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1981 Math. USSR Izv. 17 73

(http://iopscience.iop.org/0025-5726/17/1/A03)

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Сер. Матем. Том 44(1980), Вып. 4

Math. USSR Izvestija
Vol. 17(1981), No. 1

REDUCTION OF DIFFERENTIAL EQUATIONS WITH SYMMETRIES

UDC 517.9

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ABSTRACT. A method for constructing group-invariant solutions of differential equations is described. At the foundation of the method lies a reduction of the dimension of the base of a bundle of k-jets of functions $J^k(M^n, R^1)$ by means of a passage to the manifolds of orbits of the contact action of the Lie group of partial symmetries of the differential equation. Only the orbits of a certain submanifold of $J^k(M^n, R^1)$ are considered, an extension of an involutive system of first-order differential equations associated with the action of the group.

Bibliography: 7 titles.

We consider here differential equations which have a Lie group of symmetries, and we describe a reduction method (for reducing the number of independent variables) for obtaining group-invariant solutions of such equations. The main idea of this method as applied to Hamilton-Jacobi equations is contained in [1], where a reduction of the phase space and of Hamiltonian systems in the space which admit a group of symmetries is carried out. The concept of a Lagrangian manifold permits the reformulation of these results for Hamilton-Jacobi equations. By considering the contact structure instead of the symplectic structure and the objects related to it, we can carry out the reduction of contact manifolds and first-order differential equations. Higher-order differential equations are considered in this paper as submanifolds of codimension one of fibered manifolds of k-jets of functions (see [2]), and symmetries of equations are treated in the spirit of [3]. Groups of symmetries of higher-order equations are liftings of Lie groups of actions with contact manifolds of 1-jets of functions. By means of the techniques of k-jet liftings and extensions, we can carry out the reduction in this case, relying on the reduction of contact manifolds. To construct invariant solutions, it suffices to confine oneself to the partial symmetries treated in this paper.

§1. Reduction of the Hamilton-Jacobi equation. Reduction of first-order differential equations

We consider a connected symplectic manifold (M^{2n}, ω) of class C^∞, where ω is a closed nondegenerate differential form on M^{2n} of degree two (the symplectic structure). The following method for constructing new symplectic manifolds from (M^{2n}, ω) is well known [1]. Suppose that the connected Lie group G acts on M^{2n} from the left by means

1980 *Mathematics Subject Classification.* Primary 35A30, 58G99; Secondary 58A20.

of the symplectic diffeomorphisms: $g^*(\omega) = \omega$, $g \in G$. We denote by σ: $\mathfrak{g} \to H(M^{2n})$ a homomorphism of the Lie algebra $\mathfrak{g}$ of the group G, considered as an R-algebra of left-invariant vector fields over G, into the Lie algebra $H(M^{2n})$ of locally Hamiltonian vector fields over M^{2n}:

$$\sigma(A) = \lim_{t\to 0} \frac{a_t^* - 1}{t}, \tag{1}$$

where $a_t = \varphi_t(e)$, and φ_t is a one-parameter group of diffeomorphisms of the group G generated by a vector field $A \in \mathfrak{g}$. In (1), a_t is understood as a group of transformations of the manifold M^{2n}. The action of G on M^{2n} is symplectic, so $\sigma(A)$ is a locally Hamiltonian vector field.

We consider the Poisson action of a group G, in which, as is well known, 1) all fields $\sigma(A)$ have globally defined Hamilton functions H_A, and $i(\sigma(A))\omega = -dH_A$, where $i(\sigma(A))\omega$ is the inner product of $\sigma(A)$ and ω, 2) the dependence of H_A on A is linear, and 3) $H_{[A,B]} = (H_A, H_B)$, where $[A, B]$ is the commutator of the fields A and B; $(H_A, H_B) = i(\sigma(B))i(\sigma(A))\omega$ are the Poisson brackets of the functions H_A and H_B. The Poisson action permits us to define a homomorphism $\hat{\psi}$: $\mathfrak{g} \to C^\infty(M^{2n})$, $\hat{\psi}(A) = H_A$, where $C^\infty(M^{2n})$ is regarded as a Lie algebra of functions with respect to the Poisson brackets. The mapping ψ: $M^{2n} \to \mathfrak{g}^*$, $\psi(m) = m \circ \hat{\psi}$, dual to $\hat{\psi}$, where $m(f) = f(m)$, $f \in C^\infty(M^{2n})$, is called the *moment mapping*. The moment ψ is equivariant with respect to the coadjoint action Ad*: $G \to \mathfrak{g}^*$ of the group G on the space $\mathfrak{g}^*$: $\mathrm{Ad}^* g^{-1} \circ \psi = \psi \circ g$, dual to $\mathfrak{g}$.

Let μ be a regular value of the moment; then $\psi^{-1}(\mu) = M_\mu$ is a submanifold of M^{2n}, and M_μ is invariant with respect to the action of the stationary subgroup G_μ of μ. We denote the set of orbits of the points of M_μ by $F_\mu = M_\mu / G_\mu$. If the action of G_μ on M_μ is proper, and G_μ acts freely on M_μ, then it is possible to furnish F_μ with a smooth manifold structure such that the canonical projection π_μ: $M_\mu \to F_\mu$ is a submersion. The manifold F_μ is called the *reduced symplectic manifold*. There exists a unique symplectic structure ω_μ on F_μ such that $\pi_\mu^* \omega_\mu = i_\mu^* \omega$, where i_μ: $M_\mu \to M^{2n}$ is an embedding.

A submanifold E of the symplectic manifold (M^{2n}, ω) of codimension one is called a *Hamilton-Jacobi equation*. A *Lagrangian submanifold* $i: \Lambda^n \to M^{2n}$, $i^*\omega = 0$, is called a *solution* of the equation E if $\Lambda^n \subset E$. In the case $M^{2n} = T^*M^n$, E is a submanifold given by the equation $H = 0$, $H \in C^\infty(M^{2n})$, and Λ^n is diffeomorphically projected onto M^n.

These definitions reduce to the classical ones, since in this case Λ^n is the chart of the differential of some function S, a classical solution of the Hamilton-Jacobi equation.

We assume that G is the group of symmetries of the equation E: the submanifold E is an invariant submanifold of the group G. Suppose that the intersection of E and M_μ is in general position and $E \cap M_\mu$ is a submanifold of M_μ of codimension one. The submanifold of F_μ of codimension one which consists of the collection of orbits of the points of $E \cap M_\mu$ is called the *reduced Hamilton-Jacobi equation* E_μ.

THEOREM 1. *Let Λ_μ be a Lagrangian submanifold of F_μ which is a solution of the reduced equation E_μ; then the submanifold $\Lambda^n = \pi_\mu^{-1}(\Lambda_\mu)$ is a solution of the original equation.*

PROOF. The inclusion $\Lambda^n \subset E$ is obvious. We show that Λ^n is a Lagrangian submanifold. We consider an arbitrary point $Q \in \Lambda^n$ and the tangent vectors $X, Y \in T_Q\Lambda^n$. The vectors $\pi_{\mu*}X$ and $\pi_{\mu*}Y$ are tangent to Λ_μ; therefore

$$i(Y)i(X)i_\mu^*\omega = i(Y)i(X)\pi_\mu^*\omega_\mu = i(\pi_{\mu*}Y)i(\pi_{\mu*}X)\omega_\mu = 0. \quad \blacksquare$$

The method set forth in the theorem leads to solutions of the Hamilton-Jacobi equation which are invariant with respect to the group G_μ.

