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# The Hunter–Saxton equation: remarkable structures of symmetries and conserved densities

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**Abstract**

In this paper, we present extraordinary algebraic and geometrical structures for the Hunter–Saxton equation: infinitely many commuting and noncommuting $x, t$-independent higher order symmetries and conserved densities. Using a recursive relation, we explicitly generate infinitely many higher order conserved densities dependent on arbitrary parameters. We find three Nijenhuis recursion operators resulting from Hamiltonian pairs, of which two are new. They generate three hierarchies of commuting local symmetries. Finally, we give a local recursion operator depending on an arbitrary parameter.

As a by-product, we classify all anti-symmetric operators of a definite form that are compatible with the Hamiltonian operator $D_x^{-1}$.

Mathematics Subject Classification: 37K05, 37K10, 35Q53

## 1. Introduction

The Hunter–Saxton (HS) equation

$$u_{xt} = 2uu_{xx} + u_x^2 \tag{1}$$

was proposed by Hunter and Saxton as an asymptotic model for the propagation of weakly nonlinear unidirectional waves [7]. Its integrability was proved by Hunter and Zheng [8] by studying the nonlocal evolution equation

$$u_t = 2uu_x - D_x^{-1}u_x^2, \tag{2}$$

where $D_x^{-1}$ is the inverse of the total derivative $D_x$. Indeed, equation (2) can be written as

$$u_t = D_x^{-1}\,\delta_u(-uu_x^2) = (u_xD_x^{-2} - D_x^{-2}u_x)\,\delta_u(-u_x^2), \tag{3}$$

where $\delta_u$ is the variational derivative with respect to the dependent variable $u$. Operators $D_x^{-1}$ and $u_xD_x^{-2} - D_x^{-2}u_x$ are Hamiltonian and form a Hamiltonian pair. This leads to a recursion operator of the HS equation,

$$\Re = (u_xD_x^{-2} - D_x^{-2}u_x)D_x. \tag{4}$$

  

The $x$-derivative of HS equation (1), that is

$$u_{xxt} = 2uu_{xxx} + 4u_x u_{xx} \tag{5}$$

is closely related to the Camassa–Holm equation

$$u_t - u_{xxt} - 3uu_x + 2u_x u_{xx} + uu_{xxx} = 0.$$

We often write the Camassa–Holm equation in the form

$$m_t = um_x + 2mu_x, \qquad m = u_{xx} - u.$$

Equation (5) corresponds to $m = u_{xx}$ under the time scaling transformation $t \mapsto 2t$. As pointed in [8], the bi-Hamiltonian structure (3) and the Lax pair of the HS equation can be obtained from the corresponding known structures of the Camassa–Holm equation. Geometrically, the HS equation (1) describes geodesic flow associated with the right-invariant metrics on a homogeneous space [9]. It is a particular case of the Euler–Poincaré equation on the diffeomorphisms in one spatial dimension [6].

Recently, equation (2) was proposed as a model to describe shortwave perturbation in a relaxing one-dimensional medium. Its integrability was studied by introducing $v = D_x^{-1}u_x^2$. This leads to the hydrodynamic system

$$u_t = 2uu_x - v, \qquad v_t = 2uv_x,$$

which is called an integrable regularization of equation (2). We refer to [5, 13] and references there in for more details in this aspect.

In this paper, we look at the HS equation (1) in its own right instead of the traditional approach of relating it to the Korteweg–de Vries equation and the Camassa–Holm equation [9]. Quite surprisingly, we find that it possesses extraordinarily rich algebraic structures: possessing infinitely many commutative and noncommutative $x, t$-independent higher order symmetries and conservation laws. Besides the bi-Hamiltonian structure in (3), we found another two bi-Hamiltonian structures. Using these operators, we can obtain local recursion operators for the HS equation (1).

We note that equation (5) can be linearized by the transformation [1]

$$v = \frac{u_x}{u_{xx}}, \qquad y = u_x. \tag{6}$$

The linearized equation is

$$v_t = -y^2 v_y - 3yv. \tag{7}$$

However, we did not find a direct way to produce the results in this paper via the linearization. We also note that the calculation in this paper is purely algebraic. We do not justify it in analytical sense.

The arrangement of the paper is as follows: in section 2, we define the required concepts such as Hamiltonian, symplectic and Nijenhuis operators, symmetries, cosymmetries, conservation laws and recursion operators for evolution equations in the context of the variational complex. We then devote the rest of the paper to the study of symmetries and conservation laws of the HS equation (2). In section 3, we present a recursive relation to generate infinitely many conserved densities (see theorem 2). We compute their Poisson brackets with respect to the Hamiltonian operator $D_x^{-1}$ and obtain only three commuting pairs among conserved densities $T_1^{(\alpha)}$ and $T_2^{(\beta,\gamma)}$ defined in theorem 2. In section 4, we find three recursion operators corresponding to three commuting pairs obtained in the previous section (see theorem 3). To prove that these operators are Nijenhuis, we classify all anti-symmetric operators of the form

$$\mathcal{H} = f(u_x, u_{xx})D_x + D_x f + g(u_x, u_{xx}, u_{xxx})D_x^{-1}h(u_x, u_{xx}, u_{xxx}) + hD_x^{-1}g, \qquad f \neq 0,$$

which are compatible to Hamiltonian operator $D_x^{-1}$. Here $f$ is a smooth function of $u_x$ and $u_{xx}$; $g$ and $h$ are smooth functions of $u_x$, $u_{xx}$ and $u_{xxx}$. We list all five cases that arise (see theorem 4). From one of the Nijenhuis recursion operators, we construct a parameter-dependent local recursion operator, which is no longer Nijenhuis (see corollary 2). Finally, we complete the paper with some discussion in section 5.

## 2. Definitions

In this section, we sketch the basic definitions of Hamiltonian, symplectic and Nijenhuis operators following [2, 4, 17]. In the context of the variational complex we also define the some concepts for evolution equations such as symmetries, cosymmetries, conservation laws and recursion operators, which also serves to fix our notation.

### *2.1. Complex of variational calculus*

Let $x$, $t$ be the independent variables and $u$ be a (vector-valued) dependent variable. All smooth functions depending on $u$ and $x$-derivatives of $u$ up to some finite, but unspecified order form a differential ring $\mathcal{A}$ with total $x$-derivation

$$D_x = \sum_{k=0}^{\infty} u_{k+1} \frac{\partial}{\partial u_k}, \qquad \text{where } u_k = \partial_x^k u.$$

The highest order of $x$-derivative is called the order of a given function. For any element $g \in \mathcal{A}$, we define an equivalence class (or a functional) $\int g$ by saying that $g$ and $h$ are equivalent denoted $g \equiv h$ if and only if $g - h \in \operatorname{Im} D_x$. Without causing confusion we sometimes write $g$ instead of $\int g$ . The space of functionals, denoted by $\mathcal{A}'$, does not inherit the ring structure from $\mathcal{A}$.

The derivations on the ring $\mathcal{A}$ commuting with $D_x$ are known as evolutionary vector fields. They are of the form $\partial_P = \sum_{k=0}^{\infty} D_x^k P \frac{\partial}{\partial u_k}$. Let $\mathfrak{h}$ denote the space of all such $P$. The natural commutator of derivations leads to the Lie bracket on $\mathfrak{h}$, that is,

$$[P,\ Q] = D_Q[P] - D_P[Q], \qquad P, Q \in \eta, \tag{8}$$

where $D_Q = \sum_{i=0}^{\infty} \frac{\partial Q}{\partial u_i} D_x^i$ is the Fréchet derivative of $Q$.

The action of any element $P \in \mathfrak{h}$ on $\int g \in \mathcal{A}'$ can be defined as

$$P \int g = \int \partial_P(g) = \int \sum_{k=0}^{\infty} D_x^k P \frac{\partial g}{\partial u_k} = \int D_g[P]. \tag{9}$$

This action is a representation of the Lie algebra $\mathfrak{h}$. We build up a Lie algebra complex associated with it. This complex is called the complex of variational calculus. Here we give the first few steps.

We denote the space of functional $n$-forms by $\Omega^n$ starting with $\Omega^0 = \mathcal{A}'$. We now consider the space $\Omega^1$. For any vertical 1-form on the ring $\mathcal{A}$, i.e. $\omega = \sum_{k=0}^{\infty} h^k \mathrm{d}u_k$, there is a natural nondegenerate pairing with an element $P \in \mathfrak{h}$:

$$\langle \omega, P \rangle = \int \sum_{k=0}^{\infty} h^k\ D_x^k P = \int \left( \sum_{k=0}^{\infty} (-D_x)^k h^k \right) P. \tag{10}$$

Thus any element of $\Omega^1$ is completely defined by $\xi = \sum_{k=0}^{\infty} (-D_x)^k h^k$.

The pairing (10) allows us to give the definition of (formal) adjoint operators to linear (pseudo)-differential operators [12].

**Definition 1.** *Given a linear operator* $\mathcal{S}: \mathfrak{h} \to \Omega^1$, *we call the operator* $\mathcal{S}^\star: \mathfrak{h} \to \Omega^1$ *the adjoint operator of* $\mathcal{S}$ *if* $\langle \mathcal{S}P_1,\ P_2\rangle = \langle \mathcal{S}^\star P_2,\ P_1\rangle$, *where* $P_i \in \mathfrak{h}$ *for* $i = 1,\ 2$.

Similarly, we can define the adjoint operator for an operator mapping from $\Omega^1$ to $\mathfrak{h}$, from $\mathfrak{h}$ to $\mathfrak{h}$ or from $\Omega^1$ to $\Omega^1$.

The variational derivative of each functional $g \in \mathcal{A}'$ denoted by $\delta_u g \in \Omega^1$ is defined so that

$$\langle \delta_u g,\ P\rangle = \left(\mathrm{d}\int g\right)(P) = \left\langle \sum_{k=0}^{\infty}(-D_x)^k \frac{\partial g}{\partial u_k},\ P\right\rangle, \tag{11}$$

where $\mathrm{d}: \Omega^n \to \Omega^{n+1}$ is a coboundary operator. Due to the nondegeneracy of the pairing (10), we have

$$\delta_u g = \sum_{k=0}^{\infty}(-D_x)^k \frac{\partial g}{\partial u_k} \in \Omega^1.$$

In the literature one often uses E referring to the Euler operator instead of $\delta_u$.

For any $\xi \in \Omega^1$, by direct calculation we obtain $\mathrm{d}\xi = D_\xi - D_\xi^\star$. We say that the 1-form $\xi$ is closed if $\mathrm{d}\xi = 0$.

Finally, we give the formulae of Lie derivatives along any $K \in \mathfrak{h}$ using Fréchet derivatives, cf [2] for the details.

