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MATHEMATICS

On Geometry of Second-Order Parabolic Equations in Two Independent Variables[1]

A. M. Vinogradov[a, b]

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In this paper we announce some new results concerning second-order differential parabolic equation in two independent variables. Equations of Monge–Ampére type are distinguished among them. They are characterized by the fact that the associated subsidiary equations describing singularities of their multivalued solutions have, in a sense, the simplest form. The structure of these subsidiary equations allows us to subdivide the parabolic equations into four classes. Each of them can be described as a special geometrical structure on 4-dimensional manifolds introduced below. Moreover, this leads to a complete classification of the considered parabolic equations with respect to the group of contact transformations.

Our approach differs from the traditional one (see, for instance, [5]) in the fact that we focus on the corresponding subsidiary, or characteristic, equations rather than on the original ones. This leads to a noteworthy simplification. In [3, 4] this approach was used to construct scalar differential invariants of hyperbolic Monge–Ampére equations.

1. SECOND-ORDER PARABOLIC AND MONGE–AMPÉRE EQUATIONS

Let E be a 3-dimensional manifold. The manifold of kth-order jets, $k \geq 0$, of 2-dimensional submanifolds of E is denoted by $J^k(E, 2)$; and $\pi_{k,l}: J^k(E, 2) \to J^l(E, 2)$, $k \geq l$, denotes the canonical projection. If E is fibered by a map $\pi: E \to M$ over a 2-dimensional manifold M, then $J^k\pi$ stands for the kth-order jet manifold of local sections of π. $J^k\pi$ is an open subset of $J^k(E, 2)$. A kth-order differential equation on one unknown function in two independent variables is a hypersurface $\mathcal{E} \subset J^k(E, 2)$. Below, we deal with second-order equations of this kind. A chart (x, y, u) in E, where (x, y) are interpreted as independent variables and u as a dependent one, extends to the chart $(x, y, u, u_x = p, u_y = q, u_{xx} = r, u_{xy} = s, u_{yy} = t)$ in $J^2(3, 2)$, in terms of which a local description of $\mathcal{E}$ looks like

$$F(x, y, u, p, q, r, s, t) = 0. \qquad (1)$$

Equation (1) is called elliptic (parabolic, or hyperbolic) if $4F_rF_t - F_s^2 > 0$ (= 0, or < 0, respectively) at all points of $\mathcal{E}$. Intrinsically, these three types of equations are distinguished from one another by the character of singularities that their multi-valued solutions admit (see [1]). These singularities are described by subsidiary equations. Equations for which these subsidiary equations are, in a sense, the simplest form the class of Monge–Ampére (MA) equations. We have no possibility to discuss the details of this conceptual characterization of MA equations and refer to the traditional descriptive definition of MA equations as equations of the form

$$N(rt - s^2) + Ar + Bs + Ct + D = 0, \qquad (2)$$

where N, A, B, C, and D are functions of variables x, y, u, p, and q. For a coordinate-free but descriptive definition of MA equations, see, for instance, [6, 7]. The subject of this paper is parabolic MA equations, i.e., those for which

$$AC - B^2 - 4ND = 0. \qquad (3)$$

The first result concerning parabolic equations (1) is somehow surprising.

Theorem 1. *The characteristic cones of parabolic equations* (1) *are bidimensional.*

For parabolic MA equations (PMA), this cone reduces to a plane.

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[a] *Dipartimento di Matematica e Informatica, Universitá degli Studi di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA), Italy*
[b] *INFN, Gruppo Collegato di Salerno, Italy*
e-mail: vinograd@unisa.it

2. GEOMETRICAL INTERPRETATION OF PARABOLIC MONGE–AMPÉRE EQUATIONS

Recall that $J^1(E, 2)$ is canonically supplied with a contact distribution C given locally by the Pfaff equation $du - pdx - qdy = 0$. Vector fields X and Y belonging to C are called C-orthogonal if $[X, Y]$ belongs to C as well. Obviously, C-orthogonality is a $C^\infty(J^1(E, 2))$-linear condition. If C is locally given by a Pfaff equation $\omega = 0$, where ω is a 1-form, then X and Y are C-orthogonal iff $d\omega(X, Y) = 0$. A bidimensional subdistribution $D \subset C$ is called Lagrangian if any two vector fields belonging to it are C-orthogonal.

