Lagrangian submanifolds and hamiltonian systems

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This paper consists of two parts. The first part (Sections 1 to 6) is devoted to the geometry of lagrangian submanifolds of symplectic manifolds. Essentially, we assume that lagrangian submanifolds L_1 and L_2 in a symplectic manifold (P, Ω) intersect along a submanifold Σ, and we see what happens when L_1 and L_2 are slightly perturbed. Under fairly general conditions, we can prove that the perturbed lagrangian submanifolds must have points of intersection near Σ, and we can estimate the size of the intersection set in terms of the topology of Σ. In the second part of the paper (Sections 7 to 11), we apply the intersection theory to the study of hamiltonian dynamical systems. We define the notion of a canonical boundary value problem for a hamiltonian system, which includes the problem of finding periodic solutions with prescribed energy or prescribed period. Assuming that a given problem admits a manifold Σ of solutions which is non-degenerate in a certain sense, we show that, after a small perturbation of the hamiltonian function or of the boundary conditions, there remain solutions near Σ whose number can be estimated in terms of the topology of Σ.

Our results, involving critical point theory for functions on Σ, resemble those which might have been obtained were the calculus of variations to be applicable, but our functions are obtained directly from the geometry of symplectic manifolds, rather than from functionals on path spaces. Thus, we use no analysis beyond the usual tools of differential geometry: the implicit function theorem and integration of vector fields. Unfortunately, these geometrical methods are presently applicable only to problems obtained by slight perturbation from problems having manifolds of solutions. Whether this limitation is essential remains to be seen. In any case, the problems to which the present method is applicable are of considerable interest. They include the existence of periodic orbits of the second kind in the planar (unrestricted) three body problem [4] and of periodic orbits near an equilibrium point of a non-linear hamiltonian system. In particular, we prove that, near an equilibrium point in $\mathcal{R}^{2n}$ at which the linearized system has hamiltonian

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$\sum_{i=1}^n (x_i^2 + y_i^2)$, the non-linear system has at least n periodic orbits on each energy surface. See the end of Section 10 for a discussion of this and more detailed results.

In Section 11, we relate our work to the averaging method of Reeb [26] and Moser [21] and generalize their results.

Since lagrangian submanifolds are of interest for quantum mechanics [3], [18], [31], partial differential equations [7], [10], [13], and singularity theory [33], as well as for hamiltonian systems, we develop the general theory in Sections 1 through 6 in more detail than is necessary for the present applications. The reader whose primary interest is in the results concerning hamiltonian systems may wish to skip to Section 7 after reading the definitions in Section 1, referring back to Sections 2 through 6 only when it is necessary.

Some of the results in this paper were announced in [35]. The proofs here are not the same as those available when the announcement was written. The original proofs of the announced results are contained in [36], which will not be published.

Added December 21, 1972. V. I. Arnold has pointed out that his paper "Sur une propriété topologique des applications globalement canoniques de la mécanique classique", C. R. Acad. Sci. Paris **261** (1965) 3719–3722, contains the idea of representing certain lagrangian submanifolds of cotangent bundles as graphs of closed forms. I would also like to thank J. Roels, whose careful reading of this manuscript led to the correction of numerous typographical errors. It would be nice to be able to say that any errors which remain are his responsibility, but of course they are not.

PART I. INTERSECTION THEORY

1. Basic definitions

The basic category for this paper is that of finite dimensional C^∞ G-manifolds, where G is a fixed but arbitrary compact Lie group, which we refer to as the *symmetry group*. Each manifold M, therefore, is equipped with an action of G on it, and all the natural bundles over M are equipped with the natural lifting of the action on M. All mappings are equivariant; in particular, each covariant tensor field is invariant under the diffeomorphism corresponding to each element of G.

Most of the results here can be extended to infinite-dimensional manifolds modeled on Banach spaces, following the approach in [34]. See, however, the remarks after Lemma 4.2. Staying in the finite dimensional case simplifies the exposition, and I know of no significant applications to the infinite-dimensional situation. In fact, the appropriate objects for the geometric study of mechanical systems with infinitely many degrees of freedom (fluids, fields, elastic media, etc.) may very well be *weak* symplectic structures, whose properties are much more complicated than those of symplectic structures [17].

We refer the reader to [34] for further details regarding the definitions and facts about symplectic manifolds which follow.

A *symplectic manifold* is a pair (P, Ω), where P is a manifold and Ω is a closed 2-form on P which is non-singular in the sense that the bundle mapping $\tilde{\Omega}: TP \rightarrow T^*P$, defined by $\tilde{\Omega}(x) = x \lrcorner \Omega$, is an isomorphism. Ω is called a symplectic structure on P.

A subspace $V \subseteqq T_pP$ is called *isotropic* if $(x, y) \in V \times V$ implies $\Omega(x, y) = 0$. V is called *lagrangian* if it is isotropic and if there exists an isotropic W such that $V \oplus W = T_pP$. Equivalently, V is lagrangian if and only if it is isotropic and $\dim V = (1/2) \dim T_pP$.

An immersion $i: M \rightarrow P$ is called isotropic [lagrangian] if the image of $T_mi: T_mM \rightarrow T_{i(m)}P$ is isotropic [lagrangian] for each $m \in M$. Clearly, i is isotropic [lagrangian] if and only if $i^*\Omega = 0$ [and $\dim M = (1/2) \dim P$]. A submanifold $L \subseteqq P$ is called isotropic [lagrangian] if the inclusion mapping $i_L: L \rightarrow P$ is isotropic [lagrangian]. The image of an isotropic [lagrangian] immersion is a union of isotropic [lagrangian] submanifolds. It is to the lagrangian submanifolds that our attention will generally be confined.

Since we will be dealing with the behavior of lagrangian submanifolds under small perturbations, it is useful to topologize the set $\mathcal{L}(P, \Omega)$ consisting of all lagrangian submanifolds of (P, Ω). We will only be concerned with what might be called "lower" topologies. (A finer topology is defined in [10].)

Let A be a closed subset of a manifold M whose dimension is half that of P, and let $\mathcal{A}$ be an open subset in the fine C^1 topology on the space $C^\infty(A, P)$ of all mappings from A to P. (See [22] for a discussion of the topology of mapping spaces.)

The sets $\mathfrak{N}_{A,\mathcal{A}} = \{L \subseteqq \mathcal{L}(P, \Omega) \mid i(A) \subseteqq L$ for some $i \in \mathcal{A}\}$, for all A and $\mathcal{A}$, form a basis for a topology on $\mathcal{L}(P, \Omega)$, called the fine (upper) C^1 topology. The subtopology generated by those $\mathfrak{N}_{A,\mathcal{A}}$ for which A is compact is called the compact, or coarse, C^1 topology.

2. Cotangent bundles

The principal technique of this paper is the parametrization of subsets of $\mathcal{L}(P, \Omega)$ by closed 1-forms on certain manifolds. In this section we will examine in detail the special case in which P is a cotangent bundle. In Section 4, we will show how to reduce the general case to the one considered here, by the use of cotangent coordinate systems.

The cotangent bundle T^*M of a manifold M, with projection π_M, carries a natural 1-form ω_M characterized by the property that, for each section $\phi: M \to T^*M$, $\phi^*\omega_M$ is equal to ϕ itself. The 2-form $\Omega_M = -d\omega_M$ is a natural symplectic structure on T^*M. (If M is modeled on a Banach space B, Ω_M is a symplectic structure if and only if B is reflexive.) If S is an open subset of M, then T^*S may be considered as an open subset of T^*M. The pullbacks to T^*S of ω_M and Ω_M are equal to ω_S and Ω_S, respectively.

A lagrangian submanifold $L \in \mathfrak{L}(T^*M, \Omega_M)$ is *horizontal* if $\pi_M \circ i_L$ is an embedding of L onto an open subset of M. If L is horizontal, and its projection in M is the subset S, then there is a unique section $\phi: S \to T^*S \subseteqq T^*M$ such that $\phi(S) = L$. Since L is lagrangian, $\phi^*\Omega_M = 0$. But $\phi^*\Omega_M = \phi^*\Omega_S = \phi^*(-d\omega_S) = -d(\phi^*\omega_S) = -d\phi$, so ϕ is a closed 1-form on S. Conversely, if ϕ is a closed 1-form on an open subset $S \subseteqq M$, then $\phi(S) \subseteqq T^*S \subseteqq T^*M$ is a horizontal lagrangian submanifold whose projection in M is S.

In summary, for any open $S \subseteqq M$, the mapping $e_{S,M}: \phi \mapsto \phi(S)$ is a 1–1 correspondence between the space $Z^1(S)$ of closed 1-forms on S and the subset $\mathfrak{L}_S(T^*M, \Omega_M) \subseteqq \mathfrak{L}(T^*M, \Omega_M)$ consisting of the horizontal lagrangian submanifolds whose projection in M is S. $Z^1(S)$ inherits fine and coarse C^1 topologies as a subset of the space $C^\infty(S, T^*S)$, and it is clear that $e_{S,M}: Z^1(S) \to \mathfrak{L}(T^*M, \Omega_M)$ is continuous when domain and range are given C^1 topologies of the same type. Then next proposition shows how close $e_{S,M}$ comes to being a homeomorphism onto an open subset of $\mathfrak{L}(T^*M, \Omega_M)$.

PROPOSITION 2.1. (a) *If $\mathcal{B}$ is an open subset of Z^1 (S), then the set of lagrangian submanifolds which* **contain** *an element of $e_{S,M}(\mathcal{B})$ is open in $\mathfrak{L}(T^*M, \Omega_M)$ in the fine C^1 topology.*

(b) *Suppose $\bar{S}$ is compact and that $\phi \in Z^1(S)$ extends to an element $\tilde{\phi} \in Z^1(S)$, where $\tilde{S}$ contains $\bar{S}$. If $\mathcal{B}$ is any coarse C^1 neighborhood of ϕ in $Z^1(S)$, then the set of lagrangian submanifolds which contain an element of $e_{S,M}(\mathcal{B})$ is a neighborhood of $e_{\tilde{S},M}(\tilde{\phi})$ in the coarse C^1 topology.*

Proof. Let $\mathfrak{D}$ be the subset of $C^\infty(S, T^*M)$ consisting of those $f: S \to T^*M$ for which $\pi_M \circ f$ is an embedding whose image is S. We claim that $\mathfrak{D}$ is open in the fine C^1 topology. First of all, since S is open in M, the set $\mathfrak{D}_1$ of all $f: S \to T^*M$ such that $(\pi_M \circ f)(S) \subseteqq S$ is already open in the fine C^0 topology, and the mapping $f \mapsto \pi_M \circ f$ from $\mathfrak{D}_1$ to $C^\infty(S, S)$ is continuous in the fine C^1 topologies. Since the diffeomorphisms of S are open in

$C^\infty(S, S)$ (see [22]), it follows that $\mathfrak{D}$ is open in $\mathfrak{D}_1$.

Let $\mathfrak{S}$ be an open subset of $C^\infty(S, T^*M)$ such that $\mathfrak{B} = \mathfrak{S} \cap Z^1(M)$. Since the mapping $f \mapsto f\circ(\pi_M\circ f)^{-1}$ is continuous from $\mathfrak{D}$ to $C^\infty(S, T^*M)$ the set $\mathfrak{A} = \{f \in \mathfrak{D} \mid f\circ(\pi_M\circ f)^{-1} \in \mathfrak{S}\}$ is open in $C^\infty(S, T^*M)$.

The proof of (a) will be complete if we can show that $\mathfrak{N}_{S,\mathfrak{A}}$ is precisely the set of lagrangian submanifolds which contain an element of $e_{S,M}(\mathfrak{B})$. If $L \in \mathfrak{N}_{S,\mathfrak{A}}$, then L contains $f(S)$ for some $f \in \mathfrak{A}$. But $f(S)$ is equal to $f\circ(\pi_M\circ f)^{-1}(S)$, and $f\circ(\pi_M\circ f)^{-1}$ is in $\mathfrak{S}$. Since $\pi_M\circ f\circ(\pi_M\circ f)^{-1}$ is the identity on S, $f\circ(\pi_M\circ f)^{-1}$ is a 1-form on S; since $f(S)$ is lagrangian, $f\circ(\pi_M\circ f)^{-1}$ is closed. Thus, $f\circ(\pi_M\circ f)^{-1} \in Z^1(M) \cap \mathfrak{S} = \mathfrak{B}$, and L contains an element of $e_{S,M}(\mathfrak{B})$.

Conversely, suppose $L \in \mathfrak{L}(T^*M, \Omega_M)$ contains an element $e_{S,M}(\phi)$ of $e_{S,M}(\mathfrak{B})$. Since ϕ is a section, ϕ is in $\mathfrak{D}$ and $\phi\circ(\pi_M\circ\phi)^{-1} = \phi$ lies in $\mathfrak{B} \subseteqq \mathfrak{S}$, so ϕ lies in $\mathfrak{A}$. Thus, L contains $e_{S,M}(\phi) = \phi(S)$, so L lies in $\mathfrak{N}_{S,\mathfrak{A}}$.