REMARK. For the reduction method to be applicable, it suffices that $E \cap M_\mu$ be a submanifold which is invariant with respect to the group G_μ. It is possible that the invariance of the whole equation is essential for the construction of the complete integral.

We turn to the "contactization" of the method. A "maximal integrable" field of hyperplanes K (in other words, a local differential form ω such that $K = \ker \omega$ satisfies the condition: $d\omega/_K$ is nondegenerate (class $\omega = 2n + 1$)) is called a *contact structure* over a manifold M^{2n+1} of odd dimension. An integral manifold L^n of a contact structure K of dimension n is called *Legendre*. A *first-order differential equation* E is a submanifold of M^{2n+1} of codimension one, and a *solution* of E is a Legendre submanifold $L^n \subset E$. This definition is natural since 1) by Darboux's theorem, in some neighborhood W of each point of M^{2n+1} there exist coordinates (q, u, p) such that $\omega = du - pdq$, and 2) if L^n is projected diffeomorphically onto the submanifold $T^n \subset W$ defined by the system of equations $u = p_1 = \cdots = p_n = 0$, then L^n is defined by the equations $u = f(q)$, $p = \operatorname{grad} f$. Consequently the equation E is given locally by the classical relation $F(q, u, \partial u/\partial q) = 0$, where F is a function on W.

An important example of a contact manifold is furnished by the manifold $J^1(M^n, R^1)$ of 1-jets of real functions over M^n (see §2) having a globally defined contact form ω. In view of the fact that differential equations on $J^1(M^n, R^1)$ arise often in applications, we restrict ourselves to a consideration of the reduction of contact manifolds with global contact forms.

Suppose that the connected Lie group acts on M^{2n+1} on the left by means of the contact transformations ($g^*\omega = F(g)\omega$, where $F(g) \in C^\infty(M^{2n+1})$). The vector fields $\sigma(A)$ of (1) are contact vector fields: $L(\sigma(A))\omega = h(A)\omega$, where $L(X)\omega$ is the Lie derivative, and $h(A) \in C^\infty(M^{2n+1})$.

The contact vector fields X over M^{2n+1} are in one-to-one correspondence with the functions $f \in C^\infty(M^{2n+1})$, defined in the following way. We put $f = i(X)\omega$. We show that the contact field X corresponding to f is uniquely determined. We denote by X_1 the vector field of degeneracy of the form $d\omega$, normalized by the condition $i(X_1)\omega = 1$. We put $X = fX_1 + Y$, $Y \in \ker \omega$. The condition for X to be a contact field is

$$L(X)\omega \equiv i(Y)d\omega + df = h\omega. \tag{2}$$

If we calculate the inner products of the differential forms in (2) with the vector field X_1, we get that $h = X_1(f)$. Now $X_1(f)\omega - df \in (\ker \omega)^*$, since $i(X_1)(X_1(f)\omega - df) = 0$. Consequently the equality $i(Y)d\omega = X_1(f)\omega - df$ uniquely determines the vector field $Y \in \ker \omega$ corresponding to f by virtue of the fact that $d\omega$ is nondegenerate on $\ker \omega$. The contact vector field X corresponding to f is denoted by X_f. The function f is called the *contact Hamiltonian* of X_f.

This correspondence between contact vector fields and functions on M^{2n+1} allows us to introduce a Lie algebra structure in $C^\infty(M^{2n+1})$, defining the *Lagrange bracket*

$$(f, g) = i([X_f, X_g])\omega. \tag{3}$$

For a vector field $A \in \mathfrak{g}$, the field $\sigma(A)$ has contact Hamiltonian $f_A = i(\sigma(A))\omega$. The mapping $\hat{\psi}$: $\mathfrak{g} \to C^\infty(M^{2n+1})$, $\hat{\psi}(A) = f_A$, is linear, and $f_{[A,B]} = (f_A, f_B)$. In a manner similar to that of the symplectic case, it is possible to introduce the *contact moment* mapping ψ: $M^{2n+1} \to \mathfrak{g}^*$, $\psi(m) = m \circ \hat{\psi}$.

THEOREM 2. *The contact moment* ψ *is equivariant with respect to the coadjoint action of the group* G *if the transformations of* G *preserve* ω. *In the general contact case* ($g^*\omega = F(g)\omega$), *we have*

$$\psi \circ g(m) = F(g)(m)\mathrm{Ad}^*g^{-1} \circ \psi(m).$$

PROOF. We calculate the value of the left-invariant differential form $\psi \circ g$ on the vector field A. We have

$$\begin{aligned} i(A)\psi \circ g &= f_A \circ g = g^*(f_A) = g^*(i(\sigma(A))\omega) \\ &= g^*(i(\sigma(A)), g^{*-1} \circ g^*\omega) = i(g_*^{-1}\sigma(A))\, g^*\psi. \end{aligned} \tag{4}$$

In addition,

$$g_*^{-1}\sigma(A) = \frac{d}{dt}(g^{*-1}a_t g^*)|_{t=0} = \frac{d}{dt}(g a_t g^{-1})^*|_{t=0} = \sigma(\mathrm{Ad}\, g^{-1}(A)), \tag{5}$$

where the left invariance of A has been taken into account in the last equality. The theorem follows from (4) and (5). ■

COROLLARY. *If* 0 *is a regular value of the contact moment* ψ, *then* $M_0 = \psi^{-1}(0)$ *is an invariant submanifold of* G.

We denote by F_0 the set of orbits M_0/G of points of the submanifold M_0. Under the condition that G acts freely on M_0 by eigentransformations, there exists a manifold structure on F_0 such that the canonical projection π_0: $M_0 \to F_0$ is a submersion. In this case M_0 is a bundle with base F_0 and standard fiber G.

THEOREM 3. *Suppose there exists a section* s: $F_0 \to M_0$ *of the bundle* π_0; *then the differential form* $\omega_0 = s^* \circ i^*(\omega)$, *where* i: $M_0 \to M^{2n+1}$ *is an injection, is a contact structure on* F_0. *The class of integral manifolds of forms* ω_0 *does not depend on the choice of* s.

PROOF. We assume that there is a characteristic vector $\tilde{X} \in T_pF_0$ of ω_0 at some point $p \in F_0$ which is different from zero. If $m = s(p)$ and $X = s_*\tilde{X}$, then $i(X)\omega = 0$ and $i(Y)i(X)d\omega = 0$, where $Y = s_*\tilde{Y}$, $\tilde{Y} \in T_pF_0$ being an arbitrary vector which is tangent to the section at the point m. We consider the inner product $i(X)i(Z)d\omega$, where $Z \in T_mG(m)$ is an arbitrary vector tangent to the orbit $G(m)$ of m. Then we have the expansion $Z = \sum_1^q \alpha_j\sigma(A_j)(m)$, where $A_1, \dots, A_q$ is a basis for $\mathfrak{g}$, whence

$$\begin{aligned} i(X)\,i(Z)\,d\omega &= \sum_{j=1}^{q} \alpha_j i(X)\,i(\sigma(A_j))\,d\omega \\ &= \sum_{j=1}^{q} \alpha_j i(X)\,(L(\sigma(A_j))\,\omega - df_{A_j}) = \sum_{j=1}^{q} \alpha_j i(X)\,(X_1(f_{A_j})\,\omega - df_{A_j}) = 0, \end{aligned}$$

since X is tangent to M_0. Thus we have proved that $i(X)d(i^*\omega)(m) = 0$.

