Center for Economic Research and Graduate Education
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A COOK-BOOK OF MATHEMATICS

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CERGE-EI LECTURE NOTES **1**

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MATHEMATICS

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To my Teachers

He liked those literary cooks
Who skim the cream of others' books
And ruin half an author's graces
By plucking bon-mots from their places.

Hannah More, *Florio (1786)*

Introduction

This textbook is based on an extended collection of handouts I distributed to the graduate students in economics attending my summer mathematics class at the Center for Economic Research and Graduate Education (CERGE) at Charles University in Prague.

Two considerations motivated me to write this book. First, I wanted to write a short textbook, which could be covered in the course of two months and which, in turn, covers the most significant issues of mathematical economics. I have attempted to maintain a balance between being overly detailed and overly schematic. Therefore this text should resemble (in the 'ideological' sense) a "hybrid" of Chiang's classic textbook *Fundamental Methods of Mathematical Economics* and the comprehensive reference manual by Berck and Sydsæter (Exact references appear at the end of this section).

My second objective in writing this text was to provide my students with simple "cook-book" recipes for solving problems they might face in their studies of economics. Since the target audience was supposed to have some mathematical background (admittance to the program requires at least BA level mathematics), my main goal was to refresh students' knowledge of mathematics rather than teach them math 'from scratch'. Students were expected to be familiar with the basics of set theory, the real-number system, the concept of a function, polynomial, rational, exponential and logarithmic functions, inequalities and absolute values.

Bearing in mind the applied nature of the course, I usually refrained from presenting complete proofs of theoretical statements. Instead, I chose to allocate more time and space to examples of problems and their solutions and economic applications. I strongly believe that for students in economics – for whom this text is meant – the *application* of mathematics in their studies takes precedence over *das Glasperlenspiel* of abstract theoretical constructions.

Mathematics is an ancient science and, therefore, it is little wonder that these notes may remind the reader of the other text-books which have already been written and published. To be candid, I did not intend to be entirely original, since that would be impossible. On the contrary, I tried to benefit from books already in existence and adapted some interesting examples and worthy pieces of theory presented there. If the reader requires further proofs or more detailed discussion, I have included a useful, but hardly exhaustive reference guide at the end of each section.

With very few exceptions, the analysis is limited to the case of real numbers, the theory of complex numbers being beyond the scope of these notes.

Finally, I would like to express my deep gratitude to Professor Jan Kmenta for his valuable comments and suggestions, to Sufana Razvan for his helpful assistance, to Aurelia Pontes for excellent editorial support, to Natalka Churikova for her advice and, last but not least, to my students who inspired me to write this book.

All remaining mistakes and misprints are solely mine.

I wish you success in your mathematical kitchen! *Bon Appetit !*

Supplementary Reading (General):

- Arrow, K. and M.Intriligator, eds. *Handbook of Mathematical Economics,* vol. 1.
- Berck P. and K. Sydsæter. *Economist's Mathematical Manual.*
- Chiang, A. *Fundamental Methods of Mathematical Economics.*
- Ostaszewski, I. *Mathematics in Economics: Models and Methods.*
- Samuelson, P. *Foundations of Economic Analysis.*
- Silberberg, E. *The Structure of Economics: A Mathematical Analysis.*
- Takayama, A. *Mathematical Economics.*
- Yamane, T. *Mathematics for Economists: An Elementary Survey.*

Basic notation used in the text:

Statements: $A, B, C, \ldots$

True/False: all statements are either true or false.

Negation: $\neg A$ 'not A'

Conjunction: $A \wedge B$ 'A and B'

Disjunction: $A \vee B$ 'A or B'

Implication: $A \Rightarrow B$ 'A implies B'

(A is sufficient condition for B; B is necessary condition for A.)

Equivalence: $A \Leftrightarrow B$ 'A if and only if B' (A iff B, for short)

(A is necessary and sufficient for B; B is necessary and sufficient for A.)

Example 1 $(\neg A) \wedge A \Leftrightarrow$ *FALSE.*

$(\neg(A \vee B)) \Leftrightarrow ((\neg A) \wedge (\neg B))$ *(De Morgan rule).*

Quantifiers:

Existential: $\exists$ 'There exists' or 'There is'

Universal: $\forall$ 'For all' or 'For every'

Uniqueness: $\exists!$ 'There exists a unique ...' or 'There is a unique...'

The colon **:** and the vertical line | are widely used as abbreviations for 'such that'

$a \in \mathcal{S}$ means 'a is an element of (belongs to) set $\mathcal{S}$'

Example 2 (Definition of continuity)

f is continuous at x if

$((\forall \epsilon > 0)(\exists \delta > 0) : (\forall y \in |y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$

Optional information which might be helpful is typeset in footnotesize font.

The symbol ⚠ is used to draw the reader's attention to potential pitfalls.

Contents

1 Linear Algebra

1.1 Matrix Algebra

Definition 1 *An $m \times n$ matrix is a rectangular array of real numbers with m rows and n columns.*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

$m \times n$ is called the *dimension* or *order* of A. If $m = n$, the matrix is the *square* of order n.

A subscribed element of a matrix is always read as $a_{\text{row,column}}$ ⚠

A shorthand notation is $A = (a_{ij})$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, or $A = (a_{ij})_{[m \times n]}$.

A vector is a special case of a matrix, when either $m = 1$ (row vectors $v = (v_1, v_2, \dots, v_n)$) or $n = 1$ – column vectors

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}.$$

1.1.1 Matrix Operations

- Addition and subtraction: given $A = (a_{ij})_{[m \times n]}$ and $B = (b_{ij})_{[m \times n]}$

  $$A \pm B = (a_{ij} \pm b_{ij})_{[m \times n]},$$

  i.e. we simply add or subtract corresponding elements.

  Note that these operations are defined only if A and B are of the same dimension. ⚠

- Scalar multiplication:

  $$\lambda A = (\lambda a_{ij}), \qquad \text{where } \lambda \in \mathbf{R},$$

  i.e. each element of A is multiplied by the same scalar λ.

- Matrix multiplication: if $A = (a_{ij})_{[m \times n]}$ and $B = (a_{ij})_{[n \times k]}$ then

  $$A \cdot B = C = (c_{ij})_{[m \times k]}, \qquad \text{where } c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}.$$

  Note the dimensions of matrices! ⚠

Recipe 1 – How to Multiply Two Matrices:

In order to get the element c_{ij} of matrix C you need to multiply the ith row of matrix A by the jth column of matrix B.

Example 3 $\begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 9 & 10 \\ 12 & 11 \\ 14 & 13 \end{pmatrix} = \begin{pmatrix} 3\cdot 9+2\cdot 12+1\cdot 14 & 3\cdot 10+2\cdot 11+1\cdot 13 \\ 4\cdot 9+5\cdot 12+6\cdot 14 & 4\cdot 10+5\cdot 11+6\cdot 13 \end{pmatrix}$.

Example 4 *A system of m linear equations for n unknowns*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ \ldots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m, \end{cases}$$

can be written as $Ax = b$, where $A = (a_{ij})_{[m\times n]}$, and

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

1.1.2 Laws of Matrix Operations

- Commutative law of addition: $A + B = B + A$.
- Associative law of addition: $(A + B) + C = A + (B + C)$.
- Associative law of multiplication: $A(BC) = (AB)C$.
- Distributive law:
  $A(B + C) = AB + AC$ (premultiplication by A),
  $(B + C)A = BA + BC$ (postmultiplication by A).

The commutative law of multiplication is not applicable in the matrix case, $AB \neq BA$!!! ⚠

Example 5 *Let* $A = \begin{pmatrix} 2 & 0 \\ 3 & 8 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 2 \\ 6 & 3 \end{pmatrix}$.

Then $AB = \begin{pmatrix} 14 & 4 \\ 69 & 30 \end{pmatrix} \neq BA = \begin{pmatrix} 20 & 16 \\ 21 & 24 \end{pmatrix}$.

Example 6 *Let* $v = (v_1, v_2, \ldots, v_n), \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$.

Then $vu = v_1u_1 + v_2u_2 + \ldots + v_nu_n$ is a scalar, and $uv = C = (c_{ij})_{[n\times n]}$ is a n by n matrix, with $c_{ij} = u_iv_j$, $i, j = 1, \ldots, n$.

We introduce the *identity* or *unit* matrix of dimension n I_n as

$$I_n = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Note that I_n is always a square $[n\times n]$ matrix (further on the subscript n will be omitted). I_n has the following properties:

a) $AI = IA = A$,

b) $AIB = AB$ for all $A,\ B$.

In this sense the identity matrix corresponds to 1 in the case of scalars.

The *null* matrix is a matrix of any dimension for which all elements are zero:

$$\mathbf{0} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}.$$

Properties of a null matrix:

a) $A + \mathbf{0} = A$,

b) $A + (-A) = \mathbf{0}$.

Note that $AB = \mathbf{0} \not\Rightarrow A = \mathbf{0}$ or $B = \mathbf{0}$; $\quad AB = AC \not\Rightarrow B = C$. ⚠

Definition 2 *A* diagonal matrix *is a square matrix whose only non-zero elements appear on the principle (or main) diagonal.*

A triangular matrix *is a square matrix which has only zero elements above or below the principle diagonal.*

1.1.3 Inverses and Transposes

Definition 3 *We say that $B = (b_{ij})_{[n\times m]}$ is the transpose of $A = (a_{ij})_{[m\times n]}$ if $a_{ji} = b_{ij}$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$.*

Usually transposes are denoted as A' (or as A^T).

Recipe 2 – How to Find the Transpose of a Matrix:

The transpose A' of A is obtained by making the columns of A into the rows of A'.

Example 7 $$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & a & b \end{pmatrix}, \qquad A' = \begin{pmatrix} 1 & 0 \\ 2 & a \\ 3 & b \end{pmatrix}.$$

Properties of transposition:

a) $(A')' = A$

b) $(A + B)' = A' + B'$

c) $(\alpha A)' = \alpha A'$, where α is a real number.

d) $(AB)' = B'A'$

Note the order of transposed matrices! ⚠

Definition 4 *If $A' = A$, A is called symmetric.*

If $A' = -A$, A is called anti-symmetric (or skew-symmetric).

If $A'A = I$, A is called orthogonal.

If $A = A'$ and $AA = A$, A is called idempotent.

Definition 5 *The inverse matrix A^{-1} is defined as $A^{-1}A = AA^{-1} = I$.*

Note that A as well as A^{-1} are square matrices of the same dimension (it follows from the necessity to have the preceding line defined). ⚠

Example 8 *If* $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *then the inverse of A is* $A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$.
We can easily check that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Another important characteristics of inverse matrices and inversion:

- Not all square matrices have their inverses. If a square matrix has its inverse, it is called *regular* or *non-singular*. Otherwise it is called *singular* matrix.
- If A^{-1} exists, it is unique.
- $A^{-1}A = I$ is equivalent to $AA^{-1} = I$.

Properties of inversion:

a) $(A^{-1})^{-1} = A$

b) $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$, where α is a real number, $\alpha \neq 0$.

c) $(AB)^{-1} = B^{-1}A^{-1}$
Note the order of matrices! ⚠

d) $(A')^{-1} = (A^{-1})'$

1.1.4 Determinants and a Test for Non-Singularity

The formal definition of the determinant is as follows: given $n \times n$ matrix $A = (a_{ij})$,

$$\det(A) = \sum_{(\alpha_1,\ldots,\alpha_n)} (-1)^{I(\alpha_1,\ldots,\alpha_n)} a_{1\alpha_1} \cdot a_{2\alpha_2} \cdot \ldots \cdot a_{n\alpha_n}$$

where $(\alpha_1, \ldots, \alpha_n)$ are all different permutations of $(1, 2, \ldots, n)$, and $I(\alpha_1, \ldots, \alpha_n)$ is the number of inversions.

Usually we denote the determinant of A as $\det(A)$ or $|A|$.

For practical purposes, we can give an alternative recursive definition of the determinant. Given the fact that the determinant of a scalar is a scalar itself, we arrive at following

Definition 6 (Laplace Expansion Formula)

$$\det(A) = \sum_{k=1}^{n} (-1)^{l+k} a_{lk} \cdot \det(M_{lk}) \qquad \textit{for some integer } l,\ 1 \leq l \leq n.$$

Here M_{lk} is the *minor* of element a_{lk} of the matrix A, which is obtained by deleting lth row and kth column of A. $(-1)^{l+k}\det(M_{lk})$ is called *cofactor* of the element a_{lk}.

Example 9 *Given matrix*

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 4 & 5 \\ 7 & 2 & 1 \end{pmatrix}, \qquad \text{the minor of the element } a_{23} \text{ is} \quad M_{23} = \begin{pmatrix} 2 & 3 \\ 7 & 2 \end{pmatrix}.$$

Note that in the above expansion formula we expanded the determinant by elements of the lth row. Alternatively, we can expand it by elements of lth column. Thus the Laplace Expansion formula can be re-written as

$$\det(A) = \sum_{k=1}^{n} (-1)^{k+l} a_{kl} \cdot \det(M_{kl}) \qquad \text{for some integer } l,\ 1 \leq l \leq n.$$

Example 10 *The determinant of* 2×2 *matrix:*

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Example 11 *The determinant of* 3×3 *matrix:*

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$= a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \cdot \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \cdot \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} =$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Properties of the determinant:

a) $\det(A \cdot B) = \det(A) \cdot \det(B)$.

b) In general, $\det(A + B) \neq \det(A) + \det(B)$. ⚠

Recipe 3 – How to Calculate the Determinant:
We can apply the following useful rules:

1. *The multiplication of any one row (or column) by a scalar k will change the value of the determinant k-fold.*

2. *The interchange of any two rows (columns) will alter the sign but not the numerical value of the determinant.*

3. *If a multiple of any row is added to (or subtracted from) any other row it will not change the value or the sign of the determinant. The same holds true for columns. (I.e. the determinant is not affected by linear operations with rows (or columns)).*

4. *If two rows (or columns) are identical, the determinant will vanish.*

5. *The determinant of a triangular matrix is a product of its principal diagonal elements.*

Using these rules, we can simplify the matrix (e.g. obtain as many zero elements as possible) and then apply Laplace expansion.

Example 12 *Let* $A = \begin{pmatrix} 4 & -2 & 6 & 2 \\ 0 & -1 & 5 & -3 \\ 2 & -1 & 8 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

Subtracting the first row, divided by 2, from the third row, we get

$$\det A = \begin{pmatrix} 4 & -2 & 6 & 2 \\ 0 & -1 & 5 & -3 \\ 2 & -1 & 8 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 2 \cdot \det \begin{pmatrix} 2 & -1 & 3 & 1 \\ 0 & -1 & 5 & -3 \\ 2 & -1 & 8 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix} =$$

$$2 \cdot \det \begin{pmatrix} 2 & -1 & 3 & 1 \\ 0 & -1 & 5 & -3 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 2 \cdot 2 \cdot (-1) \cdot 5 \cdot 2 = -40$$

If we have a block-diagonal matrix, i.e. a partitioned matrix of the form

$$P = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}, \qquad \text{where } P_{11} \text{ and } P_{22} \text{ are square matrices,}$$

then $\det(P) = \det(P_{11}) \cdot \det(P_{22})$.

If we have a partitioned matrix

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P21 & P_{22} \end{pmatrix}, \qquad \text{where } P_{11},\ P_{22} \text{ are square matrices,}$$

then $\det(P) = \det(P_{22}) \cdot \det(P_{11} - P_{12}P_{22}^{-1}P_{21}) = \det(P_{11}) \cdot \det(P_{22} - P_{21}P_{11}^{-1}P_{12})$.

Proposition 1 (The Determinant Test for Non-Singularity)
A matrix A is non-singular $\Leftrightarrow \det(A) \neq 0$.

As a corollary, we get

Proposition 2 A^{-1} *exists* $\Leftrightarrow \det(A) \neq 0$.

Recipe 4 – How to Find an Inverse Matrix:
There are two ways of finding inverses.
Assume that matrix A is invertible, i.e. $\det(A) \neq 0$.

1. Method of adjoint matrix. *For the computation of an inverse matrix A^{-1} we use the following algorithm: $A^{-1} = (d_{ij})$, where*

$$d_{ij} = \frac{1}{\det(A)}(-1)^{i+j} \det(M_{ji}).$$

Note the order of indices at M_{ji}! ⚠

This method is called "method of adjoint" because we have to compute the so-called *adjoint of matrix A*, which is defined as a matrix $\mathrm{adj}A = C' = (|C_{ji}|)$, where $|C_{ij}|$ is the cofactor of the element a_{ij}.

2. Gauss elimination method or pivotal method. *An identity matrix is placed along side a matrix A that is to be inverted. Then, the same elementary row operations are performed on* both *matrices until A has been reduced to an identity matrix. The identity matrix upon which the elementary row operations have been performed will then become the inverse matrix we seek.*

Example 13 (method of adjoint)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Longrightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example 14 (Gauss elimination method)

Let $A = \begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix}$. *Then*

$$\begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3/2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3/2 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ -2 & 1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 3/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 1/2 & -1/4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/4 & 3/8 \\ 1/2 & -1/4 \end{pmatrix}.$$

Therefore, the inverse is

$$A^{-1} = \begin{pmatrix} -1/4 & 3/8 \\ 1/2 & -1/4 \end{pmatrix}.$$

1.1.5 Rank of a Matrix

A linear combination of vectors $a^1, a^2, \ldots, a^k$ is a sum

$$q_1 a^1 + q_2 a^2 + \ldots + q_k a^k,$$

where $q_1, q_2, \ldots, q_k$ are real numbers.

Definition 7 *Vectors $a^1, a^2, \ldots, a^k$ are linearly dependent if and only if there exist numbers $c_1, c_2, \ldots, c_k$ not all zero, such that*

$$c_1 a^1 + c_2 a^2 + \ldots + c_k a^k = \mathbf{0}.$$

Example 15 *Vectors $a^1 = (2, 4)$ and $a^2 = (3, 6)$ are linearly dependent: if, say, $c_1 = 3$ and $c_2 = -2$ then $c_1 a^1 + c_2 a^2 = (6, 12) + (-6, -12) = \mathbf{0}$.*

Recall that if we have n linearly independent vectors $e^1, e^2, \ldots, e^n$, they are said to *span* an n-dimensional vector space or to *constitute a basis* in an n-dimensional vector space. For more details see the section "Vector Spaces".

Definition 8 *The rank of a matrix A* rank(A) *can be defined as*
– the maximum number of linearly independent rows;
– or the maximum number of linearly independent columns;
– or the order of the largest non-zero minor of A.

Example 16 $\text{rank}\begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ -5 & 7 & 1 \end{pmatrix} = 2.$

The first two rows are linearly dependent, therefore the maximum number of linearly independent rows is equal to 2.

Properties of the rank:

- The column rank and the row rank of a matrix are equal.
- $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.
- $\text{rank}(A) = \text{rank}(AA') = \text{rank}(A'A)$.

Using the notion of rank, we can re-formulate the condition for non-singularity:

Proposition 3 *If A is a square matrix of order n, then*
$\text{rank}(A) = n \Leftrightarrow \det(A) \neq 0$.

1.2 Systems of Linear Equations

Consider a system of n linear equations for n unknowns $Ax = b$.

Recipe 5 – How to Solve a Linear System $Ax = b$ (general rules):

$b = \mathbf{0}$ (homogeneous case)
If $\det(A) \neq 0$ *then the system has a unique trivial (zero) solution.*
If $\det(A) = 0$ *then the system has an infinite number of solutions.*

$b \neq \mathbf{0}$ (non-homogeneous case)
If $\det(A) \neq 0$ *then the system has a unique solution.*
If $\det(A) = 0$ *then*

a) $\text{rank}(A) = \text{rank}(\tilde{A}) \Rightarrow$ *the system has an infinite number of solutions.*

b) $\text{rank}(A) \neq \text{rank}(\tilde{A}) \Rightarrow$ *the system is inconsistent.*

Here $\tilde{A}$ is a so-called *augmented matrix*,

$$\tilde{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & b_n \end{pmatrix}.$$

Recipe 6 – How to Solve the System of Linear Equations, if $b \neq \mathbf{0}$ and $\det(A) \neq 0$:

1. The inverse matrix method:

   Since A^{-1} exists, the solution x can be found as $x = A^{-1}b$.

2. Gauss method:

   We perform the same elementary row operations on matrix A and vector b until A has been reduced to an identity matrix. The vector b upon which the elementary row operations have been performed will then become the solution.

3. Cramer's rule:

We can consequently find all elements $x_1, x_2, \ldots, x_n$ of vector x using the following formula:

$$x_j = \frac{\det(A_j)}{\det(A)}, \text{ where } A_j = \begin{pmatrix} a_{11} & \ldots & a_{1j-1} & b_1 & a_{1j+1} & \ldots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \ldots & a_{nj-1} & b_n & a_{nj+1} & \ldots & a_{nn} \end{pmatrix}.$$

Example 17 *Let us solve*

$$\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \end{pmatrix}$$

for x_1, x_2 using Cramer's rule:

$$\det(A) = 2 \cdot (-1) - 3 \cdot 3 = -14,$$

$$\det(A_1) = \det \begin{pmatrix} 12 & 3 \\ 10 & -1 \end{pmatrix} = 12 \cdot (-1) - 3 \cdot 10 = -42,$$

$$\det(A_2) = \det \begin{pmatrix} 2 & 12 \\ 4 & 10 \end{pmatrix} = 2 \cdot 10 - 12 \cdot 4 = -28,$$

therefore

$$x_1 = \frac{-42}{-14} = 3, \qquad x_2 = \frac{-28}{-14} = 2.$$

Economics Application 1 (General Market Equilibrium)

Consider a market for three goods. Demand and supply for each good are given by:

$$\begin{cases} D_1 = 5 - 2P_1 + P_2 + P_3 \\ S_1 = -4 + 3P_1 + 2P_2 \end{cases}$$

$$\begin{cases} D_2 = 6 + 2P_1 - 3P_2 + P_3 \\ S_2 = 3 + 2P_2 \end{cases}$$

$$\begin{cases} D_3 = 20 + P_1 + 2P_2 - 4P_3 \\ S_3 = 3 + P_2 + 3P_3 \end{cases}$$

where P_i is the price of good i, $i = 1, 2, 3$.

The equilibrium conditions are: $D_i = S_i$, $i = 1, 2, 3$, that is

$$\begin{cases} 5P_1 & +P_2 & -P_3 & = 9 \\ -2P_1 & +5P_2 & -P_3 & = 3 \\ -P_1 & -P_2 & +7P_3 & = 17 \end{cases}$$

This system of linear equations can be solved at least in two ways.

a) Using Cramer's rule:

$$A_1 = \begin{vmatrix} 9 & 1 & -1 \\ 3 & 5 & -1 \\ 17 & -1 & 7 \end{vmatrix}, \qquad A = \begin{vmatrix} 5 & 1 & -1 \\ -2 & 5 & -1 \\ -1 & -1 & 7 \end{vmatrix}$$

$$P_1^* = \frac{A_1}{A} = \frac{356}{178} = 2.$$

Similarly $P_2^* = 2$ *and* $P_3^* = 3$. *The vector of* (P_1^*,P_2^*,P_3^*) *describes the general market equilibrium.*

b) Using the inverse matrix rule:

Let us denote

$$A = \begin{pmatrix} 5 & 1 & -1 \\ -2 & 5 & -1 \\ -1 & -1 & 7 \end{pmatrix}, \quad P = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}, \quad B = \begin{pmatrix} 9 \\ 3 \\ 17 \end{pmatrix}.$$

The matrix form of the system is:

$AP = B$, *which implies* $P = A^{-1}B$.

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} 34 & -6 & 4 \\ 15 & 34 & 7 \\ 7 & 4 & 27 \end{pmatrix},$$

$$P = \frac{1}{178}\begin{pmatrix} 34 & -6 & 4 \\ 15 & 34 & 7 \\ 7 & 4 & 27 \end{pmatrix}\begin{pmatrix} 9 \\ 3 \\ 17 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

Again, $P_1^* = 2$, $P_2^* = 2$ *and* $P_3^* = 3$.

Economics Application 2 (Leontief Input-Output Model)

This model addresses the following planning problem: Assume that n *industries produce* n *goods (each industry produces only one good) and the output good of each industry is used as an input in the other* $n-1$ *industries. In addition, each good is demanded for 'non-input' consumption. What are the efficient amounts of output each of the* n *industries should produce? ('Efficient' means that there will be no shortage and no surplus in producing each good).*

The model is based on an input matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

where a_{ij} *denotes the amount of good* i *used to produce one unit of good* j.

To simplify the model, let set the price of each good equal to \$1. *Then the value of inputs should not exceed the value of output:*

$$\sum_{i=1}^{n} a_{ij} \leq 1, \; j = 1, \ldots n.$$

If we denote an additional (non-input) demand for good i by b_i, then the optimality condition reads as follows: the demand for each input should equal the supply, that is

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + b_i,\ i = 1, \dots n,$$

or

$$x = Ax + b,$$

or

$$(I - A)x = b.$$

The system $(I - A)x = b$ can be solved using either Cramer's rule or the inverse matrix.

Example 18 (Numerical Illustration)

$$\text{Let} \quad A = \begin{pmatrix} 0.2 & 0.2 & 0.1 \\ 0.3 & 0.3 & 0.1 \\ 0.1 & 0.2 & 0.4 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ 4 \\ 5 \end{pmatrix}$$

Thus the system $(I - A)x = b$ becomes

$$\begin{cases} 0.8x_1 & -0.2x_2 & -0.1x_3 & = 8 \\ -0.3x_1 & +0.7x_2 & -0.1x_3 & = 4 \\ -0.1x_1 & -0.2x_2 & +0.6x_3 & = 5 \end{cases}$$

Solving it for $x_1,\ x_2,\ x_3$ we find the solution $(\frac{4210}{269}, \frac{3950}{269}, \frac{4260}{269})$.

1.3 Quadratic Forms

Generally speaking, a *form* is a polynomial expression, in which each term has a uniform degree (e.g. $L = ax + by + cz$ is an example of a *linear* form in three variables $x,\ y,\ z$, where $a,\ b,\ c$ are arbitrary real constants).

Definition 9 *A quadratic form Q in n variables $x_1, x_2, \dots, x_n$ is a polynomial expression in which each component term has a degree two (i.e. each term is a product of x_i and x_j, where $i, j = 1, 2, \dots, n$):*

$$Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j,$$

where a_{ij} are real numbers. For convenience, we assume that $a_{ij} = a_{ji}$. In matrix notation, $Q = x'Ax$, where $A = (a_{ij})$ is a symmetric matrix, and

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Example 19 *A quadratic form in two variables:* $Q = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$.

Definition 10 *A quadratic form Q is said to be*

$$\left.\begin{array}{l} \textit{positive definite (PD)} \\ \textit{negative definite (ND)} \\ \textit{positive semidefinite (PSD)} \\ \textit{negative semidefinite (NSD)} \end{array}\right\} \textit{if } Q = x'Ax \textit{ is } \left\{\begin{array}{ll} > 0 & \textit{for all } x \neq \mathbf{0} \\ < 0 & \textit{for all } x \neq \mathbf{0} \\ \geq 0 & \textit{for all } x \\ \leq 0 & \textit{for all } x \end{array}\right.$$

otherwise Q is called indefinite (ID).

Example 20 $Q = x^2 + y^2$ *is PD,* $Q = (x+y)^2$ *is PSD,* $Q = x^2 - y^2$ *is ID.*

Leading principal minors $D_k,\ k = 1, 2, \ldots, n$ of a matrix $A = (a_{ij})_{[n\times n]}$ are defined as

$$D_k = \det\begin{pmatrix} a_{11} & \ldots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \ldots & a_{kk} \end{pmatrix}.$$

Proposition 4

1. *A quadratic form Q id PD* $\Leftrightarrow$ $D_k > 0$ *for all* $k = 1, 2, \ldots, n$.
2. *A quadratic form Q id ND* $\Leftrightarrow$ $(-1)^k D_k > 0$ *for all* $k = 1, 2, \ldots, n$.

Note that if we replace $>$ by $\geq$ in the above statement, it does NOT give us the criteria for the semidefinite case! ⚠

Proposition 5 *A quadratic form Q is PSD (NSD)* $\Leftrightarrow$ *all the principal minors of A are* $\geq (\leq) 0$.

By definition, the principal minor

$$A\begin{pmatrix} i_1 \ldots i_p \\ i_1 \ldots i_p \end{pmatrix} = \det\begin{pmatrix} a_{i_1 i_1} & \ldots & a_{i_1 i_p} \\ \vdots & & \vdots \\ a_{i_p i_1} & \ldots & a_{i_p i_p} \end{pmatrix}, \text{ where } 1 \leq i_1 < i_2 < \ldots < i_p \leq n,\ p \leq n.$$

Example 21 *Consider the quadratic form* $Q(x, y, z) = 3x^2 + 3y^2 + 5z^2 - 2xy$. *The corresponding matrix has the form*

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Leading principal minors of A are

$$D_1 = 3 > 0, \quad D_2 = \det\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} = 8 > 0, \quad D_3 = \det\begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} = 40 > 0,$$

therefore, the quadratic form is positive definite.

1.4 Eigenvalues and Eigenvectors

Definition 11 *Any number λ such that the equation*

$$Ax = \lambda x \tag{1}$$

has a non-zero vector-solution x is called an eigenvalue (or a characteristic root) of the equation (1)

Definition 12 *Any non-zero vector x satisfying (1) is called an eigenvector (or characteristic vector) of A for the eigenvalue λ.*

Recipe 7 – How to calculate eigenvalues:

$Ax - \lambda x = 0 \Rightarrow (A - \lambda I)x = 0$. *Since x is non-zero, the determinant of $(A - \lambda I)$ should vanish. Therefore all eigenvalues can be calculated as roots of the equation (which is often called the characteristic equation or the characteristic polynomial of A)*

$$\det(A - \lambda I) = 0.$$

Example 22 *Let us consider the quadratic form from Example 21.*

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{pmatrix} =$$

$$= (5-\lambda)(\lambda^2 - 6\lambda + 8) = (5-\lambda)(\lambda-2)(\lambda-4) = 0,$$

therefore the eigenvalues are $\lambda = 2$, $\lambda = 4$ and $\lambda = 5$.

Proposition 6 (Characteristic Root Test for Sign Definiteness.)

A quadratic form Q is

$$\left.\begin{array}{l} \textit{positive definite} \\ \textit{negative definite} \\ \textit{positive semidefinite} \\ \textit{negative semidefinite} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \text{eigenvalues } \lambda_i > 0 \text{ for all } i = 1, 2, \ldots, n \\ \lambda_i < 0 \text{ for all } i = 1, 2, \ldots, n \\ \lambda_i \geq 0 \text{ for all } i = 1, 2, \ldots, n \\ \lambda_i \leq 0 \text{ for all } i = 1, 2, \ldots, n \end{array}\right.$$

A form is indefinite if at least one positive and one negative eigenvalues exist.

Definition 13 *Matrix A is diagonalizable $\Leftrightarrow$ $P^{-1}AP = D$ for a non-singular matrix P and a diagonal matrix D.*

Proposition 7 (The Spectral Theorem for Symmetric Matrices)

If A is a symmetric matrix of order n and $\lambda_1, \ldots, \lambda_n$ are its eigenvalues, there exists an orthogonal matrix U such that

$$U^{-1}AU = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Usually, U is the normalized matrix formed by eigenvectors. It has the property $U'U = I$ (i.e. U is orthogonal matrix; $U' = U^{-1}$). "Normalized" means that for any column u of the matrix U $u'u = 1$.