To prove (b), we begin by choosing an open set $U \subseteqq M$, containing $\bar{S}$, whose closure is compact and contained in $\tilde{S}$. Let $\mathfrak{C}$ be the set consisting of those $f\colon \bar{U} \to T^*M$ for which the restriction of $\pi_M\circ f$ to U is an embedding whose range contains $\bar{S}$. In the space $C^\infty(\bar{U}, M)$, the subset consisting of maps whose restriction to U is an embedding covering $\bar{S}$ is open. (To prove this "folk theorem", one may use the ideas in 1.5 and 1.6 of [12].) It follows that $\mathfrak{C}$ is open in $C^\infty(\bar{U}, T^*M)$. The remainder of the proof is similar to that of (a); the details are left to the reader. Q.E.D.

The intersection theory of horizontal lagrangian submanifolds seems rather trivial, but since this theory can be carried over to the general case, we state the results in some detail.

PROPOSITION 2.2. *Let $L_1 = \phi_1(S)$ and $L_2 = \phi_2(S)$ be in $\mathfrak{L}_S(T^*M, \Omega_M)$. Then the zero set of the form $\phi_2 - \phi_1 \in Z^1(S)$ is mapped by the embedding $(1/2)(\phi_1 + \phi_2)\colon S \to T^*M$ onto the intersection $L_1 \cap L_2$.*

The proof is trivial. The proposition says that the problem of finding intersections of L_1 and L_2 is reduced to that of finding zeros of $\phi_2 - \phi_1$. We denote the form $\phi_2 - \phi_1$ by $\Phi(L_1, L_2) \in Z^1(S)$ and the embedding $(1/2)(\phi_1 + \phi_2)$ by $E(L_1, L_2) \in C^\infty(S, T^*M)$.

The next result is an immediate consequence of Propositions 2.1 and 2.2.

COROLLARY 2.3. (a) *Let L_1 and L_2 be in $\mathfrak{L}_S(T^*M, \Omega_M)$. Let $\mathfrak{B}$ be any C^1 neighborhood of $\Phi(L_1, L_2)$ in $Z^1(S)$ and let $\mathfrak{S}$ be any C^1 neighborhood of $E(L_1, L_2)$ in $C^\infty(S, T^*M)$. Then there exists a fine C^1 neighborhood*

$\mathfrak{N}_1 \times \mathfrak{N}_2$ *of* (L_1, L_2) *in* $\mathfrak{L}(T^*M, \Omega_M) \times \mathfrak{L}(T^*M, \Omega_M)$ *and mappings* $\Phi_0 : \mathfrak{N}_1 \times \mathfrak{N}_2 \to \mathfrak{B}$ *and* $E_0 : \mathfrak{N}_1 \times \mathfrak{N}_2 \to \mathfrak{E}$ *such that, for each pair* (L_1', L_2') *in* $\mathfrak{N}_1 \times \mathfrak{N}_2$, $E_0(L_1', L_2')$ *is an embedding of* S *into* T^*M *which maps the zero set of* $\Phi_0(L_1', L_2')$ *into the intersection* $L_1' \cap L_2'$.

(b) *Suppose, in addition to the assumptions in* (a), *that* $\bar{S}$ *is compact;* L_1 *and* L_2 *are contained in* $\tilde{L}_1$ *and* $\tilde{L}_2$ *which lie in* $\mathfrak{L}_{\tilde{S}}(T^*M, \Omega_M)$, *where* $\tilde{S}$ *contains* $\bar{S}$; *and* $\mathfrak{B}$ *and* $\mathfrak{E}$ *are coarse* C^1 *neighborhoods. Then there exists a coarse* C^1 *neighborhood* $\mathfrak{N}_1 \times \mathfrak{N}_2$ *of* $(\tilde{L}_1, \tilde{L}_2)$ *in* $\mathfrak{L}(T^*M, \Omega_M) \times \mathfrak{L}(T^*M, \Omega_M)$ *such that the conclusions in* (a) *hold.*

The preceding corollary becomes useful only if we can guarantee the existence of zeros for $\Phi_0(L_1', L_2')$. Two problems arise in this respect. The first is that the manifold S_0 is generally not compact, so that it always carries functions with no critical points and, hence, closed forms without zeros. We will deal with this problem in the next section by making an assumption on $\Phi(L_1, L_2)$ $\big(=\Phi_0(L_1, L_2)\big)$ and using the fact that $\Phi_0(L_1', L_2')$ lies nearby. The second problem is that the form $\Phi_0(L_1', L_2')$, while closed, may not be exact. This problem is dealt with in Section 5, where we discuss a cohomology invariant for pairs of lagrangian submanifolds.

3. Clean intersections and non-degenerate zero manifolds

We begin with a definition in the general situation. If L_1 and L_2 are elements of $\mathfrak{L}(P, \Omega)$ and $\Sigma \subset L_1 \cap L_2$ is a closed submanifold of P, we say that L_1 and L_2 *intersect cleanly* along Σ if, for each $p \in \Sigma$, the inclusion $T_p\Sigma \subseteqq T_pL_1 \cap T_pL_2$ is an equality.

Let us interpret this definition in case $L_1 = \phi_1(S)$ and $L_2 = \phi_2(S)$ are elements of $\mathfrak{L}_S(T^*M, \Omega_M)$. Denote by 0 the zero section of T^*M. The diffeomorphism of T^*M which translates each fibre $\pi_M^{-1}(m)$ by $-\phi_1(m)$ maps L_1 and L_2 onto $0(S)$ and $(\phi_2 - \phi_1)(S) = \Phi(L_1, L_2)(S)$. It is clear that L_1 and L_2 intersect cleanly along Σ if and only if $0(S)$ and $\Phi(L_1, L_2)(S)$ intersect cleanly along the submanifold $0\big(\pi_M(\Sigma)\big)$.

The problem is reduced then to the following: given $\phi \in Z^1(S)$ containing the submanifold $\mathfrak{X} \subseteqq S$ in its zero set, when do $0(S)$ and $\phi(S)$ intersect cleanly along $0(\mathfrak{X})$? Let $s \in \mathfrak{X}$. Then $T_{0(s)}T^*M$ is naturally isomorphic to the direct sum $T_SM \oplus T_s^*M$. $T_{0(s)}0(S)$ corresponds to the summand $T_SM \oplus \{0\}$. Since ϕ is a section of T^*M, $T_{0(s)}\phi(S)$ is a set of the form $\{x \oplus D_s\phi(x) \mid x \in T_sM\}$, where $D_s\phi : T_sM \to T_s^*M$ is a linear mapping naturally determined by ϕ and called the *intrinsic derivative* of ϕ at s. It follows from the fact that $T_s\phi(S)$ is isotropic that $D_s\phi$ is symmetric; i.e., $(D_s\phi)^* = D_s\phi$ modulo the natural identification of $(T_s^*M)^*$ with T_sM. In fact, $D_s\phi$

is just the linear mapping associated with the *hessian* at s of any function defined near s whose differential is ϕ.

Now

$$T_{0(s)}0(S) \cap T_{0(s)}\phi(S) = \{x \oplus D_s\phi(x) \mid D_s\phi(x) = 0\} = \operatorname{Ker} D_s\phi \oplus \{0\} ,$$

while

$$T_{0(s)}0(\mathfrak{X}) = T_s\mathfrak{X} \oplus \{0\} .$$

Thus, $0(S)$ and $\phi(S)$ intersect cleanly along $0(\mathfrak{X})$ if and only if, for every $s \in \mathfrak{X}$, the inclusion $\operatorname{Ker} D_s\phi \subseteqq T_s\mathfrak{X}$ is an equality. We express this equality by saying that $\mathfrak{X}$ is a *non-degenerate zero manifold* for ϕ. If $\phi = df$ for some function f defined on a neighborhood of $\mathfrak{X}$ (such f always exists, by the Poincaré lemma for vector bundles; see Section 3 of [34]), then $\mathfrak{X}$ is non-degenerate as a zero manifold for ϕ if and only if $\mathfrak{X}$ is non-degenerate in the sense of Bott [6] as a critical manifold for f.

In the terminology just introduced, our result is:

PROPOSITION 3.1. *L_1 and L_2 in $\mathfrak{L}_S(T^*M, \Omega_M)$ intersect cleanly along $\Sigma \subseteqq L_1 \cap L_2$ if and only if $\pi_M(\Sigma)$ is a non-degenerate zero manifold of $\Phi(L_1, L_2)$.*

Suppose, now that ϕ is a closed 1-form on a manifold S, with $\mathfrak{X} \subseteqq S$ as a non-degenerate zero manifold. We wish to see what happens in a neighborhood of $\mathfrak{X}$ when the form ϕ is slightly perturbed. Since the symmetry group is compact, $\mathfrak{X}$ has an equivariant tubular neighborhood in S. As we will be interested only in phenomena occurring near $\mathfrak{X}$, we may assume that S itself is the tubular neighborhood, so that there is a retraction $\rho: S \to \mathfrak{X}$. For each $s \in \mathfrak{X}$, the pullback ϕ_s of ϕ to $\rho^{-1}(s)$ has $\{s\}$ as a non-degenerate zero manifold. (This is easily seen if one considers ϕ as the differential of a function f having $\mathfrak{X}$ as a non-degenerate critical manifold.) In other words, the mapping $\phi_s: \rho^{-1}(s) \to T^*\rho^{-1}(s)$ is transversal to the zero section at s. If ψ is sufficiently close to ϕ in the fine C^1 topology, there exists for each $s \in \mathfrak{X}$ a uniquely determined point $F(\psi)(s)$ near s in $\rho^{-1}(s)$ which is a zero of the pulled back form ψ_s. In fact, the mapping $F(\psi): \mathfrak{X} \to S$ is a smooth section of ρ which approaches the inclusion as ψ approaches ϕ. (A complete proof of the previous two sentences requires that one consider $\bigcup_s \phi_s$ and $\bigcup_s \psi_s$ as sections of the smooth bundle $\bigcup_s T^*\rho^{-1}(s)$. One observes that $\bigcup_s \phi_s$ is transversal to the zero section and then applies standard results in transversality theory, essentially the transversal isotopy theorem of [2].)

Now let $r = F(\psi)(s) \in F(\psi)(\mathfrak{X})$. Since $F(\psi)$ is a section of ρ, we have a natural isomorphism

$$T_r S = T_r\rho^{-1}(s) \oplus T_r F(\psi)(\mathfrak{X}) \ .$$

By the way $F(\psi)$ was defined, $\phi(r)$ annihilates the first summand. Thus, $\phi(r)$ is zero if and only if it annihilates the second summand. In other words, the zero set of the pullback of ψ to $F(\psi)(\mathfrak{X})$ is contained in the zero set of ψ. Conversely, if $\psi(r) = 0$ for some $r \in S$, then in particular $\psi_{\rho(r)}(r) = 0$. If r is sufficiently close to $\mathfrak{X}$ and ψ is sufficiently close to ϕ, then r must be equal to $F(\psi)(\rho(r))$, and the previous observation applies. Our result is summarized as follows.

PROPOSITION 3.2. *Let ϕ be a closed 1-form on the manifold S, having $\mathfrak{X}$ as a non-degenerate zero manifold. Let $\mathcal{E} \subseteqq C^\infty(\mathfrak{X}, S)$ be any C^1 neighborhood of the inclusion mapping. Then there exist*

a neighborhood S_1 of $\mathfrak{X}$ in S,

a neighborhood $\mathcal{B}$ of ϕ in $Z'(S)$,

a mapping $F: \mathcal{B} \to C^\infty(\mathfrak{X}, S_1) \cap \mathcal{E}$

such that, for each $\psi \in \mathcal{B}$, $F(\psi)$ is an embedding and the zero set of $\psi \mid S_1$ is equal to the zero set of the pullback of ψ to $F(\psi)(\mathfrak{X})$. In other words, the zero set of $\psi \mid S_1$ is the image under $F(\psi)$ of the zero set of $[F(\psi)]^*\psi$.

Combining Proposition 3.2 with Corollary 2.3, we have the following theorem, which is our fundamental result in the case of cotangent bundles.

THEOREM 3.3. *Let S be an open subset of M, and let L_1 and L_2 in $\mathfrak{L}_S(T^*M, \Omega_M)$ intersect cleanly along the closed submanifold Σ. Let $\mathcal{E} \subseteqq C^\infty(\Sigma, T^*M)$ be any C^1 neighborhood of the inclusion. Then there exist a fine C^1 neighborhood $\mathfrak{N}_1 \times \mathfrak{N}_2$ of (L_1, L_2) in $\mathfrak{L}(T^*M, \Omega_M) \times \mathfrak{L}(T^*M, \Omega_M)$ and mappings $G: \mathfrak{N}_1 \times \mathfrak{N}_2 \to \mathcal{E}$ and $\Gamma: \mathfrak{N}_1 \times \mathfrak{N}_2 \to Z^1(\Sigma)$ such that, for each pair (L_1', L_2') in $\mathfrak{N}_1 \times \mathfrak{N}_2$, $G(L_1', L_2')$ is an embedding which maps the zero set of $\Gamma(L_1', L_2')$ into the intersection $L_1' \cap L_2'$. If Σ is compact, then $\mathfrak{N}_1 \times \mathfrak{N}_2$ can be taken to be a coarse C^1 neighborhood.*

4. Cotangent coordinates

Up to now, we have assumed that the lagrangian submanifolds L_1 and L_2 were located in a cotangent bundle and that they were nicely situated there. Although, in many applications, the symplectic manifold of interest is indeed a cotangent bundle, the lagrangian submanifolds are often not horizontal. In this case, it is simpler to forget that one is in a cotangent bundle and pass to the case of an arbitrary symplectic manifold. The goal of this section is to show how to reduce the general case to the special one studied in Sections 2 and 3.