**Definition 2.** *Let* $L_K$ *denote the Lie derivative along* $K \in \mathfrak{h}$. *We have*

$$\begin{aligned}
&L_K g = \int D_g[K] \qquad \text{for } g \in \mathcal{A}';\\
&L_K h = [K, h] \qquad \text{for } h \in \mathfrak{h};\\
&L_K \xi = D_\xi[K] + D_K^\star(\xi) \qquad \text{for } \xi \in \Omega^1;\\
&L_K \Re = D_\Re[K] - D_K\Re + \Re D_K \qquad \text{for } \Re: \mathfrak{h} \to \mathfrak{h};\\
&L_K \mathcal{H} = D_\mathcal{H}[K] - D_K\mathcal{H} - \mathcal{H}D_K^\star \qquad \text{for } \mathcal{H}: \Omega^1 \to \mathfrak{h};\\
&L_K \mathcal{I} = D_\mathcal{I}[K] + D_K^\star\mathcal{I} + \mathcal{I}D_K \qquad \text{for } \mathcal{I}: \mathfrak{h} \to \Omega^1.
\end{aligned}$$

In this complex we can identify most of the important concepts in the study of integrable systems such as symmetries, cosymmetries, conservation laws and recursion operators. They are all characterized by the vanishing of the Lie derivatives with respect to a given evolution equation. This will be discussed further in section 2.3.

### *2.2. Symplectic, Hamiltonian and Nijenhuis operators*

**Definition 3.** *A linear operator* $\mathcal{S}: \mathfrak{h} \to \Omega^1$ *(or* $\Omega^1 \to \mathfrak{h}$*) is anti-symmetric if* $\mathcal{S} = -\mathcal{S}^\star$.

Given an anti-symmetric operator $\mathcal{I}: \mathfrak{h} \to \Omega^1$, there is an anti-symmetric 2-form associated with it. Namely,

$$\omega(P, Q) = \langle \mathcal{I}(P), Q\rangle = -\langle \mathcal{I}(Q), P\rangle = -\omega(Q, P), \qquad P, Q \in \mathfrak{h}. \tag{12}$$

Here the functional 2-form $\omega$ has the canonical form [12]

$$\omega = \frac{1}{2}\int \mathrm{d}u \wedge \mathcal{I}\mathrm{d}u. \tag{13}$$

**Definition 4.** *An operator* $\mathcal{I}: \mathfrak{h} \to \Omega^1$ *is called symplectic if and only if the anti-symmetric 2-form (13) is closed, i.e.* $\mathrm{d}\omega = 0$.

It is useful to know that for any $\xi \in \Omega^1$, if $\xi$ is not closed, the operator $\mathrm{d}\xi = D_\xi - D_\xi^\star$ is a symplectic operator. This can be used to generate new symplectic operators or to determine whether a given operator is symplectic or not.

For an anti-symmetric operator $\mathcal{H} : \Omega^1 \to \mathfrak{h}$, we define a bracket of two functionals $f$ and $g$ as

$$\{f, g\}_\mathcal{H} = \langle \delta_u f, \mathcal{H}\delta_u g\rangle. \tag{14}$$

**Definition 5.** *The operator $\mathcal{H}$ is Hamiltonian if the bracket defined by (14) is Poisson, that is, anti-symmetric and satisfies the Jacobi identity*

$$\{\{f, g\}_\mathcal{H}, h\}_\mathcal{H} + \{\{g, h\}_\mathcal{H}, f\}_\mathcal{H} + \{\{h, f\}_\mathcal{H}, g\}_\mathcal{H} = 0.$$

For the Jacobi identity, there are several equivalent formulae given in [2] (see theorem 5.1). In [12] (see p 443), it is formulated as the vanishing of the functional tri-vector:

$$\int \theta \wedge D_\mathcal{H}[\mathcal{H}\theta] \wedge \theta = 0.$$

We are going to use it to classify all the Hamiltonian operators of a given family of operators in section 4.

Let $\mathcal{H}$ be a Hamiltonian operator. The Hamiltonian vector fields and their Hamiltonians possess the property [2, 12]:

$$\mathcal{H}\delta_u\{f, g\}_\mathcal{H} = [\mathcal{H}\delta_u f, \mathcal{H}\delta_u g]. \tag{15}$$

The Jacobi identity is a quadratic relation for the operator $\mathcal{H}$. In general, the linear combination of two Hamiltonian operators is no longer Hamiltonian. If it is, we say that these two Hamiltonian operators form a Hamiltonian pair. Hamiltonian pairs play an important role in the theory of integrability. They naturally generate Nijenhuis operators.

**Definition 6.** *A linear operator $\Re : \mathfrak{h} \to \mathfrak{h}$ is called a Nijenhuis operator if it satisfies*

$$[\Re P, \Re Q] - \Re[\Re P, Q] - \Re[P, \Re Q] + \Re^2[P, Q] = 0, \qquad P, Q \in \mathfrak{h}. \tag{16}$$

For a Hamiltonian pair $\mathcal{H}_1$ and $\mathcal{H}_2$, if $\mathcal{H}_1$ is invertible, then operator $\mathcal{H}_2\mathcal{H}_1^{-1}$ is a Nijenhuis operator.

Using the definition of the Lie bracket (8), formula (16) is equivalent to

$$L_{\Re P}\Re = \Re L_P \Re. \tag{17}$$

The properties of Nijenhuis operators [2] provide us with the explanation how the infinitely many commuting symmetries and conservation laws of integrable equations arise. In application, there are nonlocal terms in Nijenhuis operators. A lot of work has been done to find sufficient conditions for Nijenhuis operators to produce local objects [14, 15, 18].

### *2.3. Symmetries and conserved densities of evolution equations*

To each element $K \in \mathfrak{h}$, we associate an evolution equation of the form

$$u_t = K. \tag{18}$$

Strictly speaking, one associates with the evolution equation the derivation

$$\frac{\partial}{\partial t} + \sum_{k=0}^{\infty} D_x^k K \frac{\partial}{\partial u_k}.$$

As long as the objects concerned are explicitly time independent as in this paper, there is no difference. We refer to [16, 17] for the case when objects explicitly depend on time $t$.

**Definition 7.** *Given an evolution equation (18), when the Lie derivatives of the following vanish along $K \in \mathfrak{h}$ we call: $g \in \mathcal{A}'$ a conserved density; $h \in \mathfrak{h}$ a symmetry; $\xi \in \Omega^1$ a cosymmetry; $\Re : \mathfrak{h} \to \mathfrak{h}$ a recursion operator; a Hamiltonian operator $\mathcal{H} : \Omega^1 \to \mathfrak{h}$ a Hamiltonian operator for the equation; a symplectic operator $\mathcal{I} : \mathfrak{h} \to \Omega^1$ a symplectic operator for the equation.*

From the above definitions, we can show that if $f \in \mathcal{A}'$ is a conserved density of the equation, then $\delta_u f$ is its cosymmetry. Moreover, if $\mathcal{H}$ is a Hamiltonian operator and $\mathcal{I}$ is a symplectic operator of a given equation, then $\mathcal{H}\mathcal{I}$ is a recursion operator. The operator $\mathcal{H}$ maps cosymmetries to symmetries while $\mathcal{I}$ maps symmetries to cosymmetries.

We say that the evolution equation (18) is a Hamiltonian system if for a (pseudo-differential) Hamiltonian operator $\mathcal{H}$, there exists a functional $f \in \mathcal{A}'$, called the Hamiltonian, such that $\mathcal{H}\ \delta_u f$ is a symmetry of the equation. Additionally, if for a (pseudo-differential) symplectic operator $\mathcal{I}$, which is compatible with $\mathcal{H}$, there exists a functional $g \in \mathcal{A}'$ such that

$$\mathcal{I}u_t = \mathcal{I}K = \delta_u g,$$

we say that the evolutionary equation is a (generalized) bi-Hamiltonian system.

## 3. Conserved densities of the HS equation

In this section, we give the recursive relation to generate infinitely many conserved densities for the HS equation $T_k^{(\alpha_1,\ldots,\alpha_k)}$, where $k$ is a nonnegative integer and $\alpha_i$ are parameters. In general, these conserved densities are not in involution with respect to the Hamiltonian operator $D_x^{-1}$. We show that there are only three commuting pairs among $T_1^{(\alpha)}$ and $T_2^{(\beta,\gamma)}$.

**Proposition 1.** *Equation (1) possesses a conserved density of the form $T = u_1^2(\frac{u_2}{u_1^4})^\alpha$ satisfying $D_t T = D_x(2uT)$, where $\alpha$ is a constant.*

**Proof.** According to definition 7, if $T$ is a conserved density, then its Lie derivative along equation (1) vanishes. This is equivalent to $D_t T \in \operatorname{Im} D_x$. We have

$$\begin{aligned} D_t T &= 2u_1 u_{xt}\left(\frac{u_2}{u_1^4}\right)^\alpha + \alpha u_1^2\left(\frac{u_2}{u_1^4}\right)^{\alpha-1}\left(\frac{u_{xxt}}{u_1^4} - 4\frac{u_2 u_{xt}}{u_1^5}\right) \\ &= 2u_1(2uu_2 + u_1^2)\left(\frac{u_2}{u_1^4}\right)^\alpha + \alpha u_1^2\left(\frac{u_2}{u_1^4}\right)^{\alpha-1}\left(\frac{2uu_{xxx}}{u_1^4} - \frac{8uu_2^2}{u_1^5}\right) = D_x\left(2uu_1^2\left(\frac{u_2}{u_1^4}\right)^\alpha\right) \end{aligned}$$

and thus we proved the statement. ◊

Since the HS equation is free of any parameter, we get new conservation laws by differentiating $T$ with respect to parameter $\alpha$. Thus we have

**Corollary 1.** *Expression $S = u_1^2(\frac{u_2}{u_1^4})^\alpha(\ln\frac{u_2}{u_1^4})^n$, $n \in \mathbb{N}$, is a conserved density of equation (1) satisfying $D_t S = D_x(2uS)$.*

In what follows we show how to build up more conserved densities using the above proposition and corollary. First we prove the following general result.