We assume that $X_1(m)$ does not lie in the tangent plane to M_0 at the point m. From the definition of the field X_1, we have that $i(X_1)d\omega = 0$; therefore in this case $i(X)d\omega(m) = 0$ and the vector X is characteristic for the form ω, which contradicts the equality class $\omega = 2n + 1$.

We suppose now that $X_1(m) \in T_mM_0$, i.e. $X_1(f_{A_j})(m) = 0, j = 1, \dots, q$. We show that in this case every vector $\sigma(A_j)(m)$ is characteristic for the form $i^*\omega$. In fact, $i(\sigma(A_j))\omega = f_{A_j} = 0$ on M_0. In addition,

$$i(\sigma(A_j))d(i^*\omega) = L(\sigma(A_j))i^*\omega - i^*(df_{A_j}) = i^*(X_1(f_{A_j})\omega - df_{A_j}) = 0.$$

Consequently the $q + 1$ linearly independent tangent vectors $X(m)\sigma(A_j)(m)$, $j = 1, \ldots, q$, are characteristic for the form $i^*\omega$, which contradicts the inequality class $i^*\omega \geqslant$ class $\omega -$ codim M_0.

We consider two sections s_1 and s_2 of the bundle π_0. Let $\varphi\colon N^k \to F_0$ be an integral manifold of the form $\omega_{01} = s_1^* \circ i^*(\omega)$; we show that it is also an integral manifold of the form $\omega_{02} = s_2^* \circ i^*(\omega)$. We consider an arbitrary vector $\tilde{\tilde{X}} \in T_n N^k$. The tangent vector $\tilde{X} = \varphi_* \tilde{\tilde{X}} \in T_p F_0$ satisfies the equation $i(\tilde{X})\omega_{01} = 0$, whence $i(X)\omega = 0$, where $X = s_{1*}\tilde{X} \in T_{m_1} M_0$, $m_1 = s_1(p)$. If $m_2 = s_2(p)$, then there exists an element g of the group G such that $m_2 = g(m_1)$. We denote the tangent vector $g_* X = s_{2*} \circ \varphi_* \tilde{\tilde{X}}$ by $Y \in T_{m_2} M_0$; then

$$i(Y)\omega = i(g_* X)\omega = g^{*^{-1}} \circ g^*\big(i(g_* X)\omega\big) = g^{*^{-1}}(i(X) g^*\omega) = g^{*^{-1}}(F(g) i(X)\omega) = 0.$$

Consequently, $i(\varphi_* \tilde{\tilde{X}})\omega_{02} = 0$, which completes the proof of the theorem. ■

COROLLARY 1. *In the absence of a global section of the bundle π_0, it is possible to furnish the manifold F_0 with a contact structure K_0 such that if L_0 is an integral submanifold of K_0, then $\pi_0^{-1}(L_0)$ is an integral manifold of the structure K on M^{2n+1}.*

PROOF. Everywhere in Theorem 3 we must understand s to be a local section. ■

Let G be the group of symmetries of the equation E, and suppose that the intersection $E \cap M_0$, in general position, is a submanifold of M_0 of codimension one. The *reduced equation* E_0 is the submanifold F_0 of orbits of the points of $E \cap M_0$.

COROLLARY 2. *Let L_0 be a Legendre submanifold of F_0, which is a solution of the reduced equation E_0; then the submanifold $\pi_0^{-1}(L_0)$ is a solution of the original equation.*

The class of solutions described in the corollary consists of G-invariant solutions. For the method of reduction to be applicable, it suffices that the submanifold $E \cap M_0$ be invariant with respect to G.

§2. A contact structure on a manifold of k-jets of functions

In what follows, a kth-order differential equation on the manifold M^n will be regarded as a submanifold of codimension one of the fibered manifold $J^k(M^n, R^1)$ of k-jets of functions defined on M^n. In what follows, we will denote the vector fibration $J^k(M^n, R^1)$ briefly by J^k. A point of the manifold J^k is an equivalence class of C^∞-functions which have kth-order contact at the point $m \in M^n$. The equivalence class of the function f is denoted by $j_k(f)(m)$ and is called the k-jet of f at the point m. The manifold structure on J^k can be introduced with the help of an atlas whose charts are defined in the following way. Let $(U; \tilde{q}_1, \ldots, \tilde{q}_n)$ be a chart of the manifold M^n, where $\tilde{q}_1, \ldots, \tilde{q}_n$ are coordinate functions on $U \subset M^n$. A *chart* on J^k is a set $V = \bigcup_{m \in U} J_m^k$ (disjoint sum), where J_m^k is a set of k-jets of functions at the point m, together with coordinate functions $\{q_j, u, p_j, \ldots, p_{i_1 \ldots i_\nu}, \ldots\}$, $1 \leqslant j \leqslant n$, $1 \leqslant i_1 \leqslant \cdots \leqslant i_\nu \leqslant n$, $2 \leqslant \nu \leqslant k$, defined on V whose values at $Q = j_k(f)(m) \in V$ are given by

$$q_j(Q) = \tilde{q}_j(m), \quad u(Q) = f(m),$$
$$p_j(Q) = \frac{\partial f(m)}{\partial q_j}, \ \ldots, \ p_{i_1 \ldots i_\nu}(Q) = \frac{\partial^\nu f(m)}{\partial q_{i_1} \cdots \partial q_{i_\nu}}, \ \ldots. \tag{6}$$

We define the submersion $\pi^k\colon J^k \to M^n$ by $\pi^k(j_k(f)(m)) = m$. The fibers J_m^k of π^k are furnished with a vector-space structure over R by the relation

$$\alpha j_k(f)(m) + \beta j_k(g)(m) = j_k(\alpha f + \beta g)(m).$$

The equalities (6) define a diffeomorphism $\varphi_V\colon V \to U \times R^N$, which trivializes π^k locally. It is also not difficult to see that the bundles given by the natural projections $\pi_l^k\colon J^k \to J^l$, where $\pi_l^k(j_k(f)(m)) = j_l(f)(m)$, are locally trivialized.

A section $s\colon M^n \to J^k$ such that there exists a function $g \in C^\infty(M^n)$ which satisfies the relation $s(m) = j_k(g)(m)$ for any point $m \in M^n$ is called a *k-jet* of the function g and is denoted by $j_k(g)$. The section s is l-integrable ($l \leqslant k$) at the point m if $j_l(\pi_0^k \circ s)(m) = \pi_l^k \circ s(m)$. We will call k-integrable sections of the bundle π^k-*integrable.*

We consider a point $Q \in J^k$ and all possible sections passing through Q which are integrable in a neighborhood of $\pi^k(Q)$. If we regard Q as an equivalence class of integrable sections of the bundle π^{k-1} which pass through the point $P = \pi_{k-1}^k(Q)$ and which have first-order contact there, then it becomes obvious that the mapping $\pi_{k-1*}^k \circ j_k(f)_*\colon T_mM^n \to T_PJ^{k-1}$ does not depend on the choice of $f, j_k(f)(m) = Q$, and it is uniquely determined by Q. Consequently there comes into consideration the bundle $\tilde{K}(J^k)$ induced by the mapping π_{k-1}^k, each fiber of which over Q is a subspace of the space T_PJ^{k-1} of the form $j_{k-1}(f)_*(T_mM^n)$, where $j_k(f)(m) = Q$.