A *cotangent coordinate system* on a symplectic manifold (P, Ω) is a triple (P_0, M, α) where P_0 is an open subset of P and $\alpha: P_0 \to T^*M$ is an open embedding such that $\alpha^*\Omega_M = \Omega$. We will often denote the triple by the symbol α alone. A mapping $\alpha_\sharp: \mathfrak{L}(P, \Omega) \to \mathfrak{L}(T^*M, \Omega_M)$ is defined by $\alpha_\sharp(L) = \alpha(L \cap P_0)$. This mapping is continuous between fine or coarse C^1 topologies.

The idea of this section is, given L_1 and L_2 in $\mathfrak{L}(P, \Omega)$, to try to construct a cotangent coordinate system (P_0, M, α) such that $\alpha_\sharp(L_1)$ and $\alpha_\sharp(L_2)$ are in $\mathfrak{L}_S(T^*M, \Omega_M)$ for some open $S \subseteqq M$, and then to apply the intersection theory of Sections 2 and 3.

We begin the task of constructing cotangent coordinate systems with a digression on vector bundles. A symplectic structure on a vector bundle E over a manifold is a bilinear form Ω on E such that the mapping $\tilde{\Omega}: E \to E^*$ defined by $\tilde{\Omega}(x)(y) = \Omega(x, y)$ is an isomorphism. (We make the convention that, whenever we write $\Omega(x, y)$, it is assumed that x and y are in the same fibre of E.) A symplectic structure Ω on a manifold P may be considered as a symplectic structure on the tangent bundle TP over P (but not conversely, since a symplectic structure on P must be closed). More generally, if Σ is any submanifold of P, then Ω induces a symplectic structure on the restricted tangent bundle $T_\Sigma P$.

If E is any symplectic vector bundle (i.e., a vector bundle together with a symplectic structure Ω on it), and $A \subseteqq E$ is a subbundle, then $A^\perp = \{x \in E \mid \Omega(x, y) = 0 \text{ whenever } y \in A\}$ is also a subbundle of E. It is always true that $\dim A + \dim A^\perp = \dim E$. (Here, dim means fibre dimension, of course.) If $A \cap A^\perp = 0$, or, equivalently, if $A \oplus A^\perp = E$, we call A *nonsingular*. In this case, the restriction of Ω to A is a symplectic structure on A. If $A \subseteqq A^\perp$, we call A *isotropic*. If $A = A^\perp$, or, equivalently, if A is isotropic and $\dim A = (1/2) \dim E$, we call A *lagrangian*. For example, if L is a submanifold of a symplectic manifold (P, Ω), then L is isotropic [lagrangian] if and only if TL is isotropic [lagrangian] as a subbundle of $T_L P$.

Subbundles A_1 and A_2 in E are said to intersect uniformly if $A_1 \cap A_2$ is a subbundle; i.e., if $A_1 \cap A_2$ has constant fibre dimension. For example, if the lagrangian submanifolds L_1 and L_2 in $\mathfrak{L}(P, \Omega)$ intersect cleanly along Σ, then $T_\Sigma L_1$ and $T_\Sigma L_2$ intersect uniformly in $T_\Sigma P$.

The following proposition will enable us to construct good cotangent coordinate systems near clean intersections of lagrangian submanifolds.

PROPOSITION 4.1. *If A_1 and A_2 are uniformly intersecting lagrangian subbundles of a symplectic vector bundle E, then there exists a lagrangian subbundle $B \subseteqq E$ such that $A_1 \oplus B = A_2 \oplus B = E$.*

Proof. First, consider the case in which $A_1 \oplus A_2 = E$. The mapping $\alpha: A_2 \to A_1^*$ defined by $\alpha(x)(y) = \Omega(x, y)$ is an isomorphism, because its kernel is $A_1^\perp \cap A_2 = A_1 \cap A_2 = 0$. Let $S: A_1 \to A_1^*$ be the symmetric isomorphism associated with a riemannian metric on the vector bundle A. Such a metric exists because the symmetry group is compact. Now it is easy to check that $\{x + \alpha^{-1}S(x) \mid x \in A_1\}$ is a lagrangian subbundle complementary to A_1 and A_2.

Second, consider the case $A_1 = A_2$. By Lemma 4.2 below, there exists a lagrangian complement to $A_1 = A_2$ in E. (Let Σ be the empty set.)

In the general case, we can choose subbundles C_1 and C_2 such that $A_1 = (A_1 \cap A_2) \oplus C_1$ and $A_2 = (A_1 \cap A_2) \oplus C_2$. This, again, follows from the compactness of the symmetry group. Obviously, $C_1 \cap C_2 = 0$.

We will now show that $C_1 \oplus C_2$ is non-singular. Let $x_i \in C_i$ $(i = 1, 2)$ be such that $x_1 + x_2 \in (C_1 \oplus C_2) \cap (C_1 \oplus C_2)^\perp$. For all $y_2 \in C_2$, we have

$$0 = \Omega(x_1 + x_2, y_2) = \Omega(x_1, y_2) + \Omega(x_2, y_2) = \Omega(x_1, y_2) ,$$

since C_2 is isotropic, so $x_1 \in C_2^\perp$. On the other hand,

$$x_1 \in C_1 \subseteqq A_1 = A_1^\perp \subseteqq (A_1 \cap A_2)^\perp .$$

Thus

$$x_1 \in [(A_1 \cap A_2) \oplus C_2]^\perp = A_2^\perp = A_2 .$$

But $C_1 \cap A_2 = 0$, so $x_1 = 0$. Similarly, $x_2 = 0$, so $x_1 + x_2$ must be zero.

In the symplectic vector bundle $C_1 \oplus C_2$, C_1 and C_2 are lagrangian, so we are in the first special case considered above, and there is a bundle C, lagrangian in $C_1 \oplus C_2$, such that $C_1 \oplus C_2 = C_1 \oplus C = C_2 \oplus C$.

The subbundle $(C_1 \oplus C_2)^\perp$ is also non-singular, and it contains $A_1 \cap A_2$. In fact, counting of dimensions shows that $A_1 \cap A_2$ is lagrangian in $(C_1 \oplus C_2)^\perp$, so the second special case considered above implies that there is a subbundle A, lagrangian in $(C_1 \oplus C_2)^\perp$, such that $(L_1 \cap L_2) \oplus A = (C_1 \oplus C_2)^\perp$.

Let $B = C \oplus A$. Since $E = (C_1 \oplus C_2) \oplus (C_1 \oplus C_2)^\perp$, it is easy to check that B is lagrangian and $A_1 \oplus B = A_2 \oplus B = E$. Q.E.D.

In case $A_1 = A_2$, the following relative version of Proposition 4.1 holds. We use the result in the proof of Proposition 4.1 and again in our later discussion. Incidentally, the natural common generalization of Proposition 4.1 and Lemma 4.2 is false.

LEMMA 4.2. *If A is a lagrangian subbundle of a symplectic vector bundle E over a manifold M, Σ is a submanifold of M, and the restricted bundle $A_\Sigma \subseteqq E_\Sigma$ admits a lagrangian complement B_Σ, then B_Σ extends to a lagrangian complement B of A in E.*

Proof. Choose an inner product on E_Σ in which A_Σ and B_Σ are perpendicular, and extend it to an inner product $\langle\, , \rangle$ on E. The bundle mapping $K: E \to E$ defined by $\langle K(x), y\rangle = \Omega(x, y)$ is a skew-adjoint isomorphism, so $-K^2$ is positive definite. Let P be the positive definite square root of $-K^2$, and set $J = KP^{-1}$. The arguments used in the proof of Proposition 5.1 of [34] show that $J(A)$ is lagrangian and $A \oplus J(A) = E$.

It remains to show that $J(A_\Sigma) = B_\Sigma$. Since A_Σ and B_Σ are isotropic, it is clear that $K(A_\Sigma) = B_\Sigma$ and $K(B_\Sigma) = A_\Sigma$. A_Σ, being invariant under $-K^2$, is also invariant under its square root P. Then

$$J(A_\Sigma) = KP^{-1}(A_\Sigma) = K(A_\Sigma) = B_\Sigma \, . \qquad \text{Q.E.D.}$$

We conclude this digression with some remarks. Hörmander [13] defines a cohomology invariant α for pairs of lagrangian subbundles of a symplectic vector bundle and shows that α vanishes if either the hypothesis or the conclusion of Proposition 4.1 holds. The present (though not the original) proof of Proposition 4.1 was strongly motivated by Hörmander's constructions.

Proposition 4.1 extends immediately to the category of vector bundles with Hilbert space fibres, but it fails for general Banach spaces, even if the symmetry group and the base space each reduce to a single element. If B is any reflexive Banach space, $B \oplus B^*$ carries a symplectic structure for which the summands are lagrangian [34]. If there existed a mutual lagrangian complement, B and B^* would be isomorphic. Letting B be the sequence space l^p $(p \neq 2)$, we see that this is not necessarily the case.

Proposition 4.1 may be interpreted as a result concerning projective modules over the ring of C^∞ functions on the base space. Related results for more general rings may be found in Novikov [23]. Novikov does not seem to prove Proposition 4.1, though.

Finally, we remark to readers of [35] or [36] that Proposition 4.1 is the basis for the elimination of the hypothesis of "regularity" in the present work.

Returning to the original problem, suppose we are given L_1 and L_2 in $\mathfrak{L}(P, \Omega)$, intersecting cleanly along a submanifold Σ. Since $T_\Sigma L_1$ and $T_\Sigma L_2$ intersect uniformly in $T_\Sigma P$, there exists, by Proposition 4.1, a lagrangian subbundle $B_\Sigma \subseteqq T_\Sigma P$ such that

$$T_\Sigma L_1 \oplus B_\Sigma = T_\Sigma L_2 \oplus B_\Sigma = T_\Sigma P .$$

Let M be an element of $\mathfrak{L}(P, \Omega)$, containing Σ, such that $T_\Sigma M \oplus B_\Sigma = T_\Sigma P$. (For instance, we could take M to be L_1 or L_2, but this is not necessary.) By Lemma 4.2, we can extend B_Σ to a lagrangian subbundle $B \subseteqq T_M P$ such that $T_M \oplus B = T_M P$.

We now turn our attention to the cotangent bundle T^*M with zero section 0. The restricted bundle $T_{0(M)} T^*M$ is naturally isomorphic to $TM \oplus T^*M$, both summands being lagrangian subbundles. Now, by Theorem 6.1 of [34], and its proof, there is a symplectic diffeomorphism α between a neighborhood P_0 of M in P and a neighborhood of $0(M)$ in T^*M such that $\alpha \circ i_M = 0$ and $T_M\alpha\colon T_M P \to T_{0(M)} T^*M$ maps B onto the subbundle, tangent to the fibres, corresponding to $0 \oplus T^*M$. (The equivariance is not mentioned in Theorem 6.1 of [34], but remarks in Sections 3 and 4 of that paper show that, since the symmetry group is compact, α can indeed be chosen to be equivariant.) Since α is symplectic, (P_0, M, α) is a cotangent coordinate system for (P, Ω).

Look now at the lagrangian submanifolds $\alpha_\sharp(L_1)$ and $\alpha_\sharp(L_2)$ in T^*M. For each $s \in \Sigma$, we have $\alpha(s) = 0(s)$, and $T_s\alpha\colon T_s P \to T_{0(s)} T^*M$ maps B_s onto the subspace of $T_{0(s)} T^*M$ tangent to the fibres. Let j be 1 or 2. Since $T_s L_j \cap B_s = 0$, the intersection of $T_{\alpha(s)}[\alpha_\sharp(L_j)] = T_s\alpha(T_s L_j)$ with the space tangent to the fibres is zero. Restricted to $\alpha(\Sigma)$, the mapping $\pi_M \circ i_{\alpha_\sharp(L_j)}$ is just the embedding α^{-1}. The computation just completed shows that, for each $s \in \Sigma$, the differential

$$T_{\alpha(s)}(\pi_M \circ i_{\alpha_\sharp(L_j)})\colon T_{\alpha(s)}[\alpha_\sharp(L_j)] \to T_s M$$

is an isomorphism. It follows (see, for instance, Lemma 5.7 in [22]) that there exists a neighborhood U_j of $\alpha(\Sigma)$ in $\alpha_\sharp(L_j)$ such that $\pi_M \circ i_{\alpha_\sharp(L_j)}$ is an embedding onto a neighborhood of Σ in M. Then neighborhoods U_j can be chosen such that $\pi_M \circ i_{\alpha_\sharp(L_1)}(U_1)$ and $\pi_M \circ i_{\alpha_\sharp(L_2)}(U_2)$ are the same set, which we call S. Let $L_j^0 = \alpha^{-1}(U_j)$. Then $U_j = \alpha_\sharp(L_j^0)$, and we have the following theorem.

