j', m + j' + n - 2 \\ -\ell' + j', j + j' + n - 2 \end{matrix} \middle| \sin^{-2} \theta \right), \qquad (2)$$

where $r = m' - m + \ell + p' - 2j$, $s = m - m' - p' + j' - j + \ell'$ and

$$M = \left[ \frac{(m+m'+n-2)(j+p'+n-3)!(j+m'+n-3)!}{(\ell'-m')!(\ell'-p')!(m+\ell'+n-2)!(p'-j')!} \times \frac{(\ell+j'+n-3)!(m+j'+n-3)!(\ell-j)!(\ell+\ell'+n-2)!}{(m'-j')!(m-j)!(\ell-m)!(\ell+p'+n-3)!} \right]^{1/2} \times \frac{(\ell'-j')!}{(\ell+m'+n-2)!(j+j'+n-3)!}.$$

If $j = m$ in (1), then

$$t^{n\ell\ell'}_{(jm')(pj')(jj')}(g_{n-1}(\theta)) = \left[ \frac{(\ell-p)!(\ell-j)!(p+\ell'+n-2)!(p+j'+n-2)!(j+m'+n-2)!(\ell'-j')!}{(\ell+m'+n-2)!(\ell'-m')!(j+\ell'+n-2)!(m'-j')!(\ell+j'+n-2)!(p-j)!} \right]^{1/2} \times (\sin\theta)^{p-j'-j+m'} (\cos\theta)^{p-m'-\ell+\ell'} P^{(p+m'+n-2,\ell'-m')}_{\ell-p}(\cos 2\theta). \qquad (3)$$

By setting $m = j$ into (2), we obtain

$$t^{n\ell\ell'}_{(jm')(\ell p')(jj')}(g_{n-1}(\theta)) = M \sin^a\theta \cos^b\theta F(j'-p', j'-m'; j'-\ell'; \sin^{-2}\theta), \quad (4)$$

where $a = \ell + m' + p' + j$, $b = \ell' + j' - m' - p'$, and

$$M = (-1)^{p'-j'} \left[ \frac{(j+p'+n-3)!(j+m'+n-2)!(\ell+j'+n-3)!}{(\ell'-m')!(\ell'-p')!(p'-j')!(m'-j')!(\ell'+j+n-2)!} \times \frac{(\ell+\ell'+n-2)!(\ell'-j')!^2}{(\ell+p'+n-3)!(\ell+m'+n-2)!(j+j'+n-3)!} \right]^{1/2}.$$

For $\ell' = m'$ formula (2) gives

$$t^{n\ell\ell'}_{(m\ell')(\ell p')(jj')}(g_{n-1}(\theta)) = N \sin^a\theta \cos^b\theta F(j'-p', m+j'+n-2; j+j'+n-2; \sin^{-2}\theta), \quad (5)$$

where $a = \ell - m + \ell' + p' - 2j'$, $b = m - p' - j + j'$ and

$$N = (-1)^{p;-j'} \left[ \frac{(j+p'+n-3)!(j+\ell'+n-3)!(\ell+j'+n-3)!}{(\ell'-p')!(p'-j')!(m+\ell'+n-3)!(m-j)!} \times \frac{(m+j'+n-3)!(\ell-j)!(\ell'-j')!}{(\ell-m)!(\ell+p'+n-3)!(j+j'+n-3)!^2} \right]^{1/2}.$$

Other special cases of matrix elements of the operators $T^{n\ell\ell'}(g_{n-1}(\theta))$ follow from those obtained above, if one applies the symmetry relations from Section 11.6.10.

**11.6.14. Special cases of matrix elements of the operators $T^{n\sigma k}(g'_{n-1}(t))$.** Formula (6) of Section 11.6.8 for $m \ge p$ and $p' = j'$ gives

$$t^{n\sigma k}_{(mm')(pj')(jj')}(t) = N \sinh^a t \cosh^b t \; {}_3F_2 \left( \begin{matrix} m - \frac{\sigma+k}{2}, \frac{\sigma-k}{2} + m + n - 1, m - j + 1 \\ m + m' + n - 1, m - p + 1 \end{matrix} \middle| -\sinh^2 t \right), \quad (1)$$

where $a = m - j' - p + m'$, $b = p + m - m' - k - j$ and

$$N = \frac{\Gamma\left(\frac{\sigma+k}{2} - p + 1\right)\Gamma\left(\frac{\sigma-k}{2} - j' + 1\right)}{\Gamma\left(\frac{\sigma+k}{2} - m + 1\right)\Gamma\left(\frac{\sigma-k}{2} - m' + 1\right)} \times \left[ \frac{(p+j'+n-2)!(m+j'+n-3)!(j+m'+n-3)!(m-j)!}{(p-j)!(j+j'+n-3)!(m'-j')!(m-p)!^2(m+m'+n-2)(m+m'+n-3)!^2} \right]^{1/2}.$$

For $m < p$ one has to permute $m$ and $p$ on the right hand side of (1) and to fulfill the replacement (8) of Section 11.6.8.

For $m' = j'$, $j = p$ formula (6) of Section 11.6.8 leads to

$$t^{n\sigma k}_{(mj')(jp')(jj')}(g'_{n-1}(t))$$
$$= M \sinh^a t \cosh^b t\, F\left(m - \frac{\sigma+k}{2}, \frac{\sigma-k}{2} + m + n - 1; m + p' + n - 1; -\sinh^2 t\right), \quad (2)$$

where $a = m + j + j' + p'$, $b = m + k - p'$ and

$$M = \frac{\Gamma\left(\frac{\sigma+k}{2} - j + 1\right)\Gamma\left(\frac{\sigma+k}{2} + p' + n - 1\right)}{\Gamma\left(\frac{\sigma+k}{2} - m + 1\right)\Gamma\left(\frac{\sigma+k}{2} + j' + n - 1\right)}$$
$$\times \left[\frac{(j + p' + n - 2)(m + j' + n - 2)!(j + m' + n - 3)!}{(p' - j')!(m - j)!(m + p' + n - 2)!^2}\right]^{1/2}.$$

Expression (2) for $t^{n\sigma k}_{(mj')(jp')(jj')}(g'_{n-1}(t))$ can be rewritten as

$$t^{n\sigma k}_{(mj')(jp')(jj')}(g'_{n-1}(t))$$
$$= \frac{\Gamma\left(\frac{\sigma+k}{2} - j + 1\right)}{\Gamma\left(\frac{\sigma+k}{2} + j' + n - 1\right)}\left[\frac{(j + p' + n - 2)(m + j' + n - 2)!(j + m' + n - 3)!}{(p' - j')!(m - j)!}\right]^{1/2}$$
$$\times (-1)^{m+p'-n+2}(\sinh t)^{j+j'-n+2}(\cosh t)^{2m}\mathfrak{P}^{m-(\sigma+n)/2}_{-m-p'+1,-1+(n-k)/2}(\cosh 2t). \quad (3)$$

Other special cases for matrix elements of the operators $T^{n\sigma k}(g'_{n-1}(t))$ are obtained from the above ones by means of the symmetry relations of Section 11.6.10.

**11.6.15. Special cases of matrix elements of the operators $T^{nkR}(g_r)$.** It follows from formula (11) of Section 11.6.9 that

$$t^{nR}_{(mm')(jp')(jj')}(g_r)$$
$$= \left[\frac{(m+m'+n-2)(p'-j')!(j+p'+n-2)!(m+j'+n-3)!(j+m'+n-3)!}{(m'-j')!(j+j'+n-3)!(m-j)!(p'-m')!^2(m+p'+n-2)!^2}\right]^{1/2}$$
$$\times \left(\frac{Rr}{2}\right)^{m-m'-j+p'} {}_1F_2\left(p' - j' + 1; m + p' + n - 1, p' - m' + 1; \left(\frac{Rr}{2}\right)^2\right), \quad (1)$$

where $p' \geq m'$. If $p' < m'$, one has to permute $m'$ and $p'$. If in addition $m' = j'$, then we obtain

$$t^{nR}_{(mj')(jp')(jj')}(g_r) = i^{m+p'+n-2}\left[\frac{(j + p' + n - 2)!(m + j' + n - 2)!}{(m - j)!(p' - j')!}\right]^{1/2}$$
$$\times \left(\frac{Rr}{2}\right)^{2m-2p'-j-j'+n-2} J_{m+p'+n-2}(iRr). \quad (2)$$

Other special cases of matrix elements of the operators $T^{nkR}(g_r)$ are derived by means of symmetry relations.

Note that formulas (3) of Section 11.6.9 and (1) imply the integral relation

$$\int_0^{\pi/2} J_{j-p'+m'-m}(r\cos\theta) P_{m-j}^{(j+j'+n-2,m'-m+j-j')}(\cos 2\theta)\sin^{2(j+j'+m)-5}\theta \times \cos^{p'+m'-m-2j'+j+1}\theta\, d\theta$$

$$= \frac{(p'-j')!(m+j'+n-3)!(-r/2)^{m-m'-j+p'}}{2(p'-j')!(p'-m')!(m+p'+n-2)!} \times {}_1F_2\left(p'-j'+1; m+p'+n-1, p'-m'+1; \frac{-r^2}{4}\right). \quad (3)$$

**11.6.16. Matrix elements of the operators $T^{n\ell\ell'}\left(g_{n-1}\left(\frac{\pi}{2}\right)\right)$.** The integral representation for the matrix elements $t^{n\ell\ell'}_{(mm')(pp')(00)}(g_{n-1}(\theta))$ is of the form

$$t^{n\ell\ell'}_{(mm')(pp')(00)}(g_{n-1}(\theta)) = i^{p-p'-m+m'}\left[\frac{(\ell-m)!(\ell+m'+n-2)!(\ell'+m+n-2)!(\ell'-m')!}{(\ell-p)!(\ell+p'+n-2)!(\ell'+p+n-2)!(\ell'-p')!}\right]^{1/2}$$

$$\times D^n_{mm'pp'00}\int_0^{2\pi}\int_0^{\pi/2}(\cos\theta - ie^{i\varphi}\cos\psi\sin\theta)^{\ell-\frac{p-p'}{2}}(\cos\theta - ie^{-i\varphi}\cos\psi\sin\theta)^{\ell'+\frac{p-p'}{2}}$$

$$\times\left(\frac{e^{i\varphi}\cos\varphi\cos\theta - i\sin\theta}{e^{-i\varphi}\cos\psi\cos\theta - i\sin\theta}\right)^{\frac{p-p'}{2}} e^{i(m'-m)\varphi} t^{n-1,pp'}_{00}(g_{n-2}(\psi')) t^{n-1,mm'}_{00}(g_{n-2}(\psi)) \times \sin^{2n-5}\psi\cos\psi\, d\psi\, d\varphi, \quad (1)$$

where $D^n_{mm'pp'00}$ is given by formula (3) of Section 11.6.1 and

$$\cos\psi' = \frac{(e^{i\varphi}\cos\psi\cos\theta - i\sin\theta)(e^{-i\varphi}\cos\psi\cos\theta - i\sin\theta)}{(\cos\theta - ie^{i\varphi}\cos\psi\sin\theta)(\cos\theta - ie^{-i\varphi}\cos\psi\sin\theta)}. \quad (2)$$

More general matrix elements of the operator $T^{n\ell\ell'}(g_{n-1}(\theta))$ are expressed in terms of matrix elements of form (1) by means of formula (7) of Section 11.6.11.

By setting $\theta = \frac{\pi}{2}$ into (1), we obtain

$$t^{n\ell\ell'}_{(mm')(pp')(00)}\left(g_{n-1}\left(\frac{\pi}{2}\right)\right) = 0 \quad \text{if} \quad \ell - \ell' - p + p' - m + m' \neq 0. \quad (3)$$

If $\ell-\ell'-p+p'-m+m'=0$, then

$$t^{n\ell\ell'}_{(mm')(pp')(00)}\left(g_{n-1}\left(\frac{\pi}{2}\right)\right) = i^{p-p'-m'+m}\left[\frac{(\ell-m)!(\ell+m'+n-2)!(\ell'+m+n-2)!(\ell'-m')!}{(\ell-p)!(\ell+p'+n-2)!(\ell'+p+n-2)!(\ell'-p')!}\right]^{1/2} \times D^n_{mm'pp'00}I, \quad (4)$$

where

$$I = 2\pi(-i)^{\ell+\ell'}\int_0^{\pi/2} t_{00}^{n-1,pp'}(g_{n-2}(\psi'))t_{00}^{n-1,mm'}(g_{n-2}(\psi))\sin^{2n-5}\psi\cos^{\ell+\ell'+1}\psi\,d\psi.$$

Here

$$\cos\psi' = \cos^{-1}\psi, \quad \sin\psi' = -\tan\psi.$$

Since

$$t_{00}^{n-1,pp'}(g_{n-2}(\psi')) = \cos^{p-p'}\psi\, F(-p, p'+n-2; n-2; -\tan^2\psi) = \frac{p!(n-3)!}{(p+n-3)!}\cos^{-p-p'}\psi P_p^{(n-3,-p-p'-n+2)}(\cos 2\psi),$$

$$t_{00}^{n-1,mm'}(g_{n-2}(\psi)) = \frac{m!(n-3)!}{(m+n-3)!}\cos^{m'-m}\psi P_m^{(n-3,m'-m)}(\cos 2\psi),$$

then with the help of the substitution $x=\cos 2\psi$, we obtain

$$I = \frac{(-i)^{\ell+\ell'}\pi m!p!(n-3)!^2}{(m+n-3)!(p+n-3)!}2^{2-n+(p+p'+m-m'-\ell-\ell')/2} \times \int_{-1}^{1} P_p^{(n-3,-p-p'-n+2)}(x)P_m^{(n-3,m'-m)}(x)(1-x)^{n-3}(1+x)^{(\ell+\ell'-p-p'+m'-m)/2}dx.$$

The integral is calculated by means of formula (6) of Section 10.2.5. Since $\ell-\ell'-p+p'-m+m'=0$, then for $m'\geq m$, $p'\geq p$ we obtain

$$I = \frac{(-i)^{\ell+\ell'}\pi m'!(n-3)!^2(-1)^m}{(m+n-3)!(m'-m)!}\,\frac{(\ell'-p')!(\ell'+p+n-2)!\ell!(\ell-p+p')!}{(\ell-m)!(\ell'-m')!(\ell+m'+n-2)!(\ell'+m+n-2)!} \times {}_4F_3\left(\left.\begin{matrix}-m, m'+n-2, -\ell+p, -\ell-p'-n+2\\ m'-m+1, -\ell+p-p', -\ell\end{matrix}\right|1\right).$$

Consequently, for $m' \ge m$, $p' \ge p$ and $\ell - \ell' - p + p' - m + m' = 0$, we have

$$t^{n\ell\ell'}_{(mm')(pp')(00)}\left(g_{n-1}\left(\frac{\pi}{2}\right)\right) = (-1)^{\ell'+m'} N\, {}_4F_3\left(\begin{matrix} -m, m'+n-2, -\ell+p, -\ell-p'-n+2 \\ m'-m+1, -\ell+p-p', -\ell \end{matrix} \middle| 1\right), \quad (5)$$

where

$$N = \left[\frac{\ell!^2(\ell-p+p')!^2 m'!(\ell'-p')!(\ell'+p+n-2)!(m'+n-3)!}{m!p!p'!(m'-m)!^2(m+p-3)!(\ell-m)!(\ell'-m')!} \times \frac{(p+n-3)!(p'+n-3)!(m+m'+n-2)(p+p'+n-2)}{(\ell+m'+n-2)!(\ell'+m+n-2)!(\ell-p)!(\ell+p'+n-2)!}\right]^{1/2}.$$

In order to derive from (5) expressions for $t^{n\ell\ell'}_{(mm')(pp')(00)}\left(g_{n-1}\left(\frac{\pi}{2}\right)\right)$ with other values of $m$, $m'$, $p$, $p'$, we have to use the symmetry relations from Section 11.6.10.

Expressions for $t^{n\ell\ell'}_{\cdots}\left(g_{n-1}\left(\frac{\pi}{2}\right)\right)$ can be represented in terms of Racah polynomials and in terms of Wilson polynomials.

As in the case of the groups $SO(n)$ and $SO_0(n-1,1)$ (see Section 9.5.7), matrix elements of the operators $T^{n\ell\ell'}\left(g_{n-1}\left(\frac{\pi}{2}\right)\right)$ are utilized for obtaining matrix elements of the operators $T^{n\ell\ell'}(g_{ij}(\theta))$ and $T^{n\sigma k}(g'_{in}(t))$.

## 11.7. Zonal and Associated Spherical Functions of the Groups $Sp(n)$ and $Sp(n-1,1)$

**11.7.1. The group $Sp(n)$ and spheres in the quaternion space.** Remember that the skew field $\mathbb{H}$ of quaternions over $\mathbb{R}$ consists of quaternions $q = a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$ and the basis elements $i$, $j$, $k$ satisfy the relations

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ ik = -ki = -j,\ jk = -kj = i. \quad (1)$$

Quaternions $q \in \mathbb{H}$ are represented in the form

$$q = z + wj,\ z = a + bi,\ w = c + di. \quad (2)$$

The correspondence

$$q \to \bar{q} = a - bi - cj - dk = \bar{z} - wj$$

defines the conjugation in $\mathbb{H}$. It is easy to verify that

$$q\bar{q} \equiv |q|^2 = a^2 + b^2 + c^2 + d^2. \quad (3)$$

Let $\mathbf{H}^n$ be the $n$-dimensional linear space over the skew field of quaternions, consisting of vectors $\mathbf{q} = (q_1, q_2, \ldots, q_n)$. We equip $\mathbf{H}^n$ with the scalar product

$$(\mathbf{q}, \mathbf{q}') = q_1\overline{q_1'} + \ldots + q_n\overline{q_n'}. \tag{4}$$

The value $r = |\mathbf{q}| \equiv (\mathbf{q}, \mathbf{q})^{1/2}$ is called the *length* of $\mathbf{q}$, and $|\mathbf{q} - \mathbf{q}'|$ is called the *distance* between $\mathbf{q}$ and $\mathbf{q}'$. The set of points $\mathbf{q} \in \mathbf{H}^n$, for which $|\mathbf{q}| = 1$, forms the *sphere* $S_H^{n-1}$ in $\mathbf{H}^n$. In particular, $S_H^0$ coincides with the set of quaternions $q$ such that $q\bar{q} = 1$. If $q_i = a_i + b_i i + c_i j + d_i k = z_i + w_i j$, then

$$|\mathbf{q}|^2 = \sum_{i=1}^{n}(a_i^2 + b_i^2 + c_i^2 + d_i^2) = \sum_{i=1}^{n}(|z_i|^2 + |w_i|^2), \tag{5}$$

that is, $S_H^{n-1}$ is identified with the sphere $S^{4n-1}$ in $\mathbb{R}^{4n}$ and with the sphere $S_C^{2n-1}$ in $\mathbb{C}^{2n}$. For $n = 1$ we have $S_H^0 \sim S_C^1 \sim S^3$.

The set of linear transformations of $\mathbb{H}^n$, conserving the distance between points, forms the Lie group $Sp(n)$. It can be also defined as the group of linear transformations in $\mathbb{H}^n$ transferring $S_H^{n-1}$ into $S_H^{n-1}$. In particular, $Sp(1)$ coincides with $S_H^0$. Elements of the group $Sp(n)$ are represented as quaternion $n \times n$-matrices. It is clear that $Sp(n) \supset U(n) \supset SO(n)$.

Let $\mathbf{e}_n$ denote the point from $\mathbb{H}^n$ with the coordinates $(0, \ldots, 0, 1)$. The subgroup in $Sp(n)$, leaving the point $\mathbf{e}_n$ fixed, is isomorphic to $Sp(n-1)$. Therefore,

$$S_H^{n-1} \sim Sp(n)/Sp(n-1).$$

Since $S_H^{n-1} \sim S_C^{2n-1} \sim S^{4n-1}$, then

$$Sp(n)/Sp(n-1) \sim U(2n)/U(2n-1) \sim SO(4n)/SO(4n-1).$$

Let us identify the points $\mathbf{q}u \equiv (q_1 u, \ldots, q_n u)$, $u \in Sp(1)$, in $S_H^{n-1}$ and denote the obtained space by $P_H^{n-1}$. The group $Sp(n)$ is a transitive group of motions of $P_H^{n-1}$. Let $p_n$ denote the element from $P_H^{n-1}$ consisting of the points $(0, \ldots, 0, u)$, $u \in Sp(1)$. The subgroup leaving $p_n$ fixed consists of matrices $\mathrm{diag}(g, u')$, $g \in Sp(n-1)$, $u' \in Sp(1)$. It is isomorphic to the group $Sp(n-1) \times Sp(1)$. The space $P_H^{n-1}$ is a symmetric Riemannian space of compact type (see Section 1.2.3). It is identified with $Sp(n)/(Sp(n-1) \times Sp(1))$.

Let us introduce spherical coordinates in $\mathbb{H}^n$. For this we note that every point $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{H}^n$ can be represented in the form

$$\mathbf{q} = r(q_1', \ldots, q_n') = r(u_1 x_1, \ldots, u_n x_n),$$

where $r = |\mathbf{q}|$, $(q_1', \ldots, q_n') \in S_H^{n-1}$, $u_i \in Sp(1)$ and $x_i \geq 0$. The point $\mathbf{x} = (x_1, \ldots, x_n)$ belongs to $S^{n-1}$. Introducing spherical coordinates for $\mathbf{x}$, we have

$$\left.\begin{aligned} q_1 &= r u_1 \sin\theta_{n-1} \ldots \sin\theta_2 \sin\theta_1, \\ q_2 &= r u_2 \sin\theta_{n-1} \ldots \sin\theta_2 \cos\theta_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ q_{n-1} &= r u_{n-1} \sin\theta_{n-1} \cos\theta_{n-2}, \\ q_n &= r u_n \cos\theta_{n-1}, \end{aligned}\right\} \tag{6}$$

where

$$0 \le r < \infty, \quad u_i \in Sp(1), \quad 0 \le \theta_j \le \frac{\pi}{2}. \tag{7}$$

Putting $r = 1$ into (6), we obtain the coordinates $(u_1, \dots, u_n, \theta_1, \dots, \theta_{n-1})$ on the sphere $S_H^{n-1}$.

It follows from (6) that almost every point $\mathbf{q} = (q_1, \dots, q_n) \in S_H^{n-1}$ with coordinates (6) for $r = 1$ is obtained from the point $\mathbf{e}_n = (0, \dots, 0, 1)$ as follows:

$$\mathbf{q} = g^n(\mathbf{u}, \boldsymbol{\theta})\mathbf{e}_n \equiv d_1(u_1) \cdot d_2(u_2) g_1(\theta_1) \cdot \ldots \cdot d_n(u_n) g_{n-1}(\theta_{n-1})\mathbf{e}_n, \tag{8}$$

where $d_i(u_i) = \mathrm{diag}(1, \dots, 1, u_i, 1, \dots, 1)$ ($u_i$ occupies the $i$-th position) and $g_j(\theta)$ is a rotation matrix which coincides with the matrix $g_j(\theta)$ from the subgroup $SO(n)$. Since $S_H^{n-1} = Sp(n)/Sp(n-1)$, then every matrix $g \in Sp(n)$ is represented as the product

$$g = g^n(\mathbf{u}, \theta) h, \quad h \in Sp(n-1). \tag{9}$$

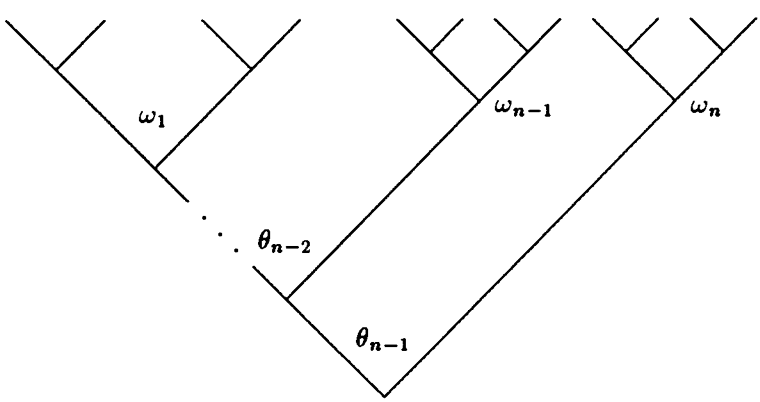
According to (2) the quaternions $u_k$ from (6) are represented in the form

$$u_k = e^{i\varphi_k} \cos\omega_k + e^{i\psi_k} \sin\omega_k \cdot j. \tag{10}$$

Therefore, every one of the equalities $q_k = r u_k \sin\theta_{n-1} \dots$ from (6) can be replaced by the two equalities

$$q_k' = r e^{i\varphi_k} \cos\omega_k \sin\theta_{n-1} \dots, q_k'' = r e^{i\psi_k} \sin\omega_k \sin\theta_{n-1} \dots . \tag{10'}$$

This leads to spherical coordinates on $S_C^{2n-1}$. Considering real and imaginary parts of the expressions in (10′), we obtain real spherical coordinates on $S_H^{n-1} \sim S^{4n-1}$. They correspond to the tree




(11)

Since $S_H^{n-1} \sim S_C^{2n-1}$, then $Sp(n)$ is a subgroup of $U(2n)$. In other words, we can imbed $Sp(n)$ into $U(2n)$. It follows from (10′) that to the matrix $d_k(u_k) \in Sp(n)$

there corresponds the matrix $\mathrm{diag}(I_2,\ldots,I_2,h,I_2,\ldots,I_2)$ in $U(2n)$, where $I_2$ is the identity $2\times 2$-matrix and[4]

$$h=\begin{pmatrix} e^{i\varphi}\cos\omega & e^{i\psi}\sin\omega \\ -e^{-i\psi}\sin\omega & e^{-i\varphi}\cos\omega \end{pmatrix}. \tag{12}$$

To the matrix $g_{n-1}(\theta)\in Sp(n)$ there corresponds the matrix $\mathrm{diag}(I_{2n-4},g(\theta))$ in $U(2n)$, where

$$g(\theta)=\begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

Since the group $U(2n)$ is imbedded into $SO(4n)$ (see Section 11.1.1), then $Sp(n)$ is also imbedded into $SO(4n)$.

Bispherical coordinates on $S_H^{n-1}$ are given by the formulas

$$\left.\begin{array}{l} q_1 = u_1\sin\theta\sin\theta_{s-1}\ldots\sin\theta_2\sin\theta_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ q_s = u_s\sin\theta\cos\theta_{s-1}, \\ q_{s+1} = u_{s+1}\cos\theta\sin\psi_{n-s-1}\ldots\sin\psi_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ q_n = u_n\cos\theta\cos\psi_{n-s-1}. \end{array}\right\} \tag{13}$$

Now instead of (8) we have

$$\mathbf{q}=g^s(\mathbf{u}',\boldsymbol{\theta})g^{n-s}(\mathbf{u}'',\boldsymbol{\psi})g_{sn}(\theta)\mathbf{e}_n, \tag{14}$$

where

$$g^s(\mathbf{u}',\boldsymbol{\theta})=d_1(u_1)\ldots d_s(u_s)g_{s-1}(\theta_{s-1}), \tag{15}$$

$$g^{n-s}(\mathbf{u}'',\boldsymbol{\psi})=d_{s+1}(u_{s+1})\ldots d_n(u_n)g_{n-1}(\psi_{n-s-1}). \tag{15'}$$

Every element $g\in Sp(n)$ is represented in the form

$$g=g^s(\mathbf{u}',\boldsymbol{\theta})g^{n-s}(\mathbf{u}'',\boldsymbol{\psi})g_{sn}(\theta)h,\quad h\in Sp(n-1). \tag{16}$$

**11.7.2. The group $Sp(n-1,1)$, the hyperboloid and the cone in the quaternion space.** We introduce in $\mathbb{H}^n$ the form

$$[\mathbf{q},\mathbf{q}']=-q_1\overline{q_1'}-\ldots-q_{n-1}\overline{q_{n-1}'}+q_n\overline{q_n'} \tag{1}$$

[4]The subgroup of matrices $d_k(u_k)$ is isomorphic to $Sp(1)$. On the other hand, the set of matrices (12) coincides with $SU(2)$. Consequently, the groups $Sp(1)$ and $SU(2)$ are isomorphic.

and denote the space $\mathbf{H}^n$ with this form by $E^H_{n-1,1}$.

The set of linear transformations of $\mathbf{H}^n$, conserving form (1), is the group $Sp(n-1,1)$. The action of $Sp(n-1,1)$ splits $E^H_{n-1,1}$ into orbits consisting of points $\mathbf{q}$, such that $[\mathbf{q},\mathbf{q}] = r$, $-\infty < r < +\infty$. The origin $\mathbf{q} = \mathbf{0}$ is a separate orbit. The orbit $[\mathbf{q},\mathbf{q}] = 1$ forms the hyperboloid $H^{n-1}_H$ in $E^H_{n-1,1}$. Since the subgroup $Sp(n-1)$ leaves the point $\mathbf{e}_n \in H^{n-1}_H$ fixed, then $H^{n-1}_H$ is identified with the homogeneous space $Sp(n-1,1)/Sp(n-1)$. We have

$$H^{n-1}_H \sim Sp(n-1,1)/Sp(n-1) \sim U(2n-2,2)/U(2n-2,1) \\ \sim SO_0(4n-4,4)/SO_0(4n-4,3). \qquad (2)$$

The group $Sp(n-1,1)$ is imbedded into $U(2n-2,2)$ and into $SO_0(4n-4,4)$.

By identifying the points $\mathbf{q}u \equiv (q_1u,\ldots,q_nu)$, $u \in Sp(1)$, in $H^{n-1}_H$ we obtain the space denoted by $\mathcal{P}^{n-1}_H$. The stabilizer of the element $p_n = \{(0,\ldots,0,u)|u \in Sp(1)\}$ coincides with $Sp(n-1) \times Sp(1)$. Therefore,

$$\mathcal{P}^{n-1}_H \sim Sp(n-1,1)/(Sp(n-1) \times Sp(1)). \qquad (3)$$

It is a symmetric Riemannian space of noncompact type. This space is dual by Cartan to $P^{n-1}_H$.

Replacing in formulas (6) of Section 11.7.1 $\cos\theta_{n-1}$ and $\sin\theta_{n-1}$ by $\cosh t$ and $\sinh t$, respectively, and setting $r = 1$, we obtain a parametrization of $H^{n-1}_H$. By means of this parametrization we find that if $\mathbf{q} \in H^{n-1}_H$, then

$$\mathbf{q} = g^n(\mathbf{u},\boldsymbol{\theta},t)\mathbf{e}_n \equiv \\ \equiv d_1(u_1)\cdot d_2(u_2)g_1(\theta_1)\cdot\ldots\cdot d_{n-1}(u_{n-1})g_{n-2}(\theta_{n-2})\cdot d_n(u_n)g'_{n-1}(t)\mathbf{e}_n, \qquad (4)$$

where $d_k(u_k)$ and $g_k(\theta_k)$ are the same as in equality (8) of Section 11.7.1, and $g'_{n-1}(t)$ is the matrix of hyperbolic rotations coinciding with the matrix $g'_{n-1}(t)$ from $SO_0(n-1,1)$. It follows from (4) that every element $g \in Sp(n-1,1)$ is representable in the form

$$g = g^n(\mathbf{u},\boldsymbol{\theta},t)h, \qquad h \in Sp(n-1). \qquad (5)$$

Replacing in formula (13) of Section 11.7.1 $\sin\theta$, $\cos\theta$, $\sin\psi_{n-s-1}$, $\cos\psi_{n-s-1}$ by $\sinh t$, $\cosh t$, $\sinh\beta$, $\cosh\beta$, respectively, we obtain cylindrical coordinates on $H^{n-1}_H$. If $\mathbf{q} \in H^{n-1}_H$, then

$$\mathbf{q} = g^s(\mathbf{u}',\boldsymbol{\theta})g^{n-s}(\mathbf{u}'',\boldsymbol{\psi},\beta)g'_{sn}(t)\mathbf{e}_n, \qquad (6)$$

where $g^{n-s}(\mathbf{u}'',\boldsymbol{\psi},\beta) \in Sp(n-s-1,1)$. Every element $g \in Sp(n-1,1)$ is representable as

$$g = g^s(\mathbf{u}',\boldsymbol{\theta})g^{n-s}(\mathbf{u}'',\boldsymbol{\psi},\beta)g'_{ns}(t)h,\ h \in Sp(n-1). \qquad (7)$$

The subgroup $K = Sp(n-1) \times Sp(1)$ is a maximal compact subgroup in $Sp(n-1,1)$. Elements $g'_{n-1}(t)$, $-\infty < t < \infty$, form the one-parameter subgroup $A$. We also separate in $Sp(n-1,1)$ the subgroup $N$ of matrices

$$n(\mathbf{q}, w) \equiv (q_1, \ldots, q_{n-2}, w) = \begin{pmatrix} I_{n-2} & -\mathbf{q}^T & \mathbf{q}^T \\ \overline{\mathbf{q}} & 1 - w - \frac{|\mathbf{q}|^2}{2} & w + \frac{|\mathbf{q}|^2}{2} \\ \overline{\mathbf{q}} & -w - \frac{|\mathbf{q}|^2}{2} & 1 + w + \frac{|\mathbf{q}|^2}{2} \end{pmatrix}, \quad (8)$$

where $w$ is a quaternion of the form $bi+cj+dk$. It is a maximal nilpotent subgroup in $Sp(n-1,1)$. If $\boldsymbol{\xi}_0$ denotes the vector from $C_H^{n-1}$ with the coordinates $(0,\ldots,0,1,1)$, then for any element $n(\mathbf{q},w) \in N$ we have $n(\mathbf{q},w)\boldsymbol{\xi}_0 = \boldsymbol{\xi}_0$. The subgroup in $Sp(n-1,1)$, leaving $\boldsymbol{\xi}_0$ fixed, is of the form $MN = NM$, where $M$ is the subgroup of matrices $g = \mathrm{diag}(h,u,u)$, $h \in Sp(n-2)$, $u \in Sp(1)$. The subgroup $M$ is isomorphic to $Sp(n-2)\times Sp(1)$. It is the centralizer of the subgroup $A$ in $K \equiv Sp(n-1)\times Sp(1)$.

For the elements $g'_{n-1}(t)$ we have $g'_{n-1}(t)\boldsymbol{\xi}_0 = e^t\boldsymbol{\xi}_0$.

**11.7.3. Invariant measures and Laplace operators.** Points of the quaternion sphere $S_H^{n-1}$ are enumerated by the elements $g^n(\mathbf{u},\boldsymbol{\theta})$ of the group $Sp(n)$ (see formula (8) of Section 11.7.1). We represent $g^n(\mathbf{u},\boldsymbol{\theta})$ in the form

$$g^n(\mathbf{u},\boldsymbol{\theta}) = g^{n-1}(\mathbf{u}',\boldsymbol{\theta}')d_n(u_n)g_{n-1}(\theta_{n-1}), \quad (1)$$

where $\mathbf{u}' = (u_1,\ldots,u_{n-1})$, $\boldsymbol{\theta}' = (\theta_1,\ldots,\theta_{n-2})$. The normalized invariant measure $d\boldsymbol{\xi}$ on $S_H^{n-1}$ is found by means of the tree (11) of Section 11.7.1 and of the results of Section 10.5.2. We have

$$d\boldsymbol{\xi} = dg^n(\mathbf{u},\boldsymbol{\theta}) = 4(n-1)(2n-1)\sin^{4n-5}\theta_{n-1}\cos^3\theta_{n-1}d\theta_{n-1}du_n d\boldsymbol{\xi}', \quad (2)$$

where $du_n$ is the normalized invariant measure on $Sp(1)$ and $d\boldsymbol{\xi}' \equiv dg^{n-1}(\mathbf{u}',\boldsymbol{\theta}')$ is the normalized invariant measure on $S_H^{n-2}$.

Points of $S_H^{n-1}$ can be also parametrized by the elements

$$g^s(\mathbf{u}',\boldsymbol{\theta})g^{n-s}(\mathbf{u}'',\boldsymbol{\psi})g_{sn}(\theta)$$

of the group $Sp(n)$ (see formula (14) of Section 11.7.1). In these parameters the normalized invariant measure $d\boldsymbol{\xi}$ on $S_H^{n-1}$ is of the form

$$d\boldsymbol{\xi} = \frac{2(2n-1)!}{(2s-1)!(2n-2s-1)!}\sin^{4s-1}\theta\cos^{4n-4s-1}\theta\, d\theta\, d\boldsymbol{\xi}'\, d\boldsymbol{\xi}'', \quad (3)$$

where $d\boldsymbol{\xi}' \equiv dg^s(\mathbf{u}',\boldsymbol{\theta})$ and $d\boldsymbol{\xi}'' \equiv dg^{n-s}(\mathbf{u}'',\boldsymbol{\psi})$.

The invariant measure on $H_H^{n-1}$ in parameters (4) of Section 11.7.2 is written down as

$$d\boldsymbol{\eta} \equiv dg^n(\mathbf{u},\boldsymbol{\theta},t) = \sinh^{4n-5}t\cosh^3 t\, dt\, du_n\, \boldsymbol{\xi}', \quad (4)$$

where $d\xi' \equiv dg^{n-1}(u_1,\ldots,u_{n-1},\theta)$ is the invariant measure on $S_H^{n-2}$, and in parameters (6) of Section 11.7.2 as

$$d\eta \equiv \sinh^{4s-1} t \cosh^{4n-4s-1} t \, dt \, dg^s(\mathbf{u}',\theta) dg^{n-s}(\mathbf{u}'',\psi,\beta). \tag{5}$$

The space $\mathbf{H}^n$ can be identified with $\mathbf{R}^{4n}$. The Laplace operator $\Delta$ on $\mathbf{H}^n$ coincides with the Laplace operator on $\mathbf{R}^{4n}$. If $q_j = x_j + y_j i + s_j j + t_j k$, then

$$\Delta = \sum_{j=1}^{n} \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial s_j^2} + \frac{\partial^2}{\partial t_j^2} \right).$$

As in the case of the group $SO(n)$ (see Section 9.1.8), it follows from here that the Laplace operator $\Delta_0^{(n)}$ on $S_H^{n-1}$ coincides with the Laplace operator on the real sphere $S^{4n-1}$. In the spherical coordinate system on $S_H^{n-1}$ we have

$$\Delta_0^{(n)} = \frac{1}{\sin^{4n-5}\theta \cos^3\theta} \frac{\partial}{\partial\theta} \sin^{4n-5}\theta \cos^3\theta \frac{\partial}{\partial\theta} + \frac{\Delta_0^{(n-1)}}{\sin^2\theta} + \frac{\Delta_0^{(1)}}{\cos^2\theta}, \tag{6}$$

where $\Delta_0^{(n-1)}$ is the Laplace operator on $S_H^{n-2}$ and $\Delta_0^{(1)}$ is the Laplace operator on $S_H^0 \sim Sp(1)$. In bispherical coordinates we obtain

$$\Delta_0^{(n)} = \frac{1}{\sin^{4s-1}\theta \cos^{4n-4s-1}\theta} \frac{\partial}{\partial\theta} \sin^{4s-1}\theta \cos^{4n-4s-1}\theta \frac{\partial}{\partial\theta} + \frac{\Delta_0^{(s)}}{\sin^2\theta} + \frac{\Delta_0^{(n-s)}}{\cos^2\theta}, \tag{7}$$

where $\Delta_0^{(s)}$ and $\Delta_0^{(n-s)}$ are the Laplace operators on $S_H^{s-1}$ and $S_H^{n-s-1}$, respectively.

In spherical coordinates on $H_H^{n-1}$ the Laplace operator is of the form

$$\Box_0^{(n)} = \frac{1}{\sinh^{4n-5} t \cosh^3 t} \frac{\partial}{\partial t} \sinh^{4n-5} t \cosh^3 t \frac{\partial}{\partial t} + \frac{\Delta_0^{(n-1)}}{\sinh^2 t} - \frac{\Delta_0^{(1)}}{\cosh^2 t} \tag{8}$$

and in cylindrical coordinates we obtain

$$\Box_0^{(n)} = \frac{1}{\sinh^{4s-1} t \cosh^{4n-4s-1} t} \frac{\partial}{\partial t} \sinh^{4s-1} t \cosh^{4n-4s-1} t \frac{\partial}{\partial t} + \frac{\Delta_0^{(s)}}{\sinh^2 t} - \frac{\Box_0^{(n-s)}}{\cosh^2 t}. \tag{9}$$

**11.7.4. Spherical functions of representations of the group $Sp(n)$.** Irreducible representations of the group $Sp(n)$, which are of class 1 relative to the subgroup $Sp(n-1)$, are realized in subspaces of the space of harmonic functions on $\mathbb{H}^n \sim \mathbb{R}^{4n}$ or in subspaces of the space of square integrable (with respect to the invariant measure) functions on $S_H^{n-1} \sim S^{4n-1}$. Let $\mathfrak{H}^{4n,m}$ be the space of homogeneous harmonic polynomials on $\mathbf{R}^{4n}$ of degree $m$. The left shift operators $L(g), g \in Sp(n) \subset SO(4n)$, realize on $\mathfrak{H}^{4n,m}$ the reducible representation $L_m$ of $Sp(n)$. The representation $L_m$ decomposes into the orthogonal sum of irreducible representations (some of them appear in the decomposition more than once). Let $T_1^n, \ldots, T_k^n$ be pairwise nonequivalent irreducible representations from this decomposition. One can show that the restriction of every one of the representations $T_1^n, \ldots, T_k^n$ onto the subgroup $Sp(n-1)$ contains (with unit multiplicity) the identity representation $T_0^{n-1}$ of $Sp(n-1)$. We restrict $T_1^n, \ldots, T_k^n$ onto the subgroup $Sp(n-1) \times Sp(1)$. These restrictions contain the irreducible representations[5] $T_0^{n-1} \times T_r^1$ where for different $T_i^n$ $(i = 1, \ldots, k)$ the representations $T_r^1$ of $Sp(1)$ are different. The representations $T_i^n$ are characterized by numbers $m$, $r$ and are denoted by $T^{n\ell\ell'}$, $\ell \geq \ell' \geq 0$, where

$$\ell + \ell' = m, \quad \ell - \ell' = r. \tag{1}$$

The set of representations $\{T_1^n, \ldots, T_k^n\}$ coincides with $\{T^{n,m-a,a} | a = 0, 1, \ldots, [m/2]\}$, where $[m/2]$ is the integral part of $m/2$. We have

$$T^{n\ell\ell'} \Big\downarrow {}^{Sp(n)}_{Sp(n-1) \times Sp(1)} = \sum_{m,m'} \sum_{s} \otimes T^{n-1,mm'} \times T_s^1, \tag{2}$$

where the first summation is over all integral values of $m$ and $m'$ such that $\ell \geq m \geq m' \geq 0$ and the second one is over the values

$$s = |\ell - \ell' - m + m'|,\ |\ell - \ell' - m + m'| + 2, \ldots, \ell + \ell' - m - m'$$

if $\ell' \leq m$ and over the values

$$s = |\ell - \ell' - m + m'|,\ |\ell - \ell' - m + m'| - 2, \ldots, \ell - \ell' + m - m'$$

if $\ell' \geq m$. The proofs of these statements can be found in [301].

It follows from (2) that the representation $T^{n\ell\ell'}$ of the group $Sp(n)$ is of class 1 relative to the subgroup $Sp(n-1) \times Sp(1)$ if and only if $\ell = \ell'$.

The dimensionality of the representation $T^{n\ell\ell'}$ is

$$\dim T^{n\ell\ell'} = \frac{(\ell + \ell' + 2n - 1)(\ell - \ell' + 1)(\ell + 2n - 2)!(\ell' + 2n - 3)!}{(\ell + 1)!\ell'!(2n - 1)!(2n - 3)!}. \tag{3}$$

The associated spherical functions of the representation $T^{n\ell\ell'}$ of $Sp(n)$ in the basis, corresponding to the subgroup chain

$$Sp(n) \supset Sp(n-1) \times Sp(1) \supset Sp(n-2) \times Sp(1) \times U(1) \supset \ldots, \tag{3'}$$

[5] The groups $Sp(1)$ and $SU(2)$ are isomorphic. Therefore, irreducible representations of these groups coincide. We denote by $T_r^1$ the irreducible representation of $Sp(1)$ which coincides with the representation $T_{r/2}$ of $SU(2)$ (see Section 6.2.1). It is clear that $r$ is an integer and that $T_r^1$ is realized in the space of homogeneous polynomials in $z_1$ and $z_2$ of degree $r$.

(where $Sp(n-2)\times Sp(1)$ is a subgroup in $Sp(n-1)$ and $U(1)$ is a subgroup in $Sp(1)$), are functions of $g^n(\mathbf{u},\boldsymbol{\theta})$ (see formula (8) of Section 11.7.1). They are of the form

$$t^{n\ell\ell'}_{MO}(g^n(\mathbf{u},\boldsymbol{\theta})) = t^{n-1,mm'}_{NO}(g^{n-1}(\mathbf{u}',\boldsymbol{\theta}'))t^{1s}_{kk'}(u_n)t^{n\ell\ell'}_{(mm',s)0}(g_{n-1}(\theta)). \tag{4}$$

Here $t^{n-1,mm'}_{NO}(g^{n-1}(\mathbf{u}',\boldsymbol{\theta}'))$ is the associated spherical function of the representation $T^{n-1,mm'}$ of the subgroup $Sp(n-1)$, $t^{1s}_{kk'}(u_n)$ is the matrix element of the representation $T^1_s$ of $Sp(1)$ and $M=(mm',s,N)$. Spherical function (4) belongs to the space $\widetilde{\mathfrak{H}}^{4n,\ell+\ell'}$ (see formula (1)). Therefore,

$$\Delta_0^{(n)}t^{n\ell\ell'}_{MO}(g^n(\mathbf{u},\boldsymbol{\theta})) = -(\ell+\ell')(\ell+\ell'+4n-2)t^{n\ell\ell'}_{MO}(g^n(\mathbf{u},\boldsymbol{\theta})). \tag{5}$$

Consequently, from formula (6) of Section 11.7.4 for $\Delta_0^{(n)}$ and from (4) we derive the differential equation for $t^{n\ell\ell'}_{(mm',s)0}(g_{n-1}(\theta))$:

$$\left[\frac{1}{\sin^{4n-5}\theta\cos^3\theta}\frac{d}{d\theta}\sin^{4n-5}\theta\cos^3\theta\frac{d}{d\theta} - \frac{(m+m')(m+m'+4n-6)}{\sin^2\theta} - \frac{s(s+2)}{\cos^2\theta} + (\ell+\ell')(\ell+\ell'+4n-2)\right]t^{n\ell\ell'}_{(mm',s)0}(g_{n-1}(\theta)) = 0. \tag{6}$$

Comparing (6) with equation (2) of Section 10.4.4 for the associated $SO(4n-4)\times SO(4)$-spherical functions $t^{SO(4n),\ell+\ell'}_{(m+m',s)0}(g_{4n-4,4n}(\theta))$ of the symmetric space $SO(4n)/SO(4n-1)$ (see formula (8) of Section 10.4.2), we find that

$$t^{Sp(n),\ell\ell'}_{(mm',s)0}(g_{n-1}(\theta)) = A^{n\ell\ell'}_{mm',s}t^{SO(4n),\ell+\ell'}_{(m+m',s)0}(g_{4n-4,4n}(\theta)). \tag{7}$$

The constant factor $A^{n\ell\ell'}_{mm',s}$ can be found in the same way as in formula (8) of Section 10.4.2. We have

$$A^{n\ell\ell'}_{mm',s} = \left[\frac{\left(\dim T^{SO(4n),\ell+\ell'}\right)\left(\dim T^{Sp(n-1),mm'}\right)\left(\dim T^{Sp(1),s}\right)}{\left(\dim T^{SO(4n-4),m+m'}\right)\left(\dim T^{SO(4),s}\right)\left(\dim T^{Sp(n),\ell\ell'}\right)}\right]^{1/2}. \tag{8}$$

Therefore,

$$t^{Sp(n),\ell\ell'}_{(mm',s)0}(g_{n-1}(\theta)) = N\sin^{m+m'}\theta\cos^s\theta P^{(m+m'+2n-3,s+1)}_{(\ell+\ell'-m-m'-s)/2}(\cos 2\theta), \tag{9}$$

where

$$N = \left[\frac{2(n-2)(2n-3)(m+m'+2n-3)(m-m'+1)(m+2n-4)!}{(m+1)!m'!(\ell-\ell'+1)(\ell+2n-2)!(\ell'+2n-3)!} \times\frac{(m'+2n-5)!(s+1)(\ell+1)!\ell'!}{\Gamma\left(\frac{1}{2}(\ell+\ell'+m+m'-s+4n-4)\right)} \times\frac{\left(\frac{1}{2}(\ell+\ell'-m-m'-s)\right)!\Gamma\left(\frac{1}{2}(\ell+\ell'+m+m'+s+4n-2)\right)}{\Gamma\left(\frac{1}{2}(\ell+\ell'-m-m'+s+4)\right)}\right]^{1/2}. \tag{10}$$

We also have

$$t^{Sp(n),\ell\ell'}_{(mm',s)0}(g_{n-1}(\theta)) = B^{n\ell\ell'}_{mm',s} t^{U(2n),\ell\ell'}_{(mm',s)0}(g_{2n-2,2n}(\theta)), \tag{11}$$

where $t^{U(2n),\ell\ell'}_{(mm',s)0}(g_{2n-2,2n}(\theta))$ is the associated $U(2n-2)\times U(2)$-spherical function of the space $U(2n)/U(2n-1)$ (see Section 11.3.12) and

$$B^{n\ell\ell'}_{mm',s} = \left[\frac{(\dim T^{U(2n),\ell\ell'})(\dim T^{Sp(n-1),mm'})}{(\dim T^{Sp(n),\ell\ell'})(\dim T^{U(2n-2),mm'})}\right]^{1/2}. \tag{12}$$

It follows from formulas (8) and (9) of Section 11.7.1 that if function (4) is invariant with respect to the left shifts by elements of $Sp(n-1)$, then $m=m'=0$, $s=\ell-\ell'$. Hence, zonal spherical functions of the space $S^{n-1}_H \sim Sp(n)/Sp(n-1)$ are of the form

$$\begin{aligned} t^{Sp(n),\ell\ell'}_{OO}(g^n(\mathbf{u},\theta)) &= t^{1,\ell-\ell'}_{kk'}(u_n) t^{n,\ell\ell'}_{(00,\ell-\ell')0}(g_{n-1}(\theta)) \\ &= \frac{\ell!(2n-3)!}{(\ell'+2n-3)!} t^{1,\ell-\ell'}_{kk'}(u_n)\cos^{\ell-\ell'}\theta P^{(2n-3,\ell-\ell'+1)}_{\ell'}(\cos 2\theta). \end{aligned} \tag{13}$$

For zonal spherical functions of the space $Sp(n)/(Sp(n-1)\times Sp(1))$ we have $\ell=\ell'$. Therefore, they are given by the formula

$$t^{Sp(n),\ell\ell'}_{00}(g_{n-1}(\theta)) = \frac{\ell!(2n-3)!}{(\ell+2n-3)!} P^{(2n-3,1)}_{\ell}(\cos 2\theta). \tag{14}$$

The associated $Sp(s)\times Sp(n-s)$-spherical function of the space $Sp(n)/Sp(n-1)$, that is, the associated spherical function of the representation $T^{n\ell\ell'}$ of $Sp(n)$ in bispherical coordinates (13) of Section 11.7.1, has the form

$$\begin{aligned} &t^{n\ell\ell'}_{PO}(g^s(\mathbf{u}',\theta)g^{n-s}(\mathbf{u}'',\psi)g_{sn}(\theta)) \\ &\quad = t^{n\ell\ell'}_{(mm',kk')0}(g_{sn}(\theta)) t^{smm'}_{MO}(g^s(\mathbf{u}',\theta)) t^{n-s,kk'}_{NO}(g^{n-s}(\mathbf{u}'',\psi)), \end{aligned} \tag{15}$$

where $t^{smm'}_{MO}(\dots)$ and $t^{n-s,kk'}_{NO}(\dots)$ are the associated spherical functions of the representations $T^{smm'}$ and $T^{n-s,kk'}$ of the subgroups $Sp(s)$ and $Sp(n-s)$, respectively, and $P=(mm',M,kk',N)$. The function $t^{n\ell\ell'}_{(mm',kk')0}(g_{sn}(\theta))$ satisfies the differential equation

$$\begin{aligned} &\left[\frac{1}{\sin^{4s-1}\theta\cos^{4n-4s-1}\theta}\frac{d}{d\theta}\sin^{4s-1}\theta\cos^{4n-4s-1}\theta\frac{d}{d\theta} - \frac{(m+m')(m+m'+4s-2)}{\sin^2\theta}\right. \\ &\quad \left. - \frac{(k+k')(k+k'+4n-4s-2)}{\cos^2\theta} + (\ell+\ell')(\ell+\ell'+4n-2)\right]F(\theta) = 0. \end{aligned} \tag{16}$$

Comparing (16) with equation (2) of Section 10.4.4 for the associated $SO(4s) \times SO(4n-4s)$-spherical function

$$t^{SO(4n),\ell+\ell'}_{(m+m',k+k')0}(g_{4s,4n}(\theta))$$

of the symmetric space $SO(4n)/SO(4n-1)$ (see formula (7) of Section 10.4.2), we find that

$$t^{Sp(n),\ell\ell'}_{(mm',kk')0}(g_{sn}(\theta)) = A^{n\ell\ell'}_{mm'kk'} t^{SO(4n),\ell+\ell'}_{(m+m',k+k')0}(g_{4s,4n}(\theta)), \tag{17}$$

where

$$A^{n\ell\ell'}_{mm'kk'} = \left[\frac{(\dim T^{SO(4n),\ell+\ell'})(\dim T^{Sp(s),mm'})(\dim T^{Sp(n-s),kk'})}{(\dim T^{SO(4s),m+m'})(\dim T^{SO(4n-4s),k+k'})(\dim T^{Sp(n),\ell\ell'})}\right]^{\frac{1}{2}}. \tag{18}$$

Therefore,

$$t^{Sp(n),\ell\ell'}_{(mm',kk')0}(g_{sn}(\theta)) = N \sin^{m+m'}\theta \cos^{k+k'}\theta \times P^{(m+m'+2s-1,k+k'+2n-2s-1)}_{(\ell+\ell'-p-p'-k-k')/2}(\cos 2\theta), \tag{19}$$

where

$$N = \left[\frac{(\ell+1)!\ell'!(m+m'+2s-1)(m-m'+1)(k+k'+2n-2s-1)}{(\ell-\ell'+1)(\ell+2n-2)!(\ell'+2n-3)!(m+1)!m'!} \times \frac{(k-k'+1)(m+2s-2)!(m'+2s-3)!(k+2n-2s-2)!}{(k+1)!k'!(2s+1)!(2s+3)!(2n-2s-3)!(2n-2s-1)!} \times \frac{(k'+2n-2s-3)!(2n-3)!^2\left(\frac{1}{2}(\ell+\ell'-m-m'-k-k')\right)!}{\Gamma\left(\frac{1}{2}(\ell+\ell'+m+m'-k-k'+4s)\right)} \times \frac{\Gamma\left(\frac{1}{2}(\ell+\ell'+m+m'+k+k'+4n-2)\right)}{\Gamma\left(\frac{1}{2}(\ell+\ell'-m-m'+k+k'+4n-4s)\right)}\right]^{1/2}.$$

**11.7.5. The representations $T^{n\sigma}$ of the group $Sp(n-1,1)$.** Let $\mathfrak{B}^{n\sigma}_H$, $\sigma \in \mathbb{C}$, be the space of infinitely differentiable complex functions $f$ on the cone $C^{n-1}_H = \{\mathbf{q} \in \mathbf{H}^n \mid [\mathbf{q},\mathbf{q}] = 0,\ \mathbf{q} \neq 0\}$ satisfying the homogeneity conditions

$$f(a\mathbf{q}) = a^\sigma f(\mathbf{q}), \quad a > 0, \tag{1}$$

$$f(\mathbf{q}u) = f(\mathbf{q}), \quad u \in Sp(1). \tag{2}$$

The operators $T^{n\sigma}(g)\colon f(\mathbf{q}) \to f(g^{-1}\mathbf{q})$, $g \in Sp(n-1,1)$, leave $\mathfrak{B}^{n\sigma}_H$ invariant. They define the representation of the group $Sp(n-1,1)$, which will be denoted by $T^{n\sigma}$.

By virtue of properties (1) and (2), functions $f \in \mathfrak{B}_H^{n\sigma}$ are uniquely defined by their values on any contour $\Gamma$, intersecting every generatrix of the cone $C_H^{n-1}$ at one point. The representation $T^{n\sigma}$ can be realized in the space of functions on this contour. As in the case of representations of the group $SO_0(n-1,1)$ (see Section 9.2.1), one shows that in this space the operators $T^{n\sigma}(g)$, $g \in Sp(n-1,1)$, are given by the formula

$$(T^{n\sigma}(g)F)(\boldsymbol{\eta}) = \alpha(\boldsymbol{\eta}, g)^\sigma F(\widehat{\boldsymbol{\eta}}), \tag{3}$$

where $\boldsymbol{\eta} \in \Gamma$, $\widehat{\boldsymbol{\eta}} \in \Gamma$ and the real function $\alpha(\boldsymbol{\eta}, g)$ is defined by the equality

$$\widehat{\boldsymbol{\eta}} = \alpha^{-1}(\boldsymbol{\eta}, g)(g^{-1}\boldsymbol{\eta})u, \quad u \in Sp(1).$$

In particular, if $\Gamma = S_H^{n-2}$ and $F(\boldsymbol{\eta}) = F(\mathbf{u}, \boldsymbol{\theta})$, then

$$(T^{n\sigma}(k)F)(\boldsymbol{\eta}) = F(k^{-1}\boldsymbol{\eta}), \quad k \in Sp(n-1),\ \boldsymbol{\eta} \in S_H^{n-2}, \tag{4}$$

and

$$(T^{n\sigma}(g'_{n-1}(t))F)(\mathbf{u}, \boldsymbol{\theta}) = |\cosh t - u_{n-1}\cos\theta_{n-2}\sinh t|^\sigma \times F(u'_1, \dots, u'_{n-1}, \theta_1, \dots, \theta_{n-3}, \theta'_{n-2}), \tag{5}$$

where

$$\cos\theta'_{n-2} = \left|\frac{u_{n-1}\cos\theta_{n-2}\cosh t - \sinh t}{\cosh t - u_{n-1}\cos\theta_{n-2}\sinh t}\right|,$$

$$\sin\theta'_{n-2} = \frac{\sin\theta_{n-2}}{|\cosh t - u_{n-1}\cos\theta_{n-2}\sinh t|}, \tag{6}$$

$$u'_{n-1} = \frac{(u_{n-1}\cos\theta_{n-2}\cosh t - \sinh t)|\cosh t - u_{n-1}\cos\theta_{n-2}\sinh t|}{|u_{n-1}\cos\theta_{n-2}\cosh t - \sinh t|(\cosh t - u_{n-1}\cos\theta_{n-2}\sinh t)}, \tag{7}$$

$$u'_s = u_s\frac{|\cosh t - u_{n-1}\cos\theta_{n-2}\sinh t|}{\cosh t - u_{n-1}\cos\theta_{n-2}\sinh t}. \tag{8}$$

If $\operatorname{Re} u_{n-1} = \cos\varphi$, then

$$|\cosh t - u_{n-1}\cos\theta_{n-2}\sinh t| = |\cosh t - e^{i\varphi}\cos\theta_{n-2}\sinh t| = (\cosh^2 t + \sinh^2 t\cos\theta_{n-2} - \sinh 2t\cos\theta_{n-2}\cos\varphi)^{1/2} \tag{9}$$

and (5) can be written as

$$(T^{n\sigma}(g'_{n-1}(t))F)(\mathbf{u}, \boldsymbol{\theta}) = (\cosh^2 t + \sinh^2 t\cos\theta_{n-2} - \sinh 2t\cos\theta_{n-2}\cos\varphi)^{\sigma/2} F(u'_1, \dots, u'_{n-1}; \theta_1, \dots, \theta_{n-3}, \theta'_{n-2}). \tag{10}$$

The restriction of $T^{n\sigma}$ onto $Sp(n-1)\times Sp(1)$ decomposes into irreducible components as follows:

$$T^{n\sigma}\Big\downarrow^{Sp(n-1,1)}_{Sp(n-1)\times Sp(1)} = \sum_{\ell\geq\ell'\geq 0}\oplus(T^{n-1,\ell\ell'}\otimes T^{1,\ell-\ell'}). \tag{11}$$

(See, for example, [300].) Consequently, $T^{n\sigma}$ is of class 1 relative to the subgroup $Sp(n-1)\times Sp(1)$.

We equip $\mathfrak{B}_H^{n\sigma}$ with the scalar product

$$(F_1,F_2) = \int\limits_{S_H^{n-2}} F_1(\boldsymbol{\xi})\overline{F_2(\boldsymbol{\xi})}d\boldsymbol{\xi}, \tag{12}$$

where $d\boldsymbol{\xi}$ is the normalized invariant measure on $S_H^{n-2}$, and complete this space with respect to the corresponding norm. We obtain the Hilbert space $\mathfrak{L}^2(S_H^{n-2})$. The operators $T^{n\sigma}(g)$, $g\in Sp(n-1,1)$, are continued to bounded operators in $\mathfrak{L}^2(S_H^{n-2})$, and we obtain the representation $T^{n\sigma}$ of $Sp(n-1,1)$ in $\mathfrak{L}^2(S_H^{n-2})$.

By using scalar product (12), one can easily verify that the representations $T^{n\sigma}$ and $T^{n,-\bar\sigma-4n+2}$ are Hermitian-adjoint. In particular, the representations $T^{n\sigma}$ with $\sigma=i\rho-2n+1$, $\rho\in\mathbb{R}$, are unitary. They form the so-called *spherical principal unitary series of representations of the group* $Sp(n-1,1)$.

**11.7.6. Spherical functions of the representations $T^{n\sigma}$.** The set of functions

$$\Xi_M^{mm'}(\mathbf{u},\boldsymbol{\theta}) \equiv (\dim T^{n-1,mm'})^{1/2}\overline{t_{MO}^{n-1,mm'}(g^{n-1}(\mathbf{u},\boldsymbol{\theta}))}, \tag{1}$$

where $t_{MO}^{n-1,mm'}(\dots)$ are associated spherical functions of the representation $T^{n-1,mm'}$, $m\geq m'\geq 0$, of the subgroup $Sp(n-1)$ (see formula (4) of Section 11.7.4), form an orthonormal basis of $\mathfrak{L}^2(S_H^{n-2})$. If an element $g\in Sp(n-1,1)$ has the form

$$g = g^n(\mathbf{u},\boldsymbol{\theta},t)h = g^{n-1}(\mathbf{u}',\boldsymbol{\theta})d_n(u_n)g'_{n-1}(t)h, \tag{2}$$
$$g^{n-1}(\mathbf{u}',\boldsymbol{\theta})\in Sp(n-1),\quad h\in Sp(n-1)$$

(see Section 11.7.2), then for the associated spherical function

$$t^{n\sigma}_{(mm',M)O}(g) \equiv (T^{n\sigma}(g)1,\Xi_M^{mm'}) \tag{3}$$

of the representation $T^{n\sigma}$ we have

$$t^{n\sigma}_{(mm',M)O}(g) = t^{n\sigma}_{(mm',M)O}(g^n(\mathbf{u},\boldsymbol{\theta},t))$$
$$= t^{n\sigma}_{(mm')0}(g'_{n-1}(t))t_{00}^{1,m-m'}(d_n(u_n))t_{MO}^{n-1,mm'}(g^{n-1}(\mathbf{u}',\boldsymbol{\theta})), \tag{4}$$

where $t^{n\sigma}_{(mm')0}(g'_{n-1}(t))$ does not depend on $M$.

As in the case of the group $U(n-1,1)$ (see Section 11.3.4), we derive that

$$\Box_0 t^{n\sigma}_{(mm',M)O}(g^n(\mathbf{u},\boldsymbol{\theta},t)) = \sigma(\sigma+4n-2)t^{n\sigma}_{(mm',M)O}(g^n(\mathbf{u},\boldsymbol{\theta},t)), \tag{5}$$

where $g^n(\mathbf{u},\boldsymbol{\theta},t)$ is considered as a point on $H^{n-1}_H$. Since

$$\Delta_0^{(n-1)} t^{n-1,mm'}_{MO}(g^{n-1}(\mathbf{u}',\boldsymbol{\theta})) = -(m+m')(m+m'+4n-4)t^{n-1,mm'}_{MO}(g^{n-1}(\mathbf{u}',\boldsymbol{\theta})),$$

then (4) and (5) imply the differential equation for $t^{n\sigma}_{(mm')0}(g'_{n-1}(\theta))$:

$$\left[\frac{1}{\sinh^{4n-5} t\cosh^3 t}\frac{d}{dt}\sinh^{4n-5} t\cosh^3 t\frac{d}{dt} - \frac{(m+m')(m+m'+4n-4)}{\sinh^2 t} + \frac{(m-m')(m-m'+2)}{\cosh^2 t} - \sigma(\sigma+4n-2)\right]u(t) = 0. \tag{6}$$

The function

$$u^{n\sigma}_{mm'}(t) = \tanh^{m+m'} t\cosh^{\sigma} t\, F\left(m-\frac{\sigma}{2}, m'-\frac{\sigma}{2}-1; m+m'+2n-2; \tanh^2 t\right)$$

is a solution of (6), which is regular at $t=0$. Therefore,

$$t^{n\sigma}_{(mm')0}(g'_{n-1}(t)) = a^{n\sigma}_{mm'}u^{n\sigma}_{mm'}(t). \tag{7}$$

In order to find the constants $a^{n\sigma}_{mm'}$, we note that formula (10) of Section 11.7.5 yields the integral representation of the matrix element $t^{n\sigma}_{(mm')0}(g'_{n-1}(t))$:

$$t^{n\sigma}_{(mm')0}(g'_{n-1}(t)) = \frac{(2n-3)(2n-4)}{\pi}(\dim T^{n-1,mm'})^{1/2} \times \int_0^{2\pi}\int_0^{\pi/2} |\cosh t - e^{i\varphi}\cos\theta\sinh t|^{\sigma}\overline{t^{n-1,mm'}_{00}(\varphi,\theta)}\sin^{4n-9}\theta\cos^3\theta\, d\theta\, d\varphi, \tag{8}$$

where $t^{n-1,mm'}_{00}(\varphi,\theta)$ is the zonal spherical function of the representation $T^{n-1,mm'}$ of the group $Sp(n-1)$ (see Section 11.7.2).

As in the case of $U(n-1,1)$ (see Section 11.3.5), by means of the formulas of Section 11.7.5 we find that

$$t^{n\sigma}_{(mm')0}(g'_{n-1}(t)) \sim (\dim T^{n-1,mm'})^{\frac{1}{2}} c_0(i(\sigma+2n-1))e^{-(\sigma+4n-2)t} \tag{9}$$

for $t \to +\infty$. Here $\mathrm{Re}\,\sigma < -2n-1$ and $\mathbf{c}_0(\lambda)$ is the Harish-Chandra $\mathbf{c}$-function of the group $Sp(n-1,1)$ which for $\mathrm{Re}\,\sigma < -2n-1$ is given by the formula

$$\mathbf{c}_0(i(\sigma+2n-1)) = \frac{2^{\sigma+4n-2}(2n-3)(2n-4)}{\pi} \int_0^{2\pi}\int_0^{\pi/2} |1-e^{i\varphi}\cos\theta|^{-\sigma-4n+2}$$
$$\times \sin^{4n-9}\theta\cos^3\theta\, d\theta\, d\varphi = \frac{2^{\sigma+4n-2}(2n-3)!\Gamma(-\sigma-2n+1)}{\Gamma\left(-\frac{\sigma}{2}\right)\Gamma\left(-\frac{\sigma}{2}-1\right)}. \quad (10)$$

Comparing the asymptotic behavior of the functions $t^{n\sigma}_{(mm')0}(g'_{n-1}(t))$ with the asymptotics of $u^{n\sigma}_{mm'}(t)$, we find that

$$t^{n\sigma}_{(mm')0}(g'_{n-1}(t))$$
$$= \left[\frac{(2n-3)!(m-m'+1)(m+2n-4)!(m'+2n-5)!}{(m+m'+2n-3)(m+1)!m'!(2n-5)!}\right]^{\frac{1}{2}} \frac{\Gamma(m-\frac{\sigma}{2})\Gamma(m'-\frac{\sigma}{2}-1)}{\Gamma\left(-\frac{\sigma}{2}\right)\Gamma\left(-\frac{\sigma}{2}-1\right)}$$
$$\times \frac{\tanh^{m+m'}t\cosh^\sigma t}{(m+m'+2n-4)!} F\left(m-\frac{\sigma}{2}, m'-\frac{\sigma}{2}-1; m+m'+2n-2; \tanh^2 t\right) \quad (11)$$

for $\mathrm{Re}\,\sigma < -2n-1$.

It follows from (8) that $t^{n\sigma}_{(mm')0}(g'_{n-1}(t))$ is an entire analytic function of $\sigma$. Therefore, expression (11) for $t^{n\sigma}_{(mm')0}(g'_{n-1}(\theta))$ is analytically continued onto all $\sigma \in \mathbf{C}$.

The zonal spherical function of the representation $T^{n\sigma}$ has the form

$$t^{n\sigma}_{00}(g'_{n-1}(t)) = \cosh^\sigma t\, F\left(-\frac{\sigma}{2}, -\frac{\sigma}{2}-1; 2n-2; \tanh^2 t\right). \quad (12)$$

It satisfies the differential equation

$$\left[\sinh^{5-4n}t\cosh^{-3}t\frac{d}{dt}\sinh^{4n-5}t\cosh^3 t\frac{d}{dt} - \sigma(\sigma+4n-2)\right]u(t) = 0. \quad (13)$$

Function (11) can be expressed in terms of $\mathfrak{P}^\tau_{mn}(\cosh 2t)$. Making use of expression (1) of Section 6.5.3 for $\mathfrak{P}^\tau_{mn}(x)$, we have

$$t^{n\sigma}_{(mm')0}(g'_{n-1}(t)) = (-1)^{m+m'}$$
$$\times \left[\frac{(m+m'+2n-3)(m-m'+1)(m+2n-4)!(m'+2n-5)!(2n-3)(2n-4)}{(m+1)!m'!}\right]^{1/2}$$
$$\times \frac{\Gamma\left(\frac{\sigma}{2}+1\right)\Gamma\left(\frac{\sigma}{2}+2\right)}{\Gamma(\frac{\sigma}{2}-m'+2)\Gamma(\frac{\sigma}{2}+m'+2n-2)} \frac{\sinh^{3-2n}t}{\cosh t}\mathfrak{P}^{\sigma/2+n-1}_{m+n-1,-m'-n+2}(\cosh 2t). \quad (14)$$

In particular,

$$t_{00}^{n\sigma}(g'_{n-1}(t)) = \frac{(2n-3)!\Gamma\left(\frac{\sigma}{2}+1\right)}{\Gamma\left(\frac{\sigma}{2}+2n-2\right)}\frac{\sinh^{3-2n}t}{\cosh t}\mathfrak{P}_{n-1,-n+2}^{\sigma/2+n-1}(\cosh 2t). \tag{15}$$

For spherical principal unitary series representations we obtain

$$t_{00}^{n,i\rho-2n+1}(g'_{n-1}(t)) = \frac{(2n-3)!\Gamma\left(\frac{i\rho+3}{2}-n\right)}{\Gamma\left(\frac{i\rho-3}{2}+n\right)}\frac{\sinh^{3-2n}t}{\cosh t}\mathfrak{P}_{n-1,-n+2}^{(i\rho-1)/2}(\cosh 2t). \tag{16}$$

By means of zonal and associated spherical functions of representations of the groups $Sp(n)$ and $Sp(n-1,1)$ derived above, one can obtain new addition and product formulas for Jacobi polynomials and functions. These formulas generalize those of Section 11.4. Because of the awkwardness of these formulas, we do not give them here.