THEOREM 4 [4]. *A section $s\colon M^n \to J^k$ is integrable at a point m if and only if*

$$\pi_{k-1*}^k \circ s_*(T_mM^n) \subset \tilde{K}_{s(m)}(J^k).$$

PROOF. The necessity is obvious from the construction of $\tilde{K}(J^k)$. To prove the sufficiency, we consider an arbitrary section $j_k(f)$ such that $j_k(f)(m) = s(m)$. From the hypothesis of the theorem it follows that the integrable sections $j_0(f)$ and $\pi_0^k \circ s$ have contact of at least the first order at m, whence $\pi_1^k \circ s(m) = j_1(\pi_0^k \circ s)(m)$, i.e. the section s is 1-integrable. The sections $\pi_1^k \circ s$ and $j_1(f)$, which are integrable at m, have contact of at least the first order at m; therefore the section s is 2-integrable, and so forth. ■

The fiber $\tilde{K}_Q(J^k)$ can be lifted to the subspace $\bar{K}_Q(J^k) \subset T_QJ^k$ by means of any k-jet $j_k(f)$ such that $j_k(f)(\pi^k(Q)) = Q$ by using the relation

$$\bar{K}_Q(J^k) = j_k(f)_*(T_{\pi^k(Q)}M^n).$$

If we fix a basis B in $T_{\pi^k(Q)}M^n$, then the bases $j_k(f)_*(B)$ in $\bar{K}_Q(J^k)$ for various f will differ by a vector from $\ker \pi_{k-1*}^k$. From this observation, it follows that the concept in the following definition is well defined.

DEFINITION. A contact structure $K(J^k)$ over the manifold J^k is called a *subbundle TJ^k with fibers $K_Q(J^k) = \bar{K}_Q(J^k) \oplus \ker \pi_{k-1*}^k(Q)$.*

COROLLARY. *Integrable sections of the bundle π^k are integral manifolds of dimension n of the contact structure $K(J^k)$.*

In a local chart of the manifold J^k it is possible to find the following basis of the $C^\infty(J^k)$-module of sections of the bundle $K(J^k)$:

$$D_j = \frac{\partial}{\partial q_j} + p_j \frac{\partial}{\partial u} + p_{\sigma(j,i)} \frac{\partial}{\partial p_i} + \cdots + p_{\sigma(j,i_1,\ldots,i_{k-1})} \frac{\partial}{\partial p_{i_1\ldots i_{k-1}}}, \tag{7}$$

$$Y_{j_1\ldots j_k} = \frac{\partial}{\partial p_{j_1\ldots j_k}}, \quad 1 \leqslant j \leqslant n, \quad 1 \leqslant j_1 \leqslant \cdots \leqslant j_k \leqslant n,$$

where $\sigma(j, i_1, \ldots, i_\nu)$ is a nondecreasing sequence of indices of the set $\{j, i_1, \ldots, i_\nu\}$. The repeated indices in (7) denote summation. Vector fields of a local basis satisfy the commutation relations

$$[D_i, D_j] = 0, \tag{8_1}$$

$$[Y_{i_1\ldots i_k}, Y_{j_1\ldots j_k}] = 0, \tag{8_2}$$

$$[Y_i, D_j] = \delta_{ij}\frac{\partial}{\partial u}, \tag{8_3}$$

$$[Y_{i_1\ldots i_k}, D_j] = Y_{i_1\ldots \hat{j}\ldots i_k}\delta_{i_1 j}\cdots\delta_{i_k j}, \tag{8_4}$$

where $i_1 \ldots \hat{j} \ldots i_k$ is a sequence of the indices $i_1, \ldots, i_k$ in which one of the indices, equal to j, is omitted, and δ_{ij} is the Kronecker symbol. The commutation relations

$$[Y_{i_1\ldots i_\nu}, D_j] = Y_{i_1\ldots \hat{j}\ldots i_\nu}\delta_{i_1 j}\cdots\delta_{i_\nu j}, \quad 2 \leqslant \nu \leqslant k-1 \tag{8_5}$$

will also be useful in what follows.

The bundle $K^*(J^k)$ dual to $K(J^k)$ is a subbundle of T^*J^k with the fibers

$$K_Q^*(J^k) = \{\omega \in T_Q^*J^k \mid i(X)\,\omega = 0 \;\; \forall X \in K_Q(J^k)\},$$

or, equivalently,

$$K_Q^*(J^k) = \{\omega \in \pi_{k-1}^{k*}(TJ^{k-1}) \mid i(X)\,\omega = 0 \;\; \forall X \in \tilde{K}_Q(J^k)\}.$$

The sections of $K_Q^*(J^k)$ form a *contact* $C^\infty(J^k)$*-module*, denoted by $\Omega(J^k)$.

THEOREM 5. *It is possible to find an invariantly defined differential form* $U_1 \in \Omega(J^k)$ *such that in each local chart of the manifold* J^k *the following differential forms constitute a local basis of the module* $\Omega(J^k)$:

$$U_1,\; L(D_j)\,U_1,\; \ldots,\; L(D_{i_1})\cdots L(D_{i_\nu})\,U_1,\; \ldots, \tag{9}$$
$$1 \leqslant j \leqslant n, \quad 1 \leqslant i_1 \leqslant \ldots \leqslant i_\nu \leqslant n, \quad 2 \leqslant \nu \leqslant k-1,$$

where $L(D_j)U_1$ *is the Lie derivative of the form* U_1 *along the field* D_j. *The forms* U_1 *and* $L(D_j)U_1$ *are horizontal for the bundle* π_1^k, *the forms* $L(D_i)L(D_j)U_1$ *are horizontal for the bundle* π_2^k, *and so forth, and the forms* $L(D_{i_1}) \cdots L(D_{i_{k-1}})U_1$ *are horizontal for the bundle* π_{k-1}^k. *The forms* $dU_1 d(L(D_j)U_1), \ldots, d(L(D_{i_1}) \cdots L(D_{i_{k-1}})U_1)$ *are horizontal for the bundles* $\pi_1^k, \ldots, \pi_{k-2}^k$, *respectively*.

PROOF. We consider first of all the case $k = 1$. We note that the section $s: M^n \to J^0$, $s = j_0(1)$, trivializes the bundle π^0; therefore $J^0 = M^n \times R^1$. Consequently $T_P J^0 = \tilde{K}_Q(J^1) \oplus TR^1$, where $P = \pi_0^1(Q)$, $Q \in J^1$. The value of the differential form U_1 at the point Q, the exterior form $U_1(Q) \in \pi_0^{1*}T_P^*J^0$, is uniquely determined by the conditions $i(X)U_1(Q) = 0$, $X \in \tilde{K}_Q(J^1)$, $i(\partial/\partial u)U_1(Q) = 1$, where $\partial/\partial u$ is a basis vector of TR^1. In local coordinates, $U_1 = du - pdq$, which proves the smoothness of the form U_1.