THEOREM 4.3. *Let L_1 and L_2 in $\mathfrak{L}(P, \Omega)$ intersect cleanly along the closed submanifold Σ. Then there exist: an open neighborhood P_0 of Σ in P; a manifold M containing Σ as a closed submanifold; a cotangent coordinate* *system* (P_0, M, α) *such that* $\alpha(s) = 0(s)$ *for all* $s \in \Sigma$; *and open neighborhoods* L_1^0, L_2^0, *and* S *of* Σ *in* $L_1 \cap P_0$, $L_2 \cap P_0$, *and* M, *respectively, such that* $\alpha_\sharp(L_1^0)$ *and* $\alpha_\sharp(L_2^0)$ *lie in* $\mathfrak{L}_S(T^*M, \Omega_M)$.

Since $\alpha_\sharp(L_1^0)$ and $\alpha_\sharp(L_2^0)$ obviously intersect cleanly along $0(\Sigma)$, and $\alpha_\sharp$ is continuous, we may combine Theorems 3.3 and 4.3 to obtain our principal result on intersections.

THEOREM 4.4. *Let* L_1 *and* L_2 *in* $\mathfrak{L}(P, \Omega)$ *intersect cleanly along the closed submanifold* Σ, *and let* $\mathcal{E} \subseteqq C^\infty(\Sigma, P)$ *be any* C^1 *neighborhood of the inclusion. Then there exist a fine* C^1 *neighborhood* $\mathfrak{N}_1 \times \mathfrak{N}_2$ *of* (L_1, L_2) *in* $\mathfrak{L}(P, \Omega) \times \mathfrak{L}(P, \Omega)$ *and mappings* $G: \mathfrak{N}_1 \times \mathfrak{N}_2 \to \mathcal{E}$ *and* $\Gamma: \mathfrak{N}_1 \times \mathfrak{N}_2 \to Z^1(\Sigma)$ *such that, for each pair* (L_1', L_2') *in* $\mathfrak{N}_1 \times \mathfrak{N}_2$, $G(L_1', L_2')$ *is an embedding which maps the zero set of* $\Gamma(L_1', L_2')$ *into the intersection* $L_1' \cap L_2'$. *If* Σ *is compact, then* $\mathfrak{N}_1 \times \mathfrak{N}_2$ *can be taken to be a coarse* C^1 *neighborhood.*

The details of the proof are left to the reader. We refer to Section 5 for a rather explicit construction of $\Gamma(L_1', L_2')$.

As a sample of the results which can be obtained from Theorem 4.4, we present the following corollary.

COROLLARY 4.5. *Let* L_1 *and* L_2 *in* $\mathfrak{L}(P, \Omega)$ *intersect cleanly along* Σ. *Suppose that* Σ *is compact and that the cohomology group* $H^1(\Sigma; \mathcal{R})$ *is zero. Then there is a coarse* C^1 *neighborhood* $\mathfrak{N}_1 \times \mathfrak{N}_2$ *of* (L_1, L_2) *in* $\mathfrak{L}(P, \Omega) \times \mathfrak{L}(P, \Omega)$ *such that, for all pairs* (L_1', L_2') *in* $\mathfrak{N}_1 \times \mathfrak{N}_2$, $L_1 \cap L_2$ *contains at least* cat (Σ) *points, where* cat (Σ) *is the Lusternik-Schnirelmann category of* Σ.

Proof. Let $\mathfrak{N}_1 \times \mathfrak{N}_2$ be as in Theorem 4.4. We must show that $\Gamma(L_1', L_2')$ has at least cat (Σ) zeros. But $H^1(\Sigma; \mathcal{R}) = 0$ implies that $\Gamma(L_1', L_2') = df$ for some $f: \Sigma \to \mathcal{R}$. The zeros of $\Gamma(L_1', L_2')$ are the critical points of f, which number at least cat (Σ) [16], [25]. Q.E.D.

Remarks. Notice that, if the symmetry group is not trivial, $H^1(\Sigma; \mathcal{R})$ means the space of invariant closed 1-forms modulo the differentials of invariant functions. Nevertheless, if an invariant 1-form is the differential of a non-invariant function, we may average over the symmetry group to make it the differential of an invariant function. In other words, if we denote Σ with trivial symmetry group by Σ_*, there is a natural injection of $H^1(\Sigma; \mathcal{R})$ into $H^1(\Sigma_*; \mathcal{R})$. If the latter cohomology group is zero, so is the former, and Corollary 4.5 applies. On the other hand, $H^1(\Sigma; \mathcal{R})$ may be zero when

$H^1(\Sigma_*; \mathcal{R})$ is not; e.g., $\Sigma =$ unit complex numbers with $G = \mathbf{Z}_2$ acting by conjugation.

As long as Σ is not a single point, cat (Σ) is at least 2. Other examples are cat $(\mathbf{C}P^n) =$ cat $(\mathcal{R}P^n) = n + 1$. Sometimes, one may use the invariance of $\Gamma(L_1', L_2')$ under the symmetry group to obtain a better estimate for the number of zeros.

5. A cohomology invariant

If L_1 and L_2 in $\mathcal{L}(P, \Omega)$ intersect cleanly along Σ, we have seen in the previous section how to associate to (L_1', L_2') near (L_1, L_2) an element $\Gamma(L_1', L_2')$ of $Z^1(\Sigma)$ whose zeros are mapped by an embedding $G(L_1', L_2')$ into the intersection $L_1' \cap L_2'$. The closed 1-form $\Gamma(L_1', L_2')$ is not invariantly defined but depends on the choice of a cotangent coordinate system around Σ. The aim of this section is to show that the cohomology class of $\Gamma(L_1', L_2')$ in $H^1(\Sigma; \mathcal{R})$ is independent of the choice of cotangent coordinate system. We will also give some sufficient conditions for the vanishing of this cohomology class. When these conditions are satisfied, the conclusion of Corollary 4.5 is true for (L_1', L_2') without the assumption that $H^1(\Sigma; \mathcal{R})$ be zero.

To prove our invariance theorem, we will look again at the way in which the form $\Gamma(L_1', L_2')$ arises from the cotangent coordinate system (P_0, M, α). First of all, by Theorem 4.3, we have $L_1^0 \subsetneqq L_1$ and $L_2^0 \subsetneqq L_2$ such that $\alpha_\sharp(L_1^0)$ and $\alpha_\sharp(L_2^0)$ are elements of $\mathcal{L}_S(T^*M, \Omega_M)$, cleanly intersecting along $0(\Sigma)$. Now, by Proposition 2.1, if (L_1', L_2') is close enough to (L_1, L_2), $\alpha_\sharp(L_1')$ and $\alpha_\sharp(L_2')$ contain open subsets U_1 and U_2 which are again in $\mathcal{L}_S(T^*M, \Omega_M)$. The form $\Phi(U_1, U_2)$ is defined as $\phi_2 - \phi_1$, where $U_j = \phi_j(S)$. Notice that $\Phi(U_1, U_2)$ is also equal to $\phi_2^*\omega_M - \phi_1^*\omega_M$, where ω_M is the fundamental 1-form on T^*M. Now $\Gamma(L_1', L_2')$ is defined to be the form $\Gamma(\alpha_\sharp(L_1'), \alpha_\sharp(L_2'))$ of Theorem 3.3, which is in turn the form $\mathcal{F}^*(\Phi(U_1, U_2))$ of Proposition 3.2, where $\mathcal{F} = F(\Phi(U_1, U_2)) \colon \Sigma \to M$. Putting these equalities together, we have

$$\Gamma(L_1', L_2') = \mathcal{F}^*(\phi_2^*\omega_M - \phi_1^*\omega_M) = (\phi_2\circ\mathcal{F})^*(\omega_M) - (\phi_1\circ\mathcal{F})^*(\omega_M) .$$

Here, $\phi_1\circ\mathcal{F}$ and $\phi_2\circ\mathcal{F}$ are mappings from Σ to T^*M which are C^1 close to the zero section. If we write ω for $\alpha^*\omega_M$, we have $d\omega = -\Omega$, and

$$\Gamma(L_1', L_2') = (\alpha^{-1}\circ\phi_2\circ\mathcal{F})^*\omega - (\alpha^{-1}\circ\phi_1\circ\mathcal{F})^*\omega .$$

Notice that the image of $\alpha^{-1}\circ\phi_j$ is contained in L_j'. We may summarize all this by the following statement.

LEMMA 5.1. $\Gamma(L_1', L_2')$, *as constructed from any cotangent coordinate* *system, is of the form* $\beta_2^*\omega - \beta_1^*\omega$; β_j *is a mapping from* Σ *into* L'_j, C^1 *close to the inclusion;* Σ *is contained in the zero set of* ω, *and* $d\omega = -\Omega$.

Now suppose we construct $\Gamma(L'_1, L'_2)$ from another cotangent coordinate system. It will be equal to $\bar\beta_2^*\bar\omega - \bar\beta_1^*\bar\omega$, where $\bar\beta_1$, $\bar\beta_2$, and $\bar\omega$ are as in Lemma 5.1. Now suppose that β_j and $\bar\beta_j$ are homotopic in L'_j, and that their images are all contained in a tubular neighborhood of Σ in P. (This will be true for (L'_1, L'_2) close enough to (L_1, L_2).) Since the form $\bar\omega - \omega$ vanishes along Σ, and $d(\bar\omega - \omega) = -\Omega + \Omega = 0$, the Poincaré lemma for vector bundles [34] implies that there is a function θ, defined on the tubular neighborhood, such that $d\theta = \bar\omega - \omega$. Now we compute the difference of the two versions of $\Gamma(L'_1, L'_2)$. It is

$$\begin{aligned}\bar\beta_2^*\omega - \bar\beta_1^*\bar\omega - \beta_2^*\omega + \beta_1^*\omega &= \bar\beta_2^*\bar\omega - \bar\beta_2^*\omega - \bar\beta_1^*\bar\omega + \bar\beta_1^*\omega + \bar\beta_2^*w \\ &\quad - \beta_2^*\omega - \bar\beta_1^*\omega + \beta_1^*\omega \\ &= d(\bar\beta_2^*\theta) - d(\bar\beta_1^*\theta) + (\bar\beta_2^*\omega - \beta_2^*\omega) \\ &\quad - (\bar\beta_1^*\omega - \beta_1^*\omega)\ .\end{aligned}$$

Now each term of the form $\bar\beta_j^*\omega - \beta_j^*\omega$ is exact because β_j and $\bar\beta_j$ are homotopic in L'_j, and ω is closed on L'_j; hence, the difference of the two versions of $\Gamma(L'_1, L'_2)$ is exact. We have thus proven the following result.

PROPOSITION 5.2. *Let* L_1 *and* L_2 *in* $\mathfrak{L}(P, \Omega)$ *intersect cleanly along* Σ. *There is a neighborhood* $\mathfrak{M}_1 \times \mathfrak{M}_2$ *of* (L_1, L_2) *in* $\mathfrak{L}(P, \Omega) \times \mathfrak{L}(P, \Omega)$ *and an invariantly defined mapping* $c: \mathfrak{M}_1 \times \mathfrak{M}_2 \to H^1(\Sigma; \mathfrak{R})$ *such that, for* $(L'_1, L'_2) \in \mathfrak{M}_1 \times \mathfrak{M}_2$, *the form* $\Gamma(L'_1, L'_2)$ *in Theorem* 4.4, *defined in terms of any cotangent coordinate system, belongs to the cohomology class* $c(L'_1, L'_2)$. *In particular, if* $c(L'_1, L'_2)$ *is zero,* $\Gamma(L'_1, L'_2)$ *is the differential of a function, and the conclusion of Corollary* 4.5 *applies.*

We will now go on to find a sufficient condition for the vanishing of $c(L'_1, L'_2)$. Notice first that the proof of Proposition 5.2 shows that, if ϕ is *any* form such that $d\phi = -\Omega$, then $\beta_2^*\phi - \beta_1^*\phi$ belongs to the cohomology class $c(L'_1, L'_2)$.

PROPOSITION 5.3. *For* $c(L'_1, L'_2)$ *to be zero, it is sufficient that* Ω *be exact on* P *and that the mappings* $\beta_j^*: H^1(L'_j; \mathfrak{R}) \to H^1(\Sigma; \mathfrak{R})$ *be zero. The latter condition is satisfied whenever* $H^1(L'_1; \mathfrak{R}) = 0 = H^1(L'_2; \mathfrak{R})$.

Proof. Since Ω is exact on P, there is a 1-form ϕ on P such that $d\phi = -\Omega$. By the remark immediately preceding this proposition, $c(L'_1, L'_2) = [\beta_2^*\phi - \beta_1^*\phi]$. (Square brackets around a closed form denote its cohomology class.) Writing i_j for the inclusion of L'_j into P, we have $\beta_j = i_j \circ \beta_j$, and

$$[\beta_2^*\phi - \beta_1^*\phi] = \beta_2^*[i_2^*\phi] - \beta_1^*[i_1^*\phi] ,$$

which is zero because the mappings β_j^* are zero on cohomology. Q.E.D.

