Elementary Differential Geometry: Lecture Notes

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Preface

These notes are for a beginning graduate level course in differential geometry. It is assumed that this is the students' first course in the subject. Thus the choice of subjects and presentation has been made to facilitate as much as possible a concrete picture. For those interested in a deeper study, a second course would take a more abstract point of view, and in particular, could go further into Riemannian geometry.

Much of the material is borrowed from the following sources, but has been adapted according to my own taste:

[1] M. P. Do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall.

[2] L. P. Eisenhart *An introduction to differential geometry with use of the tensor calculus*, Princeton University Press.

[3] W. Klingenberg, *A course in differential geometry*, Springer-Verlag.

[4] B. O'Neill *Elementary differential geometry*, Academic Press.

[5] M. Spivak, *A comprehensive introduction to Differential Geometry*, Publish or Perish.

[6] J. J. Stoker, *Differential Geometry*, Wiley & Sons.

The prerequisites for this course are: *linear algebra*, preferably with some exposure to multilinear algebra; *calculus* up to and including the inverse and implicit function theorem; the fundamental theorem of *ordinary differential equations* concerning existence of solutions, uniqueness, and continuous dependence on parameters, and some knowledge of linear systems of ordinary differential equations; linear first order *partial differential equations*; *complex analysis* including Liouville's theorem; and some elementary *topology*.

It is highly recommended for the students to complete all the exercises included in these notes.

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Birmingham, Alabama
April 2000

CHAPTER 1

Curves

1. Preliminaries

DEFINITION 1.1. A *parametrized curve* is a smooth (C^∞) function $\gamma\colon I \to \mathbb{R}^n$. A curve is *regular* if $\gamma' \neq 0$.

When the interval I is closed, we say that γ is C^∞ on I if there is an interval J and a C^∞ function β on J which agrees with γ on I.

DEFINITION 1.2. Let $\gamma\colon I \to \mathbb{R}^n$ be a parametrized curve, and let $\beta\colon J \to \mathbb{R}^n$ be another parametrized curve. We say that β is a *reparametrization* (orientation-preserving reparametrization) of γ if there is a smooth map $\tau\colon J \to I$ with $\tau' > 0$ such that $\beta = \gamma \circ \tau$.

Note that the relation β is a reparametrization of γ is an equivalence relation. A *curve* is an equivalence class of parametrized curves. Furthermore, if γ is regular then every reparametrization of γ is also regular, so we may speak of *regular curves.*

DEFINITION 1.3. Let $\gamma\colon I \to \mathbb{R}^n$ be a regular curve. For any compact interval $[a,b] \subset I$, the *arclength* of γ over $[a,b]$ is given by:

$$L_\gamma([a,b]) = \int_a^b |\gamma'|\ dt.$$

Note that if β is a reparametrization of γ then γ and β have the same length. More specifically, if $\beta = \gamma \circ \tau$, then

$$L_\gamma\left(\big[\tau(c), \tau(d)\big]\right) = L_\beta([c,d]).$$

DEFINITION 1.4. Let γ be a regular curve. We say that γ is *parametrized by arclength* if $|\gamma'| = 1$

Note that this is equivalent to the condition that for all $t \in I = [a,b]$ we have:

$$L_\gamma([a,t]) = t - a.$$

Furthermore, any regular curve can be parametrized by arclength. Indeed, if γ is a regular curve, then the function

$$s(t) = \int_a^t |\gamma'|\,,$$

is strictly monotone increasing. Thus, $s(t)$ has an inverse function $\tau(s)$ function, satisfying:

$$\frac{d\tau}{ds} = \frac{1}{|\gamma'|}.$$

It is now straightforward to check that $\beta = \gamma \circ \tau$ is parametrized by arclength.

2. Local Theory for Curves in $\mathbb{R}^3$

We will assume throughout this section that $\gamma\colon I \to \mathbb{R}^3$ is a regular curve in $\mathbb{R}^3$ parametrized by arclength and that $\gamma'' \neq 0$. Note that $\gamma' \cdot \gamma'' = 0$.

Definition 1.5. Let $\gamma\colon I \to \mathbb{R}^3$ be a curve in $\mathbb{R}^3$. The unit vector $T = \gamma'$ is called the *unit tangent* of γ. The *curvature* κ is the scalar $\kappa = |\gamma''|$. The unit vector $N = \kappa^{-1}T'$ is called the *principal normal*. The *binormal* is the unit vector $B = T \times N$. The positively oriented orthonormal frame (T, N, B) is called the Frenet frame of γ.

It is not difficult to see that $N' + \kappa T$ is perpendicular to both T and N, hence we can define the *torsion* τ of γ by: $N' + \kappa T = \tau B$. Note that the torsion, unlike the curvature, is signed. Finally, it is easy to check that $B' = -\tau N$. Let X denote the 3×3 matrix whose columns are (T, N, B). We will call X also the Frenet frame of γ. Define the rotation matrix of γ:

$$\omega := \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \tag{1.1}$$

Proposition 1.1 (Frenet frame equations). *The Frenet frame $X = (T, N, B)$ of a curve in $\mathbb{R}^3$ satisfies:*

$$X' = X\omega. \tag{1.2}$$

The Frenet frame equations, Equation (1.2), form a system of nine linear ordinary differential equations.

Definition 1.6. A rigid motion of $\mathbb{R}^3$ is a function of the form $R(x) = x_0 + Qx$ where Q is orthonormal with $\det Q = 1$.

Note that if X is the Frenet frame of γ and $R(x) = x_0 + Qx$ is a rigid motion of $\mathbb{R}^3$, then QX is the Frenet frame of $R \circ \gamma$. This follows easily from the fact that Q is preserves the inner product and orientation of $\mathbb{R}^3$.

Theorem 1.2 (Fundamental Theorem). *Let $\kappa > 0$ and τ be smooth scalar functions on the interval $[0, L]$. Then there is a regular curve γ parametrized by arclength, unique up to a rigid motion of $\mathbb{R}^3$, whose curvature is κ and torsion is τ.*

Proof. Let ω be given by (1.1). The initial value problem

$$X' = X\omega,$$
$$X(0) = I$$

can be solved uniquely on $[0, L]$. The solution X is an orthogonal matrix with $\det X = 1$ on $[0, L]$. Indeed, since ω is anti-symmetric, the matrix $A = XX^t$ is constant. Indeed,

$$A' = X\omega X^t + X\omega^t X^t = X(\omega + \omega^t)X^t = 0,$$

and since $A(0) = I$, we conclude that $A \equiv I$, and X is orthogonal. Furthermore, $\det X$ is continuous, and $\det X(0) = 1$, so $\det X = 1$ on $[0, L]$. Let (T, N, B) be the columns of X, and let $\gamma = \int T$, then (T, N, B) is orthonormal and positively oriented on $[0, L]$. Thus, γ is parametrized by arclength, $\gamma' = T$, and $N = \kappa^{-1}T'$ is the principal normal of γ. Similarly B is the binormal, and consequently, κ is the curvature of γ and τ its torsion.

Now suppose that $\tilde{\gamma}$ is another curve with curvature κ and torsion τ, and let $\tilde{X}$ be its Frenet frame. Then there is a rigid motion $R(x) = Qx + x_0$ of $\mathbb{R}^3$ such that $R\gamma(0) = \tilde{\gamma}(0)$, and $QX(0) = \tilde{X}(0)$. By the remark preceding the theorem, QX is the Frenet frame of the curve $R \circ \gamma$, and thus both QX and $\tilde{X}$ satisfy the initial value problem:

$$Y' = Y\omega,$$
$$Y(0) = QX(0).$$

By the uniqueness of solutions of the initial value problem, it follows that $QX = \tilde{X}$. In particular, $(R \circ \gamma)' = \tilde{\gamma}'$, and since $R \circ \gamma(0) = \tilde{\gamma}(0)$ we conclude $R \circ \gamma \equiv \tilde{\gamma}$. □

Assuming $\gamma(0) = 0$, the Taylor expansion of γ of order 3 at $s = 0$ is:

$$\gamma(s) = \gamma'(0)s + \frac{1}{2}\gamma''(0)s^2 + \frac{1}{6}\gamma'''(0)s^3 + O(s^4).$$

Denote $T_0 = T(0)$, $N_0 = N(0)$, $B_0 = B(0)$, $\kappa_0 = \kappa(0)$, and $\tau_0 = \tau(0)$. We have $\gamma'(0) = T_0$, $\gamma''(0) = \kappa_0 N_0$, and $\gamma'''(0) = \kappa'(0)N_0 + \kappa_0(-\kappa_0 T_0 + \tau_0 B_0)$. Substituting these into the equation above, decomposing into T, N, and B components, and retaining only the leading order terms, we get:

$$\gamma(s) = \big(s + O(s^3)\big)T + \Big(\frac{\kappa}{2}\, s^2 + O(s^3)\Big)\, N + \Big(\frac{\tau}{6}\, s^3 + O(s^4)\Big)\, B$$

The planes spanned by pairs of vectors in the Frenet frame are given special names:

(1) T and N — the *osculating* plane;
(2) N and B — the *normal* plane;
(3) T and B — the *rectifying* plane.

We see that to second order the curve stays within its osculating plane, where it traces a parabola $y = (\kappa/2)\, s^2$. The projection onto the normal plane is a cusp to third order: $x = \big((3\tau/2)\, y\big)^{2/3}$. The projection onto the rectifying plane is to second order a line, whence its name.

Here are a few simple applications of the Frenet frame.

THEOREM 1.3. *Let γ be a regular curve with $\kappa \equiv 0$. Then γ is a straight line.*

PROOF. Since $|T'| = \kappa = 0$, it follows that T is constant and γ is linear. □

THEOREM 1.4. *Let γ be a regular curve with $\kappa > 0$, and $\tau = 0$. Then γ is planar.*

PROOF. Since $B' = 0$, B is constant. Thus the function $\xi = (\gamma - \gamma(0)) \cdot B$ vanishes identically:

$$\xi(0) = 0, \qquad \xi' = T \cdot B = 0.$$

It follows that γ remains in the plane through $\gamma(0)$ perpendicular to B. □

THEOREM 1.5. *Let γ be a regular curve with κ constant and $\tau = 0$. Then γ is a circle.*

PROOF. Let $\beta = \gamma + \kappa^{-1}N$. Then

$$\beta' = T + \frac{1}{\kappa}(-\kappa T + \tau B) = 0.$$

Thus β is constant, and $|\gamma - \beta| = \kappa^{-1}$. It follows that γ lies in the intersection between a plane and a sphere, thus γ is a circle. $\square$

3. Plane Curves

3.1. Local Theory. Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a regular plane curve parametrized by arclength, and let κ be its curvature. Note that κ is signed, and in fact changes sign (but not magnitude) when the orientation of γ is reversed. The Frenet frame equations are:

$$e_1' = \kappa\, e_2, \qquad e_2' = -\kappa\, e_1$$

PROPOSITION 1.6. *Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a regular curve with $|\gamma'| = 1$. Then there exists a differentiable function $\theta\colon [a,b] \to \mathbb{R}$ such that*

$$e_1 = (\cos\theta, \sin\theta). \tag{1.3}$$

Moreover, θ is unique up to a constant integer multiple of 2π, and in particular $\theta(b) - \theta(a)$ is independent of the choice of θ. The derivative of θ is the curvature: $\theta' = \kappa$.

PROOF. Let $a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a,b]$ so that the diameter of $e_1([t_{i-1}, t_i])$ is less than 2, i.e., e_1 restricted to each subinterval maps into a semi-circle. Such a partition exists since e_1 is uniformly continuous on $[a,b]$. Choose $\theta(a)$ so that (1.3) holds at a, and proceed by induction on i: if θ is defined at t_i then there is a unique continuous extension so that (1.3) holds. If ψ is any other continuous function satisfying (1.3), then $k = (1/2\pi)(\theta - \psi)$ is a continuous integer-valued function, hence is constant. Finally, $e_2 = (-\sin\theta, \cos\theta)$ hence

$$e_1' = \kappa\, e_2 = \theta'(-\sin\theta, \cos\theta),$$

and we obtain $\theta' = \kappa$. $\square$

3.2. Global Theory.

DEFINITION 1.7. A curve $\gamma\colon [a,b] \to \mathbb{R}^n$ is *closed* if $\gamma^{(k)}(a) = \gamma^{(k)}(b)$. A closed curve $\gamma\colon [a,b] \to \mathbb{R}^n$ is *simple* if $\gamma|_{(a,b)}$ is one-to-one. The *rotation number* of a smooth closed curve is:

$$n_\gamma = \frac{1}{2\pi}\big(\theta(a) - \theta(b)\big), \tag{1.4}$$

where θ is the function defined in Proposition 1.6.

We note that the rotation number is always an integer. For reference, we also note that the rotation number of a curve is the *winding number* of the map e_1. Finally, in view of the last statement in Proposition 1.6, we have:

$$n_\gamma = \frac{1}{2\pi}\int_{[0,L]} \kappa\, ds.$$

THEOREM 1.7 (Rotation Theorem). *Let $\gamma\colon [0,L] \to \mathbb{R}^2$ be a smooth, regular, simple, closed curve. Then $n_\gamma = \pm 1$. In particular*

$$\frac{1}{2\pi}\int_{[0,L]} \kappa\, ds = \pm 1.$$

For the proof we will need the following technical lemma. We say that a set $\Delta \subset \mathbb{R}^n$ is *star-shaped with respect to $x_0 \in \Delta$* if for every $y \in \Delta$ the line segment $\overline{x_0 y}$ lies in Δ.

LEMMA 1.8. *Let $\Delta \subset \mathbb{R}^n$ be star-shaped with respect to $x_0 \in \Delta$, and let $e\colon \Delta \to \mathbb{S}^1$ be a continuous function. Then there exists a continuous function $\theta\colon \Delta \to \mathbb{R}$ such that:*

$$e = (\cos\theta, \sin\theta). \tag{1.5}$$

Moreover, if ψ is another continuous function satisfying (1.5)*, then $\theta - \psi = 2\pi k$ where k is a constant integer.*

In fact, it is sufficient to assume that Δ is simply connected, but we will not prove this more general result here.

PROOF. Define $\theta(x_0)$ so that (1.5) holds at x_0. For each $x \in \Delta$ define θ continuously along the line segment $\overline{x_0 x}$ as in the proof of Proposition 1.6. Since Δ is star-shaped with respect to x_0, this defines θ everywhere in Δ. It remains to show that θ is continuous. Let $y_0 \in \Delta$. Since $\overline{x_0 y_0}$ is compact, it is possible to choose δ small enough that the following holds: $y' \in \overline{x_0 y_0}$ and $|y - y'| < \delta$ implies $|e(y) - e(y')| < 2$ or equivalently $e(y)$ and $e(y')$ are not antipodal. Let $0 < \epsilon < \pi$. Then there exists a neighborhood $U \subset B_\delta(y_0)$ of y_0 such that $y \in U$ implies $\theta(y) - \theta(y_0) = 2\pi k(y) + \epsilon'(y)$ where $|\epsilon'(y)| < \epsilon$ and $k(y)$ is integer-valued. It remain to prove that $k \equiv 0$. Let $y \in U$ and consider the continuous function:

$$\phi(s) = \theta\big(x_0 + s(y - x_0)\big) - \theta\big(x_0 + s(y_0 - x_0)\big), \qquad 0 \leqslant s \leqslant 1.$$

Since

$$\big|\big(x_0 + s(y - x_0)\big) - \big(x_0 + s(y_0 - x_0)\big)\big| = |s(y - y_0)| < \delta,$$

it follows from our choice of δ that $e\big(x_0 + s(y - x_0)\big)$ and $e\big(x_0 + s(y_0 - x_0)\big)$ are not antipodal. Thus, $\phi(s) \neq \pi$ for all $0 \leqslant s \leqslant 1$, and since $\phi(0) = 0$ we conclude that $|\phi| < \pi$. In particular

$$|2\pi k(y) + \epsilon'(y)| = |\theta(y) - \theta(y_0)| = |\phi(1)| < \pi,$$

and it follows that

$$|2\pi k(y)| \leqslant |2\pi k(y) + \epsilon'(y)| + |\epsilon'(y)| < 2\pi.$$

Since $k(y)$ is integer-valued this implies $k(y) = 0$. □

PROOF OF THE ROTATION THEOREM. Pick a line which intersects the curve γ and pick a *last point* p on this line, i.e., a point with the property that one ray of the line from p has no other intersection points with γ. Let h be the unit vector pointing in the direction of that ray. We assume without loss of generality that γ is parametrized by arclength, $\gamma(0) = \gamma(L) = 0$. Now, let $\Delta = \{(t_1, t_2) \in \mathbb{R}^2 :\ 0 \leqslant t_1 \leqslant t_2 \leqslant L\}$, and note that Δ is star-shaped. Define the $\mathbb{S}^1$-valued function:

$$e(t_1, t_2) = \begin{cases} \gamma'(t_1) & \text{if } t_1 = t_2; \\ -\gamma'(0) & \text{if } (t_1, t_2) = (0, L); \\ \dfrac{\gamma(t_2) - \gamma(t_1)}{|\gamma(t_2) - \gamma(t_1)|} & \text{otherwise.} \end{cases}$$

It is straightforward to check that e is continuous on Δ. By the Lemma, there is a continuous function $\theta\colon \Delta \to \mathbb{R}$ such that $e = (\cos\theta, \sin\theta)$. We claim that $\theta(L, L) - \theta(0, 0) = \pm 2\pi$ which proves the theorem, since $\theta(t, t)$ is a continuous function satisfying (1.3) in Proposition 1.6, and thus can be used on the right-hand side of (1.4) to compute the rotation number.

To prove this claim, note that, for any $0 < t < L$, the unit vector

$$e(0,t) = \frac{\gamma(t) - \gamma(0)}{|\gamma(t) - \gamma(0)|}$$

is never equal to h. Hence, there is some value α such that $\theta(0,t) - \theta(0,0) \neq \alpha + 2\pi k$ for any integer k. Thus, $|\theta(0,t) - \theta(0,0)| < 2\pi$, and since $e(0,L) = -e(0,0)$ it follows that $\theta(0,L) - \theta(0,0) = \pm\pi$.

Since the curves $e(0,t)$ and $e(t,L)$ are related via a rigid motion, i.e., $e(t,L) = Re(0,t)$ where R is rotation by π, it follows that $\psi(t) = \big(\theta(t,L) - \theta(0,L)\big) - \big(\theta(0,t) - \theta(0,0)\big)$ is a constant. Since clearly $\psi(0) = 0$, we get $\theta(0,L) - \theta(0,0) = \theta(L,L) - \theta(0,L)$, and we conclude:

$$\theta(L,L) - \theta(0,0) = \big(\theta(t,L) - \theta(0,L)\big) + \big(\theta(0,t) - \theta(0,0)\big) = \pm 2\pi.$$

□

DEFINITION 1.8. A *piecewise smooth curve* is a continuous function $\gamma\colon [a,b] \to \mathbb{R}^n$ such that there is a partition of $[a,b]$:

$$a = a_0 < a_1 < \cdots < b_n = b$$

such that for each $1 \leqslant j \leqslant n$ the curve segment $\gamma_j = \gamma|_{[a_{j-1},a_j]}$ is smooth. The points $\gamma(a_j)$ are called the *corners* of γ. The directed angle $-\pi < \psi_j \leqslant \pi$ from $\gamma'(a_j-)$ to $\gamma'(a_j+)$ is called the *exterior angle* at the j-th corner. Define $\theta_j\colon [a_{j-1},a_j] \to \mathbb{R}$ as in Proposition 1.6, i.e., so that $\gamma_j' = (\cos\theta_j, \sin\theta_j)$. The *rotation number* of γ is given by:

$$n_\gamma = \frac{1}{2\pi}\sum_{j=1}^{n}\big(\theta_j(a_j) - \theta_j(a_{j-1})\big) + \frac{1}{2\pi}\sum_{j=1}^{n}\psi_j.$$

Again, n_γ is an integer, and we have:

$$n_\gamma = \frac{1}{2\pi}\int_{[a,b]}\kappa\, ds + \frac{1}{2\pi}\sum_{j=1}^{n}\psi_j.$$

The Rotation Theorem can be generalized to piecewise smooth curves provided corners are taken into account.

THEOREM 1.9. *Let $\gamma\colon [0,L] \to \mathbb{R}^2$ be a piecewise smooth, regular, simple, closed curve, and assume that none of the exterior angles are equal to π. Then $n_\gamma = \pm 1$.*

3.3. Convexity.

DEFINITION 1.9. Let $\gamma\colon [0,L] \to \mathbb{R}^2$ be a regular closed plane curve. We say that γ is *convex* if for each $t_0 \in [0,L]$ the curve lies on one side only of its tangent at t_0, i.e., if one of the following inequality holds:

$$\big(\gamma - \gamma(t_0)\big)\cdot e_2 \leqslant 0,$$
$$\big(\gamma - \gamma(t_0)\big)\cdot e_2 \geqslant 0.$$

THEOREM 1.10. *Let $\gamma\colon [0,L] \to \mathbb{R}^2$ be a regular simple closed plane curve, and let κ be its curvature. Then γ is convex if and only if either $\kappa \geqslant 0$ or $\kappa \leqslant 0$.*

We note that an orientation reversing reparametrization of γ changes $\kappa \geqslant 0$ into $\kappa \leqslant 0$ and vice versa. Thus, ignoring orientation, those two conditions are equivalent. We also note that the theorem fails if γ is not assumed simple.