For arbitrary $k \geqslant 2$ we put $U_1 = \pi_1^{k*}\tilde{U}_1$, where $\tilde{U}_1$ is the differential form on J^1 defined in the preceding section. We verify that the forms $L(D_{i_1}) \cdots L(D_{i_\nu})U_1$, $1 \leqslant i_1 \leqslant \cdots \leqslant i_\nu \leqslant n$, $1 \leqslant \nu \leqslant k-1$, belong to the module $\Omega(J^k)$. Indeed $i(Y_{i_1 \ldots i_k})U_1 = 0$, since $Y_{i_1 \ldots i_k} \in \ker \pi_{1*}^k$. In addition,

$$i(D_j)\,U_1 = i(D_j)\pi_1^{k*}\tilde{U}_1 = i\big(\pi_{1*}^k D_j\big)\tilde{U}_1 = 0.$$

Hence it follows from (8_1) that $i(D_j)L(D_{i_1})\cdots L(D_{i_\nu})U_1=0$, $2\leqslant\nu\leqslant k-1$. Using the fact that

$$i(Y)L(X)-L(X)i(Y)=i([Y,X]), \tag{10}$$

the commutation relations (8_4), and the definition of U_1, we get

$$i(Y_{i_1\ldots i_k})L(D_j)U_1=i([Y_{i_1\ldots i_k}D_j])U_1=0.$$

Using (8_3), (8_4), (8_5), and a ν-fold application of (10), we can show that $i(Y_{i_1\ldots i_k})L(D_{j_1})\cdots L(D_{j_\nu})U_1=0$, $2\leqslant\nu\leqslant k-1$. The assertion in the theorem about the horizontalness of the forms in the basis and their exterior derivatives can be proved similarly.

We suppose that

$$\alpha U_1+\alpha_jL(D_j)U_1+\cdots+\alpha_{i_1\ldots i_{k-1}}L(D_{i_1})\cdots L(D_{i_{k-1}})U_1=0. \tag{11}$$

If we calculate the inner products of the left-hand side of (11) successively with the vector fields $\partial/\partial u$, $\partial/\partial p_j,\ldots,\partial/\partial p_{i_1\ldots i_\nu}$, $1\leqslant j\leqslant n$, $1\leqslant i_1\leqslant\cdots\leqslant i_\nu\leqslant n$, $2\leqslant\nu\leqslant k-1$, we get that all of the coefficients in the left-hand side of (11) are equal to zero. This proves the independence of the collection of forms under consideration. It is not difficult to see that their number is the same as the codimension of $K_Q(J^k)$ in T_QJ^k. By the same token the collection of differential forms mentioned in the hypothesis of the theorem is a local basis of the module $\Omega(J^k)$. ■

DEFINITION. The diffeomorphism σ: $J^k\to J^k$ is called a *contact diffeomorphism* if $\sigma_*(K(J^k))\subset K(J^k)$.

We study infinitesimal contact transformations of contact vector fields. The theorem below has been stated in [3], and the proof given in [5] and [6]. An independent proof is given here.

THEOREM 6. *Every contact vector field X over the manifold J^k is compatible with the projection π_1^k, and the field π_{1*}^kX is a contact field X_f over J^1, where $f=i(X)U_1\in C^\infty(J^1)$. Any contact field X_f over J^1 can be lifted uniquely to a contact field over J^k.*

PROOF. From the definition of a contact vector field it follows that

$$[X,D_j]=\alpha_i^{(j)}D_i+\beta_{i_1\ldots i_k}^{(j)}Y_{i_1\ldots i_k}, \tag{12_1}$$

$$[X,Y_{j_1\ldots j_k}]=\gamma_i^{(j_1\ldots j_k)}D_i+\delta_{i_1\ldots i_k}^{(j_1\ldots j_k)}Y_{i_1\ldots i_k}, \tag{12_2}$$

$$L(X)U_1=aU_1+a_jL(D_j)U_1+\cdots+a_{i_1\ldots i_{k-1}}L(D_{i_1})\cdots L(D_{i_{k-1}})U_1, \tag{12_3}$$

$$L(X)L(D_j)U_1\in\Omega(J^k),\ \ldots,\ L(X)L(D_{i_1})\cdots L(D_{i_{k-1}})U_1\in\Omega(J^k) \tag{12_4}$$

(the membership relation in the last expressions must be understood locally). We show that all the coefficients on the right-hand side of (12_3), with the possible exception of a, are equal to zero. We apply the operator $L(D_i)$ to both sides of (12_3). On the left-hand side we get

$$\begin{aligned}L(D_i)L(X)U_1&=L(X)L(D_i)U_1+L([D_i,X])U_1=L([D_i,X])U_1\ (\mathrm{mod}\,\Omega(J^k))\\&=-\beta_{i_1\ldots i_k}^{(i)}L(Y_{i_1\ldots i_k})U_1\ (\mathrm{mod}\,\Omega(J^k))=0\ (\mathrm{mod}\,\Omega(J^k)),\end{aligned}$$

since the forms U_1 and ∂U_1 are horizontal for the bundle π_1^k. The right-hand side becomes

$$a_{i_1\ldots i_{k-1}}L(D_{i_1})\cdots L(D_{i_{k-1}})U_1\ (\mathrm{mod}\,\Omega(J^k));$$

therefore

$$a_{i_1\dots i_{k-1}} L(D_{i_1}) \cdots L(D_{i_{k-1}}) U_1 = 0 \pmod{\Omega(J^k)}. \tag{13}$$

The inner product of the left-hand side of (13) with the vector field $Y_{j_1 \dots j_{k-1}}$ is proportional to $a_{j_1 \dots j_{k-1}}$, whence $a_{j_1 \dots j_{k-1}} = 0$. Applying the operators $L(D_{j_1}) \cdots L(D_{j_\nu})$, $2 \leqslant \nu \leqslant k-1$, successively to (12_3) and calculating the inner products of the resulting differential forms with the vertical fields $Y_{j_1 \dots j_{k-1}}$, we arrive at the proof.