Another sufficient condition for the vanishing of $c(L'_1, L'_2)$ is given in Proposition 6.3.

6. One-parameter families

Let $\{\gamma_t\}_{t \in J}$ be a smooth family of mappings of a manifold Σ into (P, Ω), parametrized by an interval J in the real numbers. If ϕ is any form such that $d\phi = -\Omega$, we have the homotopy formula [11]

$$\frac{d}{dt}\gamma_t^*\phi = \gamma_t^*[d(\dot{\gamma}_t \lrcorner \phi) + \dot{\gamma}_t \lrcorner d\phi] = \gamma_t^*[d(\dot{\gamma}_t \lrcorner \phi) - \dot{\gamma}_t \lrcorner \Omega] ,$$

where $\dot{\gamma}_t$ is the vector field along γ_t representing the time derivative of γ_t. Suppose that $\gamma_t^*\Omega = 0$ for each $t \in J$. Then the forms $\gamma_t^*\phi$, $(d/dt)\gamma_t^*\phi$, and $\gamma_t^*(\dot{\gamma}_t \lrcorner \Omega)$ are all closed for each $t \in J$. We will use two special cases of our formula.

LEMMA 6.1. (a) *If $\gamma_t^*(\dot{\gamma}_t \lrcorner \Omega)$ is exact for each $t \in J$, then the cohomology class $[\gamma_t^*\phi]$ is independent of t.*

(b) *If $\gamma_{t_0}(\Sigma)$ is contained in the zero set of ϕ for some $t_0 \in J$, then*

$$\frac{d}{dt}\gamma_t^*\phi\,|_{t=t_0} = -\gamma_{t_0}^*(\dot{\gamma}_{t_0} \lrcorner \Omega) .$$

The right hand side is independent of the choice of ϕ.

First, we will apply Lemma 6.1 (a) to derive another sufficient condition for the vanishing of $c(L'_1, L'_2)$. Recall from Section 5 that there are embeddings $\beta_j \colon \Sigma \to L'_j$ such that, if ϕ is any form such that $d\phi = -\Omega$, $c(L'_1, L'_2) = [\beta_2^*\phi] - [\beta_1^*\phi]$.

PROPOSITION 6.2. *Suppose there exists a 1-parameter family $\{\gamma_t\}_{t \in [0,1]}$ of mappings from Σ into P such that: $\gamma_t^*\Omega = 0$ and $\gamma_t^*(\dot{\gamma}_t \lrcorner \Omega)$ is exact, for each $t \in [0, 1]$; $\gamma_0 = \beta_2$; and $[\gamma_1^*\phi] = [\beta_1^*\phi]$. Then $c(L'_1, L'_2) = 0$.*

The second goal of this chapter will be to apply Lemma 6.1 (b) to derive a result concerning 1-parameter deformations of cleanly intersecting lagrangian submanifolds.

Let J be an interval in the real numbers. Given a family $\mathbf{L} = \{L_t\}_{t \in J}$ of elements of $\mathcal{L}(P, \Omega)$, we define $\bar{\mathbf{L}} \subseteq P \times J$ to be the subset $\{(p, t) \mid t \in J$ and $p \in L_t\}$. There are natural projections $\Pi_{\mathbf{L}}^P \colon \bar{\mathbf{L}} \to P$ and $\Pi_{\mathbf{L}}^J \colon \bar{\mathbf{L}} \to J$. If $\bar{\mathbf{L}}$ is a (locally closed) submanifold of $P \times J$ and $\Pi_{\mathbf{L}}^J$ is a submersion, we call

$\mathbf{L}$ a smooth family. If $\mathbf{L}$ is a smooth family, it is not hard to see that the mapping $t \mapsto L_t$ from J to $\mathcal{L}(P, \Omega)$ is continuous if $\mathcal{L}(P, \Omega)$ is given the coarse C^1 topology.

Let 0 be contained in the interval J. Given each $p \in L_0$, choose a vector $v \in T_pP$ such that there exists a vector $w \in T_{(p,0)}\bar{\mathbf{L}}$ for which $T\Pi^J_{\mathbf{L}} w = d/dt$ and $T\Pi^P_{\mathbf{L}} w = v$. Let $v^* \in T^*_p L_0$ be defined by the equation $v^*(u) = -\Omega(v, u)$ for all $u \in T_pL_0$. In other words, v^* is the restriction to T_pL_0 of $-v \lrcorner \Omega$. If v' is another choice of the vector, then $v' = T\Pi^P_{\mathbf{L}} w'$ where $T\Pi^J_{\mathbf{L}} w' = d/dt = T\Pi^J_{\mathbf{L}} w$, The difference $w' - w$ must then be tangent to the fibre $L_0 \times \{0\}$, so that difference $v' - v = T\Pi^P_{\mathbf{L}}(w' - w)$ is tangent to L_0. Since L_0 is lagrangian, $\Omega(v' - v, u) = 0$ for all $u \in T_pL_0$, and v'^* is equal to v^*. Thus, we can associate to each p in L_0 an element of $T^*_pL_0$ which depends only on the family $\mathbf{L}$. These elements define a 1-form on L_0 which we denote by $D_0\mathbf{L}$. It represents the family $\mathbf{L}$ to first order around $t = 0$.

PROPOSITION 6.3. *Let* $\mathbf{L}$ *be a smooth family of lagrangian submanifolds of* (P, Ω), *parametrized by an interval* J *containing* 0. *Let* $\{\gamma_t\}_{t \in J}$ *be a smooth family of mappings of a manifold* Σ *into* P *such that* $\gamma_t(\Sigma) \subseteq L_t$ *for each* $t \in J$. *If* ϕ *is any* 1-*form, defined on a neighborhood of the* $\gamma_t(\Sigma)$ *for* t *near* 0, *such that* $d\phi = -\Omega$ *and such that* $\gamma_0(\Sigma)$ *is contained in the zero set of* ϕ, *then*

$$\frac{d}{dt}\gamma_t^*\phi\,|_{t=0} = \gamma_0^* D_0\mathbf{L}\ .$$

Proof. By Lemma 6.1 (b), $(d/dt)\gamma_t^*\phi|_{t=0} = -\gamma_0^*(\dot\gamma_0 \lrcorner \Omega)$. Given any $p \in \Sigma$, let w be the tangent vector at 0 of the map $\sigma: J \to \bar{\mathbf{L}}$ defined by $\sigma(t) = (\gamma_t(p), t)$. Then $T\Pi^J_{\mathbf{L}}(w) = 1$ and $T\Pi^P_{\mathbf{L}}(w) = \dot\gamma_0(p)$. By definition, the value v^* of $D_0\mathbf{L}$ at $\gamma_0(p)$ satisfies the equation $v^*(u) = -\Omega(\dot\gamma_0(p), u)$ for all $u \in T_{\gamma_0(p)}L$.

For all $x \in T_p\Sigma$, $[\gamma_0^*(-\dot\gamma_0 \lrcorner \Omega)](x)$ is, by definition, $\Omega(-\dot\gamma_0(p), T_p\gamma_0(x))$, which equals $v^*(T_p\gamma_0(x)) = [T_p^*\gamma_0(v^*)](x)$; i.e., $\gamma_0^*(-\dot\gamma_0 \lrcorner \Omega) = \gamma_0^*(D_0\mathbf{L})$.

Q.E.D.

As a first application of Proposition 6.3, we can conclude that D_0L is closed. In fact, given any $p \in L_0$, we can let Σ be a neighborhood of p and find $\{\gamma_t\}$ and ϕ satisfying the hypotheses of the proposition. (Choose a cotangent coordinate system (P_0, M, α) mapping L_0 onto the zero section of T^*M.) Since $(d/dt)\gamma_t^*\phi|_{t=0}$ is closed, so is $\gamma_0^*D_0\mathbf{L}$. The form $D_0\mathbf{L}$ is thus locally closed and, therefore, closed.

Our main application of Proposition 6.3 is to one-parameter deformations of cleanly intersecting submanifolds. The results which follow will be seen in Part II to form the basis of an extension of the averaging method of Reeb [26] and Moser [21].

Suppose that $\mathbf{L}_1$ and $\mathbf{L}_2$ are smooth families parametrized by an interval J containing 0 and that $L_{1,0}$ and $L_{2,0}$ intersect cleanly along a submanifold Σ. We apply Theorem 4.4, taking $L_1 = L_{1,0}$ and $L_2 = L_{2,0}$. Suppose that, for t sufficiently near 0 in J, the pair $(L_{1,t}, L_{2,t})$ is in the neighborhood $\mathfrak{N}_1 \times \mathfrak{N}_2$ of Theorem 4.4. By the continuity property mentioned above, this will always be true if Σ is compact. Then $\Gamma(L_{1,t}, L_{2,t})$ is defined for t sufficiently near zero. It is not hard to verify given the natural way in which $\Gamma(L_{1,t}, L_{2,t})$ is constructed, that $\Gamma(L_{1,t}, L_{2,t})$ depends smoothly on t, so that we may consider the derivative $(d/dt)\Gamma(L_{1,t}, L_{2,t})\,|_{t=0}$.

THEOREM 6.4. *Let* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *be smooth families parametrized by an interval* J *containing* 0. *Suppose that* $L_{1,0}$ *and* $L_{2,0}$ *intersect cleanly along a submanifold* Σ *and that* $\Gamma(L_{1,t}, L_{2,t})$ *is defined, via Theorem* 4.4, *for all* t *sufficiently near* 0. (*This is necessarily the case if* Σ *is compact.*) *Then*

$$\frac{d}{dt}\Gamma(L_{1,t}, L_{2,t})\,|_{t=0}$$

is equal to $i_\Sigma^* D_0\mathbf{L}_2 - i_\Sigma^* D_0\mathbf{L}_1$ *and is, therefore, independent of the cotangent coordinate system used to construct* $\Gamma(L_{1,t}, L_{2,t})$.

Proof. By Lemma 5.1, $\Gamma(L_{1,t}, L_{2,t})$ is of the form $\beta_{2,t}^*\omega - \beta_{1,t}^*\omega$, where $\beta_{j,t}$ maps Σ into $L_{j,t}$, Σ is contained in the zero set of ω, and $d\omega = -\Omega$. By construction, the families $\{\beta_{j,t}\}$ are smooth, and $\beta_{1,t}$ and $\beta_{0,t}$ both equal the inclusion i_Σ. Then

$$\frac{d}{dt}\Gamma(L_{1,t}, L_{2,t})\,|_{t=0} = \frac{d}{dt}\beta_{2,t}^*\omega\,|_{t=0} - \frac{d}{dt}\beta_{1,t}^*\omega\,|_{t=0}$$

which, by Proposition 6.3, is equal to $\beta_{2,0}^* D_0\mathbf{L}_2 - \beta_{1,0}^* D_0\mathbf{L}_2$, which equals $i_\Sigma^* D_0\mathbf{L}_2 - i_\Sigma^* D_0\mathbf{L}_1$. Q.E.D.

Besides showing that $(d/dt)\Gamma(L_{1,t}, L_{2,t})_{t=0}$ is an invariant of the families $\mathbf{L}_1$ and $\mathbf{L}_2$, Theorem 6.4 can also be used to deduce the existence of points of $L_{1,t} \cap L_{2,t}$ for small t. A component Σ_0 of the zero set of a closed form ψ on Σ is called [weakly] *stable* if, given any neighborhood U of Σ_0 in Σ, there is a neighborhood $\mathfrak{B}$ of ψ in $Z^1(\Sigma)$ such that every [exact] form ψ' in $\mathfrak{B}$ has at least one zero in U.

THEOREM 6.5. *Let* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *be as in Theorem* 6.4. *If* Σ_0 *is a stable component of the zero set of* $i_\Sigma^* D_0\mathbf{L}_2 - i_\Sigma^* D_0\mathbf{L}_1$, *then, given any neighborhood*

U of Σ_0 in P, the intersection $L_{1,t} \cap L_{2,t} \cap U$ is non-empty for t sufficiently near 0. If $c(L_{1,t}, L_{2,t})$ is known to be zero for all t, it is sufficient that Σ_0 be weakly stable.

Proof. Consider $\Gamma(L_{1,t}, L_{2,t})$ as a function of t. Since it vanishes at $t = 0$, it is divisible by t, and $(\Gamma(L_{1,t}, L_{2,t}))/t = \widetilde{\Gamma}_t$, where $\widetilde{\Gamma}_t$ depends smoothly on t and

$$\widetilde{\Gamma}_0 = \frac{d}{dt}\Gamma(L_{1,t}, L_{2,t})\,|_{t=0} \,.$$

By Theorem 6.4, $\widetilde{\Gamma}_0 = i_\Sigma^* D_0 \mathbf{L}_2 - i_\Sigma^* D_0 \mathbf{L}_1$. If Σ_0 is a stable component of the zero set of $\widetilde{\Gamma}_0$, $\Gamma(L_{1,t}, L_{2,t}) = t\widetilde{\Gamma}_t$ has at least one zero near Σ_0 for all sufficiently small t. By Theorem 4.4, this zero is mapped by the embedding $G(L_{1,t}, L_{1,t})$, which is near the inclusion, onto a point of $L_{1,t} \cap L_{2,t}$. Q.E.D.