It is essential that A be symmetrical! ⚠

Example 23 *Diagonalize the matrix*

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

First, we need to find the eigenvalues:

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} = (1-\lambda)(4-\lambda) - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5),$$

i.e. $\lambda = 0$ *and* $\lambda = 5$.
For $\lambda = 0$ *we solve*

$$\begin{pmatrix} 1-0 & 2 \\ 2 & 4-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{aligned} x_1 + 2x_2 &= 0, \\ 2x_1 + 4x_2 &= 0. \end{aligned}$$

The second equation is redundant and the eigenvector, corresponding to $\lambda = 0$, *is* $v_1 = C_1 \cdot (2, -1)'$, *where* C_1 *is an arbitrary real constant.*
For $\lambda = 5$ *we solve*

$$\begin{pmatrix} 1-5 & 2 \\ 2 & 4-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{aligned} -4x_1 + 2x_2 &= 0, \\ 2x_1 - x_2 &= 0. \end{aligned}$$

Thus the general expression for the second eigenvector is $v_2 = C_2 \cdot (1, 2)'$.

Let us normalize the eigenvectors, i.e. let us pick constants C *such that* $v_1'v_1 = 1$ *and* $v_2'v_2 = 1$. *After normalization we get* $v_1 = (2/\sqrt{5}, -1/\sqrt{5})'$, $v_2 = (1/\sqrt{5}, 2/\sqrt{5})'$. *Thus the diagonalization matrix* U *is*

$$U = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

You can easily check that

$$U^{-1}AU = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}.$$

Some useful results:

- $\det(A) = \lambda_1 \cdot \ldots \cdot \lambda_n$.
- if $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A then $1/\lambda_1, \ldots, 1/\lambda_n$ are eigenvalues of A^{-1}.
- if $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A then $f(\lambda_1), \ldots, f(\lambda_n)$ are eigenvalues of $f(A)$, where $f(\cdot)$ is a polynomial.
- the rank of a symmetric matrix is the number of non-zero eigenvalues it contains.
- the rank of any matrix A is equal to the number of non-zero eigenvalues of $A'A$.
- if we define the *trace* of a square matrix of order n as the sum of the n elements on its principal diagonal $\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$, then $\text{tr}(A) = \lambda_1 + \ldots + \lambda_n$.

Properties of the trace:

a) if A and B are of the same order, $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$;

b) if λ is a scalar, $\text{tr}(\lambda A) = \lambda \text{tr}(A)$;

c) $\text{tr}(AB) = \text{tr}(BA)$, whenever AB is square;

d) $\text{tr}(A') = \text{tr}(A)$.

e) $\text{tr}(A'A) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2$.

1.5 Appendix: Vector Spaces

1.5.1 Basic Concepts

Definition 14 *A* (real) vector space *is a nonempty set V of objects together with an additive operation $+ : V \times V \to V$, $+(u, v) = u + v$ and a scalar multiplicative operation $\cdot : R \times V \to V$, $\cdot(a, u) = au$ which satisfies the following axioms for any $u, v, w \in V$ and any $a, b \in R$ (R is the set of all real numbers):*

A1) (u+v)+w=u+(v+w)
A2) u+v=v+u
A3) 0+u=u
A4) u+(-u)=0
S1) a(u+v)=au+av
S2) (a+b)u=au+bu
S3) a(bu)=(ab)u
S4) 1u=u.

Definition 15 *The objects of a vector space V are called* vectors*, the operations $+$ and $\cdot$ are called* vector addition *and* scalar multiplication*, respectively. The element* $0 \in V$ *is the* zero vector *and* $-v$ *is the* additive inverse *of V.*

Example 24 (The n-Dimensional Vector Space R^n)

Define $R^n = \{(u_1, u_2, \ldots, u_n)' | u_i \in R, i = 1, \ldots, n\}$ (the apostrophe denotes the transpose). Consider $u, v \in R^n$, $u = (u_1, u_2, \ldots, u_n)', v = (v_1, v_2, \ldots, v_n)'$ and $a \in R$.

Define the additive operation and the scalar multiplication as follows:

$$\begin{aligned} u+v &= (u_1+v_1,\dots u_n+v_n)', \\ au &= (au_1,\dots au_n)'. \end{aligned}$$

It is not difficult to verify that R^n together with these operations is a vector space.

Definition 16 *Let V be a vector space. An* inner product *or* scalar product *in V is a function $s: V\times V\to R$, $s(u,v)=u\cdot v$ which satisfies the following properties:*

$u\cdot v = v\cdot u$,
$u\cdot(v+w)=u\cdot v+u\cdot w$,
$a(u\cdot v)=(au)\cdot v=u\cdot(av)$,
$u\cdot u\geq 0$ *and* $u\cdot u=0$ *iff* $u=0$,

Example 25 *Let $u,v\in R^n$, $u=(u_1,u_2,\dots,u_n)'$, $v=(v_1,v_2,\dots,v_n)'$. Define $u\cdot v=(u_1v_1,\dots u_nv_n)'$. Then this rule is an inner product in R^n.*

Definition 17 *Let V be a vector space and $\cdot: V\times V\to R$ an inner product in V. The* norm *of* magnitude *is a function $\|\cdot\|: V\to R$ defined as $\|v\|=\sqrt{v\cdot v}$.*

Proposition 8 *If V is a vector space, then for any $v\in V$ and $a\in R$*

i) $\|au\|=|a|\|u\|$;
ii) *(Triangle inequality)* $\|u+v\|\leq\|u\|+\|v\|$;
iii) *(Schwarz inequality)* $|u\cdot v|\leq\|u\|\|v\|$.

Example 26 *If $u\in R^n, u=(u_1,u_2,\dots,u_n)$, the norm of u can be introduced as*

$$\|u\|=\sqrt{u\cdot u}=\sqrt{u_1^2+\cdots+u_n^2}.$$

The triangle inequality and Schwarz's inequality in R^n become:

ii) $\sqrt{(u_1+v_1)^2+\cdots+(u_n+v_n)^2}\leq\sqrt{\sum_{i=0}^n u_i^2}+\sqrt{\sum_{i=0}^n v_i^2}$;
(Minkowski's inequality for sums)

iii) $(\sum_{i=0}^n u_iv_i)\leq(\sqrt{\sum_{i=0}^n u_i^2})(\sqrt{\sum_{i=0}^n v_i^2})$.
(Cauchy-Schwarz inequality for sums)

Definition 18

a) *The nonzero vectors u and v are* parallel *if there exists $a\in R$ such that $u=av$.*

b) *The vectors u and v are* orthogonal *or* perpendicular *if their scalar product is zero, that is, if $u\cdot v=0$.*

c) *The* angle *between vectors u and v is* $\arccos(\frac{uv}{\|u\|\|v\|})$.

1.5.2 Vector Subspaces

Definition 19 *A nonempty subset S of a vector space V is a* subspace *of V if for any $u, v \in S$ and $a \in R$*

$$u + v \in S \qquad \text{and} \qquad au \in S.$$

Example 27 *V is a subset of itself. $\{0\}$ is also a subset of V. These subspaces are called* proper subspaces*.*

Example 28 *$L = \{(x, y) | y = mx + n\}$ where $m, n \in R$ and $m \neq 0$ is a subspace of R^2.*

Definition 20 *Let $u_1, u_2 \ldots u_k$ be vectors in a vector space V. The set S of all linear combinations of these vectors*

$$S = \{a_1 u_1 + a_2 u_2 + \cdots + a_k u_k | a_i \in R, i = 1, \cdots k\}$$

is called the subspace generated *or* spanned *by the vectors $u_1, u_2 \ldots, u_k$ and denoted as $sp(u_1, u_2 \cdots u_k)$.*

Proposition 9 *S is a subspace of V.*

Example 29 *Let $u_1 = (2, -1, 1)', u_2 = (3, 4, 0)'$. Then the subspace of R^3 generated by u_1 and u_2 is*

$$sp(u_1, u_2) = \{au_1 + bu_2 | a, b \in R\} = \{(2a + 3b, -a + 4b, a)' | a, b \in R\}.$$

1.5.3 Independence and Bases

Definition 21 *A set $\{u_1, u_2 \ldots u_k\}$ of vectors in a vector space V is* linearly dependent *if there exists the real numbers $a_1, a_2 \ldots a_k$, not all zero, such that $a_1 u_1 + a_2 u_2 + \ldots a_k u_k = 0$.*

In other words, the set of vectors in a vector space is linearly dependent if and only if one vector can be written as a linear combination of the others. ⚠

Example 30 *The vectors $u_1 = (2, -1, 1)', u_2 = (1, 3, 4)', u_3 = (0, -7, -7)'$ are linearly dependent since $u_3 = u_1 - 2u_2$.*

Definition 22 *A set $\{u_1, u_2 \ldots u_k\}$ of vectors in a vector space V is* linearly independent *if $a_1 u_1 + a_2 u_2 + \cdots a_k u_k = 0$ implying $a_1 = a_2 = \cdots = a_k = 0$ (that is, they are not linearly dependent).*

In other words, the definition says that a set of vectors in a vector space is linearly independent if and only if none of the vectors can be written as a linear combination of the others.

Proposition 10 *Let $\{u_1, u_2 \ldots u_n\}$ be n vectors in R^n. The following conditions are equivalent:*

i) *The vectors are independent.*
ii) *The matrix having these vectors as columns is nonsingular.*
iii) *The vectors generate R^n.*

Example 31 *The vectors* $u_1 = (1, 2, -2)', u_2 = (2, 3, 1)', u_3 = (-2, 0, 1)'$ *in* R^3 *are linearly independent since* $\begin{vmatrix} 1 & 2 & -2 \\ 2 & 3 & 0 \\ -2 & 1 & 1 \end{vmatrix} = -17 \neq 0.$

Definition 23 *A set* $\{u_1, u_2 \dots u_k\}$ *of vectors in* V *is a* basis *for* V *if it, first, generates* V *(that is,* $V = sp(u_1, u_2 \dots u_k)$*), and, second, is linearly independent.*

Any set of n linearly independent vectors in R^n form a basis for R^n. ⚠

Example 32 *The vectors from the preceding example* $u_1 = (1, 2, -2)', u_2 = (2, 3, 1)', u_3 = (-2, 0, 1)'$ *form a basis for* R^3.

Example 33 *Consider the following vectors in* R^n*:* $e_i = (0, \dots, 0, 1, 0, \dots, 0)'$*, where* 1 *is in the* ith *position,* $i = 1, \dots, n$*. The set* $E_n = \{e_1, \dots, e_n\}$ *form a basis for* R^n *which is called the* standard basis.

Definition 24 *Let* V *be a vector space and* $B = \{u_1, u_2, \dots, u_k\}$ *a basis for* V*. Since* B *generates* V*, for any* $u \in V$ *there exists the real numbers* $x_1, x_2, \dots, x_n$ *such that* $u = x_1 u_1 + \dots + x_n u_n$*. The column vector* $x = (x_1, x_2, \dots, x_n)'$ *is called the* vector of coordinates *of* u *with respect to* B*.*

Example 34 *Consider the vector space* R^n *with the standard basis* E_n*. For any* $u = (u_1, \dots, u_n)'$ *we can represent* u *as* $u = u_1 e_1 + \dots u_n e_n$*; therefore,* $(u_1, \dots, u_n)'$ *is the vector of coordinates of* u *with respect to* E_n*.*

Example 35 *Consider the vector space* R^2*. Let us find the coordinate vector of* $(-1, 2)'$ *with respect to the basis* $B = (1, 1)', (2, -3)'$ *(i.e. find* $(-1, 2)'_B$*). We have to solve for* a, b *such that* $(-1, 2)' = a(1, 1)' + b(2, -3)'$*. Solving the system*

$$\begin{cases} a & +2b & = -1 \\ a & -3b & = 2 \end{cases}$$

we find $a = \frac{1}{5}, b = \frac{-3}{5}$*. Thus,* $(-1, 2)'_B = (\frac{1}{5}, \frac{-3}{5})'$*.*

Definition 25 *The* dimension *of a vector space* V $dim(V)$ *is the number of elements in any basis for* V*.*

Example 36 *The dimension of the vector space* R^n *with the standard basis* E_n *is* $dim(R^n) = n$*.*

1.5.4 Linear Transformations and Changes of Bases

Definition 26 *Let* U, V *be two vector spaces. A* linear transformation *of* U *into* V *is a mapping* $T : U \to V$ *such that for any* $u, v \in U$ *and any* $a, b \in R$

$$T(au + bv) = aT(u) + bT(v).$$

Example 37 *Let* A *be an* $m \times n$ *real matrix. The mapping* $T : R^n \to R^m$ *defined by* $T(u) = Au$ *is a linear transformation.*

Example 38 (Rotation of the plane)

The function $T_R : R^2 \to R^2$ that rotates the plane counterclockwise through a positive angle α is a linear transformation.

To check this, first note that any two-dimensional vector $u \in R^2$ can be expressed in polar coordinates as $u = (r\cos\theta, r\sin\theta)'$ where $r = \|u\|$ and θ is the angle that u makes with the x-axis of the system of coordinates.

The mapping T_R is thus defined by

$$T_R(u) = (r\cos(\theta+\alpha), r\sin(\theta+\alpha))'.$$

Therefore,

$$T_R(u) = \begin{pmatrix} r(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \\ r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \end{pmatrix},$$

or, alternatively,

$$T_R(u) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = Au,$$

where

$$A = \begin{pmatrix} \cos\alpha & -sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad \textit{(the rotation matrix).}$$

From example 37 it follows that T_R is a linear transformation.

Proposition 11 *Let U and V be two vector spaces, $B = (b_1, \ldots, b_n)$ a basis for U and $C = (c_1, \ldots, c_m)$ a basis for V.*

- *Any linear transformation T can be represented by an $m \times n$ matrix A_T whose ith column is the coordinate vector of $T(b_i)$ relative to C.*
- *If $x = (x_1, \ldots, x_n)'$ is the coordinate vector of $u \in U$ relative to B and $y = (y_1, \ldots, y_m)'$ is the coordinate vector of $T(u)$ relative to C then T defines the following transformation of coordinates:*

  $$y = A_T x \text{ for any } u \in U.$$

Definition 27 *The matrix A_T is called the* matrix representation *of T relative to bases B, C.*

Any linear transformation is uniquely determined by a transformation of coordinates. ⚠

Example 39 *Consider the linear transformation $T : R^3 \to R^2$, $T((x,y,z)') = (x-2y, x+z)'$ and bases $B = \{(1,1,1)', (1,1,0)', (1,0,0)'\}$ for R^3 and $C = \{(1,1)', (1,0)'\}$ for R^2. How can we find the matrix representation of T relative to bases B, C?*

We have:

$$T((1,1,1)') = (-1,2),\ T((1,1,0)') = (-1,1),\ T((1,0,0)') = (1,1).$$

The columns of A_T are formed by the coordinate vectors of $T((1,1,1)')$, $T((1,1,0)')$, $T((1,0,0)')$ relative to C. Applying the procedure developed in Example 35 we find

$$A_T = \begin{pmatrix} 2 & 1 & 1 \\ -3 & -2 & 0 \end{pmatrix}.$$

Definition 28 (Changes of Bases)

Let V be a vector space of dimension n, B and C be two bases for V, and $I : V \to V$ be the identity transformation ($I(v) = v$ for all $v \in V$). The change-of-basis *matrix D relative to B, C is the matrix representation of I relative to B, C.*

Example 40 *For $u \in V$, let $x = (x_1, \ldots, x_n)'$ be the coordinate vector of u relative to B and $y = (y_1, \ldots, y_n)'$ is the coordinate vector of u relative to C. If D is the change-of-basis matrix relative to B, C then $y = Cx$. The change-of-basis matrix relative to C, B is D^{-1}.*

Example 41 *Given the following bases for R^2: $B = \{(1,1)', (1,0)'\}$ and $C = \{(0,1)', (1,1)'\}$, find the change-of-basis matrix D relative to B, C.*

The columns of D are the coordinate vectors of $(1,1)', (1,0)'$ relative to C. Following Example 35 we find $D = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

Proposition 12 *Let $T : V \to V$ be a linear transformation, and let B, C be two bases for V. If A_1 is the matrix representation of T in the basis B, A_2 is the matrix representation of T in the basis C and D is the change-of-basis matrix relative to C, B then $A_2 = D^{-1} A_1 D$.*

Further Reading:

- Bellman, R. *Introduction to Matrix Analysis.*
- Fraleigh, J.B. and R.A. Beauregard. *Linear Algebra.*
- Gantmacher, F.R. *The Theory of Matrices.*
- Lang, S. *Linear Algebra.*

2 Calculus

2.1 The Concept of Limit

Definition 29 *The function $f(x)$ has a limit A (or tends to A as a limit) as x approaches a if for each given number $\varepsilon > 0$, no matter how small, there exists a positive number δ (that depends on ε) such that $|f(x) - A| < \varepsilon$ whenever $0 < |x - a| < \delta$.*

The standard notation is $\lim_{x\to a} f(x) = A$ or $f(x) \to A$ as $x \to a$.

Definition 30 *The function $f(x)$ has a left-side (or right-side) limit A as x approaches a from the left (or right),*

$$\lim_{x\to a_-} f(x) = A \qquad (\lim_{x\to a_+} f(x) = A),$$

if for each given number $\varepsilon > 0$ there exists a positive number δ such that $|f(x) - A| < \varepsilon$ whenever $a - \delta < x < a \quad (a < x < a + \delta)$.

Recipe 8 – How to Calculate Limits:

We can apply the following basic rules for limits:
if $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} g(x) = B$ then

1. *if C is constant, then $\lim_{x\to x_0} C = C$.*
2. $\lim_{x\to x_0}(f(x) + g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = A + B$.
3. $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x) = A \cdot B$.
4. $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \dfrac{A}{B}$.
5. $\lim_{x\to x_0}(f(x))^n = (\lim_{x\to x_0} f(x))^n = A^n$.
6. $\lim_{x\to x_0} f(x)^{g(x)} = e^{\lim_{x\to x_0} g(x) \ln f(x)} = e^{B \ln A}$.

Some important results:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e = 2.718281828459\ldots, \qquad \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1,$$

$$\lim_{x\to 0} \frac{(1+x)^p - 1}{x} = p, \qquad \lim_{x\to 0} \frac{e^x - 1}{x} = 1, \qquad \lim_{x\to 0} \frac{a^x - 1}{x} = \ln a, \quad a > 0.$$

Example 42

a) $$\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = \lim_{x\to 1} \frac{(x-1)(x^2 + x + 1)}{x - 1} = \lim_{x\to 1}(x^2 + x + 1) = 1 + 1 + 1 = 3.$$

b) $$\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0}(\frac{\sin x}{x} \sin x) = \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \sin x = 1 \cdot 0 = 0.$$

c) $$\lim_{x\to\infty} x^2 \ln \frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty} x^2 \ln \sqrt{\frac{x^2+1}{x^2}} = \lim_{x\to\infty} x^2 \frac{1}{2} \ln(1 + \frac{1}{x^2}) =$$

$$= \frac{1}{2} \lim_{x\to\infty} \frac{\ln(1+1/x^2)}{1/x^2} = \frac{1}{2}.$$

$$d)\ \lim_{x\to 0_+} x^{\frac{1}{1+\ln x}} = e^{\lim_{x\to 0_+} \frac{1}{1+\ln x}\ln x} = e^{\lim_{x\to 0_+} \frac{1}{1/\ln x+1}} = e^1 = e.$$

Another powerful tool for evaluating limits is *L'Hôpital's rule.*

Proposition 13 (L'Hôpital's Rule)
Suppose $f(x)$ and $g(x)$ are differentiable[1] *in an interval (a,b) around x_0 except possibly at x_0, and suppose that $f(x)$ and $g(x)$ both approach 0 when x approaches x_0. If $g'(x) \neq 0$ for all $x \neq x_0$ in (a,b) and $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = A$ then*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = A.$$

The same rule applies if $f(x) \to \pm\infty$, $g(x) \to \pm\infty$. x_0 can be either finite or infinite.

Note that L'Hôpital's rule can be applied only if we have expressions of the form $\dfrac{0}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$. ⚠

Example 43

$$a)\ \lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{6} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{6}.$$

$$b)\ \lim_{x\to 0} \frac{x-\sin 2x}{x^3} = \lim_{x\to 0} \frac{1-2\cos 2x}{3x^2} = \frac{\lim_{x\to 0}(1-2\cos 2x)}{\lim_{x\to 0} 3x^2} = \frac{-1}{\lim_{x\to 0} 3x^2} = -\infty.$$

2.2 Differentiation - the Case of One Variable

Definition 31 *A function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.*

If $f(x)$ and $g(x)$ are continuous at a then:

- $f(x) \pm g(x)$ and $f(x)g(x)$ are continuous at a;
- if $g(a) \neq 0$ then $\frac{f(x)}{g(x)}$ is continuous at a;
- if $g(x)$ is continuous at a and $f(x)$ is continuous at $g(a)$ then $f(g(x))$ is continuous at a.

[1] For the definition of differentiability see the next section.

In general, any function built from continuous functions by additions, subtractions, multiplications, divisions and compositions is continuous where defined.

If $\lim_{x\to a+} f(x) = c_1 \neq c_2 = \lim_{x\to a-} f(x)$, $|c_1|, |c_2| < \infty$, the function $f(x)$ is said to have a *jump discontinuity* at a. If $\lim_{x\to a} f(x) = \pm\infty$, we call this type of discontinuity *infinite discontinuity.*

Suppose there is a functional relationship between x and y, $y = f(x)$. One of the natural questions one may ask is: How does y change if x changes? We can answer this question using the notion of the *difference quotient*. Denoting the change in x as $\Delta x = x - x_0$, the difference quotient is defined as

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Taking the limit of the above expression, we arrive at the following definition.

Definition 32 *The derivative of $f(x)$ at x_0 is*

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

If the limit exists, f is called differentiable at x.

An alternative notation for the derivative often found in textbooks is

$$f'(x) = y' = \frac{df}{dx} = \frac{df(x)}{dx}.$$

The first and second derivatives with respect to time are usually denoted by dots ($\dot{}$ and $\ddot{}$, respectively), i.e if $z = z(t)$ then $\dot{z} = \frac{dz}{dt}$, $\quad \ddot{z} = \frac{d^2z}{dt^2}$.

The set of all continuously differentiable functions in the domain D (i.e. the argument of a function may take any value from D) is denoted by $C^{(1)}(D)$.

Geometrically speaking, the derivative represents the slope of the tangent line to f at x. Using the derivative, the equation of the tangent line to $f(x)$ at x_0 can be written as

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Note that the continuity of $f(x)$ is a necessary but NOT sufficient condition for its differentiability! ⚠

Example 44 *For instance, $f(x) = |x - 2|$ is continuous at $x = 2$ but not differentiable at $x = 2$.*

The geometric interpretation of the derivative gives us the following formula (usually called Newton's approximation method), which allows us to find an approximate root of $f(x) = 0$. Let us define a sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If the initial value x_0 is chosen such that x_0 is reasonably close to an actual root, this sequence will converge to that root.

The derivatives of higher order (2,3,...,n) can be defined in the same manner:

$$f''(x_0) = \frac{d}{dx}f'(x_0) = \lim_{x\to x_0} \frac{f'(x_0 + \Delta x) - f'(x_0)}{\Delta x} \quad \ldots \quad f^{(n)}(x_0) = \frac{d^n}{dx^n} f^{(n-1)}(x_0),$$

provided that these limits exist. If so, $f(x)$ is called n times continuously differentiable, $f \in C^{(n)}$. The symbol $\frac{d^n}{dx^n}$ denotes an operator of taking the nth derivative of a function with respect to x.

2.3 Rules of Differentiation

Suppose we have two differentiable functions of the same variable x, say $f(x)$ and $g(x)$. Then

- $(f(x) \pm g(x))' = f'(x) \pm g'(x)$;
- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$ (product rule);
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ (quotient rule);
- if $f = f(y)$ and $y = g(x)$ then $\frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx} = f'(y)g'(x)$ (chain rule or composite-function rule).

  This rule can be easily extended to the case of more than two functions involved.

  Example 45 (Application of the Chain Rule)
  Let $f(x) = \sqrt{x^2+1}$. *We can decompose as* $f = \sqrt{y}$, *where* $y = x^2 + 1$. *Therefore,*

  $$\frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx} = \frac{1}{2\sqrt{y}} \cdot 2x = \frac{x}{\sqrt{y}} = \frac{x}{\sqrt{x^2+1}}.$$

- if $x = u(t)$ and $y = v(t)$ (i.e. x and y are *parametrically* defined) then

  $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{v}(t)}{\dot{u}(t)} \qquad \text{(recall that } \dot{} \text{ means } \tfrac{d}{dt}\text{)},$$

  $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\dot{v}(t)}{\dot{u}(t)}\right) = \frac{\frac{d}{dt}\left(\frac{\dot{v}(t)}{\dot{u}(t)}\right)}{dx/dt} = \frac{\ddot{v}(t)\dot{u}(t) - \dot{v}(t)\ddot{u}(t)}{(\dot{u}(t))^3}.$$

Example 46 *If* $x = a(t - \sin t)$, $y = a(1 - \cos t)$, *where* a *is a parameter, then*

$$\frac{dy}{dx} = \frac{\sin t}{1-\cos t}, \; \frac{d^2y}{dx^2} = \frac{\cos t(1-\cos t) - \sin^2 t}{a(1-\cos t)^3} = \frac{\cos t - 1}{a(1-\cos t)^2} = -\frac{1}{a(1-\cos t)^2}.$$

Some special rules:

$f(x) = constant$	$\Rightarrow$	$f'(x) = 0$
$f(x) = x^a$ (a is constant)	$\Rightarrow$	$f'(x) = ax^{a-1}$
$f(x) = e^x$	$\Rightarrow$	$f'(x) = e^x$
$f(x) = a^x$ $(a > 0)$	$\Rightarrow$	$f'(x) = a^x \ln a$
$f(x) = \ln x$	$\Rightarrow$	$f'(x) = \frac{1}{x}$
$f(x) = \log_a x$ $(a > 0, a \neq 1)$	$\Rightarrow$	$f'(x) = \frac{1}{x}\log_a e = \frac{1}{x\ln a}$
$f(x) = \sin x$	$\Rightarrow$	$f'(x) = \cos x$
$f(x) = \cos x$	$\Rightarrow$	$f'(x) = -\sin x$
$f(x) = \mathrm{tg}x$	$\Rightarrow$	$f'(x) = \frac{1}{\cos^2 x}$
$f(x) = \mathrm{ctg}x$	$\Rightarrow$	$f'(x) = -\frac{1}{\sin^2 x}$
$f(x) = \mathrm{arcsin}x$	$\Rightarrow$	$f'(x) = \frac{1}{\sqrt{1-x^2}}$
$f(x) = \mathrm{arccos}x$	$\Rightarrow$	$f'(x) = -\frac{1}{\sqrt{1-x^2}}$
$f(x) = \mathrm{arctg}x$	$\Rightarrow$	$f'(x) = \frac{1}{1+x^2}$
$f(x) = \mathrm{arcctg}x$	$\Rightarrow$	$f'(x) = -\frac{1}{1+x^2}$

More hints:

- if $y = \ln f(x)$ then $y' = \dfrac{f'(x)}{f(x)}$.
- if $y = e^{f(x)}$ then $y' = f'(x)e^{f(x)}$.
- if $y = f(x)^{g(x)}$ then
  $y' = (f(x)^{g(x)})' = (e^{g(x)\ln f(x)})' \overset{\text{(by chain rule)}}{=} e^{g(x)\ln f(x)}(g(x)\ln f(x))' \overset{\text{(by product rule)}}{=}$
  $= e^{g(x)\ln f(x)}\left(g'(x)\ln f(x) + g(x)\dfrac{f'(x)}{f(x)}\right) = f(x)^{g(x)}\left(g'(x)\ln f(x) + g(x)\dfrac{f'(x)}{f(x)}\right)$.
- if a function f is a one-to-one mapping (or single-valued mapping),[2] it has the inverse f^{-1}. Thus, if $y = f(x)$ and $x = f^{-1}(y)$ then $\dfrac{dx}{dy} = \dfrac{1}{f'(x)} = \dfrac{1}{f'(f^{-1}(y))}$ or, in other words, $\dfrac{dx}{dy} = \dfrac{1}{dy/dx}$.

  Example 47 *Given $y = \ln x$, its inverse is $x = e^y$. Therefore* $\dfrac{dx}{dy} = \dfrac{1}{1/x} = x = e^y$.

- if a function f is a product (or quotient) of a number of other functions, logarithmic function might be helpful while taking the derivative of f. If $y = f(x) = \dfrac{g_1(x)g_2(x)\dots g_k(x)}{h_1(x)h_2(x)\dots h_l(x)}$ then after taking the (natural) logarithms of the both sides and rearranging the terms we get $\dfrac{dy}{dx} = y(x)\cdot\left(\dfrac{g'_1(x)}{g_1(x)} + \ldots + \dfrac{g'_k(x)}{g_k(x)} - \dfrac{h'_1(x)}{h_1(x)} - \ldots - \dfrac{h'_l(x)}{h_l(x)}\right)$.

Definition 33 *If $y = f(x)$ and dx is any number then the differential of y is defined as $dy = f'(x)dx$.*

The rules of differentials are similar to those of derivatives:

- If k is constant then $dk = 0$;
- $d(u \pm v) = du \pm dv$;
- $d(uv) = v \cdot du + u \cdot dv$;
- $d\left(\dfrac{u}{v}\right) = \dfrac{v \cdot du - u \cdot dv}{v^2}$.

Differentials of higher order can be found in the same way as derivatives.

[2] In the one-dimensional case the set of all one-to-one mappings coincides with the set of strictly monotonic functions.

2.4 Maxima and Minima of a Function of One Variable

Definition 34 *If $f(x)$ is continuous in a neighborhood U of a point x_0, it is said to have a local or relative maximum (minimum) at x_0 if for all $x \in U$, $x \neq x_0$, $f(x) < f(x_0)$ $(f(x) > f(x_0))$.*

Proposition 14 (Weierstrass's Theorem)
If f is continuous on a closed and bounded subset S of $\mathbf{R}^1$ (or, in the general case, of $\mathbf{R}^n$), then f reaches its maximum and minimum in S, i.e. there exist points $m, M \in S$ such that $f(m) \leq f(x) \leq f(M)$ for all $x \in S$.

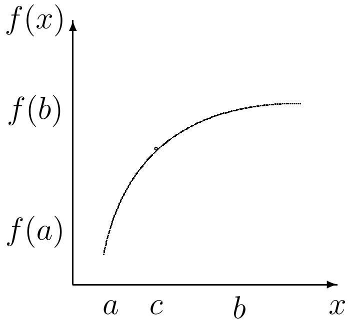
Proposition 15 (Fermat's Theorem or the Necessary Condition for Extremum)
If $f(x)$ is differentiable at x_0 and has a local extremum (minimum or maximum) at x_0 then $f'(x_0) = 0$.

Note that if the first derivative vanishes at some point, it does not imply that at this point f possesses an extremum. We can only state that f has a *stationary* point. ⚠

Some useful results:

Proposition 16 (Rolle's Theorem)
If f is continuous in $[a, b]$, differentiable in (a, b) and $f(a) = f(b)$, then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Proposition 17 (Lagrange's Theorem or the Mean Value Theorem)
If f is continuous in $[a, b]$ and differentiable in (a, b), then there exists at least one point $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



The geometric meaning of the Lagrange's theorem: the slope of a tangent line to $f(x)$ at the point c coincides with the slope of the straight line, connecting two points $(a, f(a))$ and $(b, f(b))$.