PROOF. We may assume without loss of generality that $|\gamma'| = 1$. Let $\theta\colon [0,L] \to \mathbb{R}$ be the continuous function given in Proposition 1.6 satisfying:

$$e_1 = (\cos\theta, \sin\theta),$$

and $\theta' = \kappa$.

Suppose that γ is convex. We will show that θ is weakly monotone, i.e., if $t_1 < t_2$ and $\theta(t_1) = \theta(t_2)$ then θ is constant on $[t_1, t_2]$. First, we note that since γ is simple, we have $n_\gamma = \pm 1$ by the Rotation Theorem, and it follows that e_1 is onto $\mathbb{S}^1$, see Exercise 1.6. Thus, there is $t_3 \in [0,L]$ such that

$$e_1(t_3) = -e_1(t_1) = -e_1(t_2).$$

By convexity, the three parallel tangents at t_1, t_2, and t_3 cannot be distinct, hence at least two must coincide. Let $p_1 = \gamma(s_1)$ and $p_2 = \gamma(s_2)$, $s_1 < s_2$ denote these two points, then the line $\overline{p_1 p_2}$ is contained in γ. Otherwise, if q is a point on $\overline{p_1 p_2}$ not on γ, then the line through q perpendicular to $\overline{p_1 p_2}$ intersects γ in at least two points r and s, which by convexity must lie on one side of $\overline{p_1 p_2}$. Without loss of generality, assume that r is the closer of the two to $\overline{p_1 p_2}$. Then r lies in the interior of the triangle $p_1 p_2 s$. Regardless of the inclination of the tangent at r, the three points p_1, p_2 and s, all belonging to γ, cannot all lie on one side of the tangent, in contradiction to convexity. If $\overline{p_1 p_2} \neq \{\gamma(s)\colon\ s_1 \leqslant s \leqslant s_2\}$, then $\overline{p_1 p_2} = \{\gamma(s)\colon\ s_2 \leqslant s \leqslant L\} \cap \{\gamma(s)\colon\ 0 \leqslant s \leqslant s_1\}$. However, in that case, we would have $\theta(s_2) - \theta(s_1) = \theta(L) - \theta(0) = 2\pi$, a contradiction. Thus, we have

$$\overline{p_1 p_2} = \{\gamma(s)\colon\ s_1 \leqslant s \leqslant s_2\} = \{\gamma(t)\colon\ t_1 \leqslant t \leqslant t_2\}.$$

In particular $\theta(t) = \theta(t_1) = \theta(t_2)$.

Conversely, suppose that γ is not convex. Then, there is $t_0 \in [0,L]$ such that the function $\phi = \big(\gamma - \gamma(t_0)\big) \cdot e_2$ changes sign. We will show that θ' also changes sign. Let $t_+, t_- \in [0,L]$ be such that

$$\min_{[0,L]} \phi = \phi(t_-) < 0 = \phi(t_0) = \phi(t_+) = \max_{[0,L]} \phi.$$

Note that the three tangents at t_-, t_+ and t_0 are parallel but distinct. Since $\phi'(t_-) = \phi'(t_+) = 0$, we have that $e_1(t_-)$ and $e_1(t_+)$ are both equal to $\pm e_1(t_0)$. Thus, at least two of these vectors are equal. We may assume, after reparametrization, that there exists $0 < s < L$ such that $e_1(0) = e_1(s)$. This implies that

$$\theta(s) - \theta(0) = 2\pi k, \qquad \theta(L) - \theta(s) = 2\pi k'$$

with $k, k' \in \mathbb{Z}$. By the Rotation Theorem, $n_\gamma = k + k' = \pm 1$. Since $\gamma(0)$ and $\gamma(s)$ do not lie on a line parallel to $e_1(t_0)$, it follows that θ is not constant on either $[0,s]$ or $[0,L]$. If $k = 0$ then θ' changes sign on $[0,s]$, and similarly if $k' = 0$ then θ' changes sign on $[s,L]$. If $kk' \neq 0$, then since $k + k' = \pm 1$, it follows that $kk' < 0$ and θ' changes sign on $[0,L]$. □

DEFINITION 1.10. Let $\gamma\colon [0,L] \to \mathbb{R}^2$ be a regular plane curve. A *vertex* of γ is a critical point of the curvature κ.

THEOREM 1.11 (The Four Vertex Theorem). *A regular simple convex closed curve has at least four vertices.*

PROOF. Clearly, κ has a maximum and minimum on $[0,L]$, hence γ has at least two vertices. We will assume, without loss of generality, that γ is parametrized by arclength, has its minimum at $t = 0$, its maximum at $t = t_0$ where $0 < t_0 < L$, that $\gamma(0)$ and $\gamma(t_0)$ lie on the x-axis, and that γ enters the upper-half plane in the interval $[0, t_0]$. All these properties can be achieved by reparametrizing and rotating γ.

We now claim that $p = \gamma(0)$ and $q = \gamma(t_0)$ are the only points of γ on the x-axis. Indeed, suppose that there is another point $r = \gamma(t_1)$ on the x-axis, then one of these points lies between the other two, and the tangent at that point must, by convexity, contain the other two. Thus, by the argument used in the proof of Theorem 1.10 the segment between the outer two is contained in γ, and in particular $\overline{pq}$ is contained in γ. If follows that $\kappa = 0$ at p and q where κ has its minimum and maximum, hence $\kappa \equiv 0$, a contradiction since then γ is a line and cannot be closed. We conclude that γ remains in the upper half-plane in the interval $[0, t_0]$ and remains in the lower half-plane in the interval $[t_0, L]$.

Suppose now by contradiction that $\gamma(0)$ and $\gamma(t_0)$ are the only vertices of γ. Then it follows that:

$$\kappa' \geqslant 0 \text{ on } [0, t_0], \qquad \kappa' \leqslant 0 \text{ on } [t_0, L].$$

Thus, if we write $\gamma = (x, y)$, then we have $\kappa' y \geqslant 0$ on $[0, L]$, and $x'' = -\kappa y'$, hence:

$$0 = \int_0^L x''\, ds = -\int_0^L -\kappa y'\, ds = \int_0^L \kappa' y\, ds.$$

Since the integrand in the last integral is non-negative, we conclude that $\kappa' y \equiv 0$, hence $y \equiv 0$, again a contradiction.

It follows that κ has another point where κ' changes sign, i.e., an extremum. Since extrema come in pairs, κ has at least four extrema. □

4. Fenchel's Theorem

We will use without proof the fact that the shortest path between two points on a sphere is always an arc of a great circle. We also use the notation $\gamma_1 + \gamma_2$ to denote the curve γ_1 followed by the curve γ_2.

DEFINITION 1.11. Let $\gamma\colon [0, L] \to \mathbb{R}^n$ be a regular curve parametrized by arclength. The *spherical image* of γ is the curve $\gamma'\colon [0, L] \to \mathbb{S}^{n-1}$. The *total curvature* of $\gamma\colon [0, L] \to \mathbb{R}^n$ is:

$$K_\gamma = \int_I |\gamma''|\, ds.$$

We note that the total curvature is simply the length of the spherical image.

THEOREM 1.12. *Let γ be a regular simple closed curve in $\mathbb{R}^n$ parametrized by arclength. Then the total curvature of γ is at least 2π:*

$$K_\gamma \geqslant 2\pi,$$

with equality if and only if γ is planar and convex.

The proof will follow from two lemmata which are interesting in their own right.

LEMMA 1.13. *Let $\gamma\colon [0, L] \to \mathbb{R}^n$ be a regular closed curve parametrized by arclength. Then the spherical image of γ cannot map into an open hemisphere. If γ' maps into a closed hemisphere, then γ maps into an equator.*

PROOF. Suppose, by contradiction, that there is $v \in \mathbb{S}^{n-1}$ such that $\gamma' \cdot v > 0$. Then

$$0 = \gamma \cdot v\,|_L - \gamma \cdot v\,|_0 = \int_0^L \gamma' \cdot v \; ds > 0.$$

If $\gamma' \cdot v \geqslant 0$, then the same inequality shows that $\gamma' \cdot v \equiv 0$, hence γ lies in the plane perpendicular to v through $\gamma(0)$. □

LEMMA 1.14. *Let $n \geqslant 3$, and let $\gamma\colon [0, L] \to \mathbb{S}^{n-1}$ be a regular closed curve on the unit sphere parametrized by arclength.*

(1) *If the arclength of γ is less than 2π then γ is contained in an open hemisphere.*
(2) *If the arclength of γ is equal to 2π then γ is contained in a closed hemisphere.*

PROOF. (1) First observe that no piecewise smooth curve of arclength less than 2π contains two antipodal points. Otherwise the two segments of of the curve between p and q would each have length at least π, and hence the length of the curve would have to be at least 2π. Now pick a point p on γ and let q on γ be chosen so that the two segments γ_1 and γ_2 from p to q along γ have equal length. Note that p and q cannot be antipodal. Let v be the midpoint along the shorter of the two segments of the great circle between p and q. Suppose that γ_1 intersects the equator, the great circle $v \cdot x = 0$. Let $\tilde{\gamma}_1$ be the reflection of γ with respect to v, then the length of $\gamma_1 + \tilde{\gamma}_1$ is the same as the length of γ hence is less than 2π. But $\gamma_1 + \tilde{\gamma}_1$ contains two antipodal points, a contradiction. Thus, γ_1 cannot intersect the equator. Similarly, γ_2 cannot intersect the equator, and we conclude γ stays in the open hemisphere $v \cdot x > 0$.

(2) If the arclength of γ is 2π, we refine the above argument. If p and q are antipodal, then both γ_1 and γ_2 are great semi-circle, thus, γ stays in a closed hemisphere.[1] So we can assume that p and q are not antipodal and proceed as before, defining v to be the midpoint on the shorter arc of the great circle between p and q. Now, if γ_1 crosses the equator, then $\gamma_1 + \tilde{\gamma}_1$ contains two antipodal points on the equator, and the two segments joining these points enter both hemispheres. Thus, these segments are not semi-circle, and consequently both have arclength strictly greater than π. Thus the arclength of $\gamma_1 + \tilde{\gamma}_1$ is strictly larger than 2π a contradiction. Similarly, γ_2 does not cross the equator, and we conclude that γ stays in the closed hemisphere $v \cdot x \geqslant 0$. □

PROOF OF FENCHEL'S THEOREM. Note that the total curvature is simply the arclength of the spherical image of γ. By Lemma 1.13 γ' is not contained in an open hemisphere, so by Lemma 1.14

$$K_\gamma = \int_I |\gamma''| \; ds \geqslant 2\pi.$$

If the arclength of γ' is 2π, then by Lemma 1.14, γ' is contained in a closed hemisphere, and by Lemma 1.13, γ maps into an equator. If $n > 3$, we may proceed by induction until we obtain that γ is planar. Once we have that γ is planar, the

[1] In fact, since γ is smooth, γ_1 and γ_1 are contained in the same great circle, and hence γ is itself a great circle.

Rotation Theorem gives $n_\gamma = \pm 1$. Without loss of generality,[2] we may assume that $n_\gamma = 1$. Hence

$$0 \leqslant \int_I \big(|\kappa| - \kappa\big)\, ds = K_\gamma - 2\pi = 0,$$

and it follows that $\kappa = |\kappa| \geqslant 0$, which by Theorem 1.10 implies that γ is convex. □

Exercises

EXERCISE 1.1. A regular space curve $\gamma\colon [a,b] \to \mathbb{R}^3$ is a *helix* if there is a fixed unit vector $u \in \mathbb{R}^3$ such that $e_1 \cdot u$ is constant. Let κ and τ be the curvature and torsion of a regular space curve γ, and suppose that $\kappa \neq 0$. Prove that γ is a helix if and only if $\tau = c\kappa$ for some constant c.

EXERCISE 1.2. Define a curve $\gamma\colon I \to \mathbb{R}^n$ to be *k-regular* if its first k derivatives are linearly independent. Show that if γ is k-regular, then so is any reparametrization of γ.

EXERCISE 1.3. Let $\gamma\colon I \to \mathbb{R}^n$ be an $(n-1)$-regular curve, $n > 3$. Use induction to prove the existence of a *Frenet frame*, i.e., a positively oriented orthonormal frame $X = (e_1, \dots, e_n)$ satisfying $e_1 = \gamma'$, and $X' = X\omega$, where ω is anti-symmetric and tri-diagonal with $\omega_{i,i+1} > 0$ for $i \leqslant n-2$. Define the *curvatures* of γ to be the $n-1$ functions $\kappa_i = \omega_{i,i+1}$.

EXERCISE 1.4. Prove the Fundamental Theorem for curves in $\mathbb{R}^n$: *Given functions $\kappa_1, \dots, \kappa_{n-1}$ on I with $\kappa_i > 0$ for $i = 1, \dots, n-2$, there is an $(n-1)$-regular curve γ parametrized by arclength on I such that $\kappa_1, \dots, \kappa_n$ are the curvatures of γ. Furthermore, γ is unique up to rigid motion*

EXERCISE 1.5. Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a regular plane curve with non-zero curvature $\kappa \neq 0$, and let $\beta = \gamma + \kappa^{-1} N$ be the locus of the *centers of curvature* of γ.

(1) Prove that β is regular provided that $\kappa' \neq 0$.
(2) Prove that each tangent ℓ of β intersects γ at a right angle.

A curve satisfying (1) and (2) is called an *evolute* of γ.

(3) Prove that each regular plane curve $\gamma\colon [a,b] \to \mathbb{R}^2$ has at most one evolute.

EXERCISE 1.6. A convex plane curve $\gamma\colon [a,b] \to \mathbb{R}^2$ is *strictly convex* if $\kappa \neq 0$. Prove that if $\gamma\colon [a,b] \to \mathbb{R}^2$ is a strictly convex simple closed curve, then for every $v \in \mathbb{S}^1$, there is a unique $t \in [a,b]$ such that $e_1(t) = v$.

EXERCISE 1.7. Let $\gamma\colon [0,L] \to \mathbb{R}^2$ be a strictly convex simple closed curve. The *width* $w(t)$ of γ at $t \in [0,L]$ is the distance between the tangent line at $\gamma(t)$ and the tangent line at the unique point $\gamma(t')$ satisfying $e_1(t') = -e_1(t)$ (see Exercise 1.6). A curve has *constant width* if w is independent of t. Prove that if γ has constant width then:

(1) The line between $\gamma(t)$ and $\gamma(t')$ is perpendicular to the tangent lines at those points.
(2) The curve γ has length $L = \pi w$.

[2] Reversing the orientation of γ if necessary.

EXERCISE 1.8. Let $\gamma\colon [0,L] \to \mathbb{R}^2$ be a simple closed curve. By the Jordan Curve Theorem, the complement of γ has two connected components, one of which is bounded. The *area enclosed* by γ is the area of this component, and according to Green's Theorem, it is given by:

$$A = \int_\gamma x\, dy = \int_\gamma xy'\, dt,$$

where the orientation is chosen so that the normal e_2 points into the bounded component. Let L be the length of γ, and let β be a circle of width $2r$ equal to some width of γ. Prove:

(1) $A = \frac{1}{2}\int_\gamma (xy' - yx')\, dt$.
(2) $A + \pi r^2 \leqslant Lr$.
(3) *The isoperimetric inequality*: $4\pi A \leqslant L^2$.
(4) If equality holds in (3) then γ is a circle.

EXERCISE 1.9. Prove that if a convex simple closed curve has four vertices, then it cannot meet any circle in more than four points.

CHAPTER 2

Local Surface Theory

1. Surfaces

DEFINITION 2.1. A *parametric surface patch* is a smooth mapping:

$$X: U \to \mathbb{R}^3,$$

where $U \subset \mathbb{R}^2$ is open, and the Jacobian dX is non-singular.

Write $X = (x^1, x^2, x^3)$, and each $x^i = x^i(u^1, u^2)$, then the Jacobian has the matrix representation:

$$dX = \begin{pmatrix} x_1^1 & x_2^1 \\ x_1^2 & x_2^2 \\ x_1^3 & x_2^3 \end{pmatrix}$$

where we have used the notation $f_i = f_{u^i} = \partial f / \partial u^i$. According to the definition, we are requiring that this matrix has rank 2, or equivalently that the vectors $X_1 = (x_1^1, x_1^2, x_1^3)$ and $X_2 = (x_2^1, x_2^2, x_2^3)$ are linearly independent. Another equivalent requirement is that $dX: \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

EXAMPLE 2.1. Let $U \subset \mathbb{R}^2$ be open, and suppose that $f: U \to \mathbb{R}$ is smooth. Define the *graph* of f as the parametric surface $X(u^1, u^2) = (u^1, u^2, f(u^1, u^2))$. To verify that X is indeed a parametric surface, note that:

$$dX = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_1 & f_2 \end{pmatrix}$$

so that clearly X is non-singular.

A *diffeomorphism* between open sets $U, V \subset \mathbb{R}^2$ is a map $\phi: U \to V$ which is smooth, one-to-one, and whose inverse is also smooth. If $\det(d\phi) > 0$, then we say that ϕ is an *orientation-preserving* diffeomorphism.

DEFINITION 2.2. Let $X: U \to \mathbb{R}^3$, and $\tilde{X}: \tilde{U} \to \mathbb{R}^3$ be parametric surfaces. We say that $\tilde{X}$ is *reparametrization* of X if $\tilde{X} = X \circ \phi$, where $\phi: \tilde{U} \to U$ is a diffeomorphism. If ϕ is an orientation-preserving diffeomorphism, then $\tilde{X}$ is an orientation-preserving reparametrization.

Clearly, the inverse of a diffeomorphism is a diffeomorphism. Thus, if $\tilde{X}$ is a reparametrization of X, then X is a reparametrization of $\tilde{X}$.

DEFINITION 2.3. The *tangent space* $T_u X$ of the parametric surface $X: U \to \mathbb{R}^3$ at $u \in U$ is the 2-dimensional linear subspace of $\mathbb{R}^3$ spanned by the two vectors X_1 and X_2.[1]

[1] Note that the *tangent plane* to the surface $X(U)$ at u is actually the affine subspace $X(u) + T_u X$. However, it will be very convenient to have the tangent space as a linear subspace of $\mathbb{R}^3$.

If $Y \in T_u X$, then it can be expressed as a linear combination in X_1 and X_2:

$$Y = y^1 X_1 + y^2 X_2 = \sum_{i=1}^{2} y^i X_i,$$

where $y^i \in \mathbb{R}$ are the components of the vector Y in the basis X_1, X_2 of $T_u X$. We will use the *Einstein Summation Convention*: every index which appears twice in any product, once as a subscript (*covariant*) and once as a superscript (*contravariant*), is summed over its range. For example, the above equation will be written $Y = y^i X_i$. The next proposition show that the tangent space is invariant under reparametrization, and gives the law of transformation for the components of a tangent vector. Note that covariant and contravariant indices have different transformation laws, cf. (2.1) and (2.2).

PROPOSITION 2.1. *Let $X : U \to \mathbb{R}^3$ be a parametric surface, and let $\tilde{X} = X \circ \phi$ be a reparametrization of X. Then $T_{\phi(\tilde{u})} X = T_{\tilde{u}} \tilde{X}$. Furthermore, if $Z \in T_{\phi(\tilde{u})} \tilde{X}$, and $Z = z^i X_i = \tilde{z}^j \tilde{X}_j$, then:*

$$z^i = \tilde{z}^j \frac{\partial u^i}{\partial \tilde{u}^j}, \tag{2.1}$$

where $d\phi = (\partial u^i / \partial \tilde{u}^j)$.

PROOF. By the chain rule, we have:

$$\tilde{X}_j = \frac{\partial u^i}{\partial \tilde{u}^j} X_i. \tag{2.2}$$

Thus $T_{\tilde{u}} \tilde{X} \subset T_{\phi(\tilde{u})} X$, and since we can interchange the roles of X and $\tilde{X}$, we conclude that $T_{\tilde{u}} \tilde{X} = T_{\phi(\tilde{u})} X$. Substituting (2.2) in $\tilde{z}^j \tilde{X}_j$, we find:

$$z^i X_i = \tilde{z}^j \frac{\partial u^i}{\partial \tilde{u}^j} X_i,$$

and (2.1) follows. □

DEFINITION 2.4. A *vector field* along a parametric surface $X : U \to \mathbb{R}^3$, is a smooth mapping $Y : U \to \mathbb{R}^3$[2]. A vector field Y is *tangent* to X if $Y(u) \in T_u X$ for all $u \in U$. A vector field Y is *normal* to X if $Y(u) \perp T_u X$ for all $u \in U$.

EXAMPLE 2.2. The vector fields X_1 and X_2 are tangent to the surface. The vector field $X_1 \times X_2$ is normal to the surface.

We call the unit vector field

$$N = \frac{X_1 \times X_2}{|X_1 \times X_2|}$$

the *unit normal*. Note that the triple (X_1, X_2, N), although not necessarily orthonormal, is positively oriented. In particular, we can see that the choice of an orientation on X, e.g., $X_1 \to X_2$, fixes a unit normal, and vice-versa, the choice of a unit normal fixes the orientation. Here we chose to use the orientation inherited from the orientation $u^1 \to u^2$ on U.