From the fact that only the coefficient a in (12_3) does not vanish, it follows that the coefficients γ in (12_2) are zero. In fact,

$$L([X, Y_{j_1\dots j_k}]) U_1 = [L(X), L(Y_{j_1\dots j_k})] U_1 = -Y_{j_1\dots j_k}(a) U_1.$$

At the same time

$$L([X, Y_{j_1\dots j_k}]) U_1 = \gamma_i^{(j_1\dots j_k)} L(D_i) U_1;$$

therefore

$$\gamma_i^{(j_1\dots j_k)} L(D_i) U_1 = -Y_{j_1\dots j_k}(a) U_1. \tag{14}$$

The forms in (9) are independent; consequently $\gamma_i^{(j_1 \dots j_k)} = 0$ and $Y_{j_1 \dots j_k}(a) = 0$. The fact that the coefficients γ in (12_2) are equal to zero means that the contact field X retains the fibers of π^k_{k-1} and hence is compatible with the transformation π^k_{k-1}. If $k > 2$, extending the argument, we show that the field X preserves the fibers of π^k_{k-2}, and so forth. To do this, we calculate the Lie derivative along the vector field $Y_{j_1 \dots j_k}$ of the left and right sides of (12_1). We have

$$-[Y_{j_1\dots j_k}, [X, D_j]] = [X, [D_j, Y_{j_1\dots j_k}]] + [D_j, [Y_{j_1\dots j_k}, X]]$$
$$= [X, Y_{j_1\dots \hat{j}\dots j_k}] - [D_j, \delta^{(j_1\dots j_k)}_{i_1\dots i_k} Y_{i_1\dots i_k}] = [X, Y_{j_1\dots \hat{j}\dots j_k}] \pmod{\ker \pi^k_{k-2*}},$$
$$L(Y_{j_1\dots j_k})(\alpha_i^{(j)} D_i + \beta^{(j)}_{i_1\dots i_k} Y_{i_1\dots i_k}) = Y_{j_1\dots j_k}(\alpha_i^{(j)}) D_i \pmod{\ker \pi^k_{k-2*}},$$

which gives the commutation relations

$$[X, Y_{j_1\dots j_{k-1}}] = \varepsilon_i^{(j_1\dots j_{k-1})} D_i \pmod{\ker \pi^k_{k-2*}},$$

from which it is easy to deduce by (12_3) a relation of the form (14):

$$\varepsilon_i^{(j_1\dots j_{k-1})} L(D_i) U_1 = -Y_{j_1\dots j_{k-1}}(a) U_1.$$

This last relation means that

$$\varepsilon_i^{(j_1\dots j_{k-1})} = 0, \quad Y_{j_1\dots j_{k-1}}(a) = 0.$$

If $k > 3$, then we can establish in a similar way that X is compatible with π^k_{k-3}, and that $a \in C^\infty(J^{k-3})$, and so forth. Finally we conclude that the contact field X preserves the fibers of π_1^k and that $\pi^k_{1*}X$ is a contact field over J^1, since

$$L(\pi^k_{1*}X)\tilde{U}_1 = L(X)\pi_1^{k*}\tilde{U}_1 = L(X) U_1 = aU_1,$$

where $a \in C^\infty(J^1)$.

The fact that the correspondence $X \to X_f$, where X_f is a contact field over J^1, $f = i(X)U_1$,is one-to-one follows from the simple fact that the integrable sections $j_k(g)$ are uniquely determined by their projection $j_0(g)$ in J^0, so that a nonzero vertical field of the bundle π_0^k cannot be contact.

Let X_f be an arbitrary contact field over the manifold J^1, and let σ_t be the corresponding local one-parameter group of diffeomorphisms of J^1. We define the lift $\sigma_t^{(k)}$ of σ_t to the local one-parameter group of diffeomorphisms of J^k. If the point Q belongs to J^k, then $\sigma_t^{(k)}(Q)$ is determined in the following way. We will regard Q as an equivalence class P_Q of germs of integrable sections of a bundle of J^1 at the point $\pi^k(Q)$ which have $(k-1)$th-order contact at this point. Diffeomorphisms of the group σ_t, being contact, map germs of integrable sections into germs of integrable sections (at least for sufficiently small t, perhaps depending on the germ). Therefore the definition $\sigma_t^{(k)}(Q) = \sigma_t(P_Q)$ is a proper one. The vector field X, the generator of the group $\sigma_t^{(k)}$, is the lift of X_f to a contact field over J^k. Indeed, the diffeomorphisms $\sigma_t^{(k)}$ preserve the fibers of π_{k-1}^k and (for sufficiently small t) carry germs of integrable sections into germs of integrable sections. Consequently $\sigma_{t*}^{(k)}(K(J^k)) \subset K(J^k)$. ■

§3. Symmetry and reduction of differential equations of arbitrary order

We consider a *kth-order differential equation E on a connected manifold M^n*, i.e. a submanifold of J^k of codimension one. Functions $g \in C^\infty(M^n)$ such that $j_k(g)(M^n) \subset E$ are called *classical solutions of E*. Integral submanifolds $L \subset J^k$ of the contact structure $K(J^k)$ having dimension n and belonging to E are called *solutions of E*.

Suppose that the connected Lie group G acts on the left on the manifold J^k by means of contact transformations and is the group of symmetries of the equation E (E is invariant with respect to G). If $A_1, \dots, A_q$ constitute a basis of the Lie algebra of G, then $\sigma(A_j)$, $j = 1, \dots, q$, are contact vector fields over J^k. By virtue of Theorem 6, the contact vector fields $X_j = \pi_{1*}^k\sigma(A_j)$ with contact Hamiltonians $f_j = i(\sigma(A_j))U_1$ are well defined on J^1, and in this way there arises the contact action of G on the contact manifold J^1.

Suppose that zero is a regular value of the contact moment ψ of the action of G on J^1; then $M_0 = \psi^{-1}(0) = \{Q \in J^1 | f_j(Q) = 0, j = 1, \dots, q\}$ is an invariant submanifold of G. *In what follows, we will assume that the condition*, stated in §1, *which guarantees the existence of the contact manifold $F_0 = M_0/G$, is fulfilled.* We will denote the canonical projection by $\pi_0\colon M_0 \to F_0$, an arbitrary (in the general case, local) differential form on F_0 such that $\pi_0^* U_0 = i_0^* U_1$, where $i_0\colon M_0 \to J^1$ is an injection, by U_0, and the differential form on J^1 constructed in Theorem 5 by U_1. We also put $\kappa(\mathfrak{g}) = \pi_{1*}^k\sigma(\mathfrak{g})$.

THEOREM 7. *The characteristic subspace $\kappa(\mathfrak{g})(Q) \subset T_QM_0$ is anti-orthogonal to the subspace $N(Q) = T_QM_0 \cap \ker U_1(Q)$ in the linear space* $\ker U_1(Q)$ *furnished with the symplectic form $dU_1(Q)$.*

The proof follows directly from the fact that $i(X_j)dU_1 = X_1(f_j)U_1 - df_j$. ■

COROLLARY 1. *Every Lagrangian plane $L \subset N(Q)$ contains $\kappa(\mathfrak{g})(Q)$.*

COROLLARY 2. *The contact Hamiltonians f_j belong to the involution with respect to the Lagrange brackets* (3) *on the submanifold M_0.*

Corollary 2 makes it possible to treat M_0 as an involutive system of first-order differential equations.

COROLLARY 3. *Every solution of the system M_0 (Legendre submanifold $L^n \subset M_0$) is an invariant submanifold of the group G.*

An involutive system is said to be *regular* if the restriction of the differential form U_1 to the system is different from zero.

THEOREM 8. *The involutive system M_0 is regular.*

PROOF. We assume the contrary: $T_Q M_0 \subset \ker U_1(Q)$ at some point $Q \in M_0$. The codimension of $T_Q M_0$ in $\ker U_1(Q)$ is equal to $2n - (2n + 1 - q) = q - 1$; consequently the dimension of the anti-orthogonal complement $T_Q M_0$ relative to the form $dU_1(Q)$, by virtue of its nondegeneracy, is $q - 1$. By Theorem 7, every vector of the subspace $\kappa(\mathfrak{g})(Q)$ is anti-orthogonal to $T_Q M_0$. By an assumption in §1, G acts freely on M_0; therefore $\dim \kappa(\mathfrak{g})(Q) = q$. This contradiction shows that $T_Q M_0 \not\subset \ker U_1(Q)$. ■

For the proof of the next theorem, it will be convenient to have a result established in [7].