**Theorem 1.** *Assume that $F$ is a conserved density of equation (1) satisfying $F_t = D_x(2uF)$. If for all solutions of equation (1) there exists a function $r$ such that $r_t = 2ur_x$, then both $Fr^\alpha$ and $G = F(\frac{D_x r}{u_1^2})^\alpha$ are conserved densities satisfying $D_t(Fr^\alpha) = D_x(2uFr^\alpha)$ and $D_t G = D_x(2uG)$ for any constant $\alpha$.*

**Proof.** We first check that $D_t(Fr^\alpha)$ is in the image of $D_x$. Indeed

$$D_t(Fr^\alpha) = F_t r^\alpha + \alpha F r^{\alpha-1} r_t = D_x(2uF)r^\alpha + 2u\alpha F r^{\alpha-1} r_x = D_x(2uFr^\alpha).$$

Next we show that

$$\begin{aligned} D_t\left(\left(\frac{D_x r}{u_1^2}\right)^\alpha\right) &= \alpha\left(\frac{D_x r}{u_1^2}\right)^{\alpha-1}\left(\frac{D_x r_t}{u_1^2} - 2\frac{u_{xt}D_x r}{u_1^3}\right) \\ &= \alpha\left(\frac{D_x r}{u_1^2}\right)^{\alpha-1}\left(\frac{D_x(2ur_x)}{u_1^2} - 2\frac{(2uu_2+u_1^2)D_x r}{u_1^3}\right) \\ &= 2\alpha u\left(\frac{D_x r}{u_1^2}\right)^{\alpha-1}\left(\frac{r_{xx}}{u_1^2} - 2\frac{u_2 D_x r}{u_1^3}\right) = 2uD_x\left(\frac{D_x r}{u_1^2}\right)^\alpha. \end{aligned}$$

Using the first part of the proof, we obtain that $G$ is a conserved density for any constant $\alpha$. ♢

We now search for function $r$ such that $r_t = 2ur_x$ for equation (1). Note that $r$ satisfies $r_t = 2ur_x$ if and only if $\ln r$ satisfies the same relation, that is, $(\ln r)_t = 2u(\ln r)_x$.

**Proposition 2.** *Assume that both $T$ and $Tr$ are conserved densities of equation (1) satisfying $T_t = D_x(2uT)$ and $D_t(Tr) = D_x(2uTr)$. Then $r_t = 2ur_x$.*

**Proof.** From the assumption we have

$$0 = D_t(Tr) - D_x(2uTr) = T_t r + Tr_t - D_x(2uT)r - 2uTr_x = T(r_t - 2ur_x).$$

This leads to the conclusion that $r_t = 2ur_x$. ♢

From proposition 1 and corollary 1, it follows that $r = \ln\frac{u_2}{u_1^4}$ satisfies $r_t = 2ur_x$ and so does $r = \frac{u_2}{u_1^4}$. Using theorem 1, we can now obtain the following result:

**Theorem 2.** *The functionals $T_k^{(\alpha_1,\ldots,\alpha_k)}$ generated by the recursive relation:*

$$T_0 = u_1^2, \qquad r_0 = -\frac{1}{u_1}; \tag{19}$$

$$T_1^{(\alpha_1)} = T_0 r_1^{\alpha_1}, \qquad r_1 = \frac{1}{u_1^2}D_x r_0; \tag{20}$$

$$T_2^{(\alpha_1,\alpha_2)} = T_1^{(\alpha_1)} r_2^{\alpha_2}, \qquad r_2 = \frac{1}{u_1^2}D_x r_1; \tag{21}$$

$$\cdots\cdots$$

$$T_k^{(\alpha_1,\ldots,\alpha_k)} = T_{k-1}^{(\alpha_1,\ldots,\alpha_{k-1})} r_k^{\alpha_k}, \quad r_k = \frac{1}{u_1^2}D_x r_{k-1} \tag{22}$$

$$\cdots\cdots$$

*are $(k+1)$th order conserved densities of equation (1). Moreover, $T_1^{(1)} \equiv 0$, $T_2^{(\alpha_1,1)} \equiv 0$ and when $\alpha_k \neq -1$ and $k \geqslant 2$, we have*

$$T_{k+1}^{(\alpha_1,\ldots,\alpha_k,1)} \equiv -\sum_{i=1}^{k-1}\frac{\alpha_i}{\alpha_k+1}T_k^{(\alpha_1,\ldots,\alpha_{i-1},\alpha_i-1,\alpha_{i+1}+1,\ldots,\alpha_{k-1},\alpha_k+1+\delta_{i,k-1})}. \tag{23}$$

**Proof.** We only need to prove the second part of the statement. It is easy to see that

$$T_1^{(1)} = D_x r_0 \equiv 0, \qquad T_2^{(\alpha_1,1)} = r_1^{\alpha_1} D_x r_1 \equiv 0.$$

We now show formula (23) by direct calculation:

$$\begin{aligned} T_{k+1}^{(\alpha_1,\ldots,\alpha_k,1)} &= T_k^{(\alpha_1,\ldots,\alpha_k)} r_{k+1} = r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_k^{\alpha_k} D_x r_k \\ &\equiv -\sum_{i=1}^{k-1} \frac{\alpha_i}{\alpha_k+1} r_1^{\alpha_1} \cdots r_{i-1}^{\alpha_{i-1}} r_i^{\alpha_i-1} (D_x r_i) r_{i+1}^{\alpha_{i+1}} \cdots r_{k-1}^{\alpha_{k-1}} r_k^{\alpha_k+1} \\ &= -\sum_{i=1}^{k-1} \frac{\alpha_i}{\alpha_k+1} u_1^2 r_1^{\alpha_1} \cdots r_{i-1}^{\alpha_{i-1}} r_i^{\alpha_i-1} \frac{D_x r_i}{u_1^2} r_{i+1}^{\alpha_{i+1}} \cdots r_{k-1}^{\alpha_{k-1}} r_k^{\alpha_k+1}. \end{aligned}$$

Thus we obtain the formula using the relation $r_{i+1} = \frac{D_x r_i}{u_1^2}$ and the recursive relation for $T_k$. ♢

For each $k \in \mathbb{N}$, the above $T_k^{(\alpha_1,\ldots,\alpha_k)}$ is a $(k+1)$th order conserved density with $k$ parameters. In a similar way, we can build up more conserved densities by adding more logarithms in front of $r_j$. For instance, both $T_1^{\alpha_1}(\ln r_1)^{\beta_1}$ and $T_1^{\alpha_1}(\ln r_1)^{\beta_1}(\ln\ln r_1)^{\gamma_1}$ are conserved densities.

Here all $T_k$ are local. By no means are the conserved densities we constructed above complete. For example, we have not include the conserved density $\frac{1}{u_1}$, that is,

$$D_t\left(\frac{1}{u_1}\right) = D_x\left(\frac{2u}{u_1} - 3x\right).$$

In addition, there are also nonlocal conserved densities. In [8], the authors listed some of them generated by the recursion operator (4). For example, the conserved density $2u^2u_1^2 + (D_x^{-1}u_1^2)^2$ depends on the same nonlocal term $D_x^{-1}u_1^2$ as in equation (2).

We can define the Poisson bracket of any two conserved densities with respect to a Hamiltonian operator $D_x^{-1}$ according to formula (14):

$$\{T_i^{(\alpha_1,\ldots,\alpha_i)},\ T_j^{(\beta_1,\ldots,\beta_j)}\}_{D_x^{-1}} = \langle \delta_u(T_i^{(\alpha_1,\ldots,\alpha_i)}),\ D_x^{-1}\delta_u(T_j^{(\beta_1,\ldots,\beta_j)})\rangle. \tag{24}$$

It is clear that

$$\begin{aligned} \{T_0, T_j^{(\alpha_1,\ldots,\alpha_j)}\}_{D_x^{-1}} &= \langle -2u_2,\ D_x^{-1}\delta_u(T_j^{(\alpha_1,\ldots,\alpha_j)})\rangle = \langle 2u_1,\ \delta_u(T_j^{(\alpha_1,\ldots,\alpha_j)})\rangle \\ &= \langle 2u_1,\ D^{\star}_{T_j^{(\alpha_1,\ldots,\alpha_j)}}(1)\rangle = \int 2D_x T_j^{(\alpha_1,\ldots,\alpha_j)} = 0. \end{aligned} \tag{25}$$

With the help of the computer algebra system *Maple* , we obtain that

$$\{T_1^{(\alpha)},\ T_1^{(\beta)}\}_{D_x^{-1}} = \tfrac{1}{2}\alpha\beta(\alpha-1)(\beta-1)(\alpha-\beta)\ T_2^{(\alpha+\beta-5,3)}; \tag{26}$$

$$\begin{aligned} \{T_1^{(\alpha)},\ T_2^{(\beta,\gamma)}\}_{D_x^{-1}} &= \tfrac{1}{2}\alpha\gamma(\alpha-1)(\gamma-1)(2-\gamma)\ T_3^{(\alpha+\beta-2,\gamma-3,3)} \\ &+ \tfrac{1}{2}\alpha\gamma(\alpha-1)(\gamma-1)(5\alpha-3\beta-10)\ T_3^{(\alpha+\beta-3,\gamma-1,2)} \\ &+ \alpha(\alpha-1)(\gamma-1)(6\gamma+\gamma\alpha^2+\beta-\alpha\beta\gamma+2\beta\gamma-\beta^2-5\gamma\alpha)\ T_3^{(\alpha+\beta-4,\gamma+1,1)} \\ &+ \alpha\beta(\alpha-1)(\alpha-2)(1-\beta)(\gamma-1)\ T_2^{(\alpha+\beta-5,\gamma+3)}. \end{aligned} \tag{27}$$

This implies that nontrivial $T_1^{(\alpha)}$ (i.e. $\alpha(\alpha-1) \neq 0$) and $T_1^{(\beta)}$ do not commute unless $\alpha = \beta$. Now we look at when a nontrivial $T_1^{(\alpha)}$ commutes with a nontrivial $T_2^{(\beta,\gamma)}$. From theorem 2, we

know that $T_3^{(\alpha_1,\alpha_2,1)} \equiv \frac{-\alpha_1}{\alpha_2+1} T_2^{(\alpha_1-1,\alpha_2+2)}$. Hence, the right-hand side of formula (27) vanishes if and only if

$$\begin{cases} 2-\gamma = 0; \\ 5\alpha - 3\beta - 10 = 0; \\ -\dfrac{\alpha+\beta-4}{\gamma+2}(6\gamma+\gamma\alpha^2+\beta-\alpha\beta\gamma+2\beta\gamma-\beta^2-5\gamma\alpha)+(\alpha-2)\beta(1-\beta)=0. \end{cases}$$

Solving this algebraic system, we obtain the following three solutions

$$\begin{cases} \alpha = \frac{1}{2}, \\ \beta = -\frac{5}{2}, \\ \gamma = 2, \end{cases} \qquad \begin{cases} \alpha = -1, \\ \beta = -5, \\ \gamma = 2, \end{cases} \qquad \begin{cases} \alpha = 2, \\ \beta = 0, \\ \gamma = 2. \end{cases}$$

These lead to the following statement.