DEFINITION 2.5. We call the map $N : U \to \mathbb{S}^2$ the *Gauss map*.

The Gauss map is invariant under orientation-preserving reparametrization.

[2] We often visualize $Y(u)$ as being attached at $X(u)$, i.e. belonging to the tangent space of $\mathbb{R}^3$ at $X(u)$; cf. see footnote 1.

PROPOSITION 2.2. *Let $X: U \to \mathbb{R}^3$ be a parametric surface, and let $N: U \to \mathbb{S}^2$ be its Gauss map. Let $\tilde{X} = X \circ \phi$ be an orientation-preserving reparametrization of X. Then the Gauss map of $\tilde{X}$ is $N \circ \phi$.*

PROOF. Let $v \in V$. The unit normal $\tilde{N}(v)$ of $\tilde{X}$ at v is perpendicular to $T_v\tilde{X}$. By Proposition 2.1, we have $T_{\phi(v)}X = T_v\tilde{X}$. Thus, $\tilde{N}(v)$ is perpendicular to $T_{\phi(v)}X$, as is $N(\phi(v))$. It follows that the two vectors are co-linear, and hence $\tilde{N}(v) = \pm N(\phi(v))$. But since ϕ is orientation preserving, the two pairs (X_1, X_2) and $(\tilde{X}_1, \tilde{X}_2)$ have the same orientation in the plane $T_v\tilde{X}$. Since also, the two triples $(X_1(\phi(v)), X_2(\phi(v)), N(\phi(v)))$ and $(\tilde{X}_1(v), \tilde{X}_2(v), N(v))$ have the same orientation in $\mathbb{R}^3$, it follows that $N(\phi(v)) = \tilde{N}(v)$. □

2. The First Fundamental Form

DEFINITION 2.6. A *symmetric bilinear form* on a vector space V is function $B: V \times V \to \mathbb{R}$ satisfying:

(1) $B(aX + bY, Z) = aB(X,Z) + bB(Y,Z)$, for all $X, Y \in V$ and $a, b \in \mathbb{R}$.
(2) $B(X,Y) = B(Y,X)$, for all $X, Y \in V$.

The symmetric bilinear form B is positive definite if $B(X,X) \geqslant 0$, with equality if and only if $X = 0$.

With any symmetric bilinear form B on a vector space, there is associated a quadratic form $Q(X) = B(X,X)$. Let V and W be vector spaces and let $T: V \to W$ be a linear map. If B is a symmetric bilinear form on W, we can define a symmetric bilinear form T^*Q on V by $T^*Q(X,Y) = Q(TX,TY)$. We call T^*Q the *pull-back of Q by T*. The map T is then an *isometry* between the inner-product spaces (V, T^*Q) and (W, Q).

EXAMPLE 2.3. Let $V = \mathbb{R}^3$ and define $B(X,Y) = X \cdot Y$, then B is a positive definite symmetric bilinear form. The associated quadratic form is $Q(X) = |X|^2$.

EXAMPLE 2.4. Let A be a symmetric 2×2 matrix, and let $B(X,Y) = AX \cdot Y$, then B is a symmetric bilinear form which is positive definite if and only if the eigenvalues of A are both positive.

DEFINITION 2.7. Let $X: U \to \mathbb{R}^3$ be a parametric surface. The *first fundamental form* is the symmetric bilinear form g defined on each tangent space T_uX by:

$$g(Y,Z) = Y \cdot Z, \qquad \forall Y, Z \in T_uX.$$

Thus, g is simply the restriction of the Euclidean inner product in Example 2.3 to each tangent space of X. We say that g is *induced* by the Euclidean inner product.

Let $g_{ij} = g(X_i, X_j)$, and let $Y = y^iX_i$ and $Z = z^iX_i$ be two vectors in T_uX, then

$$g(Y,Z) = g_{ij}y^iz^j. \tag{2.3}$$

Thus, the so-called *coordinate representation* of g is at each point $u_0 \in U$ an instance of Example 2.4. In fact, if $A = (g_{ij})$, and $B(\xi,\eta) = \xi \cdot A\eta$ for $\xi, \eta \in \mathbb{R}^2$ as in Example 2.4, then B is the pull-back by $dX_u: \mathbb{R}^2 \to T_uX$ of the restriction of the Euclidean inner product on T_uX.

The *classical (Gauss) notation* for the first fundamental form is $g_{11} = E$, $g_{12} = g_{21} = F$, and $G = g_{22}$, i.e.,

$$(g_{ij}) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

Clearly, $F^2 < EG$, and another condition equivalent to the condition that X_1 and X_2 are linearly independent is that $\det(g_{ij}) = EG - F^2 > 0$. The first fundamental form is also sometimes written:

$$ds^2 = g_{ij}\, du^i\, du^j = E\,(du^1)^2 + 2F\, du^1\, du^2 + G\,(du^2)^2.$$

Note that the g_{ij}'s are functions of u. The reason for the notation ds^2 is that the square root of the first fundamental form can be used to compute length of curves on X. Indeed, if $\gamma\colon [a,b] \to \mathbb{R}^3$ is a curve on X, then $\gamma = X \circ \beta$, where β is a curve in U. Let $\beta(t) = \big(\beta^1(t), \beta^2(t)\big)$, and denote time derivatives by a dot, then

$$L_\gamma([a,b]) = \int_a^b |\dot\gamma|\; dt \int_a^b \sqrt{g_{ij}\dot\beta^i\dot\beta^j}\, dt.$$

Accordingly, ds is also called the *line element* of the surface X.

Note that g contains all the *intrinsic geometric information* about the surface X. The *distance* between any two points on the surface is given by:

$$d(p,q) = \inf\{L_\gamma\colon\ \gamma \text{ is a curve on } X \text{ between } p \text{ and } q\}.$$

Also the *angle* θ between two vectors $Y, Z \in T_xX$ is given by:

$$\cos\theta = \frac{g(Y,Z)}{\sqrt{g(Y,Y)\, g(Z,Z)}},$$

and the angle between two curves β and γ on $\mathbf{X}$ is the angle between their tangents $\dot\beta$ and $\dot\gamma$. *Intrinsic geometry* is all the information which can be obtained from the three functions g_{ij} and their derivatives.

Clearly, the first fundamental form is invariant under reparametrization. The next proposition shows how the g_{ij}'s change under reparametrization.

PROPOSITION 2.3. *Let $X\colon U \to \mathbb{R}^3$ be a parametric surface, and let $\tilde X = X \circ \phi$ be a reparametrization of X. Let g_{ij} be the coordinate representation of the first fundamental form of X, and let $\tilde g_{ij}$ be the coordinate representation of the first fundamental form of $\tilde X$. Then, we have:*

$$\tilde g_{ij} = g_{kl}\, \frac{\partial u^k}{\partial \tilde u^i}\, \frac{\partial u^l}{\partial \tilde u^j}, \tag{2.4}$$

where $d\phi = (\partial u^i / \partial \tilde u^j)$.

PROOF. In view of (2.2), we have:

$$\tilde g_{ij} = g(\tilde X_i, \tilde X_j) = g\left(\frac{\partial u^k}{\partial \tilde u^i}\, X_k, \frac{\partial u^l}{\partial \tilde u^j} X_l\right) = \frac{\partial u^k}{\partial \tilde u^i}\, \frac{\partial u^l}{\partial \tilde u^j}\, g(X_k, X_l) = g_{kl}\, \frac{\partial u^k}{\partial \tilde u^i}\, \frac{\partial u^l}{\partial \tilde u^j}.$$

□

3. The Second Fundamental Form

We now turn to the second fundamental form. First, we need to prove a technical proposition. Let Y and Z be vector fields along X, and suppose that $Y = y^i X_i$ is tangential. We define the *directional derivative* of Z along Y by:

$$\partial_Y Z = y^i Z_i = y^i \frac{\partial Z}{\partial u^i}.$$

Note that the value of $\partial_Y Z$ at u depends only on the value of Y at u, but depends on the values of Z in a neighborhood of u. In addition, $\partial_Y Z$ is reparametrization invariant, but even if Z is tangent, it is not necessarily tangent. Indeed, if we write $Y = \tilde{y}^i \tilde{X}_i$, then we see that:

$$\tilde{y}^i\, \tilde{\partial}_i Z = \tilde{y}^i \frac{\partial Z}{\partial \tilde{u}^i} = y_j \frac{\partial \tilde{u}^i}{\partial u^j} \frac{\partial Z}{\partial u^k} \frac{\partial u^k}{\partial \tilde{u}^i} = y^j \partial_j Z.$$

The *commutator* of Y and Z can now be defined as the vector field:

$$[Y, Z] = \partial_Y Z - \partial_Z Y.$$

PROPOSITION 2.4. *Let $X \colon U \to \mathbb{R}^3$ be a surface, and let N be its unit normal.*

(1) *If Y and Z are tangential vector fields then $[Y, Z] \in T_u X$.*
(2) *If $Y, Z \in T_u X$ then $\partial_Y N \cdot Z = \partial_Z N \cdot Y$.*

PROOF. Note first that since X is smooth, we have $X_{ij} = X_{ji}$, where we have used the notation $X_{ij} = \partial^2 X / \partial u^i \partial u^j$. Now, write $Y = y^i X_i$ and $Z = z^j X_j$, and compute:

$$\begin{aligned} \partial_Y Z - \partial_Z Y &= y^i z^j X_{ji} + y^i \partial_i z^j X_j - y^i z^j X_{ij} - z^j \partial_j y^i X_i \\ &= \big(y^i \partial_i z^j - z^i \partial_i y^j\big) X_j. \end{aligned}$$

To prove (2), extend Y and Z to be vector fields in a neighborhood of u, and use (1):

$$\partial_Y N \cdot Z - \partial_Z N \cdot Y = -N \cdot \big(\partial_Y Z - \partial_Z Y\big) = 0.$$

□

Note that while proving the proposition, we have established the following formula for the commutator:

$$[Y, Z] = \big(y^i \partial_i z^j - z^i \partial_i y^j\big) X_j \tag{2.5}$$

DEFINITION 2.8. Let $X \colon U \to \mathbb{R}^3$ be a surface, and let $N \colon U \to \mathbb{S}^2$ be its unit normal. The *second fundamental form* of X is the symmetric bilinear form k defined on each tangent space $T_u X$ by:

$$k(Y, Z) = -\partial_Y N \cdot Z. \tag{2.6}$$

We remark that since $N \cdot N = 1$, we have $\partial_Y N \cdot N = 0$, hence $\partial_Y N$ is tangential. Thus, according to (2.6), the second fundamental form is minus the tangential directional derivative of the unit normal, and hence measures the turning of the tangent plane as one moves about on the surface. Note that part (2) of the proposition guarantees that k is indeed a symmetric bilinear form. Note that it is not necessarily positive definite. Furthermore, if we set $k_{ij} = k(X_i, X_j)$ to be the coordinate representation of the second fundamental form, then we have:

$$k_{ij} = X_{ij} \cdot N. \tag{2.7}$$

This equation leads to another representation. Consider the Taylor expansion of X at a point, say $0 \in U$:

$$X(u) = X(0) + X_i(0)\, u^i + \frac{1}{2}\, \partial_{ij} X(0)\, u^i u^j + O(|u|^3)$$

Thus, the elevation of X above its tangent plane at u is given up to second-order terms by:

$$\big(X(u) - X(0) - X_i(0) u^i\big) \cdot N = \frac{1}{2} k_{ij}(0) u^i u^j + O(|u|^3).$$

The paraboloid on the right-hand side of the equation above is called the *osculating paraboloid*. A point u of the surface is called *elliptic*, *hyperbolic*, *parabolic*, or *planar*, depending on whether this paraboloid is elliptic, hyperbolic, cylindrical, or a plane.

In classical notation the second fundamental form is:

$$\big(k_{ij}\big) = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Clearly, the second fundamental form is invariant under orientation-preserving reparametrizations. Furthermore, the k_{ij}'s, the coordinate representation of k, changes like the first fundamental form under orientation-preserving reparametrization:

$$\tilde{k}_{ij} = k(\tilde{X}_u, \tilde{X}_j) = k_{ml}\, \frac{\partial u^m}{\partial \tilde{u}^i}\, \frac{\partial u^l}{\partial \tilde{u}^j},$$

Yet another interpretation of the second fundamental form is obtained by considering curves on the surface. The following theorem is essentially due to Euler.

THEOREM 2.5. *Let $\gamma = X \circ \beta \colon [a, b] \to \mathbb{R}^3$ be a curve on a parametric surface $X \colon U \to \mathbb{R}^3$, where $\beta \colon [a, b] \to U$. Let κ be the curvature of γ, and let θ be the angle between the unit normal N of X, and the principal normal e_2 of γ. Then:*

$$\kappa \cos\theta = k(\dot{\gamma}, \dot{\gamma}). \tag{2.8}$$

PROOF. We may assume that γ is parametrized by arclength. We have:

$$\dot{\gamma} = \dot{\beta}^i\, X_i,$$

and

$$\kappa e_2 = \ddot{\gamma} = \ddot{\beta}^i\, X_i + \dot{\beta}^i \dot{\beta}^j\, X_{ij}.$$

The theorem now follows by taking inner product with N, and taking (2.7) into account. □

The quantity $\kappa \cos\theta$ is called the *normal curvature* of γ. It is particularly interesting to consider *normal sections*, i.e., curves γ on X which lie on the intersection of the surface with a normal plane. We may always orient such a plane so that the normal e_2 to γ in the plane coincide with the unit normal N of the surface. In that case, we obtain the simpler result:

$$\kappa = k(\dot{\gamma}, \dot{\gamma}).$$

Thus, the second fundamental form measures the signed curvature of normal sections in the normal plane equipped with the appropriate orientation.

Definition 2.9. Let $X: U \to \mathbb{R}^3$ be a parametric surface, and let k be its second fundamental form. Denote the unit circle in the tangent space at u by $S_uX = \{Y \in T_uX : \ |Y| = 1\}$. For $u \in U$, define the *principal curvatures* of X at u by:

$$k_1 = \min_{Y \in S_uX} k(Y,Y), \qquad k_2 = \max_{Y \in S_uX} k(Y,Y).$$

The unit vectors $Y \in S_uX$ along which the principal curvatures are achieved are called the *principal directions*. The *mean curvature* H and the *Gauss curvature* K of X at u are given by:

$$H = \frac{1}{2}(k_1 + k_2), \qquad K = k_1 k_2.$$

If we consider the tangent space T_uX with the inner product g and the unique linear transformation $\ell : T_uX \to T_uX$ satisfying:

$$g\big(\ell(Y), Z\big) = k(Y, Z), \quad \forall Z \in T_uX, \tag{2.9}$$

then $k_1 \leqslant k_2$ are the eigenvalues of ℓ and the principal directions are the eigenvectors of ℓ. If $k_1 = k_2$ then $k = \lambda g$ and every direction is a principal direction. A point where this holds is called an *umbilical* point. Otherwise, the principal directions are perpendicular. We have that H is the trace and K the determinant of ℓ. Let (g^{ij}) be the inverse of the 2×2 matrix (g_{ij}):

$$g^{im} g_{mj} = \delta^i_j.$$

Set $\ell(X_i) = \ell^j_i X_j$, then since $k_{ij} = g(\ell(X_i), X_j) = \ell^m_i g_{mj}$, we find:

$$\ell^j_i = k_{im} g^{mj}.$$

It is customary to say that g *raises the index* of k and to write the new object ${k_i}^j = k_{im} g^{mj}$. Here since k_{ij} is symmetric, it is not necessary to keep track of the position of the indices, and hence we write: $\ell^j_i = k^j_i$. In particular, we have:

$$H = \frac{1}{2}\, k^i_i, \qquad K = \frac{\det\big(k_{ij}\big)}{\det\big(g_{ij}\big)}. \tag{2.10}$$

Now, $k^{ij} = g^{im} g^{jl} k_{lm}$, and we have

$$|k|^2 = k_{ij} k^{ij} = \operatorname{tr} \ell^2 = k_1^2 + k_2^2 = 4H^2 - 2K.$$

Hence, we conclude

$$K = 2H^2 - \frac{1}{2}\, |k|^2 \tag{2.11}$$

4. Examples

In this section, we use $u^1 = u$, and $u^2 = v$ in order to simplify the notation.

4.1. Planes. Let $U \subset \mathbb{R}^2$ be open, and let $X : U \to \mathbb{R}^3$ be a linear function:

$$X(u, v) = Au + Bv,$$

with $A, B \in \mathbb{R}^3$ linearly independent. Then X is a *plane*. After reparametrization, we may assume that A and B are orthonormal. In that case, the first fundamental form is:

$$ds^2 = du^2 + dv^2.$$

Furthermore, $|A \times B| = 1$, and $N = A \times B$ is constant, hence $k = 0$. In particular, all the points of X are planar, and we have for the mean and Gauss curvatures: $H = K = 0$.

It is of interest to note that if all the points of a parametric surface are planar, then $X(U)$ is contained in a plane. We will later prove a stronger result: X has a reparametrization which is linear.

PROPOSITION 2.6. *Let $X : U \to \mathbb{R}^3$ be a parametric surface, and suppose that its second fundamental form $k = 0$. Then, there is a fixed vector A and a constant b such that $X \cdot A = b$, i.e., X is contained in a plane.*

PROOF. Let A be the unit normal N of X. Let $1 \leqslant i \leqslant 2$, and note that N_i is tangential. Indeed, $N \cdot N = 1$, and differentiating along u^i, we get $N \cdot N_i = 0$. However, since $k = 0$ it follows from (2.6) that $N_i \cdot X_j = -k_{ij} = 0$. Thus, $N_i = 0$ for $i = 1, 2$, and we conclude that N is constant. Consequently, $(X \cdot N)_i = X_i \cdot N = 0$, and $X \cdot N$ is also constant, which proves the proposition. □

4.2. Spheres. Let $U = (0, \pi) \times (0, 2\pi) \subset \mathbb{R}^2$, and let $X : U \to \mathbb{R}^2$ be given by:

$$X(u, v) = (\sin u \, \cos v, \sin u \, \sin v, \cos u).$$

The surface X is a parametric representation of the unit sphere. A straightforward calculation shows that the first fundamental form is:

$$ds^2 = du^2 + \sin^2 u \, dv^2,$$

and the unit normal is $N = X$. Thus, $N_i = X_i$, and consequently $k_{ij} = -N_i \cdot X_j = -X_i \cdot X_j = -g_{ij}$, i.e., $k = -g$. In particular, the principal curvatures are both equal to -1 and all the points are umbilical. We have for the mean and Gauss curvatures:

$$H = -1, \qquad K = 1$$

PROPOSITION 2.7. *Let $X : U \to \mathbb{R}^3$ be a parametric surface and suppose that all the points of X are umbilical. Then, $X(U)$ is either contained in a plane or a sphere.*

PROOF. By hypothesis, we have

$$N_i = \lambda X_i. \tag{2.12}$$

We first show that λ is a constant. Differentiating (2.12), we get $N_{ij} = \lambda_j X_i + \lambda X_{ij}$. Interchanging i and j, subtracting these two equations, and taking into account $N_{ij} - N_{ji} = X_{ij} - X_{ji} = 0$, we obtain $\lambda_i X_j - \lambda_j X_i = 0$, e.g.,

$$\lambda_1 X_2 - \lambda_2 X_1 = 0.$$

Since X_1 and X_2 are linearly independent, we conclude that $\lambda_1 = \lambda_2 = 0$ and it follows that λ is constant. Now, if $\lambda = 0$ then all points are planar, and by

Proposition 2.6, X is contained in a plane. Otherwise, let $A = X - \lambda^{-1}N$, then A is constant:

$$A_i = X_i - \lambda^{-1}N_i = 0,$$

and $|X - A| = |\lambda|^{-1}$ is also constant, hence X is contained in a sphere. $\square$

4.3. Ruled Surfaces. A *ruled surface* is a parametric surface of the form:

$$X(u,v) = \gamma(u) + vY(u)$$

for a curve $\gamma\colon [a,b] \to \mathbb{R}^3$, and a vector field $Y : [a,b] \to \mathbb{R}^3$ along γ. The curve γ is the *directrix*, and the lines $\gamma(u) + tY(u)$ for u fixed are the *generators* of X. We may assume that Y is a unit vector field. Provided $\dot{Y} \neq 0$. We will also assume that $\dot{Y} \neq 0$. In this case, it is possible to arrange by reparametrization that $\dot{\gamma} \cdot \dot{Y} = 0$, in which case γ is said to be a *line of striction*. Indeed, if this is not the case, then we can set $\phi = \left(\dot{\gamma} \cdot \dot{Y}\right)/|\dot{Y}|^2$, and note that the curve

$$\alpha = \gamma + \phi Y$$

lies on the surface X, and satisfies $\dot{\alpha} \cdot \dot{Y} = 0$. Consequently, the surface:

$$\tilde{X}(s,t) = \alpha(s) + tY(s)$$

is a reparametrization of X. Furthermore, there is only one line of striction on X. Indeed, if β and γ are two lines of striction, then since both β is a curve on X we may write $\beta = \gamma + \phi Y$ for some function ϕ and consequently:

$$\dot{\beta} = \dot{\gamma} + \dot{\phi} Y + \phi \dot{Y}.$$

Taking inner product with $\dot{Y}$ and using the fact that Y is a unit vector, we obtain $\phi\left|\dot{Y}\right|^2 = 0$ which implies that $\phi = 0$ and thus, $\beta = \gamma$.