LEMMA. *Let Λ^n be a Lagrange submanifold of the symplectic manifold (M^{2n}, ω), and let $T^*\Lambda^n$ be the cotangent bundle, furnished with the canonical symplectic form Ω (locally, $\Omega = dp \wedge dq$). There exist neighborhoods $U \supset \Lambda^n$ in M^{2n} and $V \supset 0(\Lambda^n)$ in $T^*\Lambda^*$, where 0 is the zero section of $T^*\Lambda^n$, and a diffeomorphism $F\colon U \to V$ such that $F^*(\Omega|_V) = \omega|_U$ and $F(\Lambda^n) = 0(\Lambda^n)$.*

THEOREM 9. *Suppose that the subspaces $\kappa(\mathfrak{g})(Q)$ are projected without degeneracy onto $T_{\pi(Q)}M^n$ at each point $Q \in M_0$; then the projection $\pi_0(Q_0) \in F_0$ of any point $Q_0 \in M_0$ is contained in a neighborhood $W \subset F_0$ furnished with a coordinate system (ξ, η, χ) such that $U_0|_W = d\eta - \chi d\xi$. Every function $\varphi(\xi)$ defines a local solution $L^n = G(s(j_1(\varphi)(T^{n-q}))) \subset M_0$ which passes through Q_0 (a local "general integral" of M_0). Here $s\colon W \to M_0$ is a local section of the bundle π_0, and T^{n-q} is a submanifold given by the system of equations $\eta = \chi_1 = \cdots = \chi_{n-q} = 0$. The solution L^n is projected diffeomorphically onto M^n in some neighborhood $\mathfrak{D} \supset s(W)$.*

PROOF. By Theorem 8, the characteristic subspace $N(Q_0) = T_{Q_0}M_0 \cap \ker U_1(Q_0)$ has codimension q in the space $\ker U_1(Q_0)$. By virtue of the way in which U_1 was constructed in Theorem 5, the vertical subspace $N_1(Q_0) = \ker \pi^1_*(Q_0) \cap \ker U_1(Q_0)$ has dimension n and is an isotropic subspace of the form $dU_1(Q_0)$. We determine the dimension of $\bar{P}(Q_0) = N(Q_0) \cap N_1(Q_0)$. From the hypotheses of the present theorem it follows that $\bar{P}(Q_0) \cap \kappa(\mathfrak{g})(Q_0) = 0$; therefore $\dim \bar{P}(Q_0) \leqslant n - q$, since otherwise $\bar{P}(Q_0) \oplus \kappa(\mathfrak{g})(Q_0)$ would be an isotropic space of the form $dU_1(Q_0)$ of dimension greater than n by Theorem 7. At the same time,

$$\dim N_1(Q_0) - \dim \bar{P}(Q_0) \leqslant \operatorname{codim} N(Q_0) = q,$$

whence $\dim \bar{P}(Q_0) = n - q$.

We consider an arbitrary local section $s\colon \mathfrak{U} \to M_0$ of the bundle π_0 passing through Q_0. The Lagrangian subspaces $P(Q) = \bar{P}(Q) \oplus \kappa(\mathfrak{g})(Q)$ are defined for the points $Q \in s(\mathfrak{U})$. Their projections $\pi_{0*}(P(Q)) \subset T_{\pi_0(Q)}F_0$ induce an involutive subbundle $S \subset T\mathfrak{U}$ on $\mathfrak{U}$ since the differential forms U_0 and dU_0 vanish on $\pi_{0*}(P(Q))$.

Let $\tilde{L}$ be an integral submanifold for S of some neighborhood $\mathfrak{U}_1 \subset \mathfrak{U}$ containing $\pi_0(Q_0)$, and let Z_1 be the vector field of degeneracies of dU_0 which satisfy the condition $i(Z_1)U_0 = 1$. We consider the quotient manifold $\mathfrak{U}_2 = \mathfrak{U}_1/\sigma_t$, where σ_t is the local one-parameter group of diffeomorphisms of the field Z_1. The form dU_0 induces a symplectic differential form ω, $\tilde{\pi}^*\omega = dU_0$, on $\mathfrak{U}_2$, where $\tilde{\pi}$ is the canonical projection of

$\mathfrak{U}_1$ on $\mathfrak{U}_2$. The field Z_1 is nowhere tangent to $\tilde{L}$; therefore $\tilde{\pi}(\tilde{L})$ is a Lagrangian submanifold of $\mathfrak{U}_2$ (if $\mathfrak{U}_1$ is sufficiently small). By virtue of the lemma, in some neighborhood $\tilde{W} \subset \mathfrak{U}_2$ of the point $\tilde{\pi} \circ \pi_0(Q_0)$ there is a system of coordinates $(\tilde{\xi}, \tilde{\chi})$ such that $\omega = -\widetilde{d\chi} \wedge \widetilde{d\xi}$, and the equations $\chi_1 = \cdots = \chi_{n-q} = 0$ give $\tilde{\pi}(\tilde{L})$ in $\tilde{W}$. Consequently in some neighborhood $W \subset \tilde{\pi}^{-1}(\tilde{W})$ it is possible to find a system of coordinates (ξ, η, χ) such that $U_0|_W = d\eta - \chi d\xi$, where $\xi = \tilde{\pi}^*(\tilde{\xi})$, $\chi = \tilde{\pi}^*(\tilde{\chi})$, $Z_1(\eta) = 1$, $\eta|_{\tilde{L}} = 0$.

Every Legendre submanifold $L_\varphi = j_1(\varphi)(T^{n-q})$, where $T^{n-q} = \tilde{L} \cap W$, is transverse to the subbundle $S \oplus RZ_1$; therefore the submanifolds $G(s(L_\varphi))$ are local solutions of the system M_0 diffeomorphically projected onto M^n in some neighborhood $\mathfrak{O} \supset s(W)$. ■

An *extension* $M_0^{(k)} \subset J^k$ *of the system of equations* M_0 is defined to be a set of k-jets $j_k(g)(m)$ of local classical solutions of M_0 at the points $m \in M^n$.

THEOREM 10. *Suppose that the hypothesis of Theorem 9 is satisfied; then the extension* $M_0^{(k)}$ *of* M_0 *is a G-invariant submanifold of* J^k *of dimension* $\dim J^k(R^{n-q}, R^1) + q$.

PROOF. The submanifold $M_0^{(k)}$ is G-invariant, as follows from Corollary 3 of Theorem 7 and the construction, in the proof of Theorem 6, of a lifting of contact vector fields on J^1 to contact vector fields on J^k.