**11.7.7. Expansion in zonal spherical functions of the group $Sp(n-1,1)$.** Formulas (5) and (6) of Section 11.4.8 give the mutually reciprocal transforms

$$f(\cosh t) = \frac{1}{2}\int_{-\infty}^{\infty} a(\rho)\mathfrak{P}_{n-1,-n+2}^{(i\rho-1)/2}(\cosh 2t)\rho\tanh\frac{\pi\rho}{2}d\rho, \tag{1}$$

$$a(\rho) = 4\int_0^{\infty} f(\cosh t)\mathfrak{P}_{n-1,-n+2}^{-(i\rho+1)/2}(\cosh 2t)\sinh t\cosh t\,dt. \tag{2}$$

Introducing the notations

$$F(t) = 4\sinh^{-2n-3}t\cosh^{-1}t f(\cosh t),\ b(\rho) = \frac{(2n-3)!\Gamma\left(\frac{-i\rho+3}{2}-n\right)}{\Gamma\left(\frac{-i\rho-3}{2}+n\right)}a(\rho)$$

and taking into account expression (16) of Section 11.7.6 for the zonal spherical functions of the representation $T^{n,i\rho-2n+1}$, we have

$$F(t) = \frac{2}{(2n-3)!^2}\int_{-\infty}^{\infty} b(\rho)t_{00}^{n,i\rho-2n+1}(g'_{n-1}(t))d\mu(\rho), \tag{3}$$

$$b(\rho) = \int_0^{\infty} F(t)t_{00}^{n,-i\rho-2n+1}(g'_{n-1}(t))\sinh^{4n-5}t\cosh^3 t\,dt, \tag{4}$$

where the Plancherel measure $d\mu(\rho)$ is given as

$$d\mu(\rho) = \left|\frac{\Gamma\left(\frac{i\rho-3}{2}+n\right)}{\Gamma\left(\frac{i\rho+3}{2}-n\right)}\right|^2\rho\tanh\frac{\pi\rho}{2}d\rho. \tag{5}$$

It is easy to derive from formula (8) of Section 11.4.8 the Plancherel formula for transforms (3) and (4):

$$\int_0^\infty |F(t)|^2 \sinh^{4n-5} t \cosh^3 t\, dt = \frac{4}{(2n-3)!^2} \int_0^\infty |b(\rho)|^2 d\mu(\rho). \tag{6}$$

**11.7.8. Decomposition of the quasi-regular representation of the group $Sp(n-1,1)$.** Let $\mathfrak{L}^2(H_H^{n-1})$ be the Hilbert space of functions on $H_H^{n-1}$ with the scalar product

$$(f_1, f_2) = \int_{H_H^{n-1}} f_1(\eta)\overline{f_2(\eta)}d\eta,$$

where $d\eta$ is the invariant measure (4) of Section 11.7.3, and $\mathfrak{L}_0^2(H_H^{n-1})$ is the subspace of $\mathfrak{L}^2(H_H^{n-1})$ consisting of functions $f$ satisfying the condition $f(\eta u) = f(\eta)$, $u \in Sp(1)$. The space $\mathfrak{L}_0^2(H_H^{n-1})$ can be identified with $\mathfrak{L}^2(\mathcal{P}_H^{n-1})$. The operators

$$(L(g)f)(\eta) = f(g^{-1}\eta), \quad g \in Sp(n-1,1),$$

define the left quasi-regular representation in $\mathfrak{L}_0^2(H_H^{n-1})$.

We introduce on $\mathfrak{L}_0^2(H_H^{n-1})$ the transform

$$b_M(\rho) = \int_{H_H^{n-1}} f(\eta) t_{MO}^{n,-i\rho-2n+1}(\eta)d\eta, \tag{1}$$

where $t_{MO}^{n\sigma}(\eta)$, $\eta = g^n(\mathbf{u}, \boldsymbol{\theta}, t)\mathbf{e}_n$, are associated spherical functions (4) of the representations $T^{n\sigma}$ from Section 11.7.6. As in the case of the quasi-regular representation of $SO_0(n-1,1)$ (see Section 10.4.7), we split this transform into two: the transform by means of the associated spherical functions $t_{MO}^{n-1,mm'}(d_n(u_n)g^{n-1}(\mathbf{u}', \boldsymbol{\theta}))$ of representations of the subgroup $Sp(n-1)\times Sp(1)$ and the transform by means of the function $t_{(mm')0}^{n,i\rho-2n+1}(g'_{n-1}(t))$. The first of these transforms is inverted with the help of the results of Section 2.3.9. The transform with the kernel $t_{(mm')0}^{n,i\rho-2n+1}({}'_{n-1}(t))$ is inverted in the same way as in Section 11.7.7 (by reducing it to transforms (5) and (6) of Section 11.4.8). As a result, we obtain the following invertion of transform (1):

$$f(\eta) = \frac{2}{(2n-3)!^2} \sum_M \int_{-\infty}^{\infty} b_M(\rho) t_{MO}^{n,i\rho-2n+1}(\eta) d\mu(\rho), \tag{2}$$

where $d\mu(\rho)$ is given by formula (5) of Section 11.7.7 and the summation is over all $M$, enumerating associated spherical functions of the representation $T^{n,i\rho-2n+1}$.

The Plancherel formula for these transforms is of the form

$$\int\limits_{H_H^{n-1}} |f(\boldsymbol{\eta})|^2 d\boldsymbol{\eta} = \frac{2}{(2n-3)!^2} \sum_M \int\limits_{-\infty}^{\infty} |b_M(\rho)|^2 d\mu(\rho). \tag{3}$$

It follows from formula (11) of Section 11.7.6 that

$$b_M(\rho) = \frac{\overline{a_{mm'}(\rho)}}{a_{mm'}(\rho)} b_M(-\rho), \tag{4}$$

where

$$a_{mm'}(\rho) = \frac{\Gamma\left(m + \frac{-i\rho+2n-1}{2}\right)\Gamma\left(m' + \frac{-i\rho+2n-3}{2}\right)}{\Gamma\left(\frac{-i\rho+2n-1}{2}\right)\Gamma\left(\frac{-i\rho+2n-3}{2}\right)}.$$

Hence

$$|b_M(\rho)|^2 = |b_M(-\rho)|^2$$

and formula (3) can be written as

$$\int\limits_{H_H^{n-1}} |f(\boldsymbol{\eta})|^2 d\boldsymbol{\eta} = \frac{4}{(2n-3)!^2} \sum_M \int\limits_0^{\infty} |b_M(\rho)|^2 d\mu(\rho). \tag{5}$$

For a fixed $\rho$ we define the Hilbert space $\mathfrak{B}^\rho$ of functions $F(\boldsymbol{\eta})$ on $H_H^{n-1}$, given by the series

$$F(\boldsymbol{\eta}) = \sum_M b_M(\rho) t_{MO}^{n,i\rho-2n+1}(\boldsymbol{\eta})$$

for which

$$\sum_M |b_M(\rho)|^2 < \infty.$$

We have

$$\mathfrak{L}_0^2(H_H^{n-1}) = \frac{4}{(2n-3)!^2} \int\limits_0^{\infty} \oplus \mathfrak{B}^\rho d\mu(\rho). \tag{6}$$

If functions from $\mathfrak{L}_0^2(H_H^{n-1})$ are transformed according to the left quasi-regular representation $L$ of the group $Sp(n-1,1)$, then functions from $\mathfrak{B}^\rho$ are transformed according to the representation $T^{n,i\rho-2n+1}$. Therefore, from (6) we derive that

$$L = \frac{4}{(2n-3)!^2} \int\limits_0^{\infty} \oplus T^{n,i\rho-2n+1} d\mu(\rho). \tag{7}$$

# Chapter 12.
# Representations of the Heisenberg Group and Special Functions

## 12.1. Representations of the Heisenberg Group, Hermite and Laguerre Polynomials

**12.1.1. The Heisenberg group.** The maximal nilpotent group $N$ in $U(n-1,1)$ consists of matrices (6) of Section 11.1.1. Replace in these matrices $\alpha$ by $-2c$. We denote the obtained matrices by

$$n(\mathbf{a},\mathbf{b},c) = \begin{pmatrix} I_{n-2} & (-\mathbf{a}+i\mathbf{b})^T & (\mathbf{a}-i\mathbf{b})^T \\ \mathbf{a}+i\mathbf{b} & 1+2ic-\frac{a^2+b^2}{2} & -2ic+\frac{a^2+b^2}{2} \\ \mathbf{a}+i\mathbf{b} & 2ic-\frac{a^2+b^2}{2} & 1-2ic+\frac{a^2+b^2}{2} \end{pmatrix}, \tag{1}$$

where

$$\mathbf{a}=(a_1,\ldots,a_{n-2}),\quad \mathbf{b}=(b_1,\ldots,b_{n-2}),\quad a_j\in\mathbb{R},\quad b_j\in\mathbb{R},$$
$$a^2=a_1^2+\ldots+a_{n-2}^2,\quad b^2=b_1^2+\ldots+b_{n-2}^2,$$

$T$ denotes the transposition and $I_{n-2}$ is the identity matrix of order $n-2$. It is clear that

$$n(\mathbf{a},\mathbf{b},c)n(\mathbf{a}',\mathbf{b}',c') = n\left(\mathbf{a}+\mathbf{a}',\mathbf{b}+\mathbf{b}',c+c'+\frac{1}{2}(\mathbf{a}\cdot\mathbf{b}'-\mathbf{b}\cdot\mathbf{a}')\right), \tag{2}$$

where $\mathbf{a}\cdot\mathbf{b}'=a_1b_1'+\ldots+a_{n-2}b_{n-2}'$. In particular,

$$n^{-1}(\mathbf{a},\mathbf{b},c)=n(-\mathbf{a},-\mathbf{b},-c). \tag{3}$$

The subgroup $Z$ consisting of elements $n(0,0,c)$, $c\in\mathbb{R}$, is the center of $N$.

The group $N$ is isomorphic to the real Heisenberg group $H(n-2,\mathbb{R})$ (see Example 2 of Section 1.1.8) consisting of the matrices

$$h(\mathbf{a},\mathbf{b},t) = \begin{pmatrix} 1 & \mathbf{a} & t \\ \mathbf{0} & I_{n-2} & \mathbf{b}^T \\ 0 & \mathbf{0} & 1 \end{pmatrix}, \tag{4}$$

where $\mathbf{a}$ and $\mathbf{b}$ are the same as in (1) and $t\in\mathbb{R}$. In fact, the multiplication in $H(n-2,\mathbb{R})$ is given by the formula

$$h(\mathbf{a},\mathbf{b},t)h(\mathbf{a}',\mathbf{b}',t') = h(\mathbf{a}+\mathbf{a}',\mathbf{b}+\mathbf{b}',t+t'+\mathbf{a}\cdot\mathbf{b}'). \tag{5}$$

Therefore,

$$h\left(\mathbf{a},\mathbf{b},c+\frac{1}{2}\mathbf{a}\cdot\mathbf{b}\right)h\left(\mathbf{a}',\mathbf{b}',c'+\frac{1}{2}\mathbf{a}'\cdot\mathbf{b}'\right)$$
$$=h\left(\mathbf{a}+\mathbf{a}',\mathbf{b}+\mathbf{b}',c+c'+\frac{1}{2}(\mathbf{a}\cdot\mathbf{b}'-\mathbf{b}\cdot\mathbf{a}')+\frac{1}{2}(\mathbf{a}+\mathbf{a}')\cdot(\mathbf{b}+\mathbf{b}')\right),$$

that is, the multiplication of the matrices $h\left(\mathbf{a},\mathbf{b},c+\frac{1}{2}\mathbf{a}\cdot\mathbf{b}\right)$ is the same as in the case of the matrices $n(\mathbf{a},\mathbf{b},c)$. Hence, the correspondence

$$n(\mathbf{a},\mathbf{b},c)\longleftrightarrow h\left(\mathbf{a},\mathbf{b},c+\frac{1}{2}\mathbf{a}\cdot\mathbf{b}\right) \tag{6}$$

is an isomorphism between $N$ and $H(n-2,\mathbb{R})$. In order to indicate the dimensionality of vectors $\mathbf{a}$ and $\mathbf{b}$ from (1) we shall use the notation $N_{n-2}$ instead of $N$.

Sometimes instead of the group $H(n-2,\mathbb{R})$ one uses the group $\widehat{N}$ of the matrices

$$g(\mathbf{z},x)=\begin{pmatrix}1 & \mathbf{z}^* & \frac{1}{2}(\mathbf{z},\mathbf{z})+ix\\ 0 & I_{n-2} & \mathbf{z}\\ 0 & \mathbf{0} & 1\end{pmatrix}, \tag{7}$$

where $\mathbf{z}=(z_1,\ldots,z_{n-2})^T$, $z_j\in\mathbb{C}$, $\mathbf{z}^*$ is the row-vector, conjugate to $\mathbf{z}$, $x\in\mathbb{R}$ and

$$(\mathbf{z},\mathbf{w})\equiv\mathbf{w}^*\cdot\mathbf{z}=z_1\bar{w}_1+\ldots+z_{n-2}\bar{w}_{n-2}.$$

The multiplication in $\widehat{N}$ is given by the equality

$$g(\mathbf{z},x)g(\mathbf{z}',x')=g(\mathbf{z}+\mathbf{z}',x+x'+\mathrm{Im}(\mathbf{z}',\mathbf{z})). \tag{8}$$

Consequently, the correspondence

$$n(\mathbf{a},\mathbf{b},c)\longleftrightarrow g(\mathbf{a}+i\mathbf{b},2c) \tag{9}$$

is an isomorphism between $N$ and $\widehat{N}$.

We introduce the measure

$$dn=d\mathbf{a}\,d\mathbf{b}\,dc,\quad d\mathbf{a}=da_1\ldots da_{n-2} \tag{10}$$

on elements $n=n(\mathbf{a},\mathbf{b},c)$ of the group $N_{n-2}$. By means of (2) one directly verifies that this measure is invariant.

**12.1.2. The Lie algebra of the Heisenberg group.** We separate in $N_{n-2}$ the one-parameter subgroups

$$n_j(t)\equiv n(\mathbf{a}_j(t),\mathbf{0},0),\ n_j'(t)\equiv n(\mathbf{0},\mathbf{b}_j(t),0),\ j=0,1,\ldots,n-2,\ n(\mathbf{0},\mathbf{0},t), \tag{1}$$

where $\mathbf{a}_j(t) \equiv \mathbf{b}_j(t) = (0, \ldots, 0, t, 0, \ldots, 0)$, and $j$ indicates that $t$ is situated on the $j$-th position. The tangent matrices $dn(t)/dt|_{t=0}$ to these one-parameter subgroups have, respectively, the forms

$$Q_j = -e_{j,n-1} + e_{jn} + e_{n-1,j} + e_{nj}, \; j = 1, 2, \ldots, n-2, \tag{2}$$

$$P_j = i(e_{j,n-1} - e_{jn} + e_{n-1,j} + e_{nj}), \; j = 1, 2, \ldots, n-2, \tag{3}$$

$$H = 2i(e_{n-1,n-1} - e_{n-1,n} + e_{n,n-1} - e_{nn}), \tag{4}$$

where $e_{ks}$ is the matrix with all zero entries except for the entry $a_{ks}$ which is equal to 1. Matrices (2)–(4) form a basis of the Lie aglebra $\mathfrak{n}_{n-2}$ of the group $N_{n-2}$. The commutation relations

$$[Q_j, P_k] = \delta_{jk} H, \; [Q_j, Q_k] = [P_j, P_k] = [Q_j, H] = [P_j, H] = 0. \tag{5}$$

hold.

Instead of $Q_j$, $P_j$, $H$ one also uses the matrices

$$Q'_j = -iQ_j, \quad P'_j = -iP_j, \quad H' = -iH \tag{6}$$

for which

$$[Q'_j, P'_j] = iH'. \tag{7}$$

**12.1.3. The exponential mapping.** For studying the exponential mapping of the Heisenberg algebra onto the Heisenberg group it is convenient to use the realization of the Heisenberg group by matrices (4) of Section 12.1.1. By virtue of isomorphism (6) of Section 12.1.1 the basis matrices $Q_j$, $P_j$, $H$ of the Heisenberg algebra in this realization are given as

$$Q_j = e_{1,j+1}, \qquad j = 1, 2, \ldots, n-2, \tag{1}$$
$$P_j = e_{j+1,n}, \qquad j = 1, 2, \ldots, n-2, \tag{2}$$
$$H = e_{1n}.$$

A simple verification shows that if $\mathbf{a} \cdot \mathbf{Q} \equiv a_1 Q_1 + \ldots + a_{n-2} Q_{n-2}$, $\mathbf{b} \cdot \mathbf{P} \equiv b_1 P_1 + \ldots + b_{n-2} P_{n-2}$, then

$$\exp(\mathbf{a} \cdot \mathbf{Q} + \mathbf{b} \cdot \mathbf{P}) = I + \mathbf{a} \cdot \mathbf{Q} + \mathbf{b} \cdot \mathbf{P} + \frac{1}{2}(\mathbf{a} \cdot \mathbf{b}) H = h\left(\mathbf{a}, \mathbf{b}, \frac{1}{2}\mathbf{a} \cdot \mathbf{b}\right).$$

Since $H$ belongs to the center of the Heisenberg algebra, then

$$\exp(\mathbf{a} \cdot \mathbf{Q} + \mathbf{a} \cdot \mathbf{P} + tH) = h\left(\mathbf{a}, \mathbf{b}, t + \frac{1}{2}\,\mathbf{a} \cdot \mathbf{b}\right). \tag{4}$$

If the parameters $a_j$, $b_j$, $t$ run over the field $\mathbb{R}$ of real numbers, then elements (4) run over the whole Heisenberg group. It is obvious from (4) that to different elements of the Heisenberg algegra there correspond different elements of the Heisenberg group. Thus, the exponential mapping gives a one-to-one correspondence between the Heisenberg algebra and the Heisenberg group. This correspondence is defined by formula (4).

If the Heisenberg group is realized by matrices (1) of Section 12.1.1, then formula (6) of Section 12.1.1 implies that in this case the exponential mapping has the form

$$\exp(\mathbf{a}\cdot\mathbf{Q}+\mathbf{b}\cdot\mathbf{P}+cH)=n(\mathbf{a},\mathbf{b},c). \tag{5}$$

**12.1.4. Unitary representations.** Let $\mathfrak{L}^2(\mathbb{R}^n)$ be the Hilbert space of functions $f(\mathbf{x})\equiv f(x_1,\ldots,x_n)$ with the scalar product

$$(f_1,f_2)=\int\limits_{\mathbb{R}^n} f_1(\mathbf{x})\overline{f_2(\mathbf{x})}d\mathbf{x},$$

where $d\mathbf{x}=dx_1\ldots dx_n$. If $\lambda$ is a fixed real number, then the operators

$$(R^\lambda(n(\mathbf{a},\mathbf{b},c))f)(\mathbf{x})=e^{i\lambda c}e^{i\mathbf{x}\cdot\mathbf{a}}e^{i\lambda\mathbf{a}\cdot\mathbf{b}/2}f(\mathbf{x}+\lambda\mathbf{b}) \tag{1}$$

give a unitary representation of the group $N_n$ in $\mathfrak{L}^2(\mathbb{R}^n)$.

The elements $n(\mathbf{0},\mathbf{0},c)$ belong to the center $Z$ of the group $N_n$ and the scalar operators

$$R^\lambda(n(\mathbf{0},\mathbf{0},c))=e^{i\lambda c}I \tag{2}$$

correspond to them. Further, $R^\lambda(n(\mathbf{a},\mathbf{0},0))$ are the operators of multiplication by a function

$$(R^\lambda(n(\mathbf{a},\mathbf{0},0))f)(\mathbf{x})=e^{i\mathbf{x}\cdot\mathbf{a}}f(\mathbf{x}) \tag{3}$$

and $R^\lambda(n(\mathbf{0},\mathbf{b},0))$ are the shift operators

$$(R^\lambda(n(\mathbf{0},\mathbf{b},0))f)(\mathbf{x})=f(\mathbf{x}+\lambda\mathbf{b}). \tag{4}$$

It follows from formulas (2)–(4) that to the matrices $Q_j$, $P_j$, $H$ of the Lie algebra $\mathfrak{n}_n$ there correspond in the representation $R^\lambda$ the operators $Q_j^\lambda$, $P_j^\lambda$, $H^\lambda$ for which

$$\left.\begin{aligned}(Q_j^\lambda f)(\mathbf{x})&=ix_jf(\mathbf{x}),\\(P_j^\lambda f)(\mathbf{x})&=\lambda\frac{\partial}{\partial x_j}f(\mathbf{x})\\(H^\lambda f)(\mathbf{x})&=i\lambda f(\mathbf{x}).\end{aligned}\right\} \tag{5}$$

The operators $Q_j^\lambda$, $P_j^\lambda$, $H^\lambda$ satisfy commutation relations (5) of Section 12.1.2.

Using formulas (5) and repeating the reasonings of Section 3.4.1, we show that for $\lambda \neq 0$ the representations $R^\lambda$ are irreducible. If $\lambda = 0$, then $R^\lambda$ is a representation of the commutative quotient group $\widetilde{N} = N_n/Z$. Therefore, the representation $R^0$ is reducible and decomposes into the integral of one-dimensional representations. We leave to the reader to derive this decomposition.

It follows from (2) that the representations $R^\lambda$, $\lambda \in \mathbb{R}$, are pairwise nonequivalent. The following theorem is of great significance.

**Theorem.** *The representations $R^\lambda$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, exhaust (up to a unitary equivalence) all irreducible unitary representations of the Heisenberg group $N_n$.*

The proof of this theorem can be found in many monographs on group representations (see, for example, [330]).

The realization of irreducible unitary representations of the Heisenberg group constructed above is called the *Schrödinger realization.* Below we shall consider other realizations.

In conclusion we note that by substituting $\lambda\mathbf{y} = \mathbf{x}$ into (1) we obtain a somewhat different realization of the representation $R^\lambda$ on functions $F(\mathbf{y}) = f(\lambda\mathbf{y})$:

$$(R^\lambda(n(\mathbf{a}, \mathbf{b}, c))F)(\mathbf{y}) = \exp\left[\lambda\left(ic + i\mathbf{y}\cdot\mathbf{a} + \frac{1}{2}i\mathbf{a}\cdot\mathbf{b}\right)\right]F(\mathbf{y}+\mathbf{b}). \tag{6}$$

The replacement $\mathbf{x} = \lambda\mathbf{y}$ transforms the scalar product $(f_1, f_2)$ for functions $f \in \mathfrak{L}^2(\mathbb{R}^n)$ into

$$\langle F_1, F_2\rangle = \lambda \int_{\mathbb{R}^n} F_1(\mathbf{y})\overline{F_2(\mathbf{y})}d\mathbf{y}. \tag{7}$$

**12.1.5. The orthonormal basis.** Let $\lambda > 0$. The functions

$$e_m(x) = (2^m m!)^{-1/2}(\pi\lambda)^{-1/4}e^{-x^2/2\lambda}H_m\left(x/\sqrt{\lambda}\right), \quad m = 0, 1, 2, \ldots, \tag{1}$$

where $H_m$ are Hermite polynomials, form an orthonormal basis of the space $\mathfrak{L}^2(\mathbb{R})$ (see Sections 5.3.6). Therefore, the set of functions

$$e_{\mathbf{m}}(\mathbf{x}) \equiv e_{m_1}(x_1)\ldots e_{m_n}(x_n), \quad \mathbf{m} = (m_1, \ldots, m_n), \quad m_j \geq 0, \tag{2}$$

is an orthonormal basis in $\mathfrak{L}^2(\mathbb{R}^n)$.

It follows from differential equation (13) of Section 5.3.6 for Hermite polynomials that functions (1) are eigenfunctions for the differential operator $\frac{\lambda^2}{2}\frac{d^2}{dx^2} - \frac{x^2}{2}$:

$$\left(\frac{\lambda^2}{2}\frac{d^2}{dx^2} - \frac{x^2}{2}\right)e_m(x) = -\lambda\left(m + \frac{1}{2}\right)e_m(x). \tag{3}$$

Hence, $e_{\mathbf{m}}(\mathbf{x})$ are eigenfunctions for the operator

$$D_1 \equiv iD = \frac{1}{2}\sum_{j=1}^{n}(P_j^2 + Q_j^2) = \sum_{j=1}^{n}\left(\frac{\lambda^2}{2}\frac{\partial^2}{\partial x_j^2} - \frac{x_j^2}{2}\right). \tag{4}$$

We have

$$D_1 e_{\mathbf{m}}(\mathbf{x}) = -\lambda\left(m_1 + \ldots + m_n + \frac{n}{2}\right) e_{\mathbf{m}}(\mathbf{x}). \tag{5}$$

If $\lambda < 0$, then instead of $e_m(x)$ we take functions (1) with $\lambda$ replaced by $-\lambda$.

**12.1.6. Matrix elements of the representations $R^\lambda$.** Let us evaluate matrix elements of the representations $R^\lambda$, $\lambda > 0$, in basis (2) of Section 12.1.5. By virtue of formula (2) of Section 12.1.4 it is sufficient to evaluate matrix elements of the operators $R^\lambda(n(\mathbf{a},\mathbf{b},0))$. We have

$$R^\lambda(n(\mathbf{a},\mathbf{b},0)) = \prod_{j=1}^{n} R^\lambda(n(\mathbf{a}_j(a_j),\mathbf{b}_j(b_j),0), \tag{1}$$

where $\mathbf{a}_j(a)$ and $\mathbf{b}_j(b)$ are the same as in formula (1) of Section 12.1.2. The element $n(\mathbf{a}_j(a_j),\mathbf{b}_j(b_j),0)$ will be denoted by $n_j(a_j,b_j,0)$. It follows from formula (1) of Section 12.1.4 that $R^\lambda(n_j(a,b,0))$ acts upon the variable $x_j$ only. Therefore,

$$\begin{aligned} r^\lambda_{\mathbf{mp}}(n(\mathbf{a},\mathbf{b},0)) &\equiv (R^\lambda(n(\mathbf{a},\mathbf{b},0))e_{\mathbf{p}}, e_{\mathbf{m}}) \\ &= \prod_{j=1}^{n}(R^\lambda(n_j(a_j,b_j,0))e_{\mathbf{p}}, e_{\mathbf{m}}) \equiv \prod_{j=1}^{n} r^\lambda_{m_j p_j}(n_j(a_j,b_j,0)) \\ &= \prod_{j=1}^{n}\int_{-\infty}^{\infty} e^{ix_j a_j} e^{i\lambda a_j b_j/2} e_{p_j}(x_j + \lambda b_j) e_{m_j}(x_j)\,dx_j. \end{aligned} \tag{2}$$

Thus, $r^\lambda_{\mathbf{mp}}(n(\mathbf{a},\mathbf{b},0))$ is equal to the product of the matrix elements $r^\lambda_{m_j p_j}(n_j(a_j,b_j,0))$ and we have

$$\begin{aligned} r^\lambda_{mp}(n_j(a,b,0)) = \left[2^{m+p} m! p! \pi\lambda\right]^{-1/2} \int_{-\infty}^{\infty} e^{i\lambda ab/2} e^{iax} \\ \times e^{-x^2/2\lambda} e^{-(x+\lambda b)^2/2\lambda} H_p\left(\frac{x+\lambda b}{\sqrt{\lambda}}\right) H_m\left(\frac{x}{\sqrt{\lambda}}\right) dx. \end{aligned} \tag{3}$$

Multiply both sides of (3) by $[2^{m+p}m!p!]^{1/2}\frac{t^m s^p}{m!p!}$ and sum with respect to $m$ and $p$ from 0 to $\infty$. Taking into account the equality

$$\sum_{m=0}^{\infty} H_m(z)\frac{t^m}{m!} = e^{2tz}e^{-t^2}$$

(see formula (8) of Section 5.3.6), we obtain

$$\sum_{m,p=0}^{\infty} \left[2^{m+p} m!p!\right]^{1/2} r^{\lambda}_{mp}(n_j(a,b,0)) \frac{t^m s^p}{m!s!}$$

$$= (\pi\lambda)^{-1/2} \exp\left(2sb\sqrt{\lambda} + \frac{1}{2} i\lambda ab - \frac{1}{2}\lambda b^2 - t^2 - s^2\right)$$

$$\times \int_{-\infty}^{\infty} \exp\left(iax - bx - \frac{x^2}{2} + \frac{2x(t+s)}{\sqrt{\lambda}}\right) dx$$

$$= \exp\frac{\lambda(a^2+b^2)}{4} \exp\left[2ts + \sqrt{\lambda}\, s(b+ia) + \sqrt{\lambda}\, t(-b+ia)\right] \tag{4}$$

(the integral has been calculated by means of the integral from Example 1 of Section 3.2.3).

In order to obtain from (4) the matrix elements $r^{\lambda}_{mp}(n_j(a,b,0))$, we use equality (4) of Section 5.5.3, which for $\alpha = m$ is of the form

$$e^{-xz}(z+1)^m = \sum_{p=0}^{\infty} L_p^{m-p}(x) z^p.$$

We set here $x = |u|^2$, $z = v/u$, multiply both sides by $u^m w^m/m!$ and sum with respect to $m$ from 0 to $\infty$. After simple manipulations we obtain

$$\exp(wv + wu - \bar{u}v) = \sum_{m,p=0}^{\infty} \frac{1}{m!} L_p^{m-p}(|u|^2) u^{m-p} v^p w^m. \tag{5}$$

Setting here $w = \sqrt{2}\, t$, $v = \sqrt{2}\, s$, $u = \sqrt{\lambda/2}\,(ia - b)$ and comparing the result with formula (4), we derive that

$$r^{\lambda}_{mp}(n_j(a,b,0))$$

$$= \exp\frac{-\lambda(a^2+b^2)}{4} \left(\frac{p!}{m!}\right)^{1/2} \left(\frac{\lambda}{2}\right)^{\frac{m-p}{2}} (ia-b)^{m-p} L_p^{m-p}\left(\frac{\lambda}{2}(a^2+b^2)\right) \tag{6}$$

for $m \geq p$. In the same way, by setting $w = \sqrt{2}\, s$, $v = \sqrt{2}\, t$, $u = \sqrt{\lambda/2}\,(ia + b)$ into (5), we find

$$r^{\lambda}_{mp}(n_j(a,b,0))$$

$$= \exp\frac{-\lambda(a^2+b^2)}{4} \left(\frac{m!}{p!}\right)^{1/2} \left(\frac{\lambda}{2}\right)^{\frac{p-m}{2}} (ia+b)^{p-m} L_m^{p-m}\left(\frac{\lambda}{2}(a^2+b^2)\right), \tag{7}$$

where $p \geq m$.

For $p < m$ we have the equality

$$L_m^{p-m}(x) = (-1)^{m-p} \frac{p!}{m!} x^{m-p} L_p^{m-p}(x). \tag{8}$$

It transfers the right hand side of (7) into the right hand side of (6). Therefore, for $p > m$ the matrix element $r^\lambda_{mp}(n_j(a,b,0))$ is also given by (6).

Setting $z = a + ib$ (see formula (9) of Section 12.1.1), we rewrite formula (6) in the form

$$r^\lambda_{mp}(n_j(a,b,0)) = \exp\left(-\frac{\lambda |z|^2}{4}\right)\left(\frac{p!}{m!}\right)^{1/2}\left(\frac{iz\sqrt{\lambda}}{\sqrt{2}}\right)^{m-p} L_p^{m-p}\left(\frac{\lambda |z|^2}{2}\right). \tag{9}$$

We suggest to the reader to prove that for $\lambda < 0$

$$r^\lambda_{mp}(n_j(a,b,0)) = \exp\left(\frac{|\lambda|\,|z|^2}{4}\right)\left(\frac{p!}{m!}\right)^{1/2}\left(\frac{i\bar{z}\sqrt{|\lambda|}}{\sqrt{2}}\right)^{m-p} L_p^{m-p}\left(-\frac{|\lambda|\,|z|^2}{2}\right). \tag{10}$$

If we introduce the parameters $r$, $\varphi$, where $a = r\cos\varphi$ and $b = r\sin\varphi$, then matrix element (9) takes the form

$$r^\lambda_{mp}(n_j(a,b,0)) = \left(\frac{p!}{m!}\right)^{1/2} \exp\left(-\frac{\lambda r^2}{4}\right)\left(\frac{ir\sqrt{\lambda}}{\sqrt{2}}\right)^{m-p} e^{i(m-p)\varphi} L_p^{m-p}\left(\frac{\lambda r^2}{2}\right). \tag{9'}$$

Hence, with the help of orthogonality relation (1) of Section 5.5.4 for Laguerre polynomials, we find

$$\frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r^\lambda_{mp}(n_j(a,b,0))\overline{r^\lambda_{st}(n_j(a,b,0))}\,da\,db$$
$$= \frac{\lambda}{2\pi}\int_0^{2\pi}\int_0^{\infty} r^\lambda_{mp}(n_j(a,b,0))\overline{r^\lambda_{st}(n_j(a,b,0))}\,r\,dr\,d\varphi = \delta_{ms}\delta_{pt}. \tag{11}$$

By virtue of (2) we have

$$\left(\frac{\lambda}{2\pi}\right)^n \int r^\lambda_{\mathbf{mp}}(n(\mathbf{a},\mathbf{b},0))\overline{r^\lambda_{\mathbf{st}}(n(\mathbf{a},\mathbf{b},0))}\,d\mathbf{a}\,d\mathbf{b} = \delta_{\mathbf{ms}}\delta_{\mathbf{pt}}, \tag{12}$$

where the integration is over the quotient group $N_n/Z$. Thus,

$$r^\lambda_{\mathbf{mp}}(n(\mathbf{a},\mathbf{b},0)) \in \mathfrak{L}^2(N_n/Z) = \mathfrak{L}^2(\mathbb{R}^{2n}). \tag{13}$$

Keeping in mind this inclusion, one says that the *representations $R^\lambda$ of the group $N_n$ are square-integrable modulo $Z$*. This square integrability is a corollary of the fact that character (2) of Section 12.1.4 for the center $Z$ defines the irreducible representation $R^\lambda$ up to a unitary equivalence. Really, the following theorem holds [161]:

**Theorem.** *Let $N$ be a nilpotent Lie group with the center $Z$ and $T$ be its irreducible unitary representation such that $T(z) = \lambda(z)I$, $z \in Z$. The representation $T$ is square integrable modulo $Z$ if and only if $T$ is the unique (up to a unitary equivalence) representation of $N$ whose restriction onto $Z$ coincides with $\lambda(z)$.*

We have shown that the functions $r^{\lambda}_{\mathbf{mp}}(n(\mathbf{a}, \mathbf{b}, 0))$ are orthogonal in $\mathfrak{L}^2(\mathbf{R}^{2n})$. We now prove that they form an orthonormal basis in this space. It is sufficient to show that every function $f(r, \varphi) \equiv F(a, b) \in \mathfrak{L}^2(\mathbf{R}^2)$ can be expanded in the functions $r^{\lambda}_{mp}(n_j(a, b, 0))$, $0 \le m, p < \infty$. We expand $f(r, \varphi)$ into the Fourier series

$$f(r, \varphi) = \sum_{q=-\infty}^{\infty} \Phi_q(r) e^{iq\varphi}.$$

Making use of the results of Section 5.5.5, one can easily show that the function $\Phi_q(r)$ is expanded in the functions

$$\exp\left(-\frac{\lambda r^2}{4}\right) r^q L^q_p\left(\frac{\lambda r^2}{2}\right), \quad p = 0, 1, 2, \ldots .$$

These expansions lead to expansion of $f(r, \varphi)$ in $r^{\lambda}_{mp}(n_j(a, b, 0))$.

**12.1.7. The Fock realization of the representations $R^{\lambda}$.** In this section elements $n(\mathbf{a}, \mathbf{b}, c)$ of $N_n$ will be denoted by $n(\mathbf{z}, c)$, $\mathbf{z} = \mathbf{a} + i\mathbf{b} \in \mathbb{C}^n$. We regard the operators $L(n(\mathbf{w}, t))$ of the left regular representation $L$ of the group $N_n$:

$$(L(n(\mathbf{w}, t))f)(n) = f(n^{--1}(\mathbf{w}, t)n), \quad n \in N_n.$$

These operators act upon the functions

$$r^{\lambda}_{\mathbf{mp}}(n(\mathbf{z}, 0))^* \equiv \overline{r^{\lambda}_{\mathbf{mp}}(n(\mathbf{z}, 0))}$$

with a fixed $\mathbf{p}$ according to the formula

$$\begin{aligned} L(n(\mathbf{w}, t)) r^{\lambda}_{\mathbf{mp}}(n(\mathbf{z}, 0))^* &= r^{\lambda}_{\mathbf{mp}}(n^{-1}(\mathbf{w}, t) n(\mathbf{z}, 0))^* \\ &= \sum_{\mathbf{q}} r^{\lambda}_{\mathbf{mq}}(n^{-1}(\mathbf{w}, t))^* r^{\lambda}_{\mathbf{qp}}(n(\mathbf{z}, 0))^* \\ &= \sum_{\mathbf{q}} r^{\lambda}_{\mathbf{qm}}(n(\mathbf{w}, t)) r^{\lambda}_{\mathbf{qp}}(n(\mathbf{z}, 0))^*. \end{aligned} \tag{1}$$

Since

$$R^{\lambda}(n(\mathbf{w}, t)) e_{\mathbf{m}}(\mathbf{x}) = \sum_{\mathbf{q}} r^{\lambda}_{\mathbf{qm}}(n(\mathbf{w}, t)) e_{\mathbf{q}}(\mathbf{x}), \tag{2}$$

then $L(n(\mathbf{w},t))$ act upon $r^{\lambda}_{\mathbf{mp}}(n(\mathbf{z},0))^*$ by the formulas which give the action of $R^{\lambda}(n(\mathbf{w},t))$ upon the basis functions $e_{\mathbf{m}}(\mathbf{x})$.

For $\mathbf{p}=\mathbf{0}$ the functions $r^{\lambda}_{\mathbf{mp}}(n(\mathbf{z},0))^*$ have the form

$$r^{\lambda}_{\mathbf{m0}}(n(\mathbf{z},0)) \equiv f_{\mathbf{m}}(\mathbf{z}) = f_{m_1}(z_1)f_{m_2}(z_1)\dots f_{m_n}(z_n), \tag{3}$$

where

$$f_m(z) = (m!)^{-1/2}\exp\left(-\frac{\lambda|z|^2}{4}\right)\left(\frac{i\bar{z}\sqrt{\lambda}}{\sqrt{2}}\right)^m. \tag{4}$$

Comparing functions (4) with the orthonormal basis $e_m(z)=z^m/\sqrt{m!}$ of the space $\mathfrak{H}$ from Section 5.5.1, we conclude that they form an orthonormal basis of the Hilbert space $\mathfrak{F}$ of entire antianalytic functions $f(z)$ on $\mathbb{C}$ (that is, of functions complex conjugate to entire analytic functions), multiplied by $\exp(-\lambda|z|^2/4)$, with the scalar product

$$\langle f_1,f_2\rangle = \frac{\lambda}{2\pi}\int_{\mathbb{C}} f_1(z)\overline{f_2(z)}dx\,dy, \qquad z=x+iy. \tag{5}$$

Consequently, functions (3) form an orthonormal basis of the Hilbert space $\mathfrak{H}^{\lambda}_0$ of entire antianalytic functions $f(\mathbf{z})$, multiplied by $\exp(-\lambda|z|^2/4)$, with the scalar product

$$\langle f_1,f_2\rangle = \left(\frac{\lambda}{2\pi}\right)^n\int_{\mathbb{C}^n} f_1(\mathbf{z})\overline{f_2(\mathbf{z})}d\mathbf{x}\,d\mathbf{y}. \tag{6}$$

The above arguments imply that the restriction $Q^{\lambda}$ of the left regular representation $L$ onto $\mathfrak{H}^{\lambda}_0$ is unitarily equivalent to $R^{\lambda}$. The equivalence is given by the operator

$$U\colon e_{\mathbf{m}}(\mathbf{x}) \to r^{\lambda}_{\mathbf{m0}}(n(\mathbf{z},0)) \equiv f_{\mathbf{m}}(\mathbf{z}). \tag{7}$$

In order to find an explicit form of the operator $U$ it is sufficient to construct $U$ in the case $n=1$, that is, the operator

$$U_1\colon e_m(x) \to r^{\lambda}_{m0}(n_j(z,0)) \equiv f_m(z), \tag{8}$$

where $f_m(z)$ is given by (4). We utilize the relation

$$\int_{-\infty}^{\infty} e^{ixz}e^{-x^2}H_n(x)dx = \sqrt{\pi}\,(iz)^n e^{-z^2/4} \tag{9}$$

(see formula (10) of Section 9.6.8). It gives

$$\int_{-\infty}^{\infty} e^{ix\bar{z}}e^{-x^2/2\lambda}e_n(x)dx = \frac{\sqrt[4]{\pi\lambda}}{\sqrt{n!}}\left(\frac{i\bar{z}\sqrt{\lambda}}{\sqrt{2}}\right)^n e^{-\bar{z}^2\lambda/4}.$$

Consequently,

$$f_n(z) = (U_1 e_n)(z) = \int_{-\infty}^{\infty} K(z,x) e_n(x) dx, \tag{10}$$

where the kernel is given by the formula

$$K(z,x) = (\pi\lambda)^{-1/4} e^{\lambda(\bar z^2 - |z|^2)/4} e^{-x^2/2\lambda} e^{ix\bar z}. \tag{11}$$

For operator (7) we have

$$f_{\mathbf{m}}(\mathbf{z}) = (U e_{\mathbf{m}})(\mathbf{z}) = \int_{\mathbb{R}^n} K(\mathbf{z},\mathbf{x}) e_{\mathbf{m}}(\mathbf{x}) d\mathbf{x}, \tag{12}$$

where

$$K(\mathbf{z},\mathbf{x}) = K(z_1,x_1)K(z_2,x_2)\dots K(z_n,x_n).$$

We now find the formula for the action of the operators $Q^\lambda(n(\mathbf{w},t))$ upon functions from $\mathfrak{H}_0^\lambda$. It follows from the first part of (1) and from equalities (2) and (3) of Section 12.1.1 that

$$L(n(\mathbf{w},t)) r^\lambda_{\mathbf{mp}}(n(\mathbf{z},0))^* = r^\lambda_{\mathbf{mp}}\left(n\left(\mathbf{z}-\mathbf{w}, -t + \frac{1}{2}\operatorname{Im}\mathbf{w}\cdot\bar{\mathbf{z}}\right)\right)^*$$
$$= \exp i\lambda\left(t - \frac{1}{2}\operatorname{Im}\mathbf{w}\cdot\bar{\mathbf{z}}\right) r^\lambda_{\mathbf{mp}}(n(\mathbf{z}-\mathbf{w}),0)^*.$$

Therefore, for functions $f \in \mathfrak{H}_0^\lambda$ we have

$$(Q^\lambda(n(\mathbf{w},t))f)(\mathbf{z}) = \exp i\lambda\left(t - \frac{1}{2}\operatorname{Im}\mathbf{w}\cdot\bar{\mathbf{z}}\right) f(\mathbf{z}-\mathbf{w}). \tag{13}$$

**Remark.** Starting from the representation

$$(\widehat{R}^\lambda(n(\mathbf{a},\mathbf{b},t))f)(\mathbf{x}) = \exp i\left(\lambda t + \mathbf{b}\cdot\mathbf{x} - \frac{1}{2}\lambda\mathbf{a}\cdot\mathbf{b}\right) f(\mathbf{x}-\mathbf{a}) \tag{14}$$

in $\mathfrak{L}^2(\mathbb{R}^n)$, we should obtain the representation $\widehat{Q}^\lambda$ in the Hilbert space $\widehat{\mathfrak{H}}$ of entire analytic functions, multiplied by $\exp(-\lambda|\mathbf{z}|^2/4)$, which is given by the formula

$$(\widehat{Q}^\lambda(n(\mathbf{w},t))f)(\mathbf{z}) = \exp i\lambda\left(t + \frac{1}{2}\operatorname{Im}\mathbf{w}\cdot\bar{\mathbf{z}}\right) f(\mathbf{z}-\mathbf{w}). \tag{15}$$

**12.1.8. Addition formulas for Laguerre polynomials.** The equality

$$\sum_{p=0}^{\infty} r^\lambda_{mp}(n_j(a,b,0)) r^\lambda_{pn}(n_j(a',b',0)) = r^\lambda_{mn}\left(n_j(a+a', b+b', \frac{1}{2}(ab'-ba'))\right)$$

leads to the addition formula for Laguerre polynomials:

$$(ia-b)^m(ia'-b')^{-n}\sum_{p=0}^{\infty}\left(\frac{ia'-b'}{ia-b}\right)^p L_p^{m-p}\left(\frac{\lambda(a^2+b^2)}{2}\right)L_n^{p-n}\left(\frac{\lambda(a'^2+b'^2)}{2}\right)$$
$$=\left(\exp\frac{-\lambda(aa'+bb')}{2}\right)(ia+ia'-b-b')^{m-n}$$
$$\times\exp\left[\frac{i\lambda}{2}(ab'-ba')\right]L_n^{m-n}\left(\frac{1}{2}(\lambda(a+a')^2+\lambda(b+b')^2)\right). \quad (1)$$

In complex variables it is represented as

$$(-z)^m(-w)^{-n}\sum_{p=0}^{\infty}\left(\frac{w}{z}\right)^p L_p^{m-p}\left(\frac{\lambda}{2}z\bar{z}\right)L_n^{p-n}\left(\frac{\lambda}{2}w\bar{w}\right)$$
$$=(-z-w)^{m-n}\exp(-\lambda\bar{w}z)L_n^{m-n}(\lambda(z+w)(\bar{z}+\bar{w})). \quad (2)$$

If $b=a'=0$, we obtain from (1) the relation

$$\sum_{p=0}^{\infty}\left(\frac{b}{ia}\right)^p L_p^{m-p}(a^2)L_n^{p-n}(b^2)=\frac{b^n}{(ia)^m}(ia+b)^{m-n}e^{iab}L_n^{m-n}(a^2+b^2). \quad (3)$$

Let us give the special cases of (2) for $n=0$ and for $m=0$:

$$\sum_{p=0}^{\infty}\left(\frac{w}{z}\right)^p L_p^{m-p}(\lambda z\bar{z})=\left(\frac{z+w}{z}\right)^m e^{-\lambda\bar{w}z}, \quad (4)$$

$$\sum_{p=0}^{\infty}\frac{(-\lambda w\bar{z})^p}{p!}L_n^{p-n}(\lambda w\bar{w})=\left(\frac{w}{z-w}\right)^n e^{-\lambda\bar{w}z}. \quad (5)$$

**12.1.9. Relations between Laguerre and Hermite polynomials.** It follows from formulas (1) of Section 12.1.4 and (2) of Section 12.1.5 that

$$R^\lambda(n_j(a,b,0))e_p(x)=\sum_{m=0}^{\infty}r_{mp}^\lambda(n_j(a,b,0))e_m(x), \quad (1)$$

where $e_n(x)$ is given by formula (1) of Section 12.1.5. Substituting the expressions for $e_n(x)$ and $r_{mp}^\lambda(n_j(a,b,0))$, after simple manipulations we have

$$\sum_{m=0}^{\infty}\frac{\lambda^{m/2}(ia-b)^m}{2^m m!}L_p^{m-p}\left(\frac{\lambda}{2}(a^2+b^2)\right)H_m\left(\frac{x}{\sqrt{\lambda}}\right)=\frac{\lambda^{p/2}(ia-b)^p}{2^p p!}$$
$$\times\exp\frac{\lambda(a^2+b^2)}{4}e^{x(ia-b)}e^{\lambda b(ia-b)/2}H_p\left(\frac{x+\lambda b}{\sqrt{\lambda}}\right). \quad (2)$$

For $b = 0$, $\lambda = 1$ this equality is rewritten as

$$\sum_{m=0}^{\infty} 2^{-m} \frac{(ia)^m}{m!} L_p^{m-p}\left(\frac{a^2}{2}\right) H_m(x) = 2^{-p} \frac{(ia)^p}{p!} e^{a^2/4} e^{iax} H_p(x). \tag{3}$$

Equating real and imaginary parts, we obtain the relations

$$\sum_{m=0}^{\infty} \frac{(-a^2)^m}{2^{2m}(2m)!!} L_p^{2m-p}\left(\frac{a^2}{2}\right) H_{2m}(x) = \frac{a^p}{2^p p!} e^{a^2/4} (\cos xa) H_p(x), \tag{4}$$

$$\sum_{m=0}^{\infty} \frac{(-a^2)^m}{a 2^{2m-1}(2m-1)!!} L_p^{2m-p-1}\left(\frac{a^2}{2}\right) H_{2m-1}(x) = \frac{a^p}{2^p p!} e^{a^2/4} (\sin xa) H_p(x), \tag{5}$$

where $k!! = k(k-2)\ldots 1$ (or 2) and $p$ is a multiple of 4.

For $a = 0$, $\lambda = 1$ relation (2) leads to the equality

$$\sum_{m=0}^{\infty} \frac{b^m}{2^m m!} L_p^{m-p}\left(\frac{b^2}{2}\right) H_m(x) = \frac{b^p}{2^p p!} e^{-b^2/4} e^{bx} H_p(x-b), \tag{6}$$

and for $\lambda = 1$, $x = 0$ we derive

$$\sum_{m=0}^{\infty} \frac{(-1)^m (2m)!(ia-b)^{2m}}{2^{3m}(m!)^2} L_p^{2m-p}\left(\frac{1}{2}(a^2+b^2)\right) = \frac{(ia-b)^2}{2^p p!} e^{(a^2+b^2)/4} e^{b(ia-b)/2} H_p(b). \tag{7}$$

We now set $\lambda = 1$ into (2), multiply both sides by $e^{-x^2} H_{m'}(x)$ and integrate with respect to $x$ from $-\infty$ to $\infty$. By virtue of the orthogonality relation for Hermite polynomials we obtain

$$\int_{-\infty}^{\infty} e^{-x^2} e^{x(ia-b)} H_p(x+b) H_m(x) dx = 2^p p! \sqrt{\pi}\, (ia-b)^{m-p} \times \exp\left[\frac{b(b-ia)}{2} - \frac{a^2+b^2}{4}\right] L_p^{m-p}\left(\frac{1}{2}(a^2+b^2)\right). \tag{8}$$

For $b = 0$ we have

$$\int_{-\infty}^{\infty} e^{-x^2} e^{iax} H_p(x) H_m(x) dx = 2^p p! \sqrt{\pi}\, (ia)^{m-p} e^{-a^2/4} L_p^{m-p}\left(\frac{a^2}{2}\right). \tag{9}$$

If $a = 0$, then this equality gives the orthogonality relation for Hermite polynomials.

One can consider formula (8) as the Fourier transform of the function $\exp(-x^2 - xb)H_p(x+b)H_m(x)$. Hence, we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-iab/2}e^{-iax}e^{-a^2/4}(ia-b)^{m-b}L_p^{m-p}\left(\frac{a^2+b^2}{2}\right)da$$
$$= [2^p p!\sqrt{\pi}]^{-1}e^{-x^2}e^{-xb}e^{-b^2/4}H_p(x+b)H_m(x). \quad (10)$$

**12.1.10. Orthogonal polynomials on $\mathbb{C}$.** The functions

$$P_{mp}(z,\bar{z}) = z^{m-p}L_p^{m-p}(z\bar{z}), \quad m,p = 0,1,2,\ldots, \quad (1)$$

are polynomials of degree $m$ in $z$ and of degree $p$ in $\bar{z}$. It follows from formula (9) of Section 12.1.6 that they are connected with the matrix elements $r^2_{mp}(n(a,b,0))$ of the representation $R^2$ of the group $N_1$:

$$r^2_{mp}(n(a,b,0)) = \left(\frac{p!}{m!}\right)^{1/2}\exp\left(-\frac{z\bar{z}}{2}\right)i^{m-p}P_{mp}(z,\bar{z}), \quad (2)$$

where $z = a+ib$. From formula (11) of Section 12.1.6 we derive the orthogonality relation for $P_{mp}(z,\bar{z})$:

$$\frac{1}{\pi}\int_{\mathbb{C}} P_{mp}(z,\bar{z})\overline{P_{st}(z,\bar{z})}\exp(-z\bar{z})dx\,dy = \frac{m!}{p!}\delta_{ms}\delta_{pt}, \; z = x+iy. \quad (3)$$

It follows from the results of Section 12.1.6 that polynomials (1) form a complete system in the Hilbert space $\mathfrak{H}$ of functions $f(z) \equiv f(x+iy)$ with the scalar product

$$(f_1,f_2) = \frac{1}{\pi}\int_{\mathbb{C}} f_1(z)\overline{f_2(z)}\exp(-z\bar{z})dx\,dy. \quad (4)$$

For every function $f \in \mathfrak{H}$ one has the expansion

$$f(z) = \sum_{m,p=0}^{\infty} a_{mp}P_{mp}(z,\bar{z}) \quad (5)$$

where

$$a_{mp} = \frac{p!}{\pi m!}\int_{\mathbb{C}} f(z)\overline{P_{mp}(z,\bar{z})}\exp(-z\bar{z})dx\,dy. \quad (6)$$

The Plancherel formula

$$\frac{1}{\pi}\int\limits_{\mathbf{C}} |f(z)|^2 \exp(-z\bar{z})dx\,dy = \sum_{m,p=0}^{\infty} \frac{m!}{p!}|a_{mp}|^2 \tag{7}$$

holds. The series in (5) converges in the mean.

The relation

$$\exp(z\bar{z} + 2ts + i\sqrt{2}tz + i\sqrt{2}s\bar{z}) = \sum_{m,p=0}^{\infty} 2^{(p+m)/2}\frac{i^{m-p}}{m!}P_{mp}(z,\bar{z})t^m s^p \tag{8}$$

which follows from equality (4) of Section 12.1.6 gives a generating function for the polynomials $P_{mp}(z,\bar{z})$.

From differentiation formula (3) of Section 5.5.2 for Laguerre polynomials we have that

$$\frac{d}{dz}P_{mp}(z,\bar{z}) = \frac{m-p}{z}P_{mp}(z,\bar{z}) - z\bar{z}P_{m,p-1}(z,\bar{z}), \tag{9}$$

$$\frac{d}{d\bar{z}}P_{mp}(z,\bar{z}) = -P_{m,p-1}(z,\bar{z}). \tag{10}$$

The recurrence relations for $P_{mp}$ follow from those for Laguerre polynomials:

$$zP_{mp}(z,\bar{z}) = P_{m+1,p}(z,\bar{z}) - P_{m,p-1}(z,\bar{z}), \tag{11}$$

$$\bar{z}P_{m+1,p}(z,\bar{z}) = (m-p+z\bar{z})P_{mp}(z,\bar{z}) - mzP_{m-1,p}(z,\bar{z}), \tag{12}$$

$$(p+1)zP_{m,p+1}(z,\bar{z}) = -(p-m+z\bar{z})P_{mp}(z,\bar{z}) - \bar{z}P_{m,p-1}(z,\bar{z}), \tag{13}$$

$$\bar{z}P_{mp}(z,\bar{z}) = mP_{m-1,p}(z,\bar{z}) - (p+1)P_{m,p+1}(z,\bar{z}). \tag{14}$$

**12.1.11. Decomposition of the regular representation of the Heisenberg group.** Let $\mathfrak{L}^2(N_n)$ be the Hilbert space of functions $f(n) \equiv f(\mathbf{a},\mathbf{b},c)$ on the Heisenberg group $N_n$ with the scalar product

$$(f_1,f_2) = \int\limits_{N_n} f_1(n)\overline{f_2(n)}dn \equiv \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\int\limits_{-\infty}^{\infty} f_1(\mathbf{a},\mathbf{b},c)\overline{f_2(\mathbf{a},\mathbf{b},c)}dc\,d\mathbf{a}\,d\mathbf{b}.$$

The formula

$$(L(n(\mathbf{a},\mathbf{b},c))f)(n) = f(n^{-1}(\mathbf{a},\mathbf{b},c)n)$$

defines the regular representation of $N_n$ in $\mathfrak{L}^2(N_n)$.

In order to decompose the representation $L$ into irreducible components we go over from functions $f(\mathbf{a}, \mathbf{b}, c)$ to their Fourier transforms in $c$:

$$F(\mathbf{a}, \mathbf{b}, \lambda) = \int_{-\infty}^{\infty} f(\mathbf{a}, \mathbf{b}, c) e^{i\lambda c} dc, \tag{1}$$

and expand functions $F(\mathbf{a}, \mathbf{b}, \lambda)$ in the matrix elements $r^{\lambda}_{mp}(n(\mathbf{a}, \mathbf{b}, 0))^*$:

$$F(\mathbf{a}, \mathbf{b}, \lambda) = \left(\frac{|\lambda|}{2\pi}\right)^n \sum_{\mathbf{m},\mathbf{p}} d_{\mathbf{mp}}(\lambda) r^{\lambda}_{\mathbf{mp}}(n(\mathbf{a}, \mathbf{b}, 0))^* \tag{2}$$

(see Section 12.1.6). We have

$$f(\mathbf{a}, \mathbf{b}, c) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \sum_{\mathbf{m},\mathbf{p}} d_{\mathbf{mp}}(\lambda) r^{\lambda}_{\mathbf{mp}}(n(\mathbf{a}, \mathbf{b}, 0))^* e^{-i\lambda c} |\lambda|^n d\lambda, \tag{3}$$

where

$$d_{\mathbf{mp}}(\lambda) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{a}, \mathbf{b}, c) r^{\lambda}_{\mathbf{mp}}(n(\mathbf{a}, \mathbf{b}, 0)) e^{i\lambda c} d\mathbf{a}\, d\mathbf{b}\, dc. \tag{4}$$

For transforms (3) and (4) one has the Plancherel formula

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(\mathbf{a}, \mathbf{b}, c)|^2 d\mathbf{a}\, d\mathbf{b}\, dc = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \sum_{\mathbf{m},\mathbf{p}} |d_{\mathbf{mp}}(\lambda)|^2 |\lambda|^n d\lambda. \tag{5}$$

Formula (3) is the expansion of functions from $\mathfrak{L}^2(N_n)$ in matrix elements of the irreducible representations $R^{\lambda}$ of the group $N_n$.

The Hilbert space of functions

$$\Phi(\mathbf{a}, \mathbf{b}, \lambda) = \sum_{\mathbf{m},\mathbf{p}} d_{\mathbf{mp}}(\lambda) r^{\lambda}_{\mathbf{mp}}(n(\mathbf{a}, \mathbf{b}, 0))^* \tag{6}$$

with the scalar product

$$(\Phi_1, \Phi_2) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi_1(\mathbf{a}, \mathbf{b}, \lambda) \overline{\Phi_2(\mathbf{a}, \mathbf{b}, \lambda)} d\mathbf{a}\, d\mathbf{b}$$

will be denoted by $\mathfrak{L}^2_{\lambda}(\mathbb{R}^{2n})$. The above arguments imply that

$$\mathfrak{L}^2(N_n) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \oplus \mathfrak{L}^2_{\lambda}(\mathbb{R}^{2n}) |\lambda|^n d\lambda. \tag{7}$$

We denote by $\widetilde{\mathfrak{H}}_{\mathbf{p}}^{\lambda}$ the subspace of $\mathfrak{L}_{\lambda}^{2}(\mathbf{R}^{2n})$ consisting of functions

$$\Phi_{\mathbf{p}}(\mathbf{a},\mathbf{b},\lambda) = \sum_{\mathbf{m}} d_{\mathbf{m}}(\lambda) r_{\mathbf{mp}}^{\lambda}(n(\mathbf{a},\mathbf{b},0))^{*} \tag{8}$$

We have

$$\mathfrak{L}^{2}(N_{n}) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \sum_{\mathbf{p}} \oplus \widetilde{\mathfrak{H}}_{\mathbf{p}}^{\lambda}\, |\lambda|^{n} d\lambda. \tag{9}$$

It follows from the results of Section 12.1.7 that left shift operators in $\widetilde{\mathfrak{H}}_{\mathbf{p}}^{\lambda}$ give a representation equivalent to $R^{\lambda}$. Thus,

$$L = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \sum_{\mathbf{p}} \oplus R_{\mathbf{p}}^{\lambda}\, |\lambda|^{n} d\lambda, \tag{10}$$

where all representations $R_{\mathbf{p}}^{\lambda}$ are equivalent to $R^{\lambda}$.

One has the equality

$$(L(n) r_{\mathbf{sp}}^{\lambda}, r_{\mathbf{mp}}^{\lambda}) = r_{\mathbf{ms}}^{\lambda}(n). \tag{11}$$

Operators of the left regular representation act in the space $\widetilde{\mathfrak{H}}_{\mathbf{p}}^{\lambda}$ by formula (13) of Section 12.1.7. Setting $\lambda = 1$, $n(a', b', 0) \equiv n(w, 0)$ and using expression (9) of Section 12.1.6 for $r_{mp}^{\lambda}(n(a,b,0))$, we derive from (11) that

$$\begin{aligned} &\frac{1}{2\pi} \int_{\mathbf{C}} \exp(-i \operatorname{Im} w\bar{z}) \exp\left(-\frac{|z-w|^{2}}{4}\right) \exp\left(-\frac{|z|^{2}}{4}\right) \\ &\quad \times \left(\frac{i(z-w)}{\sqrt{2}}\right)^{s-p} \left(\frac{iz}{\sqrt{2}}\right)^{m-p} L_{p}^{s-p}\left(\frac{|z-w|^{2}}{2}\right) L_{p}^{m-p}\left(\frac{|z|^{2}}{2}\right) dx\, dy \\ &\qquad = \frac{p!}{m!} \exp\left(-\frac{|w|^{2}}{4}\right) \left(\frac{iw}{\sqrt{2}}\right)^{m-s} L_{s}^{m-s}\left(\frac{|w|^{2}}{2}\right), \end{aligned} \tag{12}$$

where $z = x + iy$.

It follows from formula (9) of Section 12.1.6 that if functions $F(\mathbf{a},\mathbf{b},\lambda)$ and $f(\mathbf{a},\mathbf{b},c)$ are invariant with respect to rotations (i.e. if they depend on $|\mathbf{a}|^2 + |\mathbf{b}|^2$ only), then only the matrix elements $r_{mm}^{\lambda}(n(\mathbf{a},\mathbf{b},0))$ take place in expansions (2) and (3).

## 12.2. The Group of Automorphisms of the Heisenberg Group and the Weyl Representation

**12.2.1. The group of automorphisms for $N_1$.** At first we consider the Heisenberg algebra $\mathfrak{n}_1$ with the basis elements $Q$, $P$, $H$:

$$[Q, P] = H, \ [Q, H] = [P, H] = 0 \tag{1}$$

(see formulas (5) of Section 12.1.2). The linear transformation

$$\tilde{Q} = aQ + cP, \ \tilde{P} = bQ + dP, \ a, b, c, d \in \mathbf{R}, \tag{2}$$

of $P$ and $Q$, conserving commutation relations (1), generates the automorphism of $\mathfrak{n}_1$, conserving the central element $H$. We have

$$[\tilde{Q}, \tilde{P}] = [aQ + cP, bQ + dP] = (ad - bc)H.$$

Therefore, the numbers $a$, $b$, $c$, $d$ satisfy the condition

$$ad - bc = 1. \tag{3}$$

Conversely, every set of real numbers $a$, $b$, $c$, $d$ for which condition (3) is fulfilled defines transformation (2), conserving commutation relations (1). Thus, the set of linear automorphisms of the algebra $\mathfrak{n}_1$, which conserve the central element $H$, coincides with the group $SL(2, \mathbf{R})$. It is easy to verify that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$, then

$$g(\alpha Q + \beta P) = \alpha' Q + \beta' P, \tag{4}$$

where

$$\alpha' = a\alpha + b\beta, \quad \beta' = c\alpha + d\beta. \tag{4'}$$

Under the exponential mapping of a Lie algebra onto the corresponding group, an automorphism of the algebra turns into an automorphism of the group. In our case, to automorphism (4) of $\mathfrak{n}_1$ there corresponds the automorphism

$$\exp(\alpha Q + \beta P + \gamma H) \to \exp(g(\alpha Q + \beta P) + \gamma H) \tag{5}$$

of the group $N_1 = \exp \mathfrak{n}_1$. The action of the automorphism $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ upon an element $n \in N_1$ is denoted by $g \cdot n$. It follows from formula (4) of Section 12.1.3 that

$$g \cdot h\left(\alpha, \beta, \gamma + \frac{1}{2}\alpha\beta\right) = h\left(\alpha', \beta', \gamma + \frac{1}{2}\alpha', \beta'\right). \tag{6}$$

**12.2.2. The group of automorphisms for $N_n$.** We now consider the Heisenberg algebra $\mathfrak{n}_n$, generated by the matrices $Q_j$, $P_j$, $j = 1, 2, \ldots, n$, and $H$:

$$[Q_j, P_j] = H, \qquad j = 1, 2, \ldots, n \tag{1}$$

$$[Q_j, P_k] = [Q_j, Q_k] = [P_j, P_k] = [Q_j, H] = [P_j, H] = 0. \tag{2}$$

The real linear space, spanned by the matrices $Q_j$, $P_j$, $j = 1, 2, \ldots, n$, is denoted by $V$. It is clear that $\mathfrak{n}_n = V + \mathbb{R}H$. For the elements

$$\mathbf{v} = \sum_j (\alpha_j Q_j + \beta_j P_j), \;\; \mathbf{v}' = \sum_j (\alpha'_j Q_j + \beta'_j P_j)$$

of the space $V$ we have

$$[\mathbf{v}, \mathbf{v}'] = \sum_j (\alpha_j \beta'_j - \beta_j \alpha'_j) H. \tag{3}$$

The bilinear form

$$B(\mathbf{v}, \mathbf{v}') = \sum_j (\alpha_j \beta'_j - \beta_j \alpha'_j) \tag{4}$$

from (3) is non-degenerate and skew-symmetric. In particular,

$$B(Q_j, P_j) = 1, \qquad j = 1, 2, \ldots, n. \tag{5}$$

The space $V$, together with the form $B$, is said to be a *symplectic* space.

Linear transformations of $V$, leaving the form $B$ invariant, conserve commutation relation (3) and, hence, generate automorphisms of the algebra $\mathfrak{n}_n$, leaving the central element $H$ invariant. The set of these linear transformations form the group $Sp(n, \mathbb{R})$ (see Section 1.1.1). Thus, the *group of linear automorphisms of the Heisenberg algebra $\mathfrak{n}_n$, conserving the central element $H$, is isomorphic to $Sp(n, \mathbb{R})$*. If $n = 1$, then we obtain the group $Sp(1, \mathbb{R})$, isomorphic to $SL(2, \mathbb{R})$.

Let $V_1$ be the subspace of $V$, spanned by the matrices $Q_j$, $j = 1, 2, \ldots, n$, and let $V_2$ be the subspace, spanned by $P_j$, $j = 1, 2, \ldots, n$. Every element $\mathbf{v} \in V$ is uniquely represented in the form $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1 \in V_1$, $\mathbf{v}_2 \in V_2$. In $V_1$ and $V_2$ (as in $n$-dimensional linear real spaces) the rotation group $SO(n)$ acts, namely, if $k = (k_{ij}) \in SO(n)$, then

$$kQ_j = \sum_s k_{sj} Q_s, \;\; kP_j = \sum_s k_{sj} P_s.$$

Let us define the action of this group in $V$. If $k \in SO(n)$ and $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1 \in V_1$, $\mathbf{v}_2 \in V_2$, then

$$k\mathbf{v} = k\mathbf{v}_1 + k\mathbf{v}_2. \tag{6}$$

It is easy to verify that this transformation conserves values (5) and, consequently, the form $B(\ .\ ,\ .\ )$. In other words, $SO(n)$ is imbedded into $Sp(n, \mathbf{R})$.

We separate one more subgroup in $Sp(n, \mathbf{R})$. Let $W_j$ be subspace of $V$, spanned by the matrices $Q_j$ and $P_j$. Formulas (2) of Section 12.2.1 give an action of the group $SL(2, \mathbf{R})$ in $W_j$. An element $\mathbf{v} \in V$ is uniquely represented in the form

$$\mathbf{v} = \mathbf{w}_1 + \ldots + \mathbf{w}_n, \quad \mathbf{w}_j \in W_j.$$

The formula

$$h\mathbf{v} = h\mathbf{w}_1 + \ldots + h\mathbf{w}_n, \quad h \in SL(2, \mathbf{R}), \tag{7}$$

defines an action of $SL(2, \mathbb{R})$ in $V$. This action conserves the bilinear form $B(\ .\ ,\ .\ )$. Therefore, $SL(2, \mathbb{R})$ is a subgroup of $Sp(n, \mathbb{R})$. It follows from formulas (6) and (7) that for $k \in SO(n)$, $h \in SL(2, \mathbb{R})$ we have $hk\mathbf{v} = kh\mathbf{v}$, $\mathbf{v} \in V$. Thus, $Sp(n, \mathbb{R})$ contains the direct product $SO(n) \times SL(2, \mathbb{R})$.

Under the exponential mapping of the Heisenberg algebra $\mathfrak{n}_n$ onto the Heisenberg group $N_n$, an automorphism of the algebra turns into an automorphism of the group. Namely, to the automorphism

$$\mathbf{a} \cdot \mathbf{Q} + \mathbf{b} \cdot \mathbf{P} \to g(\mathbf{a} \cdot \mathbf{Q} + \mathbf{b} \cdot \mathbf{P}), \quad g \in Sp(n, \mathbb{R}),$$

of $\mathfrak{n}_n$ there corresponds the automorphism

$$\exp(\mathbf{a} \cdot \mathbf{Q} + \mathbf{b} \cdot \mathbf{P} + cH) \to \exp(g(\mathbf{a} \cdot \mathbf{Q} + \mathbf{b} \cdot \mathbf{P}) + cH) \tag{8}$$

of $N_n$. If $g(\mathbf{a} \cdot \mathbf{Q} + \mathbf{b} \cdot \mathbf{P}) = \mathbf{a}' \cdot \mathbf{Q} + \mathbf{b}' \cdot \mathbf{P}$, then we introduce the notations $\mathbf{a}' = g \cdot \mathbf{a}$, $\mathbf{b}' = g \cdot \mathbf{b}$. It follows from formula (4) of Section 12.1.3 that

$$g \cdot h\left(\mathbf{a}, \mathbf{b}, c + \frac{1}{2}\mathbf{a} \cdot \mathbf{b}\right) = h\left(g \cdot \mathbf{a}, g \cdot \mathbf{b}, c + \frac{1}{2}(g \cdot \mathbf{a}) \cdot (g \cdot \mathbf{b})\right). \tag{9}$$

In particular, if $g = k \in SO(n)$, then we have

$$k \cdot h\left(\mathbf{a}, \mathbf{b}, c + \frac{1}{2}\mathbf{a} \cdot \mathbf{b}\right) = h\left(k \cdot \mathbf{a}, k \cdot \mathbf{b}, c + \frac{1}{2}\mathbf{a} \cdot \mathbf{b}\right). \tag{10}$$

**12.2.3. The group $Sp(n, \mathbb{R}) \times N_n$.** The group $Sp(n, \mathbb{R})$ acts in $N_n$ as the group of automorphisms of $N_n$. Therefore, we can form the group $Sp(n, \mathbb{R}) \times N_n$ which is the semidirect product of the groups $Sp(n, \mathbb{R})$ and $N_n$. The group $Sp(n, \mathbb{R}) \times N_n$ consists of pairs $(g, n)$, $g \in Sp(n, \mathbb{R})$, $n \in N_n$, and the group operation is given by the formula

$$(g_1, n)(g_2, n) = (g_1 g_2, (g_2^{-1} \cdot n_1) n_2). \tag{1}$$

We suggest to the reader to verify that this operation satisfies the associativity condition

$$[(g_1,n_1)(g_2,n_2)](g_3,n_3) = (g_1,n_1)[(g_2,n_2)(g_3,n_3)]$$

(see Section 1.0.1).

The elements $(g_1,0)$, where 0 denotes the element $n(\mathbf{0},\mathbf{0},0)$ of $N_n$, form the subgroup isomorphic to $Sp(n,\mathbb{R})$. The elements $(e,n)$, where $e$ is the identity element of $Sp(n,\mathbb{R})$, form the invariant subgroup, isomorphic to $N_n$.

We are interested in the subgroups $SL(n,\mathbb{R}) \times N_n$ and $SO(n) \times N_n$ of $Sp(n,\mathbb{R}) \times N_n$. If the group $N_n$ is realized by matrices (4) of Section 12.1.1, then $SO(n) \times N_n$ is realized by the matrices

$$h(k;\mathbf{a},\mathbf{b},t) = \begin{pmatrix} 1 & \mathbf{a} & t \\ \mathbf{0} & k & k\mathbf{b}^T \\ 0 & \mathbf{0} & 1 \end{pmatrix}, \qquad k \in SO(n). \tag{2}$$

We have

$$h(k;\mathbf{a},\mathbf{b},t)h(k';\mathbf{a}',m\mathbf{b}',t') = h(kk';\mathbf{a}k'+\mathbf{a}',\mathbf{b}k'+\mathbf{b}',t+t'+\mathbf{a}k'\cdot\mathbf{b}'). \tag{3}$$

**12.2.4. The Weyl representation of the group $SL(2,\mathbb{R})$.** The Lie algebra of the Heisenberg group $N_1$ is spanned by the basis matrices $Q$, $P$, $H$. We have constructed in Section 12.1.4 the representations $R^\lambda$ of the group $N_1$. For simplicity we consider here the representation $R^1$. The operators corresponding to the matrices $Q' = -iQ$, $P' = -iP$, $H' = -iH$ (see formulas (6) of Section 12.1.2) are denoted by $\mathrm{Q}$, $\mathrm{P}$, $I$, where $I$ is the identity operator. The representation $R^1$ acts in $\mathfrak{L}^2(\mathbb{R})$ and we have

$$(\mathrm{Q}f)(x) = xf(x), \tag{1}$$

$$(\mathrm{P}f)(x) = -i\frac{d}{dx}f(x). \tag{2}$$

The operators $\mathrm{Q}$ and $\mathrm{P}$ are known to be self-adjoint in $\mathfrak{L}^2(\mathbb{R})$. It follows from the Theorem of Section 12.1.4 that any irreducible unitary representation $T$ of the group $N_1$, for which $T(n(0,0,c)) = e^{ic}I$ (see formula (2) of Section 12.1.4), is unitarily equivalent to the representation $R^1$. In other words, if to the matrices $Q'$ and $P'$ there correspond the operators $\mathrm{Q}'$ and $\mathrm{P}'$ in $T$, then there exists the unitary intertwining operator $A$:

$$A\mathrm{Q}A^{-1} = \mathrm{Q}', \quad A\mathrm{P}A^{-1} = \mathrm{P}'.$$

We use this property for constructing a representation of the group $SL(2,\mathbb{R})$ in $\mathfrak{L}^2(\mathbb{R})$.