If we denote $a = x_0$, $b = x_0 + \Delta x$, then we may re-write Lagrange theorem as $f(x_0 + \Delta x) - f(x_0) = \Delta x \cdot f'(x_0 + \theta \Delta x)$, where $\theta \in (0, 1)$.

This statement can be generalized in the following way:

Proposition 18 (Cauchy's Theorem or the Generalized Mean Value Theorem)
If f and g are continuous in $[a, b]$ and differentiable in (a, b) then there exists at least one point $c \in (a, b)$ such that $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$.

Recall the meaning of the first and the second derivatives of a function f. The sign of the first derivative tells us whether the *value* of the function increases ($f' > 0$) or decreases ($f' < 0$), whereas the sign of the second derivative tells us whether the *slope* of the function increases ($f'' > 0$) or decreases ($f'' < 0$). This gives us an insight into how to verify that at a stationary point we have a maximum or minimum.

Assume that $f(x)$ is differentiable in a neighborhood of x_0 and it has a stationary point at x_0, i.e. $f'(x_0) = 0$.

Proposition 19 (The First-Derivative Test for Local Extremum)

If at a stationary point x_0 the first derivative of a function f	*a) changes its sign from positive to negative,* *b) changes its sign from negative to positive,* *c) does not change its sign,*	*then the value of the function at x_0, $f(x_0)$, will be*	*a) a local maximum;* *b) a local minimum;* *c) neither a local maximum nor a local minimum.*

Proposition 20 (Second-Derivative Test for Local Extremum)
A stationary point x_0 of $f(x)$ will be a local maximum if $f''(x_0) < 0$ and a local minimum if $f''(x_0) > 0$.

However, it may happen that $f''(x_0) = 0$, therefore the second-derivative test is not applicable. To compensate for this, we can extend the latter result and to apply the following general test:

Proposition 21 (nth-Derivative Test)
If $f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0$, $f^{(n)}(x_0) \neq 0$ and $f^{(n)}$ is continuous at x_0 then at point x_0 $f(x)$ has
a) an inflection point if n is odd;
b) a local maximum if n is even and $f^{(n)}(x_0) < 0$;
c) a local minimum if n is even and $f^{(n)}(x_0) > 0$.

To prove this statement we can use the *Taylor series*: If f is continuously differentiable enough at x_0, it can be expanded around a point x_0, i.e. this function can be transformed into a polynomial form in which the coefficient of the various terms are expressed in terms of the derivatives values of f, all evaluated at the point of expansion x_0. More precisely,

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i, \qquad \text{where } k! = 1 \cdot 2 \cdot \ldots \cdot k,\ 0! = 1,\ f^{(0)}(x_0) = f(x_0).$$

or

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i + R_n,$$

where

$$R_n = f^{(n+1)}(x_0 + \theta(x - x_0))\frac{(x - x_0)^{n+1}}{(n+1)!}, \quad 0 < \theta < 1$$

is the remainder in the Lagrange form.

The Maclaurin series is the special case of the Taylor series when we set $x_0 = 0$.

Example 48 *Expand e^x around $x = 0$.*
Since $\frac{d^n}{dx^n}e^x = e^x$ for all n and $e^0 = 1$, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$.

The expansion of a function into Taylor series is useful as an approximation device. For instance, we can derive the following approximation:

$$f(x+dx) \approx f(x) + f'(x)dx, \qquad \text{when } dx \text{ is small.}$$

Note, however, that in general polynomial approximation is not the most efficient one. ⚠

Recipe 9 – How to Find (Global) Maximum or Minimum Points:

If $f(x)$ has a maximum or minimum in a subset S (of $\mathbf{R}^1$ or, in general, $\mathbf{R}^n$), then we need to check the following points for maximum/minimum values of f:
–interior points of S that are stationary;
–points at which f is not differentiable;
–extrema of f at the boundary of S.

Note that the first-derivative test gives us only *relative* (or *local*) extrema. It may happen (due to Weierstrass's theorem) that f reaches its global maximum at the border point. ⚠

Example 49 *Find maximum of $f(x) = x^3 - 3x$ in $[-2, 3]$.*

The first-derivative test gives two stationary points: $f'(x) = 3x^2 - 3 = 0$ at $x = 1$ and $x = -1$. The second-derivative test guarantees that $x = -1$ is a local maximum. However, $f(-1) = 2 < f(3) = 18$. Therefore the global *maximum of f in $[-2, 3]$ is reached at the border point $x = 2$.*

Recipe 10 – How to Sketch the Graph of a Function:

Given f, you should perform the following actions:

1. *Check at which points our function is continuous, differentiable, etc.*
2. *Find its asymptotics, i.e. vertical asymptotes (finite x's at which $f(x) = \infty$) and non-vertical asymptotes. By definition, $y = ax + b$ is a non-vertical asymptote for the curve $y = f(x)$ if $\lim_{x\to\infty}(f(x) - (ax + b)) = 0$.*

   How to find an asymptote for the curve $y - f(x)$ as $x \to \infty$:

   - *Examine $\lim_{x\to\infty} f(x)/x$; if the limit does not exist, there is no asymptote as $x \to \infty$.*
   - *If $\lim_{x\to\infty} f(x)/x = a$, examine $\lim_{x\to\infty} f(x) - ax$; if the limit does not exist, there is no asymptote as $x \to \infty$.*
   - *If $\lim_{x\to\infty} f(x) - ax = b$, the straight line $y = ax + b$ is an asymptote for the curve $y = f(x)$ as $x \to \infty$.*
3. *Find all stationary points and check whether they are local minima, local maxima or neither.*
4. *Find all points of inflection, i.e. points given by $f''(x) = 0$, at which the second derivative changes its sign.*

   Note that a zero first derivative value is not required for an inflection point. ⚠
5. *Find intervals in which f is increasing (decreasing).*

6. Find intervals in which f is convex (concave).

Definition 35 *f is called convex (concave) at x_0 if $f''(x_0) \geq 0$ $(f''(x_0) \leq 0)$, and strictly convex (concave) if the inequalities are strict.*

Example 50 *Sketch the graph of the function $y = \frac{2x-1}{(x-1)^2}$.*

a) $y(x)$ is defined for $x \in (-\infty, 1) \cup (1, +\infty)$.

b) $y(x)$ is continuous in $(-\infty, 1) \cup (1, +\infty)$.

c) $y(x) \to 0$ if $x \to \pm\infty$, therefore $y = 0$ is the horizontal asymptote.

d) $y(x) \to \infty$ if $x \to \pm 1$, therefore $x = 1$ is the vertical asymptote.

e) $y'(x) = -\dfrac{2x}{(x-1)^3}$. $y'(x)$ is not defined at $x = 1$, it is positive if $x \in (0, 1)$ (the function is increasing) and negative if $x \in (-\infty, 0) \cup (1, +\infty)$ (the function is decreasing).

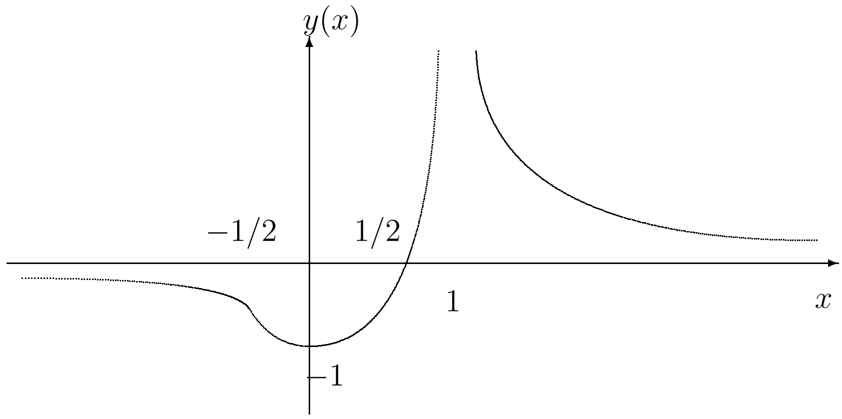
$x = 0$ is the unique extremum (minimum), $y(0) = -1$.

f) $y''(x) = 2\dfrac{2x-1}{(x-1)^4}$. It is not defined at $x = 1$, positive at $x \in (-\dfrac{1}{2}, 1) \cup (1, +\infty)$ (the function is convex) and negative at $x \in (-\infty, -\dfrac{1}{2})$ (the function is concave).

$x = -\dfrac{1}{2}$ is the unique point of inflection, $y(-\dfrac{1}{2}) = -\dfrac{8}{9}$.

g) Intersections with the axes are given by $x = 0 \Rightarrow y(x) = -1$, $y = 0 \Rightarrow x = \dfrac{1}{2}$.

Finally, the sketch of the graph looks like this:



2.5 Integration (The Case of One Variable)

Definition 36 *Let $f(x)$ be a continuous function. The indefinite integral of f (denoted by $\int f(x)dx$) is defined as*

$$\int f(x)dx = F(x) + C,$$

where $F(x)$ is such that $F'(x) = f(x)$, and C is an arbitrary constant.

Rules of integration:

- $\int (af(x) + bg(x))dx = a \int f(x)dx + b \int g(x)dx$, a, b are constants (linearity of the integral);

- $\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$ (integration by parts);
- $\int f(u(t))\frac{du}{dt}dt = \int f(u)du$ (integration by substitution).

Some special rules of integration:

$$\begin{array}{ll} \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C, & \int f'(x)e^{f(x)}dx = e^{f(x)} + C \\ \int \frac{1}{x}dx = \ln|x| + C, & \int x^a dx = \frac{x^{a+1}}{a+1} + C, \quad a \neq -1 \\ \int e^x dx = e^x + C, & \int a^x dx = \frac{a^x}{\ln a} + C \quad a > 0 \end{array}$$

See any calculus reference-book for tables of integrals of basic functions.

Example 51

a) $\int \frac{x^2+2x+1}{x}dx = \int xdx + \int 2dx + \int \frac{1}{x}dx = \frac{x^2}{2} + 2x + \ln|x| + C.$

b) $\int xe^{-x^2}dx = -\frac{1}{2}\int(-2x)e^{-x^2}dx \overset{\text{(substitution } z = x^2)}{=} -\frac{1}{2}\int e^{-z}dz = -\frac{e^{-x^2}}{2} + C.$

c) $\int xe^x dx \overset{\text{(by parts, } f(x) = x, g(x) = e^x)}{=} xe^x - \int e^x dx = xe^x - e^x + C.$

Definition 37 (the Newton-Leibniz formula).

The definite integral of a continuous function f is

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a), \tag{2}$$

for $F(x)$ such that $F'(x) = f(x)$ for all $x \in [a, b]$

The indefinite integral is a function. The definite integral is a number! ⚠

We understand the definite integral in Riemann sense:
Given a partition $a = x_0 < x_1 < \ldots < x_n = b$ and numbers $\zeta_i \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$,

$$\int_a^b f(x)dx = \lim_{\max_i(x_{i+1}-x_i)\to 0} \sum_{i=0}^{n-1} f(\zeta_i)(x_{i+1} - x_i).$$

Since every definite integral has a definite value, this value may be interpreted geometrically to be a particular area under a given curve defined by $y = f(x)$.

Note that if a curve lies below the x axis, this area will be negative. ⚠

Properties of definite integrals: For any arbitrary real numbers a, b, c, α, β

- $\int_a^b(\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$
- $\int_a^b f(x)g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$, $u = g(x)$ (change of variable)

- $\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx$

Some more useful results: If λ is a real parameter,

- $\frac{d}{d\lambda}\int_{a(\lambda)}^{b(\lambda)} f(x)dx = f(b(\lambda))b'(\lambda) - f(a(\lambda))a'(\lambda)$

  In particular, $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

  If $\int_a^b \frac{d}{d\lambda} f(x,\lambda)dx$ exists, then

- $\frac{d}{d\lambda}\int_a^b f(x,\lambda)dx = \int_a^b f'_\lambda(x,\lambda)dx$

  Example 52 *We may apply this formula when we need to evaluate a definite integral, which can not be integrated in elementary functions. For instance, let us find*

$$I(\lambda) = \int_0^1 xe^{-\lambda x}dx.$$

If we introduce another integral

$$J(\lambda) = \int_0^1 e^{-\lambda x}dx = -\frac{e^{-\lambda x}}{\lambda}|_0^1 = -\frac{e^{-\lambda}-1}{\lambda},$$

then

$$I(\lambda) = -\int_0^1 \frac{d}{d\lambda}e^{-\lambda x}dx = -\frac{dJ(\lambda)}{d\lambda} = \frac{1-e^{-\lambda}(1+\lambda)}{\lambda^2}.$$

We can combine the last two formulas to get:

- Leibniz's formula:

$$\frac{d}{d\lambda}\int_{a(\lambda)}^{b(\lambda)} f(x,\lambda)dx = f(b(\lambda),\lambda)b'(\lambda) - f(a(\lambda),\lambda)a'(\lambda) + \int_{a(\lambda)}^{b(\lambda)} f'_\lambda(x,\lambda)dx$$

Proposition 22 (Mean Value Theorem)
If $f(x)$ is continuous in $[a,b]$ then there exist at least one point $c \in (a,b)$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

Numerical methods of approximation the definite integrals:
a) The trapezoid formula: Denoting $y_i = f(a + i\frac{b-a}{n})$,

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n),$$

n is an arbitrary integer number.
b) Simpson's formula: Denoting $y_i = f(a + i\frac{b-a}{2n})$,

$$\int_a^b f(x)dx \approx \frac{b-a}{6n}(y_0 + 4\sum_{i=1}^{n} y_{2i-1} + 2\sum_{i=1}^{n-1} y_{2i} + y_{2n}).$$

Improper integrals are integrals of one of the following forms:

- $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$ or $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$.
- $\int_a^b f(x)dx = \lim_{\epsilon\to 0}\int_{a+\epsilon}^b f(x)dx$ where $\lim_{x\to a} f(x) = \infty$.
- $\int_a^b f(x)dx = \lim_{\epsilon\to 0}\int_a^{b-\epsilon} f(x)dx$ where $\lim_{x\to b} f(x) = \infty$.

If these limits exist, the improper integral is said to be *convergent*, otherwise it is called *divergent*.

Example 53 $$\int_0^1 \frac{1}{x^p}dx = \lim_{\epsilon\to 0_+}\int_\epsilon^1 \frac{1}{x^p}dx = \begin{cases} \lim_{\epsilon\to 0_+}(\ln 1 - \ln\epsilon), & p = 1 \\ \lim_{\epsilon\to 0_+}(\frac{1}{1-p} - \frac{\epsilon^{1-p}}{1-p}), & p \neq 1 \end{cases}$$
Therefore the improper integral converges if $p < 1$ and diverges if $p \geq 1$.

Economics Application 3 (The Optimal Subsidy and the Compensated Demand Curve)
Consider the consumer's dual problem (recall that the dual to utility maximization is expenditure minimization: we minimize the objective function $M = p_x x + p_y y$, subject to the utility constraint $U = U(x, y)$, where M is income level, x, y and p_x, p_y are quantities and prices of two goods, respectively).

The solution to the expenditure-minimization problem under the utility constraint is the budget line with income M^*, $M^*(p_x, p_y) = p_x x^* + p_y y^*$.

If we hold utility and p_y constant, due to Hotelling's lemma

$$\frac{dM^*(p_x)}{dp_x} = x_c^*(p_x),$$

where $x_c^*(p_x)$ is the income-compensated (or simply compensated) demand curve for good x.

By rearranging terms, $dM^*(p_x) = x_c^*(p_x)dp_x$, where dM^* is the optimal subsidy for an infinitesimal change in price, we can "add up" a series of infinitesimal changes by taking integral over those infinitesimal changes. Thus, if price increases from p_x^1 to p_x^2, the optimal subsidy S^* required to maintain a given level of utility would be the integral over all the infinitesimal income changes as price changes:

$$S^* = \int_{p_x^1}^{p_x^2} x_c^*(p_x)dp_x.$$

For example, if $p_x^1 = 1$, $p_x^2 = 4$, $x_c^*(p_x) = \frac{1}{\sqrt{p_x}}$ then

$$S^* = \int_1^4 \frac{1}{\sqrt{p_x}}dp_x = 2\sqrt{p_x}|_1^4 = 2\cdot(2-1) = 2.$$

2.6 Functions of More than One Variable

Let us consider a function $y = f(x_1, x_2, \ldots, x_n)$ where the variables x_i, $i = 1, 2, \ldots, n$ are all independent of one other, i.e. each varies without affecting others. The *partial derivative of y with respect to x_i* is defined as

$$\frac{\partial y}{\partial x_i} = \lim_{\Delta x_i\to 0}\frac{\Delta y}{\Delta x_i} = \lim_{\Delta x_i\to 0}\frac{f(x_1,\ldots,x_i+\Delta x_i,\ldots,x_n) - f(x_1,\ldots,x_i,\ldots,x_n)}{\Delta x_i}$$

for all $i = 1, 2, \ldots, n$. If these limits exist, our function is *differentiable* with respect to all arguments. Partial derivatives are often denoted with subscripts, e.g. $\frac{\partial f}{\partial x_i} = f_{x_i}$ $(= f'_{x_i})$.

Example 54 *If* $y = 4x_1^3 + x_1x_2 + \ln x_2$ *then* $\frac{\partial y}{\partial x_1} = 12x_1^2 + x_2$, $\frac{\partial y}{\partial x_2} = x_1 + 1/x_2$.

Definition 38 The total differential *of a function* f *is defined as*

$$dy = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots + \frac{\partial f}{\partial x_n}dx_n.$$

The total differential can be also used as an approximation device.
For instance, if $y = f(x_1, x_2)$ then by definition $y - y_0 = \Delta f(x_1^0, x_2^0) \approx df(x_1^0, x_2^0)$.
Therefore $f(x_1, x_2) \approx f(x_1^0, x_2^0) + df(x_1^0, x_2^0) \approx f(x_1^0, x_2^0) + \frac{\partial f(x_1^0, x_2^0)}{\partial x_1}(x_1 - x_1^0) + \frac{\partial f(x_1^0, x_2^0)}{\partial x_2}(x_2 - x_2^0)$.

If $y = f(x_1, \ldots, x_n)$, $x_i = x_i(t_1, \ldots, t_m)$, $i = 1, 2, \ldots, n$, then for all $j = 1, 2, \ldots, m$

$$\frac{\partial y}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i}\frac{\partial x_i}{\partial t_j}.$$

This rule is called *the chain rule.*

Example 55 *If* $z = f(x, y)$, $x = u(t)$, $y = v(t)$ *then*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

(a special case of the chain rule)

Example 56 *If* $z = f(x, y, t)$ *where* $x = u(t)$, $y = v(t)$ *then*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}.$$

(dz/dt is sometimes called the total derivative).

Second order partial derivative of $y = f(x_1, \ldots, x_n)$ is defined as

$$\frac{\partial^2 y}{\partial x_j \partial x_i} = f_{x_i x_j}(x_1, \ldots, x_n) = \frac{\partial}{\partial x_j} f_{x_i}(x_1, \ldots, x_n), \quad i, j = 1, 2, \ldots, n.$$

Proposition 23 (Schwarz's Theorem or Young's Theorem)
If at least one of the two partials is continuous, then

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad i, j = 1, 2, \ldots, n.$$

Similar to the case of one variable, we can expand a function of several variables in a polynomial form of Taylor series, using partial derivatives of higher order.

Definition 39 The second order total differential *of a function* $y = f(x_1, x_2, \ldots, x_n)$ *is defined as*

$$d^2 y = d(dy) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j, \quad i, j = 1, 2, \ldots, n.$$

Example 57 *Let* $z = f(x, y)$ *and* $f(x, y)$ *satisfy the conditions of Schwarz's theorem (i.e.* $f_{xy} = f_{yx}$*). Then* $d^2 z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$

2.7 Unconstrained Optimization in the Case of More than One Variable

Let $z = f(x_1, \ldots, x_n)$.

Definition 40 *z has a stationary point at $x^* = (x_1^*, \ldots, x_n^*)$ if $dz = 0$ at x^*, i.e. all f'_{x_i} should vanish at x^*.*

The conditions

$$f'_{x_i}(x^*) = 0, \quad \text{for all } i = 1, 2, \ldots, n$$

are called *the first order necessary conditions (F.O.N.C.)*.

The second-order necessary condition (S.O.N.C.) requires sign semidefiniteness of the second order total differential.

Definition 41 *$z = f(x_1, \ldots, x_n)$ has a local maximum (minimum) at $x^* = (x_1^*, \ldots, x_n^*)$ if $f(x^*) - f(x) \geq 0 (\leq 0)$ for all x in some neighborhood of x^*.*

The second-order sufficient condition (S.O.C.):
x^* is a maximum (minimum) if d^2z at x^* is negative definite (positive definite).

Recipe 11 – How to Check the Sign Definiteness of d^2z:

Let us define the symmetric Hessian matrix H of second partial derivatives as

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}.$$

Thus, the sign definiteness of d^2z is equivalent to the sign definiteness of the quadratic form H evaluated at x^*. We can apply either the principal minors test or the eigenvalue test discussed in the previous chapter.

Example 58 *Consider $z = f(x, y)$. If the first order necessary conditions are met at x^* then*

$$x^* \text{ is a (local)} \left\{ \begin{array}{c} \text{minimum} \\ \text{maximum} \end{array} \right\} \text{ if } \left\{ \begin{array}{l} f''_{xx}(x^*) > 0, \; f''_{xx}(x^*) f''_{yy}(x^*) - (f''_{xy}(x^*))^2 > 0 \\ f''_{xx}(x^*) < 0, \; f''_{xx}(x^*) f''_{yy}(x^*) - (f''_{xy}(x^*))^2 > 0 \end{array} \right\}.$$

If $f''_{xx}(x^*) f''_{yy}(x^*) - (f''_{xy}(x^*))^2 < 0$, we identify a stationary point as a saddle point.

Note the sufficiency of this test. For instance, even if $f''_{xx}(x^*) f''_{yy}(x^*) - (f''_{xy}(x^*))^2 = 0$, we still may have an extremum at (x^*). ⚠

Example 59 (Classification of Stationary Points of a $C^{(2)}$-Function[3] of n Variables)

If $x^* = (x_1^*, \ldots, x_n^*)$ is a stationary point of $f(x_1, \ldots, x_n)$ and if $|H_k(x^*)|$ is the following determinant evaluated at x^*,

$$|H_k| = \det \begin{vmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_k} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_k \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_k \partial x_k} \end{vmatrix}, \quad k = 1, 2, \ldots, n$$

then

[3] Recall that $C^{(2)}$ means 'twice continuously differentiable'.

a) if $(-1)^k|H_k(x^*)| > 0,\ k = 1, 2, \ldots, n,$ *then* x^* *is a local maximum;*

b) if $|H_k(x^*)| > 0,\ k = 1, 2, \ldots, n,$ *then* x^* *is a local minimum;*

c) if $|H_n(x^*)| \neq 0$ *and neither of the two conditions above is satisfied, then* x^* *is a saddle point.*

Example 60 *Let us find the extrema of the function* $z(x, y) = x^3 - 8y^3 + 6xy + 1$.

$$F.O.N.C.: \qquad z_x = 3(x^2 + 2y) = 0, \quad z_y = 6(-4y^2 + x) = 0.$$

Solving F.O.N.C. for $x,\ y$ *we obtain two stationary points:* $x = y = 0$ *and* $x = 1,\ y = -1/2$.

S.O.C.: The Hessian matrices evaluated at stationary points are

$$H|_{(0,0)} = \begin{pmatrix} 6x & 6 \\ 6 & -48y \end{pmatrix}_{(0,0)} = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}, \quad H|_{(1,-1/2)} = \begin{pmatrix} 6 & 6 \\ 6 & 24 \end{pmatrix},$$

therefore $(0, 0)$ *is a saddle point,* $(1, -1/2)$ *is a local minimum.*

2.8 The Implicit Function Theorem

Proposition 24 (The Two-Dimensional Case)

If $F(x, y) \in C^{(k)}$ *in a set* D *(i.e.* F *is* k *times continuously differentiable at any point in* D*) and* (x_0, y_0) *is an interior point of* D, $F(x_0, y_0) = c$ *(*c *is constant) and* $F'_y(x_0, y_0) \neq 0$ *then the equation* $F(x, y) = c$ *defines* y *as a* $C^{(k)}$*-function of* x *in some neighborhood of* (x_0, y_0)*, i.e.* $y = \psi(x)$ *and*

$$\frac{dy}{dx} = -\frac{F'_x(x, y)}{F'_y(x, y)}.$$

Example 61 *Let consider the function* $\dfrac{1 - e^{-ax}}{1 + e^{ax}}$*, where* a *is a parameter,* $a > 1$.

What happens to the extremum value of x *if* a *increases by a small number?*

To answer this question, first write down the F.O.N.C. for a stationary point:

$$f'_x = \frac{a(e^{-ax} - e^{ax} + 2)}{((1 + e^{ax})^2} = 0.$$

If this expression vanishes at, say, $x = x^*$*, we can apply the implicit function theorem to the F.O.N.C. and obtain*

$$\frac{dx}{da} = -\frac{\partial(e^{-ax} - e^{ax} + 2)/\partial a}{\partial(e^{-ax} - e^{ax} + 2)/\partial x} = -\frac{x}{a} < 0$$

in some neighborhood of (x^*, a)*. The negativity of the derivative is due to the fact that* x^* *is positive, therefore, every* x *in a small neighborhood of* x^* *should be positive as well.*

Let us formulate the general case of the implicit function theorem.

Consider the system of m equations with n *exogenous* variables, $x_1, \ldots, x_n$, and m *endogenous* variables, $y_1, \ldots, y_m$:

$$\left\{ \begin{array}{l} f^1(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0 \\ f^2(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0 \\ \qquad \ldots \ \ldots \ \ldots \\ f^m(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0 \end{array} \right. \tag{3}$$

Our objective is to find $\dfrac{\partial y_j}{\partial x_i}$, $i = 1, \ldots, n,\ j = 1, \ldots, m$.

If we take the total differentials of all f^j, we get

$$\begin{array}{rcl} \dfrac{\partial f^1}{\partial y_1} dy_1 + \ldots + \dfrac{\partial f^1}{\partial y_m} dy_m & = & -\left(\dfrac{\partial f^1}{\partial x_1} dx_1 + \ldots + \dfrac{\partial f^1}{\partial x_n} dx_n \right), \\ & \ldots & \\ \dfrac{\partial f^m}{\partial y_1} dy_1 + \ldots + \dfrac{\partial f^m}{\partial y_m} dy_m & = & -\left(\dfrac{\partial f^m}{\partial x_1} dx_1 + \ldots + \dfrac{\partial f^m}{\partial x_n} dx_n \right). \end{array}$$

Now allow only x_i to vary (i.e. $dx_i \neq 0,\ dx_l = 0,\ l = 1, 2, \ldots, i-1, i+1, \ldots, n$) and divide each remaining term in the system above by dx_i. Thus we arrive at the following linear system with respect to $\dfrac{\partial y_1}{\partial x_i}, \ldots \dfrac{\partial y_m}{\partial x_i}$ (note, that in fact $dy_1/dx_i, \ldots, dy_m/dx_i$ should be interpreted as partial derivatives with respect to x_i since we allow only x_i to vary, holding constant all other x_l):

$$\begin{array}{rcl} \dfrac{\partial f^1}{\partial y_1} \dfrac{\partial y_1}{\partial x_i} + \ldots + \dfrac{\partial f^1}{\partial y_m} \dfrac{\partial y_m}{\partial x_i} & = & -\dfrac{\partial f^1}{\partial x_i}, \\ & \ldots & \\ \dfrac{\partial f^m}{\partial y_1} \dfrac{\partial y_1}{\partial x_i} + \ldots + \dfrac{\partial f^m}{\partial y_m} \dfrac{\partial y_m}{\partial x_i} & = & -\dfrac{\partial f^m}{\partial x_i}, \end{array}$$

which can be solved for $\dfrac{\partial y_1}{\partial x_i}, \ldots \dfrac{\partial y_m}{\partial x_i}$ by, say, Cramer's rule or by the inverse matrix method.

Let define the *Jacobian matrix* of $f^1, \ldots, f^m$ with respect to $y_1, \ldots, y_m$ as

$$\mathcal{J} = \begin{pmatrix} \frac{\partial f^1}{\partial y_1} & \cdots & \frac{\partial f^1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial y_1} & \cdots & \frac{\partial f^m}{\partial y_m} \end{pmatrix}.$$

Given the definition of Jacobian, we are in a position to formulate the general result:

Proposition 25 (The General Implicit Function Theorem)
Suppose $f^1, \ldots, f^m$ are $C^{(k)}$-functions in a set $D \subset \mathbf{R}^{n+m}$.
Let $(x^*, y^*) = (x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_m^*)$ be a solution to (3) in the interior of A.

Suppose also that $\det(\mathcal{J})$ *does not vanish at* (x^*, y^*). *Then (3) defines* $y_1, \dots, y_m$ *as* $C^{(k)}$*-functions of* $x_1, \dots, x_n$ *in some neighborhood of* (x^*, y^*) *and in that neighborhood, for* $j = 1, 2, \dots, n$,

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_j} \\ \vdots \\ \frac{\partial y_m}{\partial x_j} \end{pmatrix} = -\mathcal{J}^{-1} \cdot \begin{pmatrix} \frac{\partial f^1}{\partial x_j} \\ \vdots \\ \frac{\partial f^m}{\partial x_j} \end{pmatrix}.$$

Example 62 *Given the system of two equations* $x^2 + y^2 = z^2/2$ *and* $x + y + z = 2$, *find* $x' = x'(z)$, $y' = y'(z)$ *in the neighborhood of the point* $(1, -1, 2)$.

In this setup, $f^1(x, y, z) = x^2 + y^2 - z^2/2$, $f^2(x, y, z) = x + y + z - 2$, $f^1, f^2 \in C^\infty(\mathbf{R}^3)$, $f^1(1, -1, 2) = f^2(1, -1, 2) = 0$, *and at* $(1, -1, 2)$

$$\det(\mathcal{J}) = \det \begin{pmatrix} \frac{\partial f^1}{\partial x} & \frac{\partial f^1}{\partial y} \\ \frac{\partial f^2}{\partial x} & \frac{\partial f^2}{\partial y} \end{pmatrix} = \det \begin{pmatrix} 2x & 2y \\ 1 & 1 \end{pmatrix}_{(1,-1,2)} = \det \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} = 4 \neq 0.$$

Therefore the conditions of the implicit function theorem are satisfied and

$$\begin{pmatrix} \frac{dx}{dz} \\ \frac{dy}{dz} \end{pmatrix} = -\frac{1}{2x - 2y} \begin{pmatrix} 1 & -2y \\ -1 & 2x \end{pmatrix} \begin{pmatrix} -z \\ 1 \end{pmatrix} \Longrightarrow \begin{array}{l} \frac{dx}{dz} = \frac{z+2y}{2x-2y} \\ \frac{dy}{dz} = -\frac{z+2x}{2x-2y} \end{array}$$

The Jacobian matrix can be also applied to test whether functional (linear or nonlinear) dependence exists among a set of n functions in n variables. More precisely, if we have n functions of n variables $g_i = g_i(x_1, \dots, x_n)$, $i = 1, 2, \dots, n$, then the determinant of the Jacobian matrix of $g_1, \dots, f_n$ with respect to $x_1, \dots, x_m$ will be identically zero for all values of $x_1, \dots, x_n$ if and only if the n functions $g_1, \dots, g_n$ are functionally (linearly or nonlinearly) dependent.