We have $X_u = \dot{\gamma} + v\dot{Y}$, $X_v = Y$, and $X_{vv} = 0$. Thus, the first fundamental is:

$$\left(g_{ij}\right) = \begin{pmatrix} 1 + v^2|\dot{Y}|^2 & \dot{\gamma} \cdot Y \\ \dot{\gamma} \cdot Y & 1 \end{pmatrix}$$

and

$$\det\left(g_{ij}\right) = 1 + v^2|\dot{Y}|^2 + \left(\dot{\gamma} \cdot Y\right)^2 \geqslant v^2|\dot{Y}|^2.$$

Hence, dX is non-singular except possibly on the line of striction. Furthermore, $k_{vv} = N \cdot X_{vv} = 0$, hence $\det\left(k_{ij}\right) = -k_{uv}^2$ and if $\det\left(k_{ij}\right) = 0$ then $N_v \cdot X_u = N_v \cdot X_v = 0$, i.e., N is constant along generators. We have proved the following proposition.

PROPOSITION 2.8. *Let X be a ruled surface. Then X has non-positive Gauss curvature $K \leqslant 0$, and $K(u) = 0$ if and only if N is constant along the generator through u.*

4.3.1. *Cylinders.* Let $\gamma\colon [a,b] \to \mathbb{R}^3$ be a planar curve, and A be a unit normal to the plane which contains γ. Define $X : [a,b] \times \mathbb{R} \to \mathbb{R}^3$ by:

$$X(u,v) = \gamma(u) + vA.$$

The surface X is a *cylinder*. The first fundamental form is:

$$ds^2 = du^2 + dv^2,$$

and we see that for a cylinder dX is always non-singular. After possibly reversing the orientation of A, the unit normal is $N = e_2$. Clearly, $N_v = 0$, and $N_u = -\kappa e_1$. Thus, the second fundamental form is:

$$\kappa\, du^2$$

The principal curvatures are 0 and κ. We have for the mean and Gauss curvatures:

$$H = -\frac{1}{2}\,\kappa, \qquad K = 0.$$

A surface on which $K = 0$ is called *developable*.

4.3.2. *Tangent Surfaces.* Let $\gamma\colon [a,b] \to \mathbb{R}^3$ be a curve with nonzero curvature $\kappa \neq 0$. Its *tangent surface* is the ruled surface:

$$X(u,v) = \gamma(u) + v\dot{\gamma}(u).$$

Since $\dot{\gamma}\cdot\ddot{\gamma} = 0$, the curve γ is the line of striction of its tangent surface. We have $X_u = e_1 + v\kappa\, e_2$ and $X_v = e_1$, hence the first fundamental form is:

$$\big(g_{ij}\big) = \begin{pmatrix} 1 + v^2\kappa^2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The unit normal is $N = -e_3$, and clearly $N_v = 0$. Thus,

4.3.3. *Hyperboloid.* Let $\gamma\colon (0,2\pi) \to \mathbb{R}^3$ be the unit circle in the x^1x^2-plane: $\gamma(t) = \big(\cos(t), \sin(t), 0\big)$. Define a ruled surface $X\colon (0,2\pi)\times\mathbb{R} \to \mathbb{R}^3$ by:

$$X(u,v) = \gamma(u) + v\big(\dot{\gamma}(u) + e_3\big) = \big(\cos(u) - v\sin(u), \sin(u) + v\cos(u), v\big).$$

Note that $(x^1)^2 + (x^2)^2 - (x^3)^3 = 1$ so that $X(U)$ is a hyperboloid of one sheet. A straightforward calculation gives:

$$N = \frac{1}{\sqrt{1+2v^2}}\big(\big(\cos(u) - v\sin(u), \sin(u) + v\cos(u), -v\big),$$

and

$$|N_v|^2 = \frac{2}{1 + 4v^2 + 4v^4}.$$

It follows from Proposition 2.8 that X has Gauss curvature $K < 0$.

5. Lines of Curvature

DEFINITION 2.10. A curve γ on a parametric surface X is called a *line of curvature* if $\dot{\gamma}$ is a principal direction.

The following proposition, due to Rodriguez, characterizes lines of curvature as those curves whose tangents are parallel to the tangent of their *spherical image* under the Gauss map.

PROPOSITION 2.9. *Let γ be a curve on a parametric surface X with unit normal N, and let $\beta = N\circ\gamma$ be its spherical image under the Gauss map. Then γ is a line of curvature if and only if*

$$\dot{\beta} + \lambda\dot{\gamma} = 0. \qquad (2.13)$$

PROOF. Suppose that (2.13) holds, then we have:

$$\partial_{\dot\gamma} N + \lambda \dot\gamma = 0.$$

Let ℓ be the linear transformation on $T_u X$ associated with k as defined by (2.9). Then, we have for every $Y \in T_u X$:

$$g\big(\ell(\dot\gamma), Y\big) = k\big(\dot\gamma, Y\big) = -\partial_{\dot\gamma} N \cdot Y = \lambda g\big(\lambda\dot\gamma, Y\big).$$

Thus, $\ell\big(\dot\gamma\big) = \lambda\dot\gamma$, and $\dot\gamma$ is a principal direction. The proof of the converse is similar. □

It is clear from the proof that λ in (2.13) is the associated principal curvature.

The coordinate curves of a parametric surface X are the two family of curves $\gamma_c(t) = X(t, c)$ and $\beta_c(t) = X(c, t)$. A surface is *parametrized by lines of curvature* if the coordinate curves of X are lines of curvature. We will now show that any non-umbilical point has a neighborhood in which the surface can be reparametrized by lines of curvature. We first prove the following lemma which is also of independent interest.

LEMMA 2.10. *Let $X: U \to \mathbb{R}^3$ be a parametric surface, and let Y_1 and Y_2 be linearly independent vector fields. The following statements are equivalent:*

(1) *Any point $u_0 \in U$ has a neighborhood U_0 and a reparametrization $\phi: V_0 \to U_0$ such that if $\tilde X = X \circ \phi$ then $\tilde X_i = Y_i \circ \phi$.*
(2) $[Y_1, Y_2] = 0$.

PROOF. Suppose that (1) holds. Then Equation (2.5) shows that $[\tilde X_1, \tilde X_2] = 0$. However, since the commutator is invariant under reparametrization, it follows that $[Y_1, Y_2] = 0$.

Conversely, suppose that $[Y_1, Y_2] = 0$. Express $X_i = a_i^j Y_j$ and $Y_i = b_i^j X_j$, and note that $\big(b_i^j\big)$ is the inverse of $\big(a_i^j\big)$. We now calculate:

$$\begin{aligned} 0 &= [X_i, X_j] \\ &= \big[a_i^k Y_k, a_j^l Y_l\big] \\ &= \big(a_i^l \partial_{Y_l} a_j^k - a_j^l \partial_{Y_l} a_i^k\big)\, Y_k + a_i^k a_j^l [Y_k, Y_l] \\ &= \big(a_i^l b_l^m \partial_m a_j^k - a_j^l b_l^m \partial_m a_i^k\big)\, Y_k \\ &= \big(\partial_i a_j^k - \partial_j a_i^k\big)\, Y_k. \end{aligned}$$

Since Y_1 and Y_2 are linearly independent, we conclude that:

$$\partial_i a_j^k - \partial_j a_i^k = 0. \tag{2.14}$$

Now, fix $1 \leqslant k \leqslant 2$, and consider the over-determined system:

$$\frac{\partial \tilde u^k}{\partial u^i} = a_i^k, \qquad i = 1, 2.$$

The integrability condition for this system is exactly (2.14), hence there is a solution in a neighborhood of u_0. Furthermore, since the Jacobian of the map $\psi(u^1, u^2) = (\tilde u^1, \tilde u^2)$ is $d\psi = \big(a_i^k\big)$, and $\det\big(a_i^k\big) \neq 0$, it follows from the inverse function theorem, that perhaps on yet a smaller neighborhood, ψ is a diffeomorphism. Let $\phi = \psi^{-1}$, then ϕ is a diffeomorphism on a neighborhood V_0 of $\psi(u_0)$, and if we set $\tilde X = X \circ \phi$, then:

$$\tilde X_i = X_j \frac{\partial u^j}{\partial \tilde u^i} = X_j b_i^j = Y_i.$$

□

PROPOSITION 2.11. *Let $X: U \to \mathbb{R}^3$ be a parametric surface, and let Y_1 and Y_2 be linearly independent vector fields. Then for any point $u_0 \in U$ there is a neighborhood of u_0 and a reparametrization $\tilde{X} = X \circ \phi$ such that $\tilde{X}_i = f_i Y_i \circ \phi$ for some functions f_i.*

PROOF. By Lemma 2.10 is suffices to show that there are function f_i such that $f_1 Y_1$ and $f_2 Y_2$ commute. Write $[Y_1, Y_2] = a_1 Y_1 - a_2 Y_2$, and compute:

$$[f_1 Y_1, f_2 Y_2] = f_1 f_2 \big(a_1 Y_1 - a_2 Y_2\big) + f_1 \big(\partial_{Y_1} f_2\big) Y_2 - f_2 \big(\partial_{Y_2} f_1\big) Y_1.$$

Thus, the commutator $[f_1 Y_1, f_2 Y_2]$ vanishes if and only if the following two equations are satisfied:

$$\partial_{Y_2} f_1 - a_1 f_1 = 0$$
$$\partial_{Y_1} f_2 - a_2 f_2 = 0.$$

We can rewrite those as:

$$\partial_{Y_2} \log f_1 = a_1$$
$$\partial_{Y_1} \log f_2 = a_2.$$

Each of those equation is a linear first-order partial differential equation, and can be solved for a positive solution in a neighborhood of u_0. □

In a neighborhood of a non-umbilical point, the principal directions define two orthogonal unit vector fields. Thus, we obtain the following Theorem as a corollary to the above proposition.

THEOREM 2.12. *Let $X: U \to \mathbb{R}^3$ be a parametric surface, and let u_0 be a non-umbilical point. Then there is neighborhood U_0 of u_0 and a diffeomorphism $\phi\colon \tilde{U}_0 \to U_0$ such that $\tilde{X} = X \circ \phi$ is parametrized by lines of curvature.*

If X is parametrized by lines of curvature, then the second fundamental form has the coordinate representation:

$$\big(k_{ij}\big) = \begin{pmatrix} k_1 g_{11} & 0 \\ 0 & k_2 g_{22} \end{pmatrix}$$

DEFINITION 2.11. A curve γ on a parametric surface X is called an *asymptotic line* if it has zero normal curvature, i.e., $k(\dot{\gamma}, \dot{\gamma}) = 0$.

The term *asymptotic* stems from the fact that those curve have their tangent $\dot{\gamma}$ along the asymptotes of the *Dupin indicatrix*, the conic section $k_{ij}\xi^i \xi^j = 1$ in the tangent space. Since the Dupin indicatrix has no asymptotes when $K > 0$, we see that the Gauss curvature must be non-positive along any asymptotic line.

The following Theorem can be proved by the same method as used above to obtain Theorem 2.12.

THEOREM 2.13. *Let $X: U \to \mathbb{R}^3$ be a parametric surface, and let u_0 be a hyperbolic point. Then there is neighborhood U_0 of u_0 and a diffeomorphism $\phi\colon \tilde{U}_0 \to U_0$ such that $\tilde{X} = X \circ \phi$ is parametrized by asymptotic lines.*

6. More Examples

A *surface of revolution* is a parametric surface of the form:

$$X(u,v) = \big(f(u)\cos(v), f(u)\sin(v), g(u)\big),$$

where $\big(f(t), g(t)\big)$ is a regular curve, called the *generator* , which satisfies $f(t) \neq 0$. Without loss of generality, we may assume that $f(t) > 0$. The curves

$$\gamma_v(t) = \big(f(t)\cos(v), f(t)\sin(v), g(t)\big), \quad v \text{ fixed.}$$

are called *meridians* and the curves

$$\beta_u(t) = \big(f(u)\cos(t), f(u)\sin(t), g(u)\big), \quad u \text{ fixed.}$$

are called *parallels*. Note that every meridian is a planar curve congruent to the generator and is furthermore also a normal section, and every parallel is a circle of radius $f(u)$. It is not difficult to see that parallels and meridians are lines of curvature. Indeed, let γ_v be a meridian, then choosing as in the paragraph following Theorem 2.5 the correct orientation in the plane of γ_v, its spherical image under the Gauss map is $\sigma_v = N \circ \gamma_v = e_2$, and by the Frenet equations, $\dot{\sigma}_v = -\kappa e_1 = -\kappa\dot{\gamma}_v$. Thus, using Proposition 2.9 and the comment immediately following it, we see that γ_v is a line of curvature with associated principal curvature κ. Since the parallels β_u are perpendicular to the meridians γ_v, it follows immediately that they are also lines of curvature. We derive this also follows from Proposition 2.9 and furthermore obtain the associated principal curvature. A straightforward computation gives that the spherical image of β_u under the Gauss map is:

$$\tau_u = N \circ \beta_u = c\beta_u + B$$

where $B \in \mathbb{R}^3$ and $c \in \mathbb{R}$ are constants. Thus, $\dot{\tau}_u = c\dot{\beta}_u$ and β_u is a line of curvature with associated principal curvature c.

The plane, the sphere, the cylinder, and the hyperboloid are all surfaces of revolution. We discuss one more example.

The *catenoid* is the parametric surface of revolution obtained from the generating curve $\big(\cosh(t), t\big)$:

$$X(u,v) = \big(\cosh(u)\cos(v), \cosh(u)\sin(v), u\big).$$

The normal N is easily calculated:

$$N(u,v) = \left(\frac{-\cos(v)}{\cosh(u)}, \frac{-\sin(v)}{\cosh(u)}, \frac{\sinh(u)}{\cosh(u)}\right)$$

If $\gamma_v(t)$ is a meridian, then $\sigma_v(t) = N(t,v)$ is its spherical image under the Gauss map, and differentiating with respect to t, we get the principal curvature associated with meridians: $\kappa(u,v) = -1/\cosh(u)$. Similarly, the principal curvature associated with parallels is: $1/\cosh(u)$. Thus, we conclude that

$$H = 0, \qquad K = -\frac{1}{\cosh(u)^2}.$$

Definition 2.12. A parametric surface X is *minimal* if it has vanishing mean curvature $H = 0$.

For example, the catenoid is a minimal surface. The justification for the terminology will be given in the next section. The following proposition is immediate from (2.11).

PROPOSITION 2.14. *Let X be a minimal surface. Then X has non-positive Gauss curvature $K \leqslant 0$, and $K(u) = 0$ if and only if u is a planar point.*

We will set out to construct a large class of minimal surfaces. We will use the *Weierstrass Representation.*

DEFINITION 2.13. A parametric surface X is *conformal* if the first fundamental form satisfies $g_{11} = g_{22}$ and $g_{12} = 0$. A parametric surface X is *harmonic* if $\Delta X = X_{11} + X_{22} = 0$.

PROPOSITION 2.15. *Let $X: U \to \mathbb{R}^3$ be a parametric surface which is both conformal and harmonic. Then X is a minimal surface.*

PROOF. We can write the first fundamental form $\big(g_{ij}\big)$, its inverse $\big(g^{ij}\big)$, and the second fundamental form $\big(k_{ij}\big)$ as:

$$\big(g_{ij}\big) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \big(g^{ij}\big) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \big(k_{ij}\big) = \begin{pmatrix} X_{11} \cdot N & X_{12} \cdot N \\ X_{12} \cdot N & X_{22} \cdot N \end{pmatrix}.$$

Thus, the mean curvature vanishes:

$$H = g^{ij} k_{ij} = \lambda^{-1} \big(X_{11} + X_{22}\big) \cdot N = 0.$$

□

In order to construct parametric surfaces which are both conformal and harmonic, we will use complex analysis in the domain U. Let $\zeta = u + iv$ where i denotes $\sqrt{-1}$, and let $f(\zeta)$ and $h(\zeta)$ be two complex analytic functions on U. Define

$$F_1 = f^2 - h^2, \qquad F_2 = i\big(f^2 + h^2\big), \qquad F_3 = 2fh.$$

We have:

$$\big(F_1\big)^2 + \big(F_2\big)^2 + \big(F_3\big)^2 = 0.$$

If we write $F_j = \xi_j + i\eta_j$, then this can be written as:

$$\sum_{j=1}^{3} \Big[\big(\xi_j\big)^2 - \big(\eta_j\big)^2\Big]^2 + 2i \sum_{j=1}^{3} \xi_j \eta_j = 0.$$

Now, in any simply connected subset of U, we can always find analytic functions $G_j = x_j + iy_j$ satisfying $\big(G_j\big)_\zeta = F_j$. We let $X = (x_1, x_2, x_3)$. Then X is conformal and harmonic. Indeed, x_j being the real parts of complex analytic functions, are harmonic, and hence X is harmonic. Furthermore, we have $\big(x_j\big)_u = \xi_j$, and by the Cauchy-Riemann equations $\big(x_j\big)_v = -\big(y_j\big)_u = -\eta_j$. Thus, we see that

$$X_u \cdot X_u - X_v \cdot X_v = \sum_{j=1}^{3} \Big[\big(\xi_j\big)^2 - \big(\eta_j\big)^2\Big]^2 = 0,$$

and

$$X_u \cdot X_v = -\sum_{j=1}^{3} \xi_j \eta_j = 0,$$

and hence, X is conformal.[3] Since X is real analytic, the zeroes of $\det\big(X_i \cdot X_j\big)$ are isolated. Removing the set Z of those zeroes from U, we get that $X : U \setminus Z \to \mathbb{R}^3$ is a harmonic and conformal parametric surface, hence X is a minimal surface[4].

If we carry out this procedure starting with the complex analytic functions $f(\zeta) = 1$ and $h(\zeta) = 1/\zeta$, then X is another parametrization of the catenoid, cf. 2.6.

7. Surface Area

In this section we will give interpretations of the Gauss curvature and the mean curvature. Both of these involve the concept of surface area. Before introducing the definition, we first prove a proposition which will show that the definition is reparametrization invariant.

PROPOSITION 2.16. *Let $X : U \to \mathbb{R}^3$ be a parametric surface with first fundamental form $\big(g_{ij}\big)$, and $V \subset U$. Let $\tilde{X} : \tilde{U} \to \mathbb{R}^3$ be a reparametrization of X, let $\tilde{V} = \phi^{-1}(V)$, and let $\big(\tilde{g}_{ij}\big)$ be the coordinate representation of the first fundamental form of $\tilde{X}$. Then, we have:*

$$\int_{\tilde{V}} \sqrt{\det\big(\tilde{g}_{ij}\big)}\; d\tilde{u}^1\, d\tilde{u}^2 = \int_V \sqrt{\det\big(g_{ij}\big)}\; du^1\, du^2. \tag{2.15}$$

PROOF. By (2.4) we have

$$\sqrt{\det\big(\tilde{g}_{ij}\big)} = \sqrt{\det\big(g_{ij}\big)}\; \big|\det\big(\phi^i_j\big)\big|$$

where $\phi^i_j = \partial u^i / \partial \tilde{u}^j$. Thus, for any open subset $V \subset U$, and $\tilde{V} = \phi^{-1}(V)$, we have:

$$\int_{\tilde{V}} \sqrt{\det\big(\tilde{g}_{ij}\big)}\; d\tilde{u}^1\, d\tilde{u}^2 = \int_{\tilde{V}} \sqrt{\det\big(g_{ij}\big)}\; \big|\det\big(\phi^i_j\big)\big|\; d\tilde{u}^1\, d\tilde{u}^2 = \int_V \sqrt{\det\big(g_{ij}\big)}\; du^1\, du^2$$

□

Thus, the integral on the right-hand side of (2.15) is reparametrization invariant. This justifies the following definition.

DEFINITION 2.14. Let $X : U \to \mathbb{R}^3$ be a parametric surface and let $\big(g_{ij}\big)$ be its first fundamental form. The *surface area element* of X is:

$$dA = \sqrt{\det\big(g_{ij}\big)}\; du^1\, du^2.$$

If $V \subset U$ is open then the *surface area* of X over V is:

$$A_X(V) = \int_V dA = \int_V \sqrt{\det\big(g_{ij}\big)}\; du^1\, du^2 \tag{2.16}$$

By Proposition 2.16, the surface area of X over V is reparametrization invariant, and we can thus speak of the surface area of $X(V)$.

DEFINITION 2.15. Let $X : U \to \mathbb{R}^3$ be a parametric surface, and let $V \subset U$ be open. The *total curvature* of X over V is:

$$K_X(V) = \int_V K\, dA.$$

[3] Of course, $Y = (y_1, y_2, y_3)$ is also conformal, cf. 2.5.

[4] X is also said to be a *branched* minimal surface on U. The zeroes of $\det\big(g_{ij}\big)$ are called *branched points*.