With an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, we associate the automorphism

$$g\{Q, P\} = \{dQ - bP, -cQ + aP\} \equiv \{Q_g, P_g\} \tag{3}$$

of the algebra $\mathfrak{n}_1$. This automorphism differs from automorphism (2) of Section 12.2.1. Namely, instead of the matrix $g^T$ (where $T$ denotes the transposition) one uses $g^{-1}$. Thus, matrices of linear transformations in formula (2) of Section 12.2.1 and in (3) are connected by the relation $g \leftrightarrow (g^T)^{-1}$, $g \in SL(2, \mathbb{R})$.

To the elements $Q_g$ and $P_g$ there correspond the operators

$$\mathrm{Q}_g = d\mathrm{Q} - b\mathrm{P}, \quad \mathrm{P}_g = -c\mathrm{Q} + a\mathrm{P}$$

in the representation $R^1$. The correspondence $Q' \to \mathrm{Q}_g$, $P' \to \mathrm{P}_g$, $H' \to I$ defines the representation $R^1_g$ of $N_1$. It is obtained from $R^1$ by the action of the automorphism of $N_1$ corresponding to automorphism (3) of $\mathfrak{n}_1$. This representation is equivalent to $R^1$. Therefore, there exists the unitary operator $T_g$ in $\mathfrak{L}^2(\mathbb{R})$, intertwining $R^1$ and $R^1_g$:

$$T_g \mathrm{Q} T_g^{-1} = \mathrm{Q}_g = d\mathrm{Q} - b\mathrm{P}, \tag{4}$$

$$T_g \mathrm{P} T_g^{-1} = \mathrm{P}_g = -c\mathrm{Q} + a\mathrm{P}. \tag{5}$$

If $g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2, \mathbb{R})$, then we find from (4) that

$$\begin{aligned} T_{g'}(T_g \mathrm{Q} T_g^{-1}) T_{g'}^{-1} &= d T_{g'} \mathrm{Q} T_{g'}^{-1} - b T_{g'} \mathrm{P} T_{g'}^{-1} \\ &= d(d'\mathrm{Q} - b'\mathrm{P}) - b(-c'\mathrm{Q} + a'\mathrm{P}) \\ &= (dd' + bc')\mathrm{Q} - (db' + ba')\mathrm{P} = T_{g'g} \mathrm{Q} T_{g'g}^{-1}. \end{aligned}$$

In the same way one proves that

$$T_{g'}(T_g \mathrm{P} T_g^{-1}) T_{g'}^{-1} = T_{g'g} \mathrm{P} T_{g'g}^{-1}.$$

Since the representation $R^1$ of $N_1$ is irreducible, then this equality implies that the operators $T_{g'} T_g$ and $T_{g'g}$ differ in a constant $\lambda(g', g)$, that is,

$$T_{g'} T_g = \lambda(g', g) T_{g'g}. \tag{6}$$

Because of the unitarity of $T_g$ we have $|\lambda(g', g)| = 1$.

The relation

$$(T_{g''} T_{g'}) T_g = T_{g''}(T_{g'} T_g)$$

implies the following property of $\lambda(g', g)$:

$$\lambda(g''g', g)\lambda(g'', g') = \lambda(g'', g'g)\lambda(g', g). \tag{7}$$

It is clear that $\lambda(g,e) = \lambda(e,g) = 1$.

A correspondence $g \to T_g$, for which relation (6) is fulfilled and the function $\lambda$ satisfies condition (7), is said to be a *projective representation* of a group (in our case, of $SL(2,\mathbb{R})$). The function $\lambda(g',g)$ is called *cocycle.*

One can show that the function $\lambda(g',g)$ from (6) takes two values $\pm 1$ (see, for example, Ref. [233]). We show that this fact allows us to construct from the projective representation $g \to T_g$ a usual representation for the two-fold covering group of $SL(2,\mathbb{R})$. The two-fold covering $G_2$ of $SL(2,\mathbb{R})$ consists of the pairs $(g, e^{i\pi n})$, $g \in SL(2,\mathbb{R})$, $n = 0$ or 1. The multiplication in $G_2$ is defined by the formula

$$(g, e^{i\pi n})(g', e^{i\pi n'}) = (gg', \lambda(g,g')e^{i\pi n}e^{i\pi n'}). \tag{8}$$

We leave to the reader to prove the associativity condition

$$[(g, e^{i\pi n})(g', e^{i\pi n'})](g'', e^{i\pi n''}) = (g, e^{i\pi n})[(g', e^{i\pi n'})(g'', e^{i\pi n''})] \tag{9}$$

for operation (8).

With the elements $(g, e^{i\pi n}) \in G_2$ we associate the operator

$$\widetilde{T}(g, e^{i\pi n}) = e^{i\pi n}T_g.$$

This correspondence is a representation of $G_2$. Really, it follows from (6) and (8) that

$$\begin{aligned}\widetilde{T}((g', e^{i\pi n'})(g, e^{i\pi n})) &= \widetilde{T}((g'g, \lambda(g',g)e^{i\pi n'}e^{i\pi n})) \\ &= \lambda(g',g)e^{i\pi n'}e^{i\pi n}T_{g'g} = e^{i\pi n'}e^{i\pi n}T_{g'}T_g \\ &= \widetilde{T}((g', e^{i\pi n'}))\widetilde{T}((g, e^{i\pi n})).\end{aligned}$$

An element $g$ of $SL(2,\mathbb{R})$ is covered by two elements $(g,1)$ and $(g,-1)$ of $G_2$. If with $g$ one associates the operators $T(g,1)$ and $T(g,-1)$ corresponding to these elements of the group $G_2$, then we obtain the two-valued representation $g \to \pm T_g$ of the group $SL(2,\mathbb{R})$. It is called the *Weyl representation.*

**Example 1.** The Fourier transform

$$(\mathfrak{F}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} f(p)dp$$

coincides with the operator $T_g$, $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Really, the operator $\mathfrak{F}$ posssses the following property: if $f^a(x) = f(x+a)$, then

$$(\mathfrak{F}f^a)(x) = e^{iax}(\mathfrak{F}f)(x).$$

Passing from $f^a(x)$ and from $e^{iax}f(x)$ to the derivatives with respect to $a$ at $a = 0$, we obtain

$$\mathfrak{F}\mathrm{P}\mathfrak{F}^{-1} = \mathrm{Q}, \quad \mathfrak{F}\mathrm{Q}\mathfrak{F}^{-1} = -\mathrm{P}, \tag{10}$$

where P and Q are given by formulas (1) and (2). By comparing (10) with formulas (4) and (5), we find $\mathfrak{F} = T_g$, $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

**Example 2.** The element $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generates the automorphism

$$n(a,b,c) \to n(g(a,b),c) = n(b,-a,c)$$

of the group $N_1$. Consequently,

$$\mathfrak{F}R^1(n(a,b,c))\mathfrak{F}^{-1} = R^1(n(b,-a,c)). \tag{11}$$

But formula (7) of Section 12.1.6 implies that

$$r^{\lambda}_{mp}(n(a,b,c)) = i^{p-m}r^{\lambda}_{mp}(n(b,-a,c)),$$

that is, the matrices $(r^1_{mp}(n(a,b,c)))$ and $(r^1_{mp}(n(b,-a,c)))$ are connected by the relation

$$A(r^1_{mp}(n(a,b,c)))A^{-1} = (r^1_{mp}(n(b,-a,c))), \tag{12}$$

where $A$ is the diagonal matrix with the elements $a_{mm} = i^{-m}$ on the main diagonal. Comparing (11) and (12) and taking into account the irreducibility of the representation $R^1$, we conclude that $\mathfrak{F} = \mu A$, where $\mu$ is a complex number such that $|\mu| = 1$, i.e.

$$(\mathfrak{F}e_m)(x) = \mu i^{-m}e_m(x), \tag{13}$$

where $e_m(x)$ are basis elements (1) of Section 12.1.5 with $\lambda = 1$. Calculating the Fourier transform of the function $e_0(x) = e^{-x^2/2}$ (see Example 1 of Section 3.2.3), we find that $\mu = 1$, that is,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx}e^{-p^2/2}H_m(p)dp = i^{-m}e^{-x^2/2}H_m(x). \tag{14}$$

This formula provides a new proof of the inversion formula for the Fourier transform $\mathfrak{F}$. Indeed, every function $f(x) \in \mathfrak{L}^2(\mathbb{R})$ can be expanded into the series

$$f(x) = \sum_{n=0}^{\infty} a_n e^{-x^2/2}H_n(x).$$

It follows from this expansion and from (14) that

$$F(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} f(x)dx = \sum_{n=0}^{\infty} a_n i^{-n} e^{-y^2/2} H_n(y).$$

Applying formula (14) once more, we see that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} F(y)dy = \sum_{n=0}^{\infty} a_n e^{-x^2/2} H_n(x) = f(x).$$

The inversion formula is proved. One can analogously prove the Plancherel formula for the Fourier transform.

**12.2.5. The integral form of the Weyl representation.** Let $\mathfrak{H}$ be the space of infinitely differentiable functions from $\mathfrak{L}^2(\mathbb{R})$. It is everywhere dense in $\mathfrak{L}^2(\mathbb{R})$. The function $(T_g f)(x)$, $f \in \mathfrak{L}^2(\mathbb{R})$, will be denoted by $f_g(x)$. We have

$$\begin{aligned}(T_g \mathrm{Q} f)(x) &= ((T_g \mathrm{Q} T_g^{-1}) f_g)(x) \\ &= ((d\mathrm{Q} - b\mathrm{P}) f_g)(x) = dx f_g(x) + ib\frac{d}{dx} f_g(x), \end{aligned} \tag{1}$$

$$\begin{aligned}(T_g \mathrm{P} f)(x) &= ((T_g \mathrm{P} T_g^{-1}) f_g)(x) \\ &= ((-c\mathrm{Q} + a\mathrm{P}) f_g)(x) = -cx f_g(x) = ia\frac{d}{dx} f_g(x), \end{aligned} \tag{2}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $f \in \mathfrak{H}$. By means of these equalities one derives the expression for $T_g$ in the integral form:

$$f_g(x) \equiv (T_g f)(x) = \int_{-\infty}^{\infty} f(y) K_g(x,y)dy, \quad f \in \mathfrak{H}. \tag{3}$$

From (1) and (3) we obtain the relation

$$\int_{-\infty}^{\infty} K_g(x,y) y f(y)dy = \left(d\cdot x + ib\frac{d}{dx}\right) \int_{-\infty}^{\infty} K_g(x,y) f(y)dy, \tag{4}$$

and from (2) and (3) the relation

$$-i \int_{-\infty}^{\infty} K_g(x,y) \left[\frac{d}{dy} f(y)\right] dy = -\left(cx + ia\frac{d}{dx}\right) \int_{-\infty}^{\infty} K_g(x,y) f(y)dy. \tag{5}$$

Relations (4) and (5) are fulfilled if $f(y)K_g(x,y) \to 0$ for $y \to \pm\infty$ and if $K_g(x,y)$ satisfies the differential equations

$$yK_g(x,y) = \left(d \cdot x + ib\frac{d}{dx}\right) K_g(x,y), \tag{6}$$

$$i\frac{d}{dy}K_g(x,y) = \left(cx + ia\frac{d}{dx}\right) K_g(x,y). \tag{7}$$

If $b \neq 0$, then these conditions are satisfied by the kernel

$$K_g(x,y) = c_g \exp[i(ay^2 - 2xy + dx^2)/2b], \tag{8}$$

where

$$c_g = \frac{e^{-i\pi/4}}{\sqrt{2\pi b}}.$$

The constant $c_g$ is chosen in such a way that the equality $T_{g_1 g_2} = \pm T_{g_1} T_{g_2}$ holds. The detailed calculation of $c_g$ by an analytic continuation is given in Ref. [56] (Chapter 9) of the first volume; it is shown there that different analytic continuations lead to the equalities $T_{g_1} T_{g_2} = \pm T_{g_1 g_2}$ with different signs, i.e. $T_g$ is a projective representation of $SL(2,\mathbb{R})$.

In order to obtain the kernel $K_g(x,y)$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $b = 0$ the limit is considered (see Ref. [56] (Chapter 9) of the first volume). it is shown by the analytic continuation of the parameters $a$ and $b$ into the complex domain that

$$\lim_{|b|\to 0} K_g(x,y) = \frac{1}{\sqrt{a}} e^{icx^2/2a} \delta(y - a^{-1}x), \tag{9}$$

where $\delta$ is the delta-function. Thus, if $g = \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$, then one has

$$(T_g f)(x) = \frac{1}{\sqrt{a}} \exp(icx^2/2a) f(a^{-1}x). \tag{10}$$

A direct verification shows that these operators satisfy relations (4) and (5) of Section 12.2.4.

**Example 1.** To the element $g = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ there corresponds the operator $T_g$, given by the formula

$$(T_g f)(x) = \frac{e^{-i\pi/4}}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} f(y) \exp\left(i\frac{(y-x)^2}{2b}\right) dx. \tag{11}$$

This formula defines the *imaginary Gauss-Weierstrass transform.*

**12.2.6. Infinitesimal operators of the Weyl representation.** We shall find the infinitesimal operators $A_-$, $A_+$, $A_0$ of the Weyl representation $T_g$ of the group $SL(2,\mathbb{R})$ corresponding to the one-parameter subgroups

$$g_+(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \; g_-(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \; g_0(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \tag{1}$$

respectively. We find from formulas (8) and (10) of Section 12.2.5 that

$$(A_0 f)(x) = \frac{d}{dt}(T_{g_0(t)}f)(x)\Big|_{t=0} = \left(-\frac{1}{2} - x\frac{d}{dx}\right) f(x),$$

$$A_- = i\frac{x^2}{2}, \; A_+ = \frac{i}{2}\frac{d^2}{dx^2}.$$

By comparing these operators with operators (1) and (2) of Section 12.2.4, we have

$$A_0 = -\frac{i}{2}(\mathrm{QP} + \mathrm{PQ}), \; A_- = \frac{i}{2}\mathrm{Q}^2, \; A_+ = -\frac{i}{2}\mathrm{P}^2. \tag{2}$$

Thus, the infinitesimal operators of the Weyl representation are homogeneous polynomials of the second degree in the operators Q and P.

Instead of $A_0$, $A_-$, $A_+$, one considers the operators

$$J_0 = \frac{i}{2}(A_- - A_+) = \frac{1}{4}(\mathrm{P}^2 + \mathrm{Q}^2) = \frac{1}{4}\left(x^2 - \frac{d^2}{dx^2}\right), \tag{3}$$

$$J_+ = \frac{i}{2}(A_+ + A_-) - \frac{1}{2}A_0 = \frac{1}{4}(\mathrm{P}^2 - \mathrm{Q}^2) + \frac{i}{2}(\mathrm{QP} + \mathrm{PQ}), \tag{4}$$

$$J_- = \frac{i}{2}(A_+ + A_-) + \frac{1}{2}A_0 = \frac{1}{4}(\mathrm{P}^2 - \mathrm{Q}^2) - \frac{i}{2}(\mathrm{QP} + \mathrm{PQ}) \tag{5}$$

for which

$$[J_0, J_+] = J_+, \; [J_0, J_-] = -J_-, \; [J_+, J_-] = -2J_0. \tag{6}$$

The operator $J_0$ corresponds to the one-parameter subgroup consisting of matrices $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$.

Let us calculate how $J_0$, $J_+$, $J_-$ act upon the basis functions

$$e_m(x) = (2^m m!)^{-1/2}\pi^{-1/4}e^{-x^2/2}H_m(x), \; m = 0, 1, 2, \dots, \tag{7}$$

of the space $\mathfrak{L}^2(\mathbb{R})$. The operators

$$a^- = \frac{1}{\sqrt{2}}(\mathrm{Q} + i\mathrm{P}) = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right), \tag{8}$$

$$a^+ = \frac{1}{\sqrt{2}}(\mathrm{Q} - i\mathrm{P}) = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right) \tag{9}$$

act upon the functions $e_m(x)$ by formulas

$$a^- e_m = \sqrt{m}\, e_{m-1}, \quad a^+ e_m = \sqrt{m+1}\, e_{m+1}. \tag{10}$$

Therefore, they are said to be the *annihilation operator* and the *creation operator*, respectively. They satisfy the relation $[a^-, a^+] = 1$. Since

$$J_0 = \frac{1}{4}(a^-a^+ + a^+a^-),\ J_+ = -\frac{1}{2}(a^+)^2,\ J_- = -\frac{1}{2}(a^-)^2, \tag{11}$$

then

$$\left.\begin{aligned} J_0 e_m &= \frac{1}{2}\left(m + \frac{1}{2}\right) e_m, \\ J_+ e_m &= \frac{1}{2}\sqrt{(m+1)(m+2)}\, e_{m+2}, \\ J_- e_m &= \frac{1}{2}\sqrt{(m-1)m}\, e_{m-2}, \end{aligned}\right\} \tag{12}$$

It is now clear that the space $\mathfrak{L}^2(\mathbb{R})$ decomposes into the orthogonal sum of two subspaces, denoted by $\mathfrak{L}^2_+(\mathbb{R})$ and $\mathfrak{L}^2_-(\mathbb{R})$, which are invariant with respect to the Weyl representation $T$. The subspace $\mathfrak{L}^2_+(\mathbb{R})$ is spanned by the basis functions $e_0, e_2, e_4, \ldots$, and $\mathfrak{L}^2_-(\mathbb{R})$ is spanned by $e_1, e_3, e_5, \ldots$. It follows from formulas (6) and (7) of Section 3.5.7 that $\mathfrak{L}^2_+(\mathbb{R})$ consists of even functions and that $\mathfrak{L}^2_-(\mathbb{R})$ consists of odd functions. It is easy to show by means of formulas (12) that the subspaces $\mathfrak{L}^2_+(\mathbb{R})$ and $\mathfrak{L}^2_-(\mathbb{R})$ are irreducible with respect to the Weyl representation.

Thus, *the Weyl representation of the group $SL(2, \mathbb{R})$ is reducible and consists of two irreducible components. In the first representation the spectrum of the operator $J_0$ consists of the points $n + \frac{1}{4}$, $n = 0, 1, 2, \ldots$, and in the second one the spectrum of $J_0$ consists of $n + \frac{3}{4}$, $n = 0, 1, 2, \ldots$.*

We suggest to the reader to evaluate the matrix elements of the Weyl representation in basis (7).

## 12.3. The Weyl Representation of $Sp(n, \mathbb{R})$ and Bases of the Carrier Space

**12.3.1. The Weyl representation of the group $Sp(n, \mathbb{R})$.** We now consider the representation $R^1$ of the Heisenberg group $N_n$ which acts in the space $\mathfrak{L}^2(\mathbb{R}^n)$. To the elements $Q'_j = -iQ_j$, $P'_j = -iP_j$, $j = 1, 2, \ldots, n$, of the Heisenberg algebra $\mathfrak{n}_n$ (see Section 12.1.2), there correspond the operators $\mathrm{Q}_j$, $\mathrm{P}_j$, $j = 1, 2, \ldots, n$, of the representation $R^1$, which are given by the formulas

$$(\mathrm{Q}_j f)(\mathbf{x}) = x_j f(\mathbf{x}), \tag{1}$$

$$(\mathrm{P}_j f)(\mathbf{x}) = -i\frac{d}{dx_j} f(\mathbf{x}). \tag{1'}$$

To the element $H' = -iH$ there corresponds the identity operator.

The group of automorphisms of $N_n$ is isomorphic to $Sp(n,\mathbb{R})$. We define the action of $Sp(n,\mathbb{R})$ on $N_n$ and on $\mathfrak{n}_n$ in the same way as for the group $SL(2,\mathbb{R})$ in Section 12.2.4. Namely, if $g \in Sp(n,\mathbb{R})$, $\{Q_j, P_j\} \equiv \{Q_1,\dots,Q_n,P_1,\dots,P_n\}$, then

$$g\{Q_j,P_j\} = \{Q_j^g, P_j^g\}, \tag{2}$$

where

$$Q_j^g = \sum_i g_{ji}^{(-1)} Q_i, \quad P_j^g = \sum_i g_{ji}^{(-1)} P_i$$

and $(g_{ij}^{-1})$ denotes the matrix of $g^{-1}$. Similarly to the case of the group $SL(2,\mathbb{R})$ (see Section 12.2.4), one shows that to an automorphism $g \in Sp(n,\mathbb{R})$ there corresponds the unitary operator $T_g$ such that

$$T_g\{Q_j,P_j\}T_g^{-1} = \{Q_j^g, P_j^g\}. \tag{3}$$

We have

$$T_{g'}T_g = \lambda(g',g)T_{g'g}, \ |\lambda(g',g)| = 1. \tag{4}$$

The function $\lambda(g',g)$ satisfies condition (7) of Section 12.2.4.

The operators $T_g$ define the projective representation $g \to T_g$ of the group $Sp(n,\mathbb{R})$, called the *Weyl representation.* In the same way as in the case of $SL(2,\mathbb{R})$, it defines an ordinary representation for two-fold covering group of $Sp(n,\mathbb{R})$. This covering group is called the *metaplectic group* and is denoted by $Mp(n,\mathbb{R})$.

Below we shall be interested not in the Weyl representation of the whole group $Sp(n,\mathbb{R})$, but in its restriction onto the subgroup $SO(n)\times SL(2,\mathbb{R})$. We show that the restriction of $T$ onto $SO(n)$ coincides with the quasi-regular representation of $SO(n)$ in $\mathfrak{L}^2(\mathbb{R}^n)$, i.e. that $T\downarrow^{Sp(n,\mathbb{R})}_{SO(n)} = L$, where

$$(L_k f)(\mathbf{x}) = f(k^{-1}\mathbf{x}), \quad k \in SO(n). \tag{5}$$

We have

$$\begin{aligned}(L_k Q_j L_k^{-1} f)(\mathbf{x}) &= (L_k Q_j) f(k\mathbf{x}) = L_k(x_j f(k\mathbf{x})) \\ &= (k^{-1}\mathbf{x}_j) f(\mathbf{x}) = (kQ_j) f(\mathbf{x}),\end{aligned} \tag{6}$$

where $\mathbf{x}_j = (0,\dots,0,x_j,0,\dots,0)$ and the action of $k$ upon $Q_j$ is defined by formula (2). In the same way one shows that if $k \in SO(n)$, then

$$L_k P_j L_k^{-1} = kP_j. \tag{7}$$

Thus, for $k \in SO(n)$ we have

$$L_k\{Q_j,P_j\}L_k^{-1} = \{Q_j^k, P_j^k\}. \tag{8}$$

Because of irreducibility of the representation $R^1$ of $N_n$, we obtain from (3) and (8) that $L_k = \lambda_k T_k$, $k \in SO(n)$, $|\lambda_k| = 1$. But relation (3) defines the operators $T_g$ up to the constants $\lambda_g$ (which do not change the relation (4)). Therefore, we can assume that $L_k = T_k$ for $k \in SO(n)$.

We have shown in Section 12.2.2 that the action of the subgroup $SL(2,\mathbf{R})$ of $Sp(n,\mathbf{R})$ in the space $V$ is given by means of the action of $SL(2,\mathbf{R})$ in the two-dimensional spaces $W_j$. To the action of $SL(2,\mathbb{R})$ in $W_j$ there corresponds the action of the operators $T_h$, $h \in SL(2,\mathbb{R})$, upon the variable $x_j$. To this action of $T_h$ there correspond the infinitesimal operators $A_0^{(j)}$, $A_+^{(j)}$, $A_-^{(j)}$ of one-parameter subgroups (1) of Section 12.2.6. These infinitesimal operators are given by formulas (2) of the same section:

$$A_0^{(j)} = -\frac{i}{2}(\mathrm{Q}_j\mathrm{P}_j + \mathrm{P}_j\mathrm{Q}_j),\ A_-^{(j)} = \frac{i}{2}\mathrm{Q}_j^2,\ A_+^{(j)} = -\frac{i}{2}\mathrm{P}_j^2.$$

The group $SL(2,\mathbb{R})$ acts upon all pairs $\mathrm{Q}_j$, $\mathrm{P}_j$, $j = 1,2,\ldots,n$, in the same way (see formula (7) of Section 12.2.2). Hence, the action of $SL(2,\mathbb{R})$ in the whole space $V$ implies that the operators $T_h$, $h \in SL(2,\mathbb{R})$, act upon all of the variables $x_1,\ldots,x_n$ in the same way. To these operators there correspond the infinitesimal operators

$$A_0 = \sum_{j=1}^{n} A_0^{(j)},\ A_- = \sum_{j=1}^{n} A_-^{(j)},\ A_+ = \sum_{j=1}^{n} A_+^{(j)}.$$

We introduce the operators

$$J_0 = \frac{i}{2}(A_- - A_+),\ J_+ = \frac{i}{2}(A_+ + A_-) - \frac{1}{2}A_0,\ J_- = \frac{i}{2}(A_+ + A_-) + \frac{1}{2}A_0 \qquad (9)$$

satisfying commutation relations (6) of Section 12.2.6. They can be expressed in terms of the annihilation and the creation operators

$$\begin{aligned} a_j^+ &= \frac{1}{\sqrt{2}}(\mathrm{Q}_j - i\mathrm{P}_j) = \frac{1}{\sqrt{2}}\left(x_j - \frac{\partial}{\partial x_j}\right), \\ a_j^- &= \frac{1}{\sqrt{2}}(\mathrm{Q}_j + i\mathrm{P}_j) = \frac{1}{\sqrt{2}}\left(x_j + \frac{\partial}{\partial x_j}\right), \end{aligned} \qquad (10)$$

for which

$$[a_i^-, a_j^+] = \delta_{ij},\ [a_j^+, a_k^+] = [a_j^-, a_k^-] = 0.$$

We have

$$J_0 = \frac{1}{4}\sum_{j=1}^{n}(a_j^+a_j^- + a_j^-a_j^+),\ J_+ = -\frac{1}{2}\sum_{j=1}^{n}(a_j^+)^2,\ J_- = -\frac{1}{2}\sum_{j=1}^{n}(a_j^-)^2. \qquad (11)$$

**12.3.2. New bases of $\mathfrak{L}^2(\mathbb{R}^n)$.** In Section 12.1.5 we have constructed the orthonormal basis $\{e_{\mathbf{m}}(\mathbf{x})\}$ of the space $\mathfrak{L}^2(\mathbb{R}^n)$. For $\lambda = 1$ we have

$$e_{\mathbf{m}}(\mathbf{x}) = e_{m_1}(x_1)\dots e_{m_n}(x_n),$$

where

$$e_m(x) = (2^m m!)^{-1/2}\pi^{-1/4}e^{-x^2/2}H_m(x). \tag{1}$$

The $e_{\mathbf{m}}(\mathbf{x})$ are eigenfunctions for the operator

$$\widetilde{D}_1 = -D_1 \equiv -\frac{1}{2}\sum_{s=1}^{n}(Q_s^2 + P_s^2) = \frac{1}{2}\sum_{s=1}^{n}\left(x_s^2 - \frac{\partial^2}{\partial x_s^2}\right). \tag{2}$$

For $\mathbf{m} = (m_1, \dots, m_n)$ we have

$$\widetilde{D}_1 e_{\mathbf{m}} = \left(m_1 + \dots + m_n + \frac{n}{2}\right) e_{\mathbf{m}}. \tag{3}$$

The functions $e_{\mathbf{m}}(\mathbf{x})$, for which the sum $m_1 + \dots + m_n = N$ is fixed, correspond to the same eigenvalue of $\widetilde{D}_1$, which is equal to $N + \frac{n}{2}$. The subspace belonging to this eigenvalue is denoted by $\mathfrak{L}_N$.

We go over in functions $f \in \mathfrak{L}^2(\mathbb{R}^n)$ from the Cartesian coordinates to the spherical coordinates $r$, $\theta_1, \dots, \theta_{n-1}$ corresponding to some tree $T$:

$$f(\mathbf{x}) = F(r, \theta_1 \dots, \theta_{n-1}) \equiv F(r, \boldsymbol{\xi}), \quad \boldsymbol{\xi} \in S^{n-1}.$$

The operator $\widetilde{D}_1$ in the spherical coordinates has the form

$$\widetilde{D}_1 = \frac{1}{2}\left(r^2 - \Delta_r - \frac{1}{r^2}\Delta_0\right), \tag{4}$$

where $\Delta_0$ is the Laplace operator on $S^{n-1}$, corresponding to the tree $T$ (see Section 10.5.2) and

$$\Delta_r = r^{1-n}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r}.$$

We represent the measure $d\mathbf{x} = dx_1 \dots dx_n$ in the form

$$d\mathbf{x} = r^{n-1}dr\, d\boldsymbol{\xi},$$

where $d\boldsymbol{\xi}$ is the measure on $S^{n-1}$.

Let us find eigenfunctions of $\widetilde{D}_1$ in spherical coordinates. We assume that they are representable in the form

$$\Psi(r, \boldsymbol{\xi}) = R(r)\Xi_M^{nm}(\boldsymbol{\xi}), \tag{5}$$

where $\Xi_M^{nm}(\xi)$ are the eigenfunctions of $\Delta_0$:

$$\Delta_0 \Xi_M^{nm} = m(m+n-2)\Xi_M^{nm}$$

normalized with respect to the measure $d\boldsymbol{\xi}$ (see Section 9.3.1). Substituting expression (5) for $\Psi$ into the equation

$$\widetilde{D}_1 \Psi \equiv \frac{1}{2}\left(r^2 - \Delta_r - \frac{1}{r^2}\Delta_0\right)\Psi = \left(N + \frac{n}{2}\right)\Psi,\ N = m_1 + \ldots + m_n,$$

we derive the differential equation for $R(r)$:

$$\left[\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} + (2N+n) - r^2 - \frac{m(m+n-2)}{r^2}\right] R(r) = 0. \tag{6}$$

If $N$ is such that $N \geq m$ and $N - m$ is an even number, then the substitution

$$R(r) = t^{m/2} e^{-t/2} f(t), \quad t = r^2,$$

transfers (6) into equation (21) of Section 5.5.2 for Laguerre polynomials. Therefore, (6) has the solution

$$R_{mN}(r) = \left[\frac{2\left(\frac{N-m}{2}\right)!}{\Gamma\left(\frac{N+m+n}{2}\right)}\right]^{1/2} r^m e^{-r^2/2} L_{(N-m)/2}^{m-1+n/2}(r^2). \tag{7}$$

The factor is chosen in such a way that the normalization condition

$$\int_0^\infty |R_{mN}(r)|^2 r^{n-1} dr = 1$$

is fulfilled. If $m$ is fixed, then the functions

$$R_{mN}(r), \quad N = m, m+2, m+4, \ldots, \tag{8}$$

form an orthonormal basis of the Hilbert space $\mathfrak{H}$ of functions $f(r)$ with the scalar product

$$(f_1, f_2) = \int_0^\infty f_1(r)\overline{f_2(r)} r^{n-1} dr.$$

Hence, the set of functions $\{\Psi_{NM}^m\}$, where

$$\Psi_{NM}^m(r, \boldsymbol{\xi}) = R_{mN}(r)\Xi_M^{nm}(\boldsymbol{\xi}), \tag{9}$$

forms an orthonormal basis of the space $\mathfrak{L}^2(\mathbb{R}^n)$.

**12.3.3. The restriction of the Weyl representation onto $SO(n) \times SL(2,\mathbb{R})$.** It was shown in Section 12.3.1 that restriction of the Weyl representation $T$ of $Sp(n,\mathbb{R})$ onto the subgroup $SO(n)$ coincides with representation (5) of Section 12.3.1. Consequently,

$$T_k F(r,\xi) = F(r,k^{-1}\xi), \quad k \in SO(n),$$

and the carrier space $\mathfrak{H}_N^m$ of the irreducible representation $T^{nm}$ of $SO(n)$ is spanned by the functions $\Psi_{NM}^m(r,\xi)$ with fixed $m$ and $N$.

We shall find how the operators $J_0$, $J_+$, $J_-$, (see formula (9) of Section 12.3.1) act upon $\Psi_{NM}^m(r,\xi)$. It follows from formulas (10) and (11) of Section 12.3.1 and from formula (2) of Section 12.3.2 that $J_0 = \frac{1}{2}D_1$. This means that

$$J_0\Psi_{NM}^m = \frac{1}{2}\left(N+\frac{n}{2}\right)\Psi_{NM}^m. \tag{1}$$

From formulas (10) and (11) of Section 12.3.1 we derive

$$J_+ = \frac{1}{2}\left(\widetilde{D}_1 - r^2 + \frac{n}{2} + r\frac{d}{dr}\right), \tag{2}$$

$$J_- = \frac{1}{2}\left(\widetilde{D}_1 - r^2 - \frac{n}{2} - r\frac{d}{dr}\right). \tag{3}$$

Therefore, the operators $J_0$, $J_+$, $J_-$ (and consequently the operators $T_k$, $h \in SL(2,\mathbb{R})$), act upon the functions $R_{mN}(r)$ and do not change $\Xi_M^{nm}$. Making use of formulas (5) and (6) of Section 5.5.2, we find that

$$\left.\begin{aligned} J_0 R_{mN} &= \frac{1}{2}\left(N+\frac{n}{2}\right)R_{mN}, \\ J_+ R_{mN} &= \left[\left(\frac{N}{2}-\frac{m}{2}+1\right)\left(\frac{N}{2}+\frac{n}{2}+\frac{m}{2}\right)\right]^{1/2} R_{m,N+2}, \\ J_- R_{mN} &= \left[\left(\frac{N}{2}-\frac{m}{2}\right)\left(\frac{N}{2}+\frac{n}{2}+\frac{m}{2}-1\right)\right]^{1/2} R_{m,N+2}. \end{aligned}\right\} \tag{4}$$

Consequently, functions (8) of Section 12.3.2 constitute the basis for the carrier space of the irreducible representation of the group $SL(2,\mathbb{R})$, denoted by $D_\ell$, $\ell = \frac{m}{2} + \frac{n}{4}$. The spectrum of $J_0$ in this representation consists of the points

$$\frac{n}{4} + \frac{N}{2}, \qquad N = m, m+2, m+4, \ldots .$$

We now conclude that the irreducible representation $T^{nm} \otimes D_{\ell(m)}$, $\ell(m) = \frac{m}{2}+\frac{n}{4}$, of the subgroup $SO(n)\times SL(2,\mathbb{R})$ is realized on the functions $\Psi^m_{NM}(r,\xi)$ with $m$ fixed. For the restriction of the Weyl representation of $Sp(n,\mathbb{R})$ onto $SO(n) \times SL(2,\mathbb{R})$ we have

$$T\Big\downarrow \begin{matrix} Sp(n,\mathbb{R}) \\ SO(n)\times SL(2,\mathbb{R}) \end{matrix} = \sum_{m=0}^{\infty} \otimes \left(T^{nm} \oplus D_{\ell(m)}\right). \tag{5}$$

This decomposition has the following properties: 1) multiplicities of irreducible representations of $SO(n) \times SL(2,\mathbb{R})$ in the decomposition do not exceed 1; 2) every representation $T^{nm}$ of the subgroup $SO(n)$ uniquely defines the representation $D_{\ell(m)}$ of the subgroup $SL(2,\mathbb{R})$, and, conversely, every representation $D_{\ell(m)}$ uniquely defines the representation $T^{nm}$. In this case the subgroups $SO(n)$ and $SL(2,\mathbb{R})$ are said to be *complementary* with respect to the representation $T$ of the group $Sp(n,\mathbb{R})$.

We return to the representation $D_\ell$. The basis functions $R_{mN}$ will be denoted by $f^j_k$, $j = \frac{m}{2}+\frac{n}{4}-1$, $k = \frac{N}{2}+\frac{n}{4}$. Then formulas (4) can be written as

$$J_0 f^j_k = k f^j_k \quad J_+ f^j_k = \sqrt{(k-j)(k+j+1)}\, f^j_{k+1},$$
$$J_- f^j_k = \sqrt{(k+j)(k-j-1)}\, f^j_{k-1}.$$

If $n$ is even, then by comparing these formulas with the formulas for infinitesimal operators of discrete series representations of the group $SL(2,\mathbb{R}) \sim SU(1,1)$ (see Section 6.4.6) we find that $D_\ell$ is equivalent to the discrete series representation $T^-_{-\ell}$. If $n$ is odd, then $D_\ell$ is a discrete series representation of the group $G_2$ which covers $SL(2,\mathbb{R})$ twice.

**12.3.4. Transition coefficients for the bases.** In Section 12.1.5 we have introduced the orthonormal basis $\{e_{\mathrm{m}}(\mathbf{x})\}$ of the space $\mathfrak{L}^2(\mathbb{R}^n)$ and in Section 12.3.2 we have constructed the bases $\{\Psi^m_{NM}\}$. The bases $\{e_{\mathrm{m}}(\mathbf{x})\}$ and $\{\Psi^m_{NM}\}$ are connected by a unitary matrix:

$$\Psi^m_{NM} = \sum_{m_1+\ldots+m_n=N} c^N_{\mathrm{m}(mM)} e_{\mathrm{m}}. \tag{1}$$

Let us evaluate the elements $c^N_{\mathrm{m}(mM)}$ of this matrix. The procedure of their evaluation is analogous to that for the coefficients connecting basis functions in $\mathfrak{L}^2(S^{n-1})$ which correspond to different trees $T$ (see Section 10.5.4).

The functions $\Psi^m_{NM}(r,\theta_1,\ldots,\theta_{n-1})$ with a fixed $N$ correspond to some spherical coordinate system on $S^{n-1}$, constructed with the help of some tree $T$. As in the case of bases of $\mathfrak{L}^2(S^{n-1})$, the coefficients $c^N_{\mathrm{m}(mM)}$ are obtained by a successive passage from the Cartesian coordinates to the coordinates corresponding to

$T$. As the first step, we divide $\mathbb{R}^n$ into the sum $\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$, $n = n_1 + n_2$, according to $T$. The branches outgoing from the lower node of $T$ define the spherical coordinate systems $(r_1, \boldsymbol{\theta}_1)$ and $(r_2, \boldsymbol{\theta}_2)$ in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$. Here

$$r_1 = r\,\sin\theta_{n-1}, \quad r_2 = r\,\cos\theta_{n-1}, \tag{2}$$

where $\theta_{n-1}$ is the angle corresponding to the lower node of the tree $T$. We have $\boldsymbol{\theta} \equiv (\theta_1, \dots, \theta_{n-1}) = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \theta_{n-1})$. To the coordinate systems $(r_1, \boldsymbol{\theta}_1)$ and $(r_2, \boldsymbol{\theta}_2)$ there correspond the bases $\{\Psi^p_{N_1P}\}$ and $\{\Psi^q_{N_2Q}\}$ in $\mathfrak{L}^2(\mathbb{R}^{n_1})$ and $\mathfrak{L}^2(\mathbb{R}^{n_2})$, which are analogous to the basis $\{\Psi^m_{NM}\}$ in $\mathfrak{L}^2(\mathbb{R}^n)$. Further, in accordance with two lower branches of $T$ we divide the spaces $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ into corresponding subspaces and go over from the bases $\{\Psi^p_{N_1P}\}$ and $\{\Psi^q_{N_2Q}\}$ to the bases

$$\{\Psi^u_{N_3U}(r_3, \boldsymbol{\theta}_3)\Psi^v_{N_4V}(r_4, \boldsymbol{\theta}_4)\}, \ \{\Psi^a_{N_5A}(r_5\boldsymbol{\theta}_5)\Psi^b_{N_6B}(r_6, \boldsymbol{\theta}_6)\}$$

in $\mathfrak{L}^2(\mathbb{R}^{n_1})$ and $\mathfrak{L}^2(\mathbb{R}^{n_2})$. Elementwise products of these bases give a basis of $\mathfrak{L}^2(\mathbb{R}^n)$. Continuing this procedure in accordance with $T$, after a finite number of steps we obtain the basis $\{e_{\mathbf{m}}\}$.

We shall find the coefficients for transition from the basis $\{\Psi^m_{NM}\}$ to the basis

$$\{\Psi^p_{N_1P}(r_1, \boldsymbol{\theta}_1)\Psi^q_{N_2Q}(r_2, \boldsymbol{\theta}_2)\}.$$

The functions $\Psi^m_{NM}(r, \boldsymbol{\xi})$ are represented in the form

$$\Psi^m_{NM}(r, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \theta_{n-1}) = R_{mN}(r)\Xi^{nm}_M(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \theta_{n-1}) \tag{3}$$

(see Section 12.3.2). Here

$$\Xi^{nm}_M(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \theta_{n-1}) = Y^{pq}_m(\theta_{n-1})\Xi^{n_1p}_P(\boldsymbol{\theta}_1)\Xi^{n_2q}_Q(\boldsymbol{\theta}_2), \tag{4}$$

where $m$ and $p+q$ are of the same parity, $p+q \le m$ and

$$Y^{pq}_m(\theta) = N^m_{pq}\sin^p\theta\cos^q\theta P^{(p-1+n_1/2,q-1+n_2/2)}_{(m-p-q)/2}(\cos 2\theta), \tag{5}$$

$$N^m_{pq} = \left[\frac{\Gamma(n_1/2)\Gamma(n_2/2)\Gamma\left(\frac{1}{2}(m-p-q+2)\right)\Gamma\left(\frac{1}{2}(m+p+q+n-2)\right)(2m+n-2)}{2\Gamma(n/2)\Gamma\left(\frac{1}{2}(m+p-q+n_1)\right)\Gamma\left(\frac{1}{2}(m-p+q+n_2)\right)}\right]^{1/2} \tag{6}$$

(see Section 10.5.3). The functions $\Psi^p_{N_1P}\Psi^q_{N_2Q}$ are representable as

$$\Psi^p_{N_1P}(r_1, \boldsymbol{\theta}_1)\Psi^q_{N_2Q}(r_2, \boldsymbol{\theta}_2) = R_{pN_1}(r_1)R_{qN_2}(r_2)\Xi^{n_1p}_P(\boldsymbol{\theta}_1)\Xi^{n_2q}_Q(\boldsymbol{\theta}_2). \tag{7}$$

Since the sets of functions $\{\Xi_P^{n_1 p}(\theta_1)\}$ and $\{\Xi_Q^{n_2 q}(\theta_2)\}$ are orthonormal, then it follows from (3), (4) and (7) that

$$\Psi_{NM}^m = \sum_{N_1+N_2=N} C(p,q,m;N_1,N_2,N)\Psi_{N_1P}^p\Psi_{N_2Q}^q, \tag{8}$$

where the coefficients $C(p,q,m;N_1,N_2,N) \equiv C(\mathbf{p},\mathbf{N})$, $\mathbf{p} = (p,q,m)$, $\mathbf{N} = (N_1,N_2,N)$, do not depend on $M$, $P$, $Q$. The inverse relation is of the form

$$\Psi_{N_1P}^p\Psi_{N_2Q}^q = \sum_m \overline{C(\mathbf{p},\mathbf{N})}\Psi_{NM}^m, \qquad N = N_1 + N_2. \tag{9}$$

It is clear from the above arguments that the coefficients $c_{\mathbf{m}(mM)}^N$ from (1) are products of the coefficients of the type $C(\mathbf{p},\mathbf{N})$.

Substituting expressions (3) and (7) for basis functions into (9), we have

$$R_{pN_1}(r\,\sin\theta)R_{qN_2}(r\,\cos\theta) = \sum_m \overline{C(\mathbf{p},\mathbf{N})}R_{mN}(r)Y_m^{pq}(\theta). \tag{10}$$

Since functions (5) satisfy the orthogonality relation

$$\frac{2\Gamma(n/2)}{\Gamma(n_1/2)\Gamma(n_2/2)}\int_0^{\pi/2} Y_m^{pq}(\theta)Y_{m'}^{pq}(\theta)\sin^{n_1-1}\theta\cos^{n_2-1}\theta d\theta = \delta_{mm'},$$

then

$$\overline{C(\mathbf{p},\mathbf{N})}R_{mN}(r) = \frac{2\Gamma(n/2)}{\Gamma(n_1/2)\Gamma(n_2/2)} \times \int_0^{\pi/2} R_{pN_1}(r\,\sin\theta)R_{qN_2}(r\,\cos\theta)Y_m^{pq}(\theta)\sin^{n_1-1}\cos^{n_2-1}\theta d\theta,$$

where $N = N_1 + N_2$. Passing with $r$ to infinity and replacing $\cos 2\theta$ by $x$, we derive that

$$\overline{C(\mathbf{p},\mathbf{N})} = A\int_{-1}^{1} P_{(m-p-q)/2}^{(p-1+n_1/2,q-1+n_2/2)}(x)(1-x)^{\frac{N_1+p+n_1}{2}-1}(1+x)^{\frac{N_2+q+n_2}{2}-1}dx, \tag{11}$$

where

$$A = (-1)^{(m-p-q)/2}2^{-(N+p+q+n)/2} \times \left[\frac{(N-m)!(2m+n-2)\left(\frac{m-p-q}{2}\right)!\Gamma\left(\frac{N+m-n}{2}\right)\Gamma\left(\frac{m+p+q+n-2}{2}\right)}{\left(\frac{N_1-p}{2}\right)!\left(\frac{N_2-q}{2}\right)!\Gamma\left(\frac{N_1+p+n_1}{2}\right)\Gamma\left(\frac{N_2+q+n_2}{2}\right)\Gamma\left(\frac{m+p-q+n_1}{2}\right)\Gamma\left(\frac{m-p+q+n_2}{2}\right)}\right]^{1/2}. \tag{12}$$

From formula (10) of Section 8.3.1 we find

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)(1+x)^{a-1}(1-x)^{b-1}dx = \frac{(-1)^n}{n!}\frac{\Gamma(\beta+n+1)\Gamma(a)\Gamma(b)}{\Gamma(\beta+1)\Gamma(a+b)}$$

$$\times\, {}_3F_2(-n, n+\alpha+\beta+1, a; a+b, \beta+1; 1). \quad (13)$$

Hence, $C(\mathbf{p},\mathbf{N})$ are real and

$$C(\mathbf{p},\mathbf{N}) = \tfrac{B}{2}\,{}_3F_2\left(-\tfrac{m-p-q}{2}, \tfrac{m+p+q+n}{2}-1; \tfrac{N_2+q+n_2}{2}, \tfrac{N+p+q+n}{2}, q+\tfrac{n_2}{2}; 1\right), \quad (14)$$

$$B = \frac{1}{2}\times$$

$$\left[\frac{\left(\frac{N-m}{2}\right)!(2m+n-2)\Gamma\left(\frac{N+m+n}{2}\right)\Gamma\left(\frac{m+p+q+n-2}{2}\right)\Gamma\left(\frac{m-p+q+n_2}{2}\right)\Gamma\left(\frac{N_1+p+n_1}{2}\right)\Gamma\left(\frac{N_2+q+n_2}{2}\right)}{\left(\frac{N_1-p}{2}\right)!\left(\frac{N_2-q}{2}\right)!\left(\frac{m-p-q}{2}\right)!\Gamma\left(\frac{m+p-q+n_1}{2}\right)\Gamma^2(q+\frac{n_2}{2})\Gamma^2\left(\frac{N+p+q+n}{2}\right)}\right]^{1/2}.$$

The function ${}_3F_2(\dots;1)$ from (14) can be expressed in terms of Hahn polynomials.

In the same way one evaluates the coefficients connecting the functions $\Psi_{NM}^m$ which correspond to different trees. As in Section 10.5.4, these coefficients are represented as coefficients $\mathcal{R}$ corresponding to transplantations of branches. The coefficients $\mathcal{R}$ are expressed in terms of the functions ${}_4F_3(\dots;1)$. We suggest to the reader to carry out these evaluations.

We have from (10) that

$$L_{N_1-p}^{2p+n_1-1}(r^2\sin^2\theta)L_{N_2-q}^{2q+n_2-1}(r\cos^2\theta)$$

$$= \frac{1}{2}\sum_{m=p+q}^{N}\frac{(N-m)!(2m+n-1)\Gamma(m+p+q+n-1)\Gamma(N_1+p+n_1)\Gamma(N_2+q+n_2)}{(N_1-p)!(N_2-q)!\Gamma(m+p-q+n_1)\Gamma(N+p+q+n)\Gamma(2q+n_2)}$$

$$\times\, {}_3F_2(-m+p+q, m+p+q+n-1, N_2+q+n_2; N+p+q+n, 2q+n_2; 1)$$

$$\times\, P_{m-p-q}^{(2p+n_1-1,2q+n_2-1)}(\cos 2\theta) r^{2(m-p-q)} L_{N-m}^{2m+n-1}(r^2), \quad (15)$$

where $N_1$, $N_2$, $N$, $n_1$, $n_2$, $n$, $p$, $q$, $m$ have been replaced by $2N_1$, $2N_2$, $2N$, $2n_1$, $2n_2$, $2n$, $2p$, $2q$, $2m$, respectively. This formula implies a series of interesting special cases. We propose to the reader to write them down.

**12.3.5. The transition coefficients and Clebsch-Gordan coefficients.** It was shown in Section 12.3.3 that the irreducible representation $D_{(2m+n)/4}$ of the group $G_2$ (which is the two-fold covering group for $SL(2,\mathbb{R})$) is realized on the functions $\Psi_{NM}^m$, $N = m, m+2, m+4, \dots$ (with fixed $m$ and $M$). On the functions

$\Psi^p_{N_1P}$ (with fixed $p$ and $P$) and on the functions $\Psi^q_{N_2Q}$ (with fixed $q$ and $Q$) from formula (7) of Section 12.3.4 the representations $D_{(2p+n_1)/4}$ and $D_{(2q+n_2)/4}$ of $G_2$ are realized respectively. Formula (8) of Section 12.3.4 gives the expansion of the tensor basis $\Psi^p_{N_1P}\Psi^q_{N_2Q}$ of the representation $D_{(2p+n_1)/4} \otimes D_{(2q+n_2)/4}$ of $G_2$ in the basis elements of irreducible components of this representation. In other words, $C(\mathbf{p}, \mathbf{N})$ are Clebsch-Gordan coefficients of the tensor product $D_{(2p+n_1)/4} \otimes D_{(2q+n_2)/4}$ of the representations of $G_2$. It follows from formulas (8) and (9) of Section 12.3.4 that

$$D_{(2p+n_1)/4} \otimes D_{(2q+n_2)/4} = \sum_{m=p+q}^{\infty} \oplus D_{(2m+n_1+n_2)/4}, \tag{1}$$

where the summation is over the values of $m$ which are of the same parity as $p+q$.

In the same way, one shows that the coefficients $\mathcal{R}$, mentioned in the end of Section 12.3.4, are Racah coefficients of the tensor product

$$E_{(2p+n_1)/4} \otimes D_{(2q+n_2)/4} \otimes D_{(2s+n_3)/4}$$

of the representations of $G_2$.

**12.3.6. The Weyl transform and the Heisenberg group.** Let Q and P be operators (1) and (2) of Section 12.2.4, related to the representation $R^1$ of the Heisenberg group $N_1$. By virtue of formula (4) of Section 12.1.3 we have

$$W(a, b) \equiv \exp i(a\mathrm{P} + b\mathrm{Q}) = R^1(h(-a, -b, 0))e^{iab/2}. \tag{1}$$

By making use of the multiplication rule (5) of Section 12.1.1 for $h(a, b, c)$, it is easy to derive that

$$W(a_1, b_1)W(a_2, b_2) = e^{i(a_1b_2 - a_2b_1)/2}W(a_1 + a_2, b_1 + b_2). \tag{1'}$$

Let $f(a, b)$ be a function on $\mathbb{R}^2$ with a compact support. Then the operator

$$W(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a, b)W(a, b)\,da\,db \tag{2}$$

is well defined. We say that the operator function $W(f)$ is the Weyl transform of $f$.

Repeating the arguments of Section 2.3.10, we easily obtain, by means of formula (1′), that

$$W(f_1 * f_2) = W(f_1)W(f_2), \tag{3}$$

where the convolution of $f_1$ and $f_2$ is given by

$$(f_1 * f_2)(a, b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(a_1, b_1)f_2(a - a_1, b - b_1)e^{i(a_1b - ab_1)/2}da_1\,db_1. \tag{4}$$

With the help of the expression for the operators $R_1(h(a,b,c))$ we derive that

$$(W(f)F)(x) = \int_{-\infty}^{\infty} K(x,y)F(y)dy, \tag{5}$$

where

$$K(x,y) = \int_{-\infty}^{\infty} f(y-x,t)e^{i(x+y)t/2}dt. \tag{6}$$

From (6) and from the inversion formula for the Fourier transform, we obtain

$$\operatorname{Tr} W(f) = \int_{-\infty}^{\infty} K(x,x)dx = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(0,t)e^{ixt}dt\right)dx = 2\pi f(0,0).$$

Replacing here $W(f)$ by $W(\tilde{f}) = W(a,b)^* W(f)$, we have the inversion of the Weyl transform:

$$f(a,b) = \frac{1}{2\pi}\operatorname{Tr}\left[e^{-i(a\mathrm{P}+b\mathrm{Q})}W(f)\right]. \tag{7}$$

Replacing $W(f)$ by $W(f)^* W(f)$, we obtain the Plancherel formula

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(a,b)|^2 da\, db = \frac{1}{2\pi}\operatorname{Tr}\left[W(f)^* W(f)\right]. \tag{8}$$

Formula (8) allows us to extend the Weyl transform onto the space of functions $f(a,b) \in \mathfrak{L}^2(\mathbb{R}^2)$.

In an obvious way the Weyl transform is extended onto the space of functions from $\mathfrak{L}^2(\mathbb{R}^{2n}) = \mathfrak{L}^2(\mathbb{R}^2) \times \ldots \times \mathfrak{L}^2(\mathbb{R}^2)$. It is, in fact, the product of transforms (2), taken $n$ times.

Formula (2) of Section 12.1.11 for $n = 1$, $\lambda = 1$ is, in fact, the matrix form of formula (7).

Let

$$\mathcal{D}_1 = 2\widetilde{D}_1 = -(\mathrm{Q}^2 + \mathrm{P}^2). \tag{9}$$

Then $e_m(x) = (2^m m!\sqrt{\pi})^{-1/2}e^{-x^2/2}H_m(x)$ are eigenfunctions of $\mathcal{D}_1$:

$$\mathcal{D}_1 e_m = (2m+1)e_m \tag{10}$$

(see Section 12.3.2). Therefore, if $P_m$ is the projection operator in $\mathfrak{L}^2(\mathbb{R})$ onto the one-dimensional subspace, spanned by $e_m$, then the spectral decomposition of the operator $\mathcal{D}_1$ has the form

$$\mathcal{D}_1 = \sum_{m=0}^{\infty}(2m+1)P_m. \tag{11}$$

Let the Weyl transform $W(f)$ of $f$ be a function of $\mathfrak{D}_1$ only, that is, be of the form

$$W(f) = \sum_{m=0}^{\infty} c_m P_m. \tag{12}$$

Thus, the operator $W(f)$ is diagonal in the basis $\{e_m\}$ and the matrix form of (7) is

$$f(a,b) = \frac{1}{2\pi} \sum_{m=0}^{\infty} c_m e^{-(a^2+b^2)/4} L_m^0 \left(\frac{a^2+b^2}{2}\right) \tag{13}$$

(we have taken into account the form of the matrix elements $r^1_{mm}(h(a,b,0))$ from Section 12.1.6). Hence, if $W(f)$ is a function of $\mathfrak{D}_1$ only, then $f$ is a function of $\rho = (a^2+b^2)/2$, that is, $f(a,b) = \varphi(\rho)$. It is easy to show that the converse statement is valid: if $f$ is a function of $\rho$ only, then $W(f)$ is a function of $\mathfrak{D}_1$ only.

Let $W(f)$ have the form

$$W(f) = e^{-s\mathfrak{D}_1} = \sum_{m=0}^{\infty} e^{-s(2m+1)} P_m. \tag{14}$$

Then according to (12) and (13), $f$ is of the form

$$f(a,b) = \varphi(\rho) = \frac{1}{2\pi} \sum_{m=0}^{\infty} e^{-(2m+1)s} e^{-\rho/2} L_m^0(\rho). \tag{15}$$

The function $f(a,b)$ can be also evaluated by means of the Fock realization of the representation $R^1$. Using the basis $\{f_m\}$ of the Hilbert space $\mathfrak{H}_0^1$ (see Section 12.1.7), we find

$$\begin{aligned} f(a,b) &= \frac{1}{2\pi}\mathrm{Tr}(W(a,b)^* W(f)) = \frac{1}{2\pi}\sum_{m=0}^{\infty} e^{-(2m+1)s}(W(a,b)^* f_m, f_m) \\ &= \frac{i}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\xi\bar{\xi}/4} \exp\Bigg[-s - (1-e^{-2s})w\bar{w} + \frac{w\bar{\xi}}{2} \\ &= \frac{1}{\sqrt{2}} e^{-2s}\bar{w}\xi\Bigg] dw\, d\bar{w} \\ &= \frac{1}{2\pi^2} e^{-\xi\bar{\xi}/4} e^{-s} \frac{\pi}{1-e^{-2s}} \exp\left(-\frac{1}{2}\frac{e^{-2s}\xi\bar{\xi}}{1-e^{-2s}}\right) \\ &= \frac{1}{2\pi}\frac{e^{-s}}{1-e^{-2s}} \exp\left(-\frac{\rho}{2} - \frac{e^{-2s}\rho}{1-e^{-2s}}\right) \end{aligned} \tag{16}$$

(here $\xi = a + ib$, $\rho = (a^2 + b^2)/2$). Equating the right hand sides of (15) and (16) and putting $z = e^{-2s}$, we obtain

$$\sum_{m=0}^{\infty} L_m^0(\rho) z^m = (1-z)^{-1} \exp\left(-\frac{z\rho}{1-z}\right). \tag{17}$$

Now let $W(f)$ coincides with the resolvent of the operator $\mathfrak{D}_1$, that is,

$$W(f) = \frac{1}{\mathfrak{D}_1 + p} = \int_0^{\infty} e^{-sp} e^{-s\mathfrak{D}_1}\, ds, \quad p > -1.$$

Then by using (16), we have

$$\begin{aligned} f(a,b) \equiv W^{-1}\left(\frac{1}{\mathfrak{D}_1 + p}\right) &= \int_0^{\infty} e^{-sp} W^{-1}(e^{-s\mathfrak{D}_1}) ds \\ &= \frac{1}{2\pi} \int_0^{\infty} e^{-sp} \frac{e^{-s}}{1 - e^{-sp}} \exp\left(-\frac{\rho}{2} - \frac{e^{-2s}\rho}{1 - e^{-2s}}\right) ds \\ &= \frac{1}{4\pi} e^{-\rho/2} \int_0^1 z^{(p-1)/2} (1-z)^{-1} \exp\left(-\frac{z\rho}{1-z}\right) dz \\ &= \frac{1}{4\pi} e^{-\rho/2} \Gamma\left(\frac{p+1}{2}\right) \Psi\left(\frac{p+1}{2}; 1; \rho\right) \end{aligned} \tag{18}$$

(see Section 5.3.1). On the other hand, it follows from (12) and (13) that

$$f = W^{-1}\left(\frac{1}{\mathfrak{D}_1 + p}\right) = \frac{1}{2\pi} \sum_{m=0}^{\infty} \frac{1}{2m + p + 1} e^{-\rho/2} L_m^0(\rho). \tag{19}$$

By comparing (18) and (19), we have

$$\sum_{m=0}^{\infty} \frac{1}{m+q} L_m^0(\rho) = \Gamma(q) \Psi(q; 1; \rho), \ q = \frac{p+1}{2} > 0. \tag{20}$$

Let us consider the $n$-dimensional Weyl transform on

$$\mathfrak{L}^2(\mathbb{R}^{2n}) = \mathfrak{L}^2(\mathbb{R}^2) \times \ldots \times \mathfrak{L}^2(\mathbb{R}^2).$$

Instead of $\mathfrak{D}_1$ we introduce the operator

$$\mathfrak{D}_n = \sum_{k=1}^{n} \lambda_k (\mathrm{Q}_k^2 + \mathrm{P}_k^2), \qquad \lambda_k \in \mathbb{R},$$

where $Q_k$ and $P_k$ act on the $k$-th space $\mathfrak{L}^2(\mathbf{R})$. Then the spectral decomposition of the operator $e^{-s\mathfrak{D}_n}$ is of the form

$$e^{-s\mathfrak{D}_n} = \sum_{\mathbf{m}} e^{-s\Lambda_{\mathbf{m}}} P_{\mathbf{m}},$$

where $\mathbf{m} = (m_1, \ldots, m_n)$, $\Lambda_{\mathbf{m}} = \sum_{j=1}^{n} \lambda_j(2m_j + 1)$ and $P_{\mathbf{m}}$ is the projection operator onto the one-dimensional subspace, spanned by the function $e_{\mathbf{m}}(\mathbf{x})$ from Section 12.3.2.

Let $W(f) = e^{-s\mathfrak{D}_n}$. Then, on the one hand,

$$f \equiv W^{-1}(e^{-s\mathfrak{D}_n}) = \frac{1}{(2\pi)^n} \sum_{\mathbf{m}} e^{-s\Lambda_{\mathbf{m}}} \mathfrak{L}_{\mathbf{m}}(\boldsymbol{\rho}), \tag{21}$$

where $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$ and

$$\mathfrak{L}_{\mathbf{m}}(\boldsymbol{\rho}) = e^{-(\rho_1 + \ldots + \rho_n)/2} L^0_{m_1}(\rho_1) \ldots L^0_{m_n}(\rho_n),$$

and on the other hand,

$$f = \frac{1}{(2\pi)^n} \prod_{j=1}^{n} \frac{e^{-2\lambda_j}}{1 - e^{-s\lambda_j}} \exp\left( -\frac{1}{2} \sum_{j=1}^{n} \frac{1 + e^{-2s\lambda_j}}{1 - e^{-2s\lambda_j}} \rho_j \right). \tag{22}$$

Equating the right hand sides of (21) and (22), and setting $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$, $z = e^{-2s}$, we obtain

$$\sum_{\mathbf{m}} \mathfrak{L}_{\mathbf{m}}(\boldsymbol{\rho}) z^m = e^{-\rho/2}(1 - z)^{-n} \exp \frac{z\rho}{z - 1},$$

where $m = m_1 + \ldots + m_n$, $\rho = \rho_1 + \ldots + \rho_n$. The right hand side of this equality is a generating function for the Laguerre polynomials $L_k^{n-1}(\rho)$ (see formula (4) of Section 5.5.3). Hence, by equating the expressions at $z^n$, we have

$$\sum_{m_1 + \ldots + m_n = m} L^0_{m_1}(\rho_1) \ldots L^0_{m_n}(\rho_n) = L_m^{n-1}(\rho_1 + \ldots + \rho_n). \tag{23}$$

## 12.4. Representations of the Group $U(n)N_n$ and Orthogonal Polynomials.

**12.4.1. Representations of the group $U(n)N + n$.** It was shown in Section 12.1.1 that the subgroup $N_n$ of the group $U(n_1, 1)$ is isomorphic to the group of matrices

$$\begin{pmatrix} 1 & \mathbf{w}^* & ix + \frac{1}{2}(\mathbf{w}, \mathbf{w}) \\ 0 & I_n & \mathbf{w} \\ 0 & 0 & 1 \end{pmatrix},$$

where $\mathbf{w}$ is a row consisting of $n$ complex numbers $w_1, \ldots, w_n$, $\mathbf{w}^* = (\bar{w}_1, \ldots, \bar{w}_n)$, $x \in \mathbf{R}$ and $(\mathbf{w}, \mathbf{z}) = w_1\bar{z}_1 + \ldots w_n\bar{z}_n$. The subgroup $G = U(n)N_n$ of $U(n+1, 1)$ consists of matrices

$$\begin{pmatrix} 1 & \mathbf{w}^* & ix + \frac{1}{2}(\mathbf{w}, \mathbf{w}) \\ \mathbf{0} & u & u\mathbf{w} \\ 0 & \mathbf{0} & 1 \end{pmatrix}, \qquad u \in U(n). \tag{1}$$

Matrices (1) will be denoted by $g(u, \mathbf{w}, x)$. We have

$$g(u_1, \mathbf{w}_1, x_1)g(u_2, \mathbf{w}_2, x_2) = g(u_1u_2, u_2^{-1}\mathbf{w}_1 + \mathbf{w}_2, x_1 + x_2 + \operatorname{Im}(\mathbf{w}_2, u_2^{-1}\mathbf{w}_1)). \tag{2}$$

The elements $g(e, \mathbf{0}, x)$, where $e$ is the identity matrix from $U(n)$, form the center of $G$. The invariant measure on $G$ is given by the formula $dg = dud\mathbf{w}dx$, where $du$ is the invariant measure on $U(n)$ and $d\mathbf{w} = dx_1dy_1 \ldots dx_ndy_n$, $w_j = x_j + iy_j$.

Let $\mathfrak{H}_\lambda$, $\lambda \in \mathbb{R}$, be the Hilbert space of entire analytic functions $\varphi(\mathbf{z})$, $\mathbf{z} = (z_1, \ldots, z_n)$, with the scalar product

$$(\varphi_1, \varphi_2) = \left(\frac{\lambda}{\pi}\right)^n \int\limits_{\mathbf{C}^n} \varphi_1(\mathbf{z})\overline{\varphi_2(\mathbf{z})}e^{-\lambda(\mathbf{z},\mathbf{z})}d\mathbf{z}. \tag{3}$$

In the case $n = 1$ the space $\mathfrak{H}_\lambda$ is isometric to the space $\mathfrak{H}$ from Section 5.5.1. The formula

$$(T_\lambda(g)\varphi)(\mathbf{z}) = \exp\left[i\lambda x - \frac{\lambda}{2}(\mathbf{w}, \mathbf{w}) + \lambda(u^{-1}\mathbf{z}, \mathbf{w})\right]\varphi(u^{-1}\mathbf{z} + \mathbf{w}), \tag{4}$$

where $g = g(u, \mathbf{w}, x)$, gives a representation of $G$ in $\mathfrak{H}_\lambda$. It is easy to verify that it is unitary with respect to the scalar product (3). A simple verification shows that the representations $T_\lambda$, $\lambda \neq 0$, are irreducible.

The functions $z^n/\sqrt{n!}$, $n = 0, 1, 2, \ldots$ , form an orthonormal basis of the space $\mathfrak{H}$ from Section 5.5.1. Hence, the functions

$$e_n(z) = \sqrt{\lambda^n/n!}\, z^n, \qquad n = 0, 1, 2, \ldots, \tag{5}$$

form an orthonormal basis of $\mathfrak{H}_\lambda^1$. Then

$$e_{\mathbf{m}}(\mathbf{z}) = e_{m_1}(z_1)\ldots e_{m_n}(z_n),\ \mathbf{m} = (m_1, \ldots, m_n),\ m_j = 0, 1, 2, \ldots, \tag{6}$$

is an orthonormal basis of $\mathfrak{H}_\lambda \equiv \mathfrak{H}_\lambda^n$.

If $g = g(u, \mathbf{w}, x)$, then according to (4), we have

$$t_{\mathbf{km}}^\lambda(g) \equiv (T_\lambda(g)e_{\mathbf{m}}, e_{\mathbf{k}}) = \left(\frac{\lambda}{\pi}\right)^n \left(\frac{\lambda^{k+m}}{\mathbf{k}!\mathbf{m}!}\right)^{1/2} \int e^{-\lambda(\mathbf{z},\mathbf{z})}T_\lambda(g)(\mathbf{z}^{\mathbf{m}})\overline{\mathbf{z}^{\mathbf{k}}}d\mathbf{z}$$

$$= \left(\frac{\lambda}{\pi}\right)^n \left(\frac{\lambda^{k+m}}{\mathbf{k}!\mathbf{m}!}\right)^{1/2} \exp\left[i\lambda x - \frac{\lambda}{2}(\mathbf{w}, \mathbf{w})\right] \int \exp[-\lambda(\mathbf{z}, u\mathbf{w} + \mathbf{z})](u^{-1}\mathbf{z} + \mathbf{w})^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{k}}}d\mathbf{z}, \tag{7}$$

where $\mathbf{z}^{\mathbf{m}} = z_1^{m_1} \ldots z_n^{m_n}$, $\mathbf{m}! = m_1! \ldots m_n!$, $m = m_1 + \ldots + m_n$.

For $g_x = g(e, \mathbf{0}, x)$ we have

$$(T_\lambda(g_x)\varphi)(\mathbf{z}) = e^{i\lambda x}\varphi(\mathbf{z}).$$

It follows from (4) that the operators $T_\lambda(u)$, $u = g(u, \mathbf{0}, 0) \in U(n)$, have the form

$$(T_\lambda(u)\varphi)(\mathbf{z}) = \varphi(u^{-1}\mathbf{z}). \tag{8}$$

The space $\mathfrak{H}_\lambda$ decomposes into the sum of the subspaces $\mathfrak{R}^m$, $m = 0, 1, 2,$ , consisting of homogeneous polynomials of degree $m$ in $z_1, \ldots, z_n$. The results of Section 11.2 imply that operators (8) realize on $\mathfrak{R}^m$ the irreducible representation $T^{nm0}$ of the group $U(n)$. Thus,

$$T_\lambda \Big\downarrow^{G}_{U(n)} = \sum_{m=0}^{\infty} \oplus T^{nm0}. \tag{9}$$

**12.4.2. Matrix elements of the representations $T_\lambda$.** The elements $g(e, \mathbf{w}_j, 0)$, where $\mathbf{w}_j = (0, \ldots, 0, w, 0, \ldots, 0)$ ($w$ is on $j$-th position), are denoted by $g_j(w)$. We have $g(e, \mathbf{w}, 0) = g_1(w_1) \ldots g_n(w_n)$. It follows from formula (7) of Section 12.4.1 that

$$t^\lambda_{\mathbf{km}}(g(e, \mathbf{w}, 0)) = \prod_{j=1}^{n} t^\lambda_{k_j m_j}(g_j(w_j)), \tag{1}$$

where

$$\begin{aligned} t^\lambda_{k_j m_j}(g_j(w_j)) &= t^\lambda_{\mathbf{km}}(g_j(w_j)) = (T_\lambda(g_j(w_j)e_{m_j}, e_{k_j}) \\ &= \frac{\lambda}{\pi}\left(\frac{\lambda^{k_j+m_j}}{k_j! m_j!}\right)^{1/2} e^{-\lambda|w_j|^2/2} \int_{\mathbb{C}} \exp(-\lambda|z_j|^2 - \lambda z_j \bar{w}_j)(z_j + w_j)^{m_j} \overline{z_j^{k_j}}\, dz_j \end{aligned} \tag{2}$$

(here $e_m \equiv e_m(z)$ is of the form (5) of Section 12.4.1). This matrix element can be found by evaluating the integral (2). It can be also evaluated in the same way as matrix elements of the representations $T_\chi$ of the group of complex triangular matrices have been found (see Section 5.5.1). For $w = re^{i\varphi}$ we have

$$t^\lambda_{km}(g_j(w)) = \left(\frac{k!}{m!}\right)^{1/2} e^{-\lambda r^2/2} \left(e^{i\varphi}\sqrt{\lambda}\, r\right)^{m-k} L_k^{m-k}(\lambda r^2). \tag{3}$$

Let $n = 2$. Then $u \in U(2)$ in elements $g(u, \mathbf{w}, x)$. Irreducible representations of the subgroup $U(2)$ are realized in the subspaces $\mathfrak{R}^m$ of $\mathfrak{H}_\lambda$. The subspaces $\mathfrak{R}^m$ coincide with the spaces $\mathfrak{H}_{m/2}$ introduced in Section 6.2.1. It follows from formulas (4) of Section 6.2.1 that the basis elements $\psi_k$, $-\frac{m}{2} \le k \le \frac{m}{2}$, of $\mathfrak{H}_{m/2}$ from Section 6.2.3 are associated with the basis elements $e_{\ell-k}(z_1)e_{\ell+k}(z_2)$, $\ell = \frac{m}{2}$, of $\mathfrak{H}_\ell$. Since the operators $T_\lambda(u)$, $u \in U(2)$, leave the subspaces $\mathfrak{R}^m \equiv \mathfrak{H}_{m/2}$ invariant, then non-zero matrix elements of the operator $T_\lambda(u)$ have the form $t^\lambda_{j,2\ell-j;k,2\ell-k}(u)$. It follows from the results of Sections 6.3.1 and 6.3.3 that for matrix

$$u(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{4}$$

we have

$$t^\lambda_{j,2\ell-j;k,2\ell-k}(u(\theta)) = (-1)^{k-j} t^\ell_{\ell-j,\ell-k}(\theta) = (-i)^{k-j} P^\ell_{\ell-j,\ell-k}(\cos\theta)$$
$$= (-i)^{k-j} 2^{\ell-j} \left[\frac{j!(2\ell-j)!}{k!(2k-j)!}\right]^{1/2} \left(\sin\frac{\theta}{2}\right)^{k-j} \left(\cos\frac{\theta}{2}\right)^{2\ell-k-j} P_j^{(k-j,2\ell-k-j)}(\cos\theta),$$
$$\ell = \frac{m}{2}. \tag{5}$$

**12.4.3. The addition theorem and the product formula for Laguerre polynomials.** Let $\mathbf{w}_1 = (0,\ldots,0,r_1)$, $\mathbf{w}_2 = (0,\ldots,0,r_2e^{i\varphi})$, $r_1 \geq 0$, $r_2 \geq 0$, $0 \leq \varphi < 2\pi$. The equality

$$g(e,\mathbf{w}_1,0)g(e,\mathbf{w}_2,0) = g(e,\mathbf{w}_1+\mathbf{w}_2,\mathrm{Im}(\mathbf{w}_2,\mathbf{w}_1))$$

implies

$$\sum_{\ell=0}^{\infty} t^\lambda_{k\ell}(g_n(r))t^\lambda_{\ell m}(g_n(r_2e^{i\varphi})) = \exp(i\lambda r_1r_2\sin\varphi)t^\lambda_{km}(g_n(r_1+r_2e^{i\varphi})). \tag{1}$$

Since $r_1 + r_2e^{i\varphi} = re^{i\psi}$, where

$$r^2 = r_1^2 + 2r_1r_2\cos\varphi + r_2^2, \quad e^{2i\psi} = \frac{r_1+r_2e^{i\varphi}}{r_1+r_2e^{-i\varphi}}, \tag{2}$$

then by replacing in (1) the matrix elements by their expressions (3) of Section 12.4.2, after simple manipulations we obtain the addition theorem for Laguerre polynomials

$$(re^{i\psi})^{m-k}L_k^{m-k}(r^2) = \exp(r_1r_2e^{-i\varphi})\sum_{\ell=0}^{\infty} r_1^{\ell-k}(r_2e^{i\varphi})^{m-\ell}L_k^{\ell-k}(r_1^2)L_\ell^{m-\ell}(r_2^2). \tag{3}$$

In particular, for $\varphi = 0$ we have the equality

$$\sum_{\ell=0}^{\infty}\left(\frac{r_1}{r_2}\right)^\ell L_k^{\ell-k}(r_1^2)L_\ell^{m-\ell}(r_2^2) = e^{-r_1r_2}r_1^kr_2^{-m}(r_1+r_2)^{m-k}L_k^{m-k}((r_1+r_2)^2). \tag{4}$$

Multiply both sides of (3) by $\exp(i\varphi(\ell - m_0))$ and integrate with respect to $\varphi$ from 0 to $2\pi$. As a result, we obtain

$$r_1^{\ell-k} r_2^{m-\ell} L_k^{\ell-k}(r_1^2) L_\ell^{m-\ell}(r_2^2) = \frac{1}{2\pi} \int_0^{2\pi} (r_1 + r_2 e^{i\varphi})^{m-k} L_k^{m-k}(r^2) e^{i(\ell-m)\varphi} d\varphi, \quad (5)$$

where $r = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos\varphi}$.