**Proposition 3.** *Assume that* $\alpha(\alpha-1)\gamma(\gamma-1) \neq 0$. *There are only three commuting pairs among* $T_1^{(\alpha)}$ *and* $T_2^{(\beta,\gamma)}$, *namely,*

$$\left\{u_2^{1/2}, \frac{(u_1u_3-4u_2^2)^2}{u_1^2u_2^{5/2}}\right\}_{D_x^{-1}} = \left\{\frac{u_1^6}{u_2}, \frac{u_1^8(u_1u_3-4u_2^2)^2}{u_2^5}\right\}_{D_x^{-1}} = \left\{\frac{u_2^2}{u_1^6}, \frac{(u_1u_3-4u_2^2)^2}{u_1^{12}}\right\}_{D_x^{-1}} = 0.$$

Note that the Poisson bracket of any pair of conserved densities is again a conserved density since

$$L_K\{T_i^{(\alpha_1,\ldots,\alpha_i)},\ T_j^{(\beta_1,\ldots,\beta_j)}\}_{D_x^{-1}} = \{L_K(T_i^{(\alpha_1,\ldots,\alpha_i)}),\ T_j^{(\beta_1,\ldots,\beta_j)}\}_{D_x^{-1}} + \{T_i^{(\alpha_1,\ldots,\alpha_i)},\ L_K(T_j^{(\beta_1,\ldots,\beta_j)})\}_{D_x^{-1}} = 0.$$

It is surprising to see the results are linear combinations of the conserved densities listed in theorem 2 since they are by no means complete. It seems that these conserved densities are closed under the defined Poisson bracket (24).

In next section, we show that the above three commuting pairs lie in three different commuting hierarchies generated by three Nijenhuis recursion operators. The natural question is whether there are more commuting pairs if we compute the Poisson bracket between higher order conserved densities. In appendix A, we include the formula for $\{T_1^{(\alpha)},\ T_3^{(\beta,\gamma,\mu)}\}_{D_x^{-1}}$. Based on it, we find three commuting pairs between conserved densities of second order and those of fourth order, generated by the same three Nijenhuis operators. We conjecture that there are only three commuting hierarchies starting with $T_1^{(\alpha)}$. By computing the Poisson bracket between $T_2^{(\alpha_1,\alpha_2)}$ and $T_3^{(\beta_1,\beta_2,\beta_3)}$, we discover another commuting pair. It is listed in appendix A. However, we have not found the corresponding Nijenhuis recursion operator.

## 4. Symmetries and recursion operators

We know from (3) that the operator $D_x^{-1}$ is a Hamiltonian operator for equation (2), mapping cosymmetries (the variational derivatives of conserved densities) to symmetries. Thus, we can produce infinitely many symmetries from the conserved densities listed in theorem 2. Using the property of Hamiltonian operators (15), we have

$$D_x^{-1}\delta_u\{T_i^{(\alpha_1,\ldots,\alpha_i)},\ T_j^{(\beta_1,\ldots,\beta_j)}\}_{D_x^{-1}} = [D_x^{-1}\delta_u(T_i^{(\alpha_1,\ldots,\alpha_i)}),\ D_x^{-1}\delta_u(T_j^{(\beta_1,\ldots,\beta_j)})]. \tag{28}$$

From the results in the previous section, we know some symmetries are commuting and some are not, cf formule (26) and (27). In this section, we will present some recursion operators to generate infinitely many commuting and noncommuting symmetries of the HS equation (2).

*4.1. Nijenhuis recursion operators and commuting symmetries*

The Nijenhuis recursion operators [11] are used to generate infinitely many local commuting symmetries for integrable equations. In this section, we present three Nijenhuis recursion operators for the HS equation (2), which correspond to three commuting pairs in proposition 3. We then prove that they generate infinitely many local symmetries.

**Theorem 3.** *The following three operators:*

$$\Re_1 = (2u_2^{-1}D_x + 2D_x u_2^{-1} - u_3 u_2^{-3/2} D_x^{-1} u_3 u_2^{-3/2}) D_x, \tag{29}$$

$$\Re_2 = (u_1^4 u_2^{-2} D_x + D_x u_1^4 u_2^{-2} - 8u_1 D_x^{-1} u_1) D_x, \tag{30}$$

$$\Re_3 = (u_1^{-4} D_x + D_x u_1^{-4} - 4(u_3 u_1^{-6} - 3u_2^2 u_1^{-7}) D_x^{-1} u_1 - 4u_1 D_x^{-1}(u_3 u_1^{-6} - 3u_2^2 u_1^{-7})) D_x \tag{31}$$

*are all recursion operators of equation (2).*

**Proof.** According to definition 7, we check whether $L_K \Re_i$ vanishes for each $i = 1, 2, 3$, where $K = 2uu_1 - D_x^{-1} u_1^2$. Since the calculation is similar, we only work it out for $i = 2$. To simplify the computation, we introduce some notation: $s = u_1^4 u_2^{-2}$ and $s_j = D_x^j s$. So

$$\begin{aligned}
D_K \Re_2 - \Re_2 D_K &= 2(uD_x + u_1 - D_x^{-1} u_1 D_x)(sD_x + D_x s - 8u_1 D_x^{-1} u_1) D_x \\
&\quad - 2(sD_x + D_x s - 8u_1 D_x^{-1} u_1) D_x (uD_x + u_1 - D_x^{-1} u_1 D_x) \\
&= 2(uD_x + D_x^{-1} u_2)(sD_x + D_x s - 8u_1 D_x^{-1} u_1) D_x \\
&\quad - 2(sD_x + D_x s - 8u_1 D_x^{-1} u_1)(uD_x + u_1 + u_2 D_x^{-1}) D_x \\
&= ((2us_1 - 4su_1) D_x + D_x(2us_1 - 4su_1) - 8(2uu_2 + u_1^2) D_x^{-1} u_1 \\
&\quad + D_x^{-1}(8u_1^3 - 4su_3 - 2s_1 u_2) \\
&\quad - 8u_1 D_x^{-1}(2uu_2 + u_1^2) + (8u_1^3 - 4su_3 - 2s_1 u_2) D_x^{-1}) D_x \\
&= (4(2uu_1^3 u_2^{-1} - uu_1^4 u_2^{-3} u_3 - u_1^5 u_2^{-2}) D_x \\
&\quad + 4D_x(2uu_1^3 u_2^{-1} - uu_1^4 u_2^{-3} u_3 - u_1^5 u_2^{-2}) \\
&\quad - 8(2uu_2 + u_1^2) D_x^{-1} u_1 - 8u_1 D_x^{-1}(2uu_2 + u_1^2)) D_x,
\end{aligned}$$

which equals $D_{\Re_2}[K]$. Therefore, we have $L_K \Re_2 = 0$ and thus the statement is proved. ◊

The recursion operator (4) given in [8] is the inverse operator of $\Re_1$. Indeed, we can prove the following statement:

**Proposition 4.** $\Re_1 = 8\Re^{-1} = 8D_x^{-1}(u_1 D_x^{-2} - D_x^{-2} u_1)^{-1}$.

**Proof.** Note that $u_1 D_x^{-2} - D_x^{-2} u_1 = D_x^{-1}(u_2 D_x^{-1} + D_x^{-1} u_2) D_x^{-1}$. To prove the statement, we only need to show that $8(u_2 D_x^{-1} + D_x^{-1} u_2)^{-1} = 2u_2^{-1} D_x + 2D_x u_2^{-1} - u_3 u_2^{-3/2} D_x^{-1} u_3 u_2^{-3/2}$. Indeed,

$$\begin{aligned}
&(2u_2^{-1} D_x + 2D_x u_2^{-1} - u_3 u_2^{-3/2} D_x^{-1} u_3 u_2^{-3/2})(u_2 D_x^{-1} + D_x^{-1} u_2) \\
&\quad = 8 + 2u_2^{-1} u_3 D_x^{-1} - 2u_2^{-2} u_3 D_x^{-1} u_2 - 2u_3 u_2^{-3/2} D_x^{-1}(u_2^{1/2})_x D_x^{-1} \\
&\quad\quad + 2u_3 u_2^{-3/2} D_x^{-1}(u_2^{-1/2})_x D_x^{-1} u_2 \\
&\quad = 8 + 2u_2^{-1} u_3 D_x^{-1} - 2u_2^{-2} u_3 D_x^{-1} u_2 - 2u_3 u_2^{-1} D_x^{-1} + 2u_3 u_2^{-3/2} D_x^{-1} u_2^{1/2} \\
&\quad\quad + 2u_3 u_2^{-2} D_x^{-1} u_2 - 2u_3 u_2^{-3/2} D_x^{-1} u_2^{1/2} \\
&\quad = 8,
\end{aligned}$$

and this leads to the statement. ◊

In the paper [8], the authors proved that the recursion operator (4) is the ratio of a Hamiltonian pair. Hence it is a Nijenhuis operator and so is its inverse operator $\Re_1$ [2]. We now prove that the other two recursion operators $\Re_2$ and $\Re_3$ are also the ratio of Hamiltonian pairs. To do so, we classify all anti-symmetric operators of the form

$$\mathcal{H} = f(u_1, u_2)D_x + D_x f + g(u_1, u_2, u_3)D_x^{-1}h(u_1, u_2, u_3) + hD_x^{-1}g, \qquad f \neq 0, \tag{32}$$

which are compatible with the Hamiltonian operator $D_x^{-1}$. Here $f$ is a smooth function of $u_1$ and $u_2$; $g$ and $h$ are smooth functions of $u_1$, $u_2$ and $u_3$. We give the result below and the proof in appendix B.