It is easy to show, as in the proof of Proposition 2.16 that the total curvature of X over V is invariant under reparametrization. We now introduce the signed surface area, a variant of Definition 2.14 which allows for smooth maps Y into a surface X, with Jacobian dY not necessarily everywhere non-singular, and which also accounts for multiplicity.

DEFINITION 2.16. Let $X: U \to \mathbb{R}^3$ be a parametric surface, and let $Y: U \to X(U)$ be a smooth map. Define $\sigma(u)$ to be 1, -1, or 0, according to whether the pair $Y_1(u), Y_2(u)$ has the same orientation as the pair $X_1(u), X_2(u)$, the opposite orientation, or is linearly dependent, and let $h_{ij} = Y_i \cdot Y_j$. If $V \subset U$ is open then the *signed surface area* of Y over V is:

$$\hat{A}_Y(V) = \int_V \sigma \sqrt{\det\big(h_{ij}\big)}\; du^1\, du^2$$

For a regular parametric surface, this definition reduces to Definition 2.14. Next, we prove that the total curvature of a surface X over an open set U is the area of the image of U under the Gauss map counted with multiplicity.

THEOREM 2.17. *Let $X: U \to \mathbb{R}^3$ be a parametric surface, and let $V \subset U$ be open. Let $N: U \to \mathbb{S}^2$ be the Gauss map of X, then:*

$$K_X(V) = \hat{A}_N(V).$$

PROOF. We first derive a formula which is of independent interest:

$$N_i = -k_i^j X_j \tag{2.17}$$

To verify this formula, it suffices to check that the inner product of both sides with the three linearly independent vectors X_1, X_2, N are equal. Since $N \cdot N = 1$, we have $N \cdot N_i = 0 = -k_i^j X_j \cdot N = 0$, and $-k_i^j X_j \cdot X_l = -k_i^j g_{jl} = -k_{ij} = -N_i \cdot X_k$. In particular, if $h_{ij} = N_i \cdot N_j$, then we find:

$$h_{ij} = \big(k_i^m X_m\big) \cdot \big(k_j^n X_n\big) = k_i^m k_j^n g_{mn} = k_{im} k_{jn} g^{mn}.$$

In particular,

$$\det\big(h_{ij}\big) = \frac{\big(\det\big(k_{ij}\big)\big)^2}{\det\big(g_{ij}\big)}$$

Note also that Equation (2.17) implies that the pair N_1, N_2 has the same orientation as X_1, X_2 if and only if $\det\big(k_{ij}\big) > 0$. Furthermore, since $N(u)$ is also the outward normal to the unit sphere at $N(u)$, and since X_1, X_2, N is positively oriented in $\mathbb{R}^3$, it follows that $X_1(u), X_2(u)$ also gives the positive orientation on the tangent space to the $\mathbb{S}^2$ at $N(u)$. Thus, we deduce that $\operatorname{sign}\det\big(k_{ij}\big) = \sigma$. Consequently, in view of Equation (2.10), we obtain:

$$\sigma\sqrt{\det\big(h_{ij}\big)} = \frac{\operatorname{sign}\det\big(k_{ij}\big)\,\big|\det\big(h_{ij}\big)\big|}{\sqrt{\det\big(g_{ij}\big)}} = K\sqrt{\det\big(g_{ij}\big)}$$

The proposition follows by integrating over V. □

We now turn to an interpretation of the mean curvature. Let $X: U \to$ be a parametric surface. A *variation* of X is a smooth family $F(u;t): U \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ such that $F(u;0) = X$. Note that since $dF(u;0)$ is non-singular, the same is true of $dF(u;t_0)$ for any fixed u_0, perhaps after shrinking the interval $(-\varepsilon, \varepsilon)$. Thus, all the maps $F(u;t_0)$ for t_0 close enough to 0 are parametric surfaces. The *generator* of the variation is the vector field $dF/dt(u;0)$. The variation is *compactly supported* if $F(u;t) = X(u)$ outside a compact subset of U. The smallest such compact set is called the *support* of the variation F. Clearly, if a variation is compactly supported, then the support of its generator is compact in U. We say that a variation is *tangential* if the generator is tangential; we say it is *normal* if the generator is normal. Suppose now that the closure $\overline{V}$ is compact in U. We consider the area $A_F(V)$ of $F(u;t)$ as a function of t. The next proposition shows that the derivative of this function depends only on the generator, and in fact is a linear functional in the generator.

PROPOSITION 2.18. *Let $X : U \to \mathbb{R}^3$ be a parametric surface, and let $F(u;t)$ be a variation with generator Y. Then:*

$$\left.\frac{dA_F(V)}{dt}\right|_{t=0} = \int_V g^{ij}\, X_i \cdot Y_j \, dA \tag{2.18}$$

We first need the following lemma from linear algebra. We denote by $S^{n\times n}$ the space of $n \times n$ symmetric matrices, and by $S_+^{n\times n}$ the subset of those which are positive definite.

LEMMA 2.19. *Let $B\colon (a,b) \to S_+^{n\times n}$ be continuously differentiable. Then we have:*

$$\big(\log\det B\big)' = \operatorname{tr}\left(B^{-1}B'\right). \tag{2.19}$$

PROOF. First note that (2.19) follows directly if we assume that B is diagonal. Next, suppose that B is symmetric with distinct eigenvalues. Then there is a continuously differentiable orthogonal matrix Q such that $B = Q^{-1}DQ$, where D is diagonal. Note that $dQ^{-1}/dt = -Q^{-1}(dQ/dt)Q$, hence:

$$B^{-1}B' = -Q^{-1}D^{-1}Q'Q^{-1}DQ + Q^{-1}D^{-1}D'Q + Q^{-1}Q',$$

and in view of $\operatorname{tr}\big(AB\big) = \operatorname{tr}(BA)$, we obtain:

$$\operatorname{tr}\left(B^{-1}B'\right) = \operatorname{tr}\left(D^{-1}D'\right).$$

We also have that $\det B = \det D$. Thus taking into the account that (2.19) holds for for D:

$$\big(\log\det B\big)' = \big(\log\det D\big)' = \operatorname{tr}\left(D^{-1}D'\right) = \operatorname{tr}\left(B^{-1}B'\right).$$

In order to prove the general case, it is more convenient to look at the equivalent identity:

$$\big(\det B\big)' = \operatorname{tr}\big((\det B)B^{-1}B'\big). \tag{2.20}$$

Note that by Kramer's rule, the matrix $(\det B)B^{-1}$ is the matrix of co-factors of B, hence its components being determinants of minors of B, are multivariate polynomials in the components of B. Thus, both sides of the identity (2.20) are linear polynomials

$$p(B';B) = \sum_{i,j=1}^{n} p_{ij}(B)b'_{ij}, \qquad q(B';B) = \sum_{i,j=1}^{n} q_{ij}(B)b'_{ij},$$

in the components b'_{ij} of B', whose coefficients $p_{ij}(B)$ and $q_{ij}(B)$ are themselves multivariate polynomials in the components b_{ij} of B. Since the set of matrices with distinct eigenvalues is an open set $U \subset S_+^{n\times n}$, we have already proved that $p(B';B) = q(B';B)$ holds for all values of B', and all $B \in U$. For each such

$B \in U$ the equality $p(B'; B) = q(B'; B)$ for all B' implies that $p_{ij}(B) = q_{ij}(B)$ for $i, j = 1, \ldots, n$. Since this holds for all B in an open set, we conclude that $p_{ij} = q_{ij}$, and hence $p = q$. □

We remark that the more general identity (2.20) in fact holds, as easily shown, for all square matrices B. An immediate consequence of the proposition is that:

$$\left(\sqrt{\det B}\right)' = \frac{1}{2}\operatorname{tr}\left(B^{-1}B'\right)\sqrt{\det B}, \tag{2.21}$$

for any continuously differentiable family of symmetric positive definite matrices B. We are now ready to prove the proposition.

Proof of Proposition 2.18. Differentiating the area (2.16) under the integral sign, and using (2.21), we get:

$$\frac{dA_F(V)}{dt} = \frac{1}{2}\int_V g^{ij}\,\frac{dg_{ij}}{dt}\,\sqrt{\det\left(g_{ij}\right)}\,du^1\,du^2 = \frac{1}{2}\int_V g^{ij}\,\frac{dg_{ij}}{dt}\,dA.$$

Since Y is smooth, we have at $t = 0$ that $dF_i/dt = (dF/dt)_i = Y_i$, and thus

$$g^{ij}\frac{dg_{ij}}{dt} = g^{ij}\left(Y_i \cdot X_j + X_i \cdot Y_j\right) = 2g^{ij}X_i \cdot Y_j.$$

This completes the proof of the proposition. □

Since the variation of the area $dA_F(V)/dt$ is a linear functional in the generator dF/dt of the variation, it is possible to decompose any variation into tangential and normal components. We begin by showing that the area doesn't change under a tangential variation. This is simply the infinitesimal version of Proposition (2.16).

Proposition 2.20. *Let $X\colon U \to \mathbb{R}^3$ be a parametric surface, and let $F(u;t)$ be a compactly supported tangential variation. If $V \subset U$ is open with $\overline{V}$ compact in U, and the support of F contained in V, then $dA_F(V)/dt = 0$.*

Proof. Let Y be the generator of $F(u;t)$. We will show that there is a smooth family of diffeomorphisms $\phi\colon U \times (-\delta, \delta) \to U$ such that Y is also the generator of the variation $G = X \circ \phi$. This proves the proposition since Proposition 2.16 gives that $A_G(U)$ is constant. Since Y is tangential, we can write $Y = y^i X_i$. Consider the initial value problem:

$$\frac{dv^i}{dt} = y^i(v), \qquad v^i(0) = u^i.$$

Since the y^i's are compactly supported, a solution $v = v(u;t)$ exists for all t. Defining $\phi(u;t) = v(u;t)$, then an application of the inverse function theorem shows that $\phi(u;t)$ is a diffeomorphism for t in some small interval $(-\delta, \delta)$. Finally, we see that:

$$\frac{dX \circ \phi}{dt} = X_i\frac{dv^i}{dt} = X_i y^i = Y.$$

□

Our next theorem gives an interpretation of the mean curvature as a measure of surface area variation under normal perturbations.

THEOREM 2.21. *Let $X\colon U \to \mathbb{R}^3$ be a parametric surface, and let $F(u;t)$ be a compactly supported variation with generator Y. If $V \subset U$ is open with $\overline{V}$ compact in U, and the support of F contained in V, then*

$$\frac{dA_F(V)}{dt} = -2\int_V (Y\cdot N)\,H\,dA. \tag{2.22}$$

PROOF. By Propositions 2.18 and 2.20, it suffices to consider normal variations with generator $Y = fN$. In that case, we find that $Y_j = f_j N + f N_j$, so that $g^{ij}\,X_i \cdot Y_j = f g^{ij}\,X_i \cdot N_j = -f k_i^i = -2fH$. The theorem follows by substituting into (2.18). □

DEFINITION 2.17. A parametric surface X is area minimizing if $A_X(U) \leqslant A_{\tilde{X}(U)}$ for any parametric surface $\tilde{X}$ such that $\tilde{X} = X$ on the boundary of U. A parametric surface $X\colon U \to \mathbb{R}^3$ is *locally area minimizing* if for any compactly supported variation $F(u;t)$, the area $A_F(U)$ has a local minimum at $t = 0$.

Clearly, an area-minimizing surface is locally area-minimizing. The following theorem is an immediate corollary of Theorem 2.21.

THEOREM 2.22. *A locally area minimizing surface is a minimal surface.*

Note that in general a minimal surface is only a stationary point of the area functional.

8. Bernstein's Theorem

In this section, we prove Bernstein's Theorem: *A minimal surface which is a graph over an entire plane must itself be a plane.* We say that a surface X is a *graph* over a plane $Y\colon \mathbb{R}^2 \to \mathbb{R}^3$, where Y is linear, if there is a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $X = Y + fN$ where N is the unit normal of Y.

THEOREM 2.23 (Bernstein's Theorem). *Let X be a minimal surface which is a graph over an entire plane. Then X is a plane.*

We may without loss of generality assume that X is a graph over the plane $Y(u,v) = (u,v,0)$, i.e. $X(u,v) = \big(u,v,f(u,v)\big)$ as in example 2.1. It is then straightforward to check that X is a minimal surface if and only if f satisfies the *non-parametric minimal surface equation*:

$$(1+q^2)p_u - 2pqp_v + (1+p^2)q_v = 0, \tag{2.23}$$

where we have used the classical notation: $p = f_u$, $q = f_v$. We say that a solution of a partial differential equation defined on the whole (u,v)-plane is *entire*. Thus, to prove Bernstein's Theorem, it suffices to prove that any entire solution of (2.23) is linear.

PROPOSITION 2.24. *Let f be an entire solution of* (2.23). *Then f is a linear function.*

By Exercise 2.7, if f satisfies (2.23), then p and q satisfy the following equations:

$$\frac{\partial}{\partial u}\left(\frac{1+q^2}{\sqrt{1+p^2+q^2}}\right) = \frac{\partial}{\partial v}\left(\frac{pq}{\sqrt{1+p^2+q^2}}\right), \tag{2.24}$$

$$\frac{\partial}{\partial u}\left(\frac{pq}{\sqrt{1+p^2+q^2}}\right) = \frac{\partial}{\partial v}\left(\frac{1+p^2}{\sqrt{1+p^2+q^2}}\right). \tag{2.25}$$

Since the entire plane is simply connected, Equation (2.25) implies that there exists a function ξ satisfying:

$$\xi_u = \frac{1+p^2}{\sqrt{1+p^2+q^2}}, \qquad \xi_v = \frac{pq}{\sqrt{1+p^2+q^2}},$$

and Equation (2.24) implies that there exists a function η satisfying:

$$\eta_u = \frac{pq}{\sqrt{1+p^2+q^2}}, \qquad \eta_v = \frac{1+q^2}{\sqrt{1+p^2+q^2}}.$$

Furthermore, $\xi_v = \eta_u$, hence there is a function h so that $h_u = \xi$, $h_v = \eta$. The *Hessian* of the function h is:

$$\big(h_{ij}\big) = \begin{pmatrix} h_{uu} & h_{uv} \\ h_{vu} & h_{vv} \end{pmatrix} = \begin{pmatrix} \xi_u & \xi_v \\ \eta_u & \eta_v \end{pmatrix},$$

hence h satisfies the *Monge-Ampère* equation:

$$\det\big(h_{ij}\big) = 1. \tag{2.26}$$

In addition, $h_{11} > 0$, thus $\big(h_{ij}\big)$ is positive definite, and we say that h is *convex*. Proposition 2.24 now follows from the following result due to Nitsche.

PROPOSITION 2.25. *Let $h \in C^2(\mathbb{R}^2)$ be an entire convex solution of the Monge-Ampère Equation* (2.26)*. Then h is a quadratic function.*

PROOF. The proof uses the following transformation introduced by H. Lewy:

$$\varphi\colon (u,v) \mapsto (\xi,\eta) = (u+p, v+q)$$

where $p = h_u$, and $q = h_v$. Clearly, φ is continuously differentiable, and its Jacobian is:

$$d\varphi = \begin{pmatrix} 1+r & s \\ s & 1+t \end{pmatrix},$$

where $r = h_{uu}$, $s = h_{uv}$, and $t = h_{vv}$. Since $\det\big(d\varphi\big) = 2 + r + t > 0$, it follows from the inverse function theorem that φ is a local diffeomorphism, i.e., each point has a neighborhood on which φ is a diffeomorphism. In particular, φ is open.

In view of the convexity of the function h, we have, according to Exercise 2.8:

$$\begin{aligned}
&\big(u_2-u_1\big)(\xi_2-\xi_1) + \big(v_2-v_1\big)\big(\eta_2-\eta_1\big) \\
&\quad = \big(u_2-u_1\big)^2 + \big(v_2-v_1\big)^2 + \big(u_2-u_1\big)\big(p_2-p_1\big) + \big(v_2-v_1\big)\big(q_2-q_1\big) \\
&\qquad \geqslant \big(u_2-u_1\big)^2 + \big(v_2-v_1\big)^2,
\end{aligned}$$

and therefore:

$$\big(u_2-u_1\big)^2 + \big(v_2-v_1\big)^2 \leqslant \big(\xi_2-\xi_1\big)^2 + \big(\eta_2-\eta_1\big)^2,$$

i.e., φ is an *expanding* map. This implies immediately that φ is one-to-one. According to Exercise 2.9, φ is also onto. Thus, φ has an inverse $(u,v) = \varphi^{-1}(\xi,\eta)$ which is also a diffeomorphism. Consider now the function

$$f(\xi + i\eta) = u - p - i(v-q) = 2u - \xi + i(-2v+\eta),$$

where $i = \sqrt{-1}$. In view of

$$d\varphi^{-1} = \begin{pmatrix} u_\xi & u_\eta \\ v_\xi & v_\eta \end{pmatrix} = \frac{1}{2+r+t} \begin{pmatrix} 1+t & -s \\ -s & 1+r \end{pmatrix},$$

it is straightforward to check that f satisfies the Cauchy-Riemann equations, and consequently f is analytic. In fact, f is an entire functions and so is f'. Furthermore,

$$f'(\sigma) = \frac{(t-r)+2is}{2+r+t}, \qquad |f'(\sigma)|^2 = 1 - \frac{4}{2+r+t} < 1,$$

and Liouville's Theorem gives that f' is constant. Finally, the relations:

$$r = \frac{|1-f'|^2}{1-|f'|^2}, \qquad s = \frac{-i(f'-\bar{f}')}{1-|f'|^2}, \qquad t = \frac{|1+f'|^2}{1-|f'|^2},$$

show that r, s, t are constants. □

9. Theorema Egregium

In this section, we prove that the Gauss curvature can be computed in terms of the first fundamental form and its derivatives. We then prove the Fundamental Theorem for surfaces in $\mathbb{R}^3$, analogous to Theorem 1.2 for curves, which states that a parametric surface is uniquely determined by its first and second fundamental form. Partial derivatives with respect to u^i will be denoted by a subscript i following a comma, unless there is no ambiguity in which case the comma may be omitted.

PROPOSITION 2.26. *Let $X: U \to \mathbb{R}^3$ be a parametric surface. Then the following equations hold:*

$$X_{ij} = \Gamma_{ij}^m X_m + k_{ij} N, \tag{2.27}$$

where,

$$\Gamma_{ij}^m = \frac{1}{2} g^{mn} \big(g_{ni,j} + g_{nj,i} - g_{ij,n}\big), \tag{2.28}$$

and (g_{ij}) and (k_{ij}) are the coordinate representations of its first and second fundamental form.

PROOF. Clearly, X_{ij} can be expanded in the basis X_1, X_2, N of $\mathbb{R}^3$. We already saw in Equation (2.7), that the component of X_{ij} along N is k_{ij}, hence Equation (2.27) holds with the coefficients Γ_{ij}^m given by

$$X_{ij} \cdot X_m = \Gamma_{ij}^n g_{mn}.$$

In order to derive (2.28), we differentiate $g_{ij} = X_i \cdot X_j$, and substitute the above equation to obtain:

$$g_{ij,m} = \Gamma_{im}^n g_{nj} + \Gamma_{jm}^n g_{ni}. \tag{2.29}$$

Now, permute cyclically the indices i, j, m, add the first two equations and subtract the last one:

$$g_{ij,m} + g_{mi,j} - g_{jm,i} = 2\Gamma_{jm}^n g_{ni}.$$

Multiplying by g^{il} and dividing by 2 yields (2.28). □

The coefficients Γ^m_{ij} are called the *Christoffel symbols of the second kind*.[5] It is important to note that the Christofell symbols can be computed from the first fundamental form and its first derivatives. Furthermore, they are not invariant under reparametrization.

THEOREM 2.27. *Let $X : U \to \mathbb{R}^3$ be a parametric surface. Then the following equations hold:*

$$\Gamma^m_{ij,l} - \Gamma^m_{il,j} + \Gamma^n_{ij}\Gamma^m_{nl} - \Gamma^n_{il}\Gamma^m_{nj} = g^{mn}\big(k_{ij}k_{ln} - k_{il}k_{jn}\big), \tag{2.30}$$

$$k_{ij,l} - k_{il,j} + \Gamma^m_{ij}k_{lm} - \Gamma^m_{il}k_{jm} = 0. \tag{2.31}$$

PROOF. If we differentiate (2.27), we get:

$$X_{ijl} = \big(\Gamma^m_{ij}X_m\big)_l + \big(k_{ij}N\big)_l = \Gamma^m_{ij,l}X_m + \Gamma^m_{ij}X_{ml} + k_{ij,l}N + k_{ij}N_l.$$

Substituting X_{ml} from (2.27) and N_l from (2.17), and decomposing into tangential and normal components, we obtain:

$$X_{ijl} = A^m_{ijl}X_m + B_{ijl}N,$$

where:

$$\begin{aligned} A^m_{ijl} &= \Gamma^m_{ij,l} + \Gamma^n_{ij}\Gamma^m_{nl} - g^{mn}k_{ij}k_{ln}, \\ B_{ijl} &= k_{ij,l} + \Gamma^m_{ij}k_{lm}. \end{aligned}$$

Taking note of the fact that $X_{ijl} = X_{ilj}$, we now interchange j and l and subtract to obtain (2.30) and (2.31). □

Equation (2.30) is called the *Gauss Equation*, and Equation (2.31) is called the *Codazzi Equation*. The Gauss Equation has the following corollary which has been coined *Theorema Egregium*. It's discovery marked the beginning of intrinsic geometry, the geometry of the first fundamental form.