Let D be Lagrangian. Denote by $L_{(1)}(E, 2)$ the first jet prolongation of a bidimensional submanifold $L \subset E$. The condition

$$\dim\{T_\theta(L_{(1)}) \cap D_\theta\} > 0, \quad \forall\theta \in L_{(1)} \tag{4}$$

determines a second-order differential equation imposed on bidimensional submanifolds of E. Denote it by $\mathscr{E}_D \subset J^2(E, 2)$. So, by definition, $L \subset E$ is a solution of $\mathscr{E}_D$ iff (4) holds.

Proposition 1. *The correspondence* 1. $D \mapsto \mathscr{E}_D$ *between the Lagrangian distributions on* $J^1(E, 2)$ *and the parabolic Monge–Ampére equations imposed on* 2*-dimensional submanifolds of* E *is biunique.*

We use $D_\mathscr{E}$ to denote the Lagrangian distribution corresponding to a parabolic Monge–Ampére equation $\mathscr{E} \subset J^2(E, 2)$, and $\langle X, Y\rangle$ stands for the bidimensional distribution generated by vector fields X, Y. Then, for Eq. (2), we have

$$D_\mathscr{E} = \left\langle \partial_x + p\partial_u - \frac{C}{N}\partial_p + \frac{B}{2N}\partial_q, \right.$$
$$\left. \partial_y + q\partial_u + \frac{B}{2N}\partial_p - \frac{A}{N}\partial_q \right\rangle,$$

assuming that $N \neq 0$. If $N = 0$, i.e., (2) is quasilinear, then

$$D_\mathscr{E} = \left\langle \partial_x + \frac{B}{2A}\partial_y + \left(p + \frac{B}{2A}q\right)\partial_u - \frac{D}{A}\partial_p, \frac{B}{2A}\partial_p - \partial_q \right\rangle.$$

Proposition 1 suggests the idea to define generalized PMA equations as triples of the form $\mathscr{E} = (M, C, D)$, where C is a contact distribution on a 5-fold M and D is a Lagrangian subdistribution of C. A solution of $\mathscr{E}$ is defined to be a Legendrian submanifold S of M such that

$$\dim\{T_\theta(S) \cap D_\theta\} > 0, \quad \forall\theta \in S.$$

In what follows, the term parabolic Monge–Ampére (PMA) equation will refer to such a triple.

3. DIRECTING DISTRIBUTION

A PMA equation $\mathscr{E} = (M, C, D)$ is called integrable if D is integrable.

Theorem 2. *All integrable PMA equations are locally contact equivalent to one another and, in particular, to the equation* $u_{xx} = 0$.

So, further on we concentrate on nonintegrable PMA equations. In this case the first prolongation $D_{(1)}$ of D, i.e., the span of vector fields belonging to D and their commutators, is 3-dimensional and belongs to C. The C-orthogonal complement R of $D_{(1)}$ is 1-dimensional and belongs to D. In this way, we obtain the following flag of distributions:

$$R \subset D \subset D_{(1)} \subset C.$$

R is called the directly distribution of D (alternatively, of $\mathscr{E}$).

Obviously, the distribution $D' = \{X \in D_{(1)} | [X, R] \in D_{(1)}\}$ contains D. Since $D' \subset D_{(1)}$, there are two possibilities (except the eventual singular points): either $D' = D$ or $D' = D_{(1)}$.

A PMA equation is called generic if $D' = D$ and is called special if $D' = D_{(1)}$. Since the distribution $D_{(1)}$ is the C-orthogonal complement of R in C, it is uniquely determined by R. Therefore, D' is uniquely determined by R as well. This shows that a generic PMA equation is uniquely determined by its directing distribution. On the contrary, it does not hold for special PMA equations. In this case, R is the characteristic distribution of $D_{(1)}$ and any 1-dimensional distribution $R' \in D_{(1)}$ transversal to R defines a special PMA equation $D = R \oplus R'$ for which R is the directing distribution.

4. PROJECTIVE CURVE BUNDLES AND THE ASSOCIATED PMA EQUATIONS

Let N be a 4-dimensional manifold and $p\tau^*$: $PT^*N \to N$ be the projectivization of the cotangent bundle $T^*N \to N$. By definition the fiber of $p\tau^*$ over a point $y \in N$ is the 3-dimensional projective space PT_y^*N of 1-dimensional subspaces of T_y^*N. A projective curve bundle (PCB) over N is a 1-dimensional subbundle π: $K \to N$ of $p\tau^*$. Its fiber $F_y = \pi^{-1}(y)$ is a (smooth) curve in the projective space PT_y^*N. If the curve F_y is not projectively flat at a point $\theta \in F_y$, then θ is called regular. A PCB is regular if all the points of the curves composing it are regular.