**Theorem 4.** *If an anti-symmetric operator (32) forms a Hamiltonian pair with the Hamiltonian operator $D_x^{-1}$, then the smooth functions $f$, $g$ and $h$ are one of the following five cases. Here $a, b, c, d$ are smooth functions of $u_1$ and $c_i \in \mathbb{C}$, $i = 1, 2, \ldots, 5$ are constants.*

I.
$$\begin{cases} f = \dfrac{1}{a(u_1)^2 u_2^2}, \quad a(u_1) \neq 0, \\ h = b(u_1), \\ g = c(u_1), \end{cases} \tag{33}$$

II.
$$\begin{cases} f = \dfrac{1}{(a(u_1)u_2 + c_1)^2}, \\ h = c_2 u_1 + c_3, \\ g = c_4 u_1 + c_5, \end{cases} \tag{34}$$

III.
$$\begin{cases} f = \dfrac{4}{(a(u_1)u_2 + b(u_1))^2}, \qquad a(u_1) \neq 0, \\ h = c_1 u_1 + c_2, \\ g = -\dfrac{16u_3}{(a(u_1)u_2 + b(u_1))^3} + 8\dfrac{\partial}{\partial u_1}\left(\dfrac{b + 2au_2}{a(u_1)^2(a(u_1)u_2 + b(u_1))^2}\right) + c(u_1), \\ b(u_1) = c_1 u_1^2 + 2c_2 u_1 + c_3, \\ c(u_1) = \dfrac{b'c_4 + (u_1 b' - b)c_5}{b(u_1)^2} + \dfrac{16a'b + 8ab'}{a(u_1)^3 b(u_1)^2}, \end{cases} \tag{35}$$

IV.
$$\begin{cases} f = \dfrac{4}{b(u_1)^2}, \\ h = c_1 u_1 + c_2, \\ g = -\dfrac{16u_3}{b(u_1)^3} + \dfrac{24b'u_2^2}{b^4} + c(u_1), \\ b(u_1) = c_1 u_1^2 + 2c_2 u_1 + c_3, \\ c(u_1) = \dfrac{b'c_4 + (u_1 b' - b)c_5}{b(u_1)^2}, \end{cases} \tag{36}$$

$$\text{V.}\quad \begin{cases} f = \dfrac{1}{a(u_1)u_2^2 + b(u_1)u_2 + c(u_1)}, \\ h = \dfrac{c_1 u_3}{(a(u_1)u_2^2 + b(u_1)u_2 + c(u_1))^{3/2}} \\ \quad -\dfrac{1}{2c_1c_2}\dfrac{\partial}{\partial u_1}\left(\dfrac{bu_2 + 2c}{(a(u_1)u_2^2 + b(u_1)u_2 + c(u_1))^{1/2}}\right) + d(u_1), \\ g = c_2 h - 2c_2 d(u_1), \\ d(u_1) = c_3 u_1 + c_4, \\ 4ac - b^2 = 4c_1^2 c_2, \\ \dfrac{\partial^3 c(u_1)}{\partial u_1^3} = 6c_1^2 c_2^2 c_3 d(u_1). \end{cases} \tag{37}$$

**Example 1.** *Two special cases from case II (cf (34)) lead to Hamiltonian pairs*

$$D_x + 2u_1 D_x^{-1} + 2D_x^{-1}u_1 + \lambda D_x^{-1}$$

*and*

$$D_x + 2u_1 D_x^{-1} u_1 + \lambda D_x^{-1}.$$

*These are the bi-Hamiltonian structures for potential Korteweg–de Vries equation and potential modified Korteweg–de Vries equation, respectively, since*

$$u_t = u_3 + 3u_1^2 = D_x^{-1}\delta_u(\tfrac{1}{2}u_2^2 - u_1^3) = (D_x + 2u_1 D_x^{-1} + 2D_x^{-1}u_1)\delta_u(-\tfrac{1}{2}u_1^2)$$

*and*

$$u_t = u_3 + u_1^3 = D_x^{-1}\delta_u(\tfrac{1}{2}u_2^2 - \tfrac{1}{4}u_1^4) = (D_x + 2u_1 D_x^{-1}u_1)\delta_u(-\tfrac{1}{2}u_1^2).$$

The recursion operators (30) and (31) in theorem 3 are the ratios of Hamiltonian pairs listed in theorem 4. It is obvious that the recursion operator (30) can be derived from case I (33).

For case IV, if we take $b = 2u_1^2$, $c(u_1) = 0$ and $h = 2u_1$, then we get $g = -2u_1^{-6}u_3 + 6u_1^{-7}u_2^2$. This leads to a Hamiltonian pair

$$u_1^{-4}D_x + D_x u_1^{-4} - 4(u_3 u_1^{-6} - 3u_2^2 u_1^{-7})D_x^{-1}u_1 - 4u_1 D_x^{-1}(u_3 u_1^{-6} - 3u_2^2 u_1^{-7}) + \lambda D_x^{-1},$$

from which we obtain the recursion operator $\Re_3$ (31) in theorem 3. We can also directly get the recursion operator (29) from case V by taking $a(u_1) = c(u_1) = d(u_1) = 0$ and $b(u_1) = \frac{1}{2}$. Since all three recursion operators can be obtained from Hamiltonian pairs, they are all Nijenhuis operators.

**Theorem 5.** *The recursion operators (29)–(31) are Nijenhuis operators.*

We can also view the recursion operators (29)–(31) as the products of Hamiltonian and symplectic operators. In a recent paper [18], we proved that for Nijenhuis operators that are the products of weakly nonlocal Hamiltonian and symplectic operators [10], hierarchies of commuting local symmetries and conserved densities in involution can be generated under some easily verified conditions. To be self-contained, we restate the result in [18] valid for the operators in this paper:

> Consider a Hamiltonian operator $\mathcal{H}$ of the form (32) and a symplectic operator $\mathcal{I} = D_x$ such that $\mathcal{H}\mathcal{I}$ is a Nijenhuis operator. Assume that $L_g\mathcal{I} = L_g\mathcal{H} = L_h\mathcal{I} = L_h\mathcal{H} = 0$. If there exists a closed 1-form $\xi$ satisfying $L_g\xi = L_h\xi = L_{\mathcal{H}\xi}\xi = 0$ such that $\mathcal{I}\mathcal{H}\xi$ is closed, then all $(\mathcal{I}\mathcal{H})^j\xi$ are closed 1-forms and $\mathcal{H}(\mathcal{I}\mathcal{H})^j\xi$ commute for $j = 0, 1, 2, \ldots$.

We denote the recursion operators (29)–(31) in theorem 3 as $\Re_i = \mathcal{H}_i D_x$, $i = 1, 2, 3$. Checking the conditions in this statement for all $\Re_i$, we can prove the following theorem.

**Theorem 6.** *Let* $\xi_1 = \delta_u(u_2^{1/2})$, $\xi_2 = \delta_u(\frac{u_1^6}{2u_2})$ *and* $\xi_3 = \delta_u(\frac{u_2^2}{2u_1^6})$. *Then all* $\Re_i^{\star j}\xi_i$ *are closed* 1*-forms and for each fixed* $i \in \{1, 2, 3\}$, *all* $\mathcal{H}_i\Re_i^{\star j}\xi_i$ *commute for* $j = 0, 1, 2, \ldots$.

**Proof.** We write out the proof for the recursion operator $\Re_2$. The proof for operators $\Re_1$ and $\Re_3$ is similar, and will not repeat it. Since $\xi_2$ is the variational derivative of $\frac{u_1^6}{2u_2}$, it is clearly a closed 1-form. In this case, we have $g = h = u_1$ and trivially $L_{u_1}D_x = L_{u_1}\mathcal{H}_2 = L_{u_1}\xi_2 = 0$. We only need to show that $L_{\mathcal{H}\xi_2}\xi_2 = 0$ and $D_x\mathcal{H}_2\xi_2$ is closed. Note that

$$\xi_2 = \frac{u_1^6u_4}{u_2^3} - 3\frac{u_1^6u_3^2}{u_2^4} + 12\frac{u_1^5u_3}{u_2^2} - 30u_1^4;$$

$$\mathcal{H}_2\xi_2 = 2\frac{u_1^{10}u_5}{u_2^5} - 20\frac{u_1^{10}u_3u_4}{u_2^6} + 40\frac{u_1^9u_4}{u_2^4} + 30\frac{u_1^{10}u_3^3}{u_2^7} - 120\frac{u_1^9u_3^2}{u_2^5} + 220\frac{u_1^8u_3}{u_2^3} - 320\frac{u_1^7}{u_2};$$

and $D_x\mathcal{H}_2\xi_2 = -\delta_u(\frac{u_1^8(u_1u_3-4u_2^2)^2}{u_2^5})$, which implies that $D_x\mathcal{H}_2\xi_2$ is closed. Using the computer algebra system *Maple*, we can check $L_{\mathcal{H}\xi_2}\xi_2 = D_{\xi_2}[\mathcal{H}\xi_2] + D^{\star}_{\mathcal{H}\xi_2}(\xi_2) = 0$. Thus we prove the statement for $\Re_2$. ◊

We have proved that the recursion operators (29)–(31) are Nijenhuis. Using them, we generate three hierarchies of commuting symmetries. However, the elements in the different hierarchies do not commute.

Note that the Lie derivative is a derivation. Thus the products and additions of recursion operators are recursion operators. For instance, operators $\Re_1\Re_2$ and $[\Re_1, \Re_3]$ are also recursion operators of the HS equation (2). In general, they are no longer Nijenhuis (cf (16)) and do not generate hierarchies of local symmetries.

### *4.2. Recursion operators and noncommuting symmetries*

In this section, we first prove that the adjoint operator of the recursion operator $\Re_2$ (30) acting on any cosymmetry $\delta_u(T_j^{(\alpha_1,\ldots,\alpha_j)})$ produces local cosymmetries. Since the resulting cosymmetries are not closed, we can construct symplectic operators, which depend on parameters. Further, this leads to parameter-dependent recursion operators.

**Proposition 5.** *Starting from any symmetry* $Q_0 = D_x^{-1}\delta_u(T_j^{(\alpha_1,\ldots,\alpha_j)})$, *where* $T_j^{(\alpha_1,\ldots,\alpha_j)}$) *are listed in theorem 2, all* $Q_k = \Re_2^k Q_0$ *are local symmetries* $k = 0, 1, 2, \ldots$. *Equivalently, all* $\Re_2^{\star k}\delta_u(T_j^{(\alpha_1,\ldots,\alpha_j)})$ *are local cosymmetries, where* $\Re_2^{\star}$ *is the adjoint operator of* $\Re_2$.

**Proof.** Note that $\Re_2 = (u_1^4u_2^{-2}D_x + D_xu_1^4u_2^{-2} - 8u_1D_x^{-1}u_1)D_x$. To prove that $Q_k$ are local, we only need to show that $u_2Q_k$ is in the image of $D_x$. We prove the statement by induction.