COROLLARY 2.28. *Let $X : U \to \mathbb{R}^3$ be a parametric surface. Then the Gauss curvature K of X can be computed in terms of only its first fundamental form $\big(g_{ij}\big)$ and its derivatives up to second order:*

$$K = \frac{1}{2}\, g^{ij}\big(\Gamma^m_{ij,m} - \Gamma^m_{im,j} + \Gamma^n_{ij}\Gamma^m_{nm} - \Gamma^n_{im}\Gamma^m_{nj}\big),$$

where Γ^m_{ij} are the Christoffel symbols of the first kind.

PROOF. Combine (2.30) and (2.11). □

We now show, in a manner quite analogous to Theorem 1.2, that provided they satisfy the Gauss-Codazzi Equations, the first and second fundamental form uniquely determine the parametric surface up to rigid motion.

THEOREM 2.29 (Fundamental Theorem). *Let $U \subset \mathbb{R}^2$ be open and simply-connected, let $\big(g_{ij}\big) : U \to S^{2\times 2}_{+}$ and $\big(k_{ij}\big) : U \to S^{2\times 2}$ be smooth, and suppose that they satisfy the Gauss-Codazzi Equations* (2.30)–(2.31). *Then there is a parametric surface $X : U \to \mathbb{R}^3$ such that $\big(g_{ij}\big)$ and $\big(k_{ij}\big)$ are its first and second fundamental forms. Furthermore, X is unique up to rigid motion: if $\tilde{X}$ is another parametric surface with the same first and second fundamental forms, then there is a rigid motion R of $\mathbb{R}^3$ such that $\tilde{X} = R \circ X$.*

[5]The *Christoffel symbols of the first kind* are: $\Gamma_{ijm} = \frac{1}{2}\big(g_{im,j} + g_{jm,i} - g_{ij,m}\big)$.

PROOF. We consider the following over-determined system of partial differential equations for X_1, X_2, N:[6]

$$X_{i,j} = \Gamma_{ij}^m X_m + k_{ij} N, \tag{2.32}$$

$$N_i = -k_{ij} g^{jm} X_m, \tag{2.33}$$

where Γ_{ij}^m is defined in terms of $\left(g_{ij}\right)$ by (2.28). The integrability conditions for this system are:

$$\left(\Gamma_{ij}^m X_m + k_{ij} N\right)_l = \left(\Gamma_{il}^m X_m + k_{il} N\right)_j \tag{2.34}$$

$$\left(k_{ij} g^{jm} X_m\right)_l = \left(k_{lj} g^{jm} X_m\right)_i. \tag{2.35}$$

The proof of Theorem 2.27 also shows that the Gauss-Codazzi Equations (2.30)–(2.31) imply (2.34) if X_i and N satisfy (2.32) and (2.33). We now check that (2.31) also implies (2.35). First note that since Γ_{ij}^m is defined by (2.28), we have

$$\Gamma_{ij}^m g_{mn} = \frac{1}{2}\left(g_{ni,j} + g_{nj,i} - g_{ij,n}\right).$$

Interchanging n and i and adding, we get (2.29). Now, differentiate (2.33), and taking into account that $g^{ij}_{,l} = -g^{ia} g_{ab,l} g^{bj}$, substitute (2.29) to get:

$$\begin{aligned} N_{i,l} = -k_{ij,l} g^{jm} X_m &+ k_{ij} g^{ja}\left(\Gamma_{al}^n g_{nb} + \Gamma_{bl}^n g_{na}\right) g^{bm} X_m \\ &- k_{ij} g^{jm}\left(\Gamma_{ml}^a X_a + k_{ml} g^{jm} N\right) = \left(-k_{ij,l} + k_{in}\Gamma_{jl}^n\right) g^{jm} X_m + k_{ij} k_{ml} g^{jm} N. \end{aligned}$$

Note that the last term is symmetric in i and l so that interchanging i and l, and subtracting, we get:

$$N_{i,l} - N_{l,i} = \left(-k_{ij,l} + k_{il,j} - \Gamma_{ij}^n k_{ln} + \Gamma_{il}^n k_{jn}\right) g^{jm} X_m$$

which vanishes by (2.31). Thus, it follows that (2.35) is satisfied. We conclude that given values for X_1, X_2, N at a point $u_0 \in U$ there is a unique solution of (2.32)–(2.33) in U. We can choose the initial values to that $X_i \cdot X_j = g_{ij}$, $N \cdot X_i = 0$, and $N \cdot N = 1$ at u_0. Using (2.32) and (2.33), it is straightforward to check that the functions $h_{ij} = X_i \cdot X_j$, $p_i = N \cdot X_i$ and $q = N \cdot N$, satisfy the differential equations:

$$\begin{aligned} h_{ij,l} &= \Gamma_{il}^n h_{nj} + \Gamma_{jl}^n h_{ni} + k_{il} p_j + k_{jl} p_i, \\ p_{i,j} &= -k_{jl} g^{lm} h_{mi} + \Gamma_{ij}^m p_m + k_{ij} q, \\ q_i &= -2k_{ij} g^{jm} p_m. \end{aligned}$$

However, the functions $h_{ij} = g_{ij}$, $p_i = 0$ and $q = 1$ also satisfy these equations, as well as the same initial conditions as $h_{ij} = X_i \cdot X_j$, $p_i = N \cdot X_i$ and $q = N \cdot N$ at u_0. Thus, by the uniqueness statement mentioned above, it follows that $X_i \cdot X_j = g_{ij}$, $N \cdot X_i = 0$, and $N \cdot N = 1$. Clearly, in view of (2.32) we have $X_{i,j} = X_{j,i}$, hence there is a function $X \colon U \to \mathbb{R}^3$ whose partial derivatives are X_i, cf. foonote 6. Since $\left(g_{ij}\right)$ is positive definite we have that X_1, X_2 are linearly independent, hence X is a parametric surface with first fundamental form $\left(g_{ij}\right)$. Furthermore, it is easy to see that the unit normal of X is N, and $N_i \cdot X_j = -N \cdot X_{ij} = -k_{ij}$, hence the second fundamental form of X is k_{ij}. This completes the proof of the existence statement.

[6] Here X_i is not to be understood as the derivative of X with respect to u^i until later in the proof.

Assume now that $\tilde{X}$ is another surface with the same first and second fundamental forms. Since X and $\tilde{X}$ have the same first fundamental form, it follows that there is a rigid motion $R(x) = Qx + y$ with $Q \in SO(n;\mathbb{R})$ such that $R\big(X(u_0)\big) = \tilde{X}(u_0)$, $QX_i(u_0) = \tilde{X}_i(u_0)$, $QN(u_0) = \tilde{N}(u_0)$. Let $\hat{X} = R \circ X$. Since the two triples $(\tilde{X}_1, \tilde{X}_2, \tilde{N})$ and $(\hat{X}_1, \hat{X}_2, \hat{N})$ both satisfy the same partial differential equations (2.32) and (2.33), it follows follows that they are equal everywhere, and consequently $\tilde{X} = \hat{X} = R \circ X$. □

Exercises

Exercise 2.1. Let $X\colon U \to \mathbb{R}^3$ and $\tilde{X}\colon \tilde{U} \to \mathbb{R}^3$ be two parametric surfaces. The *angle* θ between them is the angle between their unit normals: $\cos\theta = N \cdot \tilde{N}$. Let γ be a regular curve which lies on both X and $\tilde{X}$, and suppose that the angle between X and $\tilde{X}$ is constant along γ. Show that γ is a line of curvature of X if and only if it is a line of curvature of $\tilde{X}$.

Exercise 2.2. Let $X\colon U \to \mathbb{R}^3$ be a parametric surface, and let γ be an asymptotic line with curvature $\kappa \neq 0$, and torsion τ. Show that $|\tau| = \sqrt{-K}$

Exercise 2.3. Denote by $SO(n)$ the set of orthogonal $n \times n$ matrices, and by $D(n)$ the set of $n \times n$ diagonal matrices. Let $A\colon (a,b) \to S^{n\times n}$ be a C^k function, and suppose that A maps into the set of matrices with distinct eigenvalues. Show that there exist C^k functions $Q\colon (a,b) \to SO(n)$ and $\Lambda\colon (a,b) \to D(n)$ such that $Q^{-1}AQ = \Lambda$. Conclude the matrix function A has C^k eigenvector fields $e_1, \ldots, e_n\colon (a,b) \to \mathbb{R}^n$, $Ae_j = \lambda_j e_j$. Give a counter-example to show that this last conclusion can fail the eigenvalues of A are allowed to coincide.

Exercise 2.4. Let $M^{n\times n}$ be the space of all $n \times n$ matrices, and let $B\colon (a,b) \to M^{n\times n}$ be continuously differentiable. Prove that:

$$\big(\det B\big)' = \operatorname{tr}(B^* B'),$$

where B^* is the matrix of co-factors of B.

Exercise 2.5. Two harmonic surfaces $X, Y\colon U \to \mathbb{R}^3$ are called *conjugate*, if they satisfy the Cauchy-Riemann Equations:

$$X_u = Y_v, \qquad X_v = -Y_u,$$

where (u,v) denote the coordinates in U. Prove that if X is conformal then Y is also conformal. Let X and Y be conformal conjugate minimal surfaces. Prove that for any t:

$$Z = X\cos t + Y\sin t$$

is also a minimal surface. Show that all the surfaces Z above have the same first fundamental form.

Exercise 2.6. Prove that setting $f(\zeta) = 1$, $g(\zeta) = 1/\zeta$ in the Weierstrass representation, we get the catenoid. Find the conjugate harmonic surface of the catenoid.

EXERCISE 2.7. Let $U \subset \mathbb{R}^2$, let $f\colon U \to \mathbb{R}$ be a smooth function, and let $X\colon U \to \mathbb{R}^3$ be given by $(u, v, f(u, v))$, where (u, v) denote the variables in U. Show that X is a minimal surface if and only if it satisfies the non-parametric minimal surface equation:

$$(1+q^2)p_u - 2pqp_v + (1+p^2)q_v = 0,$$

where we have used the classical notation: $p = f_u$, $q = f_v$. Show that if f satisfies the equation above then the following equations are also satisfied:

$$\frac{\partial}{\partial u}\left(\frac{1+q^2}{\sqrt{1+p^2+q^2}}\right) = \frac{\partial}{\partial v}\left(\frac{pq}{\sqrt{1+p^2+q^2}}\right),$$

$$\frac{\partial}{\partial u}\left(\frac{pq}{\sqrt{1+p^2+q^2}}\right) = \frac{\partial}{\partial v}\left(\frac{1+p^2}{\sqrt{1+p^2+q^2}}\right).$$

EXERCISE 2.8. Let $f \in C^2(U)$ be a convex function defined on a convex open set U, and let $\nabla f = (p, q)\colon U \to \mathbb{R}^2$ denote the *gradient* of f. Prove that for any $u_1, u_2 \in U$ the following inequality holds:

$$\big(u_2 - u_1\big) \cdot \big(\nabla f(u_2) - \nabla f(u_1)\big) \geqslant 0.$$

EXERCISE 2.9. Let $U \subset \mathbb{R}^n$ be open. A map $\varphi\colon U \to \mathbb{R}^n$ is *expanding* if $|x - y| \leqslant |\varphi(x) - \varphi(y)|$ for all $x, y \in U$. Let $\varphi\colon U \to \mathbb{R}^n$ be an open expanding map. Show that the image of the ball $B_R(x_0)$ of radius R centered at $x_0 \in U$ contains the disk $B_R\big(\varphi(x_0)\big)$ of radius R centered at $\varphi(x_0)$. Conclude that if $U = \mathbb{R}^n$, then φ is onto $\mathbb{R}^n$.

CHAPTER 3

Local Intrinsic Geometry of Surfaces

In this chapter, we change our point of view, and study *intrinsic geometry*, in which the starting point is the first fundamental form. Thus, given a parametric surface, we will ignore all information which cannot be recovered from the first fundamental form and its derivatives only. In particular, we will ignore the Gauss map and the second fundamental form. Thanks to Gauss' Theorema Egregium, we will still be able to take the Gauss curvature into account.

1. Riemannian Surfaces

DEFINITION 3.1. Let $U \subset \mathbb{R}^2$ be open. A *Riemannian metric* on U is a smooth function $g\colon U \to \mathbb{S}_+^{2\times 2}$. A Riemannian surface patch is an open set U equipped with a Riemannian metric.

The tangent space of U at $u \in U$ is $\mathbb{R}^2$. The Riemannian metric g defines an inner-product on each tangent space by:

$$g(Y, Z) = g_{ij} y^i z^j,$$

where y^i and z^j are the components of Y and Z with respect to the standard basis of $\mathbb{R}^2$. We will write $|Y|_g^2 = g(Y, Y)$, and omit the subscript g when it is not ambiguous.

Two Riemannian surface patches (U, g) and $(\tilde{U}, \tilde{g})$ are *isometric* if there is a diffeomorphism $\phi\colon \tilde{U} \to U$ such that

$$\tilde{g}_{ij} = g_{lm} \phi_i^l \phi_j^m, \tag{3.1}$$

where $\phi_i^l = \partial u^l / \partial \tilde{u}^i$. In fact, Equation (3.1) reads:

$$d\phi^* g = \tilde{g},$$

where $d\phi^* g$ is the pull-back of g by the Jacobian of ϕ at $\tilde{u}$. We then say that ϕ is an *isometry* between (U, g) and $(\tilde{U}, \tilde{g})$. As before, we denote by g^{ij} the inverse of the matrix g_{ij}.

As in Chapter 2, we also denote the Riemannian metric:

$$ds^2 = g_{ij}\, du^i\, du^j,$$

and at times refer to it as a line element. The *arclength* of a curve $\gamma\colon [a, b] \to U$ is then given by:

$$L_\gamma = \int_a^b \sqrt{g_{ij} \dot{\gamma}^i \dot{\gamma}^j}\, dt.$$

Note that the arclength is simply the integral of $\sqrt{g(\dot{\gamma}, \dot{\gamma})}$.

EXAMPLE 3.1. Let $U \subset \mathbb{R}^2$ be open, and let (δ_{ij}) be the identity matrix, then (U, δ) is a Riemannian surface. The Riemannian metric δ will be called the *Euclidean metric*.

EXAMPLE 3.2. Let $X : U \to \mathbb{R}^3$ be a parametric surface, and let g be the coordinate representation of its first fundamental form, then (U, g) is a Riemannian surface patch. We say that the metric g is *induced* by the parametric surface X. If $\tilde{X} = X \circ \phi : \tilde{U} \to \mathbb{R}^3$ is a reparametrization of X and $\tilde{g}$ the coordinate representation of its first fundamental form, then $(\tilde{U}, \tilde{g})$ is isometric to (U, g).

EXAMPLE 3.3 (The Poincaré Disk). Let $D = \{(u, v) : u^2 + v^2 < 1\}$ be the unit disk in $\mathbb{R}^2$, and let

$$g_{ij} = \frac{4}{(1 - r^2)^2}\, \delta_{ij}$$

where $r = \sqrt{u^2 + v^2}$ is the Euclidean distance to the origin. We can write this line element also as

$$ds^2 = 4\, \frac{du^2 + dv^2}{(1 - u^2 - v^2)^2}. \tag{3.2}$$

The Riemannian surface (D, g) is called the *Poincaré Disk*. Let $U = \{(x, y) : y > 0\}$ be the upper half-plane, and let

$$h_{ij} = \frac{1}{y^2}\, \delta_{ij}.$$

Then it is not difficult to see that (D, g_{ij}) and (U, h_{ij}) are isometric with the isometry given by:

$$\phi : (u, v) \mapsto (x, y) = \left(\frac{2v}{(1 + u)^2 + v^2}, \frac{1 - u^2 - v^2}{(1 + u^2) + v^2} \right).$$

In fact, a good bookkeeping technique to check this type of identity is to compute the *differentials* :

$$dx = -4\, \frac{v(1 + u)}{\big((1 + u)^2 + v^2\big)^2}\, du + 2\, \frac{(1 + u)^2 - v^2}{\big((1 + u)^2 + v^2\big)^2}\, dv$$

$$dy = -2\, \frac{(1 + u)^2 - v^2}{\big((1 + u)^2 + v^2\big)^2}\, du + 4\, \frac{v(1 + u)}{\big((1 + u)^2 + v^2\big)^2}\, dv,$$

substitute into

$$\frac{dx^2 + dy^2}{y^2},$$

and then simplify using $du\, dv = dv\, du$ to obtain (3.2). It is not difficult to see that this is equivalent to checking (3.1).

DEFINITION 3.2. Let (U, g) be a Riemannian surface. The *Christoffel symbols of the second kind* of g are defined by:

$$\Gamma_{ij}^{m} = \frac{1}{2}\, g^{mn}\big(g_{ni,j} + g_{nj,i} - g_{ij,n}\big). \tag{3.3}$$

The *Gauss curvature* of g is defined by:

$$K = \frac{1}{2}\, g^{ij}\big(\Gamma_{ij,m}^{m} - \Gamma_{im,j}^{m} + \Gamma_{ij}^{n}\Gamma_{nm}^{m} - \Gamma_{im}^{n}\Gamma_{nj}^{m}\big). \tag{3.4}$$

If (U, g) is induced by the parametric surface $X : U \to \mathbb{R}^3$, then these definitions agree with those of Section 9.

2. Lie Derivative

In this section, we study the *Lie derivative*. We denote the standard basis on $\mathbb{R}^2$ by ∂_1, ∂_2. Let f be a smooth function on U, and let $Y = y^i\partial_i \in T_uU$ be a vector at $u \in U$. The *directional derivative* of f along Y is:

$$\partial_Y f = y^i \partial_i f = y^i f_i.$$

Since $y^i = \partial_Y u^i$ where (u^1, u^2) are the coordinates on U, we see that $Y = Z$ follows from $\partial_Y = \partial_Z$ as operators. The next proposition shows that the directional derivative of a function is reparametrization invariant.

PROPOSITION 3.1. *Let $\phi\colon \tilde{U} \to U$ be a diffeomorphism, and let $\tilde{Y}$ be a vector at $\tilde{u} \in \tilde{U}$. Then for any smooth function f on U, we have:*

$$\big(\partial_{d\phi(\tilde{Y})} f\big) \circ \phi = \partial_{\tilde{Y}}(f \circ \phi).$$

PROOF. Denoting the coordinates on U by u^j and the coordinates on $\tilde{U}$ by $\tilde{u}^i$, we let $\phi^j_i = \partial u^j / \partial \tilde{u}^i$, and we find, by the chain rule:

$$\partial_{\tilde{Y}}(f \circ \phi) = \tilde{y}^i \partial_i (f \circ \phi) = \tilde{y}^i (\partial_j f) \phi^j_i = \big(\partial_{d\phi(\tilde{Y})} f\big) \circ \phi.$$

□

We define the *commutator* of two tangent vector fields $Y = y^i\partial_i$ and $Z = z^i\partial_i$, as in Section (3), Equation (2.5):

$$[Y, Z] = \big(y^i \partial_i z^j - z^i \partial_i y^j\big)\partial_j. \tag{3.5}$$

Note that

$$\partial_{[Y,Z]} f = \partial_Y \partial_Z f - \partial_Z \partial_Y f. \tag{3.6}$$

This observation together with Proposition 3.1 are now used to show that the commutator is reparametrization invariant.

PROPOSITION 3.2. *Let $\tilde{Y}$ and $\tilde{Z}$ be vector fields on $\tilde{U}$, and let $\phi\colon \tilde{U} \to U$ be a diffeomorphism, then*

$$d\phi\big([\tilde{Y}, \tilde{Z}]\big) = \big[d\phi(\tilde{Y}), d\phi(\tilde{Z})\big].$$

PROOF. For any smooth function f on U, we have:

$$\begin{aligned} \partial_{d\phi\left([\tilde{Y},\tilde{Z}]\right)} f &= \partial_{[\tilde{Y},\tilde{Z}]}(f \circ \phi) = \partial_{\tilde{Y}} \partial_{\tilde{Z}} (f \circ \phi) - \partial_{\tilde{Z}} \partial_{\tilde{Y}} (f \circ \phi) \\ &= \partial_{\tilde{Y}} \big(\partial_{d\phi(\tilde{Z})} f\big) \circ \phi - \partial_{\tilde{Z}} \big(\partial_{d\phi(\tilde{Y})} f\big) \circ \phi = \partial_{d\phi(\tilde{Y})} \partial_{d\phi(\tilde{Z})} f - \partial_{d\phi(\tilde{Z})} \partial_{d\phi(\tilde{Y})} f \\ &= \partial_{\left[d\phi(\tilde{Y}), d\phi(\tilde{Z})\right]} f, \end{aligned} \tag{3.7}$$

and the proposition follows. □

We note for future reference that in the proofs of propositions 3.1 and 3.2, only the smoothness of the map ϕ is used, and not the fact that it is a diffeomorphism.