A diffeomorphism Φ: $N \to N'$ lifts canonically to a fibered diffeomorphism $PT^*N \to PT^*N'$ which sends a PCB over N to a PCB over N'. Such two PCBs are called equivalent (via Φ).

Let $\theta \in PT_y^*N$ and $\theta = \langle\rho\rangle$ with $\rho \in T_y^*N$. Then $W_\theta = \{\xi \in T_yN\} | \rho(\xi) = 0\}$ is a 3-dimensional subspace of T_yN. Put

$$V_\theta = \{\eta \in T_\theta K | d_\theta\pi(\eta) \in W_\theta\} \subset T_\theta K.$$

Then C_π: $\theta \mapsto V_\theta$ is a 4-dimensional distribution on K containing the distribution $\mathrm{vert}(\pi)$ of tangent lines to fibers of π.

Proposition 2. *If π is a regular PCB, then the distribution C_π is a contact structure on K.*

In view of this proposition, C_π-orthogonal to $\mathrm{vert}(\pi)$, the subdistribution of C_π, denoted by $\mathrm{vert}^\perp(\pi)$, is well-defined and we put

$$D_\pi = \{X \in \mathrm{vert}^\perp(\pi) |\ [X, \mathrm{vert}(\pi)] \subset \mathrm{vert}^\perp(\pi)\}.$$

Theorem 3. *If π is a regular* PCB, *then D_π is a Lagrangian distribution with respect to C_π and, hence, (K, C_π, D_π) is a generic parabolic Monge–Ampére equation whose directing distribution is* $\mathrm{vert}(\pi)$. *Conversely, any generic parabolic Monge–Ampére equation is locally equivalent to such one.*

Corollary 1. *The problem of local contact classification of generic PMA equations is equivalent to the problem of local classification of regular PCBs with respect to diffeomorphisms of base manifolds.*

5. SPECIAL PMA EQUATIONS AND FRINGES

Let N be a 4-dimensional manifold supplied with a 2-dimensional distribution Q. The subbundle $\varrho = \varrho_Q$: $N_Q \to N$ of $p\tau^*$ associated with Q is defined as

$$N_Q = \{\theta \in PT^*N |\ W_\theta \supset Q_{p\tau^*(\theta)}\} \subset PT^*N,$$
$$\rho = p\tau^*|_{N_Q}.$$

Note that fibers of ρ are projective lines in PT^*N and $\dim N_Q = 5$. Therefore, ϱ is an (irregular) PCB.

The projectivization $p\tau$: $PTN \to N$ of the tangent bundle of N contains a 1-dimensional subbundle $\iota = \iota_Q$: $N^Q \to N$ composed of all tangent lines to N belonging to Q. A fringe over Q is a map Ψ: $N_Q \to N^Q$ such that $\Psi(\varrho^{-1}(y)) \subset \iota^{-1}(y)$, $\forall y \in N$. A 2-dimensional distribution D_Ψ on N_Q is naturally associated with a fringe Ψ:

$$D_\Psi\colon \theta \mapsto \{\xi \in T_\theta(N_Q) |\ d_\theta\, \varrho(\xi) \in \Psi(\theta)\}.$$

Consider the 4-dimensional distribution C_ϱ on N_Q associated with PCB ϱ (see the previous section).

Proposition 3. *If Q is not integrable, then the distribution C_ϱ is a contact structure on N_Q with respect to which D_Ψ is Lagrangian for any fringe Ψ over Q and $\mathscr{E}_\Psi = (N_Q, C_\varrho, D_\Psi)$ is a special PMA equation with directing distribution* $\mathrm{vert}(\varrho)$.

Now we describe two model nonintegrable 2-dimensional distributions on 4-dimensional manifolds. Let C^k_α be the Cartan distribution on the kth-order jet bundle $J^k(\alpha)$ of the trivial bundle α: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. It is bidimensional. The 4-dimensional manifold $J^1(\alpha) \times \mathbb{R}$ possesses a natural 2-dimensional distribution $C^1_\alpha \times 0$, which is the direct product of the contact distribution C^1_α on $J^1(\alpha)$ and the zero distribution on $\mathbb{R}$. This is the first model. The second is C^2_α on $J^2(\alpha)$.