The importance of Theorem 6.5 is that, in practice, $i_\Sigma^* D_0 \mathbf{L}_2 - i_\Sigma^* D_0 \mathbf{L}_1$ may be computed much more simply than $\Gamma(L_{1,t}, L_{2,t})$. After verifying stability by transversality theory or topology, one may locate points of $L_{1,t} \cap L_{2,t}$, of which Theorem 4.4 merely guarantees the existence. In addition, $i_\Sigma^* D_0 \mathbf{L}_2 - i_\Sigma^* D_0 \mathbf{L}_1$ is sometimes invariant under more than the original symmetry group, so that it becomes easier to deduce the existence of many zeros. Finally, we remark without proof that Theorem 6.5 holds if Σ_0 is compact, even if Σ is not. In this case, $\Gamma(L_{1,t}, L_{2,t})$ might only be defined near Σ_0 for t near 0, but that is sufficient.

PART II. APPLICATIONS TO HAMILTONIAN SYSTEMS

7. Canonical relations and fixed manifolds

The intersection theorems of Part I will be applied to certain submanifolds of the cartesian square of a symplectic manifold.

If (P, Ω) is a symplectic manifold, the cartesian square $P \times P$ has a symplectic structure Ω_x, defined as $\pi_1^*\Omega - \pi_2^*\Omega$, where π_1 and π_2 are the natural projections of $P \times P$ onto P. If $\beta\colon P \to P$ is a *canonical transformation*, i.e., a diffeomorphism for which $\beta^*\Omega = \Omega$, then the graph $\gamma_\beta = \{(p, \beta(p)) \mid p \in P\}$ is easily seen to be a lagrangian submanifold of $(P \times P, \Omega_x)$. By way of generalization, we refer to any lagrangian submanifold of $(P \times P, \Omega_x)$ as a *canonical relation* on (P, Ω).

The graphs of canonical transformations furnish our first examples of canonical relations, of which the *diagonal* $\Delta_P = \{(p, p) \mid p \in P\}$ is of special importance. If R is any canonical relation, a point $p \in P$ such that $(p, p) \in R$ is called a *fixed point* of R.

If L_1 and L_2 are lagrangian submanifolds of (P, Ω), then the product $L_1 \times L_2 \subseteqq P \times P$ is a canonical relation on (P, Ω). It is interesting to note that the fixed points of $L_1 \times L_2$ are exactly the points of $L_1 \cap L_2$. (One could use this observation to reduce the intersection problems of Part I to the special case in which one of the manifolds is fixed.)

Another important class of canonical relations, arising from hamiltonian dynamical systems, will be introduced in the next section.

If R is a canonical relation on (P, Ω), a submanifold $\Sigma \subset P$ is called a *non-degenerate fixed manifold* for R if R and Δ_P intersect cleanly along $\Delta_\Sigma = \{(p, p) \mid p \in \Sigma\}$. This definition is easy to interpret in the two special cases introduced in the previous section. If $\beta: P \to P$ is a canonical transformation, Σ is a non-degenerate fixed manifold for γ_β if and only if, for every $s \in \Sigma$, $\beta(s) = s$ and the kernel of $(T_s\beta - \text{id}): T_sP \to T_sP$ is $T_s\Sigma$. If L_1 and L_2 are lagrangian submanifolds of (P, Ω), Σ is a non-degenerate fixed manifold for $L_1 \times L_2$ if and only if L_1 and L_2 intersect cleanly along Σ.

We can now obtain the following result, which shows what happens to a non-degenerate fixed manifold under small perturbations of the canonical relation.

THEOREM 7.1. *Let Σ be a non-degenerate fixed manifold for $R \in \mathfrak{L}(P \times P, \Omega_x)$, and let $\mathfrak{S} \subseteqq C^\infty(\Sigma, P)$ be any C^1 neighborhood of the inclusion. Then there exist a fine C^1 neighborhood $\mathfrak{N}$ of R in $\mathfrak{L}(P \times P, \Omega_x)$ and mappings $G: \mathfrak{N} \to \mathfrak{S}$ and $\Gamma: \mathfrak{N} \to Z^1(\Sigma)$ such that, for each $R' \in \mathfrak{N}$, $G(R')$ is an embedding which maps the zero set of $\Gamma(R')$ into the fixed point set of R'. If Σ is compact, then $\mathfrak{N}$ can be taken to be a coarse C^1 neighborhood.*

Proof. Apply Theorem 4.4 to the pair (Δ_P, R). Use the natural identification of Σ with Δ_Σ to make $G(\Delta_P, R')$ and $\Gamma(\Delta_P, R')$ defined on Σ instead of Δ_Σ. Compose $G(\Delta_P, R')$ with either π_1 or π_2 to make it map into P instead of $P \times P$. Q.E.D.

Remarks. Since the map $(L_1, L_2) \mapsto L_1 \times L_2$ from $\mathfrak{L}(P, \Omega) \times \mathfrak{L}(P, \Omega)$ to $\mathfrak{L}(P \times P, \Omega_x)$ is continuous, Theorem 4.4 may be considered as a special case of Theorem 7.1.

If Σ is compact and $H^1(\Sigma; \mathcal{R}) = 0$, we may deduce, as in Corollary 4.5, the existence of fixed points for $R' \in \mathfrak{N}$. Even if $H^1(\Sigma; \mathcal{R})$ is not zero, we may be able to establish that the cohomology class $c(\Delta_P, R')$ is zero, in which case $\Gamma(R')$ must be exact, and the conclusion still applies. Examples of this are given in this next section.

If R is the graph of a canonical transformation β, then $\gamma_{\beta'} \in \mathfrak{N}$ for all

β' sufficiently near β in the C^1 topology on $C^\infty(P, P)$. Theorem 7.1 yields, therefore, fixed point theorems for canonical transformations near one with a non-degenerate fixed manifold.

8. Hamiltonian systems and canonical boundary value problems

This section is the heart of Part II of this paper. We will present a general formulation which encompasses most of the boundary value problems considered in conjunction with hamiltonian systems and the calculus of variations. In terms of this formulation, we will apply the intersection theory of Part I to derive the existence of solutions for problems obtained by slightly perturbing problems having manifolds of solutions.

A *hamiltonian system* is a triple (P, Ω, H) where (P, Ω) is a symplectic manifold and H is a real valued function on P. H is called the hamiltonian function, or simply the hamiltonian. Though many of our results can be extended to the time-dependent situation, in which H is defined on $P \times \mathcal{R}$, this paper will be confined to the time-independent case. We refer the reader to [1] as a general reference on hamiltonian systems.

Associated with the hamiltonian system (P, Ω, H) is the hamiltonian vector field ξ_H on P, defined by the equation $\xi_H \lrcorner \Omega = dH$, or $\xi_H = \tilde{\Omega}^{-1} \circ dH$. For each $p \in P$, the hamiltonian vector field ξ_H has a maximal integral curve σ_p, with $\sigma_p(0) = p$, defined on an open interval of $\mathcal{R}$. The collection of maximal integral curves gives rise to the flow $F_H: \mathfrak{D}_H \to P$, where $\mathfrak{D}_H = \{(p, t) \in P \times \mathcal{R} \mid t \text{ is in the domain of } \sigma_p\}$, and $F_H(p, t) = \sigma_p(t)$.

We will examine the map F_H in some detail. For each $t \in \mathcal{R}$, let ${}^t\mathfrak{D}_H = \{p \in P \mid (p, t) \in D_H\}$, and define ${}^tF_H: {}^t\mathfrak{D}_H \to P$ by ${}^tF_H(p) = F_H(p, t)$. ${}^tF_H$ is a diffeomorphism of ${}^t\mathfrak{D}_H$ onto ${}^{-t}\mathfrak{D}_H$ and is called the *time t mapping* associated with the hamiltonian system (P, Ω, H). It is well known that ${}^tF_H{}^*\Omega = \Omega$, so that the graph $\{(p, {}^tF_H(p)) \mid p \in {}^t\mathfrak{D}_H\}$ is a canonical relation on (P, Ω), which we denote by ${}^tR_H$. Given $p \in P$, $\sigma_p(t) = p$ if and only if p is a fixed point of ${}^tR_H$. The set $I_p = \{t \mid \sigma_p(t) = p\}$ is either $\{0\}$, $\mathcal{R}$, or a non-trivial cyclic subgroup of $\mathcal{R}$. In the last case, we say that p is a *periodic point* for the system (P, Ω, H), and we call the positive generator of I_p the *least period* of p. Any positive element of I_p is called a period of p. If p is periodic with least period t_0, so are all the points of $\{\sigma_P(t) \mid t \in \mathcal{R}\}$. This set, which is an embedded circle in P, is called a *periodic orbit* for (P, Ω, H) with least period t_0.

The problem of determining the existence and properties of periodic orbits is of fundamental importance in the study of hamiltonian (and other) dynamical systems. It should already be evident to the reader of Section 7 that our intersection theory will be useful for deducing the existence of periodic orbits. Before doing this explicitly, we will generalize the setting of the problem.

Define the mapping $\tilde{F}_H: \mathcal{D}_H \to P \times P$ by $\tilde{F}_H(p, t) = (p, F_H(p, t))$.

The canonical relation ${}^tR_H$ is the image under $\tilde{F}_H$ of $\{(p, t) \mid p \in {}^t\mathcal{D}_H\}$. More generally, we can look for other submanifolds of $\mathcal{D}_H$ such that the restriction to them of $\tilde{F}_H$ is a lagrangian immersion. To find such manifolds, we begin by determining further properties of F_H. Denote by π and τ the projections of $P \times \mathcal{R}$ on P and $\mathcal{R}$, respectively.

LEMMA 8.1. $$F_H^*\Omega = \pi^*\Omega - d(H\circ\pi) \wedge d\tau .$$

Proof. Let $(p, t) \in \mathcal{D}_H$. The tangent space $T_{(p,t)}\mathcal{D}_H$ is naturally isomorphic to the product $T_pP \times T_t\mathcal{R}$. We may identify $T_t\mathcal{R}$ with $\mathcal{R}$ in such a way that $d\tau(x, a) = a$. For each (x, a) in $T_{(p,t)}\mathcal{D}_H$, we have

$$TF_H(x, a) = TF_H(x, 0) + TF_H(0, a) = T^tF_H(x) + a\cdot\xi_H[{}^tF_H(p)] .$$

Given (x, a) and (y, b) in $T_{(p,t)}\mathcal{D}_H$ and writing ξ for $\xi_H[{}^tF_H(x)]$, we have

$$\begin{aligned} F_H^*\Omega((x, a), (y, b)) &= \Omega(TF_H(x, a), TF_H(y, b)) \\ &= \Omega(T^tF_H(x) + a\xi, T^tF_H(y) + b\xi) \\ &= \Omega(T^tF_H(x), T^tF_H(y)) + a\Omega(\xi, T^tF_H(y)) \\ &\quad - b\Omega(\xi, T^tF_H(x)) + ab\Omega(\xi, \xi) . \end{aligned}$$

Since ${}^tF_H^*\Omega = \Omega$, $\xi_H \lrcorner \Omega = dH$, and ${}^tF_H^*(dH) = dH$, this reduces to

$$\begin{aligned} &\Omega(x, y) + a[dH(y)] - b[dH(x)] + 0 \\ &= [\pi^*\Omega]((x, a), (y, b)) + d\tau(x, a)\cdot[d(H\circ\pi)](y, b) \\ &\qquad - d\tau(y, b)\cdot[d(H\circ\pi)](x, a) \\ &[\pi^*\Omega + d\tau \wedge d(H\circ\pi)]((x, a), (y, b)) . \end{aligned}$$

Q.E.D.

COROLLARY 8.2. $$\tilde{F}_H^*\Omega_x = d(H\circ\pi) \wedge d\tau .$$

Proof. $\tilde{F}_H^*\Omega_x = \tilde{F}_H^*(\pi_1^*\Omega - \pi_2^*\Omega) = \pi^*\Omega - F_H^*\Omega = d(H\circ\pi) \wedge d\tau$. Q.E.D.

From Corollary 8.2, we see that $H\circ\pi$ and τ are functionally dependent on any submanifold of $\mathcal{D}_H$ to which the restriction of $\tilde{F}_H$ is isotropic. (This generalizes the well known dependence between energy and period for families of periodic orbits. See [8] for a recent treatment, together with further references.) With this in mind, we make the following construction. In $\mathcal{R}^2$, we think of the first coordinate as representing energy and the second as representing time. Let $C \subseteq \mathcal{R}^2$ be any curve (1-dimensional submanifold). Let $\mathcal{D}_H^C$ be the set $\{(p, t) \in \mathcal{D}_H \mid (H(p), t) \in C$ and the map $(H\circ\pi, \tau)$ is transversal to C at $(p, t)\}$. $\mathcal{D}_H^C$ is a submanifold of codimension 1 in $\mathcal{D}_H$ on which energy and time are related by C, so $d(H\circ\pi)\wedge d\tau = 0$ on $\mathfrak{D}_H^C$. Since the range of $T_{(p,t)}(H\circ\pi, \tau) = \mathcal{R}\times\mathcal{R}$ if p is not a critical point of H and is $\{0\}\times\mathcal{R}$ otherwise, the transversality condition in the definition of $\mathfrak{D}_H^C$ (which we will refer to as *condition* T) is equivalent to the negation of the statement: p is a critical point of H and the curve C has its tangent in the time direction at $(H(p), t)$.