We know that from (25) the Poisson bracket of $T_0$ and $T_j^{(\alpha_1,\ldots,\alpha_j)}$ vanishes. This implies that $u_2Q_0 \in \text{Im } D_x$. Assume that $u_2Q_{k-1} \in \text{Im } D_x$. We now show that $u_2Q_k \in \text{Im } D_x$. Indeed,

$$\begin{aligned} u_2Q_k &= u_2\Re_2Q_{k-1} = u_2(u_1^4u_2^{-2}D_x + D_xu_1^4u_2^{-2} - 8u_1D_x^{-1}u_1)D_xQ_{k-1} \\ &= 2u_1^4u_2^{-1}D_x^2Q_{k-1} - 2u_1^4u_2^{-2}u_3D_xQ_{k-1} + 4u_1^3D_xQ_{k-1} - 8u_1^2u_2Q_{k-1} + 8u_1u_2D_x^{-1}u_2Q_{k-1} \\ &= D_x(2u_1^4u_2^{-1}D_xQ_{k-1} - 4u_1^3Q_{k-1} + 4u_1^2D_x^{-1}u_2Q_{k-1}) \in \text{Im } D_x. \end{aligned}$$

Thus all $Q_k$ are local. Note that

$$u_1\Re_2^{\star k}\delta_u(T_j^{(\alpha_1,\ldots,\alpha_j)}) = u_1 D_x\mathcal{H}_2\cdots D_x\mathcal{H}_2\delta_u(T_j^{(\alpha_1,\ldots,\alpha_j)}) = u_1 D_x Q_k = D_x(u_1Q_k) - u_2Q_k.$$

We have just proved that $u_2Q_k \in \text{Im } D_x$. Thus we have $u_1\delta_u(T_j^{(\alpha_1,\ldots,\alpha_j)}) \in \text{Im } D_x$. Hence all $\Re_2^{\star k}\delta_u(T_j^{(\alpha_1,\ldots,\alpha_j)})$ are local. ◊

In general, the symmetry $Q_0$ defined in proposition 5 is not a symmetry of operator $\Re_2$, that is, $L_{Q_0}\Re_2 \neq 0$. Therefore, $[Q_0, Q_1] \neq 0$. Although $\Re_2$ is a Nijenhuis operator, the generated symmetries do not commute. Furthermore, $\Re_2^\star\delta_u(T_j^{(\alpha_1,\ldots,\alpha_j)})$ are no longer closed.

Let us look at a simple case when $j = 1$. The corresponding cosymmetry is

$$\delta_u T_1^{(\alpha)} = \delta_u(u_1^{2-4\alpha}u_2^\alpha) = (\alpha-1)D_x(\alpha u_1^{2-4\alpha}u_2^{\alpha-2}u_3 + (2-4\alpha)u_1^{1-4\alpha}u_2^\alpha).$$

Note that

$$\delta_u(u_1^{-2}u_2\ln(u_1^{-4}u_2) = D_x(u_1^{-2}u_2^{-1}u_3 - 2u_1^{-3}u_2).$$

Without losing generality, we let $\Re_2^\star$ act on the cosymmetry

$$\xi^{(1)} = D_x(\alpha u_1^{2-4\alpha}u_2^{\alpha-2}u_3 + (2-4\alpha)u_1^{1-4\alpha}u_2^\alpha),$$

where $\alpha \in \mathbb{C}$ and $\xi^{(1)}$ is closed. We have

$$\begin{aligned}\Re_2^\star\xi^{(1)} = {}& 2\alpha D_x(u_1^{6-4\alpha}u_2^{\alpha-4}u_5 + (3\alpha-7)u_1^{6-4\alpha}u_2^{\alpha-5}u_3u_4 + (8-12\alpha)u_1^{5-4\alpha}u_2^{\alpha-3}u_4\\ &+(\alpha-4)(\alpha-2)u_1^{6-4\alpha}u_2^{\alpha-6}u_3^3 - 4(3\alpha-2)(\alpha-2)u_1^{5-4\alpha}u_2^{\alpha-4}u_3^2\\ &+2(4\alpha-1)(6\alpha-5)u_1^{4-4\alpha}u_2^{\alpha-2}u_3 - 16(4\alpha-1)(\alpha-1)u_1^{3-4\alpha}u_2^\alpha),\end{aligned}$$

which is a cosymmetry of equation (2), that is, $L_{u_t}(\Re_2^\star\xi^{(1)}) = 0$. As we mentioned in section 2, $\mathrm{d}(\Re_2^\star\xi^{(1)}) \neq 0$ is a symplectic operator. We have the following proposition.

**Proposition 6.** *Operator* $\mathcal{I} = D_x(D_x + \frac{u_3}{u_2})(\frac{4u_2^2-u_1u_3}{u_1^{4\alpha-5}u_2^{5-\alpha}}D_x + D_x\frac{4u_2^2-u_1u_3}{u_1^{4\alpha-5}u_2^{5-\alpha}})(D_x - \frac{u_3}{u_2})D_x$ *is a symplectic operator for the HS equation (2) for all* $\alpha \in \mathbb{C}$.

**Proof.** By direct calculation, we have

$$\begin{aligned}\mathrm{d}(\Re_2^\star\xi^{(1)}) = D_{\Re_2^\star\xi^{(1)}} - D^\star_{\Re_2^\star\xi^{(1)}} = {}&(\alpha+1)D_x\left(D_x + \frac{u_3}{u_2}\right)\\ &\times\left(\frac{4u_2^2-u_1u_3}{u_1^{4\alpha-5}u_2^{5-\alpha}}D_x + D_x\frac{4u_2^2-u_1u_3}{u_1^{4\alpha-5}u_2^{5-\alpha}}\right)\left(D_x - \frac{u_3}{u_2}\right)D_x,\end{aligned} \tag{38}$$

which is a symplectic operator when $\alpha \neq -1$. Since the Lie derivative commutes with d, that is

$$L_{u_t}\mathrm{d}(\Re_2^\star\xi^{(1)}) = \mathrm{d}L_{u_t}(\Re_2^\star\xi^{(1)}) = 0$$

implying that it is a symplectic operator for the HS equation (2). When $\alpha = -1$, we can write

$$\mathcal{I} = \mathrm{d}(D_xS).$$

Here $S = \frac{u_1^{10}u_5}{u_2^5} - 9\frac{u_1^{10}u_3u_4}{u_2^6} + \frac{25}{2}\frac{u_1^{10}u_3^3}{u_2^7} + 16\frac{u_1^9u_4}{u_2^4} - 43\frac{u_1^9u_3^2}{u_2^5} + 66\frac{u_1^8u_3}{u_2^3} - 96\frac{u_1^7}{u_2}$ is a symmetry of the HS equation (2), which can be verified according to definition 7. By the same reason as above, we prove that $\mathcal{I}$ is a symplectic operator for the HS equation. ◊

For formula (38) when $\alpha = -1$, we have $\mathrm{d}(\Re_2^\star\xi^{(1)}) = 0$, which is consistent with the result in theorem 6. By the Leibniz rule for the Lie derivative, any product of Hamiltonian and symplectic operators of the HS equation is a recursion operator. Using its Hamiltonian operator $D_x^{-1}$ and a symplectic operator $\mathcal{I}$ as in proposition 6, we obtain the following result:

**Corollary 2.** *Differential operator* $(D_x + \frac{u_3}{u_2})(\frac{4u_2^2-u_1u_3}{u_1^{4\alpha-5}u_2^{5-\alpha}}D_x + D_x\frac{4u_2^2-u_1u_3}{u_1^{4\alpha-5}u_2^{5-\alpha}})(D_x - \frac{u_3}{u_2})D_x$ *is a recursion operator for the HS equation (2) for* $\alpha \in \mathbb{C}$.

Here we can use different Hamiltonian operators in theorem 3 instead of $D_x^{-1}$. However, the other Hamiltonian operators will bring in nonlocal terms in the recursion operators. Further study is required to determine whether the resulting operators produce local symmetries or not.

## 5. Discussion

It is well known that the integrable equations possess an infinite number of commuting conserved densities and generalized symmetries. In this paper, we present a new feature for the integrable HS equation: infinitely many noncommuting $x, t$-independent conserved densities and symmetries. We found three Nijenhuis recursion operators and a local parameter-dependent recursion operator. We believe that there are more Nijenhuis recursion operators related to the conserved densities listed in theorem 2 since we have found a new commuting pair (39) in appendix A. Note that we can define the Poisson bracket (24) with respect to $\mathcal{H}_2$ instead of $D_x^{-1}$. It will be interesting to extend the study in the paper and to see whether there are new commuting pairs.

The conserved densities in theorem 2 give rise to infinitely many cosymmetries of the HS equation (2), which are closed 1-forms. In section 4.2, we showed that the results of recursion operator $\Re_2$ (30) acting on such closed cosymmetries are no longer closed. Hence, we can generate a lot of local symplectic operators such as in proposition 6. This will lead to local recursion operators as in corollary 2. The immediate questions are the following: what are the relations among such recursion operators? Can we write down neat formulae for them? Using the computer algebra system *Maple*, it is not hard to compute these operators although the expressions are huge. The problem is to present them in a compact way, e.g. as the product of 1st order differential operators as in proposition 6.

We know the set of symmetries is a Lie algebra under the Lie bracket (8). For noncommuting symmetries of the HS equation, we can use them to generate higher order symmetries. They are different from master symmetries [3], which generate commuting symmetries.

The fundamental question is: where do such rich structures for the HS equation come from? As mentioned in the beginning of the paper, the equation is linearizable and has a trivial dispersion law. However, transformation (6) is highly nonlocal. We did not find a direct way to produce the present results from the linearized equation (7). It would be very helpful for getting a clear and complete picture if we could find the direct link.

## Appendix A.

In this appendix, we give the *Maple* result of the commutator of conserved densities between $T_1^{(\alpha)}$ and $T_3^{(\beta,\gamma,\mu)}$. We also list the new commuting pair between $T_2^{(\alpha_1,\alpha_2)}$ and $T_3^{(\beta_1,\beta_2,\beta_3)}$.