Economics Application 4 (The Profit-Maximizing Firm)

Consider that a firm has the profit function $F(l, k) = pf(l, k) - wl - rk$ *where* f *is the firm's (neoclassical) production function,* p *is the price of output,* l,k *are the amount of labor and capital employed by the firm (in units of output),* w *is the real wage and* r *is the real rental price of capital. The firm takes* p, w *and* r *as given. Assume that the Hessian matrix of* f *is negative definite.*

a) Show that if the wage increases by a small amount, then the firm decides to employ less labor.

b) Show that if the wage increases by a small amount and the firm is constrained to maintain the same amount of capital k_0, *then the firm will reduce* l *by less than it does in part a).*

Solution.

a) The firm's objective is to choose l *and* k *so that it maximizes its profits. The first-order conditions for maximization are:*

$$\left\{ \begin{array}{lcl} p\frac{\partial f}{\partial l}(l, k) - w & = & 0 \\ p\frac{\partial f}{\partial k}(l, k) - r & = & 0 \end{array} \right. \quad or, \quad \left\{ \begin{array}{lcl} F^1(l, k; w) & = & 0 \\ F^2(l, k; r) & = & 0 \end{array} \right.$$

The Jacobian of this system of equations is:

$$|J| = \begin{vmatrix} \frac{\partial F^1}{\partial l} & \frac{\partial F^1}{\partial k} \\ \frac{\partial F^2}{\partial l} & \frac{\partial F^2}{\partial k} \end{vmatrix} = p^2 \begin{vmatrix} f_{11}(l, k) & f_{12}(l, k) \\ f_{12}(l, k) & f_{22}(l, k) \end{vmatrix} = p^2 |H| > 0.$$

where f_{ij} is the second-order partial derivative of f with respect to the jth and the ith arguments and $|H|$ is the Hessian of f. The fact that H is negative definite means that $f_{11} < 0$ and $|H| > 0$ (which also implies $f_{22} < 0$).

Since $|J| \neq 0$ the implicit-function theorem can be applied; thus, l and k are implicit functions of w. The first-order partial derivative of l with respect to w is:

$$\frac{\partial l}{\partial w} = -\frac{\begin{vmatrix} -1 & pf_{12}(l,k) \\ 0 & pf_{22}(l,k) \end{vmatrix}}{|J|} = \frac{f_{22}(l,k)}{p|H|} < 0.$$

Thus, the optimal l falls in response to a small increase in w.

b) If the firm's amount of capital is fixed at k_0, the profit function is $F(l) = pf(l,k_0) - wl - rk_0$ and the first-order condition is

$$p\frac{\partial f}{\partial l}(l,k_0) - w = 0.$$

Since $f_{11} \neq 0$, the above condition defines l as an implicit function of w. Taking the derivative of the F.O.C. with respect to l, we have:

$$pf_{11}(l,k_0)\frac{\partial l}{\partial w} = 1 \text{ or}$$

$$\frac{\partial l}{\partial w} = \frac{1}{pf_{11}(l,k_0)}.$$

This change in l is smaller in absolute value than the change in l in part a) since

$$\frac{f_{22}(l,k_0)}{p|H|} < \frac{1}{pf_{11}(l,k_0)}$$

(Note that both ratios are negative.)

For another economic application see the section 'Constrained Optimization'.

2.9 Concavity, Convexity, Quasiconcavity and Quasiconvexity

Definition 42 *A function $z = f(x_1, \ldots, x_n)$ is concave if and only if for any pair of distinct points $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in the domain of f and for any $\theta \in (0,1)$*

$$\theta f(u) + (1-\theta)f(v) \leq f(\theta u + (1-\theta)v)$$

and convex if and only if

$$\theta f(u) + (1-\theta)f(v) \geq f(\theta u + (1-\theta)v).$$

If f is differentiable, an alternative definition of concavity (or convexity) is

$$f(v) \leq f(u) + \sum_{i=1}^{n} f'_{x_i}(u)(v_i - u_i) \quad (\text{or } f(v) \geq f(u) + \sum_{i=1}^{n} f'_{x_i}(u)(v_i - u_i)).$$

If we change weak inequalities $\leq$ and $\geq$ to strict inequalities, we will get the definition of strict concavity and strict convexity, respectively.

Some useful results:

- The sum of concave (or convex) functions is a concave (or convex) function. If, additionally, at least one of these functions is strictly concave (or convex), the sum is also strictly concave (or convex).
- If $f(x)$ is a (strictly) concave function then $-f(x)$ is a (strictly) convex function and vise versa.

Recipe 12 – How to Check Concavity (or Convexity):

A twice continuously differentiable function $z = f(x_1, \dots, x_n)$ is concave (or convex) if and only if d^2z is everywhere negative (or positive) semidefinite.

$z = f(x_1, \dots, x_n)$ is strictly concave (or strictly convex) if (but not only if) d^2z is everywhere negative (or positive) definite.

Again, any test for the sign definiteness of a symmetric quadratic form is applicable.

Example 63 *$z(xy) = x^2 + xy + y^2$ is strictly convex.*

The conditions for concavity and convexity are necessary and sufficient, while those for strict concavity (strict convexity) are only sufficient. ⚠

Definition 43 *A set S in $\mathbf{R}^n$ is convex if for any $x, y \in S$ and any $\theta \in [0,1]$ the linear combination $\theta x + (1-\theta)y \in S$.*

Definition 44 *A function $z = f(x_1, \dots, x_n)$ is quasiconcave if and only if for any pair of distinct points $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ in the convex domain of f and for any $\theta \in (0,1)$*

$$f(v) \geq f(u) \Longrightarrow f(\theta u + (1-\theta)v) \geq f(u)$$

and quasiconvex if and only if

$$f(v) \geq f(u) \Longrightarrow f(\theta u + (1-\theta)v) \leq f(v).$$

If f is differentiable, an alternative definition of quasiconcavity (quasiconvexity) is

$$f(v) \geq f(u) \Longrightarrow \sum_{i=1}^{n} f'_{x_i}(u)(v_i - u_i) \geq 0 \quad (f(v) \geq f(u) \Longrightarrow \sum_{i=1}^{n} f'_{x_i}(v)(v_i - u_i) \geq 0).$$

A change of weak inequalities $\leq$ and $\geq$ on the right to strict inequalities gives us the definitions of strict quasiconcavity and strict quasiconvexity, respectively.

Example 64 *The function $z(x,y) = xy, \; x, y \geq 0$ is quasiconcave.*

Note that any convex (concave) function is quasiconvex (quasiconcave), but not *vice versa.* ⚠

2.10 Appendix: Matrix Derivatives

Matrix derivatives play an important role in economic analysis, especially in econometrics.

If A is $[n \times n]$ non-singular matrix, the derivative of its determinant with respect to A is given by

$$\frac{\partial}{\partial A}(\det(A)) = (C_{ij}),$$

where (C_{ij}) is the matrix of cofactors of A.

Example 65 (Matrix Derivatives in Econometrics: How to Find the Least Squares Estimator in a Multiple Regression Model)

Consider the multiple regression model:

$$y = X\beta + \epsilon$$

where $n \times 1$ vector y is the dependent variable, X is a $n \times k$ matrix of k explanatory variables with $rank(X) = k$, β is a $k \times 1$ vector of coefficients which are to be estimated and ϵ is a $n \times 1$ vector of disturbances. We assume that the matrices of observations X and y are given.

Our goal is to find an estimator b for β using the least squares method. The least squares estimator of β is a vector b, which minimizes the expression

$$E(b) = (y - Xb)'(y - Xb) = y'y - y'Xb - b'X'y + b'X'Xb.$$

The first-order condition for extremum is:

$$\frac{dE(b)}{db} = 0.$$

To obtain the first order conditions we use the following formulas:

- $\dfrac{d(a'b)}{db} = a,$
- $\dfrac{d(b'a)}{db} = a,$
- $\dfrac{d(Mb)}{db} = M',$
- $\dfrac{d(b'Mb)}{db} = (M + M')b,$

where a, b are $k \times 1$ vectors and M is a $k \times k$ matrix.

Using these formulas we obtain:

$$\frac{dE(b)}{db} = -2X'y + 2X'Xb$$

and the first-order condition implies:

$$X'Xb = X'y$$

or

$$b = (X'X)^{-1}X'y$$

On the other hand, by the third derivation rule above, we have:

$$\frac{d^2E(b)}{db^2} = (2X'X)' = 2X'X.$$

To check whether the solution b is indeed a minimum, we need to prove the positive definiteness of the matrix $X'X$.

First, notice that $X'X$ is a symmetric matrix. The symmetry is obvious:

$$(X'X)' = X'X.$$

To prove positive definiteness, we take an arbitrary $k \times 1$ vector z, $z \neq \mathbf{0}$ and check the following quadratic form:

$$z'(X'X)z = (Xz)'Xz.$$

The assumptions $rank(X) = k$ and $z \neq 0$ imply $Xz \neq 0$. It follows that $(Xz)'Xz > 0$ for any $z \neq 0$ or, equivalently, $z'X'Xz > 0$ for any $z \neq 0$, which means that (by definition) $X'X$ is positive definite.

Further Reading:

- Bartle, R.G. *The Elements of Real Analysis.*
- Edwards, C.N and D.E.Penny. *Calculus and Analytical Geomethry.*
- Greene, W.H. *Econometric Analysis.*
- Rudin, W. *Principles of Mathematical Analysis.*

3 Constrained Optimization

3.1 Optimization with Equality Constraints

Consider the problem[4]:

$$\begin{aligned} &\text{extremize } f(x_1, \ldots, x_n) \qquad (4)\\ &\text{subject to } g^j(x_1, \ldots, x_n) = b_j, \quad j = 1, 2, \ldots, m < n. \end{aligned}$$

f is called the objective function, $g^1, g^2, \ldots, g^m$ are the constraint functions, $b_1, b_2, \ldots, b_m$ are the constraint constants. The difference $n - m$ is the number of *degrees of freedom* of the problem.

Note that n is strictly greater than m. ⚠

If it is possible to explicitly express (from the constraint functions) m independent variables as functions of the other $n - m$ independent variables, we can eliminate m variables in the objective function, thus the initial problem will be reduced to the unconstrained optimization problem with respect to $n - m$ variables. However, in many cases it is not technically feasible to explicitly express one variable as function of the others.

Instead of the substitution and elimination method, we may resort to the easy-to-use and well-defined *method of Lagrange multipliers*.

Let f and $g^1, \ldots, g^m$ be $C^{(1)}$-functions and the Jacobian $\mathcal{J} = (\frac{\partial g^j}{\partial x_i})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ have the full rank, i.e.rank$(\mathcal{J}) = m$. We introduce the *Lagrangian function* as

$$L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(x_1, \ldots, x_n) + \sum_{j=1}^{m} \lambda_j (b_j - g^j(x_1, \ldots, x_n)),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are constant (*Lagrange multipliers*).

Recipe 13 – What are the Necessary Conditions for the Solution of (4)?

Equate all partials of L with respect to $x_1, \ldots, x_n$, $\lambda_1, \ldots, \lambda_m$ to zero:

$$\frac{\partial L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m)}{\partial x_i} = \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g^j(x_1, \ldots, x_n)}{\partial x_i} = 0, \quad i = 1, 2, \ldots, n,$$

$$\frac{\partial L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m)}{\partial \lambda_j} = b_j - g^j(x_1, \ldots, x_n) = 0, \qquad j = 1, 2, \ldots, m.$$

Solve these equations for $x_1, \ldots, x_n$ and $\lambda_1, \ldots, \lambda_m$.

In the end we will get a set of stationary points of the Lagrangian. If $x^* = (x_1^*, \ldots, x_n^*)$ is a solution of (4), it should be a stationary point of L.

It is important that rank$(\mathcal{J}) = m$, and the functions are continuously differentiable. ⚠

Example 66 *This example shows that the Lagrange-multiplier method does not always provide the solution to an optimization problem. Consider the problem:*

$$\begin{aligned} &\min x^2 + y^2 \\ &\text{subject to } (x-1)^3 - y^2 = 0. \end{aligned}$$

[4] *"Extremize"* means to find either the minimum or the maximum of the objective function f.

Notice that the solution to this problem is $(1,0)$. *The restriction* $(x-1)^3 = y^2$ *implies* $x \geq 1$. *If* $x = 1$, *then* $y = 0$ *and* $x^2 + y^2 = 1$. *If* $x > 1$, *then* $y > 0$ *and* $x^2 + y^2 > 1$. *Thus, the minimum of* $x^2 + y^2$ *is attained for* $(1,0)$.

The Lagrangian function is

$$L = x^2 + y^2 - \lambda[(x-1)^3 - y^2]$$

The first-order conditions are

$$\begin{cases} \frac{\partial L}{\partial x} & = 0 \\ \frac{\partial L}{\partial y} & = 0 \\ \frac{\partial L}{\partial \lambda} & = 0 \end{cases} \quad or \quad \begin{cases} 2x - 3\lambda(x-1)^2 & = 0 \\ 2y + 2\lambda y & = 0 \\ (x-1)^3 - y^2 & = 0 \end{cases}$$

Clearly, the first equation is not satisfied for $x = 1$. *Thus, the Lagrange-multiplier method fails to detect the solution* $(1,0)$.

If we need to check whether a stationary point is a maximum or minimum, the following *local sufficient conditions* can be applied:

Proposition 26 *Let us introduce a bordered Hessian* $|\bar{H}_r|$ *as*

$$|\bar{H}_r| = \det \begin{pmatrix} 0 & \dots & 0 & \frac{\partial g^1}{\partial x_1} & \dots & \frac{\partial g^1}{\partial x_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{\partial g^m}{\partial x_1} & \dots & \frac{\partial g^m}{\partial x_r} \\ \frac{\partial g^1}{\partial x_1} & \dots & \frac{\partial g^m}{\partial x_1} & \frac{\partial^2 L}{\partial x_1 \partial x_1} & \dots & \frac{\partial^2 L}{\partial x_1 \partial x_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g^1}{\partial x_r} & \dots & \frac{\partial g^m}{\partial x_r} & \frac{\partial^2 L}{\partial x_r \partial x_1} & \dots & \frac{\partial^2 L}{\partial x_r \partial x_r} \end{pmatrix}, \qquad r = 1, 2, \dots, n.$$

Let f *and* $g^1, \dots, g^m$ *are* $C^{(2)}$*-functions and let* x^* *satisfy the necessary conditions for the problem (4). Let* $|\bar{H}_r(x^*)|$ *be the bordered Hessian determinant evaluated at* x^*. *Then*

- *if* $(-1)^m |\bar{H}_r(x^*)| > 0$, $r = m+1, \dots, n$, *then* x^* *is a local minimum point for the problem (4).*
- *if* $(-1)^r |\bar{H}_r(x^*)| > 0$, $r = m+1, \dots, n$, *then* x^* *is a local maximum point for the problem (4).*

Example 67 (Local Second-Order Conditions for the Case with one Constraint)

Suppose that $x^* = (x_1^*, \dots, x_n^*)$ *satisfies the necessary conditions for the problem*

$\max(\min) f(x_1, \dots, x_n)$ *subject to* $g(x_1, \dots, g_n) = b$,

i.e. all partial derivatives of the Lagrangian are zero at x^*. *Define*

$$|\bar{H}_r| = \det \begin{pmatrix} 0 & \frac{\partial g}{\partial x_1} & \dots & \frac{\partial g}{\partial x_r} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 L}{\partial x_1 \partial x_1} & \dots & \frac{\partial^2 L}{\partial x_1 \partial x_r} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g}{\partial x_r} & \frac{\partial^2 L}{\partial x_r \partial x_1} & \dots & \frac{\partial^2 L}{\partial x_r \partial x_r} \end{pmatrix}, \qquad r = 1, 2, \dots, n.$$

Let $|\bar{H}_r(x^*)|$ *be* $|\bar{H}_r|$ *evaluated at* x^*. *Then*

- x^* *is a local minimum point of f subject to the given constraint if* $|\bar{H}_r(x^*)| < 0$ *for all* $r = 2, \ldots, n$.

- x^* *is a local maximum point of f subject to the given constraint if* $(-1)^r|\bar{H}_r(x^*)| > 0$ *for all* $r = 2, \ldots, n$.

Economics Application 5 (Utility Maximization)

In order to illustrate how the Lagrange-multiplier method can be applied, let us consider the following optimization problem:

The preferences of a consumer over two goods x and y are given by the utility function

$$U(x, y) = (x + 1)(y + 1) = xy + x + y + 1.$$

The prices of goods x and y are 1 and 2, respectively, and the consumer's income is 30. What bundle of goods will the consumer choose?

The consumer's budget constraint is $x + 2y \leq 30$ which, together with the conditions $x \geq 0$ and $y \geq 0$, determines his budget set (the set of all affordable bundles of goods). He will choose the bundle from his budget set that maximizes his utility. Since $U(x, y)$ is an increasing function in both x and y (over the domain $x \geq 0$ and $y \geq 0$), the budget constraint should be satisfied with equality. The consumer's optimization problem can be stated as follows:

$$\begin{array}{l} \max U(x, y) \\ \text{subject to } x + 2y = 30. \end{array}$$

The Lagrangian function is

$$\begin{aligned} L &= U(x, y) - \lambda(x + 2y - 30) \\ &= xy + x + y + 1 - \lambda(x + 2y - 30). \end{aligned}$$

The first-order necessary conditions are

$$\left\{ \begin{array}{lcl} \frac{\partial L}{\partial x} &=& 0 \\ \frac{\partial L}{\partial y} &=& 0 \\ \frac{\partial L}{\partial z} &=& 0 \end{array} \right. \quad or \quad \left\{ \begin{array}{lcl} y + 1 - \lambda &=& 0 \\ x + 1 - 2\lambda &=& 0 \\ x + 2y - 30 &=& 0 \end{array} \right.$$

From the first equation, we get $\lambda = y + 1$; substituting $y + 1$ for λ in the second equation, we obtain $x - 2y - 1 = 0$. This condition and the third condition above lead to $x = \frac{31}{2}$, $y = \frac{29}{4}$.

To check whether this solution really maximizes the objective function, let us apply the second order sufficient conditions.

The second-order conditions involve the bordered Hessian:

$$\det \bar{H} = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 4 > 0.$$

Thus, ($x = \frac{31}{2}$, $y = \frac{29}{4}$) is indeed a maximum and represents the bundle demanded by the consumer.

Economics Application 6 (Applications of the Implicit-Function Theorem)

Consider the problem of maximizing the function $f(x,y) = ax + y$ subject to the constraint $x^2 + ay^2 = 1$ where $x > 0$, $y > 0$ and a is a positive parameter. Given that this problem has a solution, find how the optimal values of x and y change if a increases by a very small amount.

Solution:

Differentiation of the Lagrangian function $L = ax + y - \lambda(x^2 + ay^2 - 1)$ with respect to x, y, λ gives the first-order conditions:

$$\left\{ \begin{array}{lcl} a - 2\lambda x & = & 0 \\ 1 - 2a\lambda y & = & 0 \\ x^2 + ay^2 - 1 & = & 0 \end{array} \right. \quad or, \quad \left\{ \begin{array}{lcl} F^1(x,\lambda;a) & = & 0 \\ F^2(y,\lambda;a) & = & 0 \\ F^3(x,y;a) & = & 0 \end{array} \right. \tag{5}$$

The Jacobian of this system of three equations is:

$$J = \begin{pmatrix} \frac{\partial F^1}{\partial x} & \frac{\partial F^1}{\partial y} & \frac{\partial F^1}{\partial \lambda} \\ \frac{\partial F^2}{\partial x} & \frac{\partial F^2}{\partial y} & \frac{\partial F^2}{\partial \lambda} \\ \frac{\partial F^3}{\partial x} & \frac{\partial F^3}{\partial y} & \frac{\partial F^3}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} -2\lambda & 0 & -2x \\ 0 & -2a\lambda & -2ay \\ 2x & 2ay & 0 \end{pmatrix},$$

$$|J| = -8a\lambda(x^2 + ay^2) = -8a\lambda.$$

From the first F.O.C. we can see that $\lambda > 0$, thus $|J| < 0$ at the optimal point. Therefore, in a neighborhood of the optimal solution, given by the system (5), the conditions of the implicit-function theorem are met, and in this neighborhood x and y can be expressed as functions of a.

The first-order partial derivatives of x and y with respect to a are evaluated as:

$$\frac{\partial x}{\partial a} = -\frac{\begin{vmatrix} \frac{\partial F^1}{\partial a} & \frac{\partial F^1}{\partial y} & \frac{\partial F^1}{\partial \lambda} \\ \frac{\partial F^2}{\partial a} & \frac{\partial F^2}{\partial y} & \frac{\partial F^2}{\partial \lambda} \\ \frac{\partial F^3}{\partial a} & \frac{\partial F^3}{\partial y} & \frac{\partial F^3}{\partial \lambda} \end{vmatrix}}{|J|} = -\frac{\begin{vmatrix} 1\lambda & 0 & -2x \\ -2\lambda y & -2a\lambda & -2ay \\ y^2 & 2ay & 0 \end{vmatrix}}{|J|} =$$

$$= \frac{4ay^2(\lambda x + a)}{8a\lambda} = \frac{3xy^2}{2} > 0.$$

$$\frac{\partial y}{\partial a} = -\frac{\begin{vmatrix} \frac{\partial F^1}{\partial x} & \frac{\partial F^1}{\partial a} & \frac{\partial F^1}{\partial \lambda} \\ \frac{\partial F^2}{\partial x} & \frac{\partial F^2}{\partial a} & \frac{\partial F^2}{\partial \lambda} \\ \frac{\partial F^3}{\partial x} & \frac{\partial F^3}{\partial a} & \frac{\partial F^3}{\partial \lambda} \end{vmatrix}}{|J|} = -\frac{\begin{vmatrix} -2\lambda & 1 & -2x \\ 0 & -2\lambda y & -2ay \\ 2x & y^2 & 0 \end{vmatrix}}{|J|} =$$

$$= -\frac{-4xy(\lambda x + a) - 4\lambda y(x^2 + ay^2)}{-8a\lambda} = -\left(\frac{3xy}{4\lambda} + \frac{y}{2a}\right) < 0.$$

In conclusion, a small increase in a will lead to a rise in the optimal x and a fall in the optimal y.

Proposition 27 *Suppose that f and $g^1, \ldots, g^m$ are defined on an open convex set $S \subset \mathbf{R}^n$. Let x^* be a stationary point of the Lagrangian. Then,*

- *if the Lagrangian is concave, x^* maximizes (4);*
- *if the Lagrangian is convex, x^* minimizes (4).*

3.2 The Case of Inequality Constraints

Classical methods of optimization (the method of Lagrange multipliers) deal with optimization problems with equality constraints in the form of $g(x_1, \ldots, x_n) = c$. Nonclassical optimization, also known as *mathematical programming*, tackles problems with inequality constraints like $g(x_1, \ldots, x_n) \leq c$.

Mathematical programming includes *linear programming* and *nonlinear programming.* In linear programming, the objective function and all inequality constraints are linear. When either the objective function or an inequality constraint is nonlinear, we face a problem of nonlinear programming.

In the following, we restrict our attention to non-linear programming. A problem of linear programming – also called a *linear program* – is discussed in the Appendix to this chapter.

3.2.1 Non-Linear Programming

The nonlinear programming problem is that of choosing nonnegative values of certain variables so as to maximize or minimize a given (non-linear) function subject to a given set of (non-linear) inequality constraints.

The nonlinear programming maximum problem is

$$\begin{aligned} \max f(x_1, \ldots, x_n) \\ \text{subject to } g^i(x_1, \ldots, x_n) \leq b_i, \quad i = 1, 2, \ldots, m. \\ x_1 \geq 0, \ldots, x_n \geq 0 \end{aligned} \tag{6}$$

Similarly, the minimization problem is

$$\begin{aligned} \min f(x_1, \ldots, x_n) \\ \text{subject to } g^i(x_1, \ldots, x_n) \geq b_i, \quad i = 1, 2, \ldots, m. \\ x_1 \geq 0, \ldots, x_n \geq 0 \end{aligned}$$

First, note that there are no restrictions on the relative size of m and n, unlike the case of equality constraints. Second, note that the direction of the inequalities ($\leq$ or $\geq$) at the constraints is only a convention, because the inequality $g^i \leq b^i$ can be easily converted to the $\geq$ inequality by multiplying it by –1, yielding $-g^i \geq -b_i$. Third, note that an equality constraint, say $g^k = b_k$, can be replaced by the two inequality constraints, $g^k \leq b_k$ and $-g^k \leq -b_k$. ⚠

Definition 45 *A constraint $g^j \leq b_j$ is called binding (or active) at $x^0 = (x_1^0, \ldots, x_n^0)$ if $g^j(x^0) = b_j$.*

Example 68 *The following is an example of a nonlinear program:*

$$\begin{aligned} \textit{Max} \quad & \pi = x_1(10 - x_1) + x_2(20 - x_2) \\ \textit{subject to} \quad & 5x_1 + 3x_2 \leq 40 \\ & x_1 \leq 5 \\ & x_2 \leq 10 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Example 69 (How to Solve a Nonlinear Program Graphically)

An intuitive way to understand a low-dimensional non-linear program is to represent the feasible set and the objective function graphically.

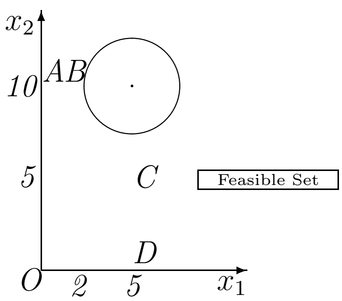
For instance, we may try to represent graphically the optimization problem from Example 68, which can be re-written as

$$\begin{aligned} \textit{Max} \quad & \pi = 125 - (x_1 - 5)^2 - (x_2 - 10)^2 \\ \textit{subject to} \quad & 5x_1 + 3x_2 \leq 40 \\ & x_1 \leq 5 \\ & x_2 \leq 10 \\ & x_1 \geq 0, x_2 \geq 0; \end{aligned}$$

or

$$\begin{aligned} \textit{Min} \quad & C = (x_1 - 5)^2 + (x_2 - 10)^2 \\ \textit{subject to} \quad & 5x_1 + 3x_2 \leq 40 \\ & x_1 \leq 5 \\ & x_2 \leq 10 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Region OABCD below shows the feasible set of the nonlinear program.



The value of the objective function can be interpreted as the square of the distance from the point $(5, 10)$ *to the point* (x_1, x_2)*. Thus, the nonlinear program requires us to find the point in the feasible region which minimizes the distance to the point* $(5, 10)$*.*

As can be seen in the figure above, the optimal point (x_1^*, x_2^*) *lies on the line* $5x_1 + 3x_2 = 40$ *and it is also the tangency point with a circle centered at* $(5, 10)$*. These two conditions are sufficient to determine the optimal solution.*

Consider $(x_1 - 5)^2 + (x_2 - 10)^2 - C = 0$*. From the implicit-function theorem, we have*

$$\frac{dx_2}{dx_1} = -\frac{x_1 - 5}{x_2 - 10}.$$

Thus, the tangent to the circle centered at $(5, 10)$ *which has the slope* $-\frac{5}{3}$ *is given by the equation*

$$-\frac{5}{3} = -\frac{x_1 - 5}{x_2 - 10}$$

or

$$3x_1 - 5x_2 + 35 = 0.$$

Solving the system of equations

$$\begin{cases} 5x_1 + 3x_2 = 40 \\ 3x_1 - 5x_2 = -35 \end{cases}$$

we find the optimal solution $(x_1^*, x_2^*) = (\frac{95}{34}, \frac{295}{34})$.

3.2.2 Kuhn-Tucker Conditions

For the purpose of ruling out certain irregularities on the boundary of the feasible set, a restriction on the constrained functions is imposed. This restriction is called *the constraint qualification.*

Definition 46 *Let* $\bar{x} = (x_1, \ldots, x_n)$ *be a point on the boundary of the feasible region of a nonlinear program.*

i) A vector $dx = (dx_1, \ldots, dx_n)$*is a* test vector *of* $\bar{x}$ *if it satisfies the conditions*

- $\bar{x}_j = 0 \Longrightarrow dx_j \geq 0$;
- $g^i(\bar{x}) = r_i \Longrightarrow dg^i(\bar{x}) \leq (\geq) 0$ *for a maximization (minimization) program.*

ii) A qualifying arc *for a test vector of* $\bar{x}$ *is an arc with the point of origin* $\bar{x}$*, contained entirely in the feasible region and tangent to the test vector.*

iii) The constraint qualification *is a condition which requires that, for any boundary point* $\bar{x}$ *of the feasible region and for any test vector of* $\bar{x}$*, there exist a qualifying arc for that test vector.*

The constraint qualification condition is often imposed in the following way:
the gradients at x^* of those g^j-constraints which are binding at x^* are linearly independent.

Assume that the constraint qualification condition is satisfied and the functions in (6) are differentiable. If x^* is the optimal solution to (6), it should satisfy the following conditions (*the Kuhn-Tucker necessary maximum conditions* or *KT conditions*):

$$\frac{\partial L}{\partial x_i} \leq 0, \; x_i \geq 0 \quad \text{and} \quad x_i \frac{\partial L}{\partial x_i} = 0, \quad i = 1, \ldots, n,$$

$$\frac{\partial L}{\partial \lambda_j} \geq 0, \; \lambda_j \geq 0 \quad \text{and} \quad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0, \quad j = 1, \ldots, m$$

where

$$L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(x_1, \ldots, x_n) + \sum_{j-1}^{m} \lambda_j (b_j - g^j(x_1, \ldots, x_n))$$

is the Lagrangian function of a non-linear program.

The minimization version of Kuhn-Tucker necessary conditions is

$$\frac{\partial L}{\partial x_i} \geq 0, \; x_i \geq 0 \quad \text{and} \quad x_i \frac{\partial L}{\partial x_i} = 0, \quad i = 1, \ldots, n,$$

$$\frac{\partial L}{\partial \lambda_j} \leq 0, \ \lambda_j \geq 0 \quad \text{and} \quad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0, \quad j = 1, \ldots, m$$

Note that, in general, the KT conditions are neither necessary nor sufficient for a local optimum. However, if certain assumptions are satisfied, the KT conditions become necessary and even sufficient. ⚠

Example 70 *The Lagrangian function of the nonlinear program in Example 68 is:*

$$L = x_1(10 - x_1) + x_2(20 - x_2) - \lambda_1(5x_1 + 3x_2 - 40) - \lambda_2(x_1 - 5) - \lambda_3(x_2 - 10).$$

The KT conditions are:

$$\begin{array}{l}
\frac{\partial L}{\partial x_1} = 10 - 2x_1 - 5\lambda_1 - \lambda_2 \leq 0 \\
\frac{\partial L}{\partial x_2} = 20 - 2x_2 - 3\lambda_1 - \lambda_3 \leq 0 \\
\frac{\partial L}{\partial \lambda_1} = -(5x_1 + 3x_2 - 40) \geq 0 \\
\frac{\partial L}{\partial \lambda_2} = -(x_1 - 5) \geq 0 \\
\frac{\partial L}{\partial \lambda_3} = -(x_2 - 10) \geq 0 \\
x_1 \geq 0, x_2 \geq 0 \\
\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0 \\
x_1 \frac{\partial L}{\partial x_1} = 0, x_2 \frac{\partial L}{\partial x_2} = 0 \\
\lambda_i \frac{\partial L}{\partial x_i} = 0, \ i = 1, 2, 3.
\end{array}$$

Proposition 28 (The Necessity Theorem)
The KT conditions are necessary conditions for a local optimum if the constraint qualification is satisfied.