The operator $Z \mapsto \mathfrak{L}_Y Z = [Y, Z]$, also called the *Lie derivative*, is a differential operator, in the sense that it is linear and satisfies a Leibniz identity: $\mathfrak{L}_Y(fZ) = (\partial_Y f)Z + f\mathfrak{L}_Y Z$. However, $\mathfrak{L}_Y Z$ depends on the values of Y in a neighborhood of a point as can be seen from the fact that it is not linear over functions in Y, but rather satisfies $\mathfrak{L}_{fY}Z = f\mathfrak{L}_Y Z - (\partial_Z f)Y$. Hence the Lie derivative cannot be used as an intrinsic directional derivative of a vector field Z, which should only depend on the direction vector Y at a single point[1].

3. Covariant Differentiation

DEFINITION 3.3. Let (U, g) be a Riemannian metric, and let Z be a vector field on U. The *covariant derivative* of Z along ∂_i is:

$$\nabla_i Z = \left(\partial_i z^j + \Gamma^j_{ik} z^k\right)\partial_j. \tag{3.8}$$

Let $Y \in T_u U$, the *covariant derivative* of Z along Y is:

$$\nabla_Y Z = y^i Z_{;i}.$$

We write the components of $\nabla_i Z$ as:

$$z^j{}_{;i} = z^j{}_{,i} + \Gamma^j_{ik} z^k, \tag{3.9}$$

so that $\nabla_Y Z = y^i z^j{}_{;i}\partial_j$. Furthermore, note that

$$\nabla_i \partial_j = \Gamma^k_{ij}\partial_k. \tag{3.10}$$

Our first task is to show that covariant differentiation is reparametrization invariant. However, since the metric g was used in the definition of the covariant derivative, it stands to reason that it would be invariant only under those reparametrization which preserve the metric, i.e., under isometries.

PROPOSITION 3.3. *Let $\phi\colon (\tilde{U}, \tilde{g}) \to (U, g)$ be an isometry. Let $\tilde{Y} \in T_{\tilde{u}}\tilde{U}$, and let $\tilde{Z}$ be a vector field on $\tilde{U}$. Then*

$$d\phi(\tilde{\nabla}_{\tilde{Y}} \tilde{Z}) = \nabla_{d\phi(\tilde{Y})} d\phi(\tilde{Z}). \tag{3.11}$$

PROOF. This proof, although tedious, is quite straightforward, and is relegated to the exercises. □

Note that on the left hand-side of (3.11), the covariant derivative $\tilde{\nabla}$ is that obtained from the metric $\tilde{g}$.

Our next observation, which follows almost immediately from (2.27), gives an interpretation of the covariant derivative when the metric g is induced by a parametric surface X.

PROPOSITION 3.4. *Let the Riemannian metric g be induced by the parametric surface X. Then the image under dX of the covariant derivative $dX(\nabla_i Z)$ is the projection of $\partial_i Z$ onto the tangent space.*

PROOF. Note that $dX(\partial_i) = X_i$. Thus, if $Z = z^j\partial_j$ then we find:

$$dX(\nabla_i Z) = z^j{}_{;i} X_j = z^j{}_{,i} X_j + \Gamma^j_{ik} z^k X_j = \partial_i\left(z^j X_j\right) - k_{ij} z^j N,$$

which proves the proposition. □

We now show that covariant differentiation is in addition well-adapted to the metric g.

[1] Indeed $\partial_Y Z$ as defined in Chapter 2 does depend only on the value of Y at a single point and satisfies $\partial_{fY} Z = f\partial_Y Z$.

PROPOSITION 3.5. *Let (U, g) be a Riemannian surface, and let Y and Z be vector fields on U. Then, we have*

$$\partial_i g(Y, Z) = g(\nabla_i Y, Z) + g(\nabla_i Y, Z). \tag{3.12}$$

PROOF. We first note that, as in the proof of Theorem 2.29, the definition of the Christoffel symbols (3.3) implies (2.29):

$$g_{ij,l} = \Gamma^k_{il} g_{kj} + \Gamma^k_{jl} g_{ki}. \tag{3.13}$$

Now, setting $Y = y^i \partial_i$ and $Z = z^i \partial_i$, we compute:

$$\begin{aligned} \partial_i g(Y, Z) = \partial_i g_{jk} y^j z^k &= \Gamma^m_{ji} g_{km} y^j z^k + \Gamma^m_{ki} g_{mj} y^j z^k + g_{jk} y^j{}_{,i} z^k + g_{jk} y^j z^k{}_{,i} \\ &= g_{jk}(y^j{}_{,i} + \Gamma^j_{mi} y^m) z^k + g_{jk} y^j (z^k{}_{,i} + \Gamma^k_{mi} z^m) = g(Y_{;i}, Z) + g(Y, Z_{;i}). \end{aligned}$$

This completes the proof of (3.12) and of the proposition. □

DEFINITION 3.4. Let $Y = y^i \partial_i$ be a vector field on the Riemannian surface (U, g). Its *divergence* is the function:

$$\operatorname{div} Y = \nabla_i y^i = \partial_i y^i + \Gamma^i_{ij} y^j.$$

Note that:

$$\Gamma^i_{ij} = \frac{1}{2} g^{im}(g_{mi,j} + g_{mj,i} - g_{ij,m}) = \frac{1}{2} g^{im} g_{im,j} = \partial_j \log \sqrt{\det g}.$$

Thus, we see that:

$$\operatorname{div} Y = \frac{1}{\sqrt{\det g}}\ \partial_i \left(\sqrt{\det g}\, y^i \right) \tag{3.14}$$

Observe that this implies

$$\int_U \operatorname{div} Y \, dA = \int_U \partial_i \left(\sqrt{\det g}\, y^i \right)\, du^1\, du^2.$$

Thus, Green's Theorem in the plane implies the following proposition.

PROPOSITION 3.6. *Let Y be a compactly supported vector field on the Riemannian surface (U, g). Then, we have:*

$$\int_U \operatorname{div} Y \, dA = 0.$$

DEFINITION 3.5. If $f \colon U \to \mathbb{R}$ is a smooth function on the Riemannian surface (U, g), its *gradient* ∇f is the unique vector field which satisfies $g(\nabla f, Y) = \partial_Y f$. The *Laplacian* of f if the divergence of the gradient of f:

$$\Delta f = \operatorname{div} \nabla f.$$

It is easy to see that $\nabla f = g^{ij} f_j \partial_j$, hence

$$\Delta f = \frac{1}{\sqrt{\det g}}\ \partial_i \left(g^{ij} \sqrt{\det g}\, f_j \right). \tag{3.15}$$

Thus, in view of Proposition 3.6, if f is compactly supported, we have:

$$\int_U \Delta f \, dA = 0.$$

4. Geodesics

DEFINITION 3.6. Let (U, g) be a Riemannian surface, and let $\gamma\colon I \to U$ be a curve. A vector field along γ is a smooth function $Y\colon I \to \mathbb{R}^2$. The *covariant derivative* of $Y = y^i\partial_i$ along γ is the vector field:

$$\nabla_{\dot\gamma}Y = \left(\dot y^i + \Gamma^i_{jk}y^j\dot\gamma^k\right)\partial_i.$$

Note that if Z is any *extension* of Y, i.e., a any vector field defined on a neighborhood V of the image $\gamma(I)$ of γ in U, then we have:

$$\nabla_{\dot\gamma}Y = \nabla_{\dot\gamma}Z = \dot\gamma^i Z_{;i}.$$

Thus, any result proved concerning the usual covariant differentiation, in particular Proposition 3.5 holds also for the covariant differentiation along a curve.

DEFINITION 3.7. A vector field Y along a curve γ is said to be *parallel* along γ if $\nabla_{\dot\gamma}Y = 0$.

Note that if Y and Z are parallel along γ, then $g(Y, Z)$ is constant. This follows from Proposition 3.5:

$$\partial_{\dot\gamma}g(Y, Z) = g(\nabla_{\dot\gamma}Y, Z) + g(Y, \nabla_{\dot\gamma}Z) = 0.$$

PROPOSITION 3.7. *Let $\gamma\colon [a, b] \to U$ be a curve into the Riemannian surface (U, g), let $u_0 \in U$, and let $Y_0 \in T_{u_0}U$. Then there is a unique vector field Y along γ which is parallel along γ and satisfies $Y(a) = Y_0$.*

PROOF. The condition that Y is parallel along γ is a pair of linear first-order ordinary differential equations:

$$\dot y^i = -\Gamma^i_{jk}(\gamma)\gamma^j y^j.$$

Given initial conditions $y^i(a) = y^i_0$, the existence and uniqueness of a solution on $[a, b]$ follows from the theory of ordinary differential equations. □

The proposition together with the comment preceding it shows that parallel translation along a curve γ is an isometry between inner-product spaces $P_\gamma : T_aU \to T_bU$.

DEFINITION 3.8. A curve γ is a *geodesic* if its tangent $\dot\gamma$ is parallel along γ:

$$\nabla_{\dot\gamma}\dot\gamma = 0.$$

If γ is a geodesic, then $|\dot\gamma|$ is constant and hence, every geodesic is parametrized proportionally to arclength. In particular, if $\beta = \gamma \circ \phi$ is a reparametrization of γ, then β is not a geodesic unless ϕ is a linear map.

PROPOSITION 3.8. *Let (U, g) be a Riemannian surface, let $u_0 \in U$ and let $0 \neq Y_0 \in T_{u_0}U$. Then there is and $\varepsilon > 0$, and a unique geodesic $\gamma\colon (-\varepsilon, \varepsilon) \to U$, such that $\gamma(0) = u_0$, and $\dot\gamma(0) = Y_0$.*

PROOF. We have:

$$\nabla_{\dot\gamma}\dot\gamma = \left(\ddot\gamma^i + \Gamma^i_{jk}\dot\gamma^j\dot\gamma^k\right)\partial_i.$$

Thus, the condition that γ is a geodesic can written as a pair of non-linear second-order ordinary differential equations:

$$\ddot\gamma^i = -\Gamma^i_{jk}(\gamma(t))\dot\gamma^j\dot\gamma^k.$$

Given initial conditions $\gamma^i(0) = u^i_0$, $\dot\gamma^i(0) = y^i_0$, there is a unique solution on defined on a small enough interval $(-\varepsilon, \varepsilon)$. □

DEFINITION 3.9. Let $\gamma\colon [a,b] \to U$ be a curve. We say that γ is *length-minimizing*, or L-minimizing, if:

$$L_\gamma \leqslant L_\beta$$

for all curves β in U such that $\beta(a) = \gamma(a)$ and $\beta(b) = \gamma(b)$.

Let $\gamma\colon [a,b] \to U$ be a curve. A *variation* of γ is a smooth family of curves $\sigma(t;s)\colon [a,b]\times(-\varepsilon,\varepsilon) \to I$ such that $\sigma(t;0) = \gamma(t)$ for all $t \in [a,b]$. For convenience, we will denote derivatives with respect to t as usual by a dot, and derivatives with respect to s by a prime. The generator of a variation σ is the vector field $Y(t) = \sigma'(t;0)$ along γ. We say that σ is a *fixed-endpoint* variation, if $\sigma(a;s) = \gamma(a)$, and $\sigma(b;s) = \gamma(b)$ for all $s \in (-\varepsilon,\varepsilon)$. Note that the generator of a fixed-endpoint variation vanishes at the end points. We say that a variation σ is *normal* if its generator Y is perpendicular to γ: $g(\dot\gamma, Y) = 0$. A curve γ is *locally L-minimizing* if

$$L_\sigma(s) = \int_a^b \sqrt{g(\dot\sigma,\dot\sigma)}\,dt$$

has a local minimum at $s = 0$ for all fixed-endpoint variations σ. Clearly, an L-minimizing curve is locally L-minimizing.

If γ is locally L-minimizing, then any reparametrization $\beta = \gamma\circ\phi$ of γ is also locally L-minimizing. Indeed, if σ is any fixed-endpoint variation of β, then $\tau(t;s) = \sigma(\phi^{-1}(t);s)$ is a fixed-endpoint variation of γ, and since reparametrization leaves arclength invariant, we see that $L_\tau(s) = L_\sigma(s)$ which implies that L_σ also has a local minimum at at $s = 0$. Thus, local minimizers of the functional L are not necessarily parametrized proportionally to arclength. This helps clarify the following comment: a locally length-minimizing curve is not necessarily a geodesic, but according to the next theorem that is only because it may not be parametrized proportionally to arclength.

THEOREM 3.9. *A locally length-minimizing curve has a geodesic reparametrization.*

To prove this theorem, we introduce the *energy* functional:

$$E_\gamma = \frac{1}{2}\int_a^b g(\dot\gamma,\dot\gamma)\,dt$$

We may now speak of *energy-minimizing* and *locally energy-minimizing* curves. Our first lemma shows the advantage of using the energy rather than the arclength functional: *minimizers of E are parametrized proportionally to arclength.*

LEMMA 3.10. *A locally energy-minimizing curve is a geodesic.*

PROOF. Suppose that γ is a locally energy-minimizing curve. We first note that if Y is any vector field along γ which vanishes at the endpoints, then setting $\sigma(t;s) = \gamma(t) + sY(t)$, we see that there is a fixed-endpoint variation of γ whose generator is Y. Since γ is locally energy-minimizing, we have:

$$E'_\sigma(0) = \int_a^b \frac{1}{2}\big(g(\dot\sigma,\dot\sigma)\big)'\big|_{s=0}\,dt = 0.$$

We now observe that:

$$\big(\dot\sigma^j\big)'\big|_{s=0} = \big(\frac{d}{dt}\sigma^j\big)'\big|_{s=0} = \frac{d}{dt}\big(\sigma^j\big)'\big|_{s=0} = \dot y^j.$$

where $Y = y^i \partial_i$ is the generator of the fixed-endpoint variation σ, and:

$$(g_{ij})'\big|_{s=0} = g_{ij,k}\,(\sigma^k)'\big|_{s=0} = g_{ij,k}\, y^k.$$

Thus, we have:

$$\frac{1}{2}\left(g(\dot\sigma,\dot\sigma)\right)'\big|_{s=0} = \frac{1}{2}\left(g_{ij}\,\dot\sigma^i\dot\sigma^j\right)'\big|_{s=0} = \frac{1}{2}(g_{ij})'\big|_{s=0}\dot\sigma^i\dot\sigma^j + g_{ij}\,\dot\sigma^i\,(\dot\sigma^j)'\big|_{s=0}$$
$$= \frac{1}{2}g_{ij,k}\,y^k\dot\gamma^i\dot\gamma^j + g_{ij}\,\dot\gamma^i\dot y^j.$$

Since Y vanishes at the endpoints, we can substitute into $E'_\sigma(0)$, and integrate by parts the second term to get:

$$E'_\sigma(0) = -\int_a^b \left[\frac{d}{dt}\big(g_{ij}\,\dot\gamma^i\big) - \frac{1}{2}g_{ik,j}\,\dot\gamma^i\dot\gamma^k\right] y^j.$$

Since:

$$\frac{d}{dt}\big(g_{ij}\,\dot\gamma^i\big) = g_{ij}\ddot\gamma^i + g_{ij,k}\,\dot\gamma^i\dot\gamma^k = g_{ij}\ddot\gamma^i + \frac{1}{2}\big(g_{ij,k} + g_{kj,i}\big)\dot\gamma^i\dot\gamma^k,$$

We now see that:

$$E'_\sigma(0) = -\int_a^b \left[g_{ij}\ddot\gamma^i + \frac{1}{2}\big(g_{mj,k} + g_{kj,m} - g_{mk,j}\big)\dot\gamma^m\dot\gamma^k\right] y^j\,dt = -\int_a^b g(\nabla_{\dot\gamma}\dot\gamma, Y)\,dt.$$

Since $E'_\sigma(0) = 0$ for all vector fields Y along γ which vanish at the endpoints, we conclude that $\nabla_{\dot\gamma}\dot\gamma = 0$, and γ is a geodesic. □

The Schwartz inequality implies the following inequality between the length and energy functional for a curve γ.

LEMMA 3.11. *For any curve γ, we have*

$$L_\gamma^2 \leqslant 2E_\gamma\ (b-a),$$

with equality if and only if γ is parametrized proportionally to arclength.

Finally, the last lemma we state to prove Theorem 3.9, exhibits the relationship between the L and E functionals.

LEMMA 3.12. *A locally energy-minimizing curve is locally length-minimizing. Furthermore, if γ is locally length-minimizing and β is a reparametrization of γ by arclength, then β is locally energy-minimizing.*

PROOF. Suppose that γ is locally energy-minimizing, and let σ be a fixed-endpoint variation of γ. For each s, let $\beta_s(t)\colon [a,b] \to U$ be a reparametrization of the curve $t \mapsto \sigma(t;s)$ proportionally to arclength. Let $\tau(t;s) = \beta_s(t)$, then it is not difficult to see, using say the theorem on continuous dependence on parameters for ordinary differential equations, that τ is also smooth. By Lemma 3.10, γ is a geodesic, hence by Lemma 3.11, $L_\gamma^2 = 2E_\gamma(b-a)$. It follows that:

$$L_\sigma^2(0) = L_\gamma^2 = 2E_\gamma(b-a) = 2E_\tau(0)(b-a) \leqslant 2E_\tau(s)(b-a) = L_\tau^2(s) = L_\sigma^2(s).$$

Thus, γ is locally length-minimizing proving the first statement in the lemma.

Now suppose that γ is locally length-minimizing, and let β be a reparametrization of γ by arclength. Then β is also locally length-minimizing, hence for any fixed-endpoint variation σ of β, we have:

$$E_\sigma(0) = E_\beta = \frac{L_\beta^2}{2(b-a)} \leqslant \frac{L_\sigma^2(s)}{2(b-a)} \leqslant E_\sigma(s).$$

Thus, β is locally energy-minimizing. □

We note that the same lemma holds if we replace locally energy-minimizing by energy-minimizing. The proof of Theorem 3.9 can now be easily completed with the help of Lemmas 3.10 and 3.12.

PROOF OF THEOREM 3.9. Let β be a reparametrization of γ by arclength. By Lemma 3.12, β is locally energy-minimizing. By Lemma 3.10, β is a geodesic. □

5. The Riemann Curvature Tensor

DEFINITION 3.10. Let X, Y, Z, W be vector fields on a Riemannian surface (U, g). The *Riemann curvature tensor* is given by:

$$R(W, Z, X, Y) = g\left(\left[\nabla_X, \nabla_Y\right] Z - \nabla_{[X,Y]} Z, W\right).$$

We first prove that R is indeed a *tensor*, i.e., it is linear over functions. Clearly, R is linear in W, additive in each of the other three variables, and anti-symmetric in X and Y. Thus, it suffices to prove the following lemma.

LEMMA 3.13. *Let X, Y, Z, W be vector fields on a Riemannian surface (U, g). Then we have:*

$$R(W, Z, fX, Y) = R(W, fZ, X, Y) = fR(W, Z, X, Y).$$

PROOF. We have:

$$\begin{aligned} \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z &= f\nabla_X \nabla_Y X - \nabla_Y\left(f\nabla_X Z\right) - \nabla_{f[X,Y]-(\partial_Y f)X} Z \\ &= f\nabla_X\nabla_Y Z - \left(\partial_Y f\right)\nabla_X Z - f\nabla_Y\nabla_X Z - f\nabla_{[X,Y]} Z + \left(\partial_Y f\right)\nabla_X Z \\ &= f\left(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z\right). \end{aligned}$$

The first identity follows by taking inner product with W. In order to prove the second identity, note that:

$$\begin{aligned} \nabla_X\nabla_Y fZ &= \nabla_X\left(\left(\partial_Y f\right) Z\right) + \nabla_X\left(f\nabla_Y Z\right) \\ &= \left(\partial_X\partial_Y f\right) Z + \left(\partial_Y f\right)\left(\nabla_X Z\right) + \left(\partial_X f\right)\left(\nabla_Y Z\right) + f\nabla_X\nabla_Y Z. \end{aligned}$$

Interchanging X and Y and subtracting we get:

$$\left[\nabla_X, \nabla_Y\right] fZ = \left(\partial_{[X,Y]} f\right) Z + f\left[\nabla_X, \nabla_Y\right] Z.$$

On the other hand, we have also:

$$\nabla_{[X,Y]} fZ = \left(\partial_{[X,Y]} f\right) Z + f\nabla_{[X,Y]} Z.$$

Thus, we conclude:

$$\left[\nabla_X, \nabla_Y\right] fZ - \nabla_{[X,Y]} fZ = f\left(\left[\nabla_X, \nabla_Y\right] Z - \nabla_{[X,Y]} Z\right).$$

The second identity now follows by taking inner product with W. □

Let

$$R_{ijkl} = R(\partial_i, \partial_j, \partial_k, \partial_l),$$

be the *components* of the Riemann tensor. The previous proposition shows that if $X = x^i\partial_i$, $Y = y^i\partial_i$, $Z = z^i\partial_i$, $W = w^i\partial_i$, then

$$R(W, Z, X, Y) = w^i z^j x^k y^l R_{ijkl},$$

that is, the value of $R(W, Z, X, Y)$ at a point u depends only on the values of W, Z, X, and Y at u.