Putting $\ell = k = m$, $r^2 = x$, $r_1^2 = x_1$, $r_2^2 = x_2$ into (5), we have

$$L_m^0(x_1) L_m^0(x_2) = \frac{1}{\pi} \int_0^{\pi} \frac{\cos\left(\sqrt{x_1 x_2}\,\sin\varphi\right)}{e^{\sqrt{x_1 x_2}\,\cos\varphi}} L_m^0(x) d\varphi, \quad (6)$$

where $x = x_1 + x_2 + 2\sqrt{x_1 x_2}\,\cos\varphi$. Replacing the integration variable $\varphi$ by $x$ and considering the obtained formula as the coefficient in expansion in $L_m^0(x)$, we derive

$$\sum_{m=0}^{\infty} L_m^0(x_1) L_m^0(x_2) L_m^0(x) = \frac{1}{2\pi} \frac{\cos\left(\sqrt{x_1 x_2}\,\sin\varphi\right)}{\sqrt{x_1 x_2}\,\sin\varphi} e^{x_1 + x_2 + \sqrt{x_1 x_2}\,\cos\varphi}. \quad (7)$$

From here we have that

$$\frac{1}{2\pi} \int_0^{\infty} \int_0^{\infty} L_m^0(x_1) L_m^0(x_2) \frac{\cos\left(\sqrt{x_1 x_2}\,\sin\varphi\right)}{\sqrt{x_1 x_2}\,\sin\varphi} e^{-(x_1^2 + x_2^2)/2} dx_1 dx_2$$
$$= e^{-x/2} L_m^0(x), \quad (8)$$

where

$$\sin\varphi = \left(1 - \frac{(x - x_1 - x_2)^2}{4x_1 x_2}\right)^{1/2}.$$

**12.4.4. Relations between Laguerre and Jacobi polynomials.** In this section we assume that $G = U(2)N_2$. If $u \in U(2)$, then

$$T_\lambda(u) T_\lambda(g(e, \mathbf{w}, 0)) T_\lambda(u)^{-1} = T_\lambda(g(e, u\mathbf{w}, 0)).$$

Consequently,

$$\sum_{\mathbf{k}} \sum_{\mathbf{p}} t_{\mathbf{jk}}^\lambda(u) t_{\mathbf{kp}}^\lambda(g(e, \mathbf{w}, 0)) t_{\mathbf{pm}}^\lambda(u^{-1}) = t_{\mathbf{jm}}^\lambda(g(e, u\mathbf{w}, 0)), \quad (1)$$

where (by virtue of formula (5) of Section 12.4.2)

$$j_1 + j_2 = k_1 + k_2, \quad p_1 + p_2 = m_1 + m_2.$$

We put $\mathbf{w} = (0, r)$, $u = u(\theta)$ into (1), where $u(\theta)$ is matrix (4) of Section 12.4.2. Then $u(\theta)\mathbf{w} = \left(-ir\sin\frac{\theta}{2}, r\cos\frac{\theta}{2}\right)$. Substituting the expressions for the matrix elements from Section 12.4.2, after simplifications we derive the equality

$$\sum_{k=0}^{\min(2\ell_1,2\ell_2)} \left[\frac{2\ell_1-k)!}{(2\ell_2-k)!}\right]^{1/2} P^{\ell_1}_{\ell_1-j,\ell_1-k}(\cos\theta)P^{\ell_2}_{\ell_2-m,\ell_2-k}(\cos\theta)L^{2\ell_2-2\ell_1}_{2\ell_1-k}(r^2)$$
$$= (-1)^{m+j}\left[\frac{j!(2\ell_1-j)!}{m!(2\ell_2-m)!}\right]^{1/2}\left(\sin\frac{\theta}{2}\right)^{m-j}\left(\cos\frac{\theta}{2}\right)^{2\ell_2-2\ell_1-m+j}$$
$$\times L^{m-j}_j\left(r^2\sin^2\frac{\theta}{2}\right)L^{2\ell_2-2\ell_1-m+j}_{2\ell_1-j}\left(r^2\cos^2\frac{\theta}{2}\right), \quad (2)$$

where $2\ell_1 = j_1 + j_2$, $2\ell_2 = m_1 + m_2$. Let us set here $j = m = 0$ and take into account that $L^0_0(x) = 1$ and

$$P^\ell_{\ell n}(\cos\theta) = \left[\frac{(2\ell)!}{(\ell-n)!(\ell+n)!}\right]^{1/2}\left(\sin\frac{\theta}{2}\right)^{\ell-n}\left(\cos\frac{\theta}{2}\right)^{\ell+n}.$$

After changing notation we conclude that for $x > 0$, $0 \le y \le 1$ the equality

$$\sum_{k=0}^{m}\frac{(p+m)!}{k!(p+m-k)!}(1-y)^k y^{m-k}L^p_{m-k}(x) = L^p_m(xy) \quad (3)$$

holds. For $\ell_2 = \ell_1 = \ell$, $j = m$ we have from (2) the relation

$$\sum_{k=0}^{2\ell}(-1)^{k-j}\left[P^\ell_{\ell-j,\ell-k}(\cos\theta)\right]^2 L^0_{2\ell-k}(r^2) = L^0_j\left(r^2\sin^2\frac{\theta}{2}\right)L^0_{2\ell-j}\left(r^2\cos^2\frac{\theta}{2}\right). \quad (4)$$

Multiply both sides of (2) by $r^{4\ell_2-4\ell_1}e^{-r^2}L^{2\ell_2-2\ell_1}_{2\ell_1-s}(r^2)$ and integrate with respect to $\rho = r^2$ from 0 to $\infty$. By virtue of the orthogonality relation for Laguerre polynomials, we have

$$\int_0^\infty L^{2\ell_2-2\ell_1}_{2\ell_1-s}(\rho)L^{m-j}_j(\rho\sin^2\theta)L^{2\ell_2-2\ell_1-m+j}_{2\ell_1-j}(\rho\cos^2\theta)\rho^{2\ell_2-2\ell_1}e^{-\rho}d\rho$$
$$= (-1)^{s-m}\left[\frac{m!(2\ell_2-m)!}{j!(2\ell_1-j)!}\right]^{1/2}\sin^{j-m}\theta\cos^{2\ell_1-2\ell_2+m-j}\theta$$
$$\times P^{\ell_1}_{\ell_1-j,\ell_1-s}(\cos 2\theta)P^{\ell_2}_{\ell_2-m,\ell_2-s}(\cos 2\theta). \quad (5)$$

The formula

$$\int_0^\infty L^0_{2\ell-s}(\rho)L^0_m(\rho\sin^2\theta)L^0_{2\ell-m}(\rho\cos^2\theta)e^{-\rho}d\rho = \left[P^\ell_{\ell-m,\ell-s}(\cos 2\theta)\right]^2 \quad (6)$$

is a special case of (5).

Multiply both sides of (3) by $y^{\beta-1}(1-y)^{\gamma-1}(1-ty)^{-\alpha}$ and integrate with respect to $y$ from 0 to 1. Due to equality (13) of Section 3.5.2, we derive

$$\int_0^1 y^{\beta-1}(1-y)^{\gamma-1}(1-ty)^{-\alpha}L_m^p(xy)dy = \frac{(p+m)!}{\Gamma(\beta+\gamma+m)}$$
$$\times \sum_{k=0}^{m} \frac{\Gamma(m+\beta-k)\Gamma(\gamma+k)}{k!\Gamma(p+m-k+1)} F(\alpha, m+\beta-k; \beta+\gamma+m; t) L_{m-k}^p(x). \quad (7)$$

Further, we multiply both sides of (4) by $\sin\theta$ and integrate with respect to $\theta$ from 0 to $\pi$. By virtue of the orthogonality relation for the functions $P_{mn}^{\ell}(x)$ (see Section 6.10.1), we obtain

$$\int_0^\pi L_j^0\left(\rho\sin^2\frac{\theta}{2}\right) L_{2\ell-j}^0\left(\rho\cos^2\frac{\theta}{2}\right)\sin\theta\, d\theta = \frac{2}{2\ell+1}\sum_{k=0}^{2\ell} L_{2\ell-k}^0(\rho). \quad (8)$$

Another relation between Jacobi and Laguerre polynomials follows from the equality

$$T_\lambda(u)T_\lambda(g(e,\mathbf{w},0)) = T_\lambda(g(e,u\mathbf{w},0))T_\lambda(u).$$

If here $u$ and $\mathbf{w}$ are the same as in (1), then we obtain

$$\sum_{m=0}^{2\ell_2}\left[\frac{j!(2\ell_1-j)!}{m!(2\ell_2-m)!}\right]^{1/2}\sin^{m-j}\theta\,\cos^{2\ell_2-2\ell_1-m+j}\theta$$
$$\times L_j^{m-j}(r^2\sin^2\theta)L_{2\ell_1-j}^{2\ell_2-2\ell_1-m+j}(r^2\cos^2\theta)P_{\ell_2-m,\ell_2-k}^{\ell_2}(\cos 2\theta)$$
$$= \left[\frac{(2\ell_1-k)!}{(2\ell_2-k)!}\right]^{1/2} P_{\ell_1-j,\ell_1-k}^{\ell_1}(\cos 2\theta)L_{2\ell_1-k}^{2\ell_2-2\ell_1}(r^2). \quad (9)$$

This equality implies a series of interesting special cases.

## 12.5. The Compact Realization of the Representation of the Heisenberg Group and Orthogonal Polynomials

**12.5.1. The Weil-Brezin operator and a new realization of the representation $R^{2\pi}$.** Along with realization (6) of Section 12.1.4 for the representations $R^\lambda$ of the Heisenberg group $N_1$, one also considers the realization

$$(\widetilde{R}^\lambda(n(a,b,c))f)(x) = \exp\left[i\lambda\left(c+bx+\frac{1}{2}ab\right)\right]f(x+a) \quad (1)$$

in the space $\mathfrak{L}^2(\mathbb{R})$ with the scalar product

$$(f_1, f_2) = \int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}dx. \tag{1'}$$

In the same way as in Section 12.1.4 one shows that the representations $\widetilde{R}^\lambda$, $\lambda \neq 0$, are irreducible. Since $\widetilde{R}^\lambda(n(0,0,c)) = e^{i\lambda c}$, then according to the Theorem from Section 12.1.4, the representations $\widetilde{R}^\lambda$ and $R^\lambda$ of $N_1$ are equivalent. We recommend to the reader to construct the operator realizing this equivalence.

Due to formula (6) of Section 12.1.1, for the matrices

$$h(a,b,t) = \begin{pmatrix} 1 & a & t \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in H(1,\mathbb{R})$$

we have

$$(\widetilde{R}(h(a,b,t))f)(x) = \exp[i\lambda(t + xb)]f(x+a). \tag{2}$$

In order to construct a realization of the representation $\widetilde{R}^\lambda$, $\lambda = 2\pi$, of the group $N_1 \sim H(1,\mathbb{R})$ in the space of functions on a compact set, we note that the set

$$D \equiv \{h(p,q,r) \in H(1,\mathbb{R}) \mid p,q,r \in \mathbb{Z}\}$$

is a discrete invariant subgroup of $H(1,\mathbb{R})$. The quotient group $D\backslash H(1,\mathbb{R})$ is homeomorphic to the cube in $\mathbb{R}^3$. If $h(0,0,r) \in D$, then $\widetilde{R}^{2\pi}(h(0,0,r))$ is the identity operator.

Let us consider the space of functions $f(h)$ on $H(1,\mathbb{R})$ such that

$$f(h(0,0,t)) = e^{2\pi it},\ f(dh) = f(h),\ d \in D. \tag{3}$$

We set $f(h(a,b,t)) \equiv F(a,b,t)$. Then

$$F(a,b,t) = e^{2\pi it}F(a,b,0). \tag{4}$$

Since

$$h(p,q,r)h(a,b,0) = h(a+p,b+q,r+pb) = h(a+p,b+q,0)h(0,0,r+pb),$$

then for $d = h(p,q,r) \in D$, $h(a,b,0) \in H(1,\mathbb{R})$ we have

$$f(dh) = e^{2\pi ipb}f(h(a+p,b+q,0)). \tag{5}$$

For functions $F(a,b,t)$ this condition is equivalent to

$$F(a+p,b+q,t+r) = e^{2\pi i(t-pb)}F(a,b,0), \quad p,q,r \in \mathbb{Z}. \tag{6}$$

Thus, functions $f(h)$ on $H(1,\mathbb{R})$ satisfying conditions (3) are in a one-to-one correspondence with functions $F(a,b,0)$ satisfying the condition

$$F(a+p,b+q,0)=e^{-2\pi ipb}F(a,b,0). \tag{6'}$$

These functions are defined by their values on the torus $\mathbf{T}^2 \equiv \left(-\frac{1}{2},\frac{1}{2}\right)\times\left(-\frac{1}{2},\frac{1}{2}\right)$. We form the Hilbert space $\mathfrak{L}$ of functions $F(a,b,0)$ with the scalar product

$$\langle F_1,F_2\rangle=\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}F_1(a,b,0)\overline{F_2(a,b,0)}da\,db. \tag{7}$$

It defines the Hilbert space of functions $f(h(a,b,t))\equiv F(a,b,t)$ on $H(1,\mathbb{R})$ satisfying condition (3). We denote it by $\mathfrak{L}$, also. The operators

$$T^{2\pi}(h_0)f(h)=f(hh_0) \tag{8}$$

give the representation $T^{2\pi}$ of the group $H(1,\mathbb{R})\sim N_1$ in $\mathfrak{L}$. On functions $F(x,y,0)$ the operators $T^{2\pi}(h)$ are given by the formula

$$\begin{aligned}(T^{2\pi}(h(a,b,t))F)(x,y,0)&=F(x+a,y+b,t+bx)\\&=e^{2\pi i(t+bx)}F(x+a,y+b,0).\end{aligned} \tag{8'}$$

A direct verification shows that these operators are unitary. We have $T^{2\pi}(h(0,0,t)) = e^{2\pi it}I$.

The representations $\widetilde{R}^{2\pi}$ and $T^{2\pi}$ are unitarily equivalent. To prove this we construct the operator $B$ realizing the equivalence.

Let $B:\mathfrak{L}^2(\mathbb{R})\to\mathfrak{L}$ be the operator given by the formula

$$\begin{aligned}(Bf)(x,y,t)&=\sum_{n\in\mathbf{Z}}(\widetilde{R}^{2\pi}(h(x,y,t))f)(n)\\&=e^{2\pi it}\sum_{n\in\mathbf{Z}}e^{2\pi iny}f(x+n),\quad f\in\mathfrak{L}^2(\mathbb{R}).\end{aligned} \tag{9}$$

It is easy to verify that the function $F(a,b,t)\equiv(Bf)(a,b,t)$ satisfies condition (6). The operator $B^{-1}$ has the form

$$(B^{-1}F)(x+n)=\int_{-1/2}^{1/2}F(x,y,0)e^{-2\pi iny}dy,\ x\in\left(-\frac{1}{2},\frac{1}{2}\right],\ n\in\mathbb{Z}. \tag{10}$$

The Plancherel equality

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |F(a,b,0)|^2 \, da\, db = \int_{-1/2}^{1/2} \left[\sum_{n\in\mathbb{Z}} |(B^{-1}F)(a+n)|^2\right] da = \int_{-\infty}^{\infty} |(B^{-1}F)(x)|^2 dx$$

holds, that is, the operators $B$ and $B^{-1}$ conserve the norm. The operator $B$ is called the ***Weil-Brezin operator.***

The Weil-Brezin operator intertwines the representations $\widetilde{R}^{2\pi}$ and $T^{2\pi}$. Indeed,

$$\left[B\widetilde{R}^{2\pi}(h(a_0,b_0,t_0))f\right](a,b,t) = e^{2\pi it}\sum_{n\in\mathbb{Z}} e^{2\pi inb} e^{2\pi i(t_0+(a+n)b_0)} \times f(a+a_0+n) = \left[T^{2\pi}(h(a_0,b_0,t_0))Bf\right](a,b,t),$$

that is, $\widetilde{R}^{2\pi}$ and $T^{2\pi}$ are unitarily equivalent.

**12.5.2. The Poisson summation formula.** The Fourier transform in $\mathfrak{L}^2(\mathbb{R})$ has the form

$$(\mathfrak{F}_{2\pi}f)(x) = \int_{-\infty}^{\infty} f(y)e^{-2\pi ixy} dy. \tag{1}$$

Then for the inverse transform $\mathfrak{F}_{2\pi}^{-1} \equiv \bar{\mathfrak{F}}_{2\pi}$ we have

$$\left(\bar{\mathfrak{F}}_{2\pi}f\right)(y) = \int_{-\infty}^{\infty} f(x)e^{2\pi ixy} dx. \tag{2}$$

Repeating the arguments of Section 12.2.4 for the representations $\widetilde{R}^{2\pi}$, we conclude that the operator $\bar{\mathfrak{F}}_{2\pi}$ coincides with the operator $T_w$, $w = \begin{pmatrix} 0 & 1 \\ -0 & 0 \end{pmatrix}$, of the corresponding Weyl representation of the group $SL(2,\mathbb{R})$.

Considering $SL(2,\mathbb{R})$ as the group of automorhisms in $N_1$ or in $H(1,\mathbb{R})$, we find that the element $w$ conserves the subgroup $D$. We denote by $W$ the operator acting in $\mathfrak{L}$ by the formula

$$(WF)(x,y,t) = F(w(x,y),t) = F(y,-x,t). \tag{3}$$

One directly verifies the equality

$$B\bar{\mathfrak{F}}_{2\pi} = WB, \tag{4}$$

where $B$ is the Weil-Brezin operator. Hence, from formula (9) of Section 12.5.1 for $B$ with $x = y = 0$ and from (4) we derive the Poisson summation formula

$$\sum_{n\in\mathbf{Z}} f(n) = \sum_{m\in\mathbf{Z}} \left(\bar{\mathfrak{F}}_{2\pi} f\right)(m). \tag{5}$$

For the convergence of both sides of (5) we assume that $f$ belongs to the space $\mathfrak{S}$ from Section 3.2.3. We recommend to the reader to write down the Poisson formula for the ordinary Fourier transform (that is, for the transform with the kernel $e^{ixy}$).

**12.5.3. Summation formulas for Hermite polynomials.** According to formula (10′) of Section 9.6.8

$$\int_{-\infty}^{\infty} e^{ixy} e^{-x^2} H_n(x)dx = \sqrt{\pi}\,(iy)^n e^{-y^2/4}.$$

Making the substitution $x = \sqrt{\pi}\,u$, $y = 2\sqrt{\pi}\,v$ and applying the Poisson summation formula, we derive

$$\sum_{n\in\mathbf{Z}} e^{-\pi n^2} H_{2k}\left(\sqrt{\pi}\,n\right) = \sum_{m\in\mathbf{Z}} (-1)^k \left(2\sqrt{\pi}\,m\right)^{2k} e^{-\pi m^2}. \tag{1}$$

In the same way, the formula

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} e^{-y^2/2} H_k(y)dy = i^k e^{-x^2/2} H_k(x) \tag{2}$$

yields

$$\sum_{n\in\mathbf{Z}} \exp\left(-\frac{\pi}{p}n^2\right) H_k\left(\sqrt{\frac{2\pi}{p}}n\right) = i^k\sqrt{p} \sum_{m\in\mathbf{Z}} \exp(-\pi p m^2) H_k\left(\sqrt{2\pi p}\,m\right), \tag{3}$$

where $p > 0$.

By shifting the argument in (2), it is easy to obtain the equality

$$i^k\sqrt{n} \sum_{p\in\mathbf{Z}} \exp\left(-\frac{\pi}{n}(np+j)^2\right) H_k\left(\sqrt{\frac{2\pi}{n}}(pn+j)\right)$$
$$= \sum_{m\in\mathbf{Z}} \exp\left(-\frac{\pi}{n}m^2\right) \exp\frac{2\pi i j m}{n} H_k\left(\sqrt{\frac{2\pi}{n}}m\right), \quad n, j \in \mathbf{Z}_+. \tag{4}$$

It can be written as

$$\sum_{p\in\mathbf{Z}} \exp\left(-\frac{\pi}{n}(np+j)^2\right) H_k\left(\sqrt{\frac{2\pi}{n}}(pn+j)\right) = \frac{(-i)^k}{\sqrt{n}}$$
$$\times \sum_{\ell=1}^{n} \exp\left(\frac{2\pi i \ell j}{n}\right) \sum_{p\in\mathbf{Z}} \exp\left(-\frac{\pi(pn+\ell^2)}{n}\right) H_k\left(\sqrt{\frac{2\pi}{n}}(pn+\ell)\right). \quad (4')$$

Introducing the functions of a discrete variable

$$F_{jk}(n) = \sum_{p\in\mathbf{Z}} \exp\left(-\frac{\pi(pn+j)^2}{n}\right) H_k\left(\sqrt{\frac{2\pi}{n}}(pn+j)\right), \quad (5)$$

$$A_{jk} = \frac{1}{\sqrt{n}} \exp\frac{2\pi i j k}{n}, \quad (6)$$

one rewrites (4′) in the form

$$\sum_{\ell=1}^{n} A_{j\ell} F_{\ell k} = i^k F_{jk}. \quad (7)$$

We introduce the unitary matrix $A = (A_{jk})$. Formula (7) shows that the vector functions $F_k = (F_{1k}, \ldots, F_{nk})$ are eigenfunctions for $A$ corresponding to the eigenvalues $i^k$. Since $F_k$ and $F_\ell$ belong to different eigenvalues if $i^k \neq i^\ell$, then

$$\sum_{j=1}^{n} F_{jk}\overline{F_{j\ell}} = 0 \quad \text{for} \quad k \neq \ell \pmod 4.$$

Substituting the values of $F_{jk}$, after simplifications we obtain the equality

$$\sum_{p,q\in\mathbf{Z}} \exp\left(-\frac{\pi}{n}(p^2+(qn+p)^2)\right) H_k\left(\sqrt{\frac{2\pi}{n}}p\right) H_\ell\left(\sqrt{\frac{2\pi}{n}}(qn+p)\right) = 0, \quad (8)$$

where $k \neq \ell \pmod 4$.

By means of the Poisson formula one derives from formula

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (2k+1-y^2)e^{-y^2/2} H_k(y) e^{ixy}\, dy = i^k x^2 e^{-x^2/2} H_k(x) \quad (9)$$

(we recommend to the reader to prove it) the equality

$$\frac{1}{\sqrt{n}}\sum_{p\in\mathbf{Z}}\left(2k+1-\frac{2\pi}{n}p^2\right)\exp\left(-\frac{\pi}{n}p^2+\frac{2\pi ipj}{n}\right)H_k\left(\sqrt{\frac{2\pi}{n}}p\right)$$
$$=i^k\sum_{r\in\mathbf{Z}}\frac{2\pi}{n}(rn+j)^2\exp\left(-\frac{\pi}{n}(rn+j)^2\right)H_k\left(\sqrt{\frac{2\pi}{n}}(rn+j)\right). \qquad (10)$$

We also note the equalities

$$\sum_{p\in\mathbf{Z}}\exp\left(-\frac{\pi}{4}(4p+1)^2\right)H_{4k+2}\left(\sqrt{\frac{\pi}{2}}(4p+1)\right)$$
$$=\sum_{m\in\mathbf{Z}}\exp(-\pi(2m+1)^2)H_{4m+2}\left(\sqrt{2\pi}(2m+1)\right)$$
$$=-\sum_{n\in\mathbf{Z}}\exp(-4\pi n^2)H_{4k+2}\left(\sqrt{8\pi}\,n\right), \qquad (11)$$

$$\sum_{p\in\mathbf{Z}}\exp\left(-\frac{\pi}{5}(5p+1)^2\right)H_{4k+2}\left(\sqrt{\frac{2\pi}{5}}(p+1)\right)$$
$$=\sum_{m\in\mathbf{Z}}\exp\left(-\frac{\pi}{5}(5m+2)^2\right)H_{4k+2}\left(\sqrt{\frac{2\pi}{5}}(5m+2)\right)$$
$$=\left(1-\sqrt{5}\right)\sum_{n\in\mathbf{Z}}\exp(-5\pi n^2)H_{4k+2}\left(\sqrt{10\pi}\,n\right) \qquad (12)$$

which can be derived from (7).

**12.5.4. The expansion in the funntions sinc.** Let us introduce the function

$$\chi(x)=\begin{cases}1 & \text{for } x\in\left(-\frac{1}{2},\frac{1}{2}\right],\\ 0 & \text{for } x\bar{\in}\left(-\frac{1}{2},\frac{1}{2}\right].\end{cases} \qquad (1)$$

For $f\in\mathfrak{L}^1(\mathbb{R})$ we have

$$\int_{-\infty}^{\infty}f(x)dx=\sum_{n\in\mathbf{Z}}\int_{-\infty}^{\infty}f(x+n)\chi(x)dx. \qquad (2)$$

If $B$ is the Weil-Brezin operator, then the function

$$(Bf)(x,0,0)=\sum_{n\in\mathbf{Z}}f(x+n)$$

belongs to $\mathfrak{L}^2\left(-\frac{1}{2},\frac{1}{2}\right)$. The coefficients of the expansion of this function into the Fourier series coincide with

$$\sum_{n\in\mathbf{Z}}\int_{-\infty}^{\infty} f(x+n)\chi(x)e^{-2\pi ikx}dx = \left(\bar{\mathfrak{F}}_{2\pi}f\right)(-k),\ k\in\mathbf{Z},$$

that is,

$$(Bf)(x,0,0) = \sum_{n\in\mathbf{Z}}\left(\bar{\mathfrak{F}}_{2\pi}f\right)(n)e^{-2\pi inx}. \tag{3}$$

By applying formula (4) of Section 12.5.2, we derive

$$\sum_{n\in\mathbf{Z}}\int_{-\infty}^{\infty} f(x+n)\chi(x)e^{2\pi ixy}dx = \sum_{n\in\mathbf{Z}}\left(\bar{\mathfrak{F}}_{2\pi}f\right)(n)\int_{-1/2}^{1/2} e^{2\pi i(y-n)x}dx. \tag{4}$$

It is known that

$$\int_{-1/2}^{1/2} e^{2\pi iux}dx = \frac{\sin\pi u}{\pi u},\quad u\in\mathbb{R},\ u\neq 0.$$

After introducing the function

$$\operatorname{sinc} u = \begin{cases}\dfrac{\sin\pi u}{\pi u} & \text{for } u\neq 0,\\ 1 & \text{for } u=0\end{cases} \tag{5}$$

we rewrite (4) in the form

$$\sum_{n\in\mathbf{Z}}\int_{n-1/2}^{n+1/2} f(x)e^{2\pi iy(x-n)}dx = \sum_{n\in\mathbf{Z}}\left(\bar{\mathfrak{F}}_{2\pi}f\right)(n)\operatorname{sinc}(y-n),\ y\in\mathbb{R}.$$

Conseqeuntly,

$$\sum_{n\in\mathbf{Z}}\left(1-e^{-2\pi iny}\right)\int_{n-1/2}^{n+1/2} f(x)e^{2\pi ixy}dx$$
$$= \left(\bar{\mathfrak{F}}_{2\pi}f\right)(y) - \sum_{n\in\mathbf{Z}}\left(\bar{\mathfrak{F}}_{2\pi}f\right)(n)\operatorname{sinc}(y-n).$$

If the function $f(x)$ vanishes outside the closed interval $\left[-\frac{1}{2},\frac{1}{2}\right]$, then the left hand side of this relation is equal to zero and we have

$$\left(\bar{\mathfrak{F}}_{2\pi}f\right)(y) = \sum_{n\in\mathbf{Z}}\left(\bar{\mathfrak{F}}_{2\pi}f\right)(n)\operatorname{sinc}(y-n),\quad y\in\mathbb{R}.$$

Applying the results of Section 3.3.1 to the function $F(w) = (\bar{\mathfrak{F}}f)(w)$, we obtain the following statement: *If $F$ is a function on $\mathbf{C}$ of exponential type $\leq \pi$, such that on $\mathbf{R}$ it belongs to $\mathfrak{L}^2(\mathbf{R})$, then we have the expansion*

$$F(w) = \sum_{n\in\mathbf{Z}} F(n)\mathrm{sinc}(w-n), \quad w \in \mathbf{C}.$$

*On compact sets from $\mathbf{C}$ the uniform convergence takes place.*

**12.5.5. Schempp's summation formula.** Let $f_1$ and $f_2$ be functions from $\mathfrak{L}^2(\mathbf{R})$. Since

$$\widetilde{R}^{2\pi}(n(a,b,c)) = B^{-1}T^{2\pi}(n(a,b,c))B, \quad n(a,b,c) \in N_1,$$

then

$$t^{2\pi}_{f_2,f_1}(n(a,b,c)) \equiv (\widetilde{R}^{2\pi}(n(a,b,c)f_1, f_2) = \left\langle T^{2\pi}(n(a,b,c))(Bf_1), Bf_2\right\rangle. \quad (1)$$

Since the action of the operators $T^{2\pi}(n(a,b,c))$ upon the function $Bf$ is defined by formula (8) of Section 12.5.1, then by virtue of formulas (6) of Section 12.1.1, (6) and (7) of Section 12.5.1, the matrix element

$$t^{2\pi}_{f_2,f_1}(n(r,s,0)) \equiv \left\langle T^{2\pi}(n(r,s,0))(Bf_1), Bf_2\right\rangle, \quad r,s \in \mathbf{Z}, \quad (2)$$

is the Fourier transform of the function $(Bf_1)(\overline{Bf_2})$. Hence, we have

$$(Bf_1)(x,y,0)(\overline{Bf_2})(x,y,0) = \sum_{r,s\in\mathbf{Z}} t^{2\pi}_{f_2,f_1}(n(r,s,0))e^{2\pi i(-sx+ry)}. \quad (3)$$

According to the Parseval equality for the Fourier transform, we obtain

$$\sum_{r,s,\in\mathbf{Z}} \left|t^{2\pi}_{f_2,f_1}(n(r,s,0))\right|^2 = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |(Bf_1)(x,y,0)|^2|(Bf_2)(x,y,0)|^2 dx\,dy. \quad (4)$$

If $f_1 = f_2$, then we have from (3) that

$$|(Bf_i)(x,y,0)|^2 = \sum_{r,s\in\mathbf{Z}} t^{2\pi}_{f_i,f_i}(n(r,s,0))e^{2\pi i(-sx+ry)}, \quad i = 1,2.$$

Substituting these expressions for $|Bf_i|^2$ into the right hand side of (4), we derive

$$\sum_{r,s,\in\mathbf{Z}} \left|t^{2\pi}_{f_2,f_1}(n(r,s,0))\right|^2 = \sum_{r,s\in\mathbf{Z}} t^{2\pi}_{f_1,f_1}(n(r,s,0))t^{2\pi}_{f_2,f_2}(n(r,s,0)). \quad (5)$$

The functions

$$e_m(x) = (2^m m!)^{-1/2} \sqrt[4]{2}\, e^{-\pi x^2} H_m\left(\sqrt{2\pi}\, x\right), \quad m = 0, 1, 2, \ldots, \tag{6}$$

form an orthonormal basis of the space $\mathfrak{L}^2(\mathbb{R})$ with the scalar product (1′) of Section 12.5.1. Repeating almost word by word the arguments of Section 12.1.6, we find that the matrix elements of the representation $\widetilde{R}^{2\pi}$ in the basis (6) are equal to

$$\tilde{r}^{2\pi}_{mp}(n(a,b,0)) \equiv t^{2\pi}_{e_m,e_p}(n(a,b,0)) = \left(\frac{p!}{m!}\right)^{1/2} \exp\left(-\frac{\pi(a^2+b^2)}{2}\right) \times \left(\sqrt{\pi}(ib-a)\right)^{m-p} L_p^{m-p}(\pi(a^2+b^2)), \tag{7}$$

if $m \geq p$, and to

$$\tilde{r}^{2\pi}_{mp}(n(a,b,0)) = \left(\frac{m!}{p!}\right)^{1/2} \exp\left(-\frac{\pi(a^2+b^2)}{2}\right) \left(\sqrt{\pi}(a+ib)\right)^{p-m} L_m^{p-m}(\pi(a^2+b^2)), \tag{8}$$

if $p \geq m$. In particular,

$$\tilde{r}^{2\pi}_{mm}(n(a,b,0)) = \exp\left(-\frac{\pi(a^2+b^2)}{2}\right) L_m^0(\pi(a^2+b^2)) \tag{9}$$

We set $f_1 = e_p$, $f_2 = e_m$ into (5) and take into account (7) and (9). If $m \geq p$, then we obtain the equality

$$\frac{p!}{m!}\pi^{m-p} \sum_{r,s\in\mathbf{Z}} (r^2+s^2)^{m-p} \exp\left(-\pi(r^2+s^2)\right) \left[L_p^{m-p}(\pi(r^2+s^2))\right]^2 = \sum_{r,s\in\mathbf{Z}} \exp\left(-\pi(r^2+s^2)\right) L_m^0\left(\pi(r^2+s^2)\right) L_p^0\left(\pi(r^2+s^2)\right). \tag{10}$$

Taking into account the expressions for the Laguerre polynomials $L_n^\alpha(x)$ when $n = 0, 1, 2$, and putting $m = 1$, $p = 0$ into (10), we derive

$$\pi \sum_{n\in\mathbf{Z}} n^2 e^{-\pi n^2} = \frac{1}{4} \sum_{n\in\mathbf{Z}} e^{-\pi n^2}. \tag{11}$$

When $m = 2$, $p = 1$, then we have

$$\pi^2 \sum_{n\in\mathbf{Z}} n^6 e^{-\pi n^2} = \frac{15}{32} \sum_{n\in\mathbf{Z}} (8\pi^2 n^4 - 1) e^{-\pi n^2}. \tag{12}$$

## 12.6. Harmonic Polynomials on the Heisenberg Group

**12.6.1. The operator $L_\gamma$.** We realize the Heisenberg group $H(n,\mathbb{R})$ by matrices (7) of Section 12.1.1 and introduce the notation

$$\tilde{g}(\mathbf{z},t) = g\left(\sqrt{2}\,\mathbf{z},t\right). \tag{1}$$

Then formula (8) of Section 12.1.1 implies that for $\tilde{g}(\mathbf{z},t)$ the multiplication is given by the formula

$$\tilde{g}(\mathbf{z},t)\tilde{g}(\mathbf{z}',t') = \tilde{g}(\mathbf{z}+\mathbf{z}',t+t'+2\operatorname{Im}\mathbf{z}\cdot\mathbf{z}'), \tag{2}$$

where $\mathbf{z}\cdot\mathbf{z}' = z_1\overline{z_1'}+\ldots+z_n\overline{z_n'}$.

Let $\mathfrak{H}$ be the space of infinitely differentiable functions $f(\tilde{g}(\mathbf{z},t)) \equiv F(\mathbf{z},t)$ on $H(n,\mathbb{R})$. The operators

$$\widetilde{R}(\tilde{g}(\mathbf{z}',t'))F(\mathbf{z},t) = f(\tilde{g}(\mathbf{z},t)\tilde{g}(\mathbf{z}',t')) = F(\mathbf{z}+\mathbf{z}',t+t'+2\operatorname{Im}\mathbf{z}\cdot\mathbf{z}') \tag{3}$$

give the right regular representation of the group $H(n,\mathbb{R})$. Let $z_j = x_j+iy_j$, $x_j, y_j \in \mathbb{R}$ and $\{\tilde{g}_j(x_j,0\}$, $\{\tilde{g}_j(iy_j,0\}$ be the one-parameter subgroups of elements $\tilde{g}(\mathbf{z},0)$, for which, respectively, $\mathbf{z} = (0,\ldots,0,x_j,0,\ldots,0)$, $\mathbf{z} = (0,\ldots,0,iy_j,0,\ldots,0)$ and $x_j$, $iy_j$ are on the $j$-th position. It follows from (3) that to the subgroups $\{\tilde{g}_j(x_j,0)\}$, $\{\tilde{g}_j(iy_j,0)\}$, $\{\tilde{g}(0,t)\}$ in the representation $\widetilde{R}$ there correspond the infinitesimal operators

$$X_j = \frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial t},\ Y_j = \frac{\partial}{\partial y_j}-2x_j\frac{\partial}{\partial t},\ T = \frac{\partial}{\partial t}. \tag{4}$$

Let us introduce the operators

$$\begin{aligned} Z_j &\equiv \frac{1}{2}(X_j - iY_j) = \frac{\partial}{\partial z_j}+i\bar{z}_j\frac{\partial}{\partial t},\\ \bar{Z} &\equiv \frac{1}{2}(X_j + iY_j) = \frac{\partial}{\partial \bar{z}_j}-iz_j\frac{\partial}{\partial t}. \end{aligned} \tag{5}$$

We have

$$[Z_j,\bar{Z}_k] = -2i\delta_{jk}T, [Z_j,Z_k] = [\bar{Z}_j,\bar{Z}_k] = [Z_j,T] = [\bar{Z}_j,T] = 0.$$

For $\gamma\in\mathbf{C}$ we form the operator

$$L_\gamma = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j+\bar{Z}_jZ_j)+i\gamma T. \tag{6}$$

With the help of (5) one finds

$$L_\gamma = -\Delta^{(C)} - |\mathbf{z}|^2 \left(\frac{\partial}{\partial t}\right)^2 + i \left[\gamma + \sum_{j=1}^{n} \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\right)\right] \frac{\partial}{\partial t}, \tag{7}$$

where $\Delta^{(C)}$ is the Laplace operator on the unitary space $E_n^C$:

$$\Delta^{(C)} = \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j}. \tag{7'}$$

Let us show that the operator $L_\gamma$, acting in the space $\mathfrak{H}$ of functions on $H(n, \mathbb{R})$, is an analog of the Laplace operator on $E_n$.

The Euclidean space $E_n$ is identified with the maximal nilpotent subgroup $N \equiv \{n(\mathbf{t}) \mid \mathbf{t} \in \mathbb{R}^n\}$ of the group $SO_0(n+1, 1)$ (see Section 9.1.2), and $H(n, \mathbb{R})$ is identified with the maximal nilpotent subgroup $N = \{n(\mathbf{z}, t)\}$ of $U(n+1, 1)$. The dilatation $\mathbf{t} \to R\mathbf{t}$, $R \in \mathbb{R}$, in $E_n$ can be defined by means of the elements $g'_{n+1}(\theta)$ of the subgroup $A'$, namely,

$$g'_{n+1}(\theta) n(\mathbf{t}) g'_{n+1}(-\theta) = n(e^\theta \mathbf{t})$$

(see formula (11) of Section 9.1.2). An analogous formula for the elements $n(\mathbf{z}, t) \in U(n+1, 1)$ has the form

$$g'_{n+1}(\theta) n(\mathbf{z}, t) g'_{n+1}(-\theta) = n(e^\theta \mathbf{z}, e^{2\theta} t)$$

(see Section 11.1.4). Therefore, in $H(n, \mathbb{R})$ the dilatation by $R \in \mathbb{R}$ is given by

$$(\mathbf{z}, t) \to (R\mathbf{z}, R^2 t) \tag{8}$$

(if elements from $H(n, \mathbb{R})$ are understood as elements of a manifold but not as elements of a group, then $(\mathbf{z}, t)$ is written instead of $\tilde{g}(\mathbf{z}, t)$).

The distance between points is homogeneous with respect to dilatations. Therefore, we define the distance from a point $(\mathbf{z}, t) \in H(n, \mathbb{R})$ to the origin $(0, 0)$ by the formula

$$\|(\mathbf{z}, t)\| = (|\mathbf{z}|^4 + t^2)^{1/4}. \tag{8'}$$

If $t = 0$ then (8′) turns into the distance in $E_n$.

Formula (12) of Section 9.1.2 defines the action of the subgroup $SO(n)$ on $E_n$. The corresponding formula for elements of the subgroups $N \subset U(n+1, 1)$ and $U(n) \subset U(n+1, 1)$ is of the form

$$m n(\mathbf{z}, t) m^{-1} = n(m\mathbf{z}, t), \qquad m \in U(n)$$

(see Section 11.1.3). Hence, the action of $U(n)$ in $H(n,\mathbb{R})$ is given by

$$m(\mathbf{z},t) = (m\mathbf{z},t). \tag{9}$$

The Laplace operator $\Delta$ in the Euclidean space $E_n$ has the following properties: 1) $\Delta$ is a second order differential operator on $E_n$; 2) $\Delta$ is homogeneous with respect to dilatations

$$\Delta f_R(\mathbf{x}) = R^2(\Delta f)(R\mathbf{x}), \quad f_R(\mathbf{x}) = f(R\mathbf{x});$$

3) $\Delta$ is invariant with respect to left and right shifts in $E_n$; 4) $\Delta$ is invariant with respect to rotations from $SO(n)$.

The operator $L_\gamma$ has analogous properties. In fact, it is easy to verify by means of formula (7) that $L_\gamma$ is a second order differential operator which is invariant with respect to the action (9) of $U(n)$ and homogeneous with respect to dilatations (8):

$$L_\gamma f(R\mathbf{z}, R^2t) = R^2(L_\gamma f)(R\mathbf{z}, R^2t).$$

Differential operators (5) are vector fields corresponding to basis elements of the Lie algebra of the group $H(n,\mathbb{R})$. They are invariant with respect to left shifts by elements of $H(n,\mathbb{R})$ (see Section 1.1.3). Therefore, $L_\gamma$ is also invariant with respect to left shifts by these elements (see also Section 12.6.2 below).

These properties do not exhaust the analogy between $\Delta$ and $L_\gamma$. It is known that the *Dirichlet problem* on the disc

$$D^n \equiv \left\{\mathbf{x} \in E_n \mid r(\mathbf{x}) \equiv (x_1^2 + \ldots + x_n^2)^{1/2} < 1\right\}$$

is solvable for the Laplace operator $\Delta$, namely, if $f$ is a continuous function on the boundary $S^{n-1}$ of $D^n$, then there exists the single continuous function $F$ on the closure of $D^n$ such that $F$ coincides with $f$ on $S^{n-1}$ and $\Delta F = 0$ on $D^n$. Jerison [168] has shown that the Dirichlet problem is also solvable for the operator $L_\gamma$ on the Heisenberg unit disc

$$D_H^n \equiv \left\{(\mathbf{z},t) \in H(n,\mathbb{R}) \mid \|(\mathbf{z},t)\| \equiv (|\mathbf{z}|^4 + t^2)^{1/4} < 1\right\};$$

namely, if $f$ is a continuous function on the boundary

$$S_H^{n-1} \equiv \{(\mathbf{z},t) \in H(n,\mathbb{R}) \mid \|(\mathbf{z},t)\| = 1\}$$

of $D_H^n$, then there exists the single continuous function $F$ on the closure of $D_H^n$ such that $F$ coincides with $f$ on $S_H^{n-1}$ and $L_\gamma F = 0$ on $D_H^n$.

The operator $\Delta$ is *hypoelliptic*, that is, if $\Delta f = F$ and $F$ is real-analytic, then $f$ is real-analytic also. Folland and Stein [111] have shown that the operator $L_\gamma$ is also hypoelliptic.

**12.6.2. The operator $L_\gamma$ and the group $U(n+1,1)$.** Instead of $U(n+1,1)$ it is convenient to consider here the group $G$, isomorphic to $U(n+1,1)$. The group $G$ is defined as the set of linear transformations in $\mathbb{C}^{n+2}$ conserving the form

$$|w_1|^2 + \ldots + |w_n|^2 - \operatorname{Im}(w_0 \bar{w}_{n+1}). \tag{1}$$

Transformations from $G$ are represented by matrices $g \in GL(n+2, \mathbb{C})$ for which $g^* J g = J$, where

$$J = \begin{pmatrix} 0 & 0 & -i/2 \\ 0 & I_n & 0 \\ i/2 & 0 & 0 \end{pmatrix}.$$

We suggest to the reader to construct the isomorphism between $U(n+1,1)$ and $G$.

When we go over from the space $E^C_{n+1,1}$ with the distance, given by the formula $[\mathbf{z}, \mathbf{w}] = -z_1 w_1 - \ldots - z_{n+1} w_{n+1} + z_{n+2} w_{n+2}$, to the space $\widetilde{E}^C_{n+1,1}$ with the distance, given by (1), the wave opertor

$$\Box = \sum_{j=1}^{n+1} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} - \frac{\partial^2}{\partial z_{n+2} \partial \bar{z}_{n+2}},$$

multiplied by $-1$, turns into the operator

$$\Box' = -\sum_{j=1}^{n} \frac{\partial^2}{\partial w_j \partial \bar{w}_j} + 2i \frac{\partial^2}{\partial w_0 \partial \bar{w}_{n+1}} - 2i \frac{\partial^2}{\partial \bar{w}_0 \partial w_{n+1}}. \tag{2}$$

Since $\Box$ is invariant with respect to the action of $U(n+1,1)$, then $\Box'$ is invariant with respect to the action of $G$.

When we go over from the group $U(n+1,1)$ to $G$, the subgroup $A' \equiv \{g'_{n+1}(\theta)\}$ turns into the subgroup $\widetilde{A}' = \{\operatorname{diag}(e^\theta, 1, \ldots, 1, e^{-\theta})\}$. The maximal compact subgroup $M$, whose elements commute with elements from $\widetilde{A}'$, consists of the matrices

$$m = \operatorname{diag}(e^{i\varphi}, u, e^{i\varphi}), \quad u \in U(n), \quad 0 \le \varphi < 2\pi. \tag{3}$$

If $g_w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}$, then $M' = M \cup g_w M$ is the normalizer of the subgroup $\widetilde{A}'$ in the maximal compact subgroup of $G$ (see Section 11.1.3). The subgroup $N$ in $G$ consists of matrices

$$\tilde{n}(\mathbf{z}, t) = \begin{pmatrix} 1 & i\mathbf{z} & t + i|\mathbf{z}|^2 \\ 0 & I_n & 2\bar{\mathbf{z}}^T \\ 0 & 0 & 1 \end{pmatrix}, \; \mathbf{z} \in \mathbb{C}^n, \; t \in \mathbb{R}. \tag{4}$$

We have

$$\tilde{n}(\mathbf{z}, t) \tilde{n}(\mathbf{z}', t') = \tilde{n}(\mathbf{z} + \mathbf{z}', t + t' + 2 \operatorname{Im} \mathbf{z} \cdot \bar{\mathbf{z}}), \tag{5}$$

that is, the multiplication rule for matrices $\tilde{n}(\mathbf{z}, t)$ is the same as in the case of matrices (1) of Section 12.6.1.

The group $G$ acts in the space $\mathfrak{B}$ of infinitely differentiable functions $f(w)$ on $\mathbb{C}^{n+2}$, namely, $(T(g)f)(\mathbf{w}) = f(g\mathbf{w})$. Let $\mathfrak{B}^{\alpha\beta}$, $\alpha+\beta=n$, $\beta-\alpha=\gamma$, be the subspace of $\mathfrak{B}$ consisting of homogeneous functions

$$f(a\mathbf{w}) = a^{-\beta}\bar{a}^{-\alpha} f(\mathbf{w}), \quad a \in \mathbb{C}. \tag{6}$$

It is clear that $\mathfrak{B}^{\alpha\beta}$ is invariant with respect to operators $T(g)$.

Let $\zeta_j = w_j/w_{n+1}$, $j = 0, 1, 2, \ldots, n$. Then

$$\begin{aligned} f(\mathbf{w}) \equiv f(w_0, w_1, \ldots, w_{n+1}) &= w_{n+1}^{-\beta}\bar{w}_{n+1}^{-\alpha} f\left(\frac{w_0}{w_{n+1}}, \ldots, \frac{w_n}{w_{n+1}}, 1\right) \\ &\equiv w_{n+1}^{-\beta}\bar{w}_{n+1}^{-\alpha} F(\zeta_0, \zeta_1, \ldots, \zeta_n). \end{aligned} \tag{7}$$

When we go over from $f(\mathbf{w})$ to $(T(g)f)(\mathbf{w}) \equiv f(g\mathbf{w})$, the function $F(\boldsymbol{\zeta}) \equiv F(\zeta_0, \ldots, \zeta_n)$ turns into

$$(\widetilde{T}(g)F)(\zeta) \equiv \alpha(g, \boldsymbol{\zeta})^{-\beta}\overline{\alpha(g, \boldsymbol{\zeta})}^{-\alpha} F(g \cdot \boldsymbol{\zeta}), \tag{8}$$

where $\alpha(g, \boldsymbol{\zeta})$ is the $n+1$-th coordinate in $g(\zeta_0, \ldots, \zeta_n, 1)$. For the element $g = k \equiv \operatorname{diag}(1, u, 1)$, $u \in U(n)$, formula (8) takes the form

$$(\widetilde{T}(k)F)(\boldsymbol{\zeta}) = F(\zeta_0, u\boldsymbol{\zeta}'), \quad \boldsymbol{\zeta}' = (\zeta_1, \ldots, \zeta_n), \tag{9}$$

and for $g_w$, $g(\theta) = \operatorname{diag}(e^\theta, 1, \ldots, 1, e^{-\theta})$, $n(\mathbf{z}, t) \in N$ we have

$$(\widetilde{T}(g_w)F)(\boldsymbol{\zeta}) = \zeta_0^{-\beta}\bar{\zeta}_0^{-\alpha} F\left(-\frac{1}{\zeta_0}, \frac{\zeta_1}{\zeta_0}, \ldots, \frac{\zeta_n}{\zeta_0}\right), \tag{10}$$

$$(\widetilde{T}(g(\theta))F)(\boldsymbol{\zeta}) = e^{(\alpha+\beta)\theta} F(e^{2\theta}\zeta_0, e^\theta\zeta_1, \ldots, e^\theta\zeta_n), \tag{11}$$

$$(\widetilde{T}(n(\mathbf{z}, t))F)(\boldsymbol{\zeta}) = F(\zeta_0', \boldsymbol{\zeta}' + \mathbf{z}), \tag{12}$$

where $\zeta_0' = \zeta_0 + t + i|\mathbf{z}|^2 + 2i \sum_{j=1}^{n} \zeta_j \bar{z}_j$.

In the coordinates $\zeta_0, \zeta_1, \ldots, \zeta_n, w_{n+1}$ operator (2) is represented as

$$\begin{aligned} \Box' = |w_{n+1}|^{-2}\Bigg[\sum_{j=1}^{n}\left(-\frac{\partial^2}{\partial\zeta_j\partial\bar{\zeta}_j} + 2i\zeta_j\frac{\partial^2}{\partial\zeta_j\partial\bar{\zeta}_0} - 2i\bar{\zeta}_j\frac{\partial^2}{\partial\bar{\zeta}_j\partial\zeta_0}\right) \\ + 2i(\zeta_0 - \bar{\zeta}_0)\frac{\partial^2}{\partial\zeta_0\partial\bar{\zeta}_0} + 2i\bar{w}_{n+1}\frac{\partial^2}{\partial\bar{w}_{n+1}\partial\zeta_0} - 2iw_{n+1}\frac{\partial^2}{\partial w_{n+1}\partial\bar{\zeta}_0}\Bigg]. \end{aligned} \tag{13}$$

Therefore, it acts upon functions (7) by the formula

$$\Box' \left[ w_{n+1}^{-\beta} \bar{w}_{n+1}^{-\alpha} F(\zeta) \right] = w_{n+1}^{-\beta-1} \bar{w}_{n+1}^{-\alpha-1} (\Box_{\alpha\beta} F)(\zeta), \tag{14}$$

where

$$\Box_{\alpha\beta} = \sum_{j=1}^{n} \left( -\frac{\partial^2}{\partial\zeta_j \partial\bar{\zeta}_j} + 2i\zeta_j \frac{\partial^2}{\partial\zeta_j \partial\bar{\zeta}_0} - 2i\bar{\zeta}_j \frac{\partial^2}{\partial\bar{\zeta}_j \partial\zeta_0} \right) + 2i(\zeta_0 - \bar{\zeta}_0) \frac{\partial^2}{\partial\zeta_0 \partial\bar{\zeta}_0} + 2i\beta \frac{\partial}{\partial\bar{\zeta}_0} - 2i\alpha \frac{\partial}{\partial\zeta_0}. \tag{15}$$

The operator $\Box_{\alpha\beta}$ acts upon functions (8) as

$$\Box_{\alpha\beta} \left[ \alpha(g,\zeta)^{-\beta} \overline{\alpha(g,\zeta)}^{-\alpha} F(g\cdot\zeta) \right] = \alpha(g,\zeta)^{-\beta-1} \overline{\alpha(g,\zeta)}^{-\alpha-1} \Box_{\alpha\beta}^{g} F(g\cdot\zeta), \tag{16}$$

where $\Box_{\alpha\beta}^{g}$ is obtained from $\Box_{\alpha\beta}$ by replacing $\zeta$ by $g\cdot\zeta$. In fact, if $\Box_g'$ is the operator obtained from $\Box'$ by replacing $\zeta$, $w_{n+1}$, respectively, by $g\cdot\zeta$, $\alpha(g,\zeta)$, then because of the invariance of $\Box'$ with respect to $G$ we have

$$\Box' \left[ \alpha(g,\zeta)^{-\beta} \overline{\alpha(g,\zeta)}^{-\alpha} F(g\cdot\zeta) \right] = \Box_g' \left[ \alpha(g,\zeta)^{-\beta} \overline{\alpha(g,\zeta)}^{-\alpha} F(g\cdot\zeta) \right]. \tag{17}$$

Relation (16) follows from (14) and (17).

Formula (8) gives the action of elements $g \in G$ upon $\zeta$. Let $\zeta_i = (i, 0, \ldots, 0)$, $i = \sqrt{-1}$. Then any element $\zeta = (\zeta_0, \ldots, \zeta_n)$ is obtained in the following way:

$$n(\zeta', t) g\left(\frac{1}{2}\log x\right) \zeta_i = (t + i(|\zeta'| + x), \zeta'), \tag{18}$$
$$\zeta_0 = t + i(|\zeta'| + x), \ \zeta' = (\zeta_1, \ldots, \zeta_n).$$

And the points $\zeta = (t + i|\zeta'|, \zeta')$ are obtained from $\zeta_0 = (0, \ldots, 0)$:

$$(t + i|\zeta'|, \zeta') = n(\zeta', t)\zeta_0, \ \zeta' = (\zeta_1, \ldots, \zeta_n). \tag{19}$$

Therefore, the subset of points $\zeta \in \mathbb{C}^{n+1}$ of the form $(t + i|\zeta'|, \zeta')$, $\zeta' \in \mathbb{C}^n$, is identified with the manifold of the group $H(n, \mathbb{R})$.

We now, according to formula (18), replace in (15) the complex variable $\zeta_0$ by the real variables $t$ and $x$:

$$\Box_{\alpha\beta} = \sum_{j=1}^{n} \left[ -\frac{\partial^2}{\partial\zeta_j \partial\bar{\zeta}_j} + i\frac{\partial}{\partial t} \left( \zeta_j \frac{\partial}{\partial\bar{\zeta}_j} - \bar{\zeta}_j \frac{\partial}{\partial\bar{\zeta}_j} \right) \right] - |\zeta'|^2 \frac{\partial^2}{\partial t^2} + i\gamma \frac{\partial}{\partial t} - x\left( \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \right). \tag{20}$$

Assume that the function $F(\zeta) \equiv F(t + i|\zeta'| + x, \zeta')$ does not depend on $x$. This means that $F$ is a function on $H(n, \mathbf{R})$:

$$F(\zeta) \equiv \Phi(t + i|\zeta'|, \zeta') \equiv f(\zeta', t).$$

Then formulas (7) of Section 12.6.1 and (20) imply that

$$\Box_{\alpha\beta} f(\zeta', t) = L_\gamma f(\zeta', t). \tag{21}$$

It follows from here and from formula (16) with $g \in U(n+1, 1)$ and $\zeta = (t + i|\zeta'|, \zeta')$ that

$$L_\gamma[\alpha(g,\zeta)^{-\beta}\overline{\alpha(g,\zeta)}^{-\alpha} f(g \cdot (\zeta', t))]$$
$$= \alpha(g,\zeta)^{-\beta-1}\overline{\alpha(g,\zeta)}^{-\alpha-1}(L_\gamma f)(g \cdot (\zeta', t)), \tag{22}$$

where $g \cdot (\zeta', t) = (\tilde{\zeta}', t')$ is defined by the action of the transformation $g$ upon $\zeta \equiv (t + i|\zeta'|, \zeta')$ according to (8):

$$g \cdot \zeta \equiv g \cdot (t + i|\zeta'|, \zeta') = (t' + i|\tilde{\zeta}'|, \tilde{\zeta}'). \tag{23}$$

The values of the function $\alpha(g, \zeta)$ and of the coordinates $\tilde{\zeta}'$ and $t'$ are defined by formulas (9)–(12). Setting $g = \mathrm{diag}(1, u, 1)$, $u \in U(n)$, and using (9), we obtain from (22) that

$$(L_\gamma f)(u\zeta', t) = (L_\gamma f)(\zeta', t). \tag{24}$$

This means that $L_\gamma$ is invariant with respect to the subgroup $U(n)$. From (11) and (22) we have

$$L_\gamma f(\tilde{n}(\mathbf{z}, t')(\zeta', t)) = L_\gamma f(\zeta', t), \tag{25}$$

that is, $L_\gamma$ is invariant with respect to left shifts by elements from $H(n, \mathbf{R})$. From (11) and (22) we derive

$$L_\gamma(f(R\zeta', R^2 t)) = R^2(L_\gamma f)(R\zeta', R^2 t)), \tag{26}$$

that is, $L_\gamma$ is invariant with respect to dilatations. According to (10) and (23) we obtain

$$g_w(\zeta', t) = \left(\frac{\zeta'}{t + i|\zeta'|^2}, -\frac{t}{t^2 + |\zeta'|^4}\right).$$

Since for $\zeta = (t + i|\zeta'|^2, \theta')$ we have $\alpha(g_w, \zeta) = t + i|\zeta'|^2$, then (22) implies

$$L_\gamma\left[(|\zeta'|^2 + it)^{-\alpha}(|\zeta'|^2 - it)^{-\beta} f\left(\frac{\zeta'}{t + i|\zeta'|^2}, \frac{-t}{t^2 + |\zeta'|^4}\right)\right]$$
$$= (|\zeta'|^2 + it)^{-\alpha-1}(|\zeta'|^2 - it)^{-\beta-1}(L_\gamma f)\left(\frac{\zeta'}{t + i|\zeta'|^2}, \frac{-t}{t^2 + |\zeta'|^4}\right). \tag{27}$$

The transformation

$$(K_\gamma f)(\boldsymbol{\zeta}', t) = (|\boldsymbol{\zeta}'|^2 + it)^{-\alpha}(|\boldsymbol{\zeta}'|^2 - it)^{-\beta} f\left(\frac{\boldsymbol{\zeta}'}{t + i|\boldsymbol{\zeta}'|^2}, \frac{-t}{t^2 + |\boldsymbol{\zeta}'|^4}\right)$$

is said to be the *Kelvin transformation* on $H(n, \mathbf{R})$. It follows from (27) that if $L_\gamma f = 0$ on $H(n, \mathbf{R})$, then $L_\gamma(K_\gamma f) = 0$ for all points from $H(n, \mathbf{R})$ except for the point $(\mathbf{0}, 0)$.

Formula (27) also implies that if

$$f_\gamma(\boldsymbol{\zeta}', t) = (|\boldsymbol{\zeta}'|^2 + it)^{-\alpha}(|\boldsymbol{\zeta}'|^2 - it)^{-\beta}, \ \alpha + \beta = n, \ \beta - \alpha = \gamma, \tag{28}$$

and $\pm\gamma \neq n,\ n+2, n+4, \ldots$, then $L_\gamma f_\gamma = 0$ for all points $(\boldsymbol{\zeta}', t) \neq (\mathbf{0}, 0)$. Folland and Stein [111] have shown that if $\pm\gamma \neq n,\ n+2, n+4, \ldots$, then for

$$\Phi_\gamma(\boldsymbol{\zeta}', t) = \Gamma(\alpha)\Gamma(\beta)2^{n-2}\pi^{-n-1} f_\gamma(\boldsymbol{\zeta}', t) \tag{29}$$

the equality

$$(L_\gamma \Phi_\gamma)(\boldsymbol{\zeta}', t) = \delta(\boldsymbol{\zeta}', t) \tag{30}$$

holds. Here $\delta$ is the delta-function on $H(n, \mathbf{R})$.

**12.6.3. Harmonic polynomials on $H(n, \mathbf{R})$.** We say that $p(\mathbf{z}, t)$ is a polynomial on $H(n, \mathbf{R})$ if it is a polynomial of $z_j$, $\bar{z}_j$, $1 \le j \le n$, and $t$. A polynomial $p$ is said to be *homogeneous* of degree $m$ if for dilatations (8) of Section 12.6.1 we have

$$p(R\mathbf{z}, R^2 t) = R^m p(\mathbf{z}, t), \quad R > 0. \tag{1}$$

Let $\mathfrak{R}^m$ be the space of homogeneous polynomials on $H(n, \mathbf{R})$ of degree $m$. It is clear that if $p \in \mathfrak{R}^m$, then $p$ is a linear combination of polynomials of the form $t^r p_{\ell\ell'}(\mathbf{z})$, where $p_{\ell\ell'} \in \mathfrak{R}_C^{n\ell\ell'}$ ($\mathfrak{R}_C^{\ell\ell'}$ is the space of homogeneous polynomials of $\mathbf{z}$ and $\bar{\mathbf{z}}$, which are of degree $\ell$ in $\mathbf{z}$ and of degree $\ell'$ in $\bar{\mathbf{z}}$) and $m = 2r + \ell + \ell'$. But

$$\mathfrak{R}_C^{n\ell\ell'} = \sum_{s=0}^{\min(\ell,\ell')} |\mathbf{z}|^{2s} \mathfrak{H}_C^{n,\ell-s,\ell'-s}, \ |\mathbf{z}| = (|z_1|^2 + \ldots + |z_n|^2)^{1/2},$$

(see Section 11.2.1), where $\mathfrak{H}_C^{n\ell\ell'}$ is the space of harmonic polynomials from $\mathfrak{R}_C^{n\ell\ell'}$. Therefore, $p \in \mathfrak{R}^m$ is a linear combination of polynomials of the form

$$t^r |\mathbf{z}|^{2s} p_{\ell\ell'}(\mathbf{z}), \ p_{\ell\ell'} \in \mathfrak{H}_C^{n\ell\ell'},$$

where $m = 2r + 2s + \ell + \ell'$. Thus, polynomials $p \in \mathfrak{R}^m$ are represented as

$$p(\mathbf{z}, t) = \sum_{\ell,\ell'} \sum_j h_{\ell\ell' j}(|\mathbf{z}|^2, t) p_{\ell\ell' j}(\mathbf{z}), \tag{2}$$

where $p_{\ell\ell' j}$, $j = 1, 2, \ldots, \dim \mathfrak{H}_C^{n\ell\ell'}$, is a basis in $\mathfrak{H}_C^{n\ell\ell'}$ and $h_{\ell\ell' j}(x,t)$ is a homogeneous polynomial of degree $\frac{1}{2}(m-\ell-\ell')$ in $x$ and $t$. The summation in (2) is carried out over the values of $\ell$ and $\ell'$ for which $m-\ell-\ell'$ are even and non-negative.

Let $h(|\mathbf{z}|^2, t)p_{\ell\ell'}(\mathbf{z})$ be a polynomial on $H(n, \mathbf{R})$ such that $p_{\ell\ell'}(z) \in \mathfrak{H}_C^{n\ell\ell'}$. Then it follows from formula (7) of Section 12.6.1 for $L_\gamma$ and from the Euler formula for homogeneous polynomials that

$$L_\gamma \left[h(|\mathbf{z}|^2, t)p_{\ell\ell'}(\mathbf{z})\right] = \tilde{h}(|\mathbf{z}|^2, t)p_{\ell\ell'}(\mathbf{z}), \tag{3}$$

where

$$\tilde{h}(x,t) = \left[-x\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}\right) - (\ell+\ell'+n)\frac{\partial}{\partial x} + i(\ell-\ell'+\gamma)\frac{\partial}{\partial t}\right] h(x,t). \tag{4}$$

It is clear that if $h(x,t)$ is a homogeneous polynomial of degree $k$, then $\tilde{h}(x,t)$ is a homogeneous polynomial of degree $k-1$.

A polynomial $p$ on $H(n, \mathbb{R})$ is said to be *$L_\gamma$-harmonic* if $L_\gamma p = 0$. The subspace of $L_\gamma$-harmonic polynomials from $\mathfrak{R}^m$ will be denoted by $\mathfrak{H}^m$. We use the representation (2) for a polynomial $p \in \mathfrak{H}^m$. Then by virtue of (3)

$$\begin{aligned}(L_\gamma p)(\mathbf{z},t) &= \sum_{\ell,\ell'}\sum_j L_\gamma \left[h_{\ell\ell' j}(|\mathbf{z}|^2, t)p_{\ell\ell' j}(\mathbf{z})\right] \\ &= \sum_{\ell,\ell'}\sum_j \tilde{h}_{\ell\ell' j}(|\mathbf{z}|^2, t)p_{\ell\ell' j}(\mathbf{z}) = 0,\end{aligned} \tag{5}$$

where $\tilde{h}_{\ell\ell' j}$ is given by formula (4). Equality (5) is valid if and only if every summand is equal to zero, that is, if $\tilde{h}_{\ell\ell' j}(|\mathbf{z}|^2, t) = 0$.

Thus, if polynomial (2) is $L_\gamma$-harmonic, then every summand is $L_\gamma$-harmonic and $h_{\ell\ell' j}(|\mathbf{z}|^2, t)$ satisfies the differential equation

$$\left[x\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}\right) + (\ell+\ell'+n)\frac{\partial}{\partial x} - i(\ell-\ell'+\gamma)\frac{\partial}{\partial t}\right] h_{\ell\ell' j}(x,t) = 0. \tag{6}$$

In order to evaluate the polynomials $h_{\ell\ell' j}$ we introduce in $H(n, \mathbb{R})$ the spherical coordinate system

$$(\mathbf{z}, t) = (\rho\boldsymbol{\xi}\sin^{1/2}\varphi,\ \rho^2\cos\varphi), \tag{7}$$

where $\boldsymbol{\xi} \in S_C^{n-1}$ (see Section 11.1.1), $0 \le \varphi < \pi$ and $\rho = (|\mathbf{z}|^4 + t^2)^{1/4}$ is the distance from $(\mathbf{z}, t)$ to $(\mathbf{0}, 0)$. Then

$$p_{\ell\ell' j}(\mathbf{z}) = \rho^{\ell+\ell'} p_{\ell\ell' j}(\boldsymbol{\xi}), \tag{8}$$

$$h_{\ell\ell' j}(|\mathbf{z}|^2, t) = h_{\ell\ell' j}(\rho^2\sin\varphi, \rho^2\cos\varphi). \tag{9}$$

Since $\sin\varphi = \frac{1}{2i}(e^{i\varphi} - e^{-i\varphi})$, $\cos\varphi = \frac{1}{2}(e^{i\varphi} - e^{-i\varphi})$, then $h_{\ell\ell' j}$ is a homogeneous polynomial of degree $\frac{1}{2}(m-\ell-\ell')$ in

$$\rho^2 e^{i\varphi} = t + i|\mathbf{z}|^2,\ \rho^2 e^{-i\varphi} = t - i|\mathbf{z}|^2.$$

We write

$$h_{\ell\ell' j}(|\mathbf{z}|^2, t) = \hat{h}_{\ell\ell' j}(t + i|\mathbf{z}|^2) \equiv \hat{h}_{l\ell' j}(\zeta).$$

We derive from (6) the differential equation for $\hat{h}_{\ell\ell' j}$. It is of the form

$$\left[(\zeta - \bar{\zeta})\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}} - (\alpha + \ell')\frac{\partial}{\partial\zeta} + (\beta + \ell)\frac{\partial}{\partial\bar{\zeta}}\right]\hat{h}_{\ell\ell' j}(\zeta) = 0, \tag{10}$$

where $\alpha + \beta = n$, $\beta - \alpha = \gamma$.

We represent $\hat{h}_{\ell\ell' j}(\zeta)$ in the form

$$\hat{h}_{\ell\ell' j}(\zeta) = \sum_{s=0}^{r} a_s \bar{\zeta}^s \zeta^{r-s},\ r = \frac{1}{2}(m-\ell-\ell'),$$

and by substituting it into (10) we obtain the recurrence relation for the coefficients $a_s$:

$$(s+1)(\beta + \ell + r - s - 1)a_{s+1} - (r-s)(\alpha + \ell' + s)a_s = 0,$$

which implies that

$$a_s = c\frac{\Gamma(\alpha + \ell' + s)\Gamma(\beta + \ell + r - s)}{s!(r-s)!\Gamma(\alpha + \ell')\Gamma(\beta + \ell)},$$

where $c$ is a constant. Therefore, $\hat{h}_{\ell\ell' j}(\zeta) = cC^{(\alpha+\ell', \beta+\ell)}_{(m-\ell-\ell')/2}(\zeta)$, where

$$C^{(\alpha,\beta)}_n(\zeta) = \sum_{s=0}^{n} \frac{\Gamma(\alpha + s)\Gamma(\beta + n - s)}{s!(n-s)!\Gamma(\alpha)\Gamma(\beta)}\bar{\zeta}^s\zeta^{n-s}. \tag{11}$$

Thus, *the polynomial*

$$C^{(\alpha+\ell', \beta+\ell)}_{(m-\ell-\ell')/2}(t + i|\mathbf{z}|^2)p_{\ell\ell'}(\mathbf{z}),\ p_{\ell\ell'}(\mathbf{z}) \in \mathfrak{H}^{n\ell\ell'}_C, \tag{12}$$

*is homogeneous and $L_\gamma$-harmonic. Any polynomial from $\mathfrak{H}^m$ is a linear combination of polynomials* (12).