Note that ⚠

- The failure of the constraint qualification signals certain irregularities at the boundary kinks of the feasible set. Only if the optimal solution occurs in such a kink may the KT conditions not be satisfied.
- If all constraints are linear, the constraint qualification is always satisfied.

Example 71 *The constraint qualification for the nonlinear program in Example 68 is satisfied since all constraints are linear. Therefore, the optimal solution* $(\frac{95}{34}, \frac{295}{34})$ *must satisfy the KT conditions in Example 70.*

Example 72 *The following example illustrates a case where the Kuhn-Tucker conditions are not satisfied in the solution of an optimization problem. Consider the problem:*

$$\begin{array}{ll}
\textit{max} & y \\
\textit{subject to} & x + (y-1)^3 \leq 0, \\
 & x \geq 0, y \geq 0.
\end{array}$$

The solution to this problem is $(0, 1)$*. (If* $y > 1$*, then the restriction* $x + (y-1)^3 \leq 0$ *implies* $x < 0$*.)*

The Lagrangian function is

$$L \ = \ y + \lambda[-x - (y-1)^3]$$

One of the Kuhn-Tucker conditions requires

$$\frac{\partial L}{\partial y} \leq 0,$$

or

$$1 - 3\lambda(y-1)^2 \leq 0.$$

As can be observed, this condition is not verified at the point $(0, 1)$.

Proposition 29 (Kuhn-Tucker Sufficiency Theorem - 1) *If*

a) f is differentiable and concave in the nonnegative orthant;

b) each constraint function is differentiable and convex in the nonnegative orthant;

c) the point x^* satisfies the Kuhn-Tucker necessary maximum conditions

then x^* gives a global maximum of f.

To adapt this theorem for minimization problems, we need to interchange the two words "concave" and "convex" in a) and b) and use the Kuhn-Tucker necessary minimum condition in c).

Proposition 30 (The Sufficiency Theorem - 2)
The KT conditions are sufficient conditions for x^* to be a local optimum of a maximization (minimization) program if the following assumptions are satisfied:

- *the objective function f is differentiable and quasiconcave (or quasiconvex);*
- *each constraint g^i is differentiable and quasiconvex (or quasiconcave);*
- *any one of the following is satisfied:*
  - *there exists j such that $\frac{\partial f(x^*)}{\partial x_j} < 0 (> 0)$;*
  - *there exists j such that $\frac{\partial f(x^*)}{\partial x_j} > 0 (< 0)$ and x_j can take a positive value without violating the constraints;*
  - *f is concave (or convex).*

The problem of finding the nonnegative vector (x^*, λ^*), $x^* = (x_1^*, \ldots, x_n^*)$, $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*)$ which satisfies Kuhn-Tucker necessary conditions and for which

$$L(x, \lambda^*) \leq L(x^*, \lambda^*) \leq L(x^*, \lambda) \quad \text{for all } x = (x_1, \ldots, x_n) \geq 0,\ \lambda = (\lambda_1, \ldots, \lambda_m) \geq 0$$

is known as the *saddle point problem.*

Proposition 31 *If (x^*, λ^*) solves the saddle point problem then (x^*, λ^*) solves the problem (6).*

Economics Application 7 (Economic Interpretation of a Nonlinear Program and Kuhn-Tucker Conditions)

A nonlinear program can be interpreted much like a linear program. A maximization program in the general form, for example, is the production problem facing a firm which has to produce n *goods such that it maximizes its revenue subject to* m *resource (factor) constraints.*

The variables have the following economic interpretations:

- x_j *is the amount produced of the* j*th product;*
- r_i *is the amount of the* i*th resource available;*
- f *is the profit (revenue) function;*
- g^i *is a function which shows how the* i*th resource is used in producing the* n *goods.*

The optimal solution to the maximization program indicates the optimal quantities of each good the firm should produce.

In order to interpret the Kuhn-Tucker conditions, we first have to note the meanings of the following variables:

- $f_j = \frac{\partial f}{\partial x_j}$ *is the marginal profit (revenue) of product* j*;*
- λ_i *is the shadow price of resource* i*;*
- $g^i_j = \frac{\partial g^i}{\partial x_j}$ *is the amount of resource* i *used in producing a marginal unit of product* j*;*
- $\lambda_i g^i_j$ *is the imputed cost of resource* i *incurred in the production of a marginal unit of product* j*.*

The KT condition $\frac{\partial L}{\partial x_j} \leq 0$ *can be written as* $f_j \leq \sum_{i=1}^{m} \lambda_i g^i_j$ *and it says that the marginal profit of the* j*th product cannot exceed the aggregate marginal imputed cost of the* j*th product.*

The condition $x_j \frac{\partial L}{\partial x_j} = 0$ *implies that, in order to produce good* j *(*$x_j > 0$*), the marginal profit of good* j *must be equal to the aggregate marginal imputed cost (*$\frac{\partial L}{\partial x_j} = 0$*). The same condition shows that good* j *is not produced (*$x_j = 0$*) if there is an excess imputation (*$x_j \frac{\partial L}{\partial x_j} < 0$*).*

The KT condition $\frac{\partial L}{\partial \lambda_i} \geq 0$ *is simply a restatement of constraint* i*, which states that the total amount of resource* i *used in producing all the* n *goods should not exceed total amount available* r_i*.*

The condition $\frac{\partial L}{\partial \lambda_i} = 0$ *indicates that if a resource is not fully used in the optimal solution (*$\frac{\partial L}{\partial \lambda_i} > 0$*), then its shadow price will be 0 (*$\lambda_i = 0$*). On the other hand, a fully used resource (*$\frac{\partial L}{\partial \lambda_i} = 0$*) has a strictly positive price (*$\lambda_i > 0$*).*

Example 73 *Let us find an economic interpretation for the maximization program given in Example 68:*

$$\begin{aligned} \textit{Max} \quad & R = x_1(10 - x_1) + x_2(20 - x_2) \\ \textit{subject to} \quad & 5x_1 + 3x_2 \leq 40 \\ & x_1 \leq 5 \\ & x_2 \leq 10 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

A firm has to produce two goods using three kinds of resources available in the amounts 40,5,10 respectively. The first resource is used in the production of both goods: five units are necessary to produce one unit of good 1, and three units to produce one unit of good 2. The second resource is used only in producing good 1 and the third resource is used only in producing good 2.

The sale prices of the two goods are given by the linear inverse demand equations $p_1 = 10 - x_1$ *and* $p_2 = 20 - x_2$. *The problem the firm faces is how much to produce of each good in order to maximize revenue* $R = x_1p_1 + x_2p_2$. *The solution* $(2, 10)$ *gives the optimal amounts the firm should produce.*

Economics Application 8 (The Sales-Maximizing Firm)

Suppose that a firm's objective is to maximize its sales revenue subject to the constraint that the profit is not less than a certain value. Let Q *denote the amount of the good supplied on the market. If the revenue function is* $R(Q) = 20Q - Q^2$, *the cost function is* $C(Q) = Q^2 + 6Q + 2$ *and the minimum profit is* 10, *find the sales-maximizing quantity.*

The firm's problem is

$$\begin{aligned}&\text{max } R(Q)\\&\text{subject to } R(Q) - C(Q) \geq 10 \text{ and } Q \geq 0\end{aligned}$$

or

$$\begin{aligned}&\text{max } R(Q)\\&\text{subject to } 2Q^2 - 14Q + 2 \leq -10 \text{ and } Q \geq 0.\end{aligned}$$

The Lagrangian function is

$$Z = 20Q - Q^2 - \lambda(2Q^2 - 14Q + 12)$$

and the Kuhn-Tucker conditions are

$$\frac{\partial Z}{\partial Q} \leq 0, \quad Q \geq 0, \quad Q\frac{\partial Z}{\partial Q} = 0, \tag{7}$$

$$\frac{\partial Z}{\partial \lambda} \geq 0, \quad \lambda \geq 0, \quad \lambda\frac{\partial Z}{\partial \lambda} = 0 \tag{8}$$

Explicitly, the first inequalities in each row above are

$$-Q + 10 - \lambda(2Q - 7) \leq 0, \tag{9}$$

$$-Q^2 + 7Q - 6 \geq 0 \tag{10}$$

The second inequality in (7) and inequality (10) imply that $Q > 0$ *(if* Q *were* 0, *(10) would not be satisfied). From the third condition in (7) it follows that* $\frac{\partial Z}{\partial Q} = 0$.

If λ *were 0, the condition* $\frac{\partial Z}{\partial Q} = 0$ *would lead to* $Q = 10$, *which is not consistent with (10); thus, we must have* $\lambda > 0$ *and* $\frac{\partial Z}{\partial \lambda} = 0$.

The quadratic equation $-Q^2 + 7Q - 6 = 0$ *has the roots* $Q_1 = 1$ *and* $Q_2 = 6$. Q_1 *implies* $\lambda = -\frac{9}{5}$, *which contradicts* $\lambda > 0$. *The solution* $Q_2 = 6$ *leads to a positive value for* λ. *Thus, the sales-maximizing quantity is* $Q = 6$.

3.3 Appendix: Linear Programming

3.3.1 The Setup of the Problem

There are two general types of linear programs:

- the maximization program in n variables subject to $m+n$ constraints:

$$\begin{array}{rll} \text{Maximize} & \pi = \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq r_i & (i = 1, 2, \ldots m) \\ \text{and} & x_j \geq 0 & (j = 1, 2, \ldots n). \end{array}$$

- the minimization program in n variables subject to $m+n$ constraints:

$$\begin{array}{rll} \text{Minimize} & C = \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \geq r_i & (i = 1, 2, \ldots m) \\ \text{and} & x_j \geq 0 & (j = 1, 2, \ldots n). \end{array}$$

Note that without loss of generality all constraints in a linear program may be written as either $\leq$ inequalities or $\geq$ inequalities because any $\leq$ inequality can be transformed into a $\geq$ inequality by multiplying both sides by -1.

A more concise way to express a linear program is by using matrix notation:

- the maximization program in n variables subject to $m+n$ constraints:

$$\begin{array}{rl} \text{Maximize} & \pi = c'x \\ \text{subject to} & Ax \leq r \\ \text{and} & x \geq 0. \end{array}$$

- the minimization program in n variables subject to $m+n$ constraints:

$$\begin{array}{rl} \text{Minimize} & C = c'x \\ \text{subject to} & Ax \geq r \\ \text{and} & x \geq 0. \end{array}$$

where $c = (c_1, \ldots, c_n)'$, $x = (x_1, \ldots, x_n)'$, $r = (r_1, \ldots, r_m)'$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Example 74 *The following is an example of a linear program:*

$$\begin{array}{rl} \textit{Max} & \pi = 10x_1 + 30x_2 \\ \textit{subject to} & 5x_1 + 3x_2 \leq 40 \\ & x_1 \leq 5 \\ & x_2 \leq 10 \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

Proposition 32 (The Globality Theorem)

If the feasible set F of an optimization problem is a closed convex set, and if the objective function is a continuous concave (convex) function over the feasible set, then

i) any local maximum (minimum) will also be a global maximum (minimum) and

ii) the points in F at which the objective function is optimized will constitute a convex set.

Note that any linear program satisfies the assumptions of the globality theorem. Indeed, the objective function is linear; thus, it is both a concave and a convex function. On the other hand, each inequality restriction defines a closed halfspace, which is also a convex set. Since the intersection of a finite number of closed convex sets is also a closed convex set, it follows that the feasible set F is closed and convex.

The theorem says that for a linear program there is no difference between a local optimum and a global optimum, and if a pair of optimal solutions to a linear program exists, then any convex combination of the two must also be an optimal solution.

Definition 47 *An* extreme point *of a linear program is a point in its feasible set which cannot be derived from a convex combination of any other two points in the feasible set.*

Example 75 *For the problem in example 74 the extreme points are* $(0,0)$*,* $(0,10)$*,* $(2,10)$*,* $(5,5)$*,* $(5,0)$*. They are just the corners of the feasible region represented in the* $x_1 0 x_2$ *system of coordinates.*

Proposition 33 *The optimal solution of a linear program is an extreme point.*

3.3.2 The Simplex Method

The next question is to develop an algorithm which will enable us to solve a linear program. Such an algorithm is called *the simplex method*.

The simplex method is a systematic procedure for finding the optimal solution of a linear program. Loosely speaking, using this method we move from one extreme point of the feasible region to another until the optimal point is attained.

The following two examples illustrate how the simplex method can be applied:

Example 76 *Consider again the maximization problem in Example 74.*

Step 1. *Transform each inequality constraint into an equality constraint by inserting a dummy variable (slack variable) in each constraint.*

After applying this step, the problem becomes:

$$\begin{aligned} \textit{Max} \quad & \pi = 10x_1 + 30x_2 \\ \textit{subject to} \quad & 5x_1 + 3x_2 + d_1 = 40 \\ & x_1 + d_2 = 5 \\ & x_2 + d_3 = 10 \\ & x_1 \geq 0, x_2 \geq 0, d_1 \geq 0, d_2 \geq 0, d_3 \geq 0. \end{aligned}$$

The extreme points of the new feasible set are called basic feasible solutions *(BFS). For each extreme point of the original feasible set there is a BFS. For example, the BFS corresponding to the extreme point* $(x_1, x_2) = (0, 0)$ *is* $(x_1, x_2, d_1, d_2, d_3) = (0, 0, 40, 5, 10)$.

Step 2. *Find a BFS to be able to start the algorithm.*

A BFS is easy to find when $(0, 0)$ *is in the feasible set, as it is in our example. Here, the BFS are* $(0, 0, 40, 5, 10)$. *In general, finding a BFS requires the introduction of additional artificial variables. Example 77 illustrates this situation.*

Step 3. *Set up a simplex tableau like the following one:*

Row	π	x_1	x_2	d_1	d_2	d_3	*Constant*
0	*1*	*-10*	*-30*	*0*	*0*	*0*	*0*
1	*0*	*5*	*3*	*1*	*0*	*0*	*40*
2	*0*	*1*	*0*	*0*	*1*	*0*	*5*
3	*0*	*0*	*1*	*0*	*0*	*1*	*10*

Row 0 *contains the coefficients of the objective function. Rows* $1, 2, 3$ *include the coefficients of the constraints as they appear in the matrix of coefficients.*

In row 0 we can see the variables which have nonzero values in the current BFS. They correspond to the zeros in row 0, in this case, the nonzero variables are d_1, d_2, d_3. *These nonzero variables always correspond to a basis in* R^m, *where* m *is the number of constraints (see the general form of a linear program). The current basis is formed by the column coefficient vectors which appear in the tableau in rows* $1, 2, 3$ *under* d_1, d_2, d_3 *(*$m = 3$*).*

The rightmost column shows the nonzero values of the current BFS (in rows $1, 2, 3$*) and the value of the objective function for the current BFS (in row* 0*). That is,* $d_1 = 40$, $d_2 = 5$, $d_3 = 10$, $\pi = 0$.

Step 4. *Choose the pivot.*

The simplex method finds the optimal solution by moving from one BFS to another BFS until the optimum is reached. The new BFS is chosen such that the value of the objective function is improved (that is, it is larger for a maximization problem and lower for a minimization problem).

In order to switch to another BFS, we need to replace a variable currently in the basis by a variable which is not in the basis. This amounts to choosing a pivot element *which will indicate both the new variable and the exit variable.*

The pivot element is selected in the following way. The new variable (and, correspondingly, the pivot column) is that variable having the lowest negative value in row 0 *(largest positive value for a minimization problem). Once we have determined the pivot column, the pivot row is found in this way:*

- *pick the strictly positive elements in the pivot column except for the element in row* 0*;*
- *select the corresponding elements in the constant column and divide them by the strictly positive elements in the pivot column;*
- *the pivot row corresponds to the minimum of these ratios.*

In our example, the pivot column is the column of x_2 (since $-30 < -10$) and the pivot row is row 3 (since min$\{\frac{40}{3}, \frac{10}{1}\} = \frac{10}{1}$); thus, the pivot element is the 1 which lies at the intersection of column of x_2 and row 3.

Step 5. *Make the appropriate transformations to reduce the pivot column to a unit vector (that is, a vector with 1 in the place of the pivot element and 0 in the rest).*

Usually, first we have to divide the pivot row by the pivot element to obtain 1 in the place of the pivot element. However, this is not necessary in our case since we already have 1 in that position.

Then the new version of the pivot row is used to get zeros in the other places of the pivot column. In our example, we multiply the pivot row by 30 and add it to row 0; then we multiply the pivot row by -3 and add it to row 1.

Hence we obtain a new simplex tableau:

Row	π	x_1	x_2	d_1	d_2	d_3	Constant
0	1	-10	0	0	0	30	300
1	0	5	0	1	0	-3	10
2	0	1	0	0	1	0	5
3	0	0	1	0	0	1	10

Step 6. *Check that row 0 still contains negative values (positive values for a minimization problem). If it does, proceed to step 4. If it does not, the optimal solution must be found.*

In our example, we still have a negative value in row 0 (-10). According to step 4,the column of x_1 is the pivot column (x_1 enters the base). Since min$\{\frac{10}{5}, \frac{5}{1}\} = \frac{10}{5}$, it follows that row 1 is the pivot row. Thus, 5 in row 1 is the new pivot element. Applying step 5, we obtain a third simplex tableau:

Row	π	x_1	x_2	d_1	d_2	d_3	Constant
0	1	0	0	2	0	24	320
1	0	1	0	$\frac{1}{5}$	0	$\frac{-3}{5}$	2
2	0	0	0	$\frac{-1}{5}$	1	$\frac{3}{5}$	3
3	0	0	1	0	0	1	10

There is no negative element left in row 0 and the current basis consists of x_1, x_2, d_2. Thus, the optimal solution can be read from the rightmost column: $x_1 = 2$, $x_2 = 10$, $d_2 = 3$, $\pi = 320$. The correspondence between the optimal values and the variables in the current basis is made using the 1's in the column vectors of x_1, x_2, d_2.

Example 77 *Consider the minimization problem:*

$$\begin{aligned} \text{Min} \quad & C = x_1 + x_2 \\ \text{subject to} \quad & x_1 - x_2 \geq 0 \\ & -x_1 - x_2 \geq -6 \\ & x_2 \geq 1 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

After applying step 1 of the simplex algorithm, we obtain:

$$\begin{array}{rl} \textit{Min} & C = x_1 + x_2 \\ \textit{subject to} & x_1 - x_2 - d_1 = 0 \\ & -x_1 - x_2 - d_2 = -6 \\ & x_2 - d_3 = 1 \\ & x_1 \geq 0, x_2 \geq 0, d_1 \geq 0, d_2 \geq 0, d_3 \geq 0. \end{array}$$

In this example, $(0,0)$ *is not in the feasible set. One way of finding a starting BFS is to transform the problem by adding 3 more artificial variables* f_1, f_2, f_3 *to the linear program as follows:*

$$\begin{array}{rl} \textit{Min} & C = x_1 + x_2 + M(f_1 + f_2 + f_3) \\ \textit{subject to} & x_1 - x_2 - d_1 + f_1 = 0 \\ & x_1 + x_2 + d_2 + f_2 = 6 \\ & x_2 - d_3 + f_3 = 1 \\ & x_1 \geq 0, x_2 \geq 0, d_1 \geq 0, d_2 \geq 0, d_3 \geq 0 \\ & f_1 \geq 0, f_2 \geq 0, f_3 \geq 0. \end{array}$$

Before adding the artificial variables, the second constraint is multiplied by -1 to obtain a positive number on the right-hand side.

As long as M *is high enough, the optimal solution will contain* $f_1 = 0,\ f_2 = 0,\ f_3 = 0$ *and thus the optimum is not affected by including the artificial variables in the linear program. In our case, we choose* $M = 100$*. (In maximization problems* M *is chosen very low.)*

Now, the starting BFS is $(x_1, x_2, d_1, d_2, d_3, f_1, f_2, f_3) = (0,0,0,0,0,0,6,1)$ *and the first simplex tableau is:*

Row	C	x_1	x_2	d_1	d_2	d_3	f_1	f_2	f_3	*Constant*
0	1	-1	-1	0	0	0	-100	-100	-100	0
1	0	1	-1	-1	0	0	1	0	0	0
2	0	1	1	0	1	0	0	1	1	6
3	0	0	1	0	0	-1	0	0	0	1

In order to obtain unit vectors in the columns of f_1*,* f_2 *and* f_3*, we add 100(row1+row2+row3) to row0:*

Row	C	x_1	x_2	d_1	d_2	d_3	f_1	f_2	f_3	*Constant*
0	1	199	99	-100	100	-100	0	0	0	700
1	0	1	-1	-1	0	0	1	0	0	0
2	0	1	1	0	1	0	0	1	1	6
3	0	0	1	0	0	-1	0	0	0	1

Proceeding with the other steps of the simplex algorithm will finally lead to the optimal solution $x_1 = 1,\ x_2 = 1,\ C = 2$*.*

Sometimes the starting BFS (or the starting base) can be found in a simpler way than by using the general method that was just illustrated. For example, our linear program *can be written as:*

$$\begin{array}{rl} \textit{Min} & C = x_1 + x_2 \\ \textit{subject to} & -x_1 + x_2 + d_1 = 0 \\ & x_1 + x_2 + d_2 = 6 \\ & x_2 - d_3 = 1 \\ & x_1 \geq 0, x_2 \geq 0, d_1 \geq 0, d_2 \geq 0, d_3 \geq 0. \end{array}$$

Taking $x_1 = 0$, $x_2 = 0$ *we get* $d_1 = 0$, $d_2 = 6$, $d_3 = -1$. *Only* d_3 *contradicts the constraint* $d_3 \geq 0$. *Thus, we need to add only one artificial variable* f *to the linear program and the variables in the starting base will be* d_1, d_2, f.

The complete sequence of simplex tableaux leading to the optimal solution is given below.

Row	C	x_1	x_2	d_1	d_2	d_3	f	*Constant*
0	1	-1	-1	0	0	0	-100	0
1	0	-1	1	1	0	0	0	0
2	0	1	1	0	1	0	0	6
3	0	0	1	0	0	-1	1	1
0	1	-1	99	0	0	-100	0	100
1	0	-1	1*	1	0	0	0	0
2	0	1	1	0	1	0	0	6
3	0	0	1	0	0	-1	1	1
0	1	98	0	-99	0	-100	0	100
1	0	-1	1	1	0	0	0	0
2	0	2	0	-1	1	0	0	6
3	0	1*	0	-1	0	-1	1	1
0	1	0	0	-1	0	-2	-98	2
1	0	0	1	0	0	-1	1	1
2	0	0	0	1	1	2	-2	4
3	0	1	0	-1	0	-1	1	1

(The asterisks mark the pivot elements.)

As can be seen from the last tableau, the optimum is $x_1 = 1$, $x_2 = 1$, $d_2 = 4$, $C = 2$.

Economics Application 9 (Economic Interpretation of a Maximization Program)

The maximization program in the general form is the production problem facing a firm which has to produce n *goods such that it maximizes its revenue subject to* m *resource (factor) constraints.*

The variables have the following economic interpretations:

- x_j *is the amount produced of the* j*th product;*
- r_i *is the amount of the* i*th resource available;*
- a_{ij} *is the amount of the* i*th resource used in producing a unit of the* j*th product;*
- c_j *is the price of a unit of product* j.

The optimal solution to the maximization program indicates the optimal quantities of each good the firm should produce.

Example 78 *Let us find an economic interpretation of the maximization program given in Example 74:*

$$\begin{aligned} \textit{Max} \quad & \pi = 10x_1 + 30x_2 \\ \textit{subject to} \quad & 5x_1 + 3x_2 \leq 40 \\ & x_1 \leq 5 \\ & x_2 \leq 10 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

A firm has to produce two goods using three kinds of resources available in the amounts 40,5,10 respectively. The first resource is used in the production of both goods: five units are necessary to produce one unit of good 1, and three units to produce 1 unit of good 2. The second resource is used only in producing good 1 and the third resource is used only in producing good 2.

The question is, how much of each good should the firm produce in order to maximize its revenue if the sale prices of goods are 10 and 30, respectively? The solution $(2, 10)$ *gives the optimal amounts the firm should produce.*

Economics Application 10 (Economic Interpretation of a Minimization Program)

The minimization program (see the general form) is also connected to the firm's production problem. This time the firm wants to produce a fixed combination of m goods using n resources (factors). The problem lies in choosing the amount of each resource such that the total cost of all resources used in the production of the fixed combination of goods is minimized.

The variables have the following economic interpretations:

- x_j *is the amount of the* j*th resource used in the production process;*
- r_i *is the amount of the good* i *the firm decides to produce;*
- a_{ij} *is the marginal productivity of resource* j *in producing good* i*;*
- c_j *is the price of a unit of resource* j*.*

The optimal solution to the minimization program indicates the optimal quantities of each factor the firm should employ.

Example 79 *Consider again the minimization program in example 77:*

$$\begin{aligned} \textit{Min} \quad & C = x_1 + x_2 \\ \textit{subject to} \quad & x_1 - x_2 \geq 0 \\ & x_1 + x_2 \leq 6 \\ & x_2 \geq 1 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

What economic interpretation can be assigned to this linear program?

A firm has to produce two goods, at most six units of the first good and one unit of the second good. Two factors are used in the production process and the price of both factors is one. The marginal productivity of both factors in producing good 1 is one and the marginal productivity of the factors in producing good 2 are zero and one, respectively (good 2 uses only the second factor). The production process also requires that the amount of the second factor not exceed the amount of the first factor.

The question is how much of each factor the firm should employ in order to minimize the total factor cost. The solution $(1, 1)$ *gives the optimal factor amounts.*

3.3.3 Duality

As a matter of fact, the maximization and the minimization programs are closely related. This relationship is called *duality*.

Definition 48 *The* dual (program) *of the maximization program*

$$\begin{aligned} \textit{Maximize} \quad & \pi = c'x \\ \textit{subject to} \quad & Ax \leq r \\ \textit{and} \quad & x \geq 0. \end{aligned}$$

is the minimization program

$$\begin{aligned} \textit{Minimize} \quad & \pi^* = r'y \\ \textit{subject to} \quad & A'y \geq c \\ \textit{and} \quad & y \geq 0. \end{aligned}$$

Definition 49 *The* dual (program) *of the minimization program*

$$\begin{aligned} \textit{Minimize} \quad & C = c'x \\ \textit{subject to} \quad & Ax \geq r \\ \textit{and} \quad & x \geq 0. \end{aligned}$$

is the maximization program

$$\begin{aligned} \textit{Maximize} \quad & C^* = r'y \\ \textit{subject to} \quad & A'y \leq c \\ \textit{and} \quad & y \geq 0. \end{aligned}$$

Definition 50 *The original program from which the dual is derived is called* the primal (program).

Example 80 *The dual of the maximization program in Example 74 is*

$$\begin{aligned} \textit{Min} \quad & C^* = 40y_1 + 5y_2 + 10y_3 \\ \textit{subject to} \quad & 5y_1 + y_2 \geq 10 \\ & 3y_1 + y_3 \geq 30 \\ & y_1 \geq 0, y_2 \geq 0, y_3 \geq 0. \end{aligned}$$

Example 81 *The dual of the minimization program in Example 77 is*

$$\begin{aligned} \textit{Max} \quad & \pi^* = -6y_2 + y_3 \\ \textit{subject to} \quad & y_1 - y_2 \leq 1 \\ & -y_1 - y_2 + y_3 \leq 1 \\ & y_1 \geq 0, y_2 \geq 0, y_3 \geq 0. \end{aligned}$$

The following duality theorems clarify the relationship between the dual and the primal.

Proposition 34 *The optimal values of the primal and the dual objective functions are always identical, if the optimal solutions exist.*

Proposition 35

1. *If a certain choice variable in a primal (dual) program is optimally nonzero then the corresponding dummy variable in the dual (primal) must be optimally zero.*

2. *If a certain dummy variable in a primal (dual) program is optimally nonzero then the corresponding choice variable in the dual (primal) must be optimally zero.*

The last simplex tableau of a linear program offers not only the optimal solution to that program but also the optimal solution to the dual program. ⚠

Recipe 14 – How to solve the dual program:
The optimal solution to the dual program can be read from row 0 of the last simplex tableau of the primal in the following way:

1. *the absolute values of the elements in the columns corresponding to the primal dummy variables are the values of the dual choice variables.*

2. *the absolute values of the elements in the columns corresponding to the primal choice variables are the values of the dual dummy variables.*

Example 82 *From the last simplex tableau in Example 74 we can infer the optimal solution to the problem in Example 80. It is* $y_1 = 2$, $y_2 = 0$, $y_3 = 24$, $\pi^* = 320$.

Example 83 *The last simplex tableau in Example 77 implies that the optimal solution to the problem in Example 81 is* $y_1 = 1$, $y_2 = 0$, $y_3 = 2$, $C = 2$.

Further reading:

- Intriligator, M. *Mathematical Optimization and Economic Theory.*
- Luenberger, D.G. *Introduction to Linear and Nonlinear Programming.*

4 Dynamics

4.1 Differential Equations

Definition 51 *An equation* $F(x, y, y', \ldots, y^{(n)}) = 0$ *which assumes a functional relationship between independent variable* x *and dependent variable* y *and includes* x, y *and derivatives of* y *with respect to* x *is called an ordinary differential equation of order* n.

The order of a differential equation is determined by the order of the highest derivative in the equation.

A function $y = y(x)$ *is said to be a solution of this differential equation, in some open interval* I, *if* $F(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0$ *for all* $x \in I$.

4.1.1 Differential Equations of the First Order

Let us consider the first order differential equations, $y' = f(x, y)$.

The solution $y(x)$ is said to solve Cauchy's problem (or the initial value problem) with the initial values (x_0, y_0), if $y(x_0) = y_0$.

Proposition 36 (The Existence Theorem)

If f *is continuous in an open domain* D, *then for any given pair* $(x_0, y_0) \in D$ *there exists a solution* $y(x)$ *of* $y' = f(x, y)$ *such that* $y(x_0) = y_0$.

Proposition 37 (The Uniqueness Theorem)

If f *and* $\frac{\partial f}{\partial y}$ *are continuous in an open domain* D, *then given any* $(x_0, y_0) \in D$ *there exists a unique solution* $y(x)$ *of* $y' = f(x, y)$ *such that* $y(x_0) = y_0$.

Definition 52 *A differential equation of the form* $y' = f(y)$ *is said to be autonomous (i.e.* y' *is determined by* y *alone).*

Note that the differential equation gives us the slope of the solution curves at all points of the region D. Thus, in particular, the solution curves cross the curve $f(x, y) = k$ (k is a constant) with slope k. This curve is called the *isocline* of slope k. If we draw the set of isoclines obtained by taking different real values of k, we can schematically sketch the family of solution curves in the $(x - y)$-plane.

Recipe 15 – How to Solve a Differential Equation of the First Order:

There is no general method of solving differential equations. However, in some cases the solution can be easily obtained:

- **The Variables Separable Case.**

  If $y' = f(x)g(y)$ *then* $\frac{dy}{g(y)} = f(x)dx$. *By integrating both parts of the latter equation, we will get a solution.*

  Example 84 *Solve Cauchy's problem* $(x^2 + 1)y' + 2xy^2 = 0$, *given* $y(0) = 1$.