We show that any local (in some neighborhood $\mathfrak{O} \supset s(W)$) solution L of M_0 of the form $L = j_1(g)(U)$, $U \subset M^n$, is representable in the form $L = G(s(j_1(\varphi)(T^{n-q})))$, where $s\colon W \to M_0$ is the section described in Theorem 9. The tangent planes $\Lambda(Q)$ to L at any point $Q \in s(W) \cap L$ are transverse in the space $\ker U_1(Q)$ to the vertical planes $\bar{P}(Q) = N(Q) \cap N_1(Q)$; therefore the planes $\pi_{0*}(\Lambda(Q))$ are transverse to the subbundle $S \oplus RZ_1$ constructed in the proof of Theorem 9. Consequently the submanifold $\pi_0(L) \cap W$ is projected in a one-to-one manner onto T^{n-q} along the integral submanifolds of the involutive subbundle $S \oplus RZ_1$. Theorem 9 allows us to identify $W \subset F_0$ with $J^1(T^{n-q}, R^1)$; hence it follows from Theorem 4 that $\pi_0(L) \cap W = j_1(\varphi)(T^{n-q})$.

This fact and Theorem 9 imply that the dimension of the fiber $M_0^{(k)}$ over the point $Q \in M_0$ is equal to the dimension of a fiber of the manifold $J^k(R^{n-q}, R^1)$ over a point of $J^1(R^{n-q}, R^1)$, whence

$$\begin{aligned}\dim M_0^k = \dim J^k(R^{n-q}, R^1) - \dim J^1(R^{n-q}, R^1)\\ + 2n + 1 - q = J^k(R^{n-q}, R^1) + q.\end{aligned}$$

The submanifold $M_0^{(k)}$ is given locally in J^k by the system of equations $f_j = 0$, $D_{i_1} \cdots D_{i_\nu} f_j = 0$, $1 \leqslant j \leqslant q$, $1 \leqslant i_1 \leqslant \cdots \leqslant i_\nu \leqslant n$, $2 \leqslant \nu \leqslant k-1$, of which $\dim J^k(R^{n-q}, R^1) + q$ are independent. ■

THEOREM 11. *Let* f *be a germ of the solution* M_0 *at the point* $m \in M^n$, *and let* $Q = j_k(f)(m) \in M_0^{(k)}$; *then* f *defines a lift* $\bar{K}_Q(J^k)$ *of the fiber* $\tilde{K}_Q(J^k)$ *of the bundle* $\tilde{K}(J^k)$ *to a subspace in* $T_Q M_0^{(k)}$.

PROOF. We put $\bar{K}_Q(J^k) = j_k(f)_*(T_{\pi^k(Q)}M^n)$; then

$$\pi^k_{k-1*}\bar{K}_Q(J^k) = \tilde{K}_Q(J^k). \quad ■$$

COROLLARY 1. *The dimension of the space Z_Q of vertical vectors of the bundle π^k_{k-1} which are tangent to $M_0^{(k)}$ at the point Q is equal to the dimension of a fiber of the bundle ρ^k_{k-1}: $J^k(R^{n-q}, R^1) \to J^{k-1}(R^{n-q}, R^1)$.*

COROLLARY 2. *The subspace $\sigma(\mathfrak{g})(Q) \subset T_Q M_0^{(k)}$ belongs to every lift $\bar{K}_Q(J^k)$.*

At each point $Q \in M_0^{(k)}$ we consider the direct sum $K_Q(M_0^{(k)}) = \bar{K}_Q(J^k) \oplus Z_Q \subset T_Q M_0^{(k)}$, where Z_Q is the linear subspace of vertical vectors of the bundle π^k_{k-1} which are tangent to a fiber of the bundle π^k_{k-1}: $M_0^{(k)} \to M_0^{(k-1)}$. It is clear that $K_Q(M_0^{(k)})$ is a subspace of $K_Q(J^k)$ and consists of the vectors of the contact structure which are tangent to the submanifold $M_0^{(k)}$ at the point Q. Since the action of G on J^k is contact, the submanifold $M_0^{(k)}$ is G-invariant; the system M_0 is involutive since

$$g_*\left(K_Q(M_0^{(k)})/\sigma(\mathfrak{g})(Q)\right) = K_{g(Q)}(M_0^{(k)})/\sigma(\mathfrak{g})(g(Q)). \tag{15}$$

We denote by $F_0^{(k)} = M_0^{(k)}/G$ the manifold of orbits of points of $M_0^{(k)}$ and by $\pi_0^{(k)}$: $M_0^{(k)} \to F_0^{(k)}$ the canonical projection. The manifold $F_0^{(k)}$ is called a *reduced manifold of k-jets of functions*, and $\dim F_0^{(k)} = \dim J^k(R^{n-q}, R^1)$ by Theorem 10.

THEOREM 12. *A reduced manifold can be furnished with a subbundle $K(F_0^{(k)}) \subset TF_0^{(k)}$ such that $(\pi_0^{(k)})^{-1}(L_0)$ is an integral submanifold of the contact structure $K(J^k)$ if L_0 is an integral submanifold of $K(F_0^{(k)})$. Here*

$$\dim K(F_0^{(k)}) = \dim K(J^k(R^{n-q}, R^1).$$

PROOF. The theorem is an immediate consequence of (15) and Corollary 1 to Theorem 11. ■

Suppose that the differential equation E intersects $M_0^{(k)}$ in general position and induces on it a submanifold $E \cap M_0^{(k)}$ of codimension one; then the *reduced equation* $E_0 \subset F_0^{(k)}$ is defined as the submanifold of orbits of points of the intersection $E \cap M_0^{(k)}$.

COROLLARY. *Suppose that L_0 is an integral submanifold of the subbundle $K(F_0^{(k)})$ belonging to E_0 (a solution of the reduced equation); then $(\pi_0^{(k)})^{-1}(L_0)$ is a G-invariant solution of the equation E.*

THEOREM 13. *The manifold $F_0^{(k)}$ is locally contact diffeomorphic to the manifold $J^k(R^{n-q}, R^1)$.*

PROOF. We consider the point $Q_0 \in M_0$. By Theorems 9 and 10 there exists a neighborhood W of the point $\pi_0(Q_0) \in F_0$ furnished with a contact system of coordinates (ξ, η, χ) such that every solution g_* of the system M_0 is generated by a function $\varphi(\xi)$. By the construction of the reduced submanifold $F_0^{(k)}$, to its points $\pi_0^{(k)}(j_k(g_\varphi)(m))$ are assigned the well-defined k-jet coordinates of the points $j_k(\varphi)(\xi)$, where ξ is the coordinate of the projection $\pi_0(j_1(g_\varphi)(m)) \in W \subset F_0$. By the same token, $F_0^{(k)}$ is locally isomorphic to $J^k(R^{n-q}, R^1)$. By its construction, the reduced contact structure $K(F_0^{(k)})$ clearly coincides locally with the canonical contact structure of the manifold $J^k(R^{n-q}, R^1)$. ■

COROLLARY. *The reduced equation E_0 is locally a differential equation.*

Theorems 9 and 13 furnish the basis for the following method for constructing G-invariant solutions of a differential equation E. We must substitute the "general integral" g_φ of the system M_0 into the equation E, and as a result we get a differential equation for φ.

The author takes this opportunity to thank R. S. Ismagilov, V. P. Maslov, and N. N. Nehorošev for a number of helpful comments.

Received 17/MAY/78; 7/DEC/79

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Translated by H. T. JONES