In the spherical coordinate system (see formula (7)) polynomials (12) take the form

$$\rho^m(\sin\varphi)^{(\ell+\ell')/2}C^{(\alpha+\ell', \beta+\ell)}_{(m-\ell-\ell')/2}(e^{i\varphi})p_{\ell\ell'}(\xi). \tag{12'}$$

**12.6.4. The polynomials $C_n^{(\alpha,\beta)}(\zeta)$.** It follows from formula (10) of the preceding section that

$$C_n^{(\alpha,\beta)}(e^{i\varphi}) = \frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)} e^{-in\varphi} F(-n,\beta;1-\alpha-n;e^{2i\varphi}) \tag{1}$$

for $\alpha \neq 0,-1,\ldots,-n+1$, and

$$C_n^{(\alpha,\beta)}(e^{i\varphi}) = \frac{\Gamma(\beta+n)}{n!\Gamma(\beta)} e^{in\varphi} F(-n,\alpha;1-\beta-n;e^{-2i\varphi}) \tag{2}$$

for $\beta \neq 0,-1,\ldots,-n+1$. Comparing these expressions with formula (4) of Section 9.3.7, we find that $C_n^{(\alpha,\beta)}(e^{i\varphi})$ with $\alpha = \beta$ coincides iwth the Gegenbauer polynomial, that is

$$C_n^{(\alpha,\beta)}(e^{i\varphi}) = C_n^{\alpha}(\cos\varphi). \tag{3}$$

We also have

$$C_n^{(\alpha,0)}(e^{i\varphi}) = \frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)} e^{-in\varphi}, \quad C_n^{(0,\beta)}(e^{i\varphi}) = \frac{\Gamma(\beta+n)}{n!\Gamma(\beta)} e^{in\varphi}, \tag{4}$$

$$C_n^{(\alpha,\beta)}(e^{i\varphi}) = (-1)^n C_n^{(\alpha,\beta)}(-e^{i\varphi}), \tag{5}$$

$$C_n^{(\alpha,\beta)}(e^{i\varphi}) = C_n^{(\beta,\alpha)}(e^{-i\varphi}). \tag{6}$$

Repeating the arguments of Section 9.3.7 with the replacement of harmonicity by $L_\gamma$-harmonicity, we obtain the generating function for $C_n^{(\alpha,\beta)}(\zeta)$, which is of the form

$$(1-t\bar{\zeta})^{-\alpha}(1-t\zeta)^{-\beta} = \sum_{n=0}^{\infty} t^n C_n^{(\alpha,\beta)}(\zeta). \tag{7}$$

The equality

$$C_n^{(\alpha+\alpha',\beta+\beta')}(\zeta) = \sum_{k=0}^{n} C_{n-k}^{(\alpha,\beta)}(\zeta) C_k^{(\alpha',\beta')}(\zeta) \tag{8}$$

is proved in the same way as equality (3) of Section 9.3.7.

One easily derives from formula (11) of Section 12.6.3 the differentiation formulas

$$\frac{\partial}{\partial\zeta} C_n^{(\alpha,\beta)}(\zeta) = \beta C_{n-1}^{(\alpha,\beta+1)}(\zeta), \tag{9}$$

$$\frac{\partial}{\partial\bar{\zeta}} C_n^{(\alpha,\beta)}(\zeta) = \alpha C_{n-1}^{(\alpha+1,\beta)}(\zeta), \tag{10}$$

$$\left(\frac{\partial}{\partial\zeta} + \frac{\partial}{\partial\bar{\zeta}}\right) C_n^{(\alpha,\beta)}(\zeta) = (\alpha+\beta+n-1) C_{n-1}^{(\alpha,\beta)}(\zeta). \tag{11}$$

The recurrence relations for the hypergeometric function imply those for $C_n^{(\alpha,\beta)}(\zeta)$. We have

$$[\bar{\zeta}(\alpha+n)+\zeta(\beta+n)]C_n^{(\alpha,\beta)}(\zeta)-(\alpha+\beta+n-1)|\zeta|^2C_{n-1}^{(\alpha,\beta)}(\zeta)=(n+1)C_{n+1}^{(\alpha,\beta)}(\zeta), \tag{12}$$

$$\frac{\alpha+\beta}{\alpha+\beta+n}C_n^{(\alpha,\beta+1)}(\zeta)-\frac{\alpha(\zeta-\bar{\zeta})}{\alpha+\beta+n}C_{n-1}^{(\alpha+1,\beta+1)}(\zeta)=C_n^{(\alpha,\beta)}(\zeta), \tag{13}$$

$$\frac{\alpha+\beta}{\alpha+\beta+n}C_n^{(\alpha+1,\beta)}(\zeta)+\frac{\beta(\zeta-\bar{\zeta})}{\alpha+\beta+n}C_{n-1}^{(\alpha+1,\beta+1)}(\zeta)=C_n^{(\alpha,\beta)}(\zeta). \tag{14}$$

It follows from the results of Section 12.6.3 that $C_n^{(\alpha,\beta)}(\zeta)$ is a solution of the differential equation

$$\left[(\zeta-\bar{\zeta})\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}-\alpha\frac{\partial}{\partial\zeta}+\beta\frac{\partial}{\partial\bar{\zeta}}\right]f=0. \tag{15}$$

**12.6.5. The polynomials $C_n^{(\alpha,\beta)}(\zeta)$ and representations of the group $SO(n)$.** The polynomials $C_n^{(\alpha,\beta)}(\zeta)$ are obtained by a "deformation" of the Gegenbauer polynomials $C_n^\alpha(\cos\varphi)$. Indeed, Gegenbauer polynomials are zonal spherical functions of the representations $T^{n\ell}$ of the group $SO(n)$:

$$\varphi^{n\ell}(\theta)\equiv\frac{(2p-1)!\ell!}{(2p+\ell-1)!}C_\ell^p(\cos\theta)=\int_{S^{n-2}}[T^{n\ell}(g_{n-1}(\theta))\Xi_O^{n-1,0}(\boldsymbol{\xi})]\overline{\Xi_O^{n-1,0}(\boldsymbol{\xi})}d\boldsymbol{\xi}, \tag{1}$$

where $p=(n-2)/2$ and $\Xi_O^{n-1,0}(\boldsymbol{\xi})\equiv 1$ (see Section 9.3.2). To obtain the polynomials $C_\ell^{(\alpha,\beta)}(e^{i\varphi})$ we deform the invariant measure on $S^{n-2}$. We take the expression

$$d\boldsymbol{\xi}=\frac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}\sin^{n-3}\varphi\,d\varphi\,d\boldsymbol{\eta},\quad \boldsymbol{\eta}\in S^{n-3},$$

for the invariant measure on $S^{n-2}$, represent $\sin^{n-3}\varphi$ in the form $2^{n-3}\sin^{n-3}\frac{\varphi}{2}\times\cos^{n-3}\frac{\varphi}{2}$ and carry out the deformation

$$\sin^{n-3}\frac{\varphi}{2}\cos^{n-3}\frac{\varphi}{2}\to\sin^{2\alpha-1}\frac{\varphi}{2}\cos^{2\beta-1}\frac{\varphi}{2}. \tag{2}$$

The measure $d\boldsymbol{\eta}$ on $S^{n-3}$ is not changed in the expression for $d\boldsymbol{\xi}$. We obtain the deformed measure

$$\tilde{d}\boldsymbol{\xi}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\left(\sin\frac{\varphi}{2}\right)^{2\alpha-1}\left(\cos\frac{\varphi}{2}\right)^{2\beta-1}d\varphi\,d\boldsymbol{\eta} \tag{3}$$

(the factor has been chosen in such a way that the measure of $S^{n-2}$ is equal to 1). The measure obtained is not invariant with respect to $SO(n-1)$.

We equip the carrier space $\mathfrak{D}^{n-1,\ell}$ of the representation $T^{n\ell}$ of $SO(n)$ (see Section 9.3.1) with the new scalar product

$$\langle f_1, f_2\rangle_{\alpha\beta} = \int\limits_{S^{n-2}} f_1(\boldsymbol{\xi})\overline{f_2(\boldsymbol{\xi})}\tilde{d}\boldsymbol{\xi}. \tag{4}$$

The space $\mathfrak{D}^{n-1,\ell}$ with this scalar product is denoted by $\mathfrak{D}^{n-1,\ell}_{\alpha\beta}$. We take the orthonormal basis $\left\{\widehat{\Xi}^{n-1,m}_M\right\}$ in $\mathfrak{D}^{n-1,\ell}_{\alpha\beta}$ by leaving in

$$\Xi^{n-1,m}_M(\boldsymbol{\xi}) = \left[\dim T^{n-1,m}\right]^{1/2} t^{n-1,m}_{m'0}(g_{n-2}(\theta))t^{n-2,m'}_{M'}(\boldsymbol{\eta})$$

(see Section 9.4.2) the factor $t^{n-2,m'}_{M'}(\boldsymbol{\eta})$ fixed. We need a part of the new basis, namely, the functions

$$\widehat{\Xi}^{n-1,m}_O(\boldsymbol{\xi}) = \left[\frac{m!\Gamma(m+\alpha+\beta-1)(2m+\alpha+\beta-1)}{\Gamma(m+\alpha)\Gamma(m+\beta)}\right]^{1/2} P^{(\alpha-1,\beta-1)}_m(\cos\theta_{n-2}),$$
$$m = 0, 1, \ldots, \ell. \tag{5}$$

In contrast to (1), we have for the zonal spherical functions $\varphi^{n\ell}_{\alpha\beta}(\theta)$ of the representation $T^{n\ell}$ of $SO(n)$ in the space $\mathfrak{D}^{n-1,\ell}_{\alpha\beta}$ the expression

$$\varphi^{n\ell}_{\alpha\beta}(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_0^\pi (\cos\theta - i\sin\theta\cos\varphi)^\ell \left(\sin\frac{\varphi}{2}\right)^{2\alpha-1}\left(\cos\frac{\varphi}{2}\right)^{2\beta-1} d\varphi. \tag{6}$$

Representing $(\cos\theta - i\sin\theta\cos\varphi)^\ell$ in the form

$$(\cos\theta - i\sin\theta\cos\varphi)^\ell = \left(e^{-i\theta}\cos^2\frac{\varphi}{2} + e^{i\theta}\sin^2\frac{\varphi}{2}\right)^\ell, \tag{6'}$$

applying the binomial formula and integrating term by term, after simple manipulations we obtain

$$\varphi^{n\ell}_{\alpha\beta}(\theta) = \frac{\Gamma(\alpha+\beta)\ell!}{\Gamma(\ell+\alpha+\beta)} C^{(\alpha,\beta)}_\ell(e^{-i\theta}). \tag{7}$$

In particular,

$$\varphi^{n\ell}_{\alpha\alpha}(\beta) = \frac{\Gamma(2\alpha)\ell!}{\Gamma(\ell+2\alpha)} C^{\alpha}_\ell(\cos\theta). \tag{7'}$$

For the matrix elements

$$\hat{t}^{n\ell}_{m0}(g_{n-1}(\theta)) \equiv \left\langle T^{n\ell}(g_{n-1}(\theta))\widehat{\Xi}^{n-1,0}_O, \widehat{\Xi}^{n-1,m}_O \right\rangle_{\alpha\beta}$$

in the same way as in Section 9.4.2, we derive the expression

$$\hat{t}^{n\ell}_{m0}(g_{n-1}(\theta)) = \left[\frac{m!\Gamma(\alpha+\beta+m)(2m+\alpha+\beta+1)}{\Gamma(\alpha+m)\Gamma(\beta+m)}\right]^{1/2} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$
$$\times \int_0^\pi (\cos\theta - i\sin\theta\cos\varphi)^\ell P_m^{(\alpha-1,\beta-1)}(\cos\varphi)\left(\sin\frac{\varphi}{2}\right)^{2\alpha-1}\left(\cos\frac{\varphi}{2}\right)^{2\beta-1} d\varphi. \quad (8)$$

The integral is calculated by means of the Rodriguez formula for Jacobi polynomials (in the same way as integral (5) of Section 9.3.5). We have

$$\hat{t}^{n\ell}_{mO}(g_{n-1}(\theta)) = (-2i)^m \left[\frac{\Gamma(\alpha+\beta+m-1)\Gamma(\alpha+m)\Gamma(\beta+m)(2m+\alpha+\beta-1)}{m!}\right]^{1/2}$$
$$\times \frac{\ell!\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\ell+m)} \sin^m\theta\, C_{\ell-m}^{(\alpha+m,\beta+m)}(e^{-i\theta}). \quad (9)$$

Formulas (6) and (7) imply the integral representation for $C_\ell^{(\alpha,\beta)}(e^{i\varphi})$, which is of the form

$$C_\ell^{(\alpha,\beta)}(e^{i\varphi}) = \frac{\Gamma(\alpha+\beta+\ell)}{\Gamma(\alpha)\Gamma(\beta)\ell!}$$
$$\times \int_0^\pi (\cos\varphi + i\sin\varphi\sin\theta)^\ell \left(\sin\frac{\theta}{2}\right)^{2\alpha-1}\left(\cos\frac{\theta}{2}\right)^{2\beta-1} d\theta. \quad (10)$$

Making the substitution $z = \cos\varphi + i\sin\varphi\cos\theta$, deforming the integration contour and setting $z = e^{-i\varphi}e^{2i\psi}$, we obtain

$$C_\ell^{(\alpha,\beta)}(e^{i\varphi}) = \frac{\Gamma(\alpha+\beta+\ell)}{\ell!\Gamma(\alpha)\Gamma(\beta)} e^{i(\alpha-\beta)\varphi/2} \sin^{-\alpha-\beta+1}\varphi$$
$$\times \int_0^\varphi \sin^{\alpha-1}(\varphi-\psi)\sin^{\beta-1}\psi\, e^{i(\ell+\frac{\alpha+\beta}{2})(2\psi-\varphi)} d\psi, \quad (11)$$

where $0 < \varphi < \pi$, $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$.

If $\ell \geq m$, then we have from (8) and (9) that

$$\int_0^\pi (\cos\theta + i\sin\theta\cos\varphi)^\ell P_m^{(\alpha-1,\beta-1)}(\cos\varphi)\left(\sin\frac{\varphi}{2}\right)^{2\alpha-1}\left(\cos\frac{\varphi}{2}\right)^{2\theta-1} d\varphi$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)\ell!\Gamma(\alpha+m)\Gamma(\beta+m)(2i)^m}{m!\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+m+\ell)} \sin^m\theta\, C_{\ell-m}^{(\alpha+m,\beta+m)}(e^{i\theta}). \quad (12)$$

Considering (12) as the coefficient of the expansion of the function $(\cos\theta + i\sin\theta \times \cos\varphi)^\ell$ in the Jacobi polynomials, we have

$$(\cos\theta + i\sin\theta\cos\varphi)^\ell = \sum_{m=0}^{\ell} \frac{\ell!\Gamma(\alpha+\beta+m)(\alpha+\beta+2m-1)}{\Gamma(\alpha+\beta+\ell+m)(\alpha+\beta+m-1)}(2i)^m \times \sin^m\theta\, C_{\ell-m}^{(\alpha+m,\beta+m)}(e^{i\theta}) P_m^{(\alpha-1,\beta-1)}(\cos\varphi). \quad (13)$$

By analogy with equation (9) of Section 9.3.8, we derive from (13)

$$C_{\ell+\ell'}^{(\alpha,\beta)}(e^{i\varphi}) = \frac{\ell!\ell'!\Gamma(\alpha+\beta+\ell+\ell')}{(\ell+\ell')!} \sum_{m=0}^{\min(\ell,\ell')} \frac{(-4)^m\Gamma(\alpha+\beta+m-1)\Gamma(\alpha+m)\Gamma(\beta+m)}{m!\Gamma(\alpha+\beta+\ell+m)\Gamma(\alpha+\beta+\ell'+m)} \times (\alpha+\beta+2m-1)\sin^{2m}\varphi\, C_{\ell-m}^{(\alpha+m,\beta+m)}(e^{i\varphi}) C_{\ell'-m}^{(\alpha+m,\beta+m)}(e^{i\varphi}). \quad (14)$$

**12.6.6. Orthogonality relations for $C_n^{(\alpha,\beta)}(\zeta)$.** We cannot use the results of Section 2.3 for derivation of orthogonality relations for the polynomials $C_n^{(\alpha,\beta)}(\zeta)$ since the operators $T^{n\ell}(g_{n-1}(\theta))$ are not unitary in the space $\mathfrak{D}_{\alpha\beta}^{n-1,\ell}$. Moreover, in the case of general $\alpha$ and $\beta$ there are no orthogonality relations for $C_n^{(\alpha,\beta)}(\zeta)$ with respect to a positive measure. To obtain orthogonality relations with respect to a complex measure, we use the formula

$$\int_0^\pi C_n^{(\alpha,\beta}(e^{i\varphi}) e^{i(\beta-\alpha+m)\varphi} \sin^{\alpha+\beta-1}\varphi\, d\varphi = \frac{\pi\Gamma(\alpha+\beta+n)\exp\left(\frac{\pi i}{2}(\beta-\alpha-1)\right)}{2^{\alpha+\beta-1}\Gamma(\beta)\Gamma(\alpha+n+1)}\delta_{m,-n-1} + \frac{\pi\Gamma(\alpha+\beta+n)\exp\left(\frac{\pi i}{2}(\beta-\alpha+1)\right)}{2^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta+n+1)}\delta_{m,n+1}. \quad (1)$$

In order to prove it we represent the integral in the form

$$\frac{\Gamma(\alpha+n)\exp\left(\frac{\pi i}{2}(\alpha+\beta-1)\right)}{2^{\alpha+\beta-1}n!\Gamma(\alpha)} \int_0^\pi F(-n,\beta;1-\alpha-n;e^{2i\varphi}) \times e^{-i(m-n-2\alpha+1)\varphi}(1-e^{2i\varphi})^{\alpha+\beta-1}d\varphi \quad (2)$$

(see formula (1) of Section 12.6.4), apply to the hypergeometric function the equality

$$F(-n,n+a+b+1;a+1;z) = \frac{z^{-a}(1-z)^{-b}\Gamma(a+1)}{\Gamma(a+n+1)}\frac{d^n}{dz^n}\left[z^{n+a}(1-z)^{n+b}\right],$$

which follows from the Rodriguez formula for Jacobi polynmials, and integrate by parts. As a result, integral (2) is expressed as a sum of integrals among which only two integrals are different from zero.

The orthogonality relation

$$\int_0^{2\pi} C_n^{(\alpha,\beta)}(e^{i\varphi})C_m^{(\alpha,\beta)}(e^{i\varphi})w(\varphi)d\varphi$$

$$= \left(\frac{c_1}{\beta+n} - \frac{c_2}{\alpha+n}\right) \frac{\pi\Gamma(\alpha+\beta+n)\exp\left(\frac{\pi i}{2}(\beta-\alpha+1)\right)}{2^{\alpha+\beta-2}\Gamma(\alpha)\Gamma(\beta)n!}\delta_{nm}, \quad (3)$$

where $c_1$, $c_2$, $\alpha$, $\beta \in \mathbf{C}$, $\mathrm{Re}(\alpha+\beta) > 0$ and the weight $w(\varphi)$ is given by the formula

$$w(\varphi) = w(\varphi+\pi) \equiv e^{i(\beta-\alpha)\varphi}(c_1 e^{i\varphi} + c_2 e^{-i\varphi})\sin^{\alpha+\beta-1}\varphi,\ 0 \le \varphi < \pi, \quad (4)$$

follows from equality (1) if one expands $C_m^{(\alpha,\beta)}(e^{i\varphi})$ by formula (11) of Section 12.6.3 and uses relation (5) of Section 12.6.4.

If $-c_1 = c_2$, then (3) yields the orthogonality relation

$$\int_0^{2\pi} C_n^{(\alpha,\beta)}(e^{i\varphi})C_n^{\alpha,\beta)}(e^{i\varphi})(1-e^{-2i\varphi})^{\alpha}(1-e^{2i\varphi})^{\beta}d\varphi$$

$$= \frac{2\pi\Gamma(\alpha+\beta+n)(2n+\alpha+\beta)}{n!\Gamma(\alpha)\Gamma(\beta)(\alpha+n)(\beta+n)}\delta_{mn}. \quad (5)$$

With the help of (1) one also proves the orthogonality

$$\int_0^{\pi} \sin^n\varphi\, C_{\ell-n}^{(\alpha+n,\beta+n)}(e^{i\varphi})\sin^m\varphi\, C_{\ell-m}^{(\alpha+m,\beta+m)}(e^{i\varphi})\sin^{\alpha+\beta-2}\varphi\, e^{i(\beta-\alpha)\varphi}d\varphi$$

$$= \frac{\pi\Gamma(\alpha+\beta+2n-1)\Gamma(\alpha+\beta+\ell+n)\exp\left(\frac{i\pi}{2}(\beta-\alpha)\right)}{2^{\alpha+\beta+2n-2}(\ell-n)!\Gamma(\alpha+n)\Gamma(\beta+n)\Gamma(\alpha+\beta+2n)}\delta_{mn}. \quad (6)$$

For $\alpha = \beta = \frac{n-2}{2}$ this equality turns into the orthogonality relation for the matrix elements $t_{m0}^{n\ell}(g_{n-1}(\theta))$ from Section 9.4.2.

By using relation (6) we derive from formula (13) of Section 12.6.5 that

$$\int_0^{\pi}(\cos\theta + i\sin\theta\cos\varphi)^{\ell}C_{\ell-m}^{(\alpha+m,\beta+m)}(e^{i\theta})\sin^{\alpha+\beta+m-2}\theta e^{i(\beta-\alpha)\theta}d\theta$$

$$= \frac{\pi(\ell-m+1)!(2i)^{-m}(-1)^{\ell-m}\Gamma(\alpha+\beta+m-1)\exp\left(\frac{i\pi}{2}(\beta-\alpha)\right)}{(\ell-1)!2^{\alpha+\beta-2}\Gamma(\alpha+m)\Gamma(\beta+m)}$$

$$\times P_m^{(\alpha-1,\beta-1)}(\cos\varphi). \quad (7)$$

**12.6.7. The addition theorem for $C_n^{(\alpha,\beta)}(\zeta)$.** Let $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{w} = (w_1, \ldots, w_n)$, $(\mathbf{z}, \mathbf{w}) = z_1\bar{w}_1 + \ldots + z_n\bar{w}_n$, $|\mathbf{z}| = (\mathbf{z}, \mathbf{z})^{1/2}$. Dunkl [89] has proved the addition theorem for the polynomials $C_n^{(\alpha,\beta)}(\zeta)$ which for $\alpha + \beta = N \in \mathbf{Z}_+$ can be written as

$$C_r^{(\alpha,\beta)}(t - s + i|\mathbf{w}|^2 + i|\mathbf{z}|^2 - 2i(\mathbf{z}, \mathbf{w})) = \frac{(N + r - 1)!}{\Gamma(\alpha)\Gamma(\beta)} \sum_{k+\ell \le r} \frac{i^{\ell-k} 2^{k+\ell}}{k!\ell!} \Gamma(\alpha + \ell)$$
$$\times \Gamma(\beta + k)(N + k + \ell - 1)!(\mathbf{z}, \mathbf{w})^k (\mathbf{w}, \mathbf{z})^\ell F\left(-k, -\ell; -\ell - k - N + 1; \frac{|\mathbf{z}|^2|\mathbf{w}|^2}{|(\mathbf{z}, \mathbf{w})|^2}\right)$$
$$\times \sum_{m=0}^{r-\ell-k} [(N + k + \ell + m - 1)!(N + r - m - 1)!]^{-1} C_m^{(\alpha+\ell,\beta+k)}(t + i|\mathbf{z}|^2)$$
$$C_{r-k-\ell-m}^{(\alpha+\ell,\beta+k)}(-s + i|\mathbf{w}|^2). \quad (1)$$

If $n = 1$, then the hypergeometric function from (1) turns into

$$F(-k, -\ell; -\ell - k - N + 1; 1) = (-1)^{k+N} \frac{(N + k - 1)!(N + \ell - 1)!}{(N - 1)!(N + \ell + k - 1)!},$$

and we have

$$C_r^{(\alpha,\beta)}(t - s + i(|w|^2 + |z|^2 - 2z\bar{w})) = \frac{(N + r - 1)!}{(N - 1)!\Gamma(\alpha)\Gamma(\beta)} \sum_{k+\ell \le r} \frac{i^{\ell-k} 2^{k+\ell}}{k!\ell!} (-1)^{N+k}$$
$$\times \Gamma(\alpha + \ell)\Gamma(\beta + k)(N + k - 1)!\Gamma(N + \ell - 1)(|z|\,|w|)^{2k+2\ell}$$
$$\times \sum_{m=0}^{r-\ell-k} [(N + k + \ell + m - 1)!(N + r - m - 1)!]^{-1} C_m^{(\alpha+\ell,\beta+k)}(t + i|z|^2)$$
$$C_{r-k-\ell-m}^{(\alpha+\ell,\beta+k)}(-s + i|w|^2). \quad (2)$$

Another addition theorem follows from the relation

$$\hat{t}_{00}^{n\ell}(g_{n-1}(\theta)) = \sum_{m=0}^{\ell} \hat{t}_{0m}^{n\ell}(g_{n-2}(\theta_1))\hat{t}_{00}^{n-1,m}(g_{n-2}(\varphi))\hat{t}_{m0}^{n\ell}(g_{n-1}(\theta_2))$$

(see Section 12.6.5), where $\cos\theta = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\varphi$. Since the operators $T^{n\ell}(g_{n-1}(\theta))$ are non-unitary in the space $\mathfrak{D}_{\alpha\beta}^{n-1,\ell}$, then the matrix element $\hat{t}_{0m}^{n\ell}(g_{n-1}(\theta))$ is not expressed in terms of $\hat{t}_{m0}^{n\ell}(g_{n-1}(\theta))$ in such a simple form as in Section 9.4.3. Therefore, one has to evaluate this matrix element by means of the integral representation

$$\hat{t}^{n\ell}_{0m}(g_{n-1}(\theta)) = \left[\frac{m!\Gamma(m+\alpha+\beta-1)(2m+\alpha+\beta-1)}{\Gamma(m+\alpha)\Gamma(m+\beta)}\right]^{1/2} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$
$$\times \int_0^\pi (\cos\theta + i\sin\theta\cos\varphi)^\ell P_m^{(\alpha-1,\beta-1)}\left(\frac{\cos\theta\cos\varphi + i\sin\theta}{\cos\theta + i\sin\theta\cos\varphi}\right)\left(\sin\frac{\varphi}{2}\right)^{2\alpha-1}$$
$$\times \left(\sin\frac{\varphi}{2}\right)^{2\beta-1} d\varphi.$$

To calculate this matrix element one uses the substitution

$$\cos\gamma = \frac{\cos\theta\cos\varphi + i\sin\theta}{\cos\theta + i\sin\theta\cos\varphi},$$

takes into account formula (6′) of Section 12.6.5 for $\cos\theta + i\sin\theta\cos\varphi$ and the analogous formula for $\cos\theta\cos\varphi + i\sin\theta$. We suggest to the reader to carry out the corresponding calculations.

# Chapter 13.
# Representations of Discrete Groups and Special Functions of Discrete Argument

## 13.1. Representations of the Symmetric Group, Krawtchouk and Hahn Polynomials

**13.1.1. Introduction.** In Chapters 6 and 8 we have studied connections of Krawtchouk and Hahn polynomials with representations of the groups $SU(2)$ and $SL(2,\mathbb{R})$. Matrix elements of representations, considered as functions of row number, and Clebsch-Gordan coefficients are expressed respectively in terms of Krawtchouk and Hahn polynomials. But there is another connection of these polynomials with the theory of group representations, namely, with irreducible representations of the symmetric group $S_n$ and of the wreath symmetric group $S(n+1,k)$. Spherical functions of these representations with respect to corresponding subgroups are expressed in terms of Krawtchouk and Hahn polynomials.

We shall further consider connections between the Chevalley groups (which are generalizations of semisimple Lie groups related to the passage to finite fields) and so-called basic special functions. Here the $q$-analogs of Krawtchouk and Hahn polynomials appear. We shall also meet the $q$-analogs of Hahn and Racah polynomials in connection with representations of so-called quantum algebras (which will be considered in the third volume). Finally, special functions, related to representations of groups of matrices over the field of $p$-adic numbers, will be our subject of study.

In what follows we shall use the normalized measure on finite sets, that is, we shall assume that the measure of every element from $X$ is equal to $|X|^{-1}$, where $|X|$ denotes the number of elements in $X$. Therefore, the scalar product of two complex functions on $X$ is defined by the formula

$$(\varphi,\psi)=\frac{1}{|X|}\sum_{x_j\in X}\varphi(x_j)\overline{\psi(x_j)}. \tag{1}$$

We denote by $\mathfrak{L}^2(X)$ the space of complex functions on a finite set $X$ with the scalar product (1).

**13.1.2. Irreducible representations of the symmetric group.** We recall that by the *symmetric group of degree n* one means the group $S_n$ of all mappings of the set $I_n=\{1,2,\dots,n\}$ onto itself. These mappings can be represented as products of *cyclic mappings*, that is, mappings $I_n\to I_n$ transferring $i_1$ into $i_2$, $i_2$ into $i_3$, $\dots, i_k$ into $i_1$ and leaving the rest of the elements fixed. Every cycle $(i_1,i_2,\dots,i_k)$ of the length $k$ is a product of transpositions (cycles of the length 2). Hence, every element from $S_n$ is generated by transpositions of the form $(k,k+1)$. Let $j$ be the number of transpositions appearing in factorization of $g\in S_n$. Then the evenness of

$j$ does not depend on the method of factorization. We set $\operatorname{sign} g = (-1)^j$. Elements $g$, for which $\operatorname{sign} g = 1$, form the *alternating subgroup* $A_n$ of $S_n$.

Let the factorization of $g \in S_n$ consist of cycles of the lengths $n_1, \ldots, n_k$, where $n_1 \geq n_2 \geq \ldots \geq n_k$. Then $n_1 + n_2 + \ldots + n_k = n$ and, therefore, $(n_1, n_2, \ldots, n_k)$ is a partition of $n$. Since conjugate elements of $S_n$ have the factorizations into cycles corresponding to the same partition of $n$, and any two such factorizations are conjugate, then the number of classes of conjugate elements in $S_n$ coincides with the number of partitions of $n$. The characteristic functions of classes of conjugate elements (that is, the functions equal to 1 on elements of the corresponding class and to 0 on other elements) form a basis in the space of functions on $S_n$ which are constant on such classes. On the other hand, according to the results of Section 2.3.11, the characters of irreducible representations of the group $S_n$ also form a basis in this space. In addition, every irreducible representation is uniquely defined by its character. Thus, we have established a one-to-one correspondence between irreducible representations of $S_n$ and partitions of $n$.

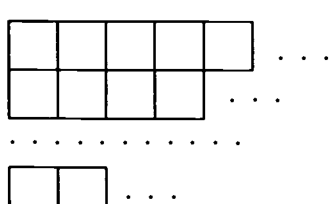
We describe one of the constructions associating with every partition $\lambda = (n_1, n_2, \ldots, n_k)$ an irreducible representation $T_\lambda$ of $S_n$. Let $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ be a splitting of the set $I_n$ into pairwise disjoint subsets such that $|A_j| = n_j$. It is called the *$\lambda$-splitting* of $I_n$. We introduce the variables $x_{ij}$, $1 \leq i, j \leq n$, and associate with the set $A_s = (j_1, \ldots, j_{n_s}) \subset I_n$, $j_1 \leq j_2 \leq \ldots \leq j_{n_s}$, the determinant $\Delta_{A_s}$ of the matrix $(x_{ij_r})$, $1 \leq i, r \leq s$. With the splitting $\mathbf{A}$ we associate the expression $\Delta_{\mathbf{A}} = \Delta_{A_1} \ldots \Delta_{A_k}$ and denote by $\mathfrak{L}_\lambda$ the linear space spanned by all expressions $\Delta_{\mathbf{A}}$, where $\mathbf{A}$ runs over all $\lambda$-splittings of $I_n$. The equality

$$T_\lambda(g)\Delta_{\mathbf{A}}(\mathbf{x}) = \Delta_{\mathbf{A}}(g^{-1}\mathbf{x}),$$

where $\mathbf{x}$ denotes the matrix $(x_{ij})$ and the action of $g$ upon $\mathbf{x}$ is reduced to a permutation of columns, gives the representation $T_\lambda$ of the group $S_n$. One can show that all these representations $T_\lambda$ are irreducible and pairwise nonequivalent. Thus, all irreducible representations of $S_n$ are constructed.

For example, to the partition $\lambda = (1, \ldots, 1)$ there corresponds the identity representation of $S_n$ and to the partition $\lambda = (n)$ there corresponds the alternating representation $T(g) = \operatorname{sign} g$.

The expressions $\Delta_{\mathbf{A}}$ are, in general, linearly dependent. A basis, that is, linearly independent elements, in the space $\mathfrak{L}_\lambda$ can be constructed in the following way. We associate with a partition $\lambda = (n_1, n_2, \ldots, n_k)$ the frame



where the first row contains $n_1$ boxes, the second row contains $n_2$ boxes and so on. We can fill in the boxes of the frame with the integers $1, 2, \ldots, n$. Then this frame is called a pattern. A pattern is said to be *admissible* if the sequences of integers appearing in each row and in each column are increasing as read, respectively, from the left to the right and from the top to the bottom. We split the set $I_n$ into the first, the second, ..., the $k$-th rows of an admissible pattern and associate the expression $\Delta_{\mathbf{A}}$ with this splitting $\mathbf{A}$. The expressions obtained form a linearly independent basis in $\mathfrak{L}_\lambda$.

To every subset $A \subset I_{m+n}$ there corresponds the subgroup $H_A$ of $S_{m+n}$, whose elements leave all $j \in A$ fixed. If $A = \{m+1, m+2, \ldots, m+n\}$, then this subgroup is isomorphic to $S_m$. In what follows $S_m \times S_n$, $m \le n$ will denote the subgroup of $S_{m+n}$, where $S_m$ consists of permutations of $1, 2, \ldots, m$, and $S_n$ consists of permutations of $m+1, m+2, \ldots, m+n$. The homogeneous space $S_{m+n}/S_m \times S_n$ is denoted by $X_{mn}$. Since $H \equiv S_m \times S_n$ is the stationary subgroup of the subset $I_m = \{1, 2, \ldots, m\}$, then $X_{mn}$ can be considered as a collection of subsets, consisting of $m$ elements of $I_{m+n}$.

**13.1.3. Zonal spherical functions on the space $X_{mn}$.** The equality

$$(L(g)f)(x) = f(g^{-1}x), \quad x \in X_{mn},\ g \in S_{m+n},$$

gives the reducible representation of the group $S_{m+n}$ in the space $\mathfrak{L}^2(X_{mn})$. With every $i \in I_{m+n}$ we associate the function $x_i$ on $X_{mn}$ such that $x_i(\xi) = 1$ if $i \in \xi \subset I_{m+n}$, $|\xi| = m$, and $x_i = 0$ otherwise. Thus, $X_{mn}$ is realized as a subset of $\mathbb{R}^{m+n}$ and the action of $S_{m+n}$ is reduced to the permutations of coordinates. We denote by $\omega$ the subset $I_m = \{1, \ldots, m\} \in X_{mn}$ and by $\omega'$ the subset $I_{m+n} \backslash I_m = \{m+1, \ldots, m+n\}$.

With every subset $A = \{i_1, \ldots, i_s\} \subset I_{m+n}$ we associate the function $x_A = x_{i_1} \ldots x_{i_s}$ on $X_{mn}$ and denote by $\mathcal{P}_s$ the space spanned by all $x_A$, $|A| = s$. It is evident that $\dim \mathcal{P}_s = (m+n)!/s!(m+n-s)!$ and $\mathcal{P}_s$ is invariant with respect to left shifts by elements from $S_{m+n}$.

The restriction of the representation $L$ onto $\mathcal{P}_s$ is reducible. We set

$$D = \sum_{i=1}^{m+n} \frac{\partial}{\partial x_i}.$$

The operator $D$ commutes with $L$. We put $V_s = \mathcal{P}_s \cap \ker D$. One can show (see Ref. [84]) that the restriction of $L$ onto $V_s$, $1 \le s \le m$, is the irreducible representation of $S_{m+n}$ given by the partition $(m+n-s, s)$ and

$$\mathfrak{L}(X_{mn}) = \sum_{s=0}^{m} \oplus V_s.$$

The zonal spherical functions $\varphi_s$ on $X_{mn}$ are functions from $V_s$, invariant with respect to shifts from $S_m \times S_n$ and such that $\varphi_s(\omega) = 1$. The $S_m \times S_n$-invariance condition shows that for $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (x_{m+1}, \ldots, x_{m+n})$ we have

$$\varphi_s = \sum_{j=0}^{s} c_j \sigma_j(\mathbf{x}) \sigma_{s-j}(\mathbf{y}),$$

where $\sigma_j(\mathbf{x})$ and $\sigma_{s-j}(\mathbf{y})$ are elementary symmetric functions of degrees $j$ and $s-j$, respectively.

Since

$$D\sigma_j(\mathbf{x}) = (m - j + 1)\sigma_{j-1}(\mathbf{x}),$$
$$D\sigma_{s-j}(\mathbf{y}) = (n - s + j + 1)\sigma_{s-j-1}(\mathbf{y}),$$

then due to the equality $D(\varphi_s(\mathbf{x}, \mathbf{y})) = 0$ we obtain

$$\varphi_s = c \sum_{j=0}^{s} (n - s + 1)_j (m - s + 1)_{s-j} (-1)^j \sigma_j(\mathbf{x}) \sigma_{s-j}(\mathbf{y}).$$

Taking into account the values of the functions $x_i(\xi)$, $\xi \in X_{mn}$,

$$\varphi_s(\xi) = Q_s(v(\xi);\ -n-1,\ -m-1;\ m), \tag{1}$$

where $v(\xi) = m - |I_m \cap \xi|$, $\xi \in X_{mn}$, and $Q_s$ is the Hahn polynomial.

The spherical functions $\varphi_s(\xi)$ are constant on the orbits of the action of the subgroup $H_m \equiv S_m \times S_n$. They are zonal spherical functions on $S_{m+n}/S_m \times S_n$. The spherical functions on $S_{m+n}/S_m \times S_n$, constant on the orbits of the action of the subgroup $H_t = S_t \times S_{m+n-t}$, $1 \le t \le m \le n$, are more general. They are of the form

$$\psi_s(\xi) = \frac{(-t)_s(-n)_s}{(-m)_s(-m-n+t)_s} Q_s(v(\xi); -n-1, -m-1; t). \tag{2}$$

This formula is proved in the same way as formula (1).

**13.1.4. Semidirect products of symmetric groups and their representations.** Let $n$ and $k$ be positive integers and $k > 1$. We denote by $(S_{n+1})^k$ the $k$-fold direct product of the group $S_{n+1}$. The group $S_k$ acts in $(S_{n+1})^k$ as permutations of "coordinates". The semidirect product of $(S_{n+1})^k$ and $S_k$ with respect to this action is denoted by $S(n+1, k)$. Thus, if $g_0, g_0' \in S_k$; $g_j, g_j' \in S_{n+1}$, $1 \le j \le k$, and $g_0' : i \to m_i$, then

$$(g_0; g_1, \ldots, g_k)(g_0'; g_1', \ldots, g_k') = (g_0 g_0'; g_{m_1} g_1', \ldots, g_{m_k} g_k').$$

The group $S(n+1,k)$ is called the ***wreath product*** of the groups $S_{n+1}$ and $S_k$.

Let $S_{n+1}$ act on the set $I_{n+1} = \{0\} \cup I_n$. We denote by $S_n$ the subgroup of $S_{n+1}$ consisting of permutations of elements of $I_n$.

With elements $j$ of $I_{n+1}$ we associate the vertices $a_j$ of the simplex $X_n$ in $\mathbb{R}^n$, inscribed into the unit sphere $S^{n-1}$. We take

$$a_0 = (1, 0, \ldots, 0), \quad a_i = (\beta_n, r_n a'_{i-1}), \quad 1 \le i \le n,$$

where

$$\beta_n = \frac{\beta_{n-1}}{1-\beta_{n-1}}, \quad r_n = \frac{(n^2-1)^{1/2}}{n}$$

and $a'_i$ are the vertices of the simplex $X_{n-1}$. Since the length of $a_i$ is equal to 1, then

$$a_1 = \left(-\frac{1}{n}, \frac{(n^2-1)^{1/2}}{n}, 0, \ldots, 0\right),$$
$$a_i = \left(-\frac{1}{n}, -\frac{(n^2-1)^{1/2}}{n(n-1)}, \ldots\right), \quad 2 \le i \le n.$$

The action of $S_{n+1}$ on $I_{n+1}$ defines the action of $S_{n+1}$ on $X_n$. Since $a_j \in S^{n-1}$, then we obtain the imbedding of $S_{n+1}$ into the orthogonal group $O(n)$.

The subgroup $S_n$ leaves the point $a_0$ of the simplex $X_n$ fixed. Therefore, $X_n$ can be identified with $S_{n+1}/S_n$.

In the sequel we shall denote the coordinate functions in $\mathbb{R}^n$ by $z_1, z_2, \ldots, z_n$. It is easy to verify that the functions $1$, $z_i$, $1 \le i \le n$, form an orthogonal basis for $\mathfrak{L}^2(X_n)$. Moreover,

$$\|1\| = 1, \quad \|z_i\|^2 = \frac{1}{n}, \quad 1 \le i \le n.$$

The correspondence $I_{n+1} \to X_n$, constructed above, allows us to consider $X_n^k$ instead of $(I_{n+1})^k$. To the point $\mathbf{0} = (0, \ldots, 0)$ from $(I_{n+1})^k$ there corresponds the point $(a_0, \ldots, a_0)$ from $X_n^k$. The group $S(n+1,k)$ acts on $X_n^k$. If $g = (g_0; g_1, \ldots, g_k)$ and $\mathbf{b} = (b_1, \ldots, b_k) \in X_n^k$, then

$$g\mathbf{b} = g_0(g_1 b_1, \ldots, g_k b_k).$$

The group $S(n+1,k)$ acts transitively on $X_n^k$ and the stationary subgroup of the point $(a_0, \ldots, a_0)$ coincides with $S(n,k)$, that is, with the wreath product of the subgroups $S_n$ and $S_k$. Therefore,

$$X_n^k \sim S(n+1,k)/S(n,k).$$

If $\mathbf{b} = (b_1, \ldots, b_k) \in X_n^k$, then $w(\mathbf{b})$ will denote the integer $|\{i \mid b_i \neq a_0\}|$. Set $z_{ij} = z_i(b_j)$. With every $\mathbf{u} = (u_1, \ldots, u_k) \in (I_{n+1})^k$ we associate the set $c(\mathbf{u}) = \{j \mid u_j \neq 0\}$ and introduce the monomials

$$z_{\mathbf{u}} = \prod_{j \in c(\mathbf{u})} z_{u_j j} = \prod_{j \in c(\mathbf{u})} z_{u_j}(b_j).$$

We recommend that the reader verifies the functions $z_{\mathbf{u}}$ form an orthonormal basis in $\mathfrak{L}^2(X_n^k)$ and

$$\|z_{\mathbf{u}}\|^2 = n^{-|c(\mathbf{u})|}.$$

The formula

$$(T(g)f)(\mathbf{b}) = f(g^{-1}\mathbf{b}), \quad f \in \mathfrak{L}^2(X_n^k),$$

gives a unitary representation of $S(n+1,k)$ in $\mathfrak{L}^2(X_n^k)$. This representation is reducible. Linear combinations of the monomials $z_{\mathbf{u}}$, for which $|c(\mathbf{u})| = s$, form the subspace $\mathfrak{P}_s$ in $\mathfrak{L}^2(X_n^k)$ of dimensionality $k!n^s/s!(n-s)!$. This subspace is invariant with respect to the representation $T$. The restriction of $T$ onto $\mathfrak{P}_s$ is denoted by $T_s$. One can easily prove (see Ref. [84]) the following statements.

*For every $s$, $0 \leq s \leq k$, the representation $T_s$ of $S(n+1,k)$ is irreducible. The zonal spherical function $\varphi_s$ of $T_s$ has the form*

$$\varphi_s(\mathbf{b}) = \frac{s!(k-s)!}{k!}\sigma_s(z_{u_1}, \ldots, z_{u_k}),$$

*where $\sigma_s$ is the elementary symmetric function of degree $s$ in $z_{u_1}, \ldots, z_{u_k}$.* Points $\mathbf{b}$ and $\mathbf{b}'$ from $X_n^k$ belong to the same orbit with respect to the subgroup $H = S(n,k)$, if $w(\mathbf{b}) = w(\mathbf{b}')$.

This statement implies that functions $f$, invariant with respect to the action of $H$, depend on $w(\mathbf{b})$ only, that is, $f(\mathbf{b}) = \tilde{f}(w(\mathbf{b}))$. Therefore,

$$\sum_{\mathbf{b} \in X_n^k} f(\mathbf{b}) = \sum_{s=0}^{k} \frac{k!n^s}{s!(k-s)!}\tilde{f}(s).$$

By applying this equality to the function $\varphi_s(\mathbf{b})$, we obtain

$$\varphi_s(\mathbf{b}) = K_s\left(w(\mathbf{b}); \frac{n}{n+1}; k\right), \quad 0 \leq s \leq k, \ \mathbf{b} \in X_n^k, \tag{1}$$

where $K_s$ is the Krawtchouk polynomial of degree $s$.

**13.1.5. Homogeneous graphs and special functions.** The homogeneous spaces $X_{mn}$ and $X_n^k$ considered above belong to the class of spaces represented by homogeneous graphs. By a *graph* we shall mean a finite set $X$ of points, some of which are joined by arcs (to every pair $x, y \in X$ there corresponds not more than one arc). A graph is said to be *connected* if any two points can be joined by a path consisting of arcs. We associate with every arc the length 1. By the distance $d(x, y)$ between points $x$ and $y$ we mean the least of lengths of the paths joining these points.

Assume that the action of a group $G$ on the connected graph $X$, metrized in the above way, is given. A graph $X$ is said to be ***homogeneous*** with respect to this action if for any pairs $(x_1, y_1)$ and $(x_2, y_2)$ of points from $X$ with $d(x_1, x_2) = d(y_1, h_2)$ there is $g \in G$ such that $y_1 = gx_1$, $y_2 = gx_2$. By setting $x_1 = x_2$, $y_1 = y_2$ we conclude that if a graph $X$ is homogeneous, then $G$ acts on $X$ transitively and $X \sim G/H$.

We fix a point $p \in X$ and denote by $H$ the stationary subgroup of $p$. Let $N$ be the diameter of the set $X$, that is, the maximal distance between points of this set. If the graph $X$ is homogeneous with respect to $G$, then $X$ splits into $H$-orbits (spheres) $\Omega_0, \Omega_1, \ldots, \Omega_N$, where

$$\Omega_k = \{x \in X \mid d(p, x) = k\}.$$

In the sequel we shall show that the left quasi-regular representation $L$ of the group $G$ in $\mathfrak{L}^2(X)$ is the direct sum of $N + 1$ irreducible pairwise nonequivalent representations (similar to the decomposition of the quasi-regular representation of the group $SO(n)$ in $\mathfrak{L}^2(S^{n-1})$ into the direct sum of irreducible nonequivalent representations $T^{n\ell}$).

In order to construct this decomposition we regard the algebra $\mathfrak{a}$ of square complex matrices $A$ of order $|X|$, which commute with the matrices giving the action of $G$ on $X$ ($X$ and $G$ can be imbedded in a natural way, respectively, into $\mathbb{R}^{|X|}$ and $O(|X|)$). The results of Section 2.2.10 imply that if $\mathfrak{a}$ is a commutative algebra, then irreducible components of the representation $L$ are pairwise nonequivalent. We show that $\mathfrak{a}$ is commutative. Since $X$ is a homogeneous graph, then one can easily check that $\mathfrak{a}$ is generated by the matrices $D_0, D_1, \ldots, D_N$, where

$$D_i(x, y) = \begin{cases} 1 & \text{for } d(x, y) = i, \\ 0 & \text{for } d(x, y) \neq i, \ \ x, y \in X. \end{cases} \tag{1}$$

The matrices $D_i$ satisfy the recurrence relations

$$D_1 D_i = c_{i+1} D_{i+1} + a_i D_i + b_{i-1} D_{i-1}, \quad 0 \le i \le N, \tag{2}$$

where the numbers $a_i$, $b_i$, $c_i$ are defined as

$$\begin{aligned} a_i &= |\{z \in X \mid d(x, z) = 1, \ d(y, z) = i\}|, \\ b_i &= |\{z \in X \mid d(x, z) = 1, \ d(y, z) = i + 1\}|, \\ c_i &= |\{z \in X \mid d(x, z) = 1, \ d(y, z) = i - 1\}|. \end{aligned}$$

Here $x$ and $y$ are fixed elements from $X$ such that $d(x,y) = i$. (We recommend to the reader to prove relations (2)). Since $c_i \neq 0$, $i = 1, 2, \dots, N$, then (2) imply that the matrices $D_i$ are polynomials of degree $i$ in $D_1$: $D_i = p_i(D_1)$. It means that $\mathfrak{a}$ is commutative. Thus, we have proved that $L$ decomposes into the sum of pairwise nonequivalent irreducible components.

Let

$$\mathfrak{L}^2(X) = V_0 \otimes V_1 \otimes \dots \otimes V_n \tag{3}$$

be the decomposition of $\mathfrak{L}^2(X)$ into the sum of $G$-irreducible subspaces. Then in every of the subspaces $V_j$ there exists the unique function $f_j$, invariant with respect to the subgroup $H$ and such that $f_j(p) = 1$. In addition, every other $H$-invariant function on $X$ is a linear combination of $f_j$ (we recommend to the reader to prove these statements by means of the results of Section 2.3). The functions $f_j$ are zonal spherical functions of the irreducible representations realized in the spaces $V_j$.

Because of the $H$-invariance of $f_j$ one can consider them as functions of the spheres $\Omega_k$:

$$f_j(x) = f_j(\Omega_k) \quad \text{if} \quad x \in \Omega_k.$$

Since $X$ is split into $N+1$ spheres, then the subspace $\mathfrak{L}^2_H(X)$ of $H$-invariant functions from $\mathfrak{L}^2(X)$ is of dimensionality $N+1$. Therefore, $n = N$ in (3).

From Theorem 2 of Section 2.3.9 we derive the following orthogonality relation for the zonal spherical functions:

$$\frac{1}{|X|} \sum_{k=0}^{N} f_i(\Omega_k) \overline{f_j(\Omega_k)} |\Omega_k| = \delta_{ij} (\dim V_j)^{-1}. \tag{4}$$

It follows from here that the matrix with the entries

$$M_{ik} = f_i(\Omega_k) \left( \frac{|X|}{(\dim V_i)|\Omega_k|} \right)^{-1/2}$$

has orthonormal rows. But then it has orthonormal columns:

$$\frac{1}{|X|} \sum_{i=0}^{N} f_i(\Omega_k) \overline{f_i(\Omega_j)} (\dim V_i) = \delta_{kj} |\Omega_k|^{-1}. \tag{5}$$

We now find the connection between the spherical functions $f_j$ and the operators $D_i$ from $\mathfrak{a}$. It is clear that $V_j$ are eigenspaces of $D_i$:

$$V_j = \{ f \in \mathfrak{L}^2(X) \mid D_i f = \lambda_{ij} f \}.$$

Since $D_i = p_i(D_1)$, then $\lambda_{ij} = p_i(\lambda_{1j})$. Therefore, the numbers $\lambda_j \equiv \lambda_{1j}$ are different. From relation (2) with $i = N$ we obtain

$$\lambda_j p_N(\lambda_j) = a_N p_N(\lambda_j) + b_{N-1} p_{N-1}(\lambda_j).$$

Since for the spherical functions $f_j$ we have

$$D_1 f_j = \lambda_j f_j,$$

then by considering this equality on $x \in \Omega_k$ and by using it in relation (2) we derive the recurrence relation

$$\lambda_j f_j(\Omega_k) = \gamma_k f_j(\Omega_{k+1}) + \alpha_k f_j(\Omega_k) + \beta_k f_j(\Omega_{k-1}), \tag{6}$$

where

$$\begin{aligned} \gamma_k &= |\{z \in X \mid d(x,z) = 1,\ d(z,p) = k+1\}|, \\ \alpha_k &= |\{z \in X \mid d(x,z) = 1,\ d(z,p) = k\}|, \\ \beta_k &= |\{z \in X \mid d(x,z) = 1,\ d(z,p) = k-1\}|. \end{aligned}$$

Here $x \in \Omega_k$ and $p$ denotes the point whose stabilizer coincides with $H$.

It follows from (6) that $f_j(\Omega_k) \equiv f_{\lambda_j}(\Omega_k)$ with a fixed $k$ is a polynomial of degree $k$ in $\lambda_j$. By comparing (2) and (6) we have

$$\begin{aligned} |\Omega_{i+1}|c_{i+1} &= |\Omega_i|\gamma_i, \\ |\Omega_i|a_i &= |\Omega_i|\alpha_i, \\ |\Omega_{i-1}|b_{i-1} &= |\Omega_i|\beta_i. \end{aligned}$$

Hence

$$|\Omega_k| f_{\lambda_j}(\Omega_k) = p_k(\lambda_j). \tag{7}$$

Thus, we have proved the following statement.

*Let $f_{\lambda_i}$ be the spherical function from the space $V_i$ and let $\Omega_k$ be the sphere of radius $k$ with center $p$. Then there exists a polynomial $p_k(x)$ of degree $k$ such that $|\Omega_k| f_{\lambda_i}(\Omega_k) = p_k(\lambda_i)$. The polynomials $p_k(x)$, $0 \le k \le N$, are pairwise orthogonal. The orthogonality relations for $p_k(\lambda_i)$ are given by relation* (5).

The equalities (2) and (6) imply the recurrence relations for $p_k(x)$. These relations can be interpreted as second order difference equations for the spherical functions. In many cases there exist polynomials $q_i(y)$, $y \in \{0, 1, \ldots, N\}$, of degree $i$ such that $f_{\lambda_i}(\Omega_k) = q_i(k)$. Equality (4) provides the orthogonality relation for these polynomials.

As it has been indicated, the spaces $X_{mn}$ and $X_n^k$ from preceding sections are homogeneous graphs. Subsets from $I_{m+n}$, consisting of $m$ points, are elements of $X_{mn}$. If two of such subsets differ in one point only, we connect them by an arc. Then we obtain the distance

$$d(\xi, \eta) = \frac{1}{2}|(\xi \backslash \eta) \cup (\eta \backslash \xi)|,$$

called the ***Johnson metric***. Evaluating the spherical functions for this homogeneous graph, we obtain formula (1) of Section 13.1.3.

By the distance between two finite sequences of the space $X_n^k \equiv \{(b_1, \ldots, b_k)\}$, one means the number of positions for which their coordinates are different:

$$d(x, y) = |\{i \mid x_i \neq y_i\}|.$$

This metric is called the *Hamming metric*. In this case Krawtchouk polynomials (1) of Section 13.1.4 are spherical functions.

We suggest that the reader writes out the recurrence formulas for Krawtchouk and Hahn polynomials following from formula (6).

**13.1.6. Addition theorems for Krawtchouk and Hahn polynomials.** The derivation of addition theorems for zonal spherical functions of the homogeneous space $X = G/H$ is reduced to the following procedure. We take the irreducible representation $T$ corresponding to the spherical function $\varphi$, restrict it onto the subgroup $H$ and decompose the resulting representation into irreducible components. We denote by $V$ the carrier space of $T$. Let the decomposition of $V$ into irreducible subspaces be of the form

$$V = \sum_\alpha \oplus V_\alpha,$$

and let the identity representation of $H$ appears in the decomposition only once. Let $P_\alpha$ be the orthogonal projection from $V$ onto $V_\alpha$. Then $P_\alpha$ commutes with the restrictionof $T$ onto $H$. Further, let $p$ be the point from $X$, invariant with respect to the action of $H$, and let $\varphi(p) = 1$. The addition theorem follows from the equality

$$\varphi(gx) = \sum_\alpha (P_\alpha(L(g^{-1})\varphi))(x), \; g \in G,$$

where $L(g^{-1})$ is an operator of the left quasi-regular representation of $G$. It is obtained, if we put $x - hg_1p$, $g_1 \in G$, $h \in H$:

$$\varphi(g_2hg_1p) = \sum_\alpha L(h^{-1})(P_\alpha(L(g_2^{-1})\varphi))(g_1p).$$

Thus, to derive the addition theorem it is necessary to find the expressions $(P_\alpha(L(g_2^{-1})\varphi))(g_1p)$. We choose an orthonormal basis $\{e_i \mid 1 \leq i \leq N\}$ in the space $V$, agreeing with the decomposition $V = \sum_\alpha \oplus V_\alpha$. Let $e_1$ be invariant with respect to the subgroup $H$ and let $t_{ij}(g) = (T(g)e_j, e_i)$. Then $\varphi(gp) = t_{11}(g)$, $g \in G$, and $\{t_{i1}(g) \mid 1 \leq i \leq N\}$ can be considered as an orthonormal basis in $V$.

We fix $\alpha$ and denote by $I$ the set of indices enumerating the basis elements in $V_\alpha$. Then every $f \in V$ is represented in the form

$$f = \sum_{j=1}^{N} c_j t_{j1}(g) \quad \text{and} \quad P_\alpha f = \sum_{j\in I} c_j t_{j1}(g).$$

Apply these statements to $L(g^{-1})\varphi$ and introduce the function

$$f_\alpha(g_1, g_2) = (P_\alpha(L(g_1^{-1})\varphi))(g_2 p), \quad g_1, g_2 \in G.$$

Then

a) $$f_\alpha(g_1, g_2) = \sum_{j\in I} t_{1j}(g_1) t_{j1}(g_2) = \sum_{j\in I} \overline{t_{j1}(g_1^{-1})} t_{j1}(g_2),$$

b) $$\sum_{g_2\in G} |f_\alpha(g_1, g_2)|^2 = \sum_{j\in I} |t_{j1}(g_1^{-1})|^2/N = f_\alpha(g_1, g_1^{-1})/N,$$

c) $$f_\alpha(h_1 g_1, g_2 h_2) = f_\alpha(g_1, g_2), \quad h_1, h_2 \in H,$$

d) $$f_\alpha(g_1, h g_2) = f_\alpha(g_1 h, g_2), \quad h \in H,$$

e) $$f_\alpha(g_1, k g_2) = f_\alpha(g_1, g_2), \quad k \in g_1^{-1} H g_1 \cap H.$$

These relations follow from the Peter-Weyl theorem, from the facts that $T$ is unitary and that $P_\alpha$ commutes with the restriction of $T$ onto $H$, and from the definition of $f_\alpha(g_1, g_2)$.

Making use of these relations, one sometimes can find $f_\alpha$. We assume that for every $g_1 \in G$ there exists a uniquely defined (up to a scalar factor) function $F(g_1, x) \in V_\alpha$, invariant with respect to the action of elements from $g_1^{-1} H g_1 \cap H$. Then e) implies that there exists a number $F_1(g_1)$, depending on $g_1$, such that

$$f_\alpha(g_1, g_2) = F_1(g_1) F(g_1, g_2 p). \tag{1}$$

Summing squares of the moduli of equality (1) with respect to $g_2$ and taking into account relation b), we derive the equation for $F_1$ which gives

$$f_\alpha(g_1, g_2) = \frac{\overline{F(g_1, g_1^{-1} p)} F(g_1, g_2 p)}{(N/|H||/|G|) \sum_{x\in X} |F(g_1, x)|^2}.$$

By applying this method Ch. Dunkl has obtained the addition theorem for Krawtchouk polynomials in the case when $G = S(n+1, k)$, $H = S(n, k)$ and the addition theorem for Hahn polynomials in the case when $G = S_{m+n}$, $H = S_m \times S_n$. We write down the results. The detailed proofs can be found in Dunkl's papers [84], [87].

Let $0 \le x \le y \le N$, $0 \le s \le \min(x, N-y)$, $0 \le r \le x-s$, $0 \le n \le N$. Then the addition theorem for Krawtchouk polynomials has the form

$$K_n(2s+r+y-x;p;N) = \sum_{k=0}^{\min(n,x)} \sum_{\ell=\max(0,k+y-N)}^{k} C_{x-\ell}^{k-\ell} C_{x-s}^{\ell} (-1)^{k+\ell}$$
$$\times \frac{(-n)_k (y-N)_{k-\ell} (n-N)_{k-\ell}}{(-N)_{2k-\ell}(k-N-1)_{k-\ell}} (2p-1)^{\ell} p^{-2k} K_{n-k}(x-k;p;N-2k+\ell)$$
$$\times K_{n-k}(y-k;p;N-2k+\ell) K_{\ell}\left(r; \frac{2p-1}{p}; x-s\right) {}_3F_2\left(\begin{matrix} \ell-k, k-N-1, -s \\ y-N, \ell-s \end{matrix} \,\middle|\, 1\right). \quad (2)$$

At present it is not clear whether this addition theorem can be obtained from formula (5) of Section 8.5.3.

For Hahn polynomials the addition theorem is of the form

$$Q_k(v+w-x-y; -a-1, -b-1, b) = \sum_{m,n} c_{mnk}(a,b)$$
$$\times E_m(x; w, b-w, v) E_n(y; w, a-w, v)$$
$$\times E_{k-m-n}(w-n; a-2n, b-2m, b-m-n)$$
$$\times E_{k-m-n}(v-n, a-2n, b-2m, b-m-n), \quad (3)$$

where $a$, $b$, $k$ are integers such that $0 \le k \le b \le a$; $v$, $w$, $x$, $y$ take integral values for which

$$0 \le v, w \le b, \qquad \max(0, w+v-b) \le x \le \min(v, w),$$
$$\max(0, w+v+a) \le y \le \min(v, w),$$

the coefficients $c_{mnk}(a,b)$ are given by the formula

$$c_{mnk}(a,b) = \frac{(-1)^{m+n}(-k)_{m+n}(k-a-b-1)_{m+n}(b-2m+1)(a-2n+1)}{n!m!(-a)_k(-b)_k(n-a)_{k-m}(m-b)_{k-n}(b-m+1)(a-n+1)}$$

and the summation is over the integral values of $m$ and $n$ such that

$$0 \le m+n \le k, \quad 0 \le m \le \frac{b}{2}, \quad 0 \le n \le \frac{a}{2}, \quad k-a \le m-n \le b-k.$$

The function $E_m(x; a, b, c)$ from (3) is defined by the formula

$$E_n(x; a, b, c) = (-1)^m (-a)_m (x-c)_m {}_3F_2\left(\begin{matrix} b-m+1, -x, -m \\ -a, c-x-m+1 \end{matrix} \,\middle|\, 1\right). \quad (4)$$

It is expressed in terms of the Hahn polynomials:

$$E_m(x;a,b,c) = (-1)^m(-a)_m(-c)_m Q_m(x;-a-1,-b-1,c)$$

if $c \le b,\ c \le a$,

$$E_m(x;a,b,c) = (-1)^m(-a)_m(-c)_m Q_m(x;-c-1,-(a+b-c)-1,a)$$

if $a \le c \le b$,

$$E_m(x;a,b,c) = (-1)^m(-b)_m(c-a-b)_m Q_m(x+b-c;-b-1,-a-1,a+b-c)$$

if $b \le c \le a$,

$$E_m(x;a,b,c) = (-1)^m(-b)_m(c-a-b)_m Q_m(x+b-c;-(a+b-c)-1,-c-1,b)$$

if $a \le c,\ b \le c$.

The orthogonality relation for $E_m(x;a,b,c)$, $0 \le m,n \le \frac{a+b}{2}$, is of the form

$$\sum_{x=\max(0,c-b)}^{\min(a,c)} C_a^x C_b^{c-x} E_m(x;a,b,c)E_n(x;a,b,c)$$
$$= \frac{m!(a+b-m)!(a+b-m+1)(-a)_m(-b)_m(-c)_m(c-a-b)_m}{c!(a+b-c)!(a+b-2m+1)}\delta_{mn}. \quad (5)$$

By using this orthogonality relation we derive from (3) the product formula

$$Q_k(v;-a-1,-b-1,b)Q_k(w;-a-1,-b-1,b)$$
$$= \sum_x \sum_y K(x,y,v,w;a,b)Q_k(v+w-x-y;-a-1,-b-1,b), \quad (6)$$

where the summation is over integral $x$ and $y$ for which

$$\max(0,w+v-b) \le x \le \min(w,v),\ \max(0,w+v-a) \le y \le \min(w,v)$$

if $b \le a \le 2b-1$, and

$$\max(0,w+v-b) \le x \le \min(w,v),\ 0 \le y \le \min(w,v)$$

otherwise.

By using spherical functions (2) of Section 13.1.3 Ch. Dunkl [86] proved the generalization of formula (6):

$$Q_n(x;-(N-t)-1,-t-1,s)Q_n(y;-(N-t)-1,-t-1,s)$$
$$= \sum_{w=0}^{s} F(x,y,w)Q_n(w;-(N-t)-1,-t-1,s), \quad (7)$$

where

$$F(x,y,w)=\frac{(-t)_n(s-N)_n}{(-s)_n(t-N)_n}\sum_{\alpha,\gamma}\frac{C^{\alpha}_{t-s}C^{\gamma}_{y}C^{s-x-\alpha-\gamma}_{s-y}C^{x-w+\alpha+\gamma}_{y+t-s}C^{w-\gamma}_{N-t-y}}{C^{s-x}_{t}C^{x+\alpha}_{N-s}} \tag{7'}$$

and the summation is over all integral $\alpha$ and $\gamma$ for which the factorials have meanings.

R. Askey and G. Gasper [19] proved an analog of formula (7) for Krawtchouk and Meixner polynomials. For Krawtchouk polynomials it has the form

$$K_n(x;p,N)K_m(x;p,N)=\sum_{i=0}^{N}F(i,m,n)K_i(x;p,N), \tag{8}$$

where

$$F(i,m,n)=\frac{m!n!(N-m)!(N-n)!}{N!p^{m+n}(1-p)^i} \times\sum_{j\geq 0}\frac{p^j(1-p)^j(2p-1)^{i+m+n-2j}}{(j-i)!(j-m)!(j-n)!(i+m+n-2j)!(N-j)!}, \tag{8'}$$

and for Meixner polynomials it has the form

$$M_n(x;\beta,c)M_k(x;\beta,c)=\sum_{i=0}^{n+k}A(i,n,k)M_i(x;\beta,c), \tag{9}$$

where

$$A(i,n,k)=\frac{n!k!(-1)^{n+k+i}(1+c)^{n+k+i}}{(\beta)_i c^{n+k}} \times\sum_{j\geq 0}\frac{c^j(1+c)^{-2j}(\beta)_j}{(j-n)!(j-k)!(j-i)!(n+k+i-2j)!}. \tag{9'}$$

Formula (8) can be obtained from equality (2) of Section 8.5.3.

By means of the formula[1]

$$\lim_{N\to\infty}(-N/a)^n K_n\left(N-u;1-\frac{a}{N},N\right)=c_n(u;a),$$

[1] Because of noncorrect type-setting there is a mistake in the definitin (1), Section 5.5.8, of Charlier polynomials. This formula must look as

$$c_n(x;a)=(-a)^{-n}n!L_n^{x-n}(a).$$

where $c_n(u;a)$ is the Charlier polynomial, we derive from (2) that

$$c_n(u+v;a)=\sum_{\ell=0}^{n}\frac{n!\Gamma(u+1)(-a)^{n-\ell}}{\ell!(n-\ell)!\Gamma(u+n-\ell+1)}c_\ell(v;a). \tag{10}$$

**13.1.7. Functions, invariant with respect to $S_n \times S_m \times S_k$, and orthogonal polynomials.** Let $N = a+b+c$, where $a$, $b$, $c$ are non-negative integers. Set

$$\eta_1=\{1,2,\ldots,a\},\ \eta_2=\{a+1,\ldots,a+b\},\ \eta_3=\{a+b+1,\ldots,a+b+c\}.$$

We shall consider the group $S_N$ acting both in $X=\{1,2,\ldots,N\}$ and in $\mathbb{R}^N$ by coordinate permutations (the $i$-th coordinate in $\mathbf{x}g$ coincides with the $(ig^{-1})$-th coordinate in $\mathbf{x}$).

As in Section 13.1.3, $\mathcal{P}_r$ denotes the linear span of $\{x_{i_1}\ldots x_{i_r} \mid 1\le i_1 < \ldots < i_r \le N\}$ and $V_r$ denotes the set $\mathcal{P}_r \cap \ker D$, where $D=\sum_{i=1}^{N}\partial/\partial x_i$. The action of $S_N$ in $V_r$ gives the irreducible representation of $S_N$ corresponding to the partition $(N-r,r)$ of $N$. We denote by $S_\eta$ the subgroup of $S_N$ leaving points of the subset $X\setminus\eta$ fixed and set $H=S_{\eta_1}\times S_{\eta_2}\times S_{\eta_3}$.

Every $H$-invariant function $p$ of $\mathcal{P}_r$ is a linear combination of linearly independent functions of the fo rm $\sigma_{i_1}(\eta_1)\sigma_{i_2}(\eta_2)\sigma_{i_3}(\eta_3)$, where

$$i_1+i_2+i_3=r,\quad 0\le i_1\le a,\quad 0\le i_2\le b,\quad 0\le i_3\le c$$

and $\sigma_i(\eta)$ are the elementary symmetric functions of degree $i$ in $x_j$, $j\in\eta$. In other words,

$$p=\sum_{i_1,i_2} f(i_1,i_2)\sigma_{i_1}(\eta_1)\sigma_{i_2}(\eta_2)\sigma_{r-i_1-i_2}(\eta_3), \tag{1}$$

where the summation is over the set

$$\mathcal{D}_r(a,b,c)=\{(i_1,i_2)\ \big|\ 0\le i_1\le a,\ 0\le i_2\le b,\ r-c\le i_1+i_2\le r\}.$$

Since $D\sigma_i(\eta)=(|\eta|-i+1)\sigma_{i-1}(\eta)$, then the equality $Dp=0$ is fulfilled if and only if the function $f$ satisfies the equation

$$(x-a)f(x+1,y)+(y-b)f(x,y+1)=(c+x+y-r+1)f(x,y), \tag{1'}$$

where $0\le x\le a$, $0\le y\le b$, $r-c-1\le x+y\le r-1$ and where $f(a+1,y)$, $f(x,b+1)$, $f(x,r-c-x-1)$ are set to be zero.

We denote by $W_r\equiv W_r(a,b,c)$ the linear space of solutions of equation (1′) in $\mathcal{D}_r(a,b,c)$. One can show that for

$$a+b\ge r,\quad b+c\ge r,\quad a+c\ge r,\quad a+b+c\ge 2r$$

the dimensionality of $W_r$ is equal to the least of the numbers

$$r,\ a,\ b,\ c,\ a+b-r,\ b+c-r,\ a+c-r,\ a+b+c-2r+1$$

and otherwise it is equal to zero. Note that $\mathcal{D}_r(a,b,c)$ is not empty for $a+b+c \geq r$.

If $a'$, $b'$, $c'$ is a permutation of the numbers $a$, $b$, $c$, then there is the natural isomorphism between $W_r(a,b,c)$ and $W_r(a',b',c')$. For example, if $h \in W_r(b,a,c)$, then $f \in W_r(a,b,c)$, where $f(x,y) = h(y,x)$, and if $h \in W_r(c,b,a)$, then $f \in W_r(a,b,c)$, where $f(x,y) = h(r-x-y,y)$.

If $c \geq r$, then

$$\mathcal{D}_r(a,b,c) = \bigcup_{j=\max(0,r-b)}^{\min(r,a)} R_j, \text{ where } R_j = \{(x,y) \mid 0 \leq x \leq j,\ 0 \leq y \leq r-j\}.$$

A simple verification shows that the function

$$f_j(x,y) = \binom{r-x-y}{j-x} \frac{(x-a)_{j-x}(y-b)_{r-j+y}}{(-1)^{r-x-y}(-c)_{r-x-y}} \tag{2}$$

$$(\max(0,r-b) \leq j \leq \min(r,a))$$

satisfies equation (1′) for $x+y \geq r-c$, is equal to 1 at the vertex $(j, r-j)$ of the rectangle $R_j$ and vanishes outside of this rectangle. It follows from here that for $f \in W_r$ one has the equality

$$f(x,y) = \sum_{j=\max(x,r-b)}^{\min(a,r-y)} f(j,r-j) f_j(x,y), \tag{3}$$

where $f_j(x,y)$ is given by (2).

Let us introduce the $S_N$-invariant "integral" in $V_r$. For this we imbed $X$ into $\Omega = \{(c_1,c_2,\ldots,c_N) \mid c_i = \pm 1,\ 1 \leq i \leq N\}$ in a natural way, extend functions $f$ from $V_r$ onto $\Omega$ by setting $f(x) = 0$ for $x \in \Omega \setminus X$ and put

$$\int_\Omega f(x)dx = \sum_{x\in\Omega} f(x).$$

We define the scalar product

$$(f,g)_\Omega = \int_\Omega f(x)\overline{g(x)}dx.$$

Both $\Omega$ and the image of $X$ in $\Omega$ are $S_N$-invariant. Therefore, this scalar product is $S_N$-invariant also.