  If we rearrange the terms, this equation becomes equivalent to

$$\frac{dy}{y^2} = -\frac{2xdx}{x^2 + 1}.$$

The integration of both parts yields

$$\frac{1}{y} = \ln(x^2+1) + C,$$

or

$$y(x) = \frac{1}{\ln(x^2+1)+C}.$$

Since we should satisfy the condition $y(0) = 1$, we can evaluate the constant C, $C = 1$.

- **Differential Equations with Homogeneous Coefficients.**

Definition 53 *A function $f(x,y)$ is called homogeneous of degree n, if for any λ $f(\lambda x, \lambda y) = \lambda^n f(x,y)$.*

If we have the differential equation $M(x,y)dx + N(x,y)dy = 0$ and $M(x,y)$, $N(x,y)$ are homogeneous of the same degree, then the change of variable $y = tx$ reduces this equation to the variables separable case.

Example 85 *Solve $(y^2 - 2xy)dx + x^2dy = 0$.*
The coefficients are homogeneous of degree 2. Note that $y \equiv 0$ is a special solution. If $y \neq 0$, let us change the variable: $y = tx$, $dy = tdx + xdt$. Thus

$$x^2(t^2 - 2t)dx + x^2(tdx + xdt) = 0.$$

Dividing by x^2 (since $y \neq 0$, $x \neq 0$ as well) and rearranging the terms, we get a variables separable equation

$$\frac{dx}{x} = -\frac{dt}{t^2 - t}.$$

Taking the integrals of both parts,

$$\int \frac{dx}{x} = \ln|x|, \quad \int -\frac{dt}{t^2-t} = \int \left(\frac{1}{t} - \frac{1}{t-1}\right) dt = \ln|t| - \ln|t-1| + C,$$

therefore

$$x = \frac{Ct}{t-1}, \text{ or, in terms of } y,\ x = \frac{Cy}{y-x}.$$

We can single out y as a function of x, $y = \dfrac{x^2}{x-C}$.

- **Exact differential equations.**

  If we have the differential equation $M(x,y)dx + N(x,y)dy = 0$ *and* $\frac{\partial M(x,y)}{\partial y} \equiv \frac{\partial N(x,y)}{\partial x}$ *then all the solutions of this equation are given by* $F(x,y) = C$, C *is a constant, where* $F(x,y)$ *is such that* $\frac{\partial F}{\partial x} = M(x,y)$, $\frac{\partial F}{\partial y} = N(x,y)$.

  Example 86 *Solve* $\dfrac{y}{x}dx + (y^3 + \ln x)dy = 0$.

  $\dfrac{\partial}{\partial y}\left(\dfrac{y}{x}\right) = \dfrac{\partial}{\partial x}(y^3 + \ln x)$, *so we have checked that we are dealing with an exact differential equation. Therefore, we need to find* $F(x,y)$ *such that*

  $$F'_x = \frac{y}{x} \Longrightarrow F(x,y) = \int \frac{y}{x}dx = y\ln x + \phi(y),$$

  and

  $$F'_y = y^3 + \ln x.$$

  To find $\phi(y)$, *note that*

  $$F'_y = y^3 + \ln x = \ln x + \phi'(y) \Longrightarrow \phi(y) = \int y^3 dy = \frac{1}{4}y^4 + C.$$

  Therefore, all solutions are given by the implicit function $4y\ln x + y^4 = C$.

- **Linear differential equations of the first order.**

  If we need to solve $y' + p(x)y = q(x)$, *then all the solutions are given by the formula*

  $$y(x) = e^{-\int p(x)dx}\left(C + \int q(x)e^{\int p(x)dx}dx\right).$$

  Example 87 *Solve* $xy' - 2y = 2x^4$.

  In other words, we need to solve $y' - \dfrac{2}{x}y = 2x^3$. *Therefore,*

  $$y(x) = e^{\int \frac{2}{x}dx}\left(C + \int 2x^3 e^{-\int \frac{2}{x}dx} sd\right) = e^{\ln x^2}\left(C + \int 2x^3 e^{\ln \frac{1}{x^2}} sd\right) = x^2(C + x^2).$$

4.1.2 Linear Differential Equations of a Higher Order with Constant Coefficients

These are the equations of the form

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}y' + a_n y = f(x).$$

If $f(x) \equiv 0$ the equation is called *homogeneous*, otherwise it is called *non-homogeneous*.

Recipe 16 How to find the general solution $y_g(x)$ of the homogeneous equation: *The general solution is a sum of basic solutions $y_1, \dots, y_n$,*

$$y_g(x) = C_1 y_1(x) + \ldots + C_n y_n(x), \qquad C_1, \dots, C_n \text{ are arbitrary constants.}$$

These arbitrary constants, however, can be defined in a unique way, once we set up the initial value Cauchy problem. Find $y(x)$ such that

$$y(x) = y_{0_0}, y'(x) = y_{0_1}, \dots, y^{(n-1)}(x) = y_{0_{n-1}} \text{ at } x = x_0,$$

where $x_0, y_{0_0}, y_{0_1}, \dots, y_{0_{n-1}}$ are given initial values (real numbers).

To find all basic solutions, we proceed as follows:

1. *Solve the characteristic equation*

$$\lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$$

for λ. The roots of this equation are $\lambda_1, \dots, \lambda_n$. Some of these roots may be complex numbers. We may also have repeated roots.

2. *If λ_i is a simple real root, the basic solution corresponding to this root is $y_i(x) = e^{\lambda_i x}$.*

3. *If λ_i is a repeated real root of degree k, it generates k basic solutions*

$$y_{i_1}(x) = e^{\lambda_i x},\ y_{i_2}(x) = xe^{\lambda_i x},\ \dots,\ y_{i_k}(x) = x^{k-1} e^{\lambda_i x}.$$

4. *If λ_j is a simple complex root, $\lambda_j = \alpha_j + i\beta_j$, $i = \sqrt{-1}$, then the complex conjugate of λ_j, say $\lambda_{j+1} = \alpha_j - i\beta_j$, is also a root of the characteristic equations. Therefore the pair λ_j, λ_{j+1} give rise to two basic solutions:*

$$y_{j_1} = e^{\alpha_j x}\cos\beta_j x, \quad y_{j_2} = e^{\alpha_j x}\sin\beta_j x.$$

5. *If λ_j is a repeated complex root of degree l, $\lambda_j = \alpha_j + i\beta_j$, then the complex conjugate of λ_j, say $\lambda_{j+1} = \alpha_j - i\beta_j$, is also a repeated root of degree l. These $2l$ roots generate $2l$ basic solutions*

$$y_{j_1} = e^{\alpha_j x}\cos\beta_j x,\ y_{j_2} = xe^{\alpha_j x}\cos\beta_j x,\ \dots,\ y_{j_l} = x^{l-1}e^{\alpha_j x}\cos\beta_j x,$$

$$y_{j_{l+1}} = e^{\alpha_j x}\sin\beta_j x,\ y_{j_{l+2}} = xe^{\alpha_j x}\sin\beta_j x,\ \dots,\ y_{j_{2l}} = x^{l-1}e^{\alpha_j x}\sin\beta_j x.$$

Recipe 17 – How to Solve the Non-Homogeneous Equation:

The general solution of the nonhomogeneous solution is $y_{nh}(x) = y_g(x) + y_p(x)$, where $y_g(x)$ is the general solution of the homogeneous equation and $y_p(x)$ is a particular solution of the nonhomogeneous equation, i.e. any function which solves the nonhomogeneous equation.

Recipe 18 – How to Find a Particular Solution of the Non-Homogeneous Equation:

1. *If $f(x) = P_k(x)e^{bx}$, $P_k(x)$ is a polynomial of degree k, then a particular solution is*

$$y_p(x) = x^s Q_k(x) e^{bx},$$

where $Q_k(x)$ is a polynomial of the same degree k. If b is not a root of the characteristic equation, $s = 0$; if b is a root of the characteristic polynomial of degree m then $s = m$.

2. *If $f(x) = P_k(x)e^{px}\cos qx + Q_k(x)e^{px}\sin qx$, $P_k(x), Q_k(x)$ are polynomials of degree k, then a particular solution can be found in the form*

$$y_p(x) = x^s R_k(x) e^{px}\cos qx + x^s T_k(x) e^{px}\sin qx,$$

where $R_k(x), T_k(x)$ are polynomials of degree k. If $p + iq$ is not a root of the characteristic equation, $s = 0$; if $p + iq$ is a root of the characteristic polynomial of degree m then $s = m$.

3. *The general method for finding a particular solution of a nonhomogeneous equation is called the* variation of parameters *or the* method of undetermined coefficients.

   Suppose we have the general solution of the homogeneous equation

$$y_g = C_1 y_1(x) + \ldots + C_n y_n(x),$$

where $y_i(x)$ are basic solutions. Treating constants $C_1, \ldots, C_n$ as functions of x, for instance, $u_1(x), \ldots, u_n(x)$, we can express a particular solution of the nonhomogeneous equation as

$$y_p(x) = u_1(x) y_1(x) + \ldots + u_n(x) y_n(x),$$

where $u_1(x). \ldots, u_n(x)$ are solutions of the system

$$\begin{aligned} u_1'(x)y_1(x) + \ldots + u_n'(x)y_n(x) &= 0, \\ u_1'(x)y_1'(x) + \ldots + u_n'(x)y_n'(x) &= 0, \\ \ldots\ \ldots\ \ldots \\ u_1'(x)y_1^{(n-2)}(x) + \ldots + u_n'(x)y_n^{(n-2)}(x) &= 0, \\ u_1'(x)y_1^{(n-1)}(x) + \ldots + u_n'(x)y_n^{(n-1)}(x) &= f(x) \end{aligned}$$

4. *If $f(x) = f_1(x) + f_2(x) + \ldots + f_r(x)$ and $y_{p1}(x), \ldots, y_{pr}(x)$ are particular solutions corresponding to $f_1(x), \ldots, f_r(x)$ respectively, then*

$$y_p(x) = y_{p1}(x) + \ldots + y_{pr}(x).$$

Example 88 *Solve* $y'' - 5y' + 6y = x^2 + e^x - 5$.

The characteristic roots are $\lambda_1 = 2$, $\lambda_2 = 3$, *therefore the general solution of the homogeneous equation is*

$$y(t) = C_1 e^{2t} + C_2 e^{3t}.$$

We search for a particular solution in the form

$$y_p(t) = at^2 + bt + c + de^t.$$

To find coefficients a, b, c, d, *let us substitute the particular solution into the initial equation:*

$$2a + de^t - 5(2at + b + de^t) + 6(at^2 + bt + c + de^t) = t^2 - 5 + e^t.$$

Equating term-by-term the coefficients at the left-hand side to the right-hand side, we obtain
$6a = 1, \quad -5 \cdot 2a + 6 \cdot b = 0, \quad 2a - 5b + 6c = -5, \quad d - 5d + 6d = 1.$
Thus $d = 1/2,\ a = 1/6, \quad b = 10/36, \quad c = -71/108.$

Finally, the general solution of the nonhomogeneous equation is

$$y_{nh}(x) = y(t) = C_1 e^{2t} + C_2 e^{3t} + \frac{x^2}{6} + \frac{5x}{18} - \frac{71}{108} + \frac{e^x}{2}.$$

Example 89 *Solve* $y'' - 2y' + y = \dfrac{e^x}{x}$.

The general solution of the homogeneous equation is $y_g(x) = C_1 x e^x + C_2 e^x$ *(the characteristic equation has the repeated root* $\lambda = 1$ *of degree 2). We look for a particular solution of the nonhomogeneous equation in the form*

$$y_p(x) = u_1(x) x e^x + u_2(x) e^x.$$

We have

$$\begin{aligned} u_1'(x) x e^x + u_2'(x) e^x &= 0, \\ u_1'(x)(e^x + x e^x) + u_2'(x) e^x &= \frac{e^x}{x}. \end{aligned}$$

Therefore $u_1'(x) = \frac{1}{x}, u_2'(x) = -1$. *Integrating, we find that* $u_1(x) = \ln|x|,\ u_2(x) = -x$. *Thus* $y_p(x) = \ln|x| x e^x + x e^x$, *and*

$$y_n(x) = y_g(x) + y_p(x) = \ln|x| x e^x + C_1 x e^x + C_2 e^x.$$

4.1.3 Systems of the First Order Linear Differential Equations

The general form of a system of the first order differential equations is:

$$\dot{x}(t) = A(t)x(t) + b(t), \quad x(0) = x_0$$

where t is the independent variable ("time"), $x(t) = (x_1(t), \ldots, x_n(t))'$ is a vector of dependent variables, $A(t) = (a_{ij}(t))_{[n \times n]}$ is a real $n \times n$ matrix with variable coefficients, $b(t) = (b_1(t), \ldots, b_n(t))'$ a variant n-vector.

In what follows, we concentrate on the case when A and b do not depend on t, which is the case in constant coefficients differential equations systems:

$$\dot{x}(t) = Ax(t) + b, \quad x(0) = x_0. \tag{11}$$

We also assume that A is non-singular.

The solution of system (11) can be determined in two steps:

- First, we consider the homogeneous system (corresponding to $b = 0$):

  $$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \tag{12}$$

  The solution $x_c(t)$ of this system is called the complementary solution.

- Second, we find a particular solution x_p of the system (11), which is called the particular integral. The constant vector x_p is simply the solution to $Ax_p = -b$, i.e. $x_p = -A^{-1}b$. [5]

Given $x_c(t)$ and x_p, the general solution of the system (11) is simply:

$$x(t) = x_c(t) + x_p$$

We can find the solution to the homogeneous system (12) in two different ways.

First, we can eliminate $n-1$ unknowns and reduce the system to one linear differential equation of degree n.

Example 90 *Let*

$$\begin{cases} \dot{x} = 2x + y, \\ \dot{y} = 3x + 4y. \end{cases}$$

Taking the derivative of the first equation and eliminating y and $\dot{y}$ ($\dot{y} = 3x + 4y = 3x + 4\dot{x} - 4 \cdot 2x$), we arrive at the second order homogeneous linear differential equation

$$\ddot{x} - 6\dot{x} + 5x = 0,$$

which has the general solution $x(t) = C_1e^t + C_2e^{5t}$. Since $y(t) = \dot{x} - 2x$, we find that $y(t) = -C_1e^t + 3C_2e^{5t}$.

The second way is to write the solution to (12) as

$$x(t) = e^{At}x_0$$

where (by definition)

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$$

[5] In a sense, x_p can be viewed as an affine shift, which restores the origin to a unique equilibrium of (11).

Unfortunately, this formula is of little practical use. In order to find a feasible formula for e^{At}, we distinguish three main cases.

Case 1: A has real and distinct eigenvalues

The fact that the eigenvalues are real and distinct implies that any corresponding eigenvectors are linearly independent. Consequently, A is diagonalizable, i.e.

$$A = P\Lambda P^{-1}$$

where $P = [v_1, v_2, \ldots, v_n]$ is a matrix composed by the eigenvectors of A and Λ is a diagonal matrix whose diagonal elements are the eigenvalues of A. Therefore, $e^A = Pe^{\Lambda}P^{-1}$.

Thus, the solution of the system (12) can be written as:

$$\begin{aligned} x(t) &= Pe^{\Lambda t}P^{-1}x_0 \\ &= Pe^{\Lambda t}c \\ &= c_1v_1e^{\lambda_1 t} + \ldots + c_nv_ne^{\lambda_n t} \end{aligned}$$

where $c = (c_1, c_2, \ldots, c_n)$ is a vector of constants determined from the initial conditions ($c = P^{-1}x_0$).

Case 2: A has real and repeated eigenvalues

First, consider the simpler case when A has only one eigenvalue λ which is repeated m times (m is called the algebraic multiplicity of λ). Generally, in this situation the maximum number of independent eigenvectors corresponding to λ is less than m, meaning that we cannot construct the matrix P of linearly independent eigenvectors and, therefore, A is not diagonalizable.[6]

The solution in this case has the form:

$$x(t) = \sum_{i=1}^{m} c_ih_i(t)$$

where $h_i(t)$ are quasipolinomials and c_i are constants determined by the initial conditions. For example, if $m = 3$, we have:

$$\begin{aligned} h_1(t) &= e^{\lambda t}v_1 \\ h_2(t) &= e^{\lambda t}(tv_1 + v_2) \\ h_3(t) &= e^{\lambda t}(t^2v_1 + 2tv_2 + 3v_3) \end{aligned}$$

where v_1, v_2, v_3 are determined by the conditions:

$$(A - \lambda I)v_i = v_{i-1}, v_0 = 0;$$

If A happens to have several eigenvalues which are repeated, the solution of (12) is obtained by finding the solution corresponding to each eigenvalue, and then adding up the individual solutions.

[6]If A is a real and symmetric matrix, then A is always diagonalizable and the formula given for case 1 can be applied.

Case 3: A has complex eigenvalues.

Since A is a real matrix, the complex eigenvalues always appear in pairs, i.e. if A has the eigenvalue $\alpha + \beta i$, then it also must have the eigenvalue $\alpha - \beta i$.

Now consider the simpler case when A has only one pair of complex eigenvalues, $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$. Let v_1 and v_2 denote the eigenvectors corresponding to λ_1 and λ_2; then $v_2 = \bar{v_1}$, where $\bar{v_1}$ is the conjugate of v_1. The solution can be expressed as:

$$
\begin{aligned}
x(t) &= e^{At}x_0 \\
&= Pe^{\Lambda t}P^{-1}x_0 \\
&= Pe^{\Lambda t}c \\
&= c_1 v_1 e^{(\alpha+\beta i)t} + c_2 v_2 e^{(\alpha-\beta i)t} \\
&= c_1 v_1 e^{\alpha t}(\cos\beta t + i\sin\beta t) + c_2 v_2 e^{\alpha t}(\cos\beta t - i\sin\beta t) \\
&= (c_1 v_1 + c_2 v_2)e^{\alpha t}\cos\beta t + i(c_1 v_1 - c_2 v_2)e^{\alpha t}\sin\beta t \\
&= h_1 e^{\alpha t}\cos\beta t + h_2 e^{\alpha t}\sin\beta t,
\end{aligned}
$$

where $h_1 = c_1 v_1 + c_2 v_2$, $h_2 = i(c_1 v_1 - c_2 v_2)$ are real vectors.

Note that if A has more pairs of complex eigenvalues, then the solution of (12) is obtained by finding the solution corresponding to each eigenvalue, and then adding up the individual solutions. The same remark holds true when we encounter any combination of the three cases discussed above.

Example 91 *Consider*

$$A = \begin{pmatrix} 5 & 2 \\ -4 & -1 \end{pmatrix}.$$

Solving the characteristic equation $det(A - \lambda I) = \lambda^2 - 4\lambda + 3 = 0$ *we find eigenvalues of* A*:* $\lambda_1 = 1$, $\lambda_2 = 3$.

From the condition $(A - \lambda_1 I)v_1 = 0$ *we find an eigenvector corresponding to* λ_1*:* $v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Similarly, the condition $(A - \lambda_2 I)v_2 = 0$ *gives an eigenvector corresponding to* λ_2*:* $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Therefore, the general solution of the system $\dot{x}(t) = Ax(t)$ *is*

$$x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

or

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -c_1 e^t - c_2 e^{3t} \\ 2c_1 e^t + c_2 e^{3t} \end{pmatrix}.$$

Example 92 *Consider*

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}.$$

det$(A - \lambda I) = \lambda^2 - 4\lambda + 4 = 0$ *gives the eigenvalue* $\lambda = 2$ *repeated twice.*
From the condition $(A - \lambda_1 I)v_1 = 0$ *we find* $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
and from $(A - \lambda_1 I)v_2 = v_1$ *we obtain* $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.
The solution of the system $\dot{x}(t) = Ax(t)$ *is*

$$\begin{aligned} x(t) &= c_1 h_1(t) + c_2 h_2(t) \\ &= c_1 v_1 e^{\lambda t} + c_2(tv_1 + v_2)e^{\lambda t} \end{aligned}$$

or

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -c_1 + c_2 - c_2 t \\ c_1 - 2c_2 + c_2 t \end{pmatrix} e^{2t}.$$

Example 93 *Consider*

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}.$$

The equation det$(A - \lambda I) = \lambda^2 - 4\lambda + 5 = 0$ *leads to the complex eigenvalues* $\lambda_1 = 2 + i$, $\lambda_2 = 2 - i$.
For λ_1 *we set the equation* $(A - \lambda_1 I)v_1 = 0$ *and obtain:*

$$v_1 = \begin{pmatrix} 1 + i \\ -1 \end{pmatrix} v_2 = \bar{v_1} = \begin{pmatrix} 1 - i \\ -1 \end{pmatrix}.$$

The solution of the system $\dot{x}(t) = Ax(t)$ *is*

$$\begin{aligned} x(t) &= c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} \\ &= c_1 v_1 e^{(2+i)t} + c_2(tv_1 + v_2)e^{(2-i)t} \\ &= c_1 v_1 e^{2t}(\cos t + i \sin t) + c_2 v_2 e^{2t}(\cos t - i \sin t). \end{aligned}$$

Performing the computations, we finally obtain:

$$x(t) = (c_1 + c_2)e^{2t} \begin{pmatrix} \cos t - \sin t \\ -\cos t \end{pmatrix} + i(c_1 - c_2)e^{2t} \begin{pmatrix} \cos t + \sin t \\ -\sin t \end{pmatrix}.$$

Now let the vector b be different from 0. To specify the solution to the system (11), we need a particular solution x_p. This is found from the condition $\dot{x} = 0$; thus, $x_p = -A^{-1}b$. Therefore, the general solution of the non-homogeneous system is

$$x(t) = x_c(t) + x_p = Pe^{\Lambda t}P^{-1}c - A^{-1}b$$

where the vector of constants c is determined by the initial condition $x(0) = x_0$ as $c = x_0 + A^{-1}b$.

Example 94 *Consider*

$$A = \begin{pmatrix} 5 & 2 \\ -4 & -1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } x_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

In Example 91 we obtained

$$x_c(t) = \begin{pmatrix} -c_1 e^t - c_2 e^{3t} \\ 2c_1 e^t + c_2 e^{3t} \end{pmatrix}.$$

In addition, we have

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = x_0 + A^{-1}b = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{-5}{3} \\ \frac{14}{3} \end{pmatrix} = \begin{pmatrix} \frac{-8}{3} \\ \frac{20}{3} \end{pmatrix}$$

and

$$-x_p = A^{-1}b = \begin{pmatrix} \frac{-1}{3} & \frac{-2}{3} \\ \frac{4}{3} & \frac{5}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{-5}{3} \\ \frac{14}{3} \end{pmatrix}.$$

The solution of the system $\dot{x}(t) = Ax(t) + b$ is

$$x(t) = x_c(t) + x_p = \begin{pmatrix} \frac{8}{3}e^t - \frac{20}{3}e^{3t} + \frac{5}{3} \\ \frac{-16}{3}e^t + \frac{20}{3}e^{3t} + \frac{-14}{3} \end{pmatrix}.$$

Economics Application 11 (General Equilibrium)

Consider a market for n goods $x = (x_1, x_2, \ldots, x_n)'$. Let $p = (p_1, p_2, \ldots, p_n)'$ denote the vector of prices and $D_i(p), S_i(p)$ demand and supply for good $i, i = 1, \ldots, n$. The general equilibrium is obtained for a price vector p^* which satisfies the conditions: $D_i(p^*) = S_i(p^*)$ for all $i = 1, \ldots, n$.

The dynamics of price adjustment for the general equilibrium can be described by a system of first order differential equations of the form:

$$\dot{p} = W[D(p) - S(p)] = WAp, \qquad p(0) = p_0$$

where $W = diag(w_i)$ is a diagonal matrix having the speeds of adjustment $w_i, i = 1, \ldots, n$, $D(p) = [D_1(p), \ldots, D_n(p)]'$ on the diagonal and $S(p) = [S_1(p), \ldots, S_n(p)]'$ are linear functions of p and A is a constant $n \times n$ matrix.

We can solve this system by applying the methods discussed above. For instance, if $WA = P\Lambda P^{-1}$, then the solution is

$$p(t) = Pe^{\Lambda t}P^{-1}p_0.$$

Economics Application 12 (The Dynamic IS-LM Model)

A simplified dynamic IS-LM model can be specified by the following system of differential equations:

$$\begin{aligned} \dot{Y} &= w_1(I - S) \\ \dot{r} &= w_2[L(Y, r) - M], \end{aligned}$$

with initial conditions

$$Y(0) = Y_0, r(0) = r_0,$$

where:

- *Y is national income;*
- *r is the interest rate;*
- *$I = I_0 - \alpha r$ is the investment function (α is a positive constant);*
- *$S = s(Y-T)+(T-G)$ is national savings; (T, G and s represents taxes, government purchases and the savings rate);*
- *$L(Y,r) = a_1 Y - a_2 r$ is money demand (a_1 and a_2 are positive constants);*
- *M is money supply;*
- *w_1, w_2 are speeds of adjustment.*

For simplicity, consider $w_1 = 1$ and $w_2 = 1$. The original system can be re-written as:

$$\begin{pmatrix} \dot{Y} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} -s & -\alpha \\ a_1 & -a_2 \end{pmatrix} \begin{pmatrix} Y \\ r \end{pmatrix} + \begin{pmatrix} I_0 - (1-s)T + G \\ -M \end{pmatrix}$$

Assume that $A = \begin{pmatrix} -s & -\alpha \\ a_1 & -a_2 \end{pmatrix}$ is diagonalizable, i.e. $A = P\Lambda P^{-1}$, and has distinct eigenvalues λ_1, λ_2 with corresponding eigenvectors v_1, v_2. (λ_1, λ_2 are the solutions of the equation $r^2 + (s+a_2)r + sa_2 + \alpha a_1 = 0$.)

The solution becomes:

$$\begin{pmatrix} Y(t) \\ r(t) \end{pmatrix} = Pe^{\Lambda t}P^{-1}(x_0 + A^{-1}b) - A^{-1}b$$

where

$$x_0 = \begin{pmatrix} Y_0 \\ r_0 \end{pmatrix}, b = \left(\begin{matrix} I_0 - (1-s)T + G \\ -M \end{matrix} \right].$$

We have

$$A^{-1} = \frac{1}{\alpha a_1 + s a_2} \begin{pmatrix} -a_2 & \alpha \\ -a_1 & -s \end{pmatrix}$$

and

$$A^{-1}b = \frac{1}{\alpha a_1 + s a_2} \begin{pmatrix} -a_2[I_0 - (1-s)T + G] - \alpha M \\ -a_1[I_0 - (1-s)T + G] + sM \end{pmatrix}$$

Assume that $P^{-1} = \begin{pmatrix} \tilde{v_1}' \\ \tilde{v_2}' \end{pmatrix}$. Then, we have:

$$Pe^{\Lambda t}P^{-1} = [v_1, v_2] \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} \tilde{v_1}' \\ \tilde{v_2}' \end{pmatrix} = v_1 \tilde{v_1}' e^{\lambda_1 t} + v_2 \tilde{v_2}' e^{\lambda_2 t}.$$

Finally, the solution is:

$$\begin{pmatrix} Y(t) \\ r(t) \end{pmatrix} = [v_1 \tilde{v_1}' e^{\lambda_1 t} + v_2 \tilde{v_2}' e^{\lambda_2 t}] \begin{pmatrix} Y_0 + \frac{-a_2 S - \alpha M}{\alpha a_1 + s a_2} \\ r_0 + \frac{-a_1 S + s M}{\alpha a_1 + s a_2} \end{pmatrix} - \begin{pmatrix} \frac{-a_2 S - \alpha M}{\alpha a_1 + s a_2} \\ \frac{-a_1 S + s M}{\alpha a_1 + s a_2} \end{pmatrix},$$

where $S = I_0 - (1-s)T + G$.

Economics Application 13 (The Solow Model)

The Solow model is the basic model of economic growth and it relies on the following key differential equation, which explains the dynamics of capital per unit of effective labor:

$$\dot{k}(t) = sf(k(t)) - (n + g + \delta)k(t), \qquad k(0) = k_0$$

where

- k *denotes capital per effective labor;*
- s *is the savings rate;*
- n *is the rate of population growth;*
- g *is the rate of technological progress;*
- δ *is the depreciation rate.*

Let us consider two types of production function.

Case 1. *The production function is linear:* $f(k(t)) = ak(t)$ *where* a *is a positive constant.*

The equation for $\dot{k}$ *becomes:*

$$\dot{k}(t) = (sa - n - g - \delta)k(t).$$

This linear differential equation has the solution

$$k(t) = k(0)e^{(sa-n-g-\delta)t} = k_0 e^{(sa-n-g-\delta)t}.$$

Case 2. *The production function takes the Cobb-Douglas form:* $f(k(t)) = [k(t)]^\alpha$ *where* $\alpha \in [0, 1]$.

In this case, the equation for $\dot{k}$ *represents a Bernoulli nonlinear differential equation:*

$$\dot{k}(t) + (n + g + \delta)k(t) = s[k(t)]^\alpha, \qquad k(0) = k_0$$

Dividing both sides of the latter equation by $[k(t)]^\alpha$*, and denoting* $m(t) = [k(t)]^{1-\alpha}$, $m(0) = (k_0)^{1-\alpha}$*, we obtain:*

$$\frac{1}{1-\alpha}\dot{m}(t) + (n + g + \delta)m(t) = s,$$

or

$$\dot{m}(t) + (1 - \alpha)(n + g + \delta)m(t) = s(1 - \alpha).$$

The solution to this first order linear differential equation is given by:

$$\begin{aligned} m(t) &= e^{-\int(1-\alpha)(n+g+\delta)dt}[A + \int s(1-\alpha)e^{\int(1-\alpha)(n+g+\delta)dt}dt] \\ &= e^{-(1-\alpha)(n+g+\delta)t}[A + s(1-\alpha)\int e^{(1-\alpha)(n+g+\delta)t}dt] \\ &= Ae^{-(1-\alpha)(n+g+\delta)t} + \frac{s}{n+g+\delta}. \end{aligned}$$

where A is determined from the initial condition $m(0) = (k_0)^{1-\alpha}$. It follows that $A = (k_0)^{1-\alpha} - \frac{s}{n+g+\delta}$ and

$$m(t) = [(k_0)^{1-\alpha} - \frac{s}{n+g+\delta}]e^{-(1-\alpha)(n+g+\delta)t} + \frac{s}{n+g+\delta},$$

$$k(t) = \{[(k_0)^{1-\alpha} - \frac{s}{n+g+\delta}]e^{-(1-\alpha)(n+g+\delta)t} + \frac{s}{n+g+\delta}\}^{\frac{1}{1-\alpha}}.$$

Economics Application 14 (The Ramsey-Cass-Koopmans Model)

This model takes capital per effective labor $k(t)$ and consumption per effective labor $c(t)$ as endogenous variables. Their behavior is described by the following system of differential equations:

$$\begin{aligned}\dot{k}(t) &= f(k(t)) - c(t) - (n+g)k(t)\\ \dot{c}(t) &= \frac{f'(k(t)) - \rho - \theta g}{\theta}c(t)\end{aligned}$$

with the initial conditions

$$k(0) = k_0, c(0) = c_0,$$

where, in addition to the variables already specified in the previous application,

- *ρ stands for the discount rate ;*
- *θ is the coefficient of relative risk aversion.*

Assume again that $f(k(t)) = ak(t)$ where $a > 0$ is a constant. The initial system reduces to a system of linear differential equations:

$$\begin{aligned}\dot{k}(t) &= (a-n-g)k(t) - c(t)\\ \dot{c}(t) &= \frac{a - \rho - \theta g}{\theta}c(t)\end{aligned}$$

The second equation implies that:

$$c(t) = c_0 e^{bt}, \text{ where } b = \frac{a-\rho-\theta g}{\theta}.$$

Substituting for c(t) in the first equation, we obtain:

$$\dot{k}(t) = (a-n-g)k(t) - c_0 e^{bt}$$

The solution of this linear differential equation takes the form:

$$k(t) = k_c(t) + k_p(t)$$

where k_c and k_p are the complementary solution and the particular integral, respectively.