Using the computer algebra system *Maple*, we obtain the following formula:

$$\begin{aligned}\{T_1^{(\alpha)},\ T_3^{(\beta,\gamma,\mu)}\}_{D_x^{-1}} = {}& \tfrac{1}{2}\alpha(\alpha-1)\mu(\mu-1)(2-\mu)\ T_4^{(\alpha+\beta-2,\gamma,\mu-3,3)}\\ & -\tfrac{3}{2}\alpha(\alpha-1)\mu(\mu-1)\gamma\ T_4^{(\alpha+\beta-2,\gamma-1,\mu-1,2)}\\ & -\tfrac{1}{2}\alpha(\alpha-1)\mu(\mu-1)(3\beta-7\alpha+14)\ T_4^{(\alpha+\beta-3,\gamma+1,\mu-2,2)}\\ & +\alpha(\alpha-1)\gamma(\gamma-1)(\mu-1)(\mu+1)\ T_4^{(\alpha+\beta-2,\gamma-2,\mu+1,1)}\end{aligned}$$

$$
\begin{aligned}
&+\alpha(\alpha-1)(2\mu^2\beta\gamma-\beta\gamma\mu+\mu^2\gamma^2\alpha-4\alpha\gamma\mu^2+8\mu^2\gamma+3\alpha\gamma\mu-2\gamma^2\mu^2-6\gamma\mu\\
&+\mu^2\beta+6\mu+\alpha\gamma+\mu\beta+2\gamma^2-6\mu^2+3\mu^2\alpha-\alpha\gamma^2-2\gamma-3\mu\alpha)\ T_4^{(\alpha+\beta-3,\gamma,\mu,1)}\\
&+\alpha\gamma(\alpha-1)(\gamma-1)(\gamma-2)(\mu-1)\ T_3^{(\alpha+\beta-2,\gamma-3,\mu+3)}\\
&+\alpha\gamma(\alpha-1)(-2\gamma^2\mu-\alpha\gamma^2+\alpha\gamma^2\mu+2\gamma^2+3\gamma\beta\mu-6\gamma+3\alpha\gamma-2\beta\gamma+6\gamma\mu\\
&-3\alpha\gamma\mu+2\alpha\mu-2\alpha+2\beta-4\mu+4)\ T_3^{(\alpha+\beta-3,\gamma-1,\mu+2)}\\
&+\alpha(\alpha-1)(72+2\gamma\beta\mu+13\beta+24\gamma-72\mu-60\alpha+4\alpha^2\gamma+\beta^2\mu+2\beta^2\gamma\mu\\
&+\alpha^2\gamma^2\mu-4\alpha^2\gamma\mu-\alpha^2\gamma^2+\alpha\beta\gamma^2\mu+4\alpha\beta\mu-\alpha\beta\gamma^2-8\alpha\beta-12\alpha^2\mu\\
&-2\alpha\beta\gamma\mu+60\alpha\mu-9\beta\mu+3\beta^2+12\alpha^2-6\gamma^2-24\gamma\mu+2\beta\gamma^2+5\alpha\gamma^2\\
&-2\beta\gamma^2\mu+20\alpha\gamma\mu-20\alpha\gamma+6\gamma^2\mu-5\alpha\gamma^2\mu)\ T_3^{(\alpha+\beta-4,\gamma+1,\mu+1)}\\
&+\alpha(\alpha-1)(240\mu-10\mu\alpha^3-8\alpha\beta-54\alpha^2-144+24\gamma+9\alpha^2\gamma+10\beta+\beta^3+\beta^2\\
&+156\alpha-260\alpha\mu+2\alpha^2\beta+6\alpha^3-2\alpha\beta^2+90\mu\alpha^2-26\alpha\gamma-\gamma\alpha^3)\ T_3^{(\alpha+\beta-5,\gamma+3,\mu)}\\
&+\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(5-\alpha)\mu\ T_3^{(\alpha+\beta-6,\gamma+5,\mu-1)}.
\end{aligned}
$$

Combining with formula (27), we can obtain the following result:

**Proposition 7.** *Assume that* $\alpha(\alpha-1)\mu(\mu-1)\neq 0$. *There are only three commuting pairs among* $T_1^{(\alpha)}$ *and the third order conserved densities, namely,*

$$
\begin{aligned}
&\{T_1^{(1/2)},\ T_3^{(-7/2,0,2)}-\tfrac{35}{16}T_2^{(-11/2,4)}\}_{D_x^{-1}}=\{T_1^{(-1)},\ T_3^{(-7,0,2)}-7\,T_2^{(-9,4)}\}_{D_x^{-1}}\\
&\qquad=\{T_1^{(2)},\ T_3^{(0,0,2)}\}_{D_x^{-1}}=0.
\end{aligned}
$$

**Proof.** We search for the linear combinations of $T_3^{(\beta,\gamma,\mu)}+\alpha_1 T_2^{(\beta_1,\gamma_1)}$, which are in involution of $T_1^{(\alpha)}$ under the Poisson bracket (24). From theorem 2, we know that

$$
\begin{aligned}
&T_3^{(\nu_1,\nu_2,1)}\equiv\frac{-\nu_1}{\nu_2+1}T_2^{(\nu_1-1,\nu_2+2)};\\
&T_4^{(\nu_1,\nu_2,\nu_3,1)}\equiv-\frac{\nu_1}{\nu_3+1}T_3^{(\nu_1-1,\nu_2+1,\nu_3+1)}-\frac{\nu_2}{\nu_3+1}T_3^{(\nu_1,\nu_2-1,\nu_3+2)}.
\end{aligned}
$$

We substitute the above formula into $\{T_1^{(\alpha)},\ T_3^{(\beta,\gamma,\mu)}+\alpha_1 T_2^{(\beta_1,\gamma_1)}\}_{D_x^{-1}}$ and write it as the combination of independent terms. Note that it vanishes if and only if all coefficients of independent terms are equal to zero. We immediately get $\mu=2$ and $\gamma=0$. The other conditions are

$$
\begin{cases}
7\alpha-3\beta-14=0;\\
\beta=\beta_1+2;\\
\gamma_1=4;\\
(-84+70\alpha+3\beta^2-4\alpha\beta+5\beta-14\alpha^2)+2\alpha_1(\gamma_1-1)(2-\gamma_1)=0;\\
336-14\alpha^3-8\alpha\beta+10\beta+\beta^3+\beta^2-364\alpha+2\alpha^2\beta-2\alpha\beta^2+126\alpha^2\\
\quad+6\alpha_1(5\alpha-3\beta_1-10)=0;\\
3\alpha_1(\alpha-2)\beta_1(1-\beta_1)-\frac{1}{6}(\alpha+\beta_1-4)(2(\alpha-2)(\alpha-3)(\alpha-4)(5-\alpha)\\
\quad+3\alpha_1(24+4\alpha^2+\beta_1-4\alpha\beta_1+8\beta_1-\beta_1^2-20\alpha))=0.
\end{cases}
$$

Solving this algebraic system, we obtain the following three solutions:

$$\begin{cases} \alpha = \frac{1}{2}, \\ \alpha_1 = -\frac{35}{16}, \\ \beta_1 = -\frac{11}{2}, \\ \gamma_1 = 4, \\ \beta = -\frac{7}{2}, \end{cases} \qquad \begin{cases} \alpha = -1, \\ \alpha_1 = -7, \\ \beta_1 = -9, \\ \gamma_1 = 4, \\ \beta = -7, \end{cases} \qquad \begin{cases} \alpha = 2, \\ \alpha_1 = 0, \\ \beta_1 = -2, \\ \gamma_1 = 4, \\ \beta = 0. \end{cases}$$

These correspond to the commuting pairs listed in the statement. ◊

Note that we get the same values of $\alpha$ as in proposition 3. In section 4.1, we show that there are three Nijenhuis recursion operators corresponding to each value of $\alpha$. These three commuting pairs can be directly found from the corresponding recursion operators. For example,

$$\Re_2^\star(\delta_u T_2^{(-5,2)}) = -2\delta_u(T_3^{(-7,0,2)} - 7\, T_2^{(-9,4)}).$$

Proposition 7 implies that there are no other commuting pairs between $T_1^{(\alpha)}$ and conserved densities generated in theorem 2 of third order. We conjecture that there are only three commuting pairs between $T_1^{(\alpha)}$ and conserved densities generated in theorem 2 of any higher order.

We compute the commutator between $T_2^{(\alpha_1,\alpha_2)}$ and $T_2^{(\alpha_1,\alpha_2,\alpha_3)}$. Beside the three pairs directly obtained from propositions 3 and 7, we also find the following new commuting pair:

$$\{T_2^{(0,1/3)}, T_3^{(0,-7/3,2)}\}_{D_x^{-1}} = \left\{ \left( \frac{u_1u_3 - 4u_2^2}{u_1} \right)^{1/3}, \frac{(u_1^2u_4 - 14u_1u_2u_3 + 28u_2^3)^2}{u_1^{5/3}(u_1u_3 - 4u_2^2)^{7/3}} \right\}_{D_x^{-1}} = 0. \tag{39}$$

We have not found the corresponding Nijenhuis recursion operator for this new case as we did in section 4.1.

## Appendix B. Proof of theorem 4

In this appendix, we give the proof of theorem 4. Let us first introduce some notation:

$$f_i = D_x^i f, \qquad f^{(j)} = \frac{\partial f}{\partial u_j}, \qquad (f\theta)_{-1} = D_x^{-1}(f\theta).$$

The same notation is also used for $g$ and $h$.

**Proof.** To prove the statement, we check when the operator

$$\mathcal{H} = f(u_1, u_2)D_x + D_x f + g(u_1, u_2, u_3)D_x^{-1}h(u_1, u_2, u_3) + hD_x^{-1}g + \lambda D_x^{-1}$$

is Hamiltonian for arbitrary constant $\lambda \in \mathbb{C}$. From chapter 7 in [12], We know $\mathcal{H}$ is Hamiltonian if and only if

$$\Pr V_{\mathcal{H}\theta}(\Theta_{\mathcal{H}}) = 0, \qquad \text{where} \quad \Theta_{\mathcal{H}} = \frac{1}{2}\int (\theta \wedge \mathcal{H}\theta)\, \mathrm{d}x$$

is the associated bi-vector of $\mathcal{H}$.

First we have

$$\mathcal{H}(\theta) = 2f\theta_1 + f_1\theta + g(h\theta)_{-1} + h(g\theta)_{-1} + \lambda\theta_{-1};$$

$$D_x(\mathcal{H}(\theta)) = 2f\theta_2 + 3f_1\theta_1 + f_2\theta + 2gh\theta + g_1(h\theta)_{-1} + h_1(g\theta)_{-1} + \lambda\theta;$$

$$D_x^2(\mathcal{H}(\theta)) = 2f\theta_3 + 5f_1\theta_2 + 4f_2\theta_1 + f_3\theta + 2gh\theta_1 + 3(gh)_1\theta + g_2(h\theta)_{-1} + h_2(g\theta)_{-1} + \lambda\theta_1;$$

$$D_x^3(\mathcal{H}(\theta)) = 2f\theta_4 + 7f_1\theta_3 + 9f_2\theta_2 + 5f_3\theta_1 + f_4\theta + 2gh\theta_2 + 5(gh)_1\theta_1 + 4(gh)_2\theta - 2g_1h_1\theta + g_3(h\theta)_{-1} + h_3(g\theta)_{-1} + \lambda\theta_2.$$

Substituting them into $\Pr V_{\mathcal{H}\theta}(\Theta_{\mathcal{H}})$, this leads to

$$\begin{aligned}\Pr_{\mathcal{H}(\theta)}\Theta_{\mathcal{H}} &= \int (\theta \wedge D_f[\mathcal{H}(\theta)] \wedge \theta_1 + \theta \wedge D_g[\mathcal{H}(\theta)] \wedge (h\theta)_{-1} + \theta \wedge D_h[\mathcal{H}(\theta)] \wedge (g\theta)_{-1})\mathrm{d}x \\ &= \int \theta \wedge (f^{(1)}D_x(\mathcal{H}(\theta)) + f^{(2)}D_x^2(\mathcal{H}(\theta))) \wedge \theta_1 \mathrm{d}x \\ &\quad + \int \theta \wedge (g^{(1)}D_x(\mathcal{H}(\theta)) + g^{(2)}D_x^2(\mathcal{H}(\theta)) + g^{(3)}D_x^3(\mathcal{H}(\theta))) \wedge (h\theta)_{-1}\mathrm{d}x \\ &\quad + \int \theta \wedge (h^{(1)}D_x(\mathcal{H}(\theta)) + h^{(2)}D_x^2(\mathcal{H}(\theta)) + h^{(3)}D_x^3(\mathcal{H}(\theta))) \wedge (g\theta)_{-1}\mathrm{d}x.\end{aligned}$$