The set $\Omega$ can be identified with the $N$-fold direct sum of the cyclic group $\mathbf{Z}_2$ of the second order. The monomials $x_{i_1}\dots x_{i_r}$, $1\le i_1<\dots<i_r\le N$, are characters of this group and, hence, are pairwise orthogonal. In addition,

$$\|\sigma_{i_1}(\eta_1)\sigma_{i_2}(\eta_2)\sigma_{i_3}(\eta_3)\|_\Omega^2 = \binom{a}{i_1}\binom{b}{i_2}\binom{c}{i_3}. \tag{4}$$

We choose in $V_r$ a basis corresponding to the decomposition of the restriction of the representation of $S_N$ in $V_r$ onto $S_{\eta_1}\times S_\eta$, $\eta=\eta_2\cup\eta_3$. Because of the invariance of the scalar product in $V_r$ we can consider that this basis is orthogonal. By virtue of formula (4) the orthogonality of functions (1) from $V_r$ means the orthogonality of corresponding functions $f(x,y)$ from $W_r$ with respect to the scalar product

$$\langle f,h\rangle = \sum_{(x,y)\in\mathfrak{D}_r}\binom{a}{x}\binom{b}{y}\binom{c}{r-x-y}f(x,y)\overline{h(x,y)}. \tag{4'}$$

We find solutions of (1′) in the form

$$f(x,y)=g(x)E_m(y;b,c,r-x),$$

where $g(x)$ and $m$ must be still defined (this form of solution is suggested by the weight function in (4′) if one compares it with the weight function in formula (5) of Section 13.1.6). Substituting this expression into (1′) and making use of the recurrence relation

$$(a-x)E_m(x+1;a,b,c+1)+(b-c+x)E_m(x;a,b,c+1)=(a+b-c-m)E_m(x;a,b,c)$$

(which follows from the recurrence relation for Hahn polynomials), we derive

$$g(x)(b+c+x-r-m+1)E_m(y;b,c,r-x-1)=g(x+1)(x-a)E_m(y;b,c,r-x-1), \tag{5}$$

where

$$\max(0,r-c-1)\le x+y\le r-1,\ 0\le x\le a,\ 0\le y\le b.$$

Analyzing (5), we conclude that this equation has non-zero solutions if and only if

$$\max(0,r-a)\le m\le \min(b,c,r,b+c-r).$$

In this range of values of the index $m$ we have

$$g(x)=\frac{(b+c-r-m+1)_x}{(-a)_x}g(0),\quad 0\le x\le r-m\le a.$$

Setting $g(0) = (r-m)!(-a)_{r-m}$ and taking into account that

$$E_c(x;a,b,c) = \frac{(b-c+1)_x}{(-a)_x} c!(-a)_c$$

for any integral values of $a$, $b$, $c$, we obtain the polynomial solution

$$g(x) = E_{r-m}(x;a,b+c-2m,r-m).$$

Hence, the basis in $W_r$ consists of the functions

$$\varphi_m(x,y) = E_{r-m}(x;a,b+c-2m,r-m)E_m(y;b,c,r-x), \tag{6}$$

where $m$ satisfies the above inequality, and $\varphi_m(x,y) = 0$ for $x > r-m$.

There are other bases in $W_r$ which are obtained by permuting the pairs $(a,x)$, $(b,y)$, $(c,r-x-y)$. In order to write them down we denote these pairs by $a$, $b$, $c$, respectively, and the function $\varphi_m$ by $\varphi_m(x,y;a,b,c)$. To the permutation $(b,a,c)$ there corresponds the basis

$$\begin{aligned}\psi_m(x,y;a,b,c) &= \varphi_m(y,x;b,a,c)\\ &= E_{r-m}(y;b,a+c-2m,r-m)E_m(x;a,c,r-y),\end{aligned} \tag{7}$$

where

$$\max(0,r-b) \le m \le \min(a,c,r,a+c-r).$$

To the permutation $(c,b,a)$ there corresponds the basis

$$\begin{aligned}\theta_m(x,y;a,b,c) &= \varphi_m(r-x-y,y;c,b,a)\\ &= E_{r-m}(r-x-y;c,b+a-2m,r-m)E_m(y;b,a,x+y),\end{aligned} \tag{8}$$

where

$$\max(0,r-c) \le m \le \min(a,b,r,a+b-r).$$

To the permutation $(a,c,b)$ there corresponds the basis

$$\begin{aligned}\varphi_m(x,r-x-y;a,c,b) &= E_{r-m}(x;a,c+b-2m,r-m)\\ &\times E_m(r-x-y;c,b,r-x) = (-1)^m\varphi_m(x,y;a,b,c).\end{aligned}$$

For derivation of the last equality we have used the relation

$$E_m(x;a,b,c) = (-1)^m E_m(c-x;b,a,c).$$

The same relation allows us to conclude that to the permutations $(b,c,a)$ and $(c,a,b)$ there correspond, respectively, the bases $\{(-1)^m\psi_m(x,y;a,b,c)\}$ and $\{(-1)^m\theta_m(x,y;a,b,c)\}$.

These bases correspond to the subgroup chains

$$S_N \supset G \supset H \equiv S_{\eta_1} \times S_{\eta_2} \times S_{\eta_3}$$

with different intermediate subgroups $G$.

The orthogonality and the normalization of the functions $E_m(x; a, b, c)$ imply that the set

$$\{\varphi_m \mid \max(0, r-a) \le m \le \min(b, c, r, b+c-r)\}$$

forms an orthogonal basis in $W_r(a, b, c)$ and

$$\sum_{(x,y)\in\mathfrak{D}_r} \binom{a}{x}\binom{b}{y}\binom{c}{r-x-y}\varphi_m(x,y)\varphi_n(x,y) = \delta_{mn}(-1)^m m!(r-m)!$$
$$\times (m-b-c)_r(-a)_{r-m}(-b)_m(-c)_m(a+b+c-2r+2)_{r-m}\frac{b+c-m+1}{b+c-2m+1}. \quad (9)$$

The bases consisting of the functions $\psi_m$ and $\theta_m$ have analogous orthogonality properties.

If $f \in W_r$, then one has the expansion

$$f = \sum_m \alpha_m \varphi_m,$$

where

$$\alpha_m = \sum_{j=\max(0,r-b)}^{\min(a,r)} f(j, r-j)\frac{(-1)^j(b+c-2m+1)}{(b+c-m+1)}$$
$$\times \frac{(-a)_j(-m)_{r-j}(m-b-c-1)_{r-j}}{(-a)_{r-m}(m-b-c)_r(-c)_m m!(r-m)!(r-j)!}. \quad (10)$$

To prove this formula we represent $\alpha_m$ as the Fourier coefficient of the expansion in Hahn polynomials, replace the double "integral" by successive integrations (at first with respect to $y$ and then with respect to $x$), use formula (3) and the relation

$$\sum_{y=0}^{n}\binom{n}{y}(-1)^y\sum_{i=0}^{m}\beta_i(-y)_i = n!\beta_n, \qquad 0 \le n \le m,$$

from the theory of finite differences.

Formula (10) allows us to establish relations between the functions $\varphi_m$, $\psi_m$, $\theta_m$. Namely, if

$$\max(0, r-b) \le k \le \min(a, c, r, a+c-r),$$

then

$$\psi_k(x,y) = \sum_m \alpha_{km}(a,b,c)\varphi_m(x,y), \tag{11}$$

where the summation is over values of $m$ from the interval

$$\max(0, r-a) \le m \le \min(b, c, r, b+c-r),$$

$$\alpha_{km}(a,b,c) = (-1)^{r+k}\frac{(-r)_m(-b)_{r-k}(a-r+1)_m(-c)_k(b+c-2m+1)}{m!(m-b-c)_r(-c)_m(b+c-m+1)}$$
$$\times r_{r-k}(\lambda(m); a-r, c-r, -b-1, -c-1)$$

if $r \le a$ and

$$\alpha_{km}(a,b,c) = (-1)^{r+k}(-r)_m(-a)_k(r-a-b)_{a-k}$$
$$\times \frac{(m-b-c-1)_{r-a}(-c)_{r-a+k}(b+c-2m+1)}{(r-a)!(-r)_k(m-b-c)_r(b+c-m+1)(-c)_m}$$
$$\times\, r_{a-k}(\lambda(m-r+a); r-a, c-r, r-a-b-1, r-a-c-1)$$

if $r \ge a$. Here

$$r_n(\lambda(x); \alpha, \beta, \gamma, \delta) = {}_4F_3\left(\begin{matrix} -n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{matrix}\middle|\, 1\right),$$

where $\lambda(x) = x(x+\gamma+\delta+1)$ and one of the numbers $\alpha+1$, $\beta+\delta+1$, $\gamma+1$ is a negative integer. It is a Racah polynomial (see Section 8.5.4).

Making use of the summation formula for the hypergeometric series (see Section 8.4.13), we find that

$$\alpha_{0m}(a,b,c) = \frac{(-1)^r(b-r+1)_{r-m}(r-a-b-c-1)_m(b+c-2m+1)}{(m-b-c)_r m!(r-m)!(b+c-m+1)}$$

for $b \ge r$ and

$$\alpha_{r-b,m}(a,b,c) = \frac{(r-a-b-c-1)_m(b+c-2m+1)}{m!(c-m+1)_b(b+c-m+1)}$$

for $b \le r$.

Besides relation (11) we have

$$\theta_k(x,y) = \sum_m (-1)^{k+m}\alpha_{km}(a,c,b)\varphi_m(x,y), \tag{12}$$

$$\theta_k(x,y) = \sum_m (-1)^m \alpha_{km}(b,c,a)\psi_m(x,y), \tag{13}$$

$$\varphi_k(x,y) = \sum_m (-1)^{k+m}\alpha_{km}(c,a,b)\theta_m(x,y), \tag{14}$$

$$\psi_k(x,y) = \sum_m (-1)^k \alpha_{km}(c,b,a)\theta_m(x,y), \tag{15}$$

$$\varphi_k(x,y) = \sum_m \alpha_{km}(b,a,c)\psi_m(x,y), \tag{16}$$

where in $\varphi_k$, $\psi_k$, $\theta_k$ we have omitted the indices $a$, $b$, $c$. Expressing $\theta_k$ in terms of $\{\psi_n\}$, then $\psi_n$ in terms of $\{\varphi_m\}$ and comparing the result with relation (12), we obtain the equality

$$(-1)^{k+m}\alpha_{km}(a,c,b) = \sum_n (-1)^n \alpha_{kn}(b,c,a)\alpha_{nm}(a,b,c), \tag{17}$$

where

$$\max(0, r-b) \le n \le \min(a, c, r, a+c-r).$$

This formula can be also obtained from relation (1) of Section 8.4.7.

The orthogonality of the bases $\{\varphi_m\}$ and $\{\psi_k\}$ implies that

$$\sum_m \alpha_{km}(a,b,c)\alpha_{\ell m}(a,b,c)\,\|\varphi_m\|^2 = \delta_{k\ell}\,\|\psi_k\|^2, \tag{18}$$

where

$$\max(0, r-a) \le m \le \min(b, c, r, b+c-r),$$
$$\max(0, r-b) \le k, \ell \le \min(a, c, r, a+c-r).$$

Let us note that $\|\varphi_m\|^2$ and $\|\psi_m\|^2$ are given by formula (9) and the permutation conditions.

We have shown in Section 8.4.1 that Racah polynomials are connected with a different grouping of factors in the tensor products. Here, as in Section 10.5.5, they appear to be connected with a different grouping of summands in $\eta_1 \cup \eta_2 \cup \eta_3$. Continuing the analogy with the results of Section 10.5, one may develop for functions of the space $V_r$ the tree method, defining and connecting different bases. These bases are products of Hahn polynomials. Functions from different bases are connected by coefficients, expressed in terms of Racah polynomials. Limiting functions for Hahn polynomials are Jacobi polynomials (see Section 8.5.1). Under this limit procedure the relations of this section turn into those of Section 10.5.5.

Now let us find the connection of representations of the group $S_N$ with Hahn polynomials of two variables. Recall that we have realized $V_r$ as the space of functions in $\mathbb{R}^N$. We choose $\mathbf{x}_0 \in \mathbb{R}^N$ and denote by $\widetilde{V}_r$ the space of functions $\tilde{f}$ on $S_N$ of the form $\tilde{f}(\pi) = f(\pi^{-1}\mathbf{x}_0)$, $f \in V_r$, $\pi \in S_N$. This space is invariant with respect to right shifts and, by Schur's lemma, either is isomorphic to $V_r$ or coincides with $\{0\}$. We denote by $G(\mathbf{x}_0)$ the stationary subgroup of the point $x_0$. Then to $H$-invariant functions from $V_r$, where $H = S_{\eta_1} \times S_{\eta_2} \times S_{\eta_3}$, there correspond functions from $\widetilde{V}_r$ which are invariant with respect to left shifts by elements from $G(\mathbf{x}_0)$ and with respect to right shifts by elements from $H$ (this double invariance is called the $G(\mathbf{x}_0), H$-invariance).

If $\mathbf{x}_0 = (1,\dots,1,0,\dots,0)$ ($M$ units and $N-M$ zeros), then $G(\mathbf{x}_0) = H_M \equiv S_M \times S_{N-M}$. To find $H_M, H$-invariant functions it is sufficient to calculate values of $H$-invariant functions from $V_r$ at points of the form $\pi^{-1}\mathbf{x}_0$. Set

$u_i(\pi) = \left|\pi^{-1}\{1,\dots,M\}\cap\eta_i\right|$, $i = 1,2,3$. Note that $u_1+u_2+u_3 = M$. Then $\sigma_{i_1}(\eta_1)\sigma_{i_2}(\eta_2)\sigma_{i_3}(\eta_3)$ take the value $\binom{u_1(\pi)}{i_1}\binom{u_2(\pi)}{i_2}\binom{u_3(\pi)}{i_3}$ at $\pi^{-1}\mathbf{x}_0$. The sum of an $H_M, H$-invariant function over $S_N$ is equal to

$$\sum_{\pi\in S_N} f(u_1(\pi),u_2(\pi),u_3(\pi)) = \sum_{u_1,u_2}\binom{a}{u_1}\binom{b}{u_2}\binom{c}{M-u_1-u_2} \times f(u_1,u_2,M-u_1-u_2),$$

where

$$0\le u_1\le a,\quad 0\le u_2\le b,\quad c-M\le u_1+u_2\le M.$$

Under the correspondence $V_r \to \widetilde{V}_r$ indicated above, to the function

$$p = \sum_{i_1,k_2}\varphi_m(i_1,i_2)\sigma_{i_1}(\eta_1)\sigma_{i_2}(\eta_2)\sigma_{r-i_1-i_2}(\eta_3)$$

there corresponds the function

$$\widetilde{\varphi}_m(u_1(\pi),u_2(\pi),u_3(\pi)) = E_{r-m}(u_1;a,b+c-2m,i_1+u_2+u_3-m)E_m(u_2;b,c,u_2+u_3), \tag{19}$$

where

$$\max(0,r-a)\le m\le \min(b,c,r,b+c-r).$$

Indeed,

$$\widetilde{\varphi}_m(u_1,u_2,u_3) = \sum_{i_1,i_2}\binom{u_1}{i_1}\binom{u_2}{i_2}\binom{u_3}{r-i_1-i_2} E_{r-m}(i_1;a,b+c-2m,r-m)E_m(i_2;b,c,r-i_1).$$

We apply twice the relation

$$\sum_{x=\max(0,c-d+y)}^{\min(c,y)}\binom{y}{x}\binom{d-y}{c-y}E_m(x;a,b,c) = \binom{d-m}{c-m}E_m(y;a,b,d)$$

which can be derived from the definition

$$E_m(x;a,b,c) = \sum_{j=1}^{m}(-1)^j\binom{m}{j}(b-m+1)_j(a-m+1)_{m-j}(-x)_j(x-c)_{m-j}$$

by means of the identity

$$\binom{y}{x}\binom{d-y}{c-x}(-x)_j(x-c)_{m-j} = (-y)_j(y-d)_{m-j}\binom{y-j}{x-j}\binom{d-y-m+j}{c-x-m+j}$$

if one sums it with respect to $x$ and takes into account the Vandermonde formula. We obtain

$$\begin{aligned}\widetilde{\varphi}_m(u_1,u_2,u_3) &= \sum_{i_1}\binom{u_1}{i_1}\binom{u_2+u_3-m}{r-m-i_1}\\ &\quad\times E_{r-m}(i_1;a,b+c-2m,r-m)E_m(u_2;b,c,u_2+u_3)\\ &= E_{r-m}(u_1;a,b+c-2m,u_1+u_2+u_3-m)E_m(u_2;b,c,u_2+u_3).\end{aligned}$$

The functions $\{\widetilde{\varphi}_m\}$ are orthogonal with respect to the weight

$$\binom{a}{u_1}\binom{b}{u_2}\binom{c}{M-u_1-u_2}$$

and $\widetilde{\varphi}_m$ belongs to the carrier space of the irreducible subrepresentation $(a,0)(b+c-m,m)$ of the subgroup $S_{\eta_1}\times S_\eta$, $\eta=\eta_2\cup\eta_3$.

When we go over from $V_r$ to $\widetilde{V}_r$, then to the function $\psi_m(x_1,x_2)$, there corresponds the function

$$\begin{aligned}\widetilde{\psi}_m(u_1,u_2,u_3) &= E_{r-m}(u_2;b,a+c-2m,u_1+u_2+u_3-m)\\ &\quad\times E_m(u_1;a,c,u_1+u_3),\end{aligned}\tag{20}$$

where

$$\max(0,r-b)\le m\le\min(a,c,r,a+c-r).$$

The function $\widetilde{\psi}_m$ belongs to the carrier space of the irreducible subrepresentation $(a+c-m,m)(b,0)$ of the subgroup $S_\eta\times S_{\eta_2}$, $\eta=\eta_1\cup\eta_3$.

Finally, to the function $\theta_m(x_1,x_2)$ there corresponds the function

$$\begin{aligned}\widetilde{\theta}_m(u_1,u_2,u_3) &= E_{r-m}(u_3;c,a+b-2m,u_1+u_2+u_3-m)\\ &\quad\times E_m(u_2;b,a,u_1+u_2),\end{aligned}\tag{21}$$

where

$$\max(0,r-c)\le m\le\min(a,b,r,a+b-r).$$

This function belongs to the carrier space of the irreducible subrepresentation $(a+b-m,m)(c,0)$ of the subgroup $S_\eta\times S_{\eta_3}$, $\eta=\eta_1\cup\eta_2$.

The mapping of $V_r$ onto $\widetilde{V}_r$ preserves the coefficients of the expansions of functions of one basis in functions of another one. This gives relations of the form

$$\begin{aligned}&E_{r-k}(u_2;b,a+c-2k,u_1+u_2+u_3-k)E_k(u_1;a,c,u_1+u_3)\\ &\quad=\sum_m \alpha_{km}(a,b,c)E_{r-m}(u_1;a,b+c-2m,u_1+u_2+u_3-m)\\ &\qquad\times E_m(u_2;b,c,u_2+u_3).\end{aligned}\tag{22}$$

The formula

$$Q\left(\begin{matrix} u_1,\ u_2,\ u_3 \\ -a-1,-b-1,-c-1 \end{matrix}\middle|\ m, r-m\right) = \frac{(-1)^m(-u_1-u_2-u_3)_r}{(-a)_m(2m-a-b)_{r-m}}\,\widetilde{\theta}(u_1,u_2,u_3;a,b,c) \tag{23}$$

defines Hahn polynomials in two variables, $x = m$ and $y = r - m$. By using the properties of $E_m(x;\dots)$ it is easy to write down the orthogonality relation for $Q$ (we recommend to the reader to derive this relation). More general Hahn polynomials in two variables are obtained from (23) by replacing the integral numbers $-a-1$, $-b-1$, $-c-1$ by the complex parameters $\alpha_1$, $\alpha_2$, $\alpha_3$ different from $-1,-2,\dots,-r$.

If $k = 0,1,\dots,r$ and $\alpha_1$, $\alpha_2$, $\alpha_3$ do not take the values $-1,-2,\dots,-r$, then by means of formula (11) it is easy to obtain the relation

$$Q\left(\begin{matrix} u_1,u_2,u_3 \\ \alpha_1,\alpha_2,\alpha_3 \end{matrix}\middle|\ k, r-k\right) = \sum_{m=0}^{r} \frac{(-1)^{m+r}(-r)_m(\alpha_2+1)_{r-k}(-r-\alpha_3)_m}{(2m+\alpha_1+\alpha_3+2)_{r-k}m!(m+\alpha_1+\alpha_2+1)_m}$$
$$\times\, {}_4F_3\left(\begin{matrix} k-r,-\alpha_1-\alpha_3-k-r-1,-m,m+\alpha_1+\alpha_2+1 \\ -r,\ \alpha_2+1,\ -\alpha_3-r \end{matrix}\middle|\ 1\right)$$
$$\times\, Q\left(\begin{matrix} u_1,u_2,u_3 \\ \alpha_1,\alpha_2,\alpha_3 \end{matrix}\middle|\ m, r-m\right). \tag{24}$$

We note that the $9j$ symbols, used in physics (see, for example, Ref. [48] of the first volume), are also connected with orthogonal polynomials in two discrete variables.

## 13.2. Groups of Linear Transformations over Finite Fields and $q$-Analogs of Special Functions

**13.2.1. Linear spaces over finite fields and Chevalley groups.** If $p$ is a prime number, then the quotient ring $\mathbb{Z}/p\mathbb{Z}$ is a field. Any finite field has a prime characteristic $p$ and is an algebraic extension of the field $\mathbb{Z}/p\mathbb{Z}$. It follows from here that the number of elements of any finite field is equal to $q = p^s$. For any $q = p^s$ there exists a unique (up to an isomorphism) field consisting of $q$ elements. It is called a *Galois field* and is denoted by $GF(q)$. If the Galois field of characteristic $p$ contains element $a$, then it contains exactly one $p$-th root from $a$. The mapping $\varphi: a \to a^p$ is an automorphism of $GF(q)$. Every other automorphism of $GF(q)$ is a power of $\varphi$. All non-zero elements of $GF(q)$ are roots of the equation $x^{q-1} - 1 = 0$. Since $q-1$ and $p$ are relatively prime integers, then non-zero elements from $GF(q)$ form a cyclic group with respect to multiplication.

For brevity $GF(q)$ is denoted by $F$. We denote by $F^n$ the $n$-dimensional linear space over $F$. Let $B(\mathrm{v},\mathrm{w})$ be a non-degenerate form over $F$ in $F^n$ and let $q = p^s$ be an odd integer. If $B$ is a skew-symmetric bilinear form, then $n$ is even, $n = 2N$, and one can choose a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N, \mathbf{e}_{-1}, \ldots, \mathbf{e}_{-N}\}$ such that for $\mathbf{v} = \sum v_i \mathbf{e}_i$, $\mathbf{w} = \sum w_i \mathbf{e}_i$ we have

$$B(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{N} (v_i w_{-i} - v_{-i} w_i). \tag{1}$$

In this case $F^n$ is said to be a space of the type $C_N$. The group of linear transformations of $F^n$, preserving form (1), is denoted by $Sp(2N, q)$.

If $B$ is a symmetric bilinear form and $n$ is an even integer, $n = 2N$, then we have two possibilities. In $F^n$ there is either a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N, \mathbf{e}_{-1}, \ldots, \mathbf{e}_{-N}\}$ for which

$$B(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{N} (v_i w_{-i} + v_{-i} w_i) \tag{2}$$

or a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_{N+1}, \mathbf{e}_{-1}, \ldots, \mathbf{e}_{-N-1}\}$ (in this case $n = 2N + 2$) for which

$$B(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{N} (v_i w_{-i} + v_{-i} w_i) + v_{N+1} w_{N+1} - \varepsilon v_{-N-1} w_{-N-1}, \tag{3}$$

where $\varepsilon$ is a fixed element from $F$ which does not coincide with square of other element from $F$. In the first case $F^n$ is said to be a space of the type $D_N$. The group of linear transformations, preserving (2), is denoted by $O(2N, q)$. In the second case $F^n$ is said to belong to the type ${}^2D_{N+1}$. The group of linear transformations of $F^n$, preserving form (3), is denoted by $O(2N + 2, q, \varepsilon)$.

If $B$ is a symmetric form and $n$ is an odd integer, $n = 2N + 1$, then in $F^n$ there exists a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N, \mathbf{e}_{-1}, \ldots, \mathbf{e}_{-N}, \mathbf{e}_0\}$ such that

$$B(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{N} (v_i w_{-i} + v_{-i} w_i) + 2 v_0 w_0. \tag{4}$$

In this case the space $F^n$ is said to belong to the type $B_N$. The corresponding transformation group is denoted by $O(2N + 1, q)$.

If $|F| = q^2$, then one can define in $F$ the involution $x \to \bar{x}$ leaving elements of the subfield $F_0$, $|F_0| = q$, fixed. Then for even $n$ there exists a Hermitian form $B(\mathbf{v}, \mathbf{w})$ and a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N, \mathbf{e}_{-1}, \ldots, \mathbf{e}_{-N}\}$ such that

$$B(\mathbf{v}, \mathbf{w}) = \varepsilon \sum_{i=1}^{N} (-1)^{i+1} (\bar{v}_i w_{-i} + \bar{v}_{-i} w_i), \tag{5}$$

where $\varepsilon$, $\varepsilon \neq 0$, is a fixed element of $F$ for which $\varepsilon + \bar{\varepsilon} = 0$. In this case $F^n$ is said to be a space of the type ${}^2A_{2N-1}$. The corresponding linear group is denoted by $U(2N, q^2)$.

If $n$ is an odd integer, $n = 2N+1$, then there exists a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N, \mathbf{e}_{-1}, \ldots, \mathbf{e}_{-N}, \mathbf{e}_0\}$ such that

$$B(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{N} (-1)^i (\bar{v}_i w_{-i} + \bar{v}_{-i} w_i) + (-1)^{N+1} \bar{v}_0 w_0. \tag{6}$$

Then $F^n$ is said to be of the type ${}^2A_{2N}$. The corresponding group is denoted by $U(2N+1, q^2)$.

The groups considered above are analogs of semi-simple Lie groups and belong to the class of so-called *Chevalley groups*. The group $SL(n,q)$ of unimodular linear transformations of $F^n$ also belongs to them. The group of non-degenerate linear transformations of $F^n$ is denoted by $GL(n,q)$.

A subspace $W \subset F^n$ is said to be *isotropic* if $B(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{v}, \mathbf{w} \in W$. In all cases considered above, the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$ generate a maximal isotropic subspace with respect to the corresponding bilinear form. It follows from the Witt theorem (which is well-known in Algebra) that all maximal isotropic subspaces of $F^n$ have the same dimension. We denote by $W^\perp$ the space of vectors $\mathbf{v}$ such that $B(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w} \in W$. A space $W$ is isotropic if and only if $W \subset W^\perp$.

By a maximal parabolic subgroup of the group $GL(n,q)$ we mean a subgroup of linear transformations leaving some subspace $\mathfrak{L}$ invariant. If $\dim \mathfrak{L} = k$, then for appropriate bases these subgroups consist of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad A \in GL(k,q), \quad D \in GL(n-k,q), \quad B \in \mathfrak{M}(k, n-k;).$$

For other groups considered above, maximal parabolic subgroups consist of linear transformations transferring some isotropic subspace, having maximal dimension, into itself.

**13.2.2. Basic hypergeometric functions and $q$-analogs of orthogonal polynomials.** Recall that the binomial coefficient $C_n^m \equiv \binom{n}{m}$ is equal to the number of $m$-subsets in the $n$-set. We generalize these coefficients denoting the number of $m$-dimensional subspaces of $F^n$, where $|F| = q$, by $\left[ {n \atop m} \right]_q$.

In order to find the expression for $\left[ {n \atop m} \right]_q$, we at first calculate the number of $m$-frames in $F^n$. It is clear that after choosing $k$ linearly independent vectors $\mathbf{e}_1, \ldots, \mathbf{e}_k$, the vector $\mathbf{e}_{k+1}$, which does not belong to the plane spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_k$, can be chosen in $q^n - q^k$ ways. It follows from there that the number of $m$-frames is equal to

$$(q^n - 1)(q^n - q) \ldots (q^n - q^{m-1}).$$

Hence, the number of non-degenerate linear transformations of $F^m$ is equal to

$$(q^m - 1)(q^m - q) \ldots (q^m - q^{m-1})$$

and, consequently,

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q^n-1)(q^n-q)\dots(q^n-q^{m-1})}{(q^m-1)(q^m-q)\dots(q^m-q^{m-1})}$$
$$= \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-m+1}-1)}{(q^m-1)(q^{m-1}-1)\dots(q-1)}. \tag{1}$$

By setting

$$(a;q)_n = \prod_{j=1}^{n}(1-aq^{j-1}) \tag{2}$$

we obtain

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}. \tag{3}$$

Therefore, the expression $(q;q)_n$ is the $q$-analog of $n!$

Formulas (2) and (3) also define $(a;q)_n$ and $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_q$ for arbitrary complex $q$, $q \neq 1$. In particular, $(a,q^{-1})_n$ is defined. A direct verification gives

$$(q^\alpha;q^{-1})_m = (q^{\alpha-m+1};q)_m = (-1)^{\alpha m - m(m-1)/2}(q^{-\alpha+m-1};q^{-1})_m. \tag{4}$$

In the sequel for brevity we shall write $(a)_n$ instead of $(a;q^{-1})_n$. We have

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q^n)_n}{(q^m)_m(q^{n-m})_{n-m}} = \frac{(q^n)_m}{(q^m)_m}. \tag{3'}$$

By induction one can easily prove that if $v$ and $w$ are non-commuting variables such that $vw = qwv$, then

$$(v+w)^n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q w^m v^{n-m} = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_{q^{-1}} v^m w^{n-m}. \tag{4'}$$

A deep analogy between $\binom{n}{m}$ and $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_q$ also appears in the fact that many of the identities for the binomial coefficients have analogs for $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_q$. To obtain these analogs one uses combinatorial models or equalities of the form (4′). The $q$-binomial identities

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ n-m \end{bmatrix}_q, \quad \begin{bmatrix} n+1 \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} n \\ m \end{bmatrix}_q, \tag{5}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \sum_{s=0}^{m} q^{(k-s)(m-j)} \begin{bmatrix} k \\ s \end{bmatrix}_q \begin{bmatrix} n-k \\ m-s \end{bmatrix}_q, \quad 0 \le k \le n, \tag{5'}$$

are examples of this analogy. The reader can easily derive these identities. When $q \to 1$ we obtain the properties of ordinary binomial coefficients.

If $|q| < 1$, then the expression

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$$

has meaning and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{6}$$

For $q > 1$ we have

$$(a)_\infty \equiv (a;q^{-1})_\infty = \lim_{n\to\infty}(a)_n.$$

Moreover, $(a)_\infty$ is an analytic function of $a$. Euler showed that

$$\sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}, \tag{6'}$$

$$\sum_{n=0}^{\infty}\frac{(-x)^n q^{n(n-1)/2}}{(q;q)_n} = (x;q)_\infty. \tag{6''}$$

These relations will be proved in the third volume.

The expression $\frac{(q^a;q)_n}{(q;q)_n}$ is an analog of $\frac{\Gamma(a+n+1)}{\Gamma(a+1)}$. Indeed, we have

$$\lim_{q\to1}\frac{(q^a;q)_n}{(q^b;q)_n} = \frac{\Gamma(a+n)}{\Gamma(a)}\frac{\Gamma(b)}{\Gamma(b+n)}.$$

Below we shall use the relations

$$(aq^n;q)_{N-n} = \frac{(a;q)_N}{(a;q)_n}, \tag{7}$$

$$(q;q)_{N-n} = \frac{(-1)^n(q;q)_N}{q^{n(N-n)}(q^{-N};q)_n}, \tag{7'}$$

$$(a;q)_{-n} = \frac{q^{n(n+1)/2}(-a)^{-n}}{(q/a;q)_n}, \tag{8}$$

$$(aq^{nk};q)_n = \frac{(a;q)_n(k+1)}{(q;q)_{kn}} \tag{9}$$

which can be proved by means of formula (2).

We suggest to the reader to prove that if $V_N$ and $W_M$ are the $N$- and $M$-dimensional spaces over the field $GF(q)$, $q = p^r$, then:

**a)** the number of pairs of $n$-dimensional subspaces $(V_n, W_n)$, $V_n \subset V_N$, $W_n \subset W_M$, is equal to

$$\begin{bmatrix} N \\ n \end{bmatrix}_q \begin{bmatrix} M \\ n \end{bmatrix}_q;$$

**b)** for every pair $(V_n, W_n)$ there exist

$$|G| = (-1)^n (q;q)_n q^{n(n-1)/2}$$

non-degenerate linear mappings of $V_n$ into $W_n$, $G = GL(n, F)$;

**c)** the number of $k$-dimensional subspaces $W_k \subset W_M$, such that $W_k \cap W_n = \{0\}$, is equal to

$$\begin{bmatrix} M-n \\ k \end{bmatrix}_q q^{kn};$$

**d)** the number of linear mappings $g$ of the space $V_n$ into $V_k$ such that $\dim(V_n g) = j$, is equal to

$$a(n,k,j) = \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j (q;q)_j q^{j(j-1)/2}. \tag{10}$$

By induction in $N$ ones proves that

**e)** the number of symmetric $N \times N$ matrices of rank $x$ is equal to

$$S(N,x) = (-1)^x \frac{(q^N; q^{-1})_x}{(q^{-2}; q^{-2})_{[x/2]}}, \tag{11}$$

where $[x/2]$ is the integral part of $x/2$;

**f)** the number of skew-symmetric $N \times N$ matrices of rank $x$ is equal to

$$A(N,2y) = \frac{(q^N; q^{-1})_{2y}}{(q^2; q^2)_y} q^{y(y-1)} (-1)^y \quad \text{if} \quad x = 2y, \tag{12}$$
$$A(N, 2y+1) = 0 \quad \text{if} \quad x = 2y+1;$$

**g)** the number of Hermitian $N \times N$ matrices of rank $x$ over the field $GF(q^2)$ is

$$H(N,x) = \frac{(q^{2N}; q^{-2})_x}{(-q;-q)_x} q^{x(x-1)/2} (-1)^{x(x+1)/2}. \tag{13}$$

We now define the *q-analogs of hypergeometric functions* (which are called *basic hypergeometric functions*) by the equality

$$ {}_m\varphi_n \left( \begin{matrix} a_1, \dots, a_m \\ b_1, \dots, b_n \end{matrix} \middle| q; z \right) = \sum_{j=0}^{\infty} \frac{(a_1;q)_j \dots (a_m;q)_j}{(b_1;q)_j \dots (b_n;q)_j} \frac{z^j}{(q;q)_j}. \tag{14}$$

This series converges for all $z$ if $m \le n$ and for $|z| < 1$ if $m = n+1$.

In the same way one defines the function

$$ {}_m\varphi_n \left( \begin{matrix} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{matrix} \middle| q^{-1}; z \right). \tag{15} $$

Basic hypergeometric functions have a number of properties which are analogous to those for the ordinary hypergeometric function and which turn into them when $q \to 1$.

It is obvious that if one of $a_k$, $1 \le k \le m$, in (14) is equal to $q^{-r}$, where $r$ is a positive integer, then the series ${}_m\varphi_n(\ldots)$ is finite and defines a polynomial in $z$.

There are relations connecting different basic hypergeometric functions. For example, we have

$$ {}_2\varphi_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| q^{-1} z \right) = \frac{(abz/c)_\infty}{(z)_\infty} {}_2\varphi_1 \left( \begin{matrix} \frac{c}{a}, \frac{c}{b} \\ c \end{matrix} \middle| q^{-1}; \frac{ab}{c} z \right), \tag{16} $$

$$ {}_3\varphi_2 \left( \begin{matrix} q^{-k}, a, b \\ c, d \end{matrix} \middle| q; q \right) = \frac{(c/a;q)_k}{(c;q)_k} a^k {}_3\varphi_2 \left( \begin{matrix} q^{-k}, a, \frac{d}{b} \\ aq^{1-k}/c, d \end{matrix} \middle| q; \frac{bq}{c} \right), \tag{17} $$

where $k \in \mathbb{Z}_+$. The last formula implies the equality

$$ {}_3\varphi_2 \left( \begin{matrix} q^{-k}, \frac{c}{a}, \frac{c}{b} \\ 0, c \end{matrix} \middle| q; q \right) = \frac{(c/a;q)_k}{(c;q)_k} \left( \frac{c}{b} \right)^k {}_2\varphi_1 \left( \begin{matrix} q^{-k}, a \\ aq^{1-k}/c \end{matrix} \middle| q; \frac{bq}{c} \right). \tag{18} $$

We also have

$$ {}_3\varphi_2 \left( \begin{matrix} q^{k}, \frac{c}{a}, \frac{c}{b} \\ 0, c \end{matrix} \middle| q^{-1}; q^{-1} \right) = \left( \frac{c}{ab} \right)^k {}_3\varphi_2 \left( \begin{matrix} q^{k}, a, b \\ 0, c \end{matrix} \middle| q^{-1}; q^{-1} \right), \tag{19} $$

$$ {}_3\varphi_2 \left( \begin{matrix} q^{-k}, a, b \\ 0, c \end{matrix} \middle| q; q \right) = \frac{(c/a;q)_k}{(c;q)_k} {}_2\varphi_1 \left( \begin{matrix} q^{-k}, a \\ aq^{1-k}/c \end{matrix} \middle| q; \frac{bq}{c} \right). \tag{20} $$

Let us note also the formula

$$ q^{m\delta} (q^{\beta-\delta})_m {}_3\varphi_2 \left( \begin{matrix} q^m, q^{\gamma-\alpha}, q^\delta \\ q^\gamma, q^{\delta-\beta+m-1} \end{matrix} \middle| q^{-1}; q^{\alpha-\beta-1} \right) $$

$$ = (q^\beta)_m {}_3\varphi_2 \left( \begin{matrix} q^m, q^\alpha, q^\delta \\ q^\gamma, q^\beta \end{matrix} \middle| q^{-1}; q^{-1} \right). \tag{21} $$

These and other relations for basic hypergeometric functions will be proved in the third volume of this book.

By means of basic hypergeometric functions one introduces orthogonal polynomials which are $q$-analogs of Hahn, Krawtchouk and Racah polynomials. The ***q-Hahn polynomial*** is defined by

$$ Q_k(q^{-x}; a, b, N; q) = {}_3\varphi_2 \left( \begin{matrix} q^{-k}, abq^{k+1}, q^{-x} \\ aq, q^{-N} \end{matrix} \middle| q; q \right), \tag{22} $$

where $k$, $N$ are non-negative integers such that $0 \le k \le N$. It is a polynomial of degree $k$ in $q^{-x}$. The orthogonality relation for $q$-Hahn polynomials is of the form

$$\sum_{x=0}^{N} Q_k(q^{-x}, a, b, N; q) Q_j(q^{-x}; a, b, N; q) w(x) = \delta_{jk} h_k^{-1}, \tag{23}$$

where

$$w(x) = \frac{(q^{-N}; q)_x (aq; q)_x}{(q; q)_x (q^{-N}/b; q)_x} (abq)^{-x},$$
$$h_k = \frac{(q^{-N}/b; q)_{N-k} (aq; q)_k (q^{-N}; q)_k}{(q^{-N-k-1}/ab; q)_{N-k} (q^{-1}; q^{-1})_k (abq^{k+1}; q)_k}.$$

The weight function $w(x)$ is positive valued if $q > 0$ and, in addition, if the numbers $a$, $b$ are positive and $a, b > q^{-1}$, $a, b > q^{-N}$ or $a, b < q^{-1}$, $a, b < q^{-N}$.

It follows from (17) that

$$Q_k(q^{-x}; a, b, N; q) = \frac{(q^{-k}/b; q)_k}{(aq; q)_k} (abq^{k+1})^k \times Q_k(q^{N-x}; b^{-1}, a^{-1}, N; q^{-1}), \tag{24}$$

$$Q_k(q^{-x}; a, b, N; q) = \frac{(bq^N; q^{-1})_x}{(a^{-1}q^{-1}; q^{-1})_x} \times Q_{N-k}(q^{-x}; q^{-N-1}b^{-1}, q^{-N-1}a^{-1}, N; q). \tag{25}$$

Instead of $q$-Hahn polynomials (22) one often considers the polynomials in $q^{-x}$ of the form

$$\widetilde{Q}_k(x; a, b, c; q^{-1}) = q^{k(k-1)/2} (q^a)_k (q^c)_k \times {}_3\varphi_2 \left( \begin{matrix} q^x, q^{a+b-k+1}, q^k \\ q^a, q^c \end{matrix} \middle| q^{-1}; q^{-1} \right), \tag{26}$$

where $a$, $b$, $c$, $k$ are integers such that $k \le \min(a, b, c, a+b-c)$. We have

$$\widetilde{Q}_k(x; -a-1, -b-1, N; q^{-1}) = q^{k(k-1)} (q^{-a-1})_k (q^N)_k \times Q_k(q^x; q^a, q^b, N; q^{-1}). \tag{26'}$$

The orthogonality relation for $\widetilde{Q}_k(x; \dots)$ is of the form

$$\sum_{x=\max(0,c-b)}^{\min(a,c)} \widetilde{Q}_k(x; a, b, c; q^{-1}) \widetilde{Q}_j(x; a, b, c; q^{-1}) \tilde{w}(x) = \delta_{kj} c_k, \tag{27}$$

where

$$\tilde{w}(x) = \begin{bmatrix} a \\ x \end{bmatrix}_q \begin{bmatrix} b \\ c-x \end{bmatrix}_q q^{(a-x)(c-x)},$$

$$c_k = (q^k)_k (q^a)_k (q^b)_k (q^{a+b-k})_k q^{k(a+c)} \frac{1-q^{a+b-k+1}}{q^m - q^{a+b-k+1}} \begin{bmatrix} a+b-2k \\ c-k \end{bmatrix}_q .$$

The relations

$$\widetilde{Q}_k(x;a,b,c;q^{-1}) = q^{-k(b-c)} \widetilde{Q}_k(x+b-c;b,a,a+b-c;q^{-1}),$$
$$\widetilde{Q}_k(x;a,b,c;q^{-1}) = (-1)^k q^{k(a+b+c)-k(k-1)/2} \widetilde{Q}_k(c-x;b,a,c;q)$$

hold.

The *dual q-Hahn polynomials* (or the *q-Eberlane polynomials*) are defined in terms of the $q$-Hahn polynomials by the formula

$$E_j(\lambda(x);a,b,N;q) = Q_x(q^{-j};a,b,N;q). \tag{28}$$

They are polynomials of degree $j$ in

$$\lambda(x) = (1-q^{-x})(1-abq^{x+1}).$$

The orthogonality relation for these polynomials is obtained from that for the $q$-Hahn polynomials by interchanging $w(x)$ and $h_j$:

$$\sum_{k=0}^{N} E_j(\lambda(k);a,b,N;q) E_i(\lambda(k);a,b,N;q) h_k$$
$$= \delta_{ij} \frac{(a^{-1}b^{-1}q^{-2};q^{-1})_N}{(q;q)_N} (w(i))^{-1}. \tag{29}$$

The $q$-Krawtchouk polynomials

$$K_j(q^{-x};c,N;q) = {}_3\varphi_2 \left( \begin{matrix} q^{-j}, -cq^{j+1}, q^{-x} \\ 0, q^{-N} \end{matrix} \middle| q;q \right), \; 0 \le j \le N, \tag{30}$$

are orthogonal on the set $x \in \{0,1,\dots,N\}$. The orthogonality relation has the form

$$\sum_{x=0}^{N} K_k(q^{-x};c,N;q) K_j(q^{-x};c,N;q) w(x) = \delta_{jk} h_k, \tag{31}$$

where

$$w(x) = \frac{(q^{-N};q)_x}{(q;q)_x}(-cq)^{-x},$$
$$h_k = \frac{(-q^{-N-k-1}c^{-1};q)_{N-k}(q^{-1};q^{-1})_k(-cq^{k+1};q)_k}{(q^{-N};q)_k}.$$

For the $q$-Krawtchouk polynomials one has the relations

$$K_j(q^{x-N};c,N;q) = (-c)^j q^{j(j+1)} K_j(q^x;c^{-1},N;q^{-1}),$$
$$K_{N-j}(q^{-x};c,N;q) = (-c)^x q^{x(N+1)} K_j(q^{-x};c^{-1}q^{-2N-2},N;q).$$

The polynomials $K_j(q^{-x};c,N;q)$ are limit for the $q$-Hahn polynomials. If we go over in $Q_j(q^{-x};a,b,N;q)$ to the limit $a \to 0$, $b \to \infty$, so that $abq = -qc$, then we obtain the $q$-Krawtchouk polynomial. Let us note also that

$$\lim_{q\to 1} K_j(q^{-x};c,N;q) = K_j(x;(1+c)^{-1},N),$$
$$\lim_{q\to 1} K_j(q^{-x};q^c,N;q) = K_j\left(x;\frac{1}{2},N\right),$$

where $K_j(x;p,N)$ are ordinary Krawtchouk polynomials (see Section 6.8.1).

After replacing in $Q_j(q^{-x};a,b,N;q)$ the parameter $a$ by $q^{-a-1}$ and turning $b$ to zero, we obtian the *affine q-Krawtchouk polynomials*, which are defined by the formula

$$K_j^{\mathrm{Aff}}(q^{-x};q^{-a},N;q) = {}_3\varphi_2\left(\begin{matrix} q^{-j},q^{-x},0 \\ q^{-a},q^{-N}\end{matrix}\,\middle|\, q;q\right) \quad 0 \le j \le N. \tag{32}$$

They are orthogonal on the set $x \in \{0,1,\dots,N\}$ with respect to the weight

$$w(x_k) = \begin{bmatrix} N \\ k \end{bmatrix}_q (q^a;q^{-1})_k(-1)^k q^{k(k-1)/2}, \tag{33}$$

where $x_k = q^{-k}$. In addition,

$$\left\|K_j^{\mathrm{Aff}}(q^{-x};q^{-a},N;q)\right\|^2 = \frac{q^{aN}}{w(x_j)}. \tag{34}$$

Sometimes it is convenient to make no distinction between $a$ and $N$ in polynomials (32). In this case one considers the polynomials

$$\widetilde{K}_j(x;a,b;q) = \frac{(q^a)_j(q^b)_j q^{-m}}{(q^{-1})_j}\,{}_2\varphi_2\left(\begin{matrix} q^{-x},q^{-j} \\ q^{-a},q^{-b}\end{matrix}\,\middle|\, q;q\right), \tag{35}$$

where $a$, $b$ are integers such that $0 \le j \le \min(a,b)$. The orthogonality relation for them is written in the form

$$\sum_{x=0}^{\min(a,b)} \frac{(q^a)_x(q^b)_x}{(q^{-1})_x} q^{-x} \widetilde{K}_j(x;a,b;q)\widetilde{K}_n(x;a,b;q) = \delta_{jn} q^{ab-n} \frac{(q^a)_n(q^b)_n}{(q^{-1})_n}. \tag{36}$$

**13.2.3. Irreducible representations of Chevalley groups and $q$-analogs of orthogonal polynomials.** Let $F$ be the field from Section 13.2.1. We denote by $X$ the set of all linear subspaces of $F^N$. For every $\zeta \in X$ we define the function $\hat{\zeta}$ on $X$ by setting $\hat{\zeta}(\eta) = 1$ if $\zeta \subset \eta$ and $\hat{\zeta}(\eta) = 0$ otherwise. It is easy to see that the functions $\hat{\zeta}$, where $\zeta \in X$, are linearly independent. For $m = 0, 1, \ldots, N$ we denote by $\mathcal{P}_m$ the subspace of $\mathcal{L}^2(X)$, generated by the functions $\hat{\zeta}$, where $\dim \zeta = m$. It is clear that $\dim \mathcal{P}_m = \begin{bmatrix} N \\ m \end{bmatrix}_q$. We define the operator $D\colon \mathcal{P}_m \to \mathcal{P}_{m-1}$ by setting

$$D\hat{\zeta} = \sum \{\hat{\eta} \mid \eta \subset \zeta, \dim \eta = \dim \zeta - 1\}$$

and by extending $D$ linearly onto $\mathcal{P}_m$.

Fix $\omega \in X$, $\dim \omega = n$, and set

$$f_{ij} = \sum \left\{\hat{\zeta} \mid \dim \zeta = i, \dim(\zeta \cap \omega) = j\right\},$$

where

$$\max(0, i+n-N) \le j \le \min(i,n).$$

It is easy to verify that

$$Df_{ij} = \frac{q^{n-j} - q^{N-i+1}}{1-q} f_{i-1,j} + \frac{1-q^{n-j+1}}{1-q} f_{i-1,j-1}. \tag{1}$$

Making use of this formula, we conclude that $D$ maps $\mathcal{P}_m$ onto $\mathcal{P}_{m-1}$ if $m \le (N+1)/2$.

Let $V_m = \mathcal{P}_m \cap \ker D$, $m = 0, 1, \ldots, [N/2]$ (where $[N/2]$ is the integral part of $N/2$). Then

$$\dim V_m = \dim \mathcal{P}_m - \dim \mathcal{P}_{m-1} = \begin{bmatrix} N \\ m \end{bmatrix}_q - \begin{bmatrix} N \\ m-1 \end{bmatrix}_q = \begin{bmatrix} N \\ m \end{bmatrix}_q \frac{q^m - q^{N-m+1}}{1-q^{N-m+1}}. \tag{2}$$

With every $g \in G \equiv GL(N,q)$ we associate the operator

$$(T(g)f)(\zeta) = f(g^{-1}\zeta), \quad f \in \mathfrak{L}^2(X).$$

Then for $\eta \subset g^{-1}\zeta$ we have $T(g)\widehat{\eta}(\zeta) = \widehat{\eta}(g^{-1}\zeta) = 1$, and if $\eta \not\subset g^{-1}\zeta$, then $T(g)\widehat{\eta}(\zeta) = 0$. Therefore, $T(g)\widehat{\eta} = (g\eta)\widehat{}$. This shows that $\mathcal{P}_m$ is invariant with respect to the action of $G$ on $\mathfrak{L}^2(X)$. Since $D$ commutes with $T(g)$, $g \in G$, then $V_m$ is also invariant with respect to $G$. The following statement holds [364]:

*The restrictions of $T$ onto the subspaces $V_m$, $0 \le m \le [N/2]$, are pairwise nonequivalent irreducible unitary representations of the group $G$. This statement remains valid if $GL(n,q)$ is replaced by $SL(n,q)$.*

We now evaluate spherical functions corresponding to one of $G$-orbits in $X$, for example, on the orbit $X_n$ consisting of linear subspaces of dimension $n$. We restrict the spaces $\mathcal{P}_m$ and $V_m$ onto $X_n$ (it is possible for $m \le n$) and choose in $X_n$ the subspace $\omega$ such that its stationary subgroup $H_n$ consists of the matrices

$$\begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix}, \; g_{11} \in GL(n,q), \; g_{22} \in GL(N-n,q), \; g_{12} \in \mathfrak{M}(n, N-n; q).$$

It is clear that $H_n$ is a maximal parabolic subgroup in $G$.

We want to evaluate zonal spherical functions on the space $X_n \sim G/H_n$, that is, $H_n$-invariant functions in the spaces $V_m$. All $H_n$-invariant functions are linear combinations of the functions $\hat{\zeta}$ over $H_n$-orbits. These sums are characterized by the integers $\dim\zeta$ and $\dim(\zeta \cap \omega)$. Hence, spherical functions are linear combinations of the functions

$$f_{mj} = \sum \left\{ \hat{\zeta} \mid \dim\zeta = m, \, \dim(\zeta \cap \omega) = j \right\},$$

where

$$0 \le m \le N, \quad \max(0, m+n-N) \le j \le \min(m,n).$$

Since we assume that $0 \le m \le n$, then $0 \le j \le m$.

Let us prove that the zonal spherical function on $G/H_n$ for the space $V_m$ coincides with

$$\varphi_m(g) = \frac{(-1)^m q^{-nm}}{(q^{N-n})_m (q^n)_m} \widetilde{Q}_m(\dim(\omega \cap g\omega); N-n, n, n; q^{-1}). \tag{3}$$

For this we find the solution of the equation $D \sum\limits_{j=0}^{m-1} \alpha_j f_{mj} = 0$. By virtue of (1) we have

$$\sum_{j=0}^{m-1} \left( \alpha_j \frac{q^{n-j} - q^{N-m+1}}{1-q} + \alpha_{j+1} \frac{1-q^{n-j}}{1-q} \right) f_{m-1,j} = 0$$

and, hence,

$$\alpha_j = q^{(N-m+1)j}\frac{(q^{n+m-N-1})_j}{(q^n)_j}\alpha_0.$$

The value of the function $f_{mj}$ at $\zeta$ is equal to the number of subspaces $\eta \subset \zeta$ such that $\dim \eta = m$, $\dim(\eta \cap \omega) = j$, and this value is $\begin{bmatrix} w \\ j \end{bmatrix}_q \begin{bmatrix} v-w \\ m-j \end{bmatrix}_q q^{(w-j)(m-j)}$, where $v = \dim \zeta$, $w = \dim(\zeta \cap \omega)$. Thus,

$$\sum_{j=0}^{m} \alpha_j f_{mj} = \alpha_0 \sum_{j=0}^{m} \frac{(q^{n+m-N-1})_j}{(q^n)_j} q^{(N-m+1)j} \begin{bmatrix} w \\ j \end{bmatrix}_q \begin{bmatrix} v-w \\ m-j \end{bmatrix}_q q^{(w-j)(m-j)}$$
$$= \alpha_0 \left[(q^n)_m (q^m)_m q^{m(m-1)/2}\right]^{-1} \tilde{Q}_m(w; N-m, n, n; q^{-1}).$$

To evaluate the value of the zonal spherical function at $g \in G$ we evaluate its value at $\zeta = g\omega$ and carry out the normalization $\varphi_m(e) = 1$. This leads to formula (3).

Analogously one finds the form of the spherical function invariant with respect to the subgroup $H_n$ on $G/H_v$. This function is a multiple of

$$\psi_m(g) = \tilde{Q}_m(\dim(g\omega_v \cap \omega), N-n, v, n; q^{-1}),$$

where $\omega_v$ is the $v$-dimensional subspace, invariant with respect to the subgroup $H_v$.

Irreducible representations of other Chevalley groups $G$, described in Section 13.2.1, are constructed in the same way with replacement of the maximal parabolic subgroup $H_n$ of $GL(N, q)$ by the corresponding subgroup of the Chevalley group $G$. Spherical functions on the spaces $G/H_n$, invariant with respect to the subgroup $H_v$ and corresponding to the irreducible representation, characterized by an integral non-negative number $m$, are given in Table 13.1 (see paper by Stanton in Ref. [3] of the first volume).

## Table 13.1

| Type of space | Group | Subgroups | Spherical functions |
|---|---|---|---|
| $A_{N-1}$ | $GL(N,q)$ | $H_n$, $H_v$, $n \leq v \leq N-n$ | $Q_m(q^{-k}; q^{n-N-1}, q^{-v-1}, n; q)$ |
| $B_N$ | $SO(2N+1,q)$ | $H_N$, $H_v$, $1 \leq v \leq N$ | $K_m(q^{-k}; q^{-N-2}, v; q)$ |
| $C_N$ | $Sp(2N,q)$ | $H_N$, $H_v$, $1 \leq v \leq N$ | $K_m(q^{-k}, q^{-N-2}, v; q)$ |
| ${}^2D_{N+1}$ | $SO^-(2N+2,q)$ | $H_N$, $H_v$, $1 \leq v \leq N$ | $K_m(q^{-k}; q^{-N-3}, v; q)$ |
| ${}^2A_{2N-1}$ | $SU(2N,q^2)$ | $H_N$, $H_v$, $1 \leq v \leq N$ | $K_m(q^{-2k}; q^{-2N-5}, v; q^2)$ |
| ${}^2A_{2N}$ | $SU(2N+1,q^2)$ | $H_N$, $H_v$, $1 \leq v \leq N$ | $K_m(q^{-2k}; q^{-2N-3}, v; q^2)$ |
| $D_N$ | $SO(2N,q)$ | $H_N$, $H_N$ | $K_m(q^{-2k}; q^{-N-1}, N; q)$ |
| $A_{N-1}$ | $H_n$ | $H \equiv GL(n,q) \times GL(N-n,q), H$ | $K_m^{\mathrm{Aff}}(q^{-k}; q^{n-N}, n; q)$ |
| $D_N$ | $H_N$ | $H \equiv GL(N,q), H$ | $K_m^{\mathrm{Aff}}(q^{-2k}; q^{1-N}, N/2; q^2)$ |
| ${}^2A_{2N-1}$ | $H_N$ | $H \equiv GL(N,q^2), H$ | $K_m^{\mathrm{Aff}}((-q)^{-k}; (-q)^N, N; -q)$ |

We denote the space $F^n$ from Section 13.2.1 by $E_N$, where $N$ is the dimensionality of a maximal isotropic subspace of $F^n$. The Chevalley groups $G$, introduced in Section 13.2.1, act in $E_N$. We denote by $X_N^N$ the space of isotropic subspaces of maximal dimensionality $N$ in $E_N$, by $\omega_N$ the isotropic subspace spanned by the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$. In the case of the group $SO(2N,q)$ we denote by $\hat{\omega}_N$ the isotropic subspace spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_{N-1}, \mathbf{e}_{-N}$. Except for the case of $SO(2N,q)$, the group $G$ acts transitively on $X_N^N$. In the case of $SO(2N,q)$ this space consists of two orbits

$$Y_1 = \left\{\alpha \in X_N^N \mid \dim(\omega_N \cap \alpha) \equiv N \pmod 2\right\},$$
$$Y_2 = \left\{\alpha \in X_N^N \mid \dim(\hat{\omega}_N \cap \alpha) \equiv N \pmod 2\right\}.$$

The space $\mathfrak{L}^2(X_N^N)$ decomposes into the direct sum of the subspaces $V_m$ on which shifts give pairwise nonequivalent representations $T_m$ of the group $G$. These subspaces consist of eigenfunctions of the operator $T$ acting on $\mathfrak{L}^2(X_N^N)$ by the formula

$$(Tf)(\alpha) = \sum_\beta \{f(\beta) \mid d(\alpha,\beta) = 1\},$$

where $f \in \mathfrak{L}^2(X_N^N)$, $\alpha \in X_N^N$ and $d(\alpha,\beta) = N - \dim(\alpha \cap \beta)$.

The number of elements of the space $X_k^N$, consisting of $k$-dimensional isotropic subspaces of $E_N$, is expressed by the formula

$$|X_k^N| = \frac{(p^N;p^{-1})_k(-cp^N;p^{-1})_k}{(p;p)_k},$$

where $p=q$ for all types of spaces except for the type ${}^2A_k$, for which $p=q^2$, and

| | | | |
|---|---|---|---|
| $c=1$ | for the types $C_N$ | and | $B_N$, |
| $c=q$ | for the types ${}^2A_{2N}$ | and | ${}^2D_{N+1}$, |
| $c=q^{-1}$ | for the types $D_N$ | and | ${}^2A_{2N-1}$. |

This equality is easily proved by induction in $k$.

The functions

$$t_{11}^m(g) \equiv f_m(\xi) = K_m(p^{-x};c^{-1}p^{-N-2},N;p),$$

where $\xi \in X_N^N$, $x = N-\dim(\xi\cap\omega_N)$ and $c$ has the above values, are the zonal spherical functions in the subspaces $V_m$. Taking into account the norm of Krawtchouk polynomials and the equality $\sum\limits_{g\in G} |t_{11}^m(g)|^2 = (\dim V_m)^{-1}$, following from the Peter-Weyl theorem, we obtain that

$$\dim V_m = \frac{(-cp;p)_N(p^{-N};p)_m}{(-cp^{-m+1};p)_{N-m}(p^{-1};p^{-1})_m(-c^{-1}p^{m-N-1};p)_m}.$$

The obvious relation

$$\sum_{m=0}^{N} \dim V_m = |X_N^N|$$

implies the summation formula for the basic hypergeometric function

$${}_6\varphi_5\left(\begin{matrix} a^2, qa, -qa, b, c, q^{-N} \\ a, -a, a^2q/b, a^2q^{N+1}\end{matrix}\,\middle|\, q, \frac{a^2q^{1+N}}{bc}\right) = \frac{(a^2;q)_N(a^2/bc;q)_N}{(a^2/b;q)_N(a^2/c;q)_N}.$$

Let us note that $V_m$ also contains functions invariant with respect to parabolic subgroups related to the isotropic subspaces $X_k^N$ of dimension $k$, $0\le k\le N$. These functions are multiples of

$$f_m^{(k)}(\xi) = K_m(p^{-x};c^{-1}p^{-N-2},k;p),$$

where $x = k-\dim(\xi\cap\omega_k)$, $\omega_k = \mathrm{span}\{\mathrm{e}_1,\dots,\mathrm{e}_k\}$.

For the group $SO(2N,q)$ we have the decomposition

$$\mathfrak{L}^2(Y_1) = \sum_{m=0}^{[N/2]} \oplus W_m$$

and the zonal spherical functions are of the form

$$f_m(\xi) = {}_3\varphi_2\left(\begin{matrix} q^{-2m}, q^{-2x}, q^{2m-2N} \\ q^{-N}, q^{1-N} \end{matrix} \,\middle|\, q^2; q^2\right),$$

where $\xi \in Y_1$, $2x = N - \dim(\xi \cap \omega_N)$. In addition, $\dim W_m = \dim V_m$ for $m < N/2$ and $2\dim W_{N/2} = \dim V_{N/2}$. These spherical functions can be expressed in terms of Krawtchouk polynomials:

$$f_m(\xi) = K_m(q^{-2x}; q^{-N-1}, N; q),\ \xi \in Y_1,\ 2x = N - \dim(\xi \cap \omega_N).$$

Comparing these two expressions for $f_m(\xi)$ and fulfilling analytical continuation in $a = q^{-N}$ and $b = q^{-m}$, we obtain

$${}_3\varphi_2\left(\begin{matrix} q^{-2x}, b, -ab^{-1} \\ 0, a \end{matrix} \,\middle|\, q; q\right) = {}_3\varphi_2\left(\begin{matrix} q^{-2x}, a^2b^{-2}, b^2 \\ a, aq \end{matrix} \,\middle|\, q^2; q^2\right).$$

It is an analog for ${}_3\varphi_2$ of the quadratic transformation for the ordinary hypergeometric function.

We also note the connection between $q$-Krawtchouk polynomials and harmonic analysis on the matrix group. We denote by $\mathfrak{M}(a,b;q)$ the space of $a \times b$ matrices with entries from the field $F \equiv GF(q)$, $q = p^s$. This set is an abelian group with respect to addition. We define the action of the group $GL(a,q) \times GL(b,q)$ on $\mathfrak{M}(a,b;q)$ by the formula

$$(g_1, g_2)m = g_1 m g_2^{-1},\ g_1 \in GL(a,q),\ g_2 \in GL(b,q).$$

By a character of an abelian group $G$ we mean a function $\chi$ on this group for which

$$|\chi(g)| = 1,\ \chi(g_1 + g_2) = \chi(g_1)\chi(g_2).$$

It is easy to check that the group of characters for $\mathfrak{M}(a,b;q)$ is isomorphic to $\mathfrak{M}(a,b;q)$ and that every character $\chi$ is given by the formula

$$\chi(T) = \alpha(\operatorname{Tr} ST), \qquad S \in \mathfrak{M}(b,a;q),$$

where $\alpha$ is the character of the additive group of the field $F$ taking the values $\exp 2\pi i r/p$, $r = 0, 1, \ldots, p-1$, only.

P. Delsarte proved the following theorem [67]: *Let $x$ and $m$ be integers such that $0 \le x, m \le \min(a,b)$. Let $S \in \mathfrak{M}(a,b;q)$ and let the rank of $S$ be equal to $x$. Then*

$$\sum \left\{\alpha(\operatorname{Tr} ST) \mid T \in \mathfrak{M}(b,a;q), \operatorname{rank} T = m\right\} = \widetilde{K}_m(x;a,b;q).$$

It is obvious that this function is invariant with respect to the action of the group $H \equiv GL(a,q) \times GL(b,q)$ on $\mathfrak{M}(a,b;q)$ and, hence, is a spherical function on the homogeneous space $G/H$, where $G$ consists of the matrices

$$\begin{pmatrix} g_1 & m \\ 0 & g_2 \end{pmatrix}, \quad g_1 \in GL(a,q), \ g_2 \in GL(b,q), \ m \in \mathfrak{M}(a,b;q).$$

**13.2.4. Functional relations for $q$-analogs of orthogonal polynomials.** Irreducible representations of Chevalley groups, considered above, are realized in the spaces which can turn into homogeneous graphs. Therefore, for the obtained spherical functions three-term recurrence relations (see Section 13.1.5) hold. They lead to recurrence formulas for the $q$-Hahn and $q$-Krawtchouk polynomials related to these spherical functions. Below we give these relations. For the $q$-Hahn polynomials $Q_j(q^{-x};q^\alpha,q^\beta,N;q)$ we have

$$A_j Q_{j+1}(q^{-x}) - (A_j + C_j - q^{-x} + 1) Q_j(q^{-x}) + C_j Q_{j-1}(q^{-x}) = 0,$$

where $Q_n(q^{-x}) \equiv Q_n(q^{-x};q^\alpha,q^\beta,N;q)$ and

$$A_j = \frac{(1-q^{n+\alpha+\beta+1})(1-q^{n+\alpha-1})(1-q^{n-N})}{(1-q^{2n+\alpha+\beta+1})(1-q^{2n+\alpha+\beta+2})},$$

$$C_j = -\frac{(1-q^n)(1-q^{n+\beta})(q^{-N-\beta-1}-q^{n+\alpha})q^{\alpha+\beta+n+1}}{(1-q^{2n+\alpha+\beta})(1-q^{2n+\alpha+\beta+1})}.$$

For the polynomials $\widetilde{Q}_k(x;a,b,c;q^{-1})$, related to $Q_j(q^x)$, the relation

$$\begin{aligned}(q^{a-x}-1)\widetilde{Q}_k(x+1;a,b,c+1;q^{-1}) + (q^{a+b-c}-q^{a-x})\widetilde{Q}_k(x;a,b,c+1;q^{-1})\\ = (q^{a+b-c}-q^k)\widetilde{Q}_k(x;a,b,c;q^{-1})\end{aligned}$$

holds. By replacing $q^{-1}$ by $q$ we obtain the polynomials $\widetilde{Q}_k(x;a,b,c;q)$. We have

$$\begin{aligned}\widetilde{Q}_k(x;a,b,c;q) &= (q^a)_k q^{-k(a+c)+k(k-1)/2} \\ &\quad \times \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(q^x)_j}{(q^a)_j} (q^{k-b-1})_j (q^{c-x})_{k-j} q^{j(c-x-k+j)} \\ &= (-1)^k q^{-k(a+b+c)+k(k-1)/2} \widetilde{Q}_k(c-x;b,a,c;q^{-1}),\end{aligned}$$

where $(a)_n = (a; q^{-1})_n$. Therefore, the orthogonality relation for $\widetilde{Q}_k(x; a, b, c; q)$ is of the form

$$\sum_{x=\max(0,c-b)}^{\min(a,c)} \begin{bmatrix} a \\ x \end{bmatrix}_q \begin{bmatrix} b \\ c-x \end{bmatrix}_q q^{x(b-c+x)} \widetilde{Q}_k(x; a, b, c; q) \widetilde{Q}_n(x; a, b, c; q)$$
$$= \begin{bmatrix} a+b-2k \\ c-k \end{bmatrix}_q (q^a)_k (q^b)_k (q^k)_k (q^{a+b-k+1})_k \times q^{-k(2a+b+c-k+2)} \delta_{kn}.$$

For the $q$-Eberlane polynomials we have

$$(1-q^k)(1-q^{k-N-\beta-1}) E_{k-1}(\lambda(x)) + q^{-N-\beta}(q^N - q^k)(q^{-\alpha-1} - q^k) E_{k+1}(\lambda(x))$$
$$- \left\{ -\lambda(x) q^{-\alpha-\beta-1} + (1-q^k)(1-q^{k-N-\beta-1}) + q^{-N-\beta}(q^N - q^k)(q^{-\alpha-1} - q^k) \right\}$$
$$\times E_k(\lambda(x)) = 0,$$

where $E_n(\lambda(x)) \equiv E_n(\lambda(x); q^\alpha, q^\beta, N; q)$.

For the $q$-Krawtchouk polynomials $K_n(q^{-x}) \equiv K_n(q^{-x}; c, N; q)$ one has the second order difference equation

$$c q^{N+1}(1-q^x) K_j(q^{1-x}) - (q^N - q^x) K_j(q^{-1-x})$$
$$- \left\{ q^x(1 - c q^{N+1}) + \lambda(j) \right\} K_j(q^{-x}) = 0, \quad 0 \le x \le N,$$

where $\lambda(j) = q^{N-j}(1 - c q^{2j+1})$.

For the affine $q$-Krawtchouk polynomials $K_n^{\mathrm{Aff}}(q^{-x}) \equiv K_n^{\mathrm{Aff}}(q^{-x}; q^{-a}, N; q)$ the recurrence relation

$$(q^N - q^k)(q^a - q^k) K_{k+1}^{\mathrm{Aff}}(q^{-x}) + q^{k-1}(q^k - 1) K_{k-1}^{\mathrm{Aff}}(q^{-x})$$
$$- \left\{ (q^N - q^k)(q^a - q^k) + q^{k-1}(q^k - 1) + \lambda(x) \right\} K_k^{\mathrm{Aff}}(q^{-x}) = 0$$

holds, where $\lambda(x) = -q^{a+N}(1 - q^{-x})$.

Applying the method described in Section 13.1.6, one derives addition theorems for $q$-Krawtchouk and $q$-Hahn polynomials. By making use of the spherical functions of the group $GL(n, q)$, the addition theorem [85] for the $q$-Hahn polynomials $\widetilde{Q}_m$ is derived.

If $a$, $b$, $k$, $x$, $y$, $v$, $w$, $z$ are integers such that

$$1 \le k \le a \le b, \; 0 \le x, y \le a, \; \max(0, x+y-a) \le v \le \min(x, y),$$
$$\max(0, 2a-b-x-y) \le w \le \min(a-x, a-y), \; 0 \le z \le \min(w, a-x-y+v),$$

then

$$\widetilde{Q}_k(v+w-z; a, b, a; q^{-1}) = \sum_{m,n,r} c_{mnrk}(a,b) q^{(n+k)(x+y)}$$
$$\times \widetilde{Q}_{k-m-n-r}(x-m; a-r-2m, b-r-2n, a-m-n-r; q^{-1})$$
$$\times \widetilde{Q}_{k-m-n-r}(y-m; a-r-2m, b-r-2n, a-m-n-r; q^{-1})$$
$$\times \widetilde{Q}_m(v; y, a-r-y, x; q^{-1}) \widetilde{Q}_m(w-r; a-r-y, b-a+y, a-r-x; q^{-1})$$
$$\times \widetilde{K}_r(z; a-x-y+v, w; q), \qquad (1)$$

where the summation is over non-negative integral values of $m$, $n$, $r$, for which

$$m+n+r \le k,\ 2m+r \le a,\ 2n+r \le b,\ k-a \le n-m \le b-k,$$

and

$$c_{mnrk}(a,b) = (-1)^{k-m-n} q^{(k-m-n-r)(3m+n+r)-a(k-m)-r+rk+n(m+r)} \times \frac{(q^k)_{m+n+r}(q^{a+b-k+1})_{m+n+r}(q^{a-r-2m+1})_1(q^{b-r-2n+1})_1}{(q^m)_m(q^n)_n(q^{a-r-m})_{k-n-r}(q^{b-r-n})_{k-m-r}(q^{a-r-m+1})_1(q^{b-r-n+1})_1}.$$

Formula (1) is also valid in the case when $b$ is a non-positive integer. In this case we use the bounds that hold for $b \ge 2a$:

$$0 \le w \le \min(a-y, a-x),\ m+n+r \le k,\ 2m+r \le a,\ k-a \le n-m.$$

When $q \to 1$, then equality (1) gives the addition theorem for ordinary Hahn polynomials (see formula (3) of Section 13.1.6).

By using the spherical functions of the representations of the group $G = SO(2N,q)$, we obtain the addition theorem for the $q$-Krawtchouk polynomials. Namely, if $k$, $a$, $b$, $c$, $x$, $N$ are non-negative integers such that

$$0 \le k \le N, \quad 0 \le N-x-a \le \min(N-x, N-b),$$
$$0 \le b-a \le x, \quad 0 \le 2c \le b-a,$$

and $p^s = q$, where $p$ is an odd prime number, then

$$\begin{aligned} K_k(q^{x-b+2a+2c}; q^{-N-1}, N; q) = \sum_{m,r} a_{mr}(k,N) q^{-rx} \\ \times K_{k-m-r}(q^{x-2r-m}; q^{-N+2m+2r-1}, N-2m-2r; q) q^{-rb} \\ \times K_{k-m-r}(q^{b-2r-m}; q^{-N+2m+2r-1}, N-2m-2r; q) \\ \times \widetilde{Q}_m(N-x-a; N-x, x-2r, N-b; q^{-1}) K_r^{\mathrm{Aff}}\left(c; q^{-2[\frac{b-a-1}{2}]-1}, \left[\frac{b-a}{2}\right]; q^2\right), \end{aligned} \tag{2}$$

where $[n/2]$ is the integral part of $n/2$,

$$a_{mr}(k,N) = \frac{(-q^{-k};q)_{m+r}(q^{-k};q)_{m+r}(-q^{k-N};q)_{m+r}(q^{k-N};q)_{m+r}}{(q;q)_m(q^{N-2r-m};q^{-1})_m(q^{-N};q)_{2m+2r}(1-q^{N-m-2r+1})} \times (q^m - q^{N-m-2r+1})(-1)^m q^{-mN+(m+2r+1)(m-2r)/2}$$

and the summation is over the domain

$$0 \le m+2r \le \min(x, b, N-m), \quad 0 \le m+r \le \min(k, N-k),$$
$$0 \le m \le \min(N-b, N-x), \quad 0 \le 2r \le b-a.$$

Using the group $U(2N,q^2)$, we derive another addition theorem. Let $k$, $a$, $b$, $c$, $x$, $N$ be non-negative integers such that

$$0 \le N-x-a \le \min(N-x, N-b),$$
$$0 \le k \le N, \quad 0 \le b-a \le x, \quad 0 \le c \le b-a$$

and $q = p^s$, where $p$ is an odd prime number. Then

$$K_k(q^{x-b+2a+c}; q^{-N-3/2}, N; q^2) = \sum_{m,r} a_{mr}(k,N) q^{-2xv}$$
$$\times K_{k-m-v}(q^{x-m-r}; q^{-N+2m+2v-3/2}, N-2m-r; q^2) q^{-2bv}$$
$$\times K_{k-m-v}(q^{b-m-r}; q^{-N+2m+2v-3/2}, N-2m-r; q^2)$$
$$\times \widetilde{Q}_m(N-x-a; N-x, x-r, N-b; q^{-2}) K_r^{\mathrm{Aff}}(c; -(-q)^{a-b}, b-a; -q), \quad (3)$$

where $v = [(r+1)/2]$,

$$a_{mr}(k,N) = \frac{(-q^{-2k-1}; q^2)_{m+[r/2]} (q^{-2k}; q^2)_{m+v} (-q^{2k-2N-1}; q^2)_{m+v}}{(q^2;q^2)_m (q^{2(N-r-m)}; q^{-2})_m (q^{-2N}; q^2)_{2m+r} (1-q^{2(N-m-r+1)})}$$
$$\times (q^{2(k-N)}; q^2)_{m+[r/2]} (q^{2m} - q^{2(N-m-r+1)}) (-1)^m q^{(m+r)(m+r+A)-2mN}.$$

Here $A = 0$ if $r$ is even, $A = 2$ if $r$ is odd, and the summation is over the domain

$$0 \le m+r \le \min(x, b, N-m), \ 0 \le m \le \min(N-b, N-x),$$
$$0 \le r \le b-a, \quad 0 \le k-m-v \le N-2m-r.$$

These addition formulas for the $q$-Krawtchouk polynomials admit an analytic continuation. Namely, formula (2) admits the analytic continuation in $q$, $q^{-N}$, $q^{-x}$, $q^{-a}$, $q^{-b}$, $q^{-c}$ and formula (3) admits the analytic continuation in $q$, $(-q)^{-x}$, $(-q)^{-N}$, $(-q)^{-a}$, $(-q)^{-b}$, $(-q)^{-c}$.