The homogeneous equation $\dot{k}(t) = (a-n-g)k(t)$ gives the complementary solution:

$$k(t) = k_0 e^{(a-n-g)t}.$$

The particular integral should be of the form $k_p(t) = me^{bt}$; the value of m can be determined by substituting k_p in the non-homogeneous equation for $\dot{k}$:

$$mbe^{bt} = (a-n-g)me^{bt} - c_0 e^{bt}$$

It follows that

$$m = \frac{c_0}{a-n-g-b}, \qquad k_p(t) = \frac{c_0}{a-n-g-b} e^{bt}$$

and

$$k(t) = k_0 e^{(a-n-g)t} + \frac{c_0}{a-n-g-b} e^{bt}.$$

4.1.4 Simultaneous Differential Equations. Types of Equilibria

Let $x = x(t),\ y = y(t)$, where t is independent variable. Then consider a two-dimensional autonomous system of simultaneous differential equations

$$\begin{cases} \dfrac{dx}{dt} = f(x, y), \\ \dfrac{dy}{dt} = g(x, y). \end{cases}$$

The point (x^*, y^*) is called an equilibrium of this system if $f(x^*, y^*) = g(x^*, y^*) = 0$.

Let $\mathcal{J}$ be the Jacobian matrix

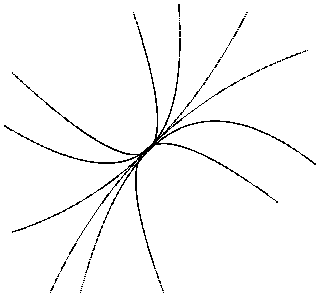
$$\mathcal{J} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

evaluated at (x^*, y^*), and let $\lambda_1,\ \lambda_2$ be the eigenvalues of this Jacobian.

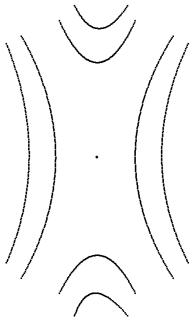
Then the equilibrium is

1. a stable (unstable) node if $\lambda_1,\ \lambda_2$ are real, distinct and both negative (positive);
2. a saddle if the eigenvalues are real and of different signs, i.e. $\lambda_1\lambda_2 < 0$;
3. a stable (unstable) focus if $\lambda_1,\ \lambda_2$ are complex and $\mathrm{Re}(\lambda_1) < 0$ ($\mathrm{Re}(\lambda_1) > 0$);
4. a center or a focus if $\lambda_1,\ \lambda_2$ are complex and $\mathrm{Re}(\lambda_1) = 0$;
5. a stable (unstable) improper node if $\lambda_1,\ \lambda_2$ are real, $\lambda_1 = \lambda_2 < 0$ ($\lambda_1 = \lambda_2 > 0$) and the Jacobian is a not diagonal matrix;
6. a stable (unstable) star node if $\lambda_1,\ \lambda_2$ are real, $\lambda_1 = \lambda_2 < 0$ ($\lambda_1 = \lambda_2 > 0$) and the Jacobian is a diagonal matrix.

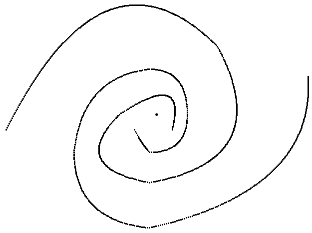
The figure below illustrates this classification:



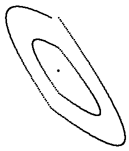
1) Node



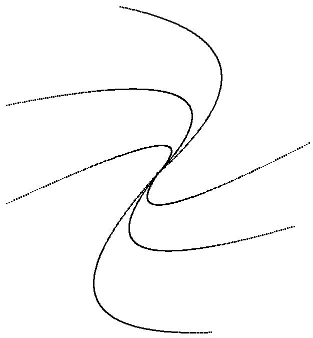
2) Saddle



3) Focus



4) Center



5) Improper node

6) Star node

Example 95 *Find the equilibria of the system and classify them*

$$\begin{cases} \dot{x} = 2xy - 4y - 8, \\ \dot{y} = 4y^2 - x^2. \end{cases}$$

Solving the system

$$2xy - 4y - 8 = 0,$$
$$4y^2 - x^2 = 0$$

for $x,\ y$*, we find two equilibria,* $(-2,-1)$ *and* $(4,2)$*.*

At $(-2,-1)$ *the Jacobian*

$$J = \begin{pmatrix} 2y & 2x-4 \\ -2x & 8y \end{pmatrix}_{(-2,-1)} = \begin{pmatrix} -2 & -8 \\ 4 & -8 \end{pmatrix}.$$

The characteristic equation for J *is* $\lambda^2 - (tr(J))\lambda + \det(J) = 0$*.*

$(tr(J))^2 - 4\det(J)) < 0 \implies \lambda_{1,2}$ *are complex.*

tr$(J) < 0 \Longrightarrow \lambda_1 + \lambda_2 < 0 \Longrightarrow Re\lambda_1 < 0 \Longrightarrow$ *the equilibrium* $(-2, -1)$ *is a stable focus.*

At $(4, 2)$

$$J = \begin{pmatrix} 4 & 4 \\ -8 & 16 \end{pmatrix}.$$

$(tr(J))^2 - 4\det(J)) > 0 \implies \lambda_{1,2}$ *are distinct and real.*

$tr(J) = \lambda_1 + \lambda_2 > 0,\ \det(J) = \lambda_1\lambda_2 > 0 \Longrightarrow \lambda_1 > 0,\ \lambda_2 > 0 \Longrightarrow$ *the equilibrium* $(4, 2)$ *is an unstable node.*

4.2 Difference Equations

Let y denote a real-valued function defined over the set of natural numbers. Hereafter y_k stands for $y(k)$ – the value of y at k, where $k = 0, 1, 2, \ldots$.

Definition 54

- *The* first difference *of* y *evaluated at* k *is* $\Delta y_k = y_{k+1} - y_k$.
- *The second difference of* y *evaluated at* k *is* $\Delta^2 y_k = \Delta(\Delta y_k) = y_{k+2} - 2y_{k+1} + y_k$.
- *In general, the* n*th difference of* y *evaluated at* k *is defined as* $\Delta^n y_k = \Delta(\Delta^{n-1} y_k)$, *for any integer* $n,\ n > 1$.

Definition 55 *A* difference equation *in the unknown* y *is an equation relating* y *and any of its differences* $\Delta y, \Delta^2 y, \ldots$ *for each value of* k, $k = 0, 1, \ldots$.

Solving a difference equation means finding all functions y which satisfy the relation specified by the equation.

Example 96 (Examples of Difference Equations)

- $2\Delta y_k - y_k = 1$ *(or, equivalently,* $2\Delta y - y = 1$*)*
- $\Delta^2 y_k + 5y_k \Delta y_k + (y_k)^2 = 2$ *(or* $\Delta^2 y + 5y\Delta y + (y)^2 = 2$*)*

Note that any difference equation can be written as

$$f(y_{k+n}, y_{k+n-1}, \ldots, y_{k+1}, y_k) = g(k),\ n \in \mathbf{N}$$

For example, the above two equations may also be expressed as follows:

- $2y_{k+1} - 3y_k = 1$
- $y_{k+2} - 2y_{k+1} + y_k + 5y_k y_{k+1} - 4(y_k)^2 = 2$

Definition 56 *A difference equation is* linear *if it takes the form*

$$f_0(k)y_{k+n} + f_1(k)y_{k+n-1} + \ldots + f_{n-1}(k)y_{k+1} + f_n(k)y_k = g(k),$$

where $f_0, f_1, \ldots, f_n, g$ *are each functions of* k *(but not of some* $y_{k+i},\ i \in \mathbf{N}$*) defined for all* $k = 0, 1, 2, \ldots$.

Definition 57 *A linear difference equation is* of order n *if both* $f_0(k)$ *and* $f_n(k)$ *are different from zero for each* $k = 0, 1, 2, \ldots$.

From now on, we focus on the problem of solving linear difference equations of order n with constant coefficients. The general form of such an equation is:

$$f_0 y_{k+n} + f_1 y_{k+n-1} + \ldots + f_{n-1} y_{k+1} + f_n y_k = g_k,$$

where, this time, $f_0, f_1, \ldots, f_n$ are real constants and f_0, f_n are different from zero.

Dividing this equation by f_0 and denoting $a_i = \frac{f_i}{f_0}$ for $i = 0, \ldots, n$, $r_k = \frac{g_k}{f_0}$, the general form of a linear difference equation of order n with constant coefficients becomes:

$$y_{k+n} + a_1 y_{k+n-1} + \ldots + a_{n-1} y_{k+1} + a_n y_k = r_k, \tag{13}$$

where a_n is non-zero.

Recipe 19 – How to Solve Difference Equation: General Procedure:

The general procedure for solving a linear difference equation of order n with constant coefficients involves three steps:

Step 1. Consider the homogeneous equation

$$y_{k+n} + a_1 y_{k+n-1} + \ldots + a_{n-1} y_{k+1} + a_n y_k = 0,$$

and find its solution Y. (First we will describe and illustrate the solving procedure for equations of order 1 *and* 2*. Following that, the theory for the case of order n is obtained as a natural extension of the theory for second-order equations.)*

Step 2. Find a particular solution y^* of the complete equation (13). The most useful technique for finding particular solutions is the method of undetermined coefficients which will be illustrated in the examples that follow.

Step 3. The general solution of equation (13) is:

$$y_k = Y + y^*.$$

4.2.1 First-order Linear Difference Equations

The general form of a first-order linear differential equation is:

$$y_{k+1} + a y_k = r_k.$$

The corresponding homogeneous equation is:

$$y_{k+1} + a y_k = 0,$$

and has the solution $Y = C(-a)^k$, where C is an arbitrary constant.

Recipe 20 – How to Solve a First-Order Non-Homogeneous Linear Difference Equation:

First, consider the case when r_k is constant over time, i.e. $r_k \equiv r$ for all k.

Much as in the case of linear differential equations, the general solution of the non-homogeneous difference equation y_t is the sum of the general solution of the homogeneous equation and a particular solution of the nonhomogeneous equation. Therefore

$$y_k = A \cdot (-a)^k + \frac{r}{1+a}, \quad a \neq -1,$$

$$y_k = A \cdot (-a)^k + rk = A + rk, \quad a = -1,$$

where A is an arbitrary constant.

If we have the initial condition $y_k = y_0$ when $k = 0$, the general solution of the nonhomogeneous equation takes the form

$$y_k = \left(y_0 - \frac{r}{1+a}\right) \times (-a)^k + \frac{r}{1+a}, \quad a \neq -1,$$

$$y_k = y_0 + rk, \quad a = -1.$$

If r now depends on k, $r = r_k$, then the solution takes the form

$$y_k = (-a)^k y_0 + \sum_{i=0}^{k-1} (-a)^{k-1-i} r_i, \quad k = 1, 2, \ldots$$

A particular solution to non-homogeneous difference equation can be found using the method of undetermined coefficients.

Example 97 $y_{k+1} - 3y_k = 2$.

The solution of the homogeneous equation $y_{k+1} - 3y_k = 0$ is: $Y = C \cdot 3^k$. The right-hand term suggests looking for a constant function as a particular solution. Assuming $y^* = A$ and substituting into the initial equation, we obtain $A = -1$; hence $y^* = -1$ and the general solution is $y_k = Y + y^* = C3^k - 1$.

Example 98 $y_{k+1} - 3y_k = k^2 + k + 2$.

The homogeneous equation is the same as in the preceding example. But the right-hand term suggests finding a particular solution of the form:

$$y^* = Ak^2 + Bk + D.$$

Substituting y^* into non-homogeneous equation, we have:

$$A(k+1)^2 + B(k+1) + D - 3Ak^2 - 3Bk - 3D = k^2 + k + 2, \text{ or}$$

$$-2Ak^2 + 2(A-B)k + A + B - 2D = k^2 + k + 2.$$

In order to satisfy this equality for each k, we must have:

$$\begin{cases} -2A = 1 \\ 2(A-B) = 1 \\ A + B - 2D = 2. \end{cases}$$

It follows that $A = -\frac{1}{2}, B = -1, D = -\frac{3}{4}$ and $y^* = -\frac{1}{2}k^2 - k - \frac{3}{4}$. The general solution is $y_k = Y + y^* = C3^k - \frac{1}{2}k^2 - k - \frac{3}{4}$.

Example 99 $y_{k+1} - 3y_k = 4e^k$.

This time we try a particular solution of the form: $y^* = Ae^k$. *After substitution, we find* $A = \frac{4}{e-3}$. *The general solution is* $y_k = Y + y^* = C3^k + \frac{4e^k}{e-3}$.

Two remarks:

- The method of undetermined coefficients illustrated by these examples applies to the general case of order n as well. The trial solutions corresponding to some simple functions r_k are given in the following table:

Right-hand term r_k	Trial solution y^*
a^k	Aa^k
k^n	$A_0 + A_1k + A_2k^2 + \ldots + A_nk^n$
$\sin ak$ or $\cos ak$	$A\sin ak + B\cos ak$

- If the function r is a combination of simple functions for which we know the trial solutions, then a trial solution for r can be obtained by combining the trial solutions for the simple functions. For example, if $r_k = 5^k k^3$, then the trial solution would be $y^* = 5^k(A_0 + A_1k + A_2k^2 + A_3k^3)$.

4.2.2 Second-Order Linear Difference Equations

The general form is:

$$y_{k+2} + a_1y_{k+1} + a_2y_k = r_k. \tag{14}$$

The solution to the corresponding homogeneous equation

$$y_{k+2} + a_1y_{k+1} + a_2y_k = 0 \tag{15}$$

depends on the solutions to the quadratic equation:

$$m^2 + a_1m + a_2 = 0,$$

which is called the auxiliary (or characteristic) equation of equation (14). Let m_1, m_2 denote the roots of the characteristic equation. Then, both m_1 and m_2 will be non-zero because a_2 is non-zero for (14) to be of second order.

Case 1. m_1 and m_2 are real and distinct.
The solution to (15) is $Y = C_1m_1^k + C_2m_2^k$, where C_1, C_2 are arbitrary constants.
Case 2. m_1 and m_2 are real and equal.
The solution to (15) is $Y = C_1m_1^k + C_2km_1^k = (C_1 + C_2k)m_1^k$.
Case 3. m_1 and m_2 are complex with the polar forms $r(\cos\theta \pm i\sin\theta)$, $r > 0$, $\theta \in (-\pi, \pi]$.
The solution to (15) is $Y = C_1r^k\cos(k\theta + C_2)$.

Any complex number $a + bi$ admits a representation in the polar (or trigonometric) form $r(\cos\theta \pm i\sin\theta)$. The transformation is given by

- $r = \sqrt{a^2 + b^2}$
- θ is the unique angle (in radians) such that $\theta \in (-\pi, \pi]$ and

$$\cos\theta = \frac{a}{\sqrt{a^2 + b^2}}\ ,\ \sin\theta = \frac{b}{\sqrt{a^2 + b^2}}$$

Example 100 $y_{k+2} - 5y_{k+1} + 6y_k = 0$

The characteristic equation is $m^2 - 5m + 6 = 0$ *and has the roots* $m_1 = 2, m_2 = 3$. *The difference equation has the solution* $y_k = C_1 2^k + C_2 3^k$.

Example 101 $y_{k+2} - 8y_{k+1} + 16y_k = 2^k + 3$

The characteristic equation $m^2 - 8m + 16 = 0$ *has the roots* $m_1 = m_2 = 4$*; therefore, the solution to the corresponding homogeneous equation is* $Y = (C_1 + C_2 k)4^k$.

To find a particular solution, we consider the trial solution $y^* = A2^k + B$. *By substituting* y^* *into the non-homogeneous equation and performing the computations we obtain* $A = \frac{1}{4}$, $B = \frac{1}{3}$. *Thus, the solution to the non-homogeneous equation is* $y_k = (C_1 + C_2 k)4^k + \frac{1}{4}2^k + \frac{1}{3}$.

Example 102 $y_{k+2} + y_{k+1} + y_k = 0$

The equation $m^2 + m + 1 = 0$ *gives the roots* $m_1 = \frac{-1-i\sqrt{3}}{2}$, $m_2 = \frac{-1+i\sqrt{3}}{2}$ *or, written in the polar form,* $m_{1,2} = \cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3}$. *Therefore, the given difference equation has the solution:* $y^* = C_1\cos(\frac{2\pi}{3}k + C_2)$.

4.2.3 The General Case of Order n

In general, any difference equation of order n can be written as:

$$y_{k+n} + a_1 y_{k+n-1} + \ldots + a_{n-1}y_{k+1} + a_n y_k = r_k. \qquad (14)$$

The corresponding characteristic equation becomes

$$m^n + a_1 m^{n-1} + \ldots + a_{n-1}m + a_n = 0$$

and has exactly n roots denoted as $m_1, \ldots, m_n$.

The general solution to the homogeneous equation is a sum of terms which are produced by the roots $m_1, \ldots, m_n$ in the following way:

- a real simple root m generates the term $C_1 m^k$.
- a real and repeated p times root m generates the term

  $$(C_1 + C_2 k + C_3 k^2 + \ldots + C_p k^{p-1})m^k.$$

- each pair of simple complex conjugate roots $r(\cos\theta \pm i\sin\theta)$ generates the term

  $$C_1 r^k \cos(k\theta + C_2).$$

- each repeated p times pair of complex conjugate roots $r(\cos\theta \pm i\sin\theta)$ generates the term

  $$r^k[C_{1,1}\cos(k\theta + C_{1,2}) + C_{2,1}k\cos(k\theta + C_{2,2}) + \ldots + C_{p,1}k^{p-1}\cos(k\theta + C_{p,2})].$$

The sum of terms will contain n arbitrary constants.

A particular solution y^* to the non-homogeneous equation again can be obtained by means of the method of undetermined coefficients (as was illustrated for the cases of first-order and second-order difference equations).

Example 103 $y_{k+4} - 5y_{k+3} + 9y_{k+2} - 7y_{k+1} + 2y_k = 0$

The characteristic equation

$$m^4 - 5m^3 + 9m^2 - 7m + 2 = 0$$

has the roots $m_1 = 1$, *which is repeated 3 times, and* $m_2 = 2$. *The solution is* $y^* = (C_1 + C_2 k + C_3 k^2) + C_4 2^k$.

Economics Application 15 (Simple and compound interest)

Let S_0 *denote an initial sum of money. There are two basic methods for computing the interest earned in a period, for example, one year.*

S_0 *earns* simple interest *at rate* r *if each period the interest equals a fraction* r *of* S_0. *If* S_k *denotes the sum accumulated after* k *periods,* S_k *is computed in the following way:*

$$S_{k+1} = S_k + rS_0.$$

This is a first-order difference equation which has the solution $S_k = S_0(1 + kr)$.

S_0 *earns* compound interest *at rate* r *if each period the interest equals a fraction* r *of the sum accumulated at the beginning of that period. In this case, the computation formula is:*

$$S_{k+1} = S_k + rS_k \text{ or } S_{k+1} = S_k(1 + r).$$

The solution to this homogeneous first-order difference equation is $S_k = (1 + r)^k S_0$.

Economics Application 16 (A dynamic model of economic growth)

Let Y_t, C_t, I_t *denote national income, consumption, and investment in period* t, *respectively. A simple dynamic model of economic growth is described by the equations:*

$$\begin{aligned} Y_t &= C_t + I_t \\ C_t &= c + mY_t \\ Y_{t+1} &= Y_t + rI_t \end{aligned}$$

where

- c *is a positive constant;*
- m *is the marginal propensity to consume,* $0 < m < 1$;
- r *is the growth factor.*

From the above system of three equations, we can express the dynamics of national income in the form of a first-order difference equation:

$$\begin{aligned} Y_{t+1} - Y_t &= rI_t \\ &= r(Y_t - C_t) \\ &= rY_t - r(c + mY_t), \end{aligned}$$

or

$$Y_{t+1} - [1 + r(1 - m)]Y_t = -rc.$$

The solution to this equation is

$$Y_t = (Y_0 - \frac{c}{1-m})[1 + r(1-m)]^t + \frac{c}{1-m}.$$

The behavior of investment can be described in the following way:

$$\begin{aligned} I_{t+1} - I_t &= (Y_{t+1} - C_{t+1}) - (Y_t - C_t) \\ &= (Y_{t+1} - Y_t) - (C_{t+1}) - C_t) \\ &= rI_t - m(Y_{t+1} - Y_t) \\ &= rI_t - mrI_t, \end{aligned}$$

or

$$I_{t+1} = [1 + r(1-m)]I_t.$$

This homogeneous difference equation has the solution $I_t = [1 + r(1-m)]^t I_0$.

4.3 Introduction to Dynamic Optimization

4.3.1 The First-Order Conditions

A typical dynamic optimization problem takes the following form:

$$\begin{array}{rrcl} \max_{c(t)} & V & = & \int_0^T f[k(t), c(t), t]dt \\ \text{subject to} & \dot{k}(t) & = & g[k(t), c(t), t], \\ & k(0) & = & k_0 > 0. \end{array}$$

where

- $[0, T]$ is the horizon over which the problem is considered (T can be finite or infinite);
- V is the value of the objective function as seen from the initial moment $t_0 = 0$;
- $c(t)$ is the *control variable* and the objective function V is maximized with respect to this variable;
- $k(t)$ is the *state variable* and the first constraint (called the *equation of motion*) describes the evolution of this state variable over time ($\dot{k}$ denotes $\frac{dk}{dt}$);

Similar to the static optimization, in order to solve the optimization program we need first to find the first-order conditions:

Recipe 21 – How to derive the First-Order Conditions:

Step 1. Construct the Hamiltonian function

$$H = f(k, c, t) + \lambda(t)g(k, c, t)$$

where $\lambda(t)$ is a Lagrange multiplier.

Step 2. Take the derivative of the Hamiltonian with respect to the control variable and set it to 0:

$$\frac{\partial H}{\partial c} = \frac{\partial f}{\partial c} + \lambda \frac{\partial g}{\partial c} = 0.$$

Step 3. Take the derivative of the Hamiltonian with respect to the state variable and set it to equal the negative of the derivative of the Lagrange multiplier with respect to time:

$$\frac{\partial H}{\partial k} = \frac{\partial f}{\partial k} + \lambda \frac{\partial g}{\partial k} = -\dot{\lambda}.$$

Step 4. Transversality condition

Case 1: *Finite horizons.*

$$\mu(T)k(T) = 0.$$

Case 2: *Infinite horizons with f of the form $f[k(t), c(t), t] = e^{-\rho t}u[k(t), c(t)]$:*

$$\lim_{t \to \infty}[\lambda(t)k(t)] = 0.$$

Case 3: *Infinite horizons with f in a form different from that specified in case 2.*

$$\lim_{t \to \infty}[H(t)] = 0.$$

Recipe 22 – The F.O.C. with More than One State and/or Control Variable:

It may happen that the dynamic optimization problem contains more than one control variable and more than one state variable. In that case we need an equation of motion for each state variable. To write the first-order conditions, the algorithm specified above should be modified in the following way:

Step 1a. The Hamiltonian includes the right-hand side of each equation of motion times the corresponding multiplier.

Step 2a. Applies for each control variable.

Step 3a. Applies for each state variable.

4.3.2 Present-Value and Current-Value Hamiltonians

In economic problems, the objective function is usually of the form

$$f[k(t), c(t), t] = e^{-\rho t} u[k(t), c(t)],$$

where ρ is a constant discount rate and $e^{-\rho t}$ is a discount factor. The Hamiltonian

$$H = e^{-\rho t} u(k, c) + \lambda(t) g(k, c, t)$$

is called the *present-value Hamiltonian* (since it represents a value at the time $t_0 = 0$).

Sometimes it is more convenient to work with the *current-value Hamiltonian*, which represents a value at the time t:

$$\hat{H} = He^{\rho t} = u(k, c) + \mu(t) g(k, c, t)$$

where $\mu(t) = \lambda(t) e^{\rho t}$. The first-order conditions written in terms of the current-value Hamiltonian appear to be a slight modification of the present-value type:

- $\frac{\partial \hat{H}}{\partial c} = 0$,
- $\frac{\partial \hat{H}}{\partial k} = \rho \mu - \dot{\mu}$,
- $\mu(T) k(T) = 0$ (The transversality condition in the case of finite horizons).

4.3.3 Dynamic Problems with Inequality Constraints

Let us add an inequality constraint to the dynamic optimization problem considered above.

$$\begin{array}{rlcl} \max_{c(t)} & V & = & \int_0^T f[k(t), c(t), t] dt \\ \text{subject to} & \dot{k}(t) & = & g[k(t), c(t), t], \\ & h(t, k, c) & \geq & 0, \\ & k(0) & = & k_0 > 0. \end{array}$$

The following algorithm allows us to write the first-order conditions:

Recipe 23 – The F.O.C. with Inequality Constraints:

Step 1. Construct the Hamiltonian

$$H = f(k, c, t) + \mu(t) g(k, c, t) + w(t) h(t, k, c)$$

where $\mu(t)$ and $w(t)$ are Lagrange multipliers.

Step 2. $$\frac{\partial H}{\partial c} = 0.$$

Step 3. $$\frac{\partial H}{\partial k} = -\dot{\mu}.$$

Step 4. $$w(t) \geq 0, \qquad w(t) h(t, k, c) = 0$$

Step 5. Transversality condition if T is finite:

$$\mu(T)k(T) = 0.$$

Economics Application 17 (The Ramsey Model)

This example illustrates how the tools of dynamic optimization can be applied to solve a model of economic growth, namely the Ramsey model (for more details on the economic illustrations presented in this section see, for instance, R. J. Barro, X. Sala-i-Martin Economic Growth*, McGraw-Hill, Inc., 1995).*

The model assumes that the economy consists of households and firms but here we further assume, for simplicity, that households carry out production directly. Each household chooses, at each moment t, the level of consumption that maximizes the present value of lifetime utility

$$U = \int_0^\infty u[A(t)c(t)]e^{nt}e^{-\rho t}dt$$

subject to the dynamic budget constraint

$$\dot{k}(t) = f[k(t)] - c(t) - (x + n + \delta)k(t).$$

and the initial level of capital $k(0) = k_0$.

Throughout the example we use the following notation:

- *$A(t)$ is the level of technology;*
- *c denotes consumption per unit of effective labor (effective labor is defined as the product of labor force $L(t)$ and the level of technology $A(t)$);*
- *k denotes the amount of capital per unit of effective labor;*
- *$f(\cdot)$ is the production function written in intensive form ($f(k) = F(K/AL, 1)$);*
- *n is the rate of population growth;*
- *ρ is the rate at which households discount future utility;*
- *x is the growth rate of technological progress;*
- *δ is the depreciation rate.*

The expression $A(t)c(t)$ in the utility function represents consumption per person and the term e^{nt} captures the effect of the growth in the family size at rate n. The dynamic constraint says that the change in the stock of capital (expressed in units of effective labor) is given by the difference between the level of output (in units of effective labor) and the sum of consumption (per unit of effective labor), depreciation and the additional effect $(x + n)k(t)$ from expressing all variables in terms of effective labor.

As can be observed, households face a dynamic optimization problem with c as a control variable and k as a state variable. The present-value Hamiltonian is

$$H = u(A(t)c(t))e^{nt}e^{-\rho t} + \mu(t)[f[k(t)] - c(t) - (x + n + \delta)k(t)]$$

and the first-order conditions state that

$$\begin{aligned}\frac{\partial H}{\partial c} &= \frac{\partial u(Ac)}{\partial c} e^{nt} e^{-\rho t} = \mu \\ \frac{\partial H}{\partial k} &= \mu[f'(k) - (x + n + \delta)] = -\dot{\mu}\end{aligned}$$

In addition the transversality condition requires:

$$\lim_{t \to \infty} [\mu(t)k(t)] = 0.$$

Eliminating μ from the first-order conditions and assuming that the utility function takes the functional form $u(Ac) = \frac{(Ac)^{(1-\theta)}}{1-\theta}$ leads us to an expression[7] *for the growth rate of c:*

$$\frac{\dot{c}}{c} = \frac{1}{\theta}[f'(k) - \delta - \rho - \theta x]$$

This equation, together with the dynamic budget constraint forms a system of differential equations which completely describe the dynamics of the Ramsey model. The boundary conditions are given by the initial condition k_0 and the transversality condition.

Economics Application 18 (A Model of Economic Growth with Physical and Human Capital)

This model illustrates the case when the optimization problem has two dynamic constraints and two control variables.

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[7] To re-construct these calculations, note first that

$$\frac{\partial u(Ac)}{\partial c} = A^{1-\theta} c^{\theta}$$

and

$$\frac{d}{dt} \frac{\partial u(Ac)}{\partial c} = \left((1-\theta)\frac{\dot{A}}{A} - \theta \frac{\dot{c}}{c} \right) \cdot \frac{\partial u(Ac)}{\partial c}$$

or, assuming that $\frac{\dot{A}}{A} = x$,

$$\frac{d}{dt} \frac{\partial u(Ac)}{\partial c} = \left((1-\theta)x - \theta \frac{\dot{c}}{c} \right) \cdot \frac{\partial u(Ac)}{\partial c}.$$

Therefore,

$$\dot{\mu} = \frac{\partial u(Ac)}{\partial c} e^{nt} e^{-\rho t} \cdot \left(n - \rho + (1-\theta)x - \theta \frac{\dot{c}}{c} \right).$$

Eliminating μ from the second equation in the F.O.C. we get

$$\theta \frac{\dot{c}}{c} - (1-\theta)x - n + \rho = f'(k) - x - n - \delta,$$

which after rearranging terms gives an expression for the growth rate of consumption c.

As was the case in the previous example, we assume that households perform production directly. The model also assumes that physical capital K and human capital H enter the production function Y in a Cobb-Douglas manner:

$$Y = AK^{\alpha}H^{1-\alpha},$$

where A denotes a constant level of technology and $0 \le \alpha \le 1$. Output is divided among consumption C, investment in physical capital I_K and investment in human capital I_H:

$$Y = C + I_K + I_H.$$

The two capital stocks change according to the dynamic equations:

$$\begin{aligned} \dot{K} &= I_K - \delta K, \\ \dot{H} &= I_H - \delta H, \end{aligned}$$

where δ denotes the depreciation rate.

The households' problem is to choose C, I_K, I_H such that they maximize the present-value of lifetime utility

$$U = \int_0^{\infty} u(C)e^{-\rho t}dt$$

subject to the above four constraints. We again assume that $u(C) = \frac{C^{(1-\theta)}}{1-\theta}$. By eliminating Y and I_K, the households' problem becomes

$$\begin{array}{rl} \text{Maximize} & U \\ \text{subject to} & \dot{K} = AK^{\alpha}H^{1-\alpha} - C - \delta K - I_H, \\ & \dot{H} = I_H - \delta H, \end{array}$$

and it contains two control variables (C, I_H) and two state variables (K, H).