PROPOSITION 3.14. *The components R_{ijkl} of the Riemann curvature tensor of any metric g satisfy the following identities:*

$$R_{ijkl} = -R_{ijlk} = -R_{jikl} = R_{klij} \tag{3.16}$$

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0. \tag{3.17}$$

PROOF. We first prove (3.17). Since $[\partial_k, \partial_l] = 0$, it suffices to prove

$$\big[\nabla_k, \nabla_l\big]\partial_j + \big[\nabla_j, \nabla_k\big]\partial_l + \big[\nabla_l, \nabla_j\big]\partial_k = 0. \tag{3.18}$$

Note that (3.10) together with the symmetry $\Gamma^m_{lj} = \Gamma^m_{jl}$ imply that $\nabla_l\partial_j = \nabla_j\partial_l$. Thus, we can write:

$$\big[\nabla_k, \nabla_l\big]\partial_j = \nabla_k\nabla_j\partial_l - \nabla_l\nabla_k\partial_j.$$

Permuting the indices cyclically, and adding, we get (3.18). The first identity in (3.22) is obvious from Definition 3.10. We now prove the identity:

$$R_{ijkl} = -R_{jikl}.$$

Using Proposition 3.5 repeatedly, we observe that:

$$\begin{aligned} g(\nabla_k\nabla_l\partial_j, \partial_i) &= \partial_k g(\nabla_l\partial_j, \partial_i) - g(\nabla_l\partial_j, \nabla_k\partial_i) \\ &= \partial_k\big(\partial_l g_{ji} - g(\partial_j, \nabla_l\partial_i)\big) - \partial_l g(\partial_j, \nabla_k\partial_i) + g(\partial_j, \nabla_l\nabla_k\partial_i) \\ &= \partial_k\partial_l g_{ji} - \partial_k g(\partial_j, \nabla_l\partial_i) - \partial_l g(\partial_j, \nabla_k\partial_i) + g(\partial_j, \nabla_l\nabla_k\partial_i). \end{aligned}$$

It is easy to see that the first term, and the next two taken together, are symmetric in k and l. Thus, interchanging k and l, and subtracting, we get:

$$R_{ijkl} = g\left(\big[\nabla_k, \nabla_l\big]\partial_j, \partial_i\right) = g\left(\partial_j, \big[\nabla_l, \nabla_k\big]\partial_i\right) = -g\left(\big[\nabla_k, \nabla_l\big]\partial_i, \partial_j\right) = -R_{jikl}.$$

The last identity in (3.16) now follows from the first two and (3.17). We prove that $B_{ijkl} = R_{ijkl} - R_{klij} = 0$. Note that B_{ijkl} satisfies (3.16) as well as $B_{ijkl} = -B_{klij}$. Now, in view of the identities already established, we see that:

$$R_{ijkl} = -R_{iljk} - R_{iklj} = -R_{likj} - R_{iklj} = R_{ljik} + R_{lkji} - R_{iklj} = B_{ljik} + R_{klij},$$

hence $B_{ijkl} = B_{ljik}$. Using the symmetries of B_{ijkl}, we can rewrite this identity as:

$$B_{ijkl} + B_{iklj} = 0. \tag{3.19}$$

We now permute the first three indices cyclically:

$$B_{kijl} + B_{kjli} = 0, \tag{3.20}$$

$$B_{jkil} + B_{jilk} = 0, \tag{3.21}$$

add (3.19) to (3.20) and subtract (3.21) to get, using the symmetries of B_{ijkl}:

$$B_{ijkl} + B_{iklj} + B_{iklj} + B_{kjli} - B_{kjli} - B_{ijkl} = 2B_{iklj} = 0.$$

This completes the proof of the proposition. □

It follows, that all the non-zero components of the Riemann tensor are determined by R_{1212}:

$$R_{1212} = -R_{2112} = R_{2121} = -R_{1221},$$

and all other components are zero. The proposition also implies that for any vectors X, Y, Z, W, the following identities hold:

$$R(W,Z,X,Y) = -R(W,Z,Y,Z) = -R(Z,W,X,Y) = R(X,Y,W,Z), \tag{3.22}$$

$$R(W,Z,X,Y) + R(W,Y,Z,X) + R(W,X,Y,Z) = 0. \tag{3.23}$$

PROPOSITION 3.15. *The components R_{ijkl} of the Riemann curvature tensor of any metric g satisfy:*

$$g^{mj} R_{imkl} = \Gamma^j_{ik,l} - \Gamma^j_{il,k} + \Gamma^n_{ik}\Gamma^j_{nl} - \Gamma^n_{il}\Gamma^j_{nk}. \tag{3.24}$$

Furthermore, we have:

$$K = \frac{R_{1212}}{\det(g)}, \tag{3.25}$$

where K is the Gauss curvature of g.

PROOF. Denote the right-hand side of (3.24) by S^j_{ikl}. We have:

$$\nabla_l \nabla_k \partial_i = \nabla_k\big(\Gamma^j_{ik}\partial_j\big) = \big(\Gamma^j_{ik,l} + \Gamma^n_{ik}\Gamma^j_{nl}\big)\partial_j,$$

or equivalently:

$$\Gamma^j_{ik,l} + \Gamma^n_{ik}\Gamma^j_{nl} = g^{jm} g(\nabla_l \nabla_k \partial_i, \partial_m).$$

Interchanging k and l and subtracting we get:

$$S^j_{ikl} = g^{jm} g\left(\big[\nabla_l, \nabla_k\big]\partial_i, \partial_m\right) = g^{jm} R_{milk} = g^{jm} R_{imkl}.$$

According to 3.4 and (3.24), we have:

$$K = \frac{1}{2} g^{ik} S^j_{ikj} = \frac{1}{2}\, g^{ik} g^{jl} R_{ijkl}.$$

In view of the comment following Proposition 3.14, the only non-zero terms in this sum are:

$$K = \frac{1}{2}\big(g^{11}g^{22}R_{1212} + g^{12}g^{21}R_{1221} + g^{21}g^{12}R_{2112} + g^{22}g^{11}R_{2121}\big) = \det\big(g^{-1}\big)R_{1212},$$

which implies (3.25) □

COROLLARY 3.16. *The Riemann curvature tensor of any metric g on a surface is given by:*

$$R_{ijkl} = K\big(g_{ik}g_{jl} - g_{il}g_{jk}\big). \tag{3.26}$$

PROOF. Denote the right-hand side of (3.26) by S_{ijkl}, and note that it satisfies (3.16). Thus, the same comment which follows Proposition 3.14 applies and the only non-zero components of S_{ijkl} are determined by S_{1212}:

$$S_{1212} = -S_{2112} = S_{2121} = -S_{1221}.$$

In view of (3.26), we have $R_{1212} = S_{1212}$, thus it follows that $R_{ijkl} = S_{ijkl}$ □

In particular, we conclude that:

$$R(Z,W,X,Y) = K\big(g(W,X)\,g(Z,Y) - g(W,Y)\,g(Z,X)\big). \tag{3.27}$$

6. The Second Variation of Arclength

In this section, we study the additional condition $E''_\sigma(0) \geqslant 0$ necessary for a minimum. This leads to the notion of Jacobi fields and conjugate points.

PROPOSITION 3.17. *Let $\gamma\colon [a,b] \to U$ be a geodesic parametrized by arclength on the Riemannian surface (U, g), and let σ be a fixed-endpoint variation of γ with generator Y. Then, we have:*

$$E''_\sigma(0) = \int_a^b \left(\left|\nabla_{\dot\gamma} Y\right|^2 - K \circ \gamma \left(|Y|^2 - g(\dot\gamma, Y)^2\right)\right) dt, \tag{3.28}$$

where K is the Gauss curvature of g.

Before we prove this proposition, we offer a second proof of the first variation formula:

$$E'_\sigma(0) = -\int_a^b g(\nabla_{\dot\gamma}\dot\gamma, Y)\, dt, \tag{3.29}$$

which is more in spirit with our derivation of the second variation formula. First note that if σ is a fixed-endpoint variation of γ with generator $\sigma' = Y$, and with $\dot\sigma = X$, then $[X, Y] = 0$. Here Y denotes the vector field σ' along σ rather than just along γ. Indeed, since $X = d\sigma(d/dt)$ and $Y = d\sigma(d/ds)$, it follows, as in Propositions 3.1 and 3.2, that for any smooth function f on U, we have

$$\partial_{[X,Y]} f = \left[\frac{d}{dt}, \frac{d}{ds}\right] f \circ \sigma = 0.$$

In view of the symmetry $\Gamma^i_{jk} = \Gamma^i_{kj}$, this implies:

$$\nabla_Y X - \nabla_X Y = [X, Y] = 0.$$

We can now calculate:

$$\begin{aligned} E'_\sigma(s) &= \frac{1}{2}\int \partial_Y g(X, X)\, dt = \int_a^b g(\nabla_Y X, X)\, dt = \int_a^b g(\nabla_X Y, X)\, dt \\ &= \int_a^b \frac{d}{dt} g(Y, X)\, dt - \int_a^b g(Y, \nabla_X X)\, dt = g(Y, X)\big|_a^b - \int_a^b g(Y, \nabla_X X)\, dt \end{aligned}$$

Setting $s = 0$, (3.29) follows.

PROOF OF PROPOSITION 3.17. We compute:

$$\begin{aligned} E''_\sigma &= \frac{1}{2}\int_a^b \partial_Y \partial_Y g(X, X)\, dt = \int_a^b \partial_Y g(\nabla_Y X, X)\, dt = \int_a^b \partial_Y g(\nabla_X Y, X)\, dt \\ &= \int_a^b \big(g(\nabla_Y \nabla_X Y, X) + g(\nabla_X Y, \nabla_Y X)\big)\, dt \\ &= \int_a^b \Big(g(\nabla_X \nabla_Y Y, X) + g\left(\left[\nabla_Y, \nabla_X\right] Y, X\right) + g(\nabla_X Y, \nabla_X Y)\Big)\, dt \\ &= \int_a^b \left(\frac{d}{dt} g(\nabla_Y Y, X) - g(\nabla_Y Y, \nabla_X X) + R(X, Y, Y, X) + g(\nabla_X Y, \nabla_X Y)\right) dt, \end{aligned}$$

where as above $X = \dot\sigma$, and $Y = \sigma'$. Now, the first term integrates to $g(\nabla_Y Y, X)\big|_a^b = 0$, and when we set $s = 0$, the second term also vanishes since $\nabla_X X = \nabla_{\dot\gamma}\dot\gamma = 0$.

Furthermore, the last term becomes $g(\nabla_{\dot\gamma}Y, \nabla_{\dot\gamma}Y)$. Hence, we conclude:

$$E''_\sigma(0) = \int_a^b \Big(\big|\nabla_{\dot\gamma}Y\big|^2 - R(X,Y,X,Y)\Big)\, dt. \tag{3.30}$$

The proposition now follows from (3.27). □

Thus, $E''_\sigma(0)$ can be viewed as a quadratic form in the generator Y. The corresponding symmetric bilinear form is called the *index form* of γ:

$$I(Y,Z) = \int_a^b \big(g(\nabla_{\dot\gamma}Y, \nabla_{\dot\gamma}Z) - K\circ\gamma\, \big(g(Y,Z) - g(\dot\gamma, Y)\, g(\dot\gamma, Z)\big)\big)\, dt.$$

It is the Hessian of the functional E, and if E has a local minimum, I is positive semi-definite. We will also write $I(Y) = I(Y,Y)$.

DEFINITION 3.11. Let γ be a geodesic parametrized by arclength on the Riemannian surface (U,g). A vector field Y along γ is called a *Jacobi field*, if it satisfies the following differential equation:

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y + K\,(Y - g(\dot\gamma, Y)\dot\gamma) = 0.$$

Two points $\gamma(a)$ and $\gamma(b)$ along a geodesic γ are called *conjugate* along γ if there is a non-zero Jacobi field along γ which vanishes at those two points.

The Jacobi field equation is a linear system of second-order differential equations. Hence given initial data specifying the initial value and initial derivative of Y, a unique solution exists along the entire geodesic γ.

PROPOSITION 3.18. *Let γ be a geodesic on the Riemannian surface (U,g). Then given two vectors $Z_1, Z_2 \in T_{\gamma(a)}U$, there is a unique Jacobi field Y along γ such that $Y(a) = Z_1$, and $\nabla_{\dot\gamma}Y(a) = Z_2$.*

In particular, any Jacobi field which is tangent to γ is a linear combination of $\dot\gamma$ and $t\dot\gamma$. The significance of Jacobi fields is seen in the following two propositions. We say that σ is a variation of γ through geodesics if the curves $t \mapsto \sigma(t;s)$ are geodesics for all s.

PROPOSITION 3.19. *Let γ be a geodesic, and let σ be a variation of γ through geodesics. Then the generator $Y = \sigma'$ of σ is a Jacobi field.*

PROOF. As before, denote $X = \dot\sigma$ and $Y = \sigma'$. We first prove the following identity:

$$\big[\nabla_Y, \nabla_X\big]X = -K\big(Y - g(X,Y)X\big).$$

Indeed, in the proof of Lemma 3.13, it was seen that the left-hand side above is a tensor, i.e., is linear over functions, and hence depends only on the values of the vector fields X and Y at one point. Fix that point. If X and Y are linearly dependent, then both sides of the equation above are zero. Otherwise, X and Y are linearly independent, and it suffices to check the inner product of the identity against X and Y. Taking inner product with X, both sides are zero, and equation (3.26) implies that the inner products with Y are equal. Since $\nabla_X X = 0$, we get:

$$0 = \nabla_Y\nabla_X X = \nabla_X\nabla_Y X + \big[\nabla_Y, \nabla_X\big]X = \nabla_X\nabla_X Y - K\big(Y - g(X,Y)X\big).$$

Thus, Y is a Jacobi field. □

We see that Jacobi fields are infinitesimal generators of variations through geodesics. If there is a non-trivial fixed endpoint variation of γ through geodesics, then the endpoints of γ are conjugate along γ. Unfortunately, the converse is not true but nevertheless, a non-zero Jacobi field which vanishes at the endpoints can be perceived as a non-trivial infinitesimal fixed-endpoint variation of γ through geodesics. This makes the next proposition all the more important.

PROPOSITION 3.20. *Let γ be a geodesic, and let Y be a Jacobi field. Then, for any vector field Z along γ, we have:*

$$I(Y,Z) = g(\nabla_{\dot\gamma} Y, Z)|_a^b. \tag{3.31}$$

In particular, if either Y or Z vanishes at the endpoints, then $I(Y,Z) = 0$.

PROOF. Multiplying the Jacobi equation by Z and integrating, we obtain:

$$\begin{aligned} 0 &= \int_a^b \big(g(\nabla_{\dot\gamma}\nabla_{\dot\gamma} Y, Z) - K\big(g(Y,Z) - g(\dot\gamma, Y)\, g(\dot\gamma, Z)\big)\big)\; dt \\ &= \int_a^b \left(\frac{d}{dt}\, g\big(\nabla_{\dot\gamma} Y, Z) - g(\nabla_{\dot\gamma} Y, \nabla_{\dot\gamma} Z) - K\big(g(Y,Z) - g(\dot\gamma, Y)\, g(\dot\gamma, Z)\big)\right)\; dt \\ &= g(\nabla_{\dot\gamma} Y, Z)|_a^b - I(Y,Z). \quad \square \end{aligned}$$

Thus, a Jacobi field which vanishes at the endpoints lies in the null space of the index form I acting on vector fields which vanish at the endpoints.

THEOREM 3.21. *Let $\gamma\colon [a,b] \to (U,g)$ be a geodesic parametrized by arclength, and suppose that there is a point $\gamma(c)$ with $a < c < b$ which is conjugate to $\gamma(a)$. Then there is a vector field Z along γ such that $I(Z) < 0$. Consequently, γ is not locally-length minimizing.*

PROOF. Define:

$$V = \begin{cases} Y & a \leqslant t \leqslant c \\ 0 & c \leqslant t \leqslant b \end{cases}$$

and let W be a vector field supported in a small neighborhood of c which satisfies $W(c) = -\nabla_{\dot\gamma} Y(c) \neq 0$. We denote the index form of γ on $[a,c]$ by I_1, and the index form on $[c,b]$ by I_2. Since V is piecewise smooth, we have, in view of (3.31):

$$I(V,W) = I_1(V,W) + I_2(V,W) = I_1(Y,W) = -|\nabla_{\dot\gamma} Y(c)|^2 < 0$$

It follows that:

$$I(V + \varepsilon W, V + \varepsilon W) = I(V) + 2\varepsilon I(V,W) + \varepsilon^2 I(W) = 2\varepsilon I(V,W) + \varepsilon^2 I(W)$$

is negative if $\varepsilon > 0$ is small enough. Although $V + \varepsilon W$ is not smooth, there is for any $\delta > 0$ a smooth vector field Z_δ, satisfying $|Y|^2 + |\nabla_{\dot\gamma} Z_\delta|^2 \leqslant C$ uniformly in $\delta > 0$, which differs from $V + \varepsilon W$ only on $(c-\delta, c+\delta)$. Since the contribution of this interval to both $I(V + \varepsilon W, V + \varepsilon W)$ and $I(Z_\delta, Z_\delta)$ tends to zero with δ, it follows that also $I(Z_\delta, Z_\delta) < 0$ for $\delta > 0$ small enough. Thus, γ is not locally energy-minimizing. Since it is parametrized by arclength, if it was locally length-minimizing, it would by Lemma 3.12 also be locally energy-minimizing. Thus, γ cannot be locally length-minimizing. $\square$

A partial converse is also true: the absence of conjugate points along γ guarantees that the index form is positive definite.

THEOREM 3.22. *Let $\gamma\colon [a,b] \to (U,g)$ be a geodesic parametrized by arclength, and suppose that no point $\gamma(t)$, $a < t \leqslant b$, is conjugate to $\gamma(a)$ along γ. Then the index form I is positive definite.*

PROOF. Let $X = \dot\sigma$, and let Y be a Jacobi field which is perpendicular to X, and vanishes at $t = a$. Note that the space of such Jacobi fields is 1-dimensional, hence Y is determined up to sign if we also require that $|\dot Y(a)| = 1$. Since Y is perpendicular to X, it satisfies the equation:

$$\nabla_X \nabla_X Y + KY = 0.$$

Furthermore, since Y never vanishes along γ, the vectors X and Y span $T_{\gamma(t)}U$ for all $t \in (a,b]$. Thus, if Z is any vector field along γ which vanishes at the endpoints, then we can write $Z = fX + hY$ for some functions f and h. Note that $f(a) = f(b) = h(b) = 0$ and $hY(a) = 0$. We then have:

$$I(Z,Z) = I(fX,fX) + 2I(fX,hY) + I(hY,hY).$$

Since $R(X,fX,X,fX) = 0$ and $\nabla_X fX = \dot f X$, it follows from (3.30) that:

$$I(fX,fX) = \int_a^b g(\dot f X, \dot f X)\,dt = \int_a^b \dot f^2\,dt.$$

Furthermore,

$$\begin{aligned} I(fX,hY) &= \int_a^b g(\dot f X, \nabla_X hY)\,dt \\ &= g(\dot f X, hY)\big|_a^b - \int_a^b g(\nabla_X \dot f X, hY)\,dt = -\int_a^b g(\ddot f X, hY)\,dt = 0. \end{aligned}$$

Finally, since $|\nabla_X hY|^2 = g(\nabla_X Y, \nabla_X h^2 Y) + \dot h^2 |Y|^2$, it follows from Proposition 3.20 that:

$$I(hY,hY) = \int_a^b \dot h^2 |Y|^2\,dt + I(Y,hY) = \int_a^b \dot h^2 |Y|^2\,dt.$$

Thus, we conclude that:

$$I(Z,Z) = \int_a^b \left(\dot f^2 + \dot h^2 |Y|^2\right) dt \geqslant 0.$$

If $I(Z,Z) = 0$, then $\dot f = 0$ and $\dot h Y = 0$ on $[a,b]$. Since $Y \neq 0$ on $(a,b]$, we conclude that $\dot h = 0$ on $(a,b]$, and in view of $h(b) = f(b) = 0$, we get that $Z = 0$. Thus, I is positive definite. □

Exercises

EXERCISE 3.1. Two Riemannian metrics g and $\tilde g$ on an open set $U \subset \mathbb{R}^2$ are *conformal* if $\tilde g = e^{2\lambda} g$ for some smooth function λ.

(1) Prove that a parametric surface $X\colon U \to \mathbb{R}^3$ is conformal if and only if its first fundamental form g is conformal to the Euclidean metric δ on U.

(2) Let $\tilde g = e^{2\lambda} g$ be conformal metrics on U, and let Γ_{ij}^k and $\tilde\Gamma_{ij}^k$ be their Christoffel symbols. Prove that:

$$\tilde\Gamma_{ij}^k = \Gamma_{ij}^k + \delta_i^k\,\lambda_j + \delta_j^k\,\lambda_i + g_{ij}\,g^{km}\,\lambda_m$$

(3) Let $\tilde{g}$ and g be two conformal metrics on U, $\tilde{g} = e^{2\lambda} g$, and let K and $\tilde{K}$ be their Gauss curvatures. Prove that:

$$\tilde{K} = e^{-2\lambda}(K - \Delta\lambda).$$

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