The addition theorems derived above imply corresponding product formulas. For example, we have the following formulas. Let $k$, $s$, $t$, $n$, $m$, $v$ be non-negative integers and $q = p^r$, where $p$ is an odd prime number. If $0 \le k, s, t \le n \le m \le v-n$, then

$$\frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} v-m \\ k \end{bmatrix}_q}{\begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} v-n \\ k \end{bmatrix}_q} q^{(m-n)k} Q_k(q^{-s}; q^{m-v-1}, q^{-m-1}, n; q)$$
$$\times Q_k(q^{-t}; q^{m-v-1}, q^{-m-1}, n; q)$$
$$= \sum_{\alpha,\beta,\gamma,\theta} \frac{\begin{bmatrix} m-n \\ \alpha \end{bmatrix}_q \begin{bmatrix} n-t \\ n-s-\alpha-\beta \end{bmatrix}_q}{\begin{bmatrix} m \\ n-s \end{bmatrix}_q \begin{bmatrix} v-n \\ s+\alpha \end{bmatrix}_q} \begin{bmatrix} t \\ \beta \end{bmatrix}_q \begin{bmatrix} m-n+t \\ s+\alpha-\gamma \end{bmatrix}_q \begin{bmatrix} v-m-t \\ \gamma \end{bmatrix}_q$$
$$\times \begin{bmatrix} t-\beta \\ \theta \end{bmatrix}_q \begin{bmatrix} s+\alpha-\gamma \\ \theta \end{bmatrix}_q (q;q)_\theta q^{\theta(\theta-1)/2} (-1)^\theta q^A Q_k(q^{-\theta-\beta-\gamma}; q^{m-v-1}, q^{-m-1}, n; q),$$

where

$$A = (s+\alpha)(2\beta+\alpha-\gamma-t)+\beta(\beta-t)-\gamma(\beta-\gamma-2t+n-m).$$

Let $k$, $s$, $t$, $n$, $m$, $v$ be integers, $q = p^r$, where $p$ is an odd prime number, and

$$0 \le k, s, t \le n \le m \le n-v.$$

Then

$$\begin{aligned}
&Q_k(q^{-s}; q^{n-v-1}, q^{-n-1}, n; q) Q_k(q^{-t}; q^{m-v-1}, q^{-m-1}, n; q) \\
&\qquad = \sum_{\alpha,\beta,\gamma} \frac{\begin{bmatrix} n-t \\ n-s-\alpha \end{bmatrix}_q \begin{bmatrix} t \\ \alpha \end{bmatrix}_q \begin{bmatrix} m-n+t \\ s-\beta \end{bmatrix}_q}{\begin{bmatrix} n \\ s \end{bmatrix}_q \begin{bmatrix} v-n \\ s \end{bmatrix}_q} \begin{bmatrix} v-m-t \\ \beta \end{bmatrix}_q \\
&\qquad \times \begin{bmatrix} t-\alpha \\ \gamma \end{bmatrix}_q \begin{bmatrix} s-\beta \\ \gamma \end{bmatrix}_q (q;q)_\gamma (-1)^\gamma q^{\gamma(\gamma-1)/2} q^A \\
&\qquad\qquad \times Q_k(q^{-\alpha-\beta-\gamma}; q^{-m-v-1}, q^{-m-1}, n; q),
\end{aligned}$$

where

$$A = (\alpha+s-\beta)(s+\alpha-t)+\beta(m+t-n+\beta)-s^2,$$

and

$$\begin{aligned}
&\frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} v-m \\ k \end{bmatrix}_q}{\begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} v-n \\ k \end{bmatrix}_q} q^{(m-n)k} Q_k(q^{-s}; q^{m-v-1}, q^{-m-1}, n; q) \\
&\qquad \times Q_k(q^{-t}; q^{m-v-1}, q^{-m-1}, n; q) \\
&\qquad = \sum_{\alpha,\beta,\gamma} \frac{\begin{bmatrix} n-t \\ n-s-\alpha \end{bmatrix}_q \begin{bmatrix} m-n+t \\ \alpha \end{bmatrix}_q}{\begin{bmatrix} m \\ n-s \end{bmatrix}_q \begin{bmatrix} v-m \\ s \end{bmatrix}_q} \begin{bmatrix} t \\ s-\beta \end{bmatrix}_q \begin{bmatrix} v-m-t \\ \beta \end{bmatrix}_q \\
&\qquad \times \begin{bmatrix} m-n+t-\alpha \\ \gamma \end{bmatrix}_q \begin{bmatrix} s-\beta \\ \gamma \end{bmatrix}_q (q;q)_\gamma (-1)^\gamma q^{(\gamma-1)\gamma/2} q^A \\
&\qquad\qquad \times Q_k(q^{-\alpha-\beta-\gamma}; q^{n-v-1}, q^{-n-1}, n; q),
\end{aligned}$$

where

$$A = (\alpha+s-\beta)(\alpha+s-t)+\beta(m-n+t+\beta)-s(m-n+s).$$

Taking into account the orthogonality relation for the polynomials $K_r^{\mathrm{Aff}}$, we derive from (2) the relation

$$\sum_t K_k(q^{x-b-2a-2t};q^{-N-1},N;q)K_\ell^{\mathrm{Aff}}\left(q^{-t};q^{-1-2[(b-a-1)/2]},\begin{bmatrix}b-a\\2\end{bmatrix};q^2\right)$$
$$\times \begin{bmatrix}N\\t\end{bmatrix}_{q^2}(q^{2+2(b-a-1)};q^2)_t(-1)^t q^{t(t-1)}q^{-N-2N[(b-a-1)/2]}$$
$$=\sum_m a_{mr}(k,N)q^{-rx}K_{k-m-r}(q^{x-2r-m};q^{-N+2m+2r-1},N-2m-2r;q)$$
$$\times q^{-rb}K_{k-m-r}(q^{b-2r-m};q^{-N+2m+2r-1},N-2m-2r;q)$$
$$\times \frac{1}{h_\ell}\widetilde{Q}_m(N-x;x-2r,N-b,N-x-a;q^{-1}),$$

where

$$h_\ell=\begin{bmatrix}N\\\ell\end{bmatrix}_q(q^{1+2[(b-a-1)/2]};q^{-1})_\ell(-1)^\ell q^{(\ell-1)\ell/2}.$$

An analogous formula follows from (3).

Zonal spherical functions of irreducible representations of a group $G$ with respect to a subgroup $H$ satisfy the relation

$$\varphi_{11}^m(g_1)\varphi_{11}^m 9g_2)=|H|^{-1}\sum_{h\in H}\varphi_{11}^m(g_1hg_2).$$

By applying this formula we obtain the following product formulas for the $q$-Krawtchouk polynomials. If $s$, $t$, $m$, $N$ are integers, $q=p^n$, where $p$ is an odd prime number, and $0\le s,t,m\le N$, then

$$K_m(q^{-s};q^{-N-2},N;q)K_m(q^{-t};q^{-N-2},N;q)$$
$$=\sum_{x,y}A(x,y,s,t,N,q)K_m(q^{-(s-t+2x+y)};q^{-N-2},N;q),$$

where

$$A(x,y,s,t,N,q)=\frac{\begin{bmatrix}N-t\\N-s-t\end{bmatrix}_q\begin{bmatrix}t\\x\end{bmatrix}_q(q^{t-x};q^{-1})_y}{\begin{bmatrix}N\\s\end{bmatrix}_q(q^{-2};q^{-2})_{[y/2]}}$$
$$\times(-1)^y q^{xs+\frac{1}{2}(x(x+1)-t(t+1))},$$

$$K_m(q^{-s};q^{-N-1},N;q)K_m(q^{-t};q^{-N-1},N;q)$$
$$=\sum_{x,y}A(x,y,s,t,N,q)K_m(q^{-(s-t+2x+2y)};q^{-N-1},N;q),$$

where

$$A(x,y,s,t,N,q) = \frac{\left[{N-t \atop N-s-t}\right]_q \left[{t \atop x}\right]_q (q^{t-x};q^{-1})_{2y}}{\left[{N \atop s}\right]_q (q^2;q^2)_y} \times (-1)^y q^{xs+y(y-1)+\frac{1}{2}(x(x-1)-t(t-1))},$$

and

$$K_m(q^{-2s};q^{-2N-3},N;q^2)K_m(q^{-2t};q^{-2N-3},N;q^2) = \sum_{x,y} A(x,y,s,t,N,q)K_m(q^{-2(s-t+2x+y)};q^{-2N-3},N;q^2),$$

where

$$A(x,y,s,t,N,q) = \frac{\left[{N-t \atop N-s-t}\right]_{q^2} \left[{t \atop x}\right]_{q^2} (q^{t-x};q^{-2})_y}{\left[{N \atop s}\right]_{q^2} (-q;-q)_y} \times (-1)^{y(y+1)/2} q^{2xs-t^2+x^2+y(y-1)/2}.$$

The problems on Clebsch-Gordan and Racah coefficients for Chevalley groups and on their relations to basic hypergeometric functions is now in the initial stage of study. We do not consider them.

**13.2.5. Functions invariant with respect to subgroups of block-triangular matrices and $q$-analogs of orthogonal polynomials in two variables.** By analogy with the polynomials studied in Section 13.1.7, we consider polynomials on the group $G = GL(N,F)$, $F = GF(q)$, which are invariant with respect to the subgroup $H$ of block-triangular matrices of the form

$$\begin{matrix} & a & b & c \\ a & * & * & * \\ b & 0 & * & * \\ c & 0 & 0 & * \end{matrix} \quad (1)$$

We shall also study the connection of these polynomials with polynomials invariant with respect to the subgroups of block-triangular matrices of the form

$$\begin{matrix} & a & b & c \\ a & * & * & * \\ b & 0 & * & * \\ c & 0 & * & * \end{matrix} \quad \text{and} \quad \begin{matrix} & a & b & c \\ a & * & * & * \\ b & * & * & * \\ c & 0 & 0 & * \end{matrix} \quad (2)$$

(the indices $a$, $b$, $c$ indicate the dimensionalities of blocks in matrices).

Let $q = p^{n}$, where $p$ is a prime number. We denote by $\Omega$ the set of all linear subspaces of $F^N$, $F = GF(q)$, and by $\mathfrak{L}(\Omega)$ the space of functions on $\Omega$. Further, we set $\Omega_M = \{\omega \in \Omega \mid \dim \omega = M\}$ and denote by $\mathfrak{L}(\Omega_M)$ the space of all functions on $\Omega_M$. This space is invariant with respect to the action of $G = GL(N, F)$. If $\omega \in \Omega_M$, then $\hat{\omega}(\xi)$ will denote the function from $\mathfrak{L}(\Omega)$ which is equal to 1 if $\omega \subset \xi$ and to 0 otherwise. Then $\mathfrak{L}(\Omega) = \sum_{r=0}^{N} \oplus \mathcal{P}_r$, where $\mathcal{P}_r$ is the linear subspace generated by functions $\hat{\omega}$ for which $\dim \omega = r$. Let us note that $\lambda(g)\hat{\omega} = (g\omega)\hat{}$, where $\lambda(g)f(x) = f(g^{-1}x)$. For $\xi \in \Omega$ we set $d\xi = \sum\{\hat{\eta} \mid \eta \subset \xi, \dim \eta = \dim \xi - 1\}$ and extend $d$ onto $\mathfrak{L}(\Omega)$ by linearity. Then $V_r = \mathcal{P}_r \cap \ker d$, $0 \le r \le N/2$, are the carrier spaces of the irreducible representations obtained under restriction of $\lambda$ onto $V_r$.

We shall find functions in $V_r$ invariant with respect to the action of the subgroup of matrices of type (1). Let $a$, $b$, $c$ be non-negative integers such that $a + b + c = N$. We choose $\zeta_1, \zeta_2 \in \Omega$ such that $\zeta_1 \subset \zeta_2$, $\dim \zeta_1 = a$, $\dim \zeta_2 = a + b$ and set $H_{abc} = \{g \in G \mid g\zeta_i = \zeta_i,\ i = 1, 2\}$. In addition, we set $H_{a,b+c} = \{g \in G \mid g\zeta_1 = \zeta_1\}$, $H_{a+b,c} = \{g \in G \mid g\zeta_2 = \zeta_2\}$.

As in Section 13.1.7, we prove that the space of $H_{abc}$-invariant functions in $\mathcal{P}_r$ $(0 \le r \le N)$ has the basis $\{g^r_{xy} \mid (x, y) \in D_r\}$, where

$$g^r_{xy} = \sum\{\hat{\omega} \mid \dim \omega = r, \dim(\omega \cap \zeta_1) = x, \dim(\omega \cap \zeta_2) = r - y\}$$

and

$$D_r = \{(x, y) \mid x, y \in \mathbb{Z};\ 0 \le x \le a,\ 0 \le y \le c,\ r - b \le x + y \le r\}.$$

For $(x, y) \in D_r$ we have

$$g^r_{xy}(\xi) = \begin{bmatrix} u_1 \\ x \end{bmatrix}_q \begin{bmatrix} u_2 \\ r - x - y \end{bmatrix}_q \begin{bmatrix} u_3 \\ y \end{bmatrix}_q q^{(u_1 - x)(r - x - y) + y(u_1 + u_2 - r + y)},\ \xi \in \Omega,$$

where $u_1 = \dim(\xi \cap \zeta_1)$, $u_2 = \dim(\xi \cap \zeta_2) - u_1$, $u_3 = \dim \xi - u_1 - u_2$. In addition, $g^r_{xy}$ is different from zero if and only if the conditions $x \le u_1$, $y \le u_3$, $r - y - x \le u_2$ are fulfilled. If $\dim \xi = r$, then $g^r_{xy}(\xi) = 0$ except for the case $x = u_1$, $y = u_3$ when $g^r_{xy}(\xi) = 1$. In the sequel we put $g^r_{xy} = 0$ for $(x, y) \bar{\in} D_r$.

One has the equality

$$dg^r_{xy} = \frac{1}{q - 1}\Big[(q^{a+b-r+y+1} - q^{a-x})g^{r-1}_{xy} + (q^{a-x+1} - 1)g^{r-1}_{x-1,y} + q^{a+b-r+1}(q^c - q^{y-1})g^{r-1}_{x,y-1}\Big]$$

which follows from combinatorial reasons. If $f \in \mathfrak{L}(D_r)$, then $\sum_{(xy) \in D_r} f(x, y)g^r_{xy} \in V_r$ if and only if $f$ satisfies the $q$-difference equation

$$(q^{a-x} - 1)f(x + 1, y) + q^{a-x}(q^{b-r+x+y+1} - 1)f(x, y) + (q^c - q^y)q^{a+b-r+1}f(x, y + 1) = 0, \qquad (3)$$

where $0 \le x \le a$, $0 \le y \le c$, $r-b-1 \le x+y \le r-1$ and $f(a+1,y)$, $f(x,c+1)$, $f(x,r-b-x-1)$ have been set to be zero.

We denote by $W_r$ the space of solutions of equation (3). One can prove that

$$\dim W_r = \min(r,a,b,c,a+b-r,b+c-r,a+c-r,a+b+c-2r)+1$$

for $r \le \min(a+b, a+c, b+c, N/2)$ and $W_r = \{0\}$ otherwise.

The following formula provides the expression for a function $f(x,y) \in W_r$ in terms of its boundary values:

$$f(x,y) = \sum_{j=\max(0,r-c)}^{\min(a,r)} f(j,r-c) \begin{bmatrix} r-x-y \\ j-x \end{bmatrix}_q \frac{(q^{a-x})_{j-x}(q^{c-y})_{r-j-y}}{(q^b)_{r-x-y}} \times (-1)^{r-x-y} q^{(j-a-1)(j-x)+b(r-j-y)-(r-x-y)(r-x-y-1)/2}, \quad (4)$$

where, we recall, $(a)_n = (a;q^{-1})_n$. If $b \ge r$, then the values $f(j,r-j)$ can be chosen arbitrarily.

The functions $g^r_{xy}$ form an orthogonal basis in $\mathfrak{L}^2(\Omega_r)$ with respect to the scalar product

$$\langle f_1, f_2 \rangle = \sum_{\omega \in \Omega_r} f_1(\omega)\overline{f_2(\omega)}$$

and $\|g^r_{xy}\|^2$ is equal to the number $m^r_{xy}$ of subspaces $\omega \in \Omega_r$ with

$$\dim(\omega \cap \zeta_1) = x, \ \dim(\omega \cap \zeta_2) = r-y,$$

that is,

$$m^r_{xy} = \begin{bmatrix} a \\ x \end{bmatrix}_q \begin{bmatrix} b \\ r-x-y \end{bmatrix}_q \begin{bmatrix} c \\ y \end{bmatrix}_q q^{(a-x)(r-x-y)+y(b+a-r+y)}. \quad (5)$$

Therefore, we define the scalar product and the norm in $W_r$ as follows:

$$(f_1, f_2) = \sum_{(x,y)\in D_r} m^r_{xy} f_1(x,y)\overline{f_2(x,y)}, \quad \|f\| = (f_1,f_2)^{1/2}.$$

In the space $V_r$ there are orthogonal bases, agreeing with the decomposition of $V_r$ into nonequivalent irreducible subspaces, invariant with respect to the action of $H_{a+b,c}$ or of $H_{a,b+c}$. Moreover, every one of these subspaces contains not more than one $H_{abc}$-invariant (up to a constant).

Taking into account the form of the weight function and the case analyzed in Section 13.1.7, we represent the solution $f(x,y)$ of equation (3) in the form

$$f(x,y) = g(y)\widetilde{Q}_m(x;a,b,r-y;q^{-1}).$$

By using the equality

$$(q^{a-x}-1)\widetilde{Q}_m(x+1;a,b,c+1;q^{-1})+(q^{a+b-c}-q^{a-x})\widetilde{Q}_m(x;a,b,c+1;q^{-1}) = (q^{a+b-c}-q^m)\widetilde{Q}_m(x;a,b,c;q^{-1})$$

(see Section 13.2.4), we conclude that $g(y)$ satisfies the equation

$$g(y+1)=\frac{(1-q^{m-a-b+r-y-1})}{(1-q^{c-y})}g(y).$$

Therefore, from the equality

$$\widetilde{Q}_m(x;a,b,m;q)=(q^a)_m(q^m)_m q^{-m(a+m)+m(m-1)/2}\frac{(q^{m-b-1})_x}{(q^a)_x}$$

we derive that $g(y)=\widetilde{Q}_{r-m}(y;c,a+b-2m;r-m;q)$. As in Section 13.1.7, one proves that nontrivial solutions exist if and only if

$$\max(0,r-c)\le m\le \min(a,b,r,a+b-r).$$

Thus, we have the solution

$$\varphi_{rm}(x,y)=\widetilde{Q}_{r-m}(y;c,a+b-2m,r-m;q)\widetilde{Q}_m(x;a,b,r-y;q^{-1}) \qquad (6)$$

(one could express both factors in terms of $\widetilde{Q}_n(x;\alpha,\beta,\gamma;q^{-1})$, but (6) agrees with the expressions from Section 13.1.7).

The orthogonality relations for $\widetilde{Q}_m(x;a,b,c;q^{-1})$ and $\widetilde{Q}_m(x;a,b,c;q)$ imply the orthogonality relation for $\varphi_{rm}$:

$$(\varphi_{rm},\varphi_{rn})=(q^m)_m(q^{r-m})_{r-m}(q^a)_m(q^b)_m(q^{a+b-m+1})_m(q^c)_{r-m} \times (q^{a+b-2m})_{r-m}(q^{a+b+c-m-r+1})_{r-m}q^M\delta_{mn}, \qquad (7)$$

where $M=m(2a+b+2c+3r-2m+1)-r(a+b+2c+2)$.

Since difference equation (3) and the weight $m^r_{xy}$ from (5) are invariant (up to a factor) with respect to the simultaneous replacements $a\leftrightarrow c$, $x\leftrightarrow y$, $q\leftrightarrow q^{-1}$, then (6) gives another orthogonal system of solutions for (3):

$$\psi_{rk}(x,y)=\widetilde{Q}_{r-k}(x;a,b+c-2k,r-k;q^{-1})\widetilde{Q}_k(y;c,b,r-x;q), \qquad (8)$$

where

$$\max(0,r-a)\le k\le \min(b,c,r,b+c-r).$$

The orthogonality relation has the form

$$(\psi_{rk},\psi_{rn}) = (q^k)_k(q^{r-k})_{r-k}(q^b)_k(q^c)_k(q^{b+c-k+1})_k \times (q^a)_{r-k}(q^{b+c-2k})_{r-k}(q^{a+b+c-k-r+1})_{r-k}q^M\delta_{kn}, \qquad (9)$$

where $M = -k(b+2c+3r-2k+1)+r(a+r-1)$.

If $f\in W_r$, then $f=\sum_m \alpha_m\varphi_{rm}$, where, by virtue of the orthogonality relation (7), we have

$$\alpha_m = \sum_x \begin{bmatrix} a \\ x \end{bmatrix}_q \begin{bmatrix} b \\ r-x-y \end{bmatrix}_q q^{(a-x)(r-x-y)} f(x,y)\widetilde{Q}_m(x;a,b,r-y;q^{-1}).$$

In the same way as in Section 13.1.7, this double sum can be replaced by one sum. We have

$$\alpha_m = (-1)^r \frac{q^{r(a+b+c+1)-m(a+c+2r-m)}}{(q^c)_{r-m}(q^m)_m(q^b)_m(q^{a+b-m+1})_m(q^{a+b-2m})_{r-m}(q^{r-m})_{r-m}} \times \sum_{j=\max(0,r-c)}^{\min(a,r)} f(j,r-j)(-1)^j q^{-j(a+b+1)+j(j-1)/2}(q^c)_{r-j}(q^m)_j(q^{a+b-m+1})_j/(q^{-1})_j,$$

where $\max(0,r-c)\le m\le \min(a,b,r,a+b-r)$.

By applying this formula to the functions $\psi_{rk}$, we derive that for

$$\max(0,r-a)\le k\le \min(b,c,r,b+c-r)$$

the equality

$$\psi_{rk} = \sum_m \alpha_{km}\varphi_{rm}$$

holds. Here

$$\alpha_{km} = (-1)^{r+k+m}\frac{q^A(q^a)_{r-k}(q^r)_m(q^b)_k(q^{a+b+c-r+1})_m}{(q^m)_m(q^{a+b-m+1})_m(q^{a+b-2m})_{r-m}(q^a)_m} \times {}_4\varphi_3\left(\begin{matrix} q^m, q^{a+b-m+1}, q^k, q^{b+c-k+1} \\ q^r, q^b, q^{a+b+c-r+1}\end{matrix} \,\middle|\, q^{-1}, q^{-1}\right) \qquad (10)$$

and $A=(r-k-m)\left(a+b+c+2r+\frac12\right)+\frac32(m^2+k^2-r^2)+ak$. Let us note that ${}_4\varphi_3$ from this formula can be considered either as a polynomial of degree $k$ in $q^m+q^{a+b-m+1}$ or as a polynomial of degree $m$ in $q^k+q^{b+c-k+1}$.

One can prove that if $k=0$, $a\ge r$, or if $k=r-a$, $r>a$, or if $k=r$, $b\ge r$, or if $k=b$, $b<r$, then the expression for ${}_4\varphi_3$ has no summation. Namely, ${}_4\varphi_3$ is equal to 1 if $k=0$, is equal to

$$q^{m(r-a)}(q^a)_m(q^{a+b-r})_m/(q^r)_m(q^b)_m$$

if $k = r - a$, is equal to

$$q^{m(c+1)}(q^{a+b-r})_m(q^{r-c-1})_m/(q^r)_m(q^{a+b+c-r+1})_m$$

if $k = b$, is equal to

$$q^{m(b+c-r+1)}(q^a)_m(q^{r-c-1})_m/(q^b)_m(q^{a+b+c-r+1})_m$$

if $k = r$.

For $\max(0, r-c) \le m \le \min(a, b, r, a+b-r)$ we have

$$\varphi_{rm} = \sum_k \beta_{mk}\psi_{rk},$$

where

$$\beta_{mk} = (-1)^{r+m+k}\frac{q^B(q^r)_k(q^b)_m(q^c)_{r-m}(q^{a+b+c-r+1})_k}{(q^k)_k(q^c)_k(q^{b+c-k+1})_k(q^{b+c-2k})_{r-k}} \times {}_4\varphi_3\left(\begin{matrix} q^m, q^{a+b-m+1}, q^k, q^{b+c-k+1} \\ q^r, q^b, q^{a+b+c-r+1}\end{matrix}\,\middle|\, q^{-1}; q^{-1}\right) \quad (11)$$

and $B = (k+m-r)\left(a+c+r+\frac{1}{2}\right) + \frac{1}{2}(r^2 - m^2 - k^2) - ak$.

Finally, we give the equality

$$\sum_m \|\varphi_{rm}\|^2 \alpha_{km}\alpha_{\ell m} = \delta_{k\ell}\|\psi_{rk}\|^2, m \quad (12)$$

where

$$\max(0, r-c) \le m \le \min(a, b, r, a+b-r),$$
$$\max(0, r-a) \le k, \ell \le \min(b, c, r, b+c-r).$$

It follows from the fact that both sides are equal to $(\psi_{rk}, \psi_{r\ell})$.

Now we establish the connection of the funtions, considered above, with $q$-analogs of Hahn polynomials in two variables. To every $f \in W_r$ there corresponds the function

$$\hat{f} = \sum_{(x,y)\in D_r} f(x,y)g_{xy}^r$$

on $\Omega$. In addition, the values $g_{xy}^r(\xi)$, $\xi \in \Omega$, depend on integers $u_1$, $u_2$, $u_3$, where

$$u_1 = \dim(\xi \cap \zeta_1),\ u_2 = \dim(\xi \cap \zeta_2) - u_1,\ u_3 = \dim \xi - u_1 - u_2.$$

Under this correspondence to the functions $\varphi_{rm}$ and $\psi_{rk}$, there correspond the functions

$$\hat{\varphi}_{rm}(u_1,u_2,u_3) = q^{(r-m)(u_1+u_2+u_3-r)}\widetilde{Q}_{r-m}(u_3;c,a+b-2m,u_1+u_2+u_3-m;q) \times \widetilde{Q}_m(u_1;a,b,u_1+u_2;q^{-1}) \quad (13)$$

and

$$\hat{\psi}_{rk}(u_1,u_2,u_3) = q^{k(u_1+u_2+u_3-r)}\widetilde{Q}_{r-k}(u_1;a,b+c-2k,u_1+u_2+u_3-k;q^{-1}) \times \widetilde{Q}_k(u_3;c,b,u_2+u_3;q). \quad (14)$$

It follows from here that

$$\hat{\psi}_{rk} = \sum_m \alpha_{km}\hat{\varphi}_{rm}, \quad \hat{\varphi}_{rm} = \sum_k \beta_{mk}\hat{\psi}_{rk}.$$

Every set $\Omega_M$ is a homogeneous space for the group $G = GL(N,F)$ corresponding to a subgroup $BW_{J_M}B$, where $B$ is the subgroup of $G$ consisting of upper triangular matrices and $W_{J_M}$ is a subgroup generated by transpositions. By the Schur lemma the restriction of $V_r$ onto $\Omega_M$ is isomorphic to $V_r$ if $r \le \min(M, N-M)$ and coincides with $\{0\}$ otherwise. Therefore, we have two orthonormal bases $\{\hat{\varphi}_{rm}\}$ and $\{\hat{\psi}_{rk}\}$ for the space of $H_{abc}$-invariant functions in $\mathfrak{L}(\Omega_M)$. For the first basis

$$0 \le r \le \min(M, N-M), \ \max(0, r-a) \le m \le \min(b,c,r,b+c-r)$$

and for the second one

$$0 \le r \le \min(M, N-M), \ \max(0, r-c) \le k \le \min(a,b,r,a+b-r).$$

The orthogonality for $r \neq s$ follows from the fact that the corresponding representations are nonequivalent.

In order to obtain orthogonal polynomials in two variables we make the substitution $u_3 = M - u_1 - u_3$. Then the function $\hat{\varphi}_{rm}$ is a polynomial in $q^{u_1}$ and in $q^{-u_3}$ of total degree $r$ and of degree $m$ in $q^{u_1}$ (it is the result of Gram's orthogonalization process applied to the functions 1, $q^{-u_3}$, $q^{u_1}$, $q^{-2u_3}$, $q^{u_1-u_3}$, $q^{2u_1}, \dots$). Analogously, $\hat{\psi}_{rk}$ is a polynomial in $q^{u_1}$ and in $q^{-u_3}$ of total degree $r$ and of degree $k$ in $q^{-u_3}$.

In the coordinates $(u_1,u_3)$ the weight for $\mathfrak{L}(\Omega_M)$ is written in the form

$$\begin{bmatrix} a \\ u_1 \end{bmatrix}_q \begin{bmatrix} b \\ M-u_1-u_3 \end{bmatrix}_q \begin{bmatrix} c \\ u_3 \end{bmatrix}_q q^A,$$

where $A = (a-u_1)(M-u_1-u_3)+u_3(a+b-M+u_3)$. It is equal to the number of $M$-dimensional subspaces $\omega$ for which

$$\dim(\omega\cap\zeta_1)=u_1,\ \dim(\omega\cap\zeta_2)=M-u_3.$$

In conclusion, we consider the $q$-analog of Jacobi polynomials in two variables. For this we use for $q>1$ the limit

$$\lim_{M\to\infty} q^{-mM}\widetilde{Q}_m(M-x;a,b,M;q^{-1}) = (-1)^m(q^a)_m {}_2\varphi_1\left(\begin{matrix} q^m, q^{a+b-m+1} \\ q^a \end{matrix}\,\middle|\, q^{-1};q^{-x-1}\right). \tag{15}$$

The right hand side is a polynomial of degree $m$ in $q^{-x}$. For $m=0,1,2,\dots$ we set

$$p_m(q^{-x};\alpha,\beta\mid q^{-1}) = {}_2\varphi_1\left(\begin{matrix} q^m, \alpha\beta q^{-m-1} \\ \alpha q^{-1} \end{matrix}\,\middle|\, q^{-1};q^{-x-1}\right), \tag{16}$$

where $\alpha\neq q^n$, $n=1,2,3,\dots$ . These are the *little q-Jacobi polynomials* of degree $m$ in $q^{-x}$. They are orthogonal with respect to the weight

$$(\beta q^{-1})_x(\alpha q^{-1})^x/(q^{-1})_x,\quad x=0,1,2,\dots,\ .$$

This weight is positive valued and summable if either $0<\alpha<q$, $\beta<q$ (infinite support) or $\beta=q^{N+1}$, $N=0,1,2,\dots$ , and $\alpha<0$ (finite $m\le N$). In the second case these polynomials are connected with the $q$-Krawtchouk polynomials by the equality

$$p_m\left(q^{-x};\alpha,q^{N+1}\mid q^{-1}\right) = \frac{(q^N)_m}{(\alpha q^{-m})_m}K_m\left(q^{-x};\frac{1}{\alpha q^{N+1}},N;q\right). \tag{17}$$

In the case of infinite support the orthogonality relation is of the form

$$\sum_{x=0}^{\infty}\frac{(\beta q^{-1})_x}{(q^{-1})_x}\alpha^x q^{-x}p_n(q^{-x};\alpha,\beta\mid q^{-1})p_m(q^{-x};\alpha,\beta\mid q^{-1}) = \delta_{mn}\frac{(\alpha\beta q^{-2})_\infty}{(\alpha q^{-1})_\infty}\frac{(q^{-1})_m(\beta q^{-1})_m(1-\alpha\beta q^{-m-1})}{(\alpha q^{-1})_m(\alpha\beta q^{-2})_m(1-\alpha\beta q^{-2m-1})}\alpha^m q^{-m}. \tag{18}$$

Let us note the formulas

$$p_m(1;\alpha,\beta\mid q^{-1}) = (-1)^m\alpha^m q^{-m-m(m-1)/2}\frac{(\beta q^{-1})_m}{(\alpha q^{-1})_m},$$

$$\lim_{M\to\infty} q^{-mM}\widetilde{Q}_m(M-x;a,b,M;q^{-1}) = (-1)^m(q^a)_m p_m(q^{-x};q^{a+1},q^{b+1}\mid q^{-1}),$$

$$\lim_{M\to\infty} \widetilde{Q}_m(x;a,b,M;q) = q^{-m(a+b)+m(m-1)/2}(q^b)_m p_m(q^{-x};q^{b-1},q^{a-1} \mid q^{-1}).$$

We now apply the same limit process to the functions $\hat{\varphi}_{rm}$ and $\hat{\psi}_{rk}$. We obtain

$$\lim_{M\to\infty} q^{-rM}\hat{\varphi}_{rm}(M-x,x-y,y) = (-1)^m(q^a)_m(q^{a+b-2m})_{r-m} \times q^A\Phi_{rm}(x,y;q^{a+1},q^{b+1},q^{c+1} \mid q^{-1}),$$

$$\lim_{M\to\infty} q^{-rM}\hat{\psi}_{rk}(M-x,x-y,y) = (-1)^r(q^a)_{r-k}(q^b)_k \times q^{-k(b+c+2r-2k+1)}\Psi_{rk}(x,y;q^{a+1},q^{b+1},q^{c+1} \mid q^{-1}),$$

where

$$A = -r(a+b+c+r) + \frac{r(r-1)}{2} + m(a+b+c-2m+2r+1) + \frac{m(m-1)}{2},$$

$$\Phi_{rm}(x,y;\alpha,\beta,\gamma \mid q^{-1}) = p_{r-m}(q^{-y};\alpha\beta q^{-2m-1},\gamma \mid q^{-1}) \times p_m(q^{y-x};\alpha,\beta \mid q^{-1})q^{-my}, \quad (19)$$

$$\Psi_{rk}(x,y;\alpha,\beta,\gamma \mid q^{-1}) = p_{r-k}(q^{k-x};\alpha,\beta\gamma q^{-2k-1} \mid q^{-1}) \times q^{-kx}(q^x)_k Q_k(q^{x-y};\beta,\gamma,x;q^{-1}) \quad (20)$$

and

$$Q_k(q^x;\alpha,\beta,N;q^{-1}) = {}_3\varphi_2\left(\begin{matrix} q^k,\alpha\beta q^{-k-1},q^x \\ \alpha q^{-1},q^N \end{matrix} \middle| q^{-1};q^{-1}\right) \quad (21)$$

are $q$-Hahn polynomials. If $q > 0$, $0 \le y \le x$, we obtain the weight functions by means of the limit process. Namely, if $x$, $y$ are integers such that $0 \le y \le x$, then

$$w_{\alpha\beta\gamma}(x,y) = \frac{(\beta q^{-1})_{x-y}(\gamma q^{-1})_y}{(q^{-1})_{x-y}(q^{-1})_y}\alpha^x\beta^y q^{-x-y}$$

is a positive weight function in the following cases:

1) $0 < \alpha < q$, $0 < \beta < q$, $\gamma < q$,
2) $0 < \alpha < q$, $-q < \beta < 0$, $\gamma = q^{c+1}$,
3) $\alpha < 0$, $\beta = q^{b+1}$, $\gamma = q^{c+1}$,

where $b, c \in \mathbf{Z}$ and $b \ge 0$, $c \ge 0$. The collection of functions $\{\Phi_{rm}\}$ is a complete system of orthogonal polynomials with respect to the weight $w_{\alpha\beta\gamma}$ and we have

$$\sum_{0\le y\le x} w_{\alpha\beta\gamma}(x,y)\Phi_{rm}(x,y;\alpha,\beta,\gamma \mid q^{-1})\Phi_{sn}(x,y;\alpha,\beta,\gamma \mid q^{-1}) = \delta_{rs}\delta_{mn}h_{\alpha\beta\gamma}, \quad (22)$$

where

$$h_{\alpha\beta\gamma} = (-1)^r q^{m(m-r+1)-3r-r(r-1)/2} \frac{(\alpha\beta\gamma q^{-r-3})_\infty (1-\alpha\beta\gamma q^{-r-2})}{(\alpha q^{-1})_\infty (1-\alpha\beta\gamma q^{-2r-2})}$$
$$\times \alpha^r \beta^{r-m} \frac{(q^m)_m (q^{r-m})_{r-m} (\beta q^{-1})_m (\gamma q^{-1})_{r-m} (\alpha\beta q^{-m-1})_m}{(\alpha q^{-1})_m (\alpha\beta\gamma q^{-r-2})_m (\alpha\beta q^{-2m-2})_{r-m}}. \quad (22')$$

In the case 1) this system is infinite and $0 \le m \le r$. In the case 2) the system is infinite and $\max(0, r-c) \le m \le r$. In the case 3) the system is finite and $\max(0, r-c) \le m \le \min(b, r)$.

Analogously, the set $\{\Psi_{rk}\}$ is a complete system of orthogonal polynomials with respect to the weight $w_{\alpha\beta\gamma}$ and

$$\sum_{0\le y\le x} w_{\alpha\beta\gamma}(x,y)\Psi_{rk}(x,y;\alpha,\beta,\gamma \mid q^{-1})\Psi_{sn}(x,y;\alpha,\beta,\gamma \mid q^{-1})$$
$$= \delta_{rs}\delta_{kn} h'_{\alpha\beta\gamma}, \quad (23)$$

where

$$h'_{\alpha\beta\gamma} = (-1)^r q^{k(r-2k)-2r-r(r-1)/2} \frac{(\alpha\beta\gamma q^{-r-3})_\infty (1-\alpha\beta\gamma q^{-r-2})}{(\alpha q^{-1})_\infty (1-\alpha\beta\gamma q^{-2r-2})}$$
$$\times \alpha^r \beta^k \frac{(q^k)_k (q^{r-k})_{r-k} (\gamma q^{-1})_k (\beta\gamma q^{-k-1})_k (\beta\gamma q^{-2k-2})_{r-k}}{(\beta q^{-1})_k (\alpha q^{-1})_{r-k} (\alpha\beta\gamma q^{-r-2})_k}. \quad (23')$$

In the case 1) the system is infinite and $0 \le k \le r$. In the case 2) it is infinite and $0 \le k \le \min(c, r)$. In the case 3) the system is finite and $0 \le k \le \min(b, c, r, b+c-r)$.

Let us note that the set $\{\Phi_{rm}\}$ is obtained by application of Gram's orthogonalization process to the functions $1, q^{-y}, q^{-x}, q^{-2y}, q^{-x-y}, q^{-2x}, \ldots$, and $\{\Psi_{rk}\}$ is obtained by application of this process to the functions $1, q^{-x}, q^{-y}, q^{-2x}, q^{-x-y}, q^{-2y}, \ldots$.

By taking the limits we obtain from the relations for the functions $\varphi_{rm}$ and $\psi_{rk}$ that

$$\Psi_{rk}(x,y;\alpha,\beta,\gamma \mid q^{-1}) = \sum_{m=0}^{r} A_{km} C_{km} \Phi_{rm}(x,y;\alpha,\beta,\gamma \mid q^{-1}), \quad (24)$$

$$\Phi_{rm}(x,y;\alpha,\beta,\gamma \mid q^{-1}) = \sum_{k=0}^{r} B_{mk} C_{km} \Psi_{rk}(x,y;\alpha,\beta,\gamma \mid q^{-1}), \quad (25)$$

where

$$A_{km} = (-1)^k q^{-k(k-1)/2} \frac{(q^r)_m(\alpha\beta\gamma q^{-r-1})_m}{(q^m)_m(\alpha\beta q^{-m-1})_m},$$

$$B_{mk} = (-1)^k \beta^{r-m-k} q^{-m(r-m-1)-r-k(r-k-1)+k(k-1)/2}$$
$$\times \frac{(q^r)_k(\beta q^{-1})_m(\beta q^{-1})_k(\alpha q^{-1})_{r-k}(\gamma q^{-1})_{r-m}(\alpha\beta\gamma q^{-r-2})_k}{(q^k)_k(\alpha q^{-1})_m(\gamma q^{-1})_k(\beta\gamma q^{-k-1})_k(\beta\gamma q^{-2k-2})_{r-k}(\alpha\beta q^{-2m-2})_{r-m}},$$

$$C_{km} = {}_4\varphi_3\left(\begin{matrix} q^m, \alpha\beta q^{-m-1}, q^k, \beta\gamma q^{-k-1} \\ q^r, \beta q^{-1}, \alpha\beta\gamma q^{-r-1} \end{matrix}\,\middle|\, q^{-1}; q^{-1}\right).$$

Multiplying both sides of (24) by $(\beta q^{-1})_{x-y}(\alpha q^{-1})^{x-y}/(q^{-1})_{x-y}$ and summing over the values $x = y + n$, $n = 0, 1, 2, \ldots$, for $k = 0$ we obtain

$$p_r(q^{-y}; \alpha\beta q^{-1}, \gamma \mid q^{-1})$$
$$= \frac{(\alpha q^{-1})_\infty}{(\alpha\beta q^{-2})_\infty} \sum_{x=y}^{\infty} \frac{(\beta q^{-1})_{x-y}}{(q^{-1})_{x-y}} (\alpha q^{-1})^{x-y} p_r(q^{-x}; \alpha, \beta\gamma q^{-1} \mid q^{-1}). \quad (26)$$

In the same way, (25) leads to the equality

$$\frac{p_r(q^{-x}; \alpha, \beta\gamma q^{-1} \mid q^{-1})}{p_r(1, \alpha, \beta\gamma q^{-1} \mid q^{-1})} = \frac{(q^{-1})x(\beta q^{-1})^x}{(\gamma\beta q^{-2})_x}$$
$$\times \sum_{y=0}^{x} \frac{(\beta q^{-1})_{x-y}(\gamma q^{-1})_y}{(q^{-1})_{x-y}(q^{-1})_y} (\beta q^{-1})^{y-x} \frac{p_r(q^{-y}; \alpha\beta q^{-1}, \gamma \mid q^{-1})}{p_r(1; \alpha\beta q^{-1}, \gamma \mid q^{-1})} \quad (27)$$

for $m = 0$ and to the equality

$$\frac{p_r(q^{-x}; \alpha, \beta\gamma q^{-1} \mid q^{-1})}{p_r(1; \alpha, \beta\gamma q^{-1} \mid q^{-1})} = \frac{(q^{-1})_x(\beta q^{-1})^x}{(\gamma\beta q^{-2})_x}$$
$$\times \sum_{y=0}^{x} \frac{(\beta q^{-1})_{x-y}(\gamma q^{-1})_y}{(q^{-1})_{x-y}(q^{-1})_y} (\beta q^{-1})^{y-x} q^{-ry} \frac{p_r(q^{y-x}; \alpha, \beta \mid q^{-1})}{p_r(1; \alpha, \beta \mid q^{-1})} \quad (28)$$

for $m = r$. Similarly, one can derive relations for $q$-Hahn polynomials.

## 13.3. Special Functions, Related to Locally Compact Totally Disconnected Fields, and Group Representations

**13.3.1. Characters of locally compact abelian groups.** Let $G$ be a locally compact abelian group. Recall that by a *character* of $G$ we mean a continuous function $\chi$ on $G$ such that $\chi(g_1 + g_2) = \chi(g_1)\chi(g_2)$. In other words, a character of a locally compact abelian group $G$ is its one-dimensional representation.

Usually one considers unitary characters, that is, characters for which $|\chi(g)| = 1$ for all $g \in G$. In the sequel, unless otherwise stipulated, we shall regard only unitary characters of abelian groups.

Since

$$(\chi_1\chi_2)(g_1 + g_2) = \chi_1(g_1 + g_2)\chi_2(g_1 + g_2)$$
$$= \chi_1(g_1)\chi_1(g_2)\chi_2(g_1)\chi_2(g_2) = \chi_1\chi_2(g_1)\chi_1\chi_2(g_2),$$

then the product of two characters is a character of the same group. Besides, $\chi^{-1}$ is also a character. Thus, the set $X$ of characters of a group $G$ is a group with respect to multiplication.

We equip $X$ with the topology for which the complete system of neighborhoods of a character $\chi_0$ consists of all sets $U(A, \varepsilon, \chi_0)$, where $\varepsilon > 0$, $A$ is a compact subset of $G$ and

$$U(A, \varepsilon, \chi_0) = \left\{\chi \in X \;\middle|\; |\chi(g) - \chi_0(g)| < \varepsilon \quad \text{for all} \quad g \in A\right\}.$$

One can show that the group $X$ is locally compact in this topology. In particular, if $G$ is a discrete group, then $X$ is compact, and if $G$ is compact, then $X$ is discrete. If $|G| = q$, $q < \infty$, then $|X| = q$. In this case the groups $G$ and $X$ are isomorphic.

The following theorem is the main statement of the theory of characters, constructed by L. S. Pontryagin. *If $G$ is a locally compact abelian group and $X$ is its group of characters, then the group of characters of $X$ is isomorphic to $G$.* This isomorphism is given in the following way: to an element $g \in G$ there corresponds the character $\tilde{g}$ on $X$, given by the formula $\tilde{g}(\chi) \equiv \chi(g)$. The groups $G$ and $X$ are called *dual by Pontryagin.*

**Example 1.** Let $\mathbb{Z}_n$ be the cyclic group of order $n$ (see Section 1.0.1). A character $\chi$ of $\mathbb{Z}_n$ is defined by a number from $\mathbb{Z}_n$ denoted by the same symbol $\chi$. The character $\chi$ has the form

$$\chi(a) = \exp\frac{2\pi i\chi a}{n}, \qquad a \in \mathbb{Z}_n.$$

**Example 2.** We denote by $\mathbb{Z}_n^\infty$ the group of sequences $\mathbf{a} = (a_1, a_n, \dots)$, $a_k \in \mathbb{Z}_n$, with the group operation

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1,\ a_2 + b_2, \dots).$$

The topology in $\mathbb{Z}_n^\infty$ is given by neighborhoods $U_m(\mathbf{a})$, $\mathbf{a} \in \mathbb{Z}_n^\infty$, $m = 0, 1, 2, \dots$. The neighborhood $U_m(\mathbf{a})$ of a sequence $\mathbf{a}$ consists of all sequences whose first $m$ coordinates coincide with those in $\mathbf{a}$. A character $\boldsymbol{\chi}$ of the group $\mathbb{Z}_n^\infty$ has the form $\boldsymbol{\chi} = (\chi_1, \chi_2, \dots)$ where $\chi_k$, $k = 1, 2, \dots$, are characters of the group $\mathbb{Z}_n$, and all $\chi_k$, except for a finite number, coincide with the identity character $\chi_0 \equiv 1$. We have

$$\boldsymbol{\chi}(\mathbf{a}) = \prod_{k=1}^{\infty} \exp\frac{2\pi i\chi_k a_k}{n}.$$

In addition, $\boldsymbol{\chi}'\boldsymbol{\chi}'' = (\chi_1'\chi_1'', \chi_2'\chi_2'', \dots)$.

The set of characters $\chi \in X$, having the value 1 on a subgroup $H$ of a group $G$, forms the subgroup of $X$, denoted by $H^{\perp}$. It is called the ***annihilator*** of $H$. It is easy to show that $H^{\perp}$ is isomorphic to the group of characters for $G/H$, and $X/H^{\perp}$ is isomorphic to the group of characters for $H$.

**Example 3.** If characters $\chi_\lambda$ of the group $\mathbb{R}$ are given by the formula $\chi_\lambda(x) = \exp i\lambda x$, then the annihilator of the subgroup $\mathbf{Z} \subset \mathbb{R}$ coincides with the subgroup $2\pi\mathbf{Z}$ and, hence, the group $2\pi\mathbf{Z} \sim \mathbf{Z}$ is the group of characters for $\mathbb{R}/\mathbf{Z} \sim \mathbf{T}$.

Harmonic analysis on locally compact abelian groups is constructed analogously to classical harmonic analysis. With every continuous finite function $f$ on $G$ one associates the function

$$\hat{f}(\chi) = \int_G f(g)\chi(g)dg, \qquad \chi \in X, \tag{1}$$

where $dg$ is the invariant measure on $G$. The function $\hat{f}$ is called the ***Fourier transform*** of $f$. Under corresponding normalization of the invariant measure on $X$ the mapping $f \to \hat{f}$ is extended to an isometric mapping of $\mathfrak{L}^2(G)$ onto $\mathfrak{L}^2(X)$ and we obtain the Fourier transform of functions from $\mathfrak{L}^2(G)$. The inversion formula for this transform is of the form

$$f(g) = \int_X \hat{f}(\chi)\overline{g(\chi)}d\chi, \tag{2}$$

where $g(\chi) = \chi(g)$. In addition, the Plancherel formula

$$\int_G |f(g)|^2 dg = \int_X |\hat{f}(\chi)|^2 d\chi \tag{3}$$

holds.

All main concepts and assertions of classical harmonic analysis (for example, the convolution theorem, Bochner theorem on positively definite functions) with the corresponding changes are valid for the Fourier transforms on abelian groups. In particular, we note that if $f_{g_0}(g) \equiv f(g + g_0)$, then

$$\hat{f}_{g_0}(\chi) = \overline{\chi(g_0)}\hat{f}(\chi), \tag{4}$$

if $f^*(g) = \overline{f(-g)}$, then

$$\hat{f}^*(\chi) = \overline{\hat{f}(\chi)}, \tag{5}$$

if

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(g-h)\,dh,$$

then

$$(f_1 * f_2)\hat{}(\chi) = \hat{f}_1(\chi)\hat{f}_2(\chi), \tag{6}$$

and so on.

Let $G$ be a locally compact commutative group and $X$ be its group of characters. Further, let $H$ be a closed subgroup in $G$ and $H^{\perp}$ be the annihilator of $H$. We normalize the invariant measures on $G$ and on $X$ in such a way that equality (3) holds. Then the Haar measures on the groups $H$ and $H^{\perp}$ can be normalized in such a way that for every function $f(g)$, $g \in G$, for which the projection onto $K = G/H$ belongs to $\mathfrak{L}^1(K) \cap \mathfrak{L}^{\infty}(K)$ and the Fourier transform of this projection belongs to the space $\mathfrak{L}^1(H^{\perp}) \cap \mathfrak{L}^{\infty}(H^{\perp})$, the equality

$$\int_H f(h)\,dh = \int_{H^{\perp}} \hat{f}(\chi)\,d\chi \tag{6'}$$

takes place.

The Poisson summation formula (5) of Section 12.5.2 is a special case of this statement which is obtained when

$$G = X = \mathbb{R}, \qquad H = \mathbb{Z}, \qquad H^{\perp} = 2\pi\mathbb{Z}.$$

If $G$ is a totally disconnected (in other words, zero-dimensional and periodic) group, then harmonic analysis is simply constructed. A locally compact abelian group $G$ is said to be ***totally disconnected*** if any neighborhood $U$ of zero in $G$ contains an open subgroup. A group $G$ is called (topologically) ***periodic*** if a closure of any cyclic subgroup of $G$ is compact.

If a totally disconnected group $G$ has a denumerable fundamental system of neighborhoods of zero, then it has a fundamental system of open compact subgroups

$$U_0 \supset U_1 \supset \ldots \supset U_n \supset \ldots\ , \qquad \bigcap_n U_n = \{0\}. \tag{7}$$

Without losing generality, we can assume that every quotient group $U_n/U_{n+1}$ is the cyclic group of order $p_n$. If we set $\mu(U_0) = 1$, then $\mu(U_k) = (p_1 p_2 \ldots p_{k-1})^{-1}$. All cosets $a + U_k$ have the same measure. This defines the invariant measure on $G$. For periodic groups the system of subgroups (7) can be extended to the left

$$\ldots \supset U_{-k} \supset \ldots \supset U_{-1} \supset U_0 \supset U_1 \supset \ldots \supset U_k \supset \ldots \tag{7'}$$

in such a way that $\bigcup_n U_n = G$ and all $U_n/U_{n+1}$ are cyclic subgroups of orders $p_n$. The annihilators of subgroups (7′) form the analogous chain in $X$:

$$\ldots \supset U_k^\perp \supset \ldots \supset U_1^\perp \supset U_0^\perp \supset U_{-1}^\perp \supset \ldots \supset U_{-k}^\perp \supset \ldots, \tag{8}$$

where $\bigcap_n U_n^\perp = \{0\}$, $\bigcup_n U_n^\perp = X$.

Let us note the following property of characters which will be used in the sequel. If a character $\chi$ of a group $G$ does not belong to the annihilator of a compact subgroup $H$, then

$$\int_{a+H} \chi(h)dh = 0. \tag{9}$$

Indeed, in this case there exists $h_0 \in H$ such that $\chi(h_0) \neq 1$. Then

$$\int_{a+H} \chi(h)dh = \int_{a+H} \chi(h+h_0)dh = \chi(h_0) \int_{a+H} \chi(h)dh.$$

Since $\chi(h_0) \neq 1$, then these equalities imply (9).

**13.3.2. The field of $p$-adic numbers and other totally disconnected locally compact fields.** Let $p$ be a prime number. By a *rational $p$-adic number* one means an infinite to both sides sequence

$$\mathbf{a} = (\ldots, a_{-n}, \ldots, a_0, \ldots, a_n, \ldots), \tag{1}$$

for which $a_k \in \{0, 1, \ldots, p-1\}$ and there exists $N \in \mathbb{Z}_+$ such that $a_{-n} = 0$ for $n \geq N$.

Sequence (1), in which only a finite number of coordinates differ from zero, is said to be *finite*. With every finite sequence $\mathbf{a}$ we associate the rational number $\sum_{n=-\infty}^{\infty} a_n p^n$. This allows us to reduce addition and multiplication of finite $p$-adic numbers to operations over rational numbers. In order to extend these operations onto the whole set $Q_p$ of rational $p$-adic numbers we equip $Q_p$ with the topology. We denote by $U_n(\mathbf{a})$ the set of $p$-adic numbers $\mathbf{b}$ from $Q_p$, for which $b_i = a_i$ if $i > n$. It is easy to verify that $\{U_n(0)\}$ gives a fundamental system of neighborhoods of zero in $Q_p$. Therefore, $\{U_n(0)\}$ defines a topology in $Q_p$. One can prove that $Q_p$ is a locally compact space in this topology.

The sum $\sum_{k=-\infty}^{n} a_k p^k$ is called the *$n$-th approximation of the number* $\mathbf{a}$ and is denoted by $\mathbf{a}^{(n)}$. It is obvious that $\mathbf{a} = \lim_{n\to\infty} \mathbf{a}^{(n)}$. We set

$$\mathbf{a} + \mathbf{b} = \lim_{n\to\infty} (\mathbf{a}^{(n)} + \mathbf{b}^{(n)}),$$

$$\mathbf{ab} = \lim_{n\to\infty} \mathbf{a}^{(n)}\mathbf{b}^{(n)}.$$

These relations define operations in $Q_p$ which are continuous in the topology introduced. The field axioms are easily verified.

In the sequel we shall denote elements $\mathbf{a} \in Q_p$ by

$$\mathbf{a} = \sum_{n=-\infty}^{\infty} a_n p^n.$$

It is clear that $Q_p$ contains the field of rational numbers as an everywhere dense subset.

The fields $Q_p$ are examples of locally compact totally disconnected non-discrete fields. *Every locally compact totally disconnected field $K$ of characteristic 0 is either one of the fields of rational p-adic numbers or a finite extension of such field. If the characteristic of a locally compact totally disconnected non-discrete field $K$ is equal to p, then $K$ is isomorphic to the field of formal power series (of the Laurent type) over a finite field $F$ of characteristic p:*

$$\left\{ \sum_{k=-\infty}^{\infty} a_k t^k \mid a_k \in F \text{ and almost all } a_{-k},\ k \in \mathbb{Z}_+, \text{ are equal to } 0 \right\}.$$

We have two groups related to a field $K$. They are the additive group $K^+$ and the multiplicative group $K^*$. We denote the invariant measure on $K^+$ by $dx$ and the invariant measure on $K^*$ by $d^*x$:

$$d(x+a) = dx, \quad d^*(ax) = d^*(x).$$

It is clear that $d(ax)$ differs from $dx$ only in a positive factor depending on $a$. This factor is denoted by $|a|$ and is called the *module* of $a$: $d(ax) = |a|dx$. One easily checks that

$$d^*x = |x|^{-1}dx.$$

One can directly verify the following properties of $|x|$:

a) $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$;

b) $|xy| = |x| \cdot |y|$,

c) $|x+y| \leq \max(|x|, |y|)$ and $|x+y| = \max(|x|, |y|)$ if $|x| \neq |y|$.

Elements $x \in K$ such that $|x| \leq 1$ form the subring $\mathfrak{O}$ in $K$. The set $\mathfrak{O}$ is compact and open. If $x \in \mathfrak{O}$, $|x| = 1$, then $x^{-1} \in \mathfrak{O}$. Elements of $\mathfrak{O}$ are called *integral elements* of $K$. The measure $dx$ on $K^+$ will be normalized in such way that the measure of $\mathfrak{O}$ is equal to 1.

We denote by $\mathfrak{P}$ the ideal in $\mathfrak{O}$ consisting of elements $x$, for which $|x| < 1$. It is the single maximal ideal of $\mathfrak{O}$; moreover, it is generated by one element, denoted by $\mathfrak{p}$. In the case $K = Q_p$ the ring $\mathfrak{O}$ consists of elements of the form $\sum_{k=0}^{\infty} a_k p^k$, the ideal $\mathfrak{P}$ consists of elements of the form $\sum_{k=1}^{\infty} a_k p^k$ and $\mathfrak{p}$ coincides with $p$.

It is easy to verify that $F = \mathfrak{O}/\mathfrak{P}$ is a finite field and, hence, contains $q = p^s$ elements, where $p$ is a prime number. In addition, $|\mathfrak{p}| = q^{-1}$ and for all $x \in K$ either $|x| = 0$ (for $x = 0$) or $|x| = q^n$, where $n$ is an integer. It follows from here that the measure of $\mathfrak{P}^n$ is equal to $q^{-n}$, $n \geq 1$, and the measure of the set $\{x \mid |x| = q^n\}$ is equal to $q^n/q'$, where $\frac{1}{q} + \frac{1}{q'} = 1$. For $n \in \mathbf{Z}$ we have $\mathfrak{P}^n = \{x \mid |x| \leq q^{-n}\}$. Then the chain

$$\mathfrak{P} \equiv \mathfrak{P}^1 \supset \mathfrak{P}^2 \supset \ldots \supset \mathfrak{P}^n \supset \ldots$$

forms a fundamental sequence of compact neighborhoods of zero element of $K^+$.

The group $K^+$ has a character $\chi$ which is trivial on $\mathfrak{O}$ and nontrivial on $\mathfrak{P}^{-1}$, that is, $\chi \in \mathfrak{O}^{\perp} \backslash (\mathfrak{P}^{-1})^{\perp}$. It is easy to prove that any character of $K^+$ is of the form $\chi_u(x) = \chi(ux)$, where $u \in K$. Moreover, $\chi_u \neq \chi_{u'}$, if $u \neq u'$. Therefore, the isomorphism $u \to \chi_u$ between the group $K^+$ and the group $\widehat{K}^+$ of its characters is given. In other words, the group $K^+$ is *dual by Pontryagin to itself* (as in the case of the additive group $\mathbf{R}$). The Fourier transform for this group is of the form

$$\hat{f}(u) = \int_K f(x)\chi(ux)dx. \tag{2}$$

It follows from equality (9) of Section 13.3.1 that if the support supp $f$ of a function $f$ belongs to $\mathfrak{P}^n$ and $f$ is constant on cosets (in $K^+$) with respect to $\mathfrak{P}^m$, then supp $\hat{f}$ belongs to $\mathfrak{P}^{-m}$ and $\hat{f}$ is constant on cosets with respect to $\mathfrak{P}^{-n}$. In order to prove this statement it is sufficient to split integral (2) into the integrals over cosets with respect to $\mathfrak{P}^m$ and to use the fact that $(\mathfrak{P}^n)^{\perp} = \mathfrak{P}^{-n}$.

We denote the characteristic function of the set $\mathfrak{P}^n$ by $\Phi_n$. It is clear that $\widehat{\Phi}_n = q^{-n}\Phi_{-n}$ and, in particular, $\widehat{\Phi}_0 = \Phi_0$. In addition, the functions $q^n\Phi_n$, $n = 0, 1, 2, \ldots$, form a $\delta$-shaped sequence in the space $S$ of functions $f$ on $K$ which have compact support and are constant on cosets with respect to $\mathfrak{P}^n$ for some $n$ ($n$ depends on a function).

Let $f$ be a locally integrable function on $K^+$. We denote by $[f]_n$ the function coinciding with $f$ on the set $\{x \mid q^{-n} \leq |x| \leq q^n\}$ and vanishing outside of this set. We define the principal value of the integral $\int_K f(x)dx$ by the formula

$$P\int_K f(x)dx = \lim_{n\to\infty} \int [f]_n(x)dx. \tag{3}$$

The set $\mathfrak{U} = \{x \mid |x| = 1\}$ forms a subgroup with respect to multiplication. It is called the *group of units* in $K^*$. In $\mathfrak{U}$ there exists an element $\varepsilon$ of order $q - 1$

such that the elements $0, 1, \varepsilon, \dots, \varepsilon^{q-2}$ belong to different cosets from $\mathfrak{U}/\mathfrak{P}$ (one element in every coset).

We denote by $\mathfrak{A}$ the set $\{x \mid |1-x| < 1\}$. It is a compact subgroup in $K^*$. One can show that every element $x \in K^*$ is uniquely represented in the form

$$x = \mathfrak{p}^n \varepsilon^k a, \quad |x| = q^{-n}, \quad 0 \le k \le q-2, \quad a \in \mathfrak{A}.$$

Consequently,

$$K^* = \mathbf{Z} \times \mathbf{Z}_{q-1} \times \mathfrak{A},$$

where the subgroups $\mathbf{Z}$, $\mathbf{Z}_{q-1}$ are defined as

$$\mathbf{Z} = \{\mathfrak{p}^n \mid -\infty < n < \infty\},\ \mathbf{Z}_{q-1} = \{\varepsilon^k \mid 0 \le k \le q-2\}.$$

The group $\mathfrak{U}$ coincides with $\mathbf{Z}_{q-1} \times \mathfrak{A}$. By setting $\mathfrak{A}_0 = \mathfrak{U}$, $\mathfrak{A}_1 = \mathfrak{A} = 1 + \mathfrak{P}$, $\mathfrak{A}_n = 1 + \mathfrak{P}^n$, $n > 1$, we obtain a fundamental system of neighborhoods of the identity element in $K^*$.

The equality $K^* = \mathbf{Z} \times \mathbf{Z}_{q-1} \times \mathfrak{A}$ implies that the group $\widehat{K}^*$ of characters for $K^*$ is of the form $\mathbf{T} \times \mathbf{Z}_{q-1} \times \widehat{\mathfrak{A}}$, where $\mathbf{T} \sim SO(2)$ and $\widehat{\mathfrak{A}}$ is a discrete periodic group. In other words, every multiplicative character $\pi$ of the field $K$ is uniquely represented as the product of: 1) a unitary character $\pi_\alpha$ depending on $|x|$ only, that is, such that

$$\pi_\alpha(x) = |x|^{i\alpha},\ -\frac{\pi}{\ln q} \le \alpha \le \frac{\pi}{\ln q},$$

2) a unitary character $\pi_m$ depending on $\varepsilon^k$ only, that is, such that

$$\pi_m(x) = \exp \frac{2\pi i m k}{q-1},$$

if $x = \mathfrak{p}^n \varepsilon^k a$, 3) a character from the periodic discrete subgroup $\widehat{\mathfrak{A}}$.

In the sequel we shall write down a (multiplicative and, possibly, non-unitary) character of the group $K$ in the form $\pi(x) = \pi_1(x)|x|^\beta$, where $\pi_1$ is a unitary character and $\beta$ is a complex number. This character can be rewritten as $\pi(x) = \pi^*(x)|x|^\gamma$, where $\pi^*$ is a character of the group $\mathfrak{U} = \mathbf{Z}_{q-1} \times \mathfrak{A}$. In this case the unitarity of $\pi$ is equivalent to the equality $\operatorname{Re}\gamma = 0$. Two characters

$$\pi_1(x) = \pi_1^*(x)|x|^{\gamma_1}, \quad \pi_2(x) = \pi_2^*(x)|x|^{\gamma_2}$$

are equal if and only if

$$\pi_1^* = \pi_2^*, \quad \gamma_1 - \gamma_2 \equiv 0 \pmod{2\pi i \ln q}.$$

Therefore, it is sufficient to consider values of $\gamma$ from the strip

$$-\frac{\pi}{\ln q} < \operatorname{Im}\gamma \le \frac{\pi}{\ln q}.$$

A character $\pi$ of the group $K^*$ is said to be of ramification degree $n$ if $\pi \in \mathfrak{A}_n^\perp$ and $\pi \bar{\in} \mathfrak{A}_{n-1}^\perp$, $n \geq 1$. If $\pi \in \mathfrak{A}_0^\perp \equiv \mathfrak{U}^\perp$, then $\pi$ is called an ***unramified character*** (this means that $\pi$ is of ramification degree 0).

According to what has been said above, we can consider $\widehat{K}^*$ as the union of a denumerable collection of circles and every circle is given by a character $\pi^*$ of the group $\mathfrak{U}$. The integration over $\widehat{K}^*$ is reduced to the ordinary integration over the circle and consequent summation:

$$\int_{\widehat{K}^*} f(\pi)d\pi = \sum_{\pi^*} \frac{1}{a} \int_{-\pi/\ln q}^{\pi/\ln q} f(\pi^*|x|^{i\alpha})d\alpha. \tag{3'}$$

Here $\pi^* \in \widehat{\mathfrak{U}}$ and the factor $a^{-1}$ will be defined below.

Let $S$ be the space of functions on $K$ introduced above. By giving sequences $\{f_k\}$ converging to zero function, we define a topology in $S$. A sequence $\{f_k\}$ converges to 0 if all the functions $f_k$ vanish outside a fixed compact set and are constant on cosets with respect to a fixed subgroup $\mathfrak{P}^n$ and if $\{f_k\}$ tends to zero uniformly. In this topology the space $S$ is complete. Linear continuous functionals on $S$ form the space $S'$ of generalized functions.

We also introduce the space $S^*$ of functions on $K^*$ with compact supports, such that for every $\varphi \in S^*$ there exists $k$ for which $\varphi(x\mathfrak{A}_k) = \varphi(x)$. Let us show that $S = S^*$. If $\varphi$ belongs to $S$ and is constant on cosets with respect to $\mathfrak{P}^n$, then for all integers $k \leq n$ this function is constant on cosets with respect to $\mathfrak{A}_{n-k}$ in $\mathfrak{P}^k$. And if $\varphi$ belongs to $S^*$ and is constant on cosets with respect to $\mathfrak{A}_n$, $n \geq 0$, then for all integers $k$ the function $\varphi$ is constant on cosets with respect to $\mathfrak{P}^{n+k}$ in $K\backslash\mathfrak{P}^{k+1}$ (the complement to $\mathfrak{P}^{k+1}$ in $K$). We omit the proof of these statements.

The support of every function $\varphi$ of the space $\widehat{S}^*$ on $\widehat{K}^*$ consists of a finite number of circles. On every circle these functions are trigonometrical polynomials:

$$\varphi(\pi) = \varphi(\pi^*|x|^{i\alpha}) = \sum_{\nu=-m}^{m} a_\nu(\pi^*)q^{i\nu\alpha}.$$

The space $\widehat{S}^{*\prime}$, conjugate (in the corresponding topology) to $\widehat{S}^*$, is the space of generalized functions on $\widehat{K}^*$.

In the sequel $\Psi_n$ will denote the characteristic function for $\mathfrak{A}_n$, $\mathfrak{A}_n'$ will denote the set of characters $\pi^*$ with ramification degree less than or equal to $n$, and $\Lambda_n$ will denote the characteristic function of circles from $\widehat{K}^*$, corresponding to these characters.

The Fourier transforms on the groups $K^*$ and $\widehat{K}^*$ are called the *Mellin transforms*. The Mellin transform of a function $f$ is denoted by $\tilde{f}$. If $f \in \mathfrak{L}^1(K^*, d^*x)$, then

$$\tilde{f}(\pi) = \int_{K^*} f(x)\pi(x)d^*x, \tag{4}$$

and if $f \in \mathfrak{L}^1(\widehat{K}^*, d\pi)$, then

$$\tilde{f}(x) = \int_{\widehat{K}^*} f(\pi)\pi^{-1}(x)d\pi. \tag{5}$$

In an ordinary way the Mellin transform is extended onto the Hilbert spaces $\mathfrak{L}^2(K^*, d^*x)$ and $\mathfrak{L}^2(\widehat{K}^*, d\pi)$ and then onto spaces of generalized functions.

One can show that the support of a function $\varphi \in S^*$ lies in the set $\{x \mid q^{-n} \le |x| \le q^n\}$ and this function is constant on cosets with respect to $\mathfrak{A}_m$ (in $K^*$) if and only if $\widetilde{\varphi} \in \widehat{S}^*$, the support of $\widetilde{\varphi}$ belongs to $\mathfrak{A}'_m$ and the restrictions of $\widetilde{\varphi}$ onto every disk $\mathbb{T}_{\pi^*}$ are of ramification degree less than or equal to $n$.

It is easy to check that

$$q'\widetilde{\Psi}_0 = \Lambda_0, \; q^n\widetilde{\Psi}_n = \Lambda_n.$$

Therefore, the measure of $\mathfrak{A}'_0$ is equal to $q'$ and the measure of $\mathfrak{A}'_n$, $n \ge 1$, is equal to $q^n$. It follows from here that normalized factor $a$ in (3′) has the form $a = q' \ln \frac{q}{2\pi}$. Further, one can easily verify that the number of unramified characters for $\mathfrak{U}$ is equal to 1. The number of characters with ramification degree 1 is equal to $(q - q')/q' = q - 2$, and $(q^n - q^{n-1})/q' = q^{n-2}(q-1)^2$ characters have ramification degree $n$, $n \ge 2$.

**13.3.3. The Gamma- and the Beta-functions, related to a field $K$.** The ordinary Gamma-function may be defined by the formula

$$\int_{-\infty}^{\infty} e^{ix}|x|^{\rho-1}dx = \Gamma(\rho)\cos\frac{\pi\rho}{2}.$$

Here $\chi(x) = e^{ix}$ is a unitary character of the additive group $\mathbb{R}$ and $\pi(x) = |x|^\rho$ is a character (possibly, non-unitary) of the multiplicative group $\mathbb{R} \setminus \{0\}$.

In a similar way one defines the Gamma-function for a locally compact totally disconnected non-discrete field $K$. It is a function of characters $\pi = \pi^*|x|^\alpha$ (possibly, non-unitary), given by the formula

$$\Gamma(\pi) \equiv \Gamma_{\pi^*}(\alpha) = P\int_K \chi(x)\pi^*(x)|x|^{\alpha-1}dx, \tag{1}$$

where $\chi$ is a fixed unitary character of the group $K^+$ which is trivial on $\mathfrak{O}$ and nontrivial on $\mathfrak{P}^{-1}$, and the symbol $P$ means that the integral is taken in the meaning of the principal value. This value exists if the character $\pi$ has a positive ramification degree. If $\pi = |x|^\alpha$ the formula

$$\Gamma_1(\alpha) = P\int_K \chi(x)|x|^{\alpha-1}dx \tag{1'}$$

defines the Gamma-function for $\mathrm{Re}\,\alpha > 0$. For $\mathrm{Re}\,\alpha < 0$ the function $\Gamma_1(\alpha)$ is defined by analytic continuation in $\alpha$.