It needs to vanish for all $\lambda$. So the coefficient of $\lambda$ should be zero, that is

$$\begin{aligned}0 &= \int \theta \wedge (g^{(2)}\theta_1 + g^{(3)}\theta_2) \wedge (h\theta)_{-1}\mathrm{d}x + \int \theta \wedge (h^{(2)}\theta_1 + h^{(3)}\theta_2) \wedge (g\theta)_{-1}\mathrm{d}x \\ &= \int (\theta \wedge (g^{(2)} - g_1^{(3)})\theta_1 \wedge (h\theta)_{-1} + \theta \wedge (h^{(2)} - h_1^{(3)})\theta_1 \wedge (g\theta)_{-1})\,\mathrm{d}x.\end{aligned}$$

This leads to $h^{(2)} = h_1^{(3)}$ and $g^{(2)} = g_1^{(3)}$. The implies that

$$h(u_1, u_2, u_3) = p_h(u_1, u_2)u_3 + q_h(u_1, u_2), \qquad g(u_1, u_2, u_3) = p_g(u_1, u_2)u_3 + q_g(u_1, u_2) \tag{40}$$

and

$$q_h^{(2)} = p_h^{(1)}u_2, \qquad q_g^{(2)} = p_g^{(1)}u_2. \tag{41}$$

The rest of the terms, i.e. the terms without $\lambda$ in $\Pr V_{\mathcal{H}\theta}(\Theta_{\mathcal{H}})$ should also vanish.

$$\begin{aligned}0 &= \int f^{(1)}\theta \wedge (2f\theta_2 + g_1(h\theta)_{-1} + h_1(g\theta)_{-1}) \wedge \theta_1\,\mathrm{d}x \\ &\quad + \int f^{(2)}\theta \wedge (2f\theta_3 + 5f_1\theta_2 + g_2(h\theta)_{-1} + h_2(g\theta)_{-1}) \wedge \theta_1\mathrm{d}x \\ &\quad + \int g^{(1)}\theta \wedge (2f\theta_2 + 3f_1\theta_1 + h_1(g\theta)_{-1}) \wedge (h\theta)_{-1}\mathrm{d}x \\ &\quad + \int g^{(2)}\theta \wedge (2f\theta_3 + 5f_1\theta_2 + 4f_2\theta_1 + 2gh\theta_1 + h_2(g\theta)_{-1}) \wedge (h\theta)_{-1}\mathrm{d}x \\ &\quad + \int g^{(3)}\theta \wedge (2f\theta_4 + 7f_1\theta_3 + 9f_2\theta_2 + 5f_3\theta_1 + 2gh\theta_2 + 5(gh)_1\theta_1 + h_3(g\theta)_{-1}) \wedge (h\theta)_{-1}\,\mathrm{d}x \\ &\quad + \int h^{(1)}\theta \wedge (2f\theta_2 + 3f_1\theta_1 + g_1(h\theta)_{-1}) \wedge (g\theta)_{-1}\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+\int h^{(2)}\theta\wedge(2f\theta_3+5f_1\theta_2+4f_2\theta_1+2gh\theta_1+g_2(h\theta)_{-1})\wedge(g\theta)_{-1}\mathrm{d}x\\
&+\int h^{(3)}\theta\wedge(2f\theta_4+7f_1\theta_3+9f_2\theta_2+5f_3\theta_1+2gh\theta_2+5(gh)_1\theta_1+g_3(h\theta)_{-1})\wedge(g\theta)_{-1}\,\mathrm{d}x\\
=&\int(2D_x(f^{(2)}f)-5f_1f^{(2)}-2ff^{(1)}+2fhg^{(3)}+2fgh^{(3)})\theta\wedge\theta_1\wedge\theta_2\mathrm{d}x\\
&+\int(-D_x^3(2fg^{(3)})+D_x^2(2fg^{(2)}+7f_1g^{(3)})+D_x(-2fg^{(1)}-5f_1g^{(2)}-9f_2g^{(3)}-2ghg^{(3)})\\
&+4f_2g^{(2)}+2ghg^{(2)}+5f_3g^{(3)}+5(gh)_1g^{(3)}+3f_1g^{(1)}-g_1f^{(1)}-g_2f^{(2)})\theta\wedge\theta_1\wedge(h\theta)_{-1}\,\mathrm{d}x\\
&+\int(-D_x^3(2fh^{(3)})+D_x^2(2fh^{(2)}+7f_1h^{(3)})+D_x(-2fh^{(1)}-5f_1h^{(2)}-9f_2h^{(3)}-2ghh^{(3)})\\
&+4f_2h^{(2)}+2ghh^{(2)}+5f_3h^{(3)}+5(gh)_1h^{(3)}+3f_1h^{(1)}-h_1f^{(1)}-h_2f^{(2)})\theta\wedge\theta_1\wedge(g\theta)_{-1}\,\mathrm{d}x\\
&+\int(g^{(1)}h_1+g^{(2)}h_2+g^{(3)}h_3-h^{(1)}g_1-h^{(2)}g_2-h^{(3)}g_3)\theta\wedge(g\theta)_{-1}\wedge(h\theta)_{-1}\mathrm{d}x\\
&+\int(D_x(4fg^{(3)})-7f_1g^{(3)}-2fg^{(2)})\theta_1\wedge\theta_2\wedge(h\theta)_{-1}\mathrm{d}x\\
&+\int(D_x(4fh^{(3)})-7f_1h^{(3)}-2fh^{(2)})\theta_1\wedge\theta_2\wedge(g\theta)_{-1}\,\mathrm{d}x.
\end{aligned}
$$

This implies that every coefficient should be equal to zero, that is,

$$
\begin{cases}
2D_x(f^{(2)}f)-5f_1f^{(2)}-2ff^{(1)}+2fhg^{(3)}+2fgh^{(3)}=0;\\
-D_x^3(2fg^{(3)})+D_x^2(2fg^{(2)}+7f_1g^{(3)})+D_x(-2fg^{(1)}-5f_1g^{(2)}-9f_2g^{(3)}-2ghg^{(3)})\\
\quad+4f_2g^{(2)}+2ghg^{(2)}+5f_3g^{(3)}+5(gh)_1g^{(3)}+3f_1g^{(1)}-g_1f^{(1)}-g_2f^{(2)}=0;\\
-D_x^3(2fh^{(3)})+D_x^2(2fh^{(2)}+7f_1h^{(3)})+D_x(-2fh^{(1)}-5f_1h^{(2)}-9f_2h^{(3)}-2ghh^{(3)})\\
\quad+4f_2h^{(2)}+2ghh^{(2)}+5f_3h^{(3)}+5(gh)_1h^{(3)}+3f_1h^{(1)}-h_1f^{(1)}-h_2f^{(2)}=0;\\
g^{(1)}h_1+g^{(2)}h_2+g^{(3)}h_3-h^{(1)}g_1-h^{(2)}g_2-h^{(3)}g_3=0;\\
4fg^{(3)})-7f_1g^{(3)}-2fg^{(2)}=0;\\
4fh^{(3)})-7f_1h^{(3)}-2fh^{(2)}=0.
\end{cases}
$$

Substituting (40) into the above formulae and combining with (41), we obtain over-determined partial differential equations for the functions $f(u_1,u_2)$, $p_h(u_1,u_2)$, $q_h(u_1,u_2)$, $g_h(u_1,u_2)$ and $q_h(u_1,u_2)$. With the help of the package *diffalg* in Maple, we obtain the five cases listed in theorem 4. ◊

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## References


[1] Calogero F 1984 A solvable nonlinear wave equation *Stud. Appl. Math.* **70** 189–99

[2] Dorfman I 1993 *Dirac Structures and Integrability of Nonlinear Evolution Equations* (Chichester: Wiley)

[3] Fuchssteiner B 1983 Mastersymmetries, higher order time-dependent symmetries and conserved densities of nonlinear evolution equations *Prog. Theor. Phys.* **70** 1508–22

[4] Gel'fand I M and Dorfman I Y 1979 Hamiltonian operators and algebraic structures related to them *Funct. Anal. Appl.* **13** 248–62

[5] Golenia J, Pavlov M V, Popowicz Z and Prykarpatsky A K 2010 On a nonlocal Ostrovsky–Whitham type dynamical system, its Riemann type inhomogeneous regularizations and their integrability *SIGMA* **6** 002

[6] Holm D D 2006 Peakons *Encyclopedia of Mathematical Physics* vol 4 ed J P Francoise *et al* (Oxford: Elsevier) pp 12–20

[7] Hunter J K and Saxton R A 1991 Dynamics of director fields *SIAM J. Appl. Math.* **51** 1498–521

[8] Hunter J K and Zheng Y 1994 On a completely integrable nonlinear hyperbolic variational equation *Physica* D **79** 361–86

[9] Khesin B and Misiołek G 2003 Euler equations on homogeneous spaces and Virasoro orbits *Adv. Math.* **176** 116–44

[10] Maltsev A Y and Novikov S P 2001 On the local systems Hamiltonian in the weakly nonlocal Poisson brackets *Physica D: Nonlinear Phenom.* **156** 53–80

[11] Olver P J 1977 Evolution equations possessing infinitely many symmetries *J. Math. Phys.* **18** 1212–5

[12] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (*Graduate Texts in Mathematics* vol 107) 2nd edn (New York: Springer)

[13] Sakovich S 2009 On a Whitham-type equation *SIGMA* **5** 101

[14] Sergyeyev A 2005 Why nonlocal recursion operators produce local symmetries: new results and applications *J. Phys. A: Math. Gen.* **38** 3397–407

[15] Sanders J A and Wang J P 2001 Integrable systems and their recursion operators *Nonlinear Anal.* **47** 5213–40

[16] Sanders J A and Wang J P 2001 On recursion operators *Physica* D **149** 1–10

[17] Wang J P 1998 Symmetries and conservation laws of evolution equations *PhD Thesis* (Amsterdam: Vrije Universiteit/Thomas Stieltjes Institute)

[18] Wang J P 2009 Lenard scheme for two-dimensional periodic volterra chain *J. Math. Phys.* **50** 023506