For example, if C is a curve given by setting the time equal to a constant, then condition T is always satisfied. In this case $\mathfrak{D}_H^C$ is essentially what we called ${}^t\mathfrak{D}_H$ above. On the other hand, if C is given by setting the energy equal to a constant, then condition T is just that p not be a critical point of H. In any case, we have the following result.

LEMMA 8.3. *If C is any curve in $\mathcal{R}^2$, then the restriction $\tilde{F}_H^C$ of $\tilde{F}_H$ to $\mathfrak{D}_H^C$ is a lagrangian immersion.*

Proof. Since we have already seen that $\tilde{F}_H(\Omega_x)$ is zero on $\mathfrak{D}_H^C$, we have only to show that $\tilde{F}_H^C$ is an immersion. It suffices to show that, for $(p, t)\in\mathfrak{D}_H^C$, the kernel of $T_{(p,t)}\tilde{F}_H$ has zero intersection with $T_{(p,t)}\mathfrak{D}_H^C$. It follows from the calculation of TF_H in the proof of Lemma 8.1 that the kernel of $T_{(p,t)}\tilde{F}_H^C$ is zero if ξ_H does not vanish at p and is $\{0\}\times\mathcal{R}$ otherwise. If ξ_H does vanish at p, though, it follows from condition T that $\{0\}\times\mathcal{R}$ has zero intersection with $T_{(p,t)}\mathfrak{D}_H^C$. Q.E.D.

We now define a *canonical boundary value problem* to be a quintuple $\mathcal{P} = (P, \Omega, H, C, R)$ where (P, Ω, H) is a hamiltonian system, C is a curve in $\mathcal{R}^2$, and R is a canonical relation on (P, Ω). A *solution* of $\mathcal{P}$ is a pair $(p, t)\in\mathfrak{D}_H^C$ such that $\tilde{F}_H^C(p, t)\in R$.

A submanifold $\Sigma\subset\mathfrak{D}_H^C$ is called a *non-degenerate solution manifold* for $\mathcal{P}$ if: $\tilde{F}_H^C(\Sigma)\subseteqq R$; $\tilde{F}_H^C$ is an embedding on Σ; for each $(p, t)\in\Sigma$, the inverse image under $T_{(p,t)}\tilde{F}_H^C$ of TR is equal to $T_{(p,t)}\Sigma$. The second condition implies that $\tilde{F}_H^C$ is an embedding on a neighborhood U of Σ in $\mathfrak{D}_H^C$; the third condition then says that $\tilde{F}_H^C(U)$ and R intersect cleanly along $\tilde{F}_C^H(\Sigma)$.

For example, if $R = \Delta_P$, then (p, t) is a solution of $\mathcal{P}$ if and only if p is a periodic point with period t, $(H(p), t)\in C$, and condition T holds. (If condition T does not hold, then ξ_H vanishes at p, and p is a "trivial" periodic point.) In case C is a curve of constant energy or constant time, we can describe in a simple way the condition that a manifold Σ of solutions be non-degenerate. First of all, $\tilde{F}_H^C$ is an embedding on Σ and, hence, on a neighborhood U of Σ, if and only if π is an embedding on Σ. In the constant time case, this is always true. In the constant energy case, there must be a real valued function θ on $\pi(\Sigma)$ such that $\Sigma = \{(p, \theta(p)) \mid p\in\pi(\Sigma)\}$.

As far as the cleanliness of intersection with Δ_P is concerned, in the case of constant time t, the condition is just that $\pi(\Sigma)$ be a non-degenerate fixed manifold for ${}^tF_H$. In the case of constant energy E, the condition is that, for each $(p, t) \in \Sigma$ and each $v \in T_p[H^{-1}(E)]$, $[T_p{}^tF_H](v) - v$ is zero modulo $\xi_H(p)$ if and only if $v \in T_p[\pi(\Sigma)]$. One consequence of non-degeneracy in either case is that $\pi(\Sigma)$ contains the entire periodic orbit of each of its points.

We note without proof that, if (P, Ω, H) is the geodesic flow on the cotangent bundle of a riemannian manifold M, then a non-denegerate solution manifold for the constant energy periodic orbit problem corresponds exactly to a non-degenerate critical manifold for the energy integral on the free loop space of M. In this situation, all the results obtained in this paper can also be obtained through the calculus of variations on the loop space. In a sense, then, what we accomplish in this paper is to extend the applicability of critical point theory to non-riemannian hamiltonian systems. In making this extension, we must, apparently, restrict ourselves to perturbations of the periodic situation. It would be extremely interesting to see if one could avoid this restriction.

In riemannian geometry, one is also interested in orbits (geodesics) which connect two given points in the riemannian manifold. In the general case, we may consider canonical boundary problems of the form $(P, \Omega, H, C, L_1 \times L_2)$, where L_1 and L_2 are lagrangian submanifolds of (P, Ω). If P is a cotangent bundle, L_1 and L_2 might be fibres, normal bundles of submanifolds, or the entire zero section. We leave to the reader the problem of interpreting the definition of a non-degenerate solution manifold in this situation. In the special case where (P, Ω, H) is a geodesic flow, C is a constant energy or constant time curve, and L_1 and L_2 are fibres, a single solution of $(P, \Omega, H, C, L_1 \times L_2)$ is non-degenerate if and only if the points connected by the geodesic arc it represents are not conjugate along the arc.

The space of all canonical boundary value problems (P, Ω, H, C, R), where (P, Ω) is a fixed symplectic manifold, is denoted by $\mathcal{B}(P, \Omega)$. It may be identified with the product $C^\infty(P, \mathcal{R}) \times \mathcal{L}(\mathcal{R}^2, d\,(\text{energy}) \wedge d\,(\text{time})) \times \mathcal{L}(P \times P, \Omega_x)$, since every curve in $\mathcal{R}^2$ is a lagrangian submanifold. We give the compact-open C^2 topology to $C^\infty(P, \mathcal{R})$ and the compact-open C^1 topologies to the other two factors.

THEOREM 8.4. *Let Σ be a compact, non-degenerate solution manifold for the canonical boundary value problem $\mathcal{P} = (P, \Omega, H, C, R)$ and let $\mathcal{E} \subseteq C^\infty(\Sigma, P \times \mathcal{R})$ be any C^1 neighborhood of the inclusion. Then there exist a neighborhood $\mathfrak{N}$ of $\mathcal{P}$ in $\mathcal{B}(P, \Omega)$ and mappings $G\colon \mathfrak{N} \to \mathcal{E}$ and $\Gamma\colon \mathfrak{N} \to Z^1(\Sigma)$*

such that, for each

$$\mathcal{P}' = (P, \Omega, H', C', R') \in \mathfrak{N} ,$$

$G(\mathcal{P}')$ *is an embedding which maps the zero set of* $\Gamma(\mathcal{P}')$ *into the set of solutions of* $\mathcal{P}'$.

Proof. Let U be a relatively compact neighborhood of Σ in $\mathfrak{D}_H^C$ such that $\widetilde{F}_H^C$ is an embedding on a neighborhood of U. By the definition of non-degenerate solution manifold, $\widetilde{F}_H^C(U)$ and R intersect cleanly along $\widetilde{F}_H^C(\Sigma)$. There exists a neighborhood $\mathfrak{N}_1$ of $\mathcal{P}$ in $\mathcal{B}(P, \Omega)$ and a neighborhood $\tilde{\mathcal{E}}$ of the inclusion in $C^\infty(\widetilde{F}_H^C(\Sigma), P \times P)$ such that, for

$$\mathcal{P}' = (P, \Omega, H', C', R')$$

in $\mathfrak{N}_1$ and $j \in \tilde{\mathcal{E}}$, the following holds: there is an open subset $U' \subseteq \mathfrak{D}_{H'}^{C'}$, near U, such that $\widetilde{F}_{H'}^{C'}$ is an embedding on U' whose range contains $j(\widetilde{F}_H^C(\Sigma))$; the mapping $\mathcal{P}' \mapsto \widetilde{F}_{H'}^{C'}(U')$ from $\mathfrak{N}_1$ to $\mathcal{L}(P \times P, \Omega_x)$ is continuous in the coarse C^1 topologies; and $(\widetilde{F}_{H'}^{C'})^{-1} \circ j \circ \widetilde{F}_H^C \colon \Sigma \to P \times \mathcal{R}$ lies in $\mathcal{E}$. All this follows from several applications of the implicit function theorem and the fact that F_H in the C^1 topology depends continuously upon H in the C^2 topology.

Applying Theorem 4.4, we can find a coarse C^1 neighborhood $\tilde{\mathfrak{N}}_1 \times \tilde{\mathfrak{N}}_2$ of $(\widetilde{F}_H^C(U), R)$ in $\mathcal{L}(P \times P, \Omega_x) \times \mathcal{L}(P \times P, \Omega_x)$ and mappings $\tilde{G} \colon \tilde{\mathfrak{N}}_1 \times \tilde{\mathfrak{N}}_2 \to \tilde{\mathcal{E}}$ and $\tilde{\Gamma} \colon \tilde{\mathfrak{N}}_1 \times \tilde{\mathfrak{N}}_2 \to Z^1(F_H^C(\Sigma))$ such that, for each pair (L', R') in $\tilde{\mathfrak{N}}_1 \times \tilde{\mathfrak{N}}_2$, $\tilde{G}(L', R')$ is an embedding which maps the zero set of $\tilde{\Gamma}(L', R')$ into the intersection $L' \cap R'$. Now there is a neighborhood $\mathfrak{N}$ of $\mathcal{P}$ in $\mathfrak{N}_1$ such that, for

$$\mathcal{P}' = (P, \Omega, H', C', R') \in \mathfrak{N} ,$$

the pair $(\widetilde{F}_{H'}^{C'}(U'), R')$ lies in $\tilde{\mathfrak{N}}_1 \times \tilde{\mathfrak{N}}_2$. Define $G \colon \mathfrak{N} \to C^\infty(\Sigma, P \times R)$ by

$$G(\mathcal{P}') = (\widetilde{F}_{H'}^{C'})^{-1} \circ \tilde{G}(\widetilde{F}_{H'}^{C'}(U'), R')$$

and define $\Gamma \colon \mathfrak{N} \to Z^1(\Sigma)$ by

$$\Gamma(\mathcal{P}') = (\widetilde{F}_H^C)^* \tilde{\Gamma}(\widetilde{F}_{H'}^{C'}(U'), R') .$$

It is now straightforward to verify that $\mathfrak{N}$, $\mathcal{G}$, and Γ have the required properties. Q.E.D.

Theorem 8.4 gains power from the fact that we can prove in some quite general situations that the closed 1-form $\Gamma(\mathcal{P}')$ is exact. In this case, we can apply critical point theory to estimate the number of solutions for the perturbed problem $\mathcal{P}'$ in terms of the topology of Σ.

PROPOSITION 8.5. *With notation as in Theorem* 8.4, *suppose there exists* *a 1-form ω_x on $P \times P$ such that $d\omega_x = -\Omega_x$ and ω_x pulls back to zero on Δ_P and R'. Suppose, in addition, that C is a curve of constant time or constant energy. Then the closed 1-form $\Gamma(\mathcal{P}')$ is exact.*

Proof. By the discussion in Section 5, there are mappings $\beta_1: \Sigma \to \tilde{F}_{H'}^{C'}(\mathfrak{D}_{H'}^{C'})$ and $\beta_2: \Sigma \to R'$ such that $\Gamma(\mathcal{P}')$ is cohomologous to $\beta_2^*\omega_x - \beta_1^*\omega_x$. The first term is zero because ω_x pulls back to zero on R'. As for β_1, it is homotopic through mappings $\Sigma \to \tilde{F}_{H'}^{C'}(\mathfrak{D}_{H'}^{C'})$ to $\tilde{F}_H \mid \Sigma$, which we call β. Since ω_x is closed on $\tilde{F}_{H'}^{C'}(\mathfrak{D}_{H'}^{C'})$, the homotopy invariance of induced mappings on cohomology implies that $\beta_1^*\omega_x$ is cohomologous to $\beta^*\omega_x$. Now, for each $(p, t) \in \Sigma$, $\beta(p, t) = \tilde{F}_H(p, t)$. Define $\gamma_s: \Sigma \to P \times P$ by $\gamma_s(p, t) = \tilde{F}_H(p, st)$. Then

$$\dot{\gamma}_s(p, t) = \big(0, t \cdot \xi_H[F_H(p, st)]\big) ,$$

and $\gamma_s^*(\dot{\gamma}_s \lrcorner \Omega_x) = -\tau \cdot d[H \circ \pi]$. Now, if C is a curve of constant time, $\gamma_s^*(\dot{\gamma}_s \lrcorner \Omega_x) = -d[\tau \cdot H \circ \pi]$. If C is a curve of constant energy $\gamma_s^*(\dot{\gamma}_s \lrcorner \Omega_x) = 0$. In either case, $\gamma_s^*(\dot{\gamma}_s \lrcorner \Omega_x)$ is exact, so, by Lemma 6.1 (a), $\beta^*\omega_x$ is cohomologous to $\gamma_0^*\omega_x$. But γ_0 maps Σ into Δ_P, on which ω_x is zero. Q.E.D.