It is easy to show that if the ramification degree $h$ of the character $\pi$ is positive and if for $u \in K$ we have $|u| \neq q^h$, then

$$\int\limits_{|x|=1} \chi(ux)\pi(x)dx = 0. \tag{2}$$

Further,

$$\int\limits_{|x|=q^k} \chi(x)dx = \begin{cases} q^k/q' & \text{if } k \leq 0, \\ -1 & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases} \tag{3}$$

If the ramification degree $h$ of $\pi$ is positive, then

$$\int\limits_{\mathfrak{A}_k \backslash \mathfrak{A}_{k+1}} \pi(x)dx = \begin{cases} 0 & \text{if } 0 \leq k < h-1, \\ -q^{-h} & \text{if } k = h-1, \\ q^{-k}/q' & \text{if } k \geq h. \end{cases} \tag{4}$$

The proof of these statements is reduced to equality (9) of Section 13.3.1.

The following theorem holds.

**Theorem 1.** (a) *If the ramification degree $h$ of a character $\pi$ is positive, then $\Gamma_{\pi^*}(\alpha) = C_{\pi^*} q^{h(\alpha-1/2)}$, where $C_{\pi^*}$ satisfies the conditions $|C_{\pi^*}| = 1$, $C_{(\pi^*)^{-1}} C_{\pi^*} = \pi^*(-1)$.*

(b) *For $\alpha \neq 0$ we have $\Gamma_1(\alpha) = (1 - q^{\alpha-1})/(1 - q^{-\alpha})$. The function $\Gamma_1(\alpha)$ has a simple pole at $\alpha = 0$ with residue $(q' \ln q)^{-1}$ and the function $(\Gamma_1(\alpha))^{-1}$ has a simple pole at $\alpha = 1$ with the residue $-(q' \ln q)^{-1}$. The function $\Gamma_1(\alpha)$ has no other poles and zeroes.*

(c) *For all $\pi$ we have*

$$\Gamma_{\pi^*}(\alpha) = \pi^*(-1)\overline{\Gamma_{(\pi^*)^{-1}}(\bar{\alpha})},\ \Gamma_{\pi^*}(\alpha)\Gamma_{(\pi^*)^{-1}}(1-\alpha) = \pi^*(-1)$$

*and, hence,*

$$\Gamma_{\pi^*}(\alpha)\overline{\Gamma_{\pi^*}(1-\bar{\alpha})} = 1.$$

*Proof:* Let $\pi$ have the positive ramification degree $h$. Then formula (2) implies

$$\Gamma_{\pi^*}(\alpha) = \int\limits_{|x|=q^h} \chi(x)\pi(x)|x|^{-1}dx = q^{h(\alpha-1/2)}$$

$$\times \int\limits_{|x|=q^h} \chi(x)\pi^*(x)|x|^{-1/2}dx = \Gamma_{\pi^*}\left(\frac{1}{2}\right)q^{h(\alpha-1/2)} = C_{\pi^*} q^{h(\alpha-1/2)},$$

where $C_{\pi^*} = \Gamma_{\pi^*}\left(\frac{1}{2}\right)$.

We now assume that the character $\pi$ is unramified, that is, $\pi(x) = |x|^\alpha$. According to formula (3) we have

$$\int_K \left[\chi(x)|x|^{\alpha-1}\right]_n dx = -q^{\alpha-1} + \frac{1}{q'}\sum_{k=0}^{n} q^{-k\alpha}.$$

Therefore,

$$P\int_K \chi(x)|x|^{\alpha-1}dx = (1-q^{\alpha-1})/(1-q^{-\alpha}).$$

The functions $\Gamma_1(\alpha)$ and $(\Gamma_1(\alpha))^{-1}$ are meromorphic on the left half-plane and all statements follow from the explicit form of these functions.

Statement (c) in the case of an unramified character follows from the explicit form of $\Gamma_1(\alpha)$. We now assume that the ramification degree $h$ is positive. Then the equality $\Gamma_{\pi^*}(\alpha) = \pi^*(-1)\overline{\Gamma_{(\pi^*)^{-1}}(\bar{\alpha})}$ easily follows from the integral representation by means of change of a variable.

Let $f(x) = \pi(x)|x|^{-1}$ for $|x| = 1$ and $f(x) = 0$ for $|x| \neq 1$. Then $f \in S$ and by using formula (2) we derive that $\hat{f}(u) = \Gamma(\pi)\pi^{-1}(u)$ if $|u| = q^h$ and $\hat{f}(u) = 0$ otherwise. Using again formula (2) and the inversion formula for the Fourier transform, we have

$$\Gamma_{\pi^*}(\alpha)\Gamma_{(\pi^*)^{-1}}(1-\alpha) = \Gamma(\pi)\int_{|u|=q^h} \pi^{-1}(u)|u|\chi(u)|u|^{-1}du$$

$$= \int_{|u|=q^h} \Gamma(\pi)\pi^{-1}(u)\chi(u)du = \int_K \hat{f}(u)\chi(u)du = f(-1) = \pi(-1).$$

The properties of $C_{\pi^*}$ from statement (a) follow directly from (c). The theorem is proved.

If the multiplicative characters $\pi = \pi^*|x|^\alpha$ and $\lambda = \lambda^*|x|^\beta$ are ramified, then there exists a connection between $C_{\pi^*}$, $C_{\lambda^*}$ and $C_{\pi^*\lambda^*}$.

**Theorem 2.** *Let $\pi^*$ and $\lambda^*$ are characters on $\mathfrak{U}$ with positive ramification degrees $h_1$ and $h_2$, respectively, and let $h_3$ be the ramification degree for $\pi^*\lambda^*$. Then*

$$C_{\pi^*}C_{\lambda^*} = C_{\pi^*\lambda^*}q^{(2h_1-h_2)/2}\int_{|x|=q^{h_2-h_1}} \pi^*(1-x)\lambda^*(x)dx \tag{5}$$

*for $h_1 > h_2$,*

$$C_{\pi^*}C_{\lambda^*} = C_{\pi^*\lambda^*}\,q^{(3h_3-2h_1)/2}\int_{|x|=q^{h_1-h_3}} \pi^*(x)\lambda^*(1-x)dx \tag{6}$$

*for* $h_1 = h_2$, $0 < h_3 < h_1$,

$$C_{\pi^*} C_{\lambda^*} = C_{\pi^*\lambda^*} q^{h_1/2} \int\limits_{\substack{|x|=1 \\ |1-x|=1}} \pi^*(x)\lambda^*(1-x)dx \tag{7}$$

*for* $h_1 = h_2 = h_3$. *If* $h_3 = 0$, *then* $\lambda^* = (\pi^*)^{-1}$ *and*

$$C_{\pi^*} C_{\lambda^*} = \pi^*(-1). \tag{8}$$

*Proof:* At first we prove formula (5). If $h_1 > h_2$, then the ramification degree of $\pi^*\lambda^*$ is $h_1$. Then formula (2) and Theorem 1 give

$$\begin{aligned} C_{\pi^*\lambda^*} & \int\limits_{|x|=q^{h_2-h_1}} \pi^*(1-x)\lambda^*(x)dx \\ &= \Big( \int\limits_{|u|=q^{h_1}} \pi^*\lambda^*(u)\chi(u)|u|^{-1/2}du \Big) \Big( \int\limits_{|x|=q^{h_2-h_1}} \pi^*(1-x)\lambda^*(x)dx \Big) \\ &= q^{-3h_1/2} \int\limits_{|u|=q^{h_1}} \chi(u) \int\limits_{|x|=q^{h_2}} \pi^*(u-x)\lambda^*(x)dx\, du = q^{-h_1+h_2/2} \\ &\times \Big( \int\limits_{|x|=q^{h_2}} \lambda^*(x)\chi(x)|x|^{-1/2}dx \Big) \Big( \int\limits_{|x|=q^{h_1}} \pi^*(u)\chi(u)|u|^{-1/2}du \Big) \\ &= q^{-h_1+h_2/2} C_{\pi^*} C_{\lambda^*}. \end{aligned}$$

The rest of the statements are proved similarly. In the proof of (7) the equality

$$\int\limits_{\substack{|x|=1 \\ |1-x|=1}} \pi^*(x)\lambda^*(1-x)d\pi = \int\limits_{|x|=1} \pi^*(x)\lambda^*(1-x)dx$$

is utilized.

Let us note that the Gamma-function introduced above is related to representations of the group of homogeneous linear transformations $x \to ax + b$ of $K$ in just the same way as the ordinary Gamma-function is related to representations of the analogous group for $\mathbb{R}$. Omitting the detailed analysis of this problem, we only note that this connection is based on the equality

$$(\pi|x|^{-1})\hat{} = \Gamma(\pi)\pi^{-1}. \tag{9}$$

Let us proceed to considering the Beta-functions $B$, related to $K$. They are defined as follows. Let $\pi = \pi^*|x|^\alpha$ and $\lambda = \lambda^*|x|^\beta$ be characters of the group $K^*$. Then

$$B(\pi,\lambda) = \frac{\Gamma(\pi)\Gamma(\lambda)}{\Gamma(\pi\lambda)} = \frac{\Gamma_{\pi^*}(\alpha)\Gamma_{\lambda^*}(\beta)}{\Gamma_{\pi^*\lambda^*}(\alpha+\beta)}. \tag{10}$$

It follows from (10) that if $\pi$ is unramified and the ramification degree $h$ for $\lambda$ is positive, then

$$B(\pi,\lambda) = q^{-h\alpha}\Gamma_1(\alpha).$$

If the ramification degrees for $\pi$ and $\lambda$ are equal to $h_1 \geq 1$ and $h_2 \geq 1$, respectively, and $h_1 < h_2$, then

$$B(\pi,\lambda) = (C_{\pi^*}C_{\lambda^*}/C_{\pi^*\lambda^*})q^{h_1(\alpha-1/2)-h_2\alpha}. \tag{11}$$

Finally, if $\pi$, $\lambda$ and $\pi\lambda$ have the same ramification degree $h$, $h \geq 1$, then

$$B(\pi,\lambda) = (C_{\pi^*}C_{\lambda^*}/C_{\pi^*\lambda^*}q^{-h/2}. \tag{11'}$$

The Beta-function introduced has an integral representation, analogous to the integral representation

$$B(\lambda,\mu) = \int_0^1 x^{\lambda-1}(1-x)^{\mu-1}dx$$

for the classical Beta-function. Namely, the following theorem holds.

**Theorem 3.** *Let $\pi = \pi^*|x|^\alpha$, $\lambda = \lambda^*|x|^\beta$. If $0 < \operatorname{Re}\alpha$, $\operatorname{Re}\beta$, $\operatorname{Re}(\alpha+\beta) < 1$, then*

$$B(\pi,\lambda) = \int_K \pi(x)|x|^{-1}\lambda(1-x)|1-x|^{-1}dx. \tag{12}$$

*Besides,*

$$\lim_{n\to\infty}\left[\pi|x|^{-1}\right]_n * \lambda|x|^{-1} = B(\pi,\lambda)\pi\lambda|x|^{-1}, \tag{13}$$

*where $*$ denotes the additive convolution. If $\pi$, $\lambda$ and $\pi\lambda$ are ramified, then*

$$B(\pi,\lambda) = P\int_K \pi(x)|x|^{-1}\lambda(1-x)|1-x|^{-1}dx \tag{14}$$

*for all $\alpha,\beta \in \mathbb{C}$. If $\pi$ and $\lambda$ are ramified and $\pi\lambda$ is unramified, then*

$$B(\pi,\lambda) = P\int_K \pi(x)|x|^{-1}\lambda(1-x)|1-x|^{-1}dx, \tag{15}$$

*where* $\operatorname{Re}(\alpha+\beta) < 1$. *If* $\pi$ *is ramified and* $\lambda$ *is unramified or if* $\lambda$ *is ramified and* $\pi$ *is unramified, then*

$$B(\pi,\lambda) = P\int_K \pi(x)|x|^{-1}\lambda(1-x)|1-x|^{-1}\,dx, \tag{16}$$

*where* $\operatorname{Re}\beta > 0$ *and* $\alpha \in \mathbb{C}$ *or* $\operatorname{Re}\alpha > 0$ *and* $\beta \in \mathbb{C}$, *respectively.*

Concerning the proof of this theorem, see Ref. [465].

Splitting the integral

$$I \equiv \int_K |x|^{\lambda-1}|1-x|^{\mu-1}\,dx,$$

which expresses the Beta-function when a ramification is absent, into the integrals over the domains $|x| < 1$, $|x| > 1$ and $|x| = 1$ and taking into account that $|1-x| = 1$ if $|x| < 1$ and that $|1-x| = |x|$ if $|x| > 1$, we obtain

$$I = \int_{|x|<1} |x|^{\lambda-1}dx + \int_{|x|>1} |x|^{\lambda+\mu-2}dx + \int_{|x|=1} |1-x|^{\mu-1}dx.$$

One has

$$\int_{|x|<1} |x|^{\lambda-1}dx = \sum_{k=1}^{\infty} q^{-k(\lambda-1)} \int_{|x|=q^{-k}} dx = (1-q^{-1})\sum_{k=1}^{\infty} q^{-\lambda k} = \frac{(1-q^{-1})q^{-\lambda}}{1-q^{-\lambda}}.$$

Analogously,

$$\int_{|x|>1} |x|^{\lambda+\mu-2}dx = \frac{(1-q^{-1})q^{\lambda+\mu-1}}{1-q^{\lambda+\mu-1}}.$$

Finally,

$$\int_{|x|=1} |1-x|^{\mu-1}dx = 1-\frac{2}{q} + \int_{\substack{|x|=1\\|1-x|<1}} |1-x|^{\mu-1}dx = 1-\frac{2}{q}+\frac{(1-q^{-1})q^{-\mu}}{1-q^{-\mu}}$$

(in this integral we have used the replacement $x = 1 + py$). Thus,

$$I = 1-\frac{2}{q}+(1-q^{-1})\left(\frac{1}{q^{\lambda}-1}+\frac{1}{q^{\mu}-1}-\frac{q^{\lambda+\mu-1}}{q^{\lambda+\mu-1}-1}\right). \tag{17}$$

For $\pi(x) = |x|^\lambda$ we derive from here the duplication formula

$$\Gamma(\pi^2) = \frac{\Gamma^2(\pi)}{B(\pi,\pi)} = \left(1 - \frac{2}{q} + \frac{2(1-q^{-1})}{q^\lambda - 1} - \frac{(1-q^{-1})q^{2\lambda-1}}{q^{2\lambda-1}-1}\right)^{-1} \Gamma^2(\pi). \qquad (17')$$

**Theorem 4.** *Let $\pi = \pi^*|x|^\alpha$ and $\lambda = \lambda^*|x|^\beta$ be multiplicative characters such that*

$$0 < \operatorname{Re}\alpha,\ \operatorname{Re}\beta,\ \operatorname{Re}(\alpha+\beta) < 1. \qquad (18)$$

*In addition, let the ramification degree for $\pi$ be $h_1 \geq 0$, for $\lambda$ be $h_2 \geq 0$ and for $\pi\lambda$ be $h_3$. If*

$$\varkappa(u) = (\pi|x|^{-1} * \lambda|x|^{-1})(u) \equiv \int_K \pi(x)|x|^{-1}\lambda(u-x)|u-x|^{-1}dx, \qquad (19)$$

*then*

$$\varkappa(u) = u^{\alpha-1} \int_{|x|=|u|} \pi^*(x)\lambda^*(u-x)|u-x|^{\beta-1}dx \qquad (20)$$

*for $h_1 > h_2 \geq 0$,*

$$\varkappa(u) = |u|^{\alpha-1} \int_{|x|=|u|q^{h_2-h_1}} \pi^*(u-x)|x|^{\beta-1}\lambda^*(x)dx \qquad (21)$$

*for $h_2 > h_1 \geq 1$,*

$$\varkappa(u) = \int_{|x|=|u|q^{h_1-h_3}} \pi^*(x)\lambda^*(u-x)|x|^{\alpha+\beta-2}dx \qquad (22)$$

*for $h_1 = h_2 > h_3 \geq 1$,*

$$\varkappa(u) = \int_{\substack{|x|=|u| \\ |x-u|=|u|}} \pi^*(x)\lambda^*(u-x)|x|^{\alpha+\beta-2}dx \qquad (23)$$

*for $h_1 = h_2 = h_3 \geq 1$,*

$$\varkappa(u) = \int_{|x|=|u|q^{h_1-1}} \pi^*(x)\lambda^*(u-x)|x|^{\alpha+\beta-2}dx \qquad (24)$$

*for* $h_1 = h_2 > 1,\ h_3 = 0,$ *and*

$$\varkappa(u) = \int\limits_{\substack{|x|\geq|u| \\ |x-u|=|x|}} \pi^*(x)\lambda^*(u-x)|x|^{\alpha+\beta-2}dx \tag{25}$$

*for* $h_1 = h_2 = 1,\ h_3 = 0.$

*Proof:* We shall prove formula (20) only; the rest of the formulas can be proved analogously. We have

$$\varkappa(u) = |u|^{\beta-1} \int\limits_{|x|<|u|} \pi^*(x)\lambda^*(u-x)|x|^{\alpha-1}dx$$
$$+ \int\limits_{|x|>|u|} \pi^*(x)\lambda^*(u-x)|x|^{\alpha+\beta-2}dx + |u|^{\alpha-1} \int\limits_{|x|=|u|} \pi^*(x)\lambda^*(u-x)|u-x|^{\beta-1}dx. \tag{26}$$

If $\lambda$ is unramified, that is, $h_2 = 0$, then the result follows from (2). Assume now that $h_2 > 0$. Let $q^k < |u|$. We define the functions $f_1$ and $f_2$ by setting $f_1(x) = \pi^*(x)$ for $|x| = q^k$, $f_1(x) = 0$ for $|x| \neq q^k$, $f_2(x) = \lambda^*(x)$ for $|x| = |u|$ and $f_2(x) = 0$ for $|x| \neq |u|$. Then

$$(f_1 * f_2)(u) = \int\limits_{|x|=q^k} \pi^*(x)\lambda^*(u-x)dx.$$

It follows from (2) that

$$\hat{f}_1(v) = \int\limits_{|x|=q^k} \chi(vx)\pi^*(x)dx = 0$$

if $|v| \neq q^{h_1-k}$ and

$$\hat{f}_2(v) = \int\limits_{|x|=|u|} \chi(vx)\lambda^*(x)dx = 0$$

if $|v| \neq q^{h_2}|u|^{-1}$. From here we have $\hat{f}_1(v)\hat{f}_2(v) = (f_1 * f_2)\hat{}(v) = 0$ and hence $f_1 * f_2 = 0$. This shows that the first integral in (26) is equal to 0. If $q^k > |u|$, then we take the same function $f_1(x)$ and set $f_2(x) = \lambda^*(x)$ if $|x| = q^k$ and $f_2(x) = 0$ if $|x| \neq q^k$. Then $\hat{f}_1(v) = 0$ for $|v| \neq q^{h_1-h_2}$ and $\hat{f}_2(v) = 0$ for $|v| \neq q^{h_2-k}$. As in the preceding case, this implies that the second integral in (26) is equal to 0 and we obtain the required formula.

Using the change of variable in (19), we find that

$$\varkappa(u) = \pi\lambda(u)|u|^{-1}\varkappa(1). \tag{27}$$

By applying (9) we obtain

$$\hat{\varkappa} = \varkappa(1)\Gamma(\pi\lambda)(\pi\lambda)^{-1}. \tag{28}$$

Simple (and long) calculations (see Ref. [465]) show that under conditions (18) we have

$$\varkappa(1) = B(\pi, \lambda). \tag{29}$$

In conclusion, we note that under conditions (18) the formula

$$\frac{|x|^{\alpha-1}}{\Gamma_1(\alpha)} * \frac{|x|^{\beta-1}}{\Gamma_1(\beta)} = \frac{|x|^{\alpha+\beta-1}}{\Gamma_1(\alpha+\beta)} \tag{30}$$

holds.

**13.3.4. Quadratic extensions of a field $K$.** The field $\mathbb{C}$ of complex numbers is obtained from the field $\mathbb{R}$ of real numbers by joining the square root of the element $-1$ which is not the square of another element from $\mathbb{R}$. In the same way one constructs quadratic extensions of the field $Q_p$ of $p$-adic numbers.

The order of the quotient group $K^*/(K^*)^2$ is equal to 4. One can choose the elements $\mathfrak{p}$, $\mathfrak{p}\varepsilon$ and $\varepsilon$ (see Section 13.3.2) as representatives of cosets. Hence, there are three quadratic extensions of $K$; namely, $K(\sqrt{\mathfrak{p}})$, $K(\sqrt{\mathfrak{p}\varepsilon})$ and $K(\sqrt{\varepsilon})$. The first and the second extensions are called *ramified* and the third one is called *unramified.* In the sequel the elements $\mathfrak{p}$, $\mathfrak{p}\varepsilon$ and $\varepsilon$ will be denoted by the symbol $\tau$ and quadratic extensions of $K$ by $K(\sqrt{\tau})$.

Let $K(\sqrt{\tau})$ be a quadratic extension of the field $K$. The elements $z = a + b\sqrt{\tau}$ and $\bar{z} = a - b\sqrt{\tau}$, $a, b, \in K$, are said to be *conjugate*. The element $z\bar{z} = a^2 - b^2\tau$ is called the *norm* of $z$ with respect to $K$. The set of norms $\{z\bar{z} \mid z \neq 0\}$ is the subgroup of $K^*$, denoted by $K_\tau^*$. It is obvious that $(K^*)^2 \subset K_\tau^*$. It is easy to show that the subgroup $K_\tau^*$ has index 2 in $K^*$, that is, $(K^*)^2 \neq K_\tau^* \neq K^*$.

We introduce the function $\text{sign}_\tau x$ on $K^*$ by setting $\text{sign}_\tau x = -1$ if $x \bar{\in} K_\tau^*$ and $\text{sign}_\tau x = 1$ if $x \in K_\tau^*$. Since $K_\tau^*$ is a subgroup of index 2 in $K^*$, then $\text{sign}_\tau x$ is a character on $K^*$, that is,

$$\text{sign}_\tau x \, \text{sign}_\tau y = \text{sign}_\tau xy.$$

One can show that the functions 1, $\text{sign}_{\mathfrak{p}} x$, $\text{sign}_{\mathfrak{p}\varepsilon} x$ and $\text{sign}_\varepsilon x$ are linearly independent and form a complete system of characters on $K^*/(K^*)^2$.

The set $\{z \mid z \in K(\sqrt{\tau}),\ z\bar{z} = c \neq 0\}$ is called a *circle* in $K(\sqrt{\tau})$. The radius of this circle is "real" if $c \in (K^*)^2$ and is "imaginary" if $c \bar{\in} (K^*)^2$. Elements of the unit circle form the group, denoted by $C_\tau$. It is easy to prove that if $a + b\sqrt{\tau} \in C_\tau$, then $|a| \leq 1$, $|b| \leq 1$. This implies that $C_\tau$ and, consequently, other circles are compact.

The set $C_\tau^{(0)} = \{z \in C_\tau \mid |1-z| < 1\}$ is a subgroup of index 2 in $C_\tau$. Besides $C_\tau = C_\tau^{(0)} \cup (-1)C_\tau^{(0)}$. The mapping

$$\varphi\colon x \longrightarrow \frac{1+x\sqrt{\tau}}{1-x\sqrt{\tau}} = \frac{1+\tau x^2}{1-\tau x^2} + 2\sqrt{\tau}\,\frac{x}{1-\tau x^2}$$

is a homeomorphism of the ring $\mathfrak{O}$ into $C_\tau^{(0)}$. This homeomorphism maps cosets with respect to $\mathfrak{P}^n$ in $\mathfrak{O}$ onto cosets with respect to $C_\tau^{(n)}$ in $C_\tau^{(0)}$, where the subgroups $C_\tau^{(n)}$ are defined by the equalities

$$C_\tau^{(n)} = \left\{a + b\sqrt{\tau} \in C_\tau^{(0)} \mid a \in \mathfrak{A}_{2n+1}, b \in \mathfrak{P}^n\right\}, \quad n > 0.$$

Since the index of the subgroup $\mathfrak{P}^{n+1}$ in $\mathfrak{P}^n$, $n \geq 0$, is equal to $q$, then the subgroup $C_\tau^{(n+1)}$ has index $q$ in $C_\tau^{(n)}$. We also have

$$\int\limits_{C_\tau^{(0)}} f(t)dt = \int\limits_{\mathfrak{O}} f(\varphi(x))dx,$$

where $dt$ is the invariant measure on $C_\tau^{(0)}$, normalized as

$$\int\limits_{C_\tau^{(0)}} dt = 1.$$

We introduce Cartesian and polar coordinates into $K(\sqrt{\tau})$. A pair $(x,y)$, $x,y \in K$, is said to be *Cartesian coordinates* of the element $z = x + y\sqrt{\tau} \in K(\sqrt{\tau})$. Let $z\bar{z} = c$. If $c = r^2$, $r \in K$, then by *polar coordinates* of $z$ we mean the elements $\rho = r \in K$ and $t = \rho^{-1}z \in C_\tau$. Note that $\rho$ is defined up to a sign and, hence, polar coordiantes of an element $z$ are defined up to a sign. If $c \bar{\in} (K^*)^2$ we fix in $K(\sqrt{\tau})$ an element $\nu$ such that $\nu \bar{\in} (K^*)^2$ and represent $c$ in the form $c = (\nu r)(\bar{\nu} r)$, $r \in K$. In this case by *polar coordinates* of $z$ we mean the numbers $\rho = \nu r$ and $t = \rho^{-1}z \in C_\tau$. As in the first case, the pairs $(\rho, t)$ and $(-\rho, -t)$ give the same point $z$.

The invariant measures $dz$ and $d^*z$, $z = x + y\sqrt{\tau}$, on $K(\sqrt{\tau})$ are defined by the formulas

$$dz = dxdy, \quad d^*z = \frac{dxdy}{|z\bar{z}|} = \frac{dxdy}{|x^2 - \tau y^2|}. \tag{1}$$

In polar coordinates they can be written as

$$dz = d(\rho\bar{\rho})d^*t = d(z\bar{z})d^*t, \tag{2}$$

$$d^*z = \frac{d(z\bar{z})d^*t}{|z\bar{z}|}, \tag{3}$$

where $d(z\bar{z})$ is the invariant measure on $K^+$ and $d^*t$ is the invariant measure on the multiplicative group $C_\tau$, normalized by the condition that the measure of the whole group $C_\tau$ is equal to 1.

Along with the polar coordinate system on $K(\sqrt{\tau})$ one considers another system in which $C_\tau$ is replaced by $N_0 = C_\tau \cap (1 + \widetilde{\mathfrak{P}})$, where $\widetilde{\mathfrak{P}} = \sqrt{\tau}\,\mathfrak{O}_\tau$, $\mathfrak{O}_\tau = \{z \in K(\sqrt{\tau}) \mid |z| \leq 1\}$, and $K^*$ is replaced by $\Gamma = K^* \cup \sqrt{\tau}\,K^*$. In this case the invariant measure on $N_0$ is defined in such a way that the measure of $N_0$ is equal to 1. The measure on $\Gamma$ is introduced in the following way. If $r \in K$, then $dr$ is the normalized invariant measure on $K$, and if $r = \sqrt{\tau}\,y$, $y \in K$, then $dr = q^{-1/2}dy$, where $dy$ is the normalized invariant measure on $K$. Then $|r|^{-1}dr$ is the invariant measure on the multiplicative group $\Gamma$.

The equality

$$K(\sqrt{\tau})^* = \Gamma N_0, \quad \Gamma \cap N_0 = 1$$

holds. If $z = rn$, where $z \in K(\sqrt{\tau})^*$, $r \in \Gamma$, $n \in N_0$, then $dz = |r|drdn$. Note that $N_0$ is a subgroup of index 2 in $C_\tau$.

Additive characters on $K(\sqrt{\tau})$ are defined by the formula

$$\chi(z) = \chi_1(x)\chi_2(y), \quad z = x + y\sqrt{\tau},$$

where $\chi_1$, $\chi_2$ are additive characters on $K$.

Let $\pi(z)$ be a multiplicative character on $K(\sqrt{\tau})$. We denote by $\pi_1$ and $\pi_2$ its restrictions onto $K^*$ and $C_\tau$. If $z = rt$, where $r \in K^*$, $t \in C_\tau$, then

$$\pi(z) = \pi(rt) = \pi_1(r)\pi_2(t).$$

Since $rt = (-r)(-t)$, then the relation

$$\pi_1(-1) = \pi_2(-1) \tag{4}$$

holds. And if $z = \nu rt$, where $r_0 = \nu\bar{\nu} \bar{\in} (K^*)^2$, then $\nu^2 = r_0 t_0$, where $t_0 \in C_\tau$. Consequently, $\pi(\nu^2) = \pi_1(r_0)\pi_2(t_0)$, that is,

$$\pi^2(\nu) = \pi_1(\nu\bar{\nu})\pi_2\left(\frac{\nu}{\bar{\nu}}\right). \tag{5}$$

Conversely, let $\pi_1$ and $\pi_2$ be multiplicative characters on $K$ and $C_\tau$, respectively, connected by relation (4). We choose $\nu \in K(\sqrt{\tau})$ such that $\nu\bar{\nu} \bar{\in} (K^*)^2$ and define $\pi(\nu)$ for which condition (5) holds. Then the function $\pi$, given by the equalities

$$\pi(rt) = \pi_1(r)\pi_2(t),$$
$$\pi(\nu rt) = \pi(\nu)\pi_1(r)\pi_2(t),$$

is a multiplicative character for $K(\sqrt{\tau})$.

Let $\chi$ be a unitary character of the group $K^+$ such that $\chi \in \mathcal{O}^\perp \backslash (\mathfrak{P}^{-1})^\perp$ (see Section 13.3.2). We define the unitary character $\psi$ on the additive group $K(\sqrt{\tau})^+$ of the field $K(\sqrt{\tau})$ by setting

$$\psi(z) = \chi\left(z\sqrt{\tau}^{-1} + \overline{z\sqrt{\tau}^{-1}}\right) = \chi(2y), \tag{6}$$

where $x + y\sqrt{\tau}$, $x, y \in K$. Then $\psi$ is trivial on $\mathcal{O}_\tau = \{z \in K(\sqrt{\tau}) \mid |z| \le 1\}$ and non-trivial on $(\sqrt{\tau}\,\mathcal{O}_\tau)^{-1}$. All additive unitary characters on $K(\sqrt{\tau})$ have the form $\psi(uz)$, $u \in K(\sqrt{\tau})$.

Now we can continue the evaluation of the Beta-function. Let us calculate the integral

$$I = \int_K |x|^{\lambda-1}\mathrm{sign}_\tau x \cdot |1-x|^{\mu-1}\,dx.$$

As in Section 13.3.3, we divide the integration domain into the parts where $|x| < 1$, $|x| > 1$ and $|x| = 1$. If $\tau = \mathfrak{p}$ or $\tau = \mathfrak{p}\varepsilon$, then

$$\int_{|x|<1} |x|^{\lambda-1}\mathrm{sign}_\tau x\,dx = \int_{|x|>1} |x|^{\lambda+\mu-2}\mathrm{sign}_\tau x\,dx = 0,$$

since

$$\int_{|x|=\mathrm{const}} \mathrm{sign}_\tau x\,dx = 0.$$

We divide the domain $|x| = 1$ into the parts where $|1-x| = 1$ and $|1-x| < 1$. The integral over the first part is equal to $-q^{-1}$ and over the second one to $(1-q^{-1})/(q^\mu - 1)$. Hence, if $\tau = \mathfrak{p}$ or $\tau = \mathfrak{p}\varepsilon$, then

$$I = -q^{-1} + \frac{1-q^{-1}}{q^\mu - 1}. \tag{7}$$

And if $\tau = \varepsilon$, then

$$I = 1 - \frac{2}{q} + (1-q^{-1})\left(-\frac{1}{q^\lambda + 1} + \frac{1}{q^\mu - 1} - \frac{q^{\lambda+\mu-1}}{q^{\lambda+\mu-1}+1}\right). \tag{8}$$

One analogously evaluates the integral

$$\int_K |x|^{\lambda-1}\mathrm{sign}_\tau x \cdot |1-x|^{\mu-1}\mathrm{sign}_\tau(1-x)\,dx.$$

If $\pi(x) = |x|^{1/2}$ or $\pi(x) = |x|^{1/2}\text{sign}_\tau x$, then the functional relation

$$\Gamma(\pi)\Gamma(\pi_0\pi^{-1}) = \pi(-1),$$

where $\pi_0(x) = |x|$, gives $\Gamma^2(\pi) = \pi(-1)$, that is,

$$\Gamma(\pi) = \pm 1 \quad \text{if} \quad \pi(-1) = 1,$$
$$\Gamma(\pi) = \pm i \quad \text{if} \quad \pi(-1) = -1.$$

The sign for $\Gamma(\pi)$ depends on the choice of the character $\chi$. We do not consider this problem.

For the character $\pi_k(x) = |x|^{k/2}\text{sign}_\tau x$, $k \in \mathbb{Z}_+$, the recurrence formula

$$\Gamma(\pi_{k+1}) = \frac{\Gamma(\pi_k)\Gamma(\pi_0^{1/2})}{B(\pi_k, \pi_0^{1/2})} \tag{8'}$$

holds, where $\pi_0^{1/2}(x) = |x|^{1/2}$. The value $B(\pi_k, \pi_0^{1/2})$ is given by formulas (7) and (8). Therefore, (8′) defines $\Gamma(\pi_k)$ up to a sign.

In conclusion of this section, we note that

$$\int_{K(\sqrt{\tau})} \chi(ut\bar{t})dt = c_\chi^{-1}\frac{\text{sign}_\tau u}{|u|} + \frac{1}{2}\delta(u), \tag{9}$$

where

$$c_\chi^{-1} = \frac{1}{2}\Gamma(\pi) \equiv \frac{1}{2}\int \chi(x)\text{sign}_\tau x dx.$$

**13.3.5. The motion group of the plane $K(\sqrt{\tau})$ and its representations.** The group $C_\tau$ acts on $K(\sqrt{\tau})$ as multiplications. We denote by $G$ the semidirect product of the additive group $K(\sqrt{\tau})^+$ and $C_\tau$. By the *motion group of the plane* $K(\sqrt{\tau})$, $\tau = \mathfrak{p}$ or $\tau = \mathfrak{p}\varepsilon$, we mean the subgroup $G_0 = N_0 K(\sqrt{\tau})^+$ of $G$. This group can be identified with the group of matrices of the form

$$g = \begin{pmatrix} 1 & z \\ 0 & n \end{pmatrix}, \qquad \text{where} \qquad n \in N_0,\ z \in K(\sqrt{\tau})^+.$$

Let $\rho$ be a non-zero element from $K(\sqrt{\tau})^+$. We denote by $\psi_\rho$ the character $\psi(\rho z)$ of the group $K(\sqrt{\tau})^+$ (see formula (6) of Section 13.3.4), and by $T_\rho$ the unitary representation of $G_0$, induced by the character $\psi_\rho$. This representation acts in the Hilbert space $\mathfrak{L}^2(N_0)$ and has the form

$$(T_\rho(g_0)f)(n) = \psi(\rho z_0 n_0^{-1} n^{-1})f(nn_0),$$

where $f \in \mathfrak{L}^2(N_0)$, $n \in N_0$, $g_0 = \begin{pmatrix} 1 & z_0 \\ 0 & n_0 \end{pmatrix} \in G_0$.

It is easy to show that all representations $T_\rho$ are irreducible and that $T_\rho$ and $T_\sigma$ are unitarily equivalent if and only if $\rho^{-1}\sigma \in N_0$. In addition, the representations $T_\rho$, together with one-dimensional unitary representations, exhaust all unitary representations of the group $G_0$. These assertions follow from the general theory of induced representations.

Every $T_\rho$ is a representation of class 1 relative to the subgroup $N_0$ and the zonal spherical function $\Phi_\rho$, corresponding to $T_\rho$, has the form

$$\Phi_\rho(z) = \int_{N_0} \psi(zn)dn, \qquad z \in K\left(\sqrt{\tau}\right). \tag{1}$$

For a character $\theta$ of $N_0$ we set

$$J_\theta(z) = \int_{N_0} \psi(zn)\overline{\theta(n)}dn, \qquad z \in K\left(\sqrt{\tau}\right). \tag{2}$$

The function $J_\theta(z)$ is called a $\mathfrak{P}$*-adic Bessel function with index* $\theta$.

Equality (1) leads to

$$\Phi_\rho(z) = J_1(\rho z).$$

Further, formula (2) implies

$$\psi(zn) = \sum_\theta J_\theta(z)\theta(n). \tag{3}$$

It is obvious from (3) that $\psi(zn)$ is a generating function for the Bessel functions $J_\theta(z)$.

By the *Fourier transform* of a function $f$ on $K\left(\sqrt{\tau}\right)$, one means the function $\hat{f}$ given by the formula

$$\hat{f}(w) = \int_{K(\sqrt{\tau})} f(z)\psi(zw)dz. \tag{4}$$

The inversion formula

$$f(z) = \int_{K(\sqrt{\tau})} \hat{f}(w)\overline{\psi(zw)}dw \tag{5}$$

is valid.

A function $f$ on $K\left(\sqrt{\tau}\right)$ is said to be *radial* if $f(zn) = f(z)$ for all $n \in N_0$. Radial functions can be identified with functions on $\Gamma \cup \{0\}$ or with functions on $G_0$, constant on two-sided cosets with respect to $N_0$.

Let $f$ be a continuous radial finite function. Then for $z = rn$, $r \in \Gamma$, $n \in N_0$, we have

$$\hat{f}(w) = \int\limits_{K(\sqrt{\tau})} f(z)\psi(zw)dz = \int\limits_{\Gamma} f(r)\left[\int\limits_{N_0} \psi(rnw)dn\right]|r|dr$$
$$= \int\limits_{\Gamma} f(r)J_1(rw)|r|dr. \tag{6}$$

Since the function $J_1$ is radial and tends to zero at the infinity, then $\hat{f}(w)$ has the same properties:

$$\hat{f}(s) = \int\limits_{\Gamma} f(r)J_1(rs)|r|dr, \qquad s \in \Gamma. \tag{7}$$

Therefore,

$$f(0) = \int\limits_{K(\sqrt{\tau})} \hat{f}(w)dw = \int\limits_{\Gamma} \hat{f}(s)|s|ds. \tag{8}$$

The following statement is valid:

**Theorem 1.** *If $f$ is a continuous finite function on $\Gamma$, then*

$$f(r) = \int\limits_{\Gamma} \hat{f}(s)\overline{J_1(rs)}|s|ds, \tag{9}$$

*where*

$$\hat{f}(s) = \int\limits_{\Gamma} f(r)J_1(rs)|r|dr. \tag{10}$$

*In addition, the analog of the Plancherel formula*

$$\int\limits_{\Gamma} |f(r)|^2|r|dr = \int\limits_{\Gamma} |\hat{f}(s)|^2|s|ds \tag{11}$$

*holds.*

The proof of this theorem easily follows from the above arguments if one uses the properties

$$\Phi_s(g^{-1}) = \overline{\Phi_s(g)},$$
$$\int\limits_{N_0} \Phi_s(g_1ng_2)dn = \Phi_s(g_1)\Phi_s(g_2)$$

of the zonal spherical functions $\Phi_s(g) = J_1(rs)$. We suggest to the reader to carry out this proof.

**13.3.6. Evaluation of the $\mathfrak{P}$-adic Bessel functions.** We denote by $N_k$, $k \in \mathbf{Z}_+$, the set of elements from $N_0$ of the form $1+\tau^{2k+1}a+\sqrt{\tau}\,\tau^k b$, where $a, b \in \mathcal{O}$. It is easy to verify that $N_k$, $k > 0$, coincides with every one of the following sets:

a) $N_0 \cap \left(1 + \widetilde{\mathfrak{P}}^{2k}\right)$,

b) $N_0 \cap \left(1 + \widetilde{\mathfrak{P}}^{2k+1}\right)$,

c) $\left\{n \in N_0 \mid n = 1 + a' + \sqrt{\tau}\,\tau^k b;\ a', b \in \mathcal{O}\right\}$,

d) $\left\{n \in N_0 \mid n = 1 + \tau^{2k+1}a + \sqrt{\tau}\,b';\ a, b' \in \mathcal{O}\right\}$.

In particular, $N_k$ is an open subgroup in $N_0$.

Indeed, it is obvious that

$$N_k \subset N_0 \cap \left(1 + \widetilde{\mathfrak{P}}^{2k+1}\right) \subset N_0 \cap \left(1 + \widetilde{\mathfrak{P}}^{2k}\right).$$

On the other hand, we take the element $n = 1 + \tau^k a' + \sqrt{\tau}\,\tau^k b'$ ($a', b' \in \mathcal{O}$) in $N_0 \cap \left(1 + \widetilde{\mathfrak{P}}^{2k}\right)$. Since $n\bar{n} = 1$, then

$$2a' + \tau^k (a')^2 - \tau^{k+1}(b')^2 = 0.$$

We have from here that $a' \in \mathfrak{P}^k$ and, consequently, $a' \in \mathfrak{P}^{2k+1}$. Thus, $N_k \supset N_0 \cap \left(1 + \widetilde{\mathfrak{P}}^{2k}\right)$. Let, further, $n = 1 + a' + \sqrt{\tau}\,\tau^k b \in N_0$, where $a' \in \mathfrak{P}$, $b \in \mathcal{O}$. Then we have $2a' + (a')^2 + \tau^{2k+1}b^2 = 0$. Since $a' \in \mathfrak{P}$, then $a' \in \mathfrak{P}^2$. Continuing these arguments, we conclude that $a' \in \mathfrak{P}^{2k+1}$ and, hence,

$$n = 1 + \tau^{2k+1}a + \sqrt{\tau}\,b' \in N_0, \quad \text{where} \quad a, b' \in \mathcal{O}.$$

Therefore, we have $2\tau^{2k}a + \tau^{4k+1}a^2 - (b')^2 = 0$. This implies that $b' \in \mathfrak{P}^k$. Thus, we have proved that the above sets coincide.

In the same way one proves the following statement. Let $k$ be a non-negative integer and let

$$n = 1 + \tau^{2k+1}a + \sqrt{\tau}\,\tau^k b \in N_k, \quad \text{where} \quad a, b \in \mathcal{O}.$$

With the element $n$ we associate the element $b \in \mathcal{O}$. This correspondence is a one-to-one mapping of $N_k$ onto $\mathcal{O}$ which transfers $N_{k+1}$ onto $\mathfrak{P}$. In particular, $N_{k+1}$ has index $q$ in $N_k$. Besides, $2a \equiv b^2 \pmod{\mathfrak{P}}$.

We go over to evaluation of $J_\theta(z)$. At first we evaluate $J_1(z)$. Since $J_1(z)$ is a radial function, it is sufficient to evaluate

$$J_1(z) = \int_{N_0} \psi(zn)\,dn$$

for $z \in \Gamma \cup \{0\}$.

**Theorem 1.** *If $z \in K$, then*

$$J_1(z) = \begin{cases} 1 & \text{for } z \in \mathfrak{O}, \\ 0 & \text{for } z \bar{\in} \mathfrak{O}. \end{cases}$$

*If $z = \sqrt{\tau}\, y$, $y \in K$, then $J_1(z) = 1$ for $y \in \mathfrak{O}$,*

$$J_1(z) = q^{-\ell}\psi(z)$$

*for $|y| = q^{2\ell+1}$, $\ell \geq 0$,*

$$J_1(z) = q^{-\ell}\psi(z)G(q,u)$$

*for $|y| = q^{2\ell}$, $\ell > 0$. In the last formula $y = \tau^{-2\ell}u$, where $u$ belongs to the subgroup $\mathfrak{U}$ of units of $K^*$,*

$$G(q,u) = \sum_{b \in \mathfrak{O} \pmod{\mathfrak{P}}} \chi(\tau^{-1}ub^2)$$

*is the Gauss sum for the field $\mathfrak{O}/\mathfrak{P}$, well-known in number theory, and $\chi$ is the character on the additive group $K$ from formula (6) of Section 13.3.4.*

For the proof of this theorem we need the following lemma:

**Lemma 1.** *Let $0 \leq i < j$, $i + j = k - 1$. Then for the sum*

$$I = \sum_{n \in N_i \pmod{N_j}} \psi(zn)$$

*we have the expression $I = \psi(z)G(q,u)$ if $i + 1 = j$, the expression $I = q\psi(z)$ if $i + 2 = j$, and the expression*

$$I = q \sum_{n \in N_{i+1} \pmod{N_{j-1}}} \psi(zn)$$

*if $i + 2 < j$.*

*Proof of* Lemma 1. Set $n = 1 + \tau^{2i+1}a + \sqrt{\tau}\,\tau^i b$, where $a, b \in \mathfrak{O}$. Then for $i + 1 = j$ we have

$$I = \psi(z) \sum_{n \in N_i \pmod{N_j}} \chi(\tau^{-1}2ua)$$
$$= \psi(z) \sum_{b \in \mathfrak{O} \pmod{\mathfrak{P}}} \chi(\tau^{-1}ub^2) = \psi(z)G(q,u).$$

Let now $i+2 \le j$. If $m$ is an element from $N_{j-1} \pmod{N_j}$, then we set $m = 1+\tau^{2j-1}c+\sqrt{\tau}\,\tau^{j-1}d$, where $c,d \in \mathfrak{O}$. Then

$$\sum_{m\in N_{j-1} \pmod{N_j}} \psi(znm) = \psi(zn) \sum_{d\in\mathfrak{O} \pmod{\mathfrak{P}}} \psi(\tau^{j-k}und)$$
$$= \psi(zn) \sum_{d\in\mathfrak{O} \pmod{\mathfrak{P}}} \chi(\tau^{-1}2ubd) = q\psi(zn)$$

for $n \in N_{i+1}$. If $n \bar{\in} N_{i+1}$, then this sum vanishes. Hence, for $i+2=j$ we have

$$I = \sum_{n\in N_i \pmod{N_{j-1}}} \sum_{m\in N_{j-1} \pmod{N_j}} \psi\left(\sqrt{\tau}\,\tau^{-k}unm\right) = q\psi(z)$$

and for $i+2<j$ we have

$$I = \sum_{n\in N_i \pmod{N_{j-1}}} \sum_{m\in N_{j-1} \pmod{N_j}} \psi(znm) = q \sum_{n\in N_{i+1} \pmod{N_{j-1}}} \psi(zn).$$

Lemma 1 is proved.

*Proof of* Theorem 1. We first consider the case when $z \in \mathfrak{O}_\tau$. Then $\psi(zn) = 1$ for all $n \in N_0$. Hence, $J_1(z) = 1$. Let now $z \in K$ and $z \bar{\in} \mathfrak{O}$. We set $z = \tau^{-k}u$, where $k>0$, $u \in \mathfrak{U}$. Then

$$J_1(z) = \int_{N_0} \psi(\tau^{-k}un)dn = q^{-k} \sum_{n\in N_0 \pmod{N_k}} \psi(\tau^{-k}un).$$

If $k=1$, then

$$J_1(z) = q^{-1} \sum_{n\in N_0 \pmod{N_1}} \psi(\tau^{-1}un) = q^{-1} \sum_{b\in\mathfrak{O} \pmod{\mathfrak{P}}} \chi(\tau^{-1}2ub) = 0,$$

where we have replaced $n$ by $1+\tau a+\sqrt{\tau}\,b$, $a,b \in \mathfrak{O}$. And if $k>1$, then

$$J_1(z) = q^{-k} \sum_{n\in N_0 \pmod{N_{k-1}}} \sum_{m\in N_{k-1} \pmod{N_k}} \psi(\tau^{-k}unm).$$

We set $m = 1+\tau^{2k-1}c+\sqrt{\tau}\,\tau^{k-1}d$, where $c,d \in \mathfrak{O}$. Then

$$\sum_{m\in N_{k-1} \pmod{N_k}} \psi(\tau^{-k}unm) = \psi(\tau^{-k}um) \sum_{d\in\mathfrak{O} \pmod{\mathfrak{P}}} \chi(\tau^{-1}2ud) = 0.$$

This implies $J_1(z) = 0$.

Further, let $z = \sqrt{\tau}\, y$, where $y \in K$ and $y \bar{\in} \mathfrak{O}$. Let us set $y = \tau^{-k} u$, where $k > 0$, $u \in \mathfrak{U}$. As above, we obtain

$$J_1(z) = q^{-k} \sum_{n \in N_0 \pmod{N_k}} \psi(zn).$$

If $k = 1$, then

$$J_1(z) = q^{-1} \sum_{n \in N_0 \pmod{N_1}} \psi(\tau^{-1} un) = \psi(z).$$

And if $k > 1$, then the equality

$$\sum_{m \in N_{k-1} \pmod{N_k}} \psi(znm) = q\psi(zn)$$

gives that

$$J_1(z) = q^{1-k} \sum_{n \in N_0 \pmod{N_{k-1}}} \psi(zn).$$

By using Lemma 1 for $k = 2\ell + 1$, $\ell > 0$, we conclude that

$$J_1(z) = q^{1-k} \sum_{n \in N_0 \pmod{N_{k-1}}} \psi(zn) = q^{-\ell-1} \sum_{n \in N_{\ell-1} \pmod{N_{\ell+1}}} \psi(zn) = q^{-\ell}\psi(z).$$

If $k = 2\ell$, $\ell > 0$, then by the same lemma we have

$$J_1(z) = q^{1-k} \sum_{n \in N_0 \pmod{N_{k-1}}} \psi(zn)$$

$$= q^{-\ell} \sum_{n \in N_{\ell-1} \pmod{N_\ell}} \psi(zn) = q^{-\ell}\psi(z) G(q,u).$$

The theorem is proved.

One analogously evaluates the Bessel functions $J_\theta$ with other indices. We omit calculations and formulate the result. But first we note that the mapping $n \to b$, where

$$n = 1 + \tau^{2k-1} a + \sqrt{\tau}\, \tau^{k-1} b, \qquad a, b \in \mathfrak{O},$$

induces an isomorphism between $N_{k-1}/N_k$ and $\mathfrak{O}/\mathfrak{P}$. In particular, a character $\theta$ of the group $N_{k-1}$, trivial on $N_k$, induces a character of the group $\mathfrak{O}$, trivial on $\mathfrak{P}$.

Let $\theta$ be a non-trivial character on $N_0$ such that $\theta \in N_h^\perp \backslash N_{h-1}^\perp$. There is an element $v$ in $\mathfrak{O}$, defined modulo $\mathfrak{P}$, for which $\theta(n) = \chi(\tau^{-1} vb)$ for all $n \in N_{h-1}$ such that

$$n = 1 + \tau^{2h-1} a + \sqrt{\tau}\, \tau^{h-1} b, \qquad a, b \in \mathfrak{O}.$$

In the following theorem $\theta$, $h$ and $v$ have the above meaning.

**Theorem 2.** *For $z \in K$ we have*

$$J_\theta(z) = 0 \quad \text{if} \quad |z| \neq q^h,$$
$$J_\theta(z) = 1 \quad \text{if} \quad z = \tau^{-1}u,\ u \in \mathfrak{U},\ 2u \equiv v \pmod{\mathfrak{P}},$$
$$J_\theta(z) = 0 \quad \text{if} \quad z = \tau^{-h}u,\ u \in \mathfrak{U},\ 2u \not\equiv v \pmod{\mathfrak{P}},$$
$$J_\theta(z) = q^{1-h} \sum_{n \in N_0 \pmod{N_{h-1}}} \psi(zn)\overline{\theta(n)}$$

*if $z = \tau^{-h}u$, $h > 0$, $u \in \mathfrak{U}$, $2u \equiv v \pmod{\mathfrak{P}}$. For the element $z = \sqrt{\tau}\,\tau^{-k}u$, $u \in \mathfrak{U}$, from $\sqrt{\tau}\,K$ we have*

$$J_\theta(z) = 0 \quad \text{if} \quad k \le h,$$
$$J_\theta(z) = q^{1-h} \sum_{n \in N_{k-h} \pmod{N_{h-1}}} \psi(zn_0n)\overline{\theta(n_0n)}$$

*if $h < k < 2h-1$,*

$$J_\theta(z) = q^{1-h}\psi(zn_0)\overline{\theta(n_0)} \quad \text{if} \quad k = 2h-1,$$
$$J_\theta(z) = q^{-h}\psi(z)\chi\left(-\frac{1}{4}\tau^{-1}u^{-1}v^2\right)G(q,u) \quad \text{if} \quad k = 2h,$$
$$J_\theta(z) = q^{-\ell}\psi(z) \quad \text{if} \quad 2h < k,\ k = 2\ell + 1,$$
$$J_\theta(z) = q^{-\ell}\psi(z)G(q,u) \quad \text{if} \quad 2h < k,\ k = 2\ell,$$

*where $n_0$ is an element from $N_0$ of the form $1 + \tau a + \frac{1}{2}\sqrt{\tau}\,u^{-1}v$, $a \in \mathfrak{O}$.*

Matrix elements of the representations $T_\rho$ of the group $G_0$ are also expressed in terms of $J_\theta(z)$. For example, matrix elements of the representation $T_1$ corresponding to the unit character are of the form

$$t^1_{\theta_1,\theta_2}(z) = J_{\theta_1\theta_2^{-1}}(z).$$

By using this relation it is easy to write down the addition and product theorems for the functions $J_\theta(z)$.

**13.3.7. Irreducible representations of the group of unimodular matrices of the second order over the field $K$.** We denote by $G$ the group $SL(2,K)$ of matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with entries from the field $K$ such that $\alpha\delta - \beta\gamma = 1$. Irreducible unitary representations of $G$, belonging to the continuous series, are given by unitary multiplicative characters $\pi$ of the group $K^*$. They are constructed in the space $\mathfrak{L}^2(K)$ of complex-valued functions $\varphi(x)$, $x \in K$, for which

$$(\varphi,\varphi) \equiv \int_K |\varphi(x)|^2 dx < \infty.$$

With the matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$ one associates the operator

$$(T_\pi(g)\varphi)(x) = \pi(\beta x + \delta)|\beta x + \delta|^{-1}\varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right). \tag{1}$$

It is easy to verify that $T_\pi(g_1 g_2) = T_\pi(g_1)T_\pi(g_2)$. The representations $T_\pi$ are unitary.

Another realization of the same representations is obtained by passing from functions $\varphi(x)$ to their Fourier transforms

$$\hat{\varphi}(u) = \int \varphi(x)\chi(ux)dx.$$

In the same way as in Section 7.6.1, we obtain that

$$(T_\pi(g)\hat{\varphi})(u) = \iint \chi\left(ux - v\frac{\alpha x + \gamma}{\beta x + \delta}\right)\pi(\beta x + \delta)|\beta x + \delta|^{-1}\hat{\varphi}(v)dvdx$$

or else

$$(T_\pi(g)\hat{\varphi})(u) = \int K_\pi^{33}(u, v; g)\hat{\varphi}(v)dv,$$

where

$$K_\pi^{33}(u, v; g) = \int \chi\left(ux - v\frac{\alpha x + \gamma}{\beta x + \delta}\right)\pi(\beta x + \delta)|\beta x + \delta|^{-1}dx.$$

If $\beta \neq 0$, then by means of the substitution $\beta x + \delta = t$ we derive

$$K_\pi^{33}(u, v; g) = |\beta|^{-1}\chi\left(-\frac{1}{\beta}(u\delta + v\alpha)\right)\int \chi\left(\frac{1}{\beta}\left(ut + \frac{v}{t}\right)\right)\pi(t)|t|^{-1}dt. \tag{2}$$

The factor before the integral is the same for all $T_\rho$ and the integral is reduced essentially to the Bessel function from the preceding section with the only difference that the integration is not over the "circle" $N_0$ but is over the "straight line" $K$.

If $\beta = 0$, then the kernel of the operator $T_\pi(g)$ has the form[2]

$$\begin{aligned} K_\pi^{33}(u, v; g) &= \pi(\delta)|\delta|^{-1}\chi\left(-\frac{v\gamma}{\delta}\right)\int \chi\left(\left(u - \frac{v\alpha}{\delta}\right)x\right)dx \\ &= \pi(\delta)|\delta|^{-1}\chi\left(-\frac{v\gamma}{\delta}\right)\boldsymbol{\delta}\left(u - \frac{v\alpha}{\delta}\right). \end{aligned}$$

Since $\delta = 1/\alpha$ in our case, then

$$K_\pi^{33}(u, v; g) = \pi^{-1}(\alpha)|\alpha|\chi(-v\alpha\gamma)\boldsymbol{\delta}(u - v\alpha^2).$$

[2] In order to distinguish the notations for the $\delta$-function and for the element $\delta$ of $g$, we use bold-face $\boldsymbol{\delta}$ for the $\delta$-function.

Along with this realization of $T_\pi$, there is another one when functions $\varphi$ are replaced by their Mellin transforms

$$\widetilde{\varphi}(\pi_1) = \int \varphi(x)\pi_1(x)|x|^{-1/2}dx,$$

where $\pi_1$ runs over the set of unitary multiplicative characters, that is, $\pi_1 \in \widehat{K}^*$. It follows from the inversion formula for the Mellin transform that

$$\varphi(x) = \int \widetilde{\varphi}(\pi_2)\pi_2^{-1}(x)|x|^{-1/2}d\pi_2,$$

$$\int |\varphi(x)|^2 dx = \int |\widetilde{\varphi}(\pi_1)|^2 d\pi_1.$$

In this realization we have

$$\begin{aligned}(T_\pi(g)\widetilde{\varphi})(\pi_1) &= \int [(T_\pi(g)\varphi)(x)]\,\pi_1(x)|x|^{-1/2}dx \\ &= \int K_\pi^{22}(\pi_1,\pi_2;g)\widetilde{\varphi}(\pi_2)d\pi_2,\end{aligned}$$

where

$$\begin{aligned}K_\pi^{22}(\pi_1,\pi_2;g) = \int & \pi\pi_2(\beta x+\delta)|\beta x+\delta|^{-1/2} \\ & \times \pi_2^{-1}(\alpha x+\gamma)|\alpha x+\gamma|^{-1/2}\pi_1(x)|x|^{-1/2}dx.\end{aligned}$$

These functions are *analogs of the classical hypergeometric function.*

Finally, we can use the mapping $x \to \frac{1+x\sqrt{\tau}}{1-x\sqrt{\tau}}$, which allows us to construct a denumerable orthogonal basis in $\mathfrak{L}^2(K)$ with the help of the system of characters $\{\theta_n \mid n \in N_0\}$ for $C_\tau$. In this basis the operators of representations are given by infinite matrices, analogous to the matrices $(K^{11}(m,n;\chi,g)) \equiv (t^\chi_{mn}(g))$ from Chapter 7.

It is clear that along with these kernels one can consider the kernels corresponding to mixed bases. Addition and product formulas for $K^{ij}$ are derived in the same way as those from Chapter 7. For this it is necessary to separate corresponding one-parameter subgroups in $SL(2,K)$ (for example, the subgroup $SO(2,K)$).

The following statements are valid (we omit their proofs):

a) the representations $T_\pi$ and $T_{\pi^{-1}}$ are equivalent,

b) all representations $T_\pi$, except for the cases $\pi(x) = \operatorname{sign}_\tau x$, $\tau = \mathfrak{p}$, $\mathfrak{p}\varepsilon$, $\varepsilon$, are irreducible. In the exclusive cases the representation $T_\pi$ decomposes into two irreducible components.

The formula

$$T(g)f(x_1,x_2) = f(\alpha x_1 + \gamma x_2, \beta x_1 + \delta x_2)$$

gives the quasi-regular representation $T$ of the group $SL(2,K)$ in the Hilbert space $\mathfrak{L}^2(K\times K)$. In order to decompose $T$ into irreducible components, we associate with every multiplicative character $\pi$ and with every function $f(x_1,x_2)$ the homogeneous component of this function:

$$f_\pi(x_1,x_2) = \int f(tx_1,tx_2)\pi^{-1}(t)dt.$$

It is clear that

$$f_\pi(sx_1,sx_2) = \pi(s)|s|^{-1}f_\pi(x_1,x_2).$$

By the inversion formula for the Mellin transform, we have

$$f(tx_1,tx_2) = |t|^{-1}\int f_\pi(x_1,x_2)\pi(t)d\pi.$$

For $t=1$ we obtain

$$f(x_1,x_2) = \int f_\pi(x_1,x_2)d\pi.$$

Because of the homogeneity, the function $f_\pi(x_1,x_2)$ is uniquely defined by its values on the line $x_2 = 1$. We set $\varphi_\pi(x) = f_\pi(x,1)$. Then to $T$ there corresponds the representation $T_\pi$ in the space of functions $\varphi_\pi(x)$. Thus, the decomposition of the quasi-regular representation $T$ into irreducible components is derived.

The formula

$$\int |f(x_1,x_2)|^2 dx_1 dx_2 = \int |\varphi_\pi(x)|^2 dx d\pi \tag{3}$$

is valid. Indeed, by the Plancherel formula for the Mellin transform, we have

$$\int |f(tx_1,tx_2)|^2|t|dt = \int |f_\pi(x_1,x_2)|^2 d\pi.$$

By setting $x_2 = 1$, we derive

$$\int |f(xt,t)|^2|t|dt = \int |\varphi_\pi(x)|^2 d\pi.$$

Integrating both sides of this equality with respect to $x$ and setting $tx = y$, we obtain (3).

The group $SL(2,K)$ also has the complementary series of unitary representations. If characters $\pi$ in (1) are nonunitary, we obtain nonunitary representations of $SL(2,K)$.

**13.3.8. Bessel functions of the second kind on $K$.** We have noted that integral (2) of Section 13.3.7 looks like the Bessel functions considered above. We set

$$J_\pi(u,v) = P\int_K \chi\left(ux+\frac{v}{x}\right)\pi(x)|x|^{-1}dx,$$

where $\chi \in \mathfrak{O}^\perp \backslash (\mathfrak{P}^{-1})^\perp$. The following properties of these functions are obvious:

a) $J_\pi(u,v) = J_{\pi^{-1}}(v,u)$,
b) $\pi(u)J_\pi(u,v) = \pi(v)J_\pi(v,u)$,
c) $J_\pi(u,v) = \overline{J_{\pi^{-1}}(-u,-v)} = \pi(-1)\overline{J_{\pi^{-1}}(u,v)}$,
d) If $\pi(-1)=1$, then the values of $J_\pi(u,u)$ are real, and if $\pi(-1)=-1$, then these values are pure imaginary.

Let $k \in \mathbb{Z}_+$, $\pi \in \widehat{K}^*$ and $v \in K^*$. We set

$$F_\pi(k,v) = \int_{|x|=q^k} \chi(x)\chi\left(\frac{v}{x}\right)\pi(x)|x|^{-1}dx.$$

The following assertion is fulfilled:

**Lemma 1.** *Let $|v| = q^m$ and $1 \le k < m$. Then for unramified characters $\pi$ we have $F_\pi(k,v) \neq 0$ if and only if $m$ is even and $k = m/2$. If the ramification degree $h$ for $\pi$ is positive, then $F_\pi(k,v) \neq 0$ only in one of the following cases:*

*a) $m$ is even, $m \ge 2h$ and $k = m/2$,*
*b) $m$ is even, $m < 2h < 2m$ and $k$ coincides with one of the values $h$, $m-h$, $m/2$,*
*c) $m$ is odd, $m < 2h < 2m$ and $k$ coincides with $h$ or $m-h$.*

*Proof:* We introduce the functions $f_1$ and $f_2$ by setting $f_1(x) = \pi(x)\chi(x)$ if $|x| = q^k$, $f_1(x) = 0$ if $|x| \neq q^k$, $f_2(x) = \chi(x)$ if $|x| = q^{m-k}$, and $f_2(x) = 0$ if $|x| \neq q^{m-k}$. Then $F_\pi(k,v) = (f_1 * f_2)(v)$, where the convolution is taken with respect to the multiplicative structure of $K^*$ and with respect to the measure $d^*x$. We consider the Mellin transforms

$$\tilde{f}_1(\pi') = \int_{|x|=q^k} \pi'\pi(x)\chi(x)|x|^{-1}dx,$$

$$\tilde{f}_2(\pi') = \int_{|x|=q^{m-k}} \pi'(x)\chi(x)|x|^{-1}dx.$$

Since $(f_1 * f_2)^{\sim} = \tilde{f}_1 \tilde{f}_2$, then $F_\pi(k,v) \neq 0$ if and only if $\tilde{f}_1(\pi')\tilde{f}_2(\pi') \neq 0$ for some $\pi'$. But the results of Section 13.3.3 (formulas (2), (3), (4)) imply that $\tilde{f}_1(\pi') \neq 0$ either if $\pi'\pi$ is unramified and $k = 1$ or if the ramification degree of $\pi'\pi$ is equal to $k$. Analogously, $\tilde{f}_2(\pi') \neq 0$ either if the character $\pi'$ is unramified and $k = m - 1$ or if the ramification degree of $\pi'$ is equal to $m - k$. This gives all possibilities of $\pi'$ which imply the statement of Lemma 1.

We also note that

$$|F_\pi(k,v)| \leq \int_{|x|=q^k} |x|^{-1} dx = \frac{1}{q'}.$$

One has the following expressions for $J_\pi(u,v)$ in terms of the Gamma-functions and of the functions $F_\pi(k,v)$. If a character $\pi \in \widehat{K}^*$ is unramified and $\pi \not\equiv 1$, then for $u, v \in K^*$ we have

$$J_\pi(u,v) = \pi(v)\Gamma(\pi^{-1}) + \pi^{-1}(u)\Gamma(\pi)$$

if $|uv| \leq q$,

$$J_\pi(u,v) = \pi^{-1}(u) F_\pi\left(\frac{m}{2}, uv\right)$$

if $|uv| = q^m$, $m$ is even and $m > 1$,

$$J_\pi(u,v) = 0$$

if $|uv| = q^m$, $m$ is odd and $m > 0$. If $\pi$ has the positive ramification degree $h$ and $u, v \in K^*$, then

$$J_\pi(u,v) = \pi(v)\Gamma(\pi^{-1}) + \pi^{-1}(u)\Gamma(\pi)$$

for $|uv| \leq q^h$,

$$J_\pi(u,v) = \pi^{-1}(u) F_\pi\left(\frac{m}{2}, uv\right)$$

for $|uv| = q^m$, where $m$ is even and $m \geq 2h$,

$$J_\pi(u,v) = 0$$

for $|uv| = q^m$, where $m$ is odd and $m > 2h$,

$$J_\pi(u,v) = \pi^{-1}(u)\left[F_\pi(h,uv) + F_\pi(m-h,uv) + F_\pi\left(\frac{m}{2}, uv\right)\right]$$

for $|uv| = q^m$, where $m$ is even and $m \leq 2h \leq 2m$,

$$J_\pi(u,v) = \pi^{-1}(u)[F_\pi(h,uv) + F_\pi(m-h,uv)]$$

for $|uv| = q^m$, where $m$ is odd and $m \le 2h \le 2m$.

As an example we prove the first part of this assertion. Let $\pi(x) = |x|^\alpha$. For $|uv| = q^{-m} \le q$, we have

$$J_\pi(u,v) = \pi^{-1}(u)P \int_{|x|\le 1} \chi\left(\frac{uv}{x}\right) |x|^{\alpha-1}dx + \pi^{-1}(u)P \int_{|x|>1} \chi(x)|x|^{\alpha-1}dx.$$

Formula (3) of Section 13.3.3 leads to

$$\pi^{-1}(u)P \int_{|x|>1} \chi(x)|x|^{\alpha-1}dx = -\pi^{-1}(u)q^{\alpha-1}.$$

It follows from the same formula that

$$\pi^{-1}(u)P \int_{|x|\le 1} \chi\left(\frac{uv}{x}\right) |x|^{\alpha-1}dx = \pi(v) \int_{q^{-m}\le|x|\le q} \chi(x)|x|^{-\alpha-1}dx = \pi(v)\left\{\frac{1}{q'}\sum_{k=0}^{m} q^{k\alpha} - q^{-\alpha-1}\right\}.$$

If $\alpha \ne 0$, then the right hand side of this equality can be rewritten as

$$\pi(v)\left[\Gamma(\pi^{-1}) + \pi^{-1}(uv)q^{\alpha-1}(1-q)/(1-q^\alpha)\right] = \pi(v)\Gamma(\pi^{-1}) + \pi^{-1}(u)(1-q^{-1})/(1-q^{-\alpha}).$$

This implies the expression for $J_\pi(u,v)$ for $|uv| \le q$. If $|uv| > q$, then

$$J_\pi(u,v) = \pi^{-1}(u)P \int_{|x|\le 1} \chi\left(\frac{uv}{x}\right) |x|^{\alpha-1}dx + \pi^{-1}(x) \times P \int_{|x|\ge|uv|} \chi(x)|x|^{\alpha-1}dx + \pi^{-1}(u) \int_{1<|x|<|uv|} \chi(x)\chi\left(\frac{uv}{x}\right) |x|^{\alpha-1}dx.$$

Formula (3) of Section 13.3.3 shows that the first and the second integrals vanish. The expression for $J_\pi(u,v)$ follows from Lemma 1.

**13.3.9. Discrete series of irreducible unitary representations of the group $SL(2,K)$.** Let $K(\sqrt{\tau})$ be a quadratic extension of the field $K$ and let $\pi$ be a multiplicative character on $K(\sqrt{\tau})$. With every matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, K)$ we associate the operator

$$(T_\pi(g)\varphi)(u) = \int K_\pi(u, v; g)\varphi(v)dv$$

in $\mathfrak{L}^2(K)$, where

$$K_\pi(u, v; g) = c_\chi \frac{\mathrm{sign}_\tau \beta}{|\beta|} \chi\left(-\frac{1}{\beta}(u\delta - v\alpha)\right) \mathrm{sign}_\tau u \times \int_{t\bar{t} = vu^{-1}} \chi\left(\frac{1}{\beta}\left(ut + \frac{v}{t}\right)\right) \pi(t) d^*(t)$$

for $\beta \neq 0$ and $\mathrm{sign}_\tau u = \mathrm{sign}_\tau v$,

$$K_\pi(u, v; g) = 0$$

for $\beta \neq 0$ and $\mathrm{sign}_\tau u \neq \mathrm{sign}_\tau v$, and

$$K_\pi(u, v; g) = (\mathrm{sign}_\tau \alpha)\pi^{-1}(\alpha)|\alpha|\chi(-v\alpha\gamma)\delta(u - v\alpha^2)$$

for $\beta = 0$. Here $d^*t$ denotes the measure on the circle $t\bar{t} = vu^{-1}$ which is uniquely defined from the conditions $d^*(tt_0) = d^*t$ for all $t_0$, such that $t_0\bar{t}_0 = 1$, and $\int d^*t = 1$. The coefficient $c_\chi$ is given by the formula

$$c_\chi^{-1} = \int \chi(t\bar{t})dt,$$

where the integration is over the whole plane $K(\sqrt{\tau})$ (see formula (9) of Section 13.3.4). Direct calculations show that the correspondence $g \to T_\pi(g)$ is a unitary representation of the group $SL(2, K)$. These representations are essentially defined by the same formulas as continuous series representations with the only difference that the integration is not over the "straight line" $K$ and is over the circle $t\bar{t} = vu^{-1}$. The representations $T_{\pi_1}$ and $T_{\pi_2}$ are equivalent if the corresponding characters $\pi_1$ and $\pi_2$ coincide on the circle $C_\tau$. Since the group $C_\tau$ is compact, then the group $\widehat{C}_\tau$ of its characters is discrete. From here the name "discrete series" appears. To every one from three extensions $K(\sqrt{\tau})$, $\tau = \mathfrak{p}$, $\mathfrak{p}\varepsilon$, $\varepsilon$, of the field $K$ there corresponds a discrete series of representations of $SL(2, K)$.

It is easy to show that every discrete series representation $T_\pi$ decomposes into the direct sum of two representations $T_\pi^+$ and $T_\pi^-$. The first representation $T_\pi^+$ acts in the space $\mathfrak{H}^+$ of functions $\varphi(u) \in \mathfrak{L}^2(K)$ such that $\varphi(u) = 0$ for $\mathrm{sign}_\tau u = -1$, and $T_\pi^-$ acts in the space $\mathfrak{H}^-$ of functions $\varphi(u)$ such that $\varphi(u) = 0$ for $\mathrm{sign}_\tau u = 1$.

Let us formulate the equivalence conditions for the discrete series representations: 1) if $\pi_1$ and $\pi_2$ coincide on $C_\pi$, then $T^+_{\pi_1} \sim T^+_{\pi_2}$ and $T^-_{\pi_1} \sim T^-_{\pi_2}$; 2) if $\pi_1(t) = \pi_2^{-1}(t)$, then $T^+_{\pi_1} \sim T^+_{\pi_2}$ and $T^-_{\pi_1} \sim T^-_{\pi_2}$. The representations $T^+_{\pi_1}$ and $T^-_{\pi_2}$ are nonequivalent for any $\pi_1$ and $\pi_2$.

Writing down discrete series representations in different bases of the space $\mathfrak{L}^2(K)$, we obtain properties of special functions, related to these representations.

Let us note that the group $SL(2, F)$, where $F$ is a finite field of characteristic $p$, $p \neq 2$, also has continuous and discrete series of representations. There is only quadratic extension of such a field $F$. Therefore, there is the only discrete series of representations.

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