The Hamiltonian associated with this dynamic problem is:

$$\mathcal{H} = u(C) + \mu_1(AK^{\alpha}H^{1-\alpha} - C - \delta K - I_H) + \mu_2(I_H - \delta H)$$

and the first-order conditions require

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial C} &= u'(C)e^{-\rho t} - \mu_1 = 0; && (16) \\ \frac{\partial \mathcal{H}}{\partial I_H} &= -\mu_1 + \mu_2 = 0; && (17) \\ \frac{\partial \mathcal{H}}{\partial K} &= -\mu_1(\alpha AK^{\alpha-1}H^{1-\alpha} - \delta) = -\dot{\mu}_1; && (18) \\ \frac{\partial \mathcal{H}}{\partial H} &= -\mu_1(1-\alpha)AK^{\alpha}H^{-\alpha} - \mu_2\delta = -\dot{\mu}_2. && (19) \end{aligned}$$

Eliminating μ_1 from (16) and (18) we obtain a result for the growth rate of consumption:

$$\frac{\dot{C}}{C} = \frac{1}{\theta}\left[\alpha A\left(\frac{K}{H}\right)^{-(1-\alpha)} - \delta - \rho\right].$$

Since $\mu_1 = \mu_2$ (from (17)), conditions (18) and (19) imply

$$\alpha A\Big(\frac{K}{H}\Big)^{-(1-\alpha)} - \delta = (1-\alpha)A\Big(\frac{K}{H}\Big)^{\alpha} - \delta,$$

which enables us to find the ratio:

$$\frac{K}{H} = \frac{\alpha}{1-\alpha}.$$

After substituting this ratio in the result for $\frac{\dot{C}}{C}$, it turns out that consumption grows at a constant rate given by:

$$\frac{\dot{C}}{C} = \frac{1}{\theta}[A\alpha^{\alpha}(1-\alpha)^{(1-\alpha)} - \delta - \rho].$$

The substitution of $\frac{K}{H}$ into the production function implies that Y is a linear function of K:

$$Y = AK(\frac{1-\alpha}{\alpha})^{(1-\alpha)}.$$

(Therefore, such a model is also called an AK model.)

Formulas for Y and the K/H-ratio suggest that K, H and Y grow at a constant rate. Moreover, it can be shown that this rate is the same as the growth rate of C found above.

Economics Application 19 (Investment-Selling Decisions)

Finally, let us exemplify the case of a dynamic problem with inequality constraints.

Assume that a firm produces, at each moment t, a good which can either be sold or reinvested to expand the productive capacity $y(t)$ of the firm. The firm's problem is to choose at each moment the proportion $u(t)$ of the output $y(t)$ which should be reinvested such that it maximizes total sales over the period $[0, T]$.

Expressed in mathematical terms, the firm's problem is:

$$\begin{array}{rl} \max & \int_0^T [1-u(t)]y(t)dt \\ \text{subject to} & \dot{y}(t) = u(t)y(t) \\ & y(0) = y_0 > 0 \\ & 0 \le u(t) \le 1. \end{array}$$

where u is the control and y is the state variable.

The Hamiltonian takes the form:

$$H = (1-u)y + \lambda uy + w_1(1-u) + w_2 u.$$

The optimal solution satisfies the conditions:

$$\begin{array}{ll} H_u = (\lambda - 1)y + w_2 - w_1 = 0, & \\ \dot{\lambda} = u - 1 - u\lambda, & \\ w_1 \ge 0, & w_1(1-u) = 0, \\ w_2 \ge 0, & w_2 u = 0, \\ \lambda(T) = 0. & \end{array}$$

These conditions imply that

$$\lambda > 1, \quad u = 1 \text{ and } \dot{\lambda} = -\lambda \tag{20}$$

or

$$\lambda < 1, \quad u = 0 \text{ and } \dot{\lambda} = -1; \tag{21}$$

thus, $\lambda(t)$ *is a decreasing function over* $[0, T]$*. Since* $\lambda(t)$ *is also a continuous function, there exists* t^* *such that (21) holds for any* t *in* $[t^*, T]$*. Hence*

$$u(t) = 0,$$
$$\lambda(t) = T - t,$$
$$x(t) = x(t^*),$$

for any t *in* $[t^*, T]$*.*

t^* *must satisfy* $\lambda(t^*) = 1$*, which implies that* $t^* = T - 1$ *if* $T \geq 1$*. If* $T \leq 1$*, we have* $t^* = 0$*.*

If $T > 1$*, then for any* t *in* $[0, T-1]$ *(20) must hold. Using the fact that* y *is continuous over* $[0, T]$ *(and, therefore, continuous at* $t^* = T - 1$*), we obtain*

$$u(t) = 1,$$
$$\lambda(t) = e^{T-t-1},$$
$$x(t) = y_0 e^t,$$

for any t *in* $[0, T - 1]$*.*

In conclusion, if $T > 1$ *then it is optimal for the firm to invest all the output until* $t = T - 1$ *and to sell all the output after that date. If* $T < 1$ *then it is optimal to sell all the output.*

Further Reading:

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5 Exercises

5.1 Solved Problems

1. Evaluate the Vandermonde determinant

$$\det \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}.$$

2. Find the rank of the matrix A,

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 1 & -3 & -1 \\ -2 & 0 & -4 \\ 4 & 6 & 14 \end{pmatrix}.$$

3. Solve the system $Ax = b$, given

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 1 & 3 & 4 \end{pmatrix}, \qquad b = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}.$$

4. Find the unknown matrix X from the equation

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} X = \begin{pmatrix} 4 & -6 \\ 2 & 1 \end{pmatrix}.$$

5. Find all solutions of the system of linear equations

$$\begin{cases} 2x + y - z &= 3, \\ 3x + 3y + 2z &= 7, \\ 7x + 5y &= 13. \end{cases}$$

6. a) given $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ find the quadratic form $v'Av$ if $A = \begin{pmatrix} 3 & 4 & 6 \\ 4 & -2 & 0 \\ 6 & 0 & 1 \end{pmatrix}$.

   b) Given the quadratic form $x^2 + 2y^2 + 3z^2 + 4xy - 6yz + 8xz$

   find matrix A of $v'Av$.

7. Find the eigenvalues and corresponding eigenvectors of the following matrix:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

   Is this matrix positive definite?

8. Show that the matrix

$$\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$$

is positive definite if and only if

$$a > 0, \quad ab - d^2 > 0, \quad abc + 2edf > af^2 + be^2 + cd^2.$$

9. Find the following limits:

a) $\lim_{x \to -2} \dfrac{x^2 + x - 2}{x^2 + 2x}$

b) $\lim_{x \to 1} \dfrac{x - 1}{\ln x}$

c) $\lim_{x \to +\infty} x^n \cdot e^{-x}$

d) $\lim_{x \to 0} x^x$.

10. Show that a parabola $y = x^2/2e$ is tangent to the curve $y = \ln x$ and find the tangent point.

11. Find $y'(x)$ if

a) $y = \ln \sqrt{\dfrac{e^{4x}}{e^{4x} + 1}}$

b) $y = \ln(\sin x + \sqrt{1 + \sin^2 x})$

c) $y = x^{1/x}$.

12. Show that the function $y(x) = x(\ln x - 1)$ is a solution to the differential equation $y''(x)x \ln x = y'(x)$.

13. Given $f(x) = 1/(1 - x^2)$, show that $f^{(n)}(0) = \begin{cases} n!, & n \text{ is even}, \\ 0, & n \text{ is odd}. \end{cases}$

14. Find $\dfrac{d^2y}{dx^2}$, given

a) $\begin{cases} x = t^2, \\ y = t + t^3. \end{cases}$

b) $\begin{cases} x = e^{2t}, \\ y = e^{3t}. \end{cases}$

15. Can we apply Rolle's Theorem to the function $f(x) = 1 - \sqrt[3]{x^2}$, where $x \in [-1, 1]$? Explain your answer and show it graphically.

16. Given $y(x) = x^2|x|$, find $y'(x)$ and draw the graph of the function.

17. Find the approximate value of $\sqrt[5]{33}$.

18. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$, given

a) $z = x^3 + 3x^2y - y^3$

b) $z = \dfrac{xy}{x - y}$

c) $z = xe^{-xy}$.

19. Show that the function $z(x, y,) = xe^{-y/x}$ is a solution to the differential equation

$$x\frac{\partial^2 z}{\partial x \partial y} + 2\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = y\frac{\partial^2 z}{\partial y^2}.$$

20. Find the extrema of the following functions:

a) $z = y\sqrt{x} - y^2 - x + 6y$

b) $z = e^{x/2}(x + y^2)$.

21. Evaluate the following indefinite integrals:

a) $\displaystyle\int \frac{(x^2+1)^2}{x^3}dx$

b) $\displaystyle\int a^x(1 + a^{-x}/\sqrt{x^3})dx$

c) $\displaystyle\int e^{x^3}x^2dx$

d) $\displaystyle\int x\sqrt{x^2+1}dx$

e) $\displaystyle\int x\ln(x-1)dx$

22. Evaluate the following definite integrals:

a) $\displaystyle\int_1^2 (x^2 + 1/x^4)dx$

b) $\displaystyle\int_4^9 \frac{dx}{\sqrt{x} - 1}$ (Hint: substitution $x = t^2$)

23. Find the area bounded by the curves:

a) $y = x^3,\ y = 8,\ x = 0$

b) $4y = x^2,\ y^2 = 4x$

24. Check whether the improper integral

$$\int_0^{+\infty} \frac{dx}{(x-3)^2}$$

converges or diverges.

25. Find extrema of the functions

a) $z = 3x + 6y - x^2 - xy - y^2$

b) $z = 3x^2 - 2x\sqrt{y} + y - 8x + 8$

26. Find the maximum and minimum values of the function $f(x, y) = x^3 + y^3 - 3xy$ in the domain $\{0 \le x \le 2,\ -1 \le y \le 2\}$. (Hint: Don't forget to test the function on the boundary of the domain)

27. Find dy/dx, given

a) $x^2 + y^2 - 4x + 6y =$

b) $xe^{2y} - ye^{2x} = 0$

28. Given the equation $2\sin(x + 2y - 3z) = x + 2y - 3z$, show that $z_x + z_y = 1$.

29. Check that the Cobb-Douglas production function $z = x^a y^b$ for $0 < a, b < 1,\ a + b \le 1$ is concave for $x, y > 0$.

30. Show that $z = (x^{1/2} + y^{1/2})^2$ is a concave function.

31. For which values of x, y is the function $f(x, y) = x^3 + xy + y^2$ convex?

32. Find dx/dz and dy/dz given

a) $x + y + z = 0,\ x^2 + y^2 + z^2 = 1$

b) $x^2 + y^2 - 2z = 1,\ x + xy + y + z = 1$

33. Find the conditional extrema of the following functions and classify them:

a) $u = x + y + z^2,\quad \text{s.t. } z - x = 1,\ y - xz = 1$

b) $u = x + y,\quad \text{s.t. } 1/x^2 + 1/y^2 = 1/a^2$

c) $u = (x + y)z,\quad \text{s.t. } 1/x^2 + 1/y^2 + 2/z^2 = 4$

d) $u = xyz,\quad \text{s.t. } x^2 + y^2 + z^2 = 3$

e) $u = xyz,\quad \text{s.t. } x^2 + y^2 + z^2 = 1,\ x + y + z = 0$

f) $u = x - 2y + z,\quad \text{s.t. } x + y^2 - z^2 = 1.$

34. A consumer is known to have a Cobb-Douglas utility of the form $u(x,y) = x^\alpha y^{1-\alpha}$, where the parameter α is unknown. However, it is known that when faced with the utility maximization problem

$$\max u(x,y) \text{ subject to } x+y=3,$$

the consumer chooses $x=1, y=2$. Find the value of α.

35. Solve the following differential equations:

a) $x(1+y^2) + y(1+x^2)y' = 0$

b) $y' = xy(y+2)$

c) $y' + y = 2x+1$

d) $y^2 + x^2y' = xyy'$

e) $2x^3y' = y(2x^2 - y^2)$

f) $x^2y' + xy + 1 = 0$

g) $y = x(y' - x\cos x)$

h) $xy' - 2y = 2x^4$

i) $2y'' - 5y' + 2y = 0$

j) $y'' - 2y' = 0$

k) $y''' - 3y' + 2y = 0$

l) $y'' - 2y' + y = 6xe^x$

m) $y'' + 2y' + y = 3e^{-x}\sqrt{x+1}$

36. Solve the following systems of differential equations:

a) $\dot{x} = 5x + 2y, \quad \dot{y} = -4x - y$

b) $\dot{x} = 2y, \quad \dot{y} = 2x$

c) $\dot{x} = 2x + y, \quad \dot{y} = x + 2y$

Answers:

1. $(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.
2. 2.

3. $$x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

4. $$X = \begin{pmatrix} 16 & -32 \\ -6 & 13 \end{pmatrix}.$$

5. The third line in the matrix is a linear combination of the first two, therefore we have to solve the system of two equations with three variables. Taking one of the variables, say z, as a parameter, we find the solution to be

$$x = \frac{2+5z}{3}, \quad y = \frac{5-7z}{3}.$$

6. a) $3x^2 - 2y^2 + z^2 + 8xy + 12xz$.

   b)

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 2 & -3 \\ 4 & -3 & 3 \end{pmatrix}.$$

7. Eigenvalues are $\lambda = 0,\ 1,\ 3$.

   Corresponding eigenvectors are $const \cdot (1, -1, -1)',\ const \cdot (1, -1, 0)',\ const \cdot (1, 1, 2)'$.

   The matrix is non-negative definite.

9. a) $3/2$ b) 1 c) 0 d) 1.

10. $x_0 = \sqrt{e},\ y_0 = 1/2$.

11. a) $2/(e^{4x} + 1)$ b) $\cos x/\sqrt{1 + \sin^2 x}$ c) $x^{1/x} \frac{1-\ln x}{x^2}$.

14. a) $(t^2 - 1)/4t^3$ b) $3/4e^t$.

17. 2.0125

18. a) $3x(x + 2y),\ 3(x^2 - y^2)$ b) $-y^2/(x - y)^2,\ x^2/(x - y)^2$ c) $e{-}xy(1 - xy),\ -x^2e{-}xy$.

20. a) $z_{max} = 12$ at $x = y = 4$ b) $z_{min} = -2/e$ at $x = -2,\ y = 0$.

21. a) $x^2/2 + 2\ln|x| - 1/2x^2 + C$ b) $a^x/\ln a - 2/\sqrt{x} + C$ c) $e^{x^3}/3 + C$ d) $\sqrt{(x^2+1)^3}/3 + C$ e) $x^2 \ln|x - 1|/2 - (x^2/2 + x + \ln|x - 1|)/2 + C$

22. a) $10/8$ b) $2(1 + \ln 2)$

23. a) 12 b) $16/3$

24. Diverges.

25. a) $z_{max} = 9$ at $x = 0,\ y = 3$ b) $z_{min} = 0$ at $x = 2,\ y = 4$

26. $f_{min} = -1$ at $x = y = 1,\ f_{max} = 13$ at $x = 2,\ y = -1$.

27. a) $\frac{2-x}{y+3}$ b) $\frac{2ye^{2x} - e^{2y}}{2xe^{2y} - e^{2x}}$

31. For any y and $x \geq 1/12$.

32. a) $x' = (y-z)/(x-y),\ y' = (z-x)/(x-y)$.
b) $y' = -x' = 1/(y-x)$.

33. a) minimum at $x=-1,\ y=1, z=0$
b) minimum at $x=y=-\sqrt{2}a$, maximum at $x=y=\sqrt{2}a$
c) maximum at $x=y=1,\ z=1$, minimum at $x=y=1,\ z=-1$
d) minimum at points $(-1,1,1),\ (1,-1,1),\ (1,1,-1),\ (-1,-1,-1)$,
maximum at points $(1,1,1),\ (-1,-1,1),\ (-1,1,-1),\ (1,-1,-1)$
e) minimum at points $(a,a,-2a),\ (a,-2a,a),\ (-2a,a,a)$,
maximum at points $(-a,-a,2a),\ (-a,2a,-a),\ (2a,-a,-a)$,
where $a = 1/\sqrt{6}$
f) no extrema

34. $\alpha = 1/3$

35. a) $(1+x^2)(1+y^2) = C$
b) $y = \frac{2Ce^{x^2}}{1-Ce^{x^2}}$
c) $y = 2x - 1 + Ce^{-x}$
d) $y = Ce^{y/x}$
e) $x = \pm y\sqrt{\ln Cx}$
f) $xy = C - \ln|x|$
g) $y = x(C + \sin x)$
h) $y = Cx^2 + x^4$
i) $y = C_1e^{2x} + C_2e^{x/2}$
j) $y = C_1 + C_2e^{2x}$
k) $y = e^x(C_1 + C_2x) + C_3e^{-2x}$
l) $y = (C_1 + C_2x)e^x + x^3e^x$
m) $y = e^{-x}(\frac{4}{5}(x+1)^{5/2} + C_1 + C_2x)$

36. a) $x = C_1e^t + C_2e^{3t},\quad y = -2C_1e^t - C_2e^{3t}$
b) $x = C_1e^{2t} + C_2e^{-2t},\quad y = C_1e^{2t} - C_2e^{-2t}$
c) $x = C_1e^t + C_2e^{3t},\quad y = -C_1e^t + C_2e^{3t}$

5.2 More Problems

1. Find maximum and minimum values of the function

$$f(x,y,z) = \sqrt{x^2+y^2+z^2}e^{-(x^2+y^2+z^2)}.$$

(Hint: introduce the new variable $u = x^2+y^2+z^2$; note that $u \geq 0$.)

2. a) In the method of least squares of regression theory the curve $y = a + bx$ is fit to the data $(x_i, y_i),\ i = 1, 2, \ldots, n$ by minimizing the sum of squared errors:

$$S(a,b) = \sum_{i=1}^{n}(y_i - (a + bx_i))^2$$

by choice of two parameters a (the intercept) and b (the slope).

Determine the necessary condition for minimizing $S(a,b)$ by choice of a and b.

Solve for a and b and show that the sufficient conditions are met.

b) The continuous analog of the discrete case of the method of least squares can be set up as follows:

Given a continuously differentiable function $f((x)$, find $a,\ b$ which solve the problem

$$\min_{a,b} \int_0^1 (f(x) - (a + bx))^2 dx.$$

Again, find the optimal $a,\ b$ and show that the sufficient conditions are met.

c) Compare your results.

3. This result is known in economic analysis as *Le Chatelier Principle*. But rise to the challenge and figure out the answer yourself!

   Consider the problem

$$\max_{x_1,x_2} F(x_1, x_2) = f(x_1, x_2) - w_1 x_1 - w_2 x_2,$$

   where the Hessian matrix of the second order partial derivatives $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right),\ i, j = 1, 2$, is assumed negative definite and $w_1,\ w_2$ are given positive parameters.
   a) What are the first order conditions for a maximum?
   b) Show that x_1 can be solved as a function of $w_1,\ w_2$ and $\frac{\partial x_1}{\partial w_1} < 0$.
   c) Suppose that to the problem is added the linear constraint $x_2 = b$, where b is a given nonzero parameter. Find the new equilibrium and show that:

$$\left(\frac{\partial x_1}{\partial w_1}\right)_{\text{without added constraint}} \leq \left(\frac{\partial x_1}{\partial w_1}\right)_{\text{with added constraint}} < 0.$$

4. Consider the following minimization problem:
   $\min(x - u)^2 + (y - v)^2$ s.t. $\ xy = 1,\ u + 2v = 1.$
   a) Find the solution.
   b) How can you interpret this problem geometrically? Illustrate your results graphically.
   c) What is the solution to this minimization problem if the second constraint is replaced with the constraint $u + 2v = 3$? In this case, do you really need to use the Lagrange-multiplier method?

   (Hint: Given two points (x_1, y_1) and (x_2, y_2) in the Euclidean space, what is the meaning of the expression $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$?)

5. Solve the integral equation

$$\int_0^x (x - t)y(t)dt = \int_0^x y(t)dt + 2x.$$

   (Hint: take the derivatives of both parts with respect to x and solve the differential equation).

6. a) Show that Chebyshev's differential equation

$$(1 - x^2)y'' - xy' + n^2 y = 0$$

(where n is constant and $|x| < 1$) can be reduced to the linear differential equation with constant coefficients

$$\frac{d^2y}{dt^2} + n^2y = 0$$

by substituting $x = \cos t$.

b) Prove that the linear differential equation of the second order

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0$$

where $a(x),\ b(x),\ c(x)$ are continuous functions and $a(x) \neq 0$, can be reduced to the equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x) = 0.$$

(Hint: in other words, you need to prove the existence of a function $\mu(x)$ such that $p(x) = \mu(x)a(x),\ q(x) = \mu(x)c(x)$ and $\frac{d}{dx}(\mu(x)a(x)) = \mu(x)b(x)$.)

7. Find the equilibria of the system of differential equations

$$\dot{x} = \sin x,\ \dot{y} = \sin y,$$

and classify them.

Draw the phase portrait of the system.

5.3 Sample Tests

Sample Test – I

1. Easy problems:

   a) Check whether the function $f(x, y) = 4x^2 - 2xy + 3y^2$, defined for all $(x, y) \in \mathbf{R}^2$, is concave, convex, strictly concave, strictly convex or neither. At which point(s) does the function reach its maximum (minimum)?

   b) Evaluate the indefinite integral $\int x^3 e^{-x^2} dx$.

   c) You are given the matrix A and the real number $x,\ x \in \mathbf{R}$:

   $$A = \begin{pmatrix} 0 & 0 & 2 & 4 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 3 & x & 0 & 0 \end{pmatrix}.$$

   For which values of x is the matrix non-singular?

   d) The function $f(x, y) = \sqrt{xy}$ is to be maximized under the constraint $x + y = 5$. Find the optimal x and y.

2. Using the inverse matrix, solve the system of linear equations

$$\begin{cases} x + 2y + 3z &= 1, \\ 3x + y + 2z &= 2, \\ 2x + 3y + z &= 3. \end{cases}$$

3. Find the eigenvalues and the corresponding eigenvectors of the following matrix:

$$\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}.$$

4. Sketch the graph of the function $y(x) = \dfrac{2-x}{x^2+x-2}$. Be as specific as you can.

5. Given the function $\dfrac{1-e^{-ax}}{1+e^{ax}}$, $a > 1$

   a) Write down the F.O.C. of an extremum point.

   b) Determine whether an extremum point is maximum or minimum.

   c) Find $\dfrac{dx}{da}$ and determine the sign of $\dfrac{dx}{da}$.

6. The function $f(x, y) = ax + y$ is to be maximized under the constraints $x^2 + ay^2 \leq 1,\ x \geq 0,\ y \geq 0$, where a is a positive real parameter.

   a) Use the Lagrange-multiplier method to write down the F.O.C. for the maximum.

   b) Taking as given that this problem has a solution, find how the optimal values of x and y change if a increases by a very small number.

7. a) Solve the differential equation $y''(x) + 2y'(x) - 3y(x) = 0$.

   b) Solve the differential equation $y''(x) + 2y'(x) - 3y(x) = x - 1$.

8. Solve the difference equation $x_t - 5x_{t-1} = 3$.

9. Consider the optimal control problem

$$\max_{u(t)} I(u(t)) = \int_0^{+\infty} (x^2(t) - u^2(t))e^{-rt}dt,$$

subject to

$$\frac{dx}{dt} = au(t) - bx(t),$$

where $a,\ b$ and r are positive parameters.

Set up the Hamiltonian function associated with this problem and write down the first-order conditions.

Sample Test – II

1. Easy problems:

   a) For what values of p and q is the function

$$f(x, y) = \frac{1}{x^p} + \frac{1}{y^q}$$

   defined in the region $x > 0,\ y > 0$
   a) convex,
   b) concave?

b) Find the area between the curves $y = x^2$ and $y = \sqrt{x}$.

c) Apply the characteristic root test to check whether the quadratic form $Q = -2x^2 - 2y^2 - z^2 + 4xy$ is positive (negative) definite.

d) A consumer is known to have a Cobb-Douglas utility of the form $u(x, y) = x^\alpha y^{1-\alpha}$, where the parameter α is unknown. However, it is known that when faced with the utility maximization problem

$\max u(x, y)$ subject to $x + y = 3$,

the consumer chooses $x = 1, y = 2$. Find the value of α.

2. Find the unknown matrix X from the equation

$$X \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 3 \\ 4 & 3 & 2 \\ 1 & -2 & 5 \end{pmatrix}.$$

3. Find local extrema of the function $z = (x^2 + y^2)e^{-(x^2+y^2)}$ and classify them.

4. A curve in the $(x - y)$ plane is given parametrically as $x = a(t - \sin t),\ y = a(1 - \cos t)$, a is a real parameter, $t \in [0, 2\pi]$.

   Draw the graph of this curve and find the equation of the tangent line at $t = \pi/2$.

5. In the method of least squares of regression theory the curve $y = a + bx$ is fit to the data $(x_i, y_i),\ i = 1, 2, \dots, n$ by minimizing the sum of squared errors

$$S(a, b) = \sum_{i=1}^{n} (y_i - (a + bx_i))^2$$

   by the choice of two parameters, a (the intercept) and b (the slope).

   Determine the necessary condition for minimizing $S(a, b)$ by the choice of a and b.

   Solve for a and b and show that the sufficient conditions are met.

6. Use the Lagrange multiplier method to write the first order conditions for the maximum of the function

$$f(x, y) = \ln x + \ln y - x - y, \text{ subject to } x + cy = 2,$$

   where c is a real parameter, $c > 1$.

   If c is such that these conditions describe the maximum, show what happens with x and y if c increases by a very small number.

7. a) Solve the differential equation $y''(x) - 2y'(x) + y(x) = 0$.

   b) Solve the differential equation $y''(x) - 2y'(x) + y(x) = xe^x$.

8. Solve the difference equation $x_{t+1} - x_t = b_t, \quad t = 0, 1, 2, \dots$.

9. Write down the Kuhn-Tucker conditions for the following problem:

$$\max_{x_1, x_2} 3x_1x_2 - x_2^3,$$

$$\text{subject to } 2x_1 + 5x_2 \geq 20, \quad x_1 - 2x_2 = 5, \quad x_1 \geq 0, x_2 \geq 0.$$

Sample Test – III

1. Answer the following questions true, false or it depends.

   Provide a complete explanation for all your answers.

   a) If A is $n \times m$ matrix and B is $m \times n$ matrix, then $\text{trace}(AB) = \text{trace}(BA)$.

   b) If x^* is a local maximum of $f(x)$ then $f''(x^*) < 0$.

   c) If A is $n \times n$ matrix and $\text{rank}(A) < n$ then the system of linear equations $Ax = b$ has no solutions.

   d) If $f(x) = -f(-x)$ for all x then for any constant a $\int_{-a}^{a} f(x)dx = 0$.

   e) If $f(x)$ and $g(x)$ are continuously differentiable at $x = a$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

2. Solve the system of linear equations

$$\begin{cases} x + y + cz &= 1 \\ x + cy + z &= 1 \\ cx + y + z &= 1 \end{cases}$$

   where c is a real parameter. Consider all possible cases!

3. Given the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & \alpha \\ 2 & 0 & 1 \end{pmatrix},$$

   a) find all values of α, for which this matrix has three distinct real eigenvalues.

   b) let $\alpha = 1$. If I tell you that one eigenvalue $\lambda_1 = 2$, find two other eigenvalues of A and the eigenvector, corresponding to λ_1.

4. You are given the function $f(x) = xe^{1-x} + (1-x)e^x$.

   a) Find $\lim_{x\to-\infty} f(x)$, $\lim_{x\to+\infty} f(x)$

   b) Write down the first order necessary condition for maximum.

   c) Using this function, prove that there exists a real number c in the interval $(0, 1)$, such that $1 - 2c = \ln c - \ln(1-c)$.

5. Find the area, bounded by the curves $y^2 = (4-x)^3$, $x = 0$, $y = 0$.

6. Let $z = f(x, y)$ be homogeneous function of degree n, i.e. $f(tx, ty) = t^n f(x, y)$ for any t. Prove that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$$

7. Find local extrema of the function

$$u(x, y, z) = 2x^2 - xy + 2xz - y + y^3 + z^2$$

   and classify them.

8. Show that the implicit function $z = z(x, y)$, defined as $x - mz = \phi(y - nz)$ (where $m,\ n$ are parameters, ϕ is an arbitrary differentiable function), solves the differential equation

$$m\frac{\partial z}{\partial x} + n\frac{\partial z}{\partial y} = 1.$$

Sample Test – IV

1. Answer the following questions true, false or it depends.

   Provide a complete explanation for all your answers.

   a) If matrices $A,\ B$ and C are such that $AB = AC$ then $B = C$.
   b) $\det(A + B) = \det A + \det B$.
   c) A concave function $f(x)$ defined in the closed interval $[a, b]$ reaches its maximum at the interior point of (a, b).
   d) If $f(x)$ is a continuous function in $[a, b]$, then the condition $\int_a^b f^2(x)dx = 0$ implies $f(x) \equiv 0$ in $[a, b]$.
   e) If a matrix A is triangular, then the elements on its principal diagonal are the eigenvalues of A.

2. Easy problems.

   a) Find maxima and minima of the function $x^2 + y^2$, subject to $x^2 + 4y^2 = 1$
   b) (back to microeconomics...)
      A demand for some good X is described by the linear demand function $D = 6 - 4x$. The industry supply of this good equals $S = \sqrt{x} + 1$. Find the value of the consumer and producer surpluses in the equilibrium.
      (Hint: The surplus is defined as the area in the positive quadrant, bounded by the demand and supply curves)
   c) Let $f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$ be convex functions. Prove that the function

   $$F(x_1, \ldots, x_n) = \sum_{i=1}^{m} \alpha_i f_i(x_1, \ldots, x_n)$$

      is convex, if $\alpha_i \geq 0,\ i = 1, 2, \ldots, m$.

3. Find the absolute minimum and maximum of the function

   $$z(x, y) = x^2 + y^2 - 2x - 4y + 1$$

   in the domain, defined as $x \geq 0,\ y \geq 0,\ 5x + 3y \leq 15$ (a triangle with vertices in (0,0), (3,0) and (0,5)).

   (Hint: Find local extrema, then check the function on the boundary and compare).

4. Consider the following optimization problem:

   extremize $f(x, y) = \frac{1}{2}x^2 + e^y$ subject to $x + y = a$, where a is a real parameter.

   Write down the first order necessary conditions.

   Check whether the extremum point is minimum or maximum.

   What happens with optimal values of x and y if a increases by a very small number?

   (Hint: Don't waste your time trying to solve the first order conditions.)

5. Solve the differential equation $y''(x) - 6y'(x) + 9y = xe^{ax}$, where a is a real parameter. Consider all possible cases!

6. a) Solve the system of linear differential equations $\dot{x} = Ax$, where

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

b) Solve the system of differential equations $\dot{x} = Ax + b$ where A is as in part a) and $b = (1, 2, 3)'$.

c) What is the solution to the problem in part b) if you know that

$$x_1(0) = x_2(0) = x_3(0) = 0?$$

7. Solve the difference equation $3y_{k+2} + 2y_{k+1} - y_k = 2^k + k$.

8. Find the equilibria of the system of non-linear differential equations and classify them:

$$\begin{aligned} \dot{x} &= \ln(1 - y + y^2), \\ \dot{y} &= 3 - \sqrt{x^2 + 8y}. \end{aligned}$$