Theorem 4. *Any special PMA equation is locally contact equivalent to $\mathscr{E}_\Psi$ with Ψ being a fringe either over $C^1_\alpha \times 0$ or over C^2_α and vice versa.*

Corollary 2. *The problem of local contact classification of special PMA equations is equivalent to that of local classification of fringes over the model distributions $J^1(\alpha) \times \mathbb{R}$ and C^2_α with respect to diffeomorphisms preserving these distributions.*

It is not difficult to see that these diffeomorphisms are either fiberwise contact diffeomorphisms of the bundle $J^1(\alpha) \times \mathbb{R} \to \mathbb{R}$ in the first case or, lifted to $J^2(\alpha)$, contact diffeomorphisms of the contact manifold $(J^1(\alpha), C^1_\alpha)$ in the second case.

Corollary 3. *Any special PMA equation is locally contact equivalent to a quasilinear one.*

6. DIFFERENTIAL INVARIANTS AND CONTACT CLASSIFICATION

It follows from Theorems 3 and 4 that nonintegrable PMA equations are subdivided into 3 classes: first, generic equations; then, special ones associated with fringes over C^2_α; and, finally, special equations associated with fringes over $C^1_\alpha \times 0$. We refer to them as G, SG, and SI, respectively. Canonical models of PMA equations described in these theorems immediately suggest a construction of scalar differential invariants that turn out to be sufficient for a complete classification of PMA equations on the basis of the principle of n-invariants (see [10, 11]). A general idea of how it can be done in each of these three cases is as follow.

I. Type *G*. In this case we look for (scalar) differential invariants of PCBs with respect to diffeomorphisms of base manifolds. Let $\mathscr{I}$ be a scalar projective differential invariant of curves in $\mathbb{R}P^3$, say, the projective curvature (see [8, 9]); $\Theta \in K$; and $y = \pi(\Theta)$ (see Section 4). The value of this invariant for the curve $\pi^{-1}(y)$ in PT^*_y is a function on this curve. Denote it by $\mathscr{I}_{\pi, y}$ and put $\mathscr{I}_\pi(\Theta) = \mathscr{I}_{\pi, y}(\Theta)$. Then $\mathscr{I}_\pi \in C^\infty(K)$ is a differential invariant of the PCB π and, as such, of the PMA equation associated with π.

II. Type *SG*. In this case we are interested in differential invariants of fringes with respect to the group of diffeomorphisms of $J^2(\alpha)$ preserving the distribution C^2_α. Let $\mathscr{I}$ be a (scalar) differential invariant of maps $\mathbb{R}P^1 \to \mathbb{R}P^1$ with respect to a natural action of the group $SL(2) \times SL(2)$ on them. Denote by Ψ_y: $\varrho^{-1}(y) \to \iota^{-1}(y)$ the restriction of the fringe Ψ: $N_Q \to N^Q$ over Q (see Section 5) to $\varrho^{-1}(y)$, $y \in J^2(\alpha)$. Ψ_y is a map of one projective line to another. Put $\mathcal{I}_\Psi(\Theta) = \mathcal{I}_{\Psi,\,y}(\Theta)$ with $\mathcal{I}_{\Psi,\,y}$ being the value of the invariant $\mathcal{I}$ for Ψ_y. Then $\mathcal{I}_\Psi \in C^\infty(N_Q)$ is a differential invariant of Ψ with respect to contact transformations of $J^2(\alpha)$.

III. Type *SI*. In this case, the construction of invariants of the type $\mathcal{I}_\Psi$ is identical to the preceding case.

Theorem 5. *The differential invariants of one of the forms $\mathcal{I}_\pi$, $\mathcal{I}_\Psi$ are sufficient for a complete classification of generic and special PMA equations, respectively, on the basis of the principle of n-invariants.*

Concerning the principle of *n*-invariants, we refer the reader to [10, 11]. A detailed description of these and some more delicate invariants constructed on the basis of the proposed geometrical interpretation of PMA equations will be given in a joint paper by D. Catalano Ferraioli and the author. This interpretation has a number of other applications to the theory of PMA equations, which will be discussed elsewhere.

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