The hypothesis of Proposition 8.5 is satisfied if there is a 1-form ω on P such that $d\omega = -\Omega$ and R' is either Δ_p or $L_1 \times L_2$, where L_1 and L_2 are lagrangian submanifolds of P on which ω pulls back to zero. This is true, for example, if $P = T^*M$ and L_1 and L_2 are homogeneous, i.e., invariant under scalar multiplication. (See [13].) In particular, the normal bundles of submanifolds of M are homogeneous.

An application of Theorem 8.4 and Proposition 8.5 to the three-body problem may be found in [4].

9. Reversible systems

If the symmetry group is not trivial, we can often use its presence to improve our estimate of the number of solutions of a canonical boundary value problem. Even more interesting is the fact that we can sometimes introduce a symmetry group where it does not exist in the original problem.

Consider, for example, the case of a *classical mechanical system*

$$(T^*M, \Omega_M, V \circ \pi_M + K) ,$$

where the *potential energy* V is a function on M and the *kinetic energy* K is homogeneous of degree two on each fibre of T^*M. Denote by $\mu: T^*M \to T^*M$ the mapping which multiplies each cotangent vector by -1. The hamiltonian function $V \circ \pi_M + K$ is invariant under the action of $\mathbf{Z}_2$ on T^*M generated by μ, but μ is not an automorphism of (T^*M, Ω_M). In fact, it is easy to verify that $\mu^*\Omega_M = -\Omega_M$. We can obtain from μ a symplectic automorphism, not of (T^*M, Ω_M), but of $(T^*M \times T^*M, (\Omega_M)_x)$, by defining $\hat{\mu}: T^*M \times T^*M \to T^*M \times T^*M$ by $\hat{\mu}(x, y) = (\mu y, \mu x)$.

In general, if (P, Ω) is a symplectic manifold, a mapping $\mu: P \to P$ is called an *antisymplectic automorphism*, or *anticanonical transformation*, if $\mu^*\Omega = -\Omega$. The product of any two anticanonical transformations is a canonical transformation.

An important example of an anticanonical transformation is the *exchange* transformation σ_P, defined on a cartesian product $(P \times P, \Omega_x)$ by the rule $\sigma_P(x, y) = (y, x)$. If there is already an anticanonical transformation μ of (P, Ω), then $\mu \times \mu$ is an anticanonical transformation of $(P \times P, \Omega_x)$ which commutes with σ_P, and the composition $(\mu \times \mu)\circ\sigma_P$ is a canonical transformation of $(P \times P, \Omega_x)$.

A hamiltonian system (P, Ω, H), together with an anticanonical involution μ (i.e., $\mu^2 =$ identity) of (P, Ω) leaving H invariant will be called a *reversible hamiltonian system*. The hamiltonian vector field ξ_H has the property $T\mu\circ\xi_H = -\xi_H\circ\mu$. Thus, if $t \mapsto \sigma(t)$ is an integral curve of ξ_H, so is $t \mapsto \mu\circ\sigma(-t)$. (In the case of a classical mechanical system this fact expresses the well known reversibility in time of the equations of motion.)

Suppose that $F_H(p, t) = q$. Then $F_H(\mu(p), -t) = \mu(q)$, and $F_H(\mu(q), t) = \mu(p)$. In other words, if $\tilde{F}_H(p, t) = (p, q)$, then

$$\tilde{F}_H(\mu(q), t) = (\mu(q), \mu(p)) = [(\mu \times \mu)\circ\sigma_P](p, q) .$$

If C is any curve in $\mathcal{R}^2$, then $(p, t) \in \mathfrak{D}_H^C$ if and only if $(\mu(p), t) \in \mathfrak{D}_H^C$, because μ leaves H invariant; hence, the image $\tilde{F}_H^C(\mathfrak{D}_H^C)$ is invariant under $(\mu \times \mu)\circ\sigma_P$.

Assuming that the symmetry group is initially trivial, we may now introduce a symmetry group $\mathbf{Z}_2$ in the following manner. (If the symmetry group is initially G, it becomes $G \times \mathbf{Z}_2$.) The generator of $\mathbf{Z}_2$ acts on $P \times P$ by the involution $(\mu \times \mu)\circ\sigma_P$. The generator acts on $P \times \mathcal{R}$ by the involution $(p, t) \mapsto (\mu F_H(p, t), t)$. We have seen that $\mathfrak{D}_H^C$ is invariant under $\mathbf{Z}_2$ and that $\tilde{F}_H^C$ is equivariant.

We now define a *reversible canonical boundary value problem* as a sextuple $(P, \Omega, H, C, R, \mu)$, where (P, Ω, H, μ) is a reversible hamiltonian system, (P, Ω, H, C, R) is a canonical boundary value problem, and R is invariant under $(\mu \times \mu)\circ\sigma_P$. As examples of R, we can take $R = \Delta_P$, or, if L_1 and L_2 are lagrangian submanifolds of (P, Ω) such that $\mu(L_1) = L_2$, we can take $R = L_1 \times L_2$.

A solution manifold Σ of $(P, \Omega, H, C, R, \mu)$ will be called reversible if it is a $\mathbf{Z}_2$-invariant subset of $\mathfrak{D}_H^C$. Using the $\mathbf{Z}_2$ symmetry in applying Theorem 4.4, we have the following result.

THEOREM 9.1. *In Theorem* 8.4, *if* $(P, \Omega, H, C, R, \mu)$ *and* $(P, \Omega, H^1, C^1, R^1, \mu)$ *are reversible, then the form* $\Gamma(\mathcal{P}^1) \in Z^1(\Sigma)$ *can be chosen to be invariant under the* $\mathbf{Z}_2$ *action.*

Theorem 9.1 enables us to get much better estimates for the size of the zero set of $\Gamma(\mathcal{P}^1)$. For instance, if Σ is S^n, its category is only 2. If $\mathbf{Z}_2$ acts freely on S^n, then the quotient space has category $n+1$, so any $\mathbf{Z}_2$-invariant function has at least $n+1$ critical points.

It is remarkable that to any hamiltonian system we can associate a reversible system. In this way, we can prove the existence of more solutions to certain boundary value problems. The next section is devoted to a study of this construction.

10. The reversible square of a hamiltonian system

Let (P, Ω, H) be a hamiltonian system. On $(P \times P, \Omega_x)$, with the exchange transformation σ_P, we may consider the σ_P-invariant function $H_x = (1/2)(H \circ \pi_1 + H \circ \pi_2)$. $(P \times P, \Omega_x, H_x, \sigma_P)$ is a reversible hamiltonian system whose properties reflect those of (P, Ω, H).

At a point $(p, q) \in P \times P$, the value $\xi_{H_x}(p, q)$ of the hamiltonian vector field associated with H_x is simply $\big((1/2)\xi_H(p), -(1/2)\xi_H(q)\big)$, where $T_{(p,q)}(P \times P)$ is identified with $T_pP \times T_qP$. It follows that $(p, q, t) \in \mathfrak{D}_{H_x}$ if and only if $\big(p, (1/2)t\big)$ and $\big(q, -(1/2)t\big)$ are in $\mathfrak{D}_H$, in which case we have

$$F_{H_x}(p, q, t) = \left(F_H\left(p, \frac{1}{2}t\right), F_H\left(q, -\frac{1}{2}t\right)\right) .$$

Notice that $q = F_H(p, t)$ if and only if $F_H\big(p, (1/2)t\big) = F_H\big(q, -(1/2)t\big)$; in other words, $q = F_H(p, t)$ if and only if $F_{H_x}(p, q, t)$ lies on the diagonal Δ_P.

Let R be any canonical relation on P. Whenever $\widetilde{F}_H(p, t) = (p, q)$ lies in R, we have $F_{H_x}(p, q, t) \in \Delta_P$. In this circumstance, since $H(q) = H(p)$, $H_x(p, q)$ equals $H(p)$, so for any curve C in $\mathcal{R}^2$, $\big(H_x(p, q), t\big) \in C$ if and only if $\big(H(p), t\big) \in C$.

In other words, there is a 1–1 correspondence between solutions of the canonical boundary value problems $\mathcal{P} = (P, \Omega, H, C, R)$ and

$$\mathcal{P}_x = (P \times P, \Omega_x, H_x, C, R \times \Delta_P) .$$

In fact, one can check that a solution manifold of $\mathcal{P}$ is non-degenerate if and only if the corresponding solution manifold for $\mathcal{P}_x$ is non-degenerate.

It is of obvious interest to determine when

$$(P \times P, \Omega_x, H_x, C, R \times \Delta_P, \sigma_P)$$

is a reversible problem. We must check when $R \times \Delta_P$ is invariant under the involution $(\sigma_p \times \sigma_p)\circ\sigma_{p\times p}$ of $(P \times P) \times (P \times P)$. We have

$$[(\sigma_P \times \sigma_P)\circ\sigma_{P\times P}](R \times \Delta_P) = (\sigma_P \times \sigma_P)(\Delta_P \times R) = \Delta_P \times \sigma_P(R) \; .$$

This is equal to $R \times \Delta_P$ if and only if $R = \Delta_P$; i.e., if and only if the original boundary value problem $\mathcal{P}$ was a problem of finding periodic orbits.

Let Σ be a solution manifold of the boundary value problem $\mathcal{P} = (P, \Omega, H, C, \Delta_P)$. We will now determine when the corresponding solution manifold Σ_x of $\mathcal{P}_x$ is reversible, and we will describe the action of $\mathbf{Z}_2$ on Σ_x.

Let $(p, t) \in \Sigma$. Then $F_H(p, t) = p$, and the corresponding point of Σ_x is (p, p, t). The generator of $\mathbf{Z}_2$ takes

$$\begin{aligned}
&(p, p, t) \text{ to } \big((\sigma_P \times \sigma_P)(p, p, t), t\big) \\
&= \Big((\sigma_P \times \sigma_P)\Big(F_H\Big(p, \frac{1}{2}t\Big), F_H\Big(p, -\frac{1}{2}t\Big)\Big), t\Big) \\
&= \Big(F_H\Big(p, -\frac{1}{2}t\Big), F_H\Big(p, \frac{1}{2}t\Big), t\Big) \\
&= (q, q, t) \; ,
\end{aligned}$$

where q is the point halfway around the closed orbit from p. We have remarked in Section 8 that, if Σ is non-degenerate, $\pi(\Sigma)$ contains the entire periodic orbit of each of its points.

The following theorem represents the application of Theorem 9.1 to this situation.

THEOREM 10.1. *In Theorem* 8.4, *if R and R' are equal to Δ_P, then the form $\Gamma(\mathcal{P}')$ can be chosen to be invariant under the involution* $(p, t) \mapsto (F_H(p, (1/2)t), t)$.

Notice that, by Proposition 8.5, if C is a curve of constant time or energy, the form $\Gamma(\mathcal{P}')$ is exact if Ω is exact. In fact, we only need Ω to be exact on a neighborhood of $\pi(\Sigma) \subseteq P$, because we can replace P by that neighborhood.

For example, we may apply Theorem 10.1 to the case where Σ is an n-sphere and C' is a curve of constant energy. Since the transversality condition requires that ξ_H be nowhere-vanishing on Σ, n must be odd. Now $H^2(S^n; \mathcal{R}) = 0$ for odd n, so Ω is exact on a tubular neighborhood of $\pi(\Sigma)$. If we write $\Gamma(\mathcal{P}') = df$, the critical point set of f is a disjoint union of embedded circles which correspond to periodic orbits of (P, Ω, H'). The function f is invariant under the involution $\partial: (p, t) \mapsto (F_H(p, (1/2)t), t)$. For each $(p, t) \in \Sigma$, the least period of p is of the form $t/k(p)$, where $k(p)$ is a positive integer. If $k(p)$ is odd for all p, the involution ∂ is fixed point-free, and we may consider f as a function on $\bar{f}$, the manifold $\Sigma/\mathbf{Z}_2$. The quotient manifold has the cohomology ring of real projective n-space and, hence, has category $n + 1$ [16]. Since the category of a circle is 2 it follows from the theorem in § 4 of [16] (see also [24]) that if the critical set of $\bar{f}$ consists of r circles, than $2r \geqq n + 1$, or $r \geqq (1/2)(n + 1)$.

In the case where $k(p)$ is even for some values of p, it remains an open problem to determine the minimum number of periodic orbits which remain near Σ after the hamiltonian function is perturbed. One approach to this problem would be to study, in addition to Σ, the submanifolds of Σ corresponding to different values of $k(p)$. These, like Σ, are non-degenerate solution manifolds for the problem of finding periodic orbits with prescribed energy.

By ignoring the $\mathbf{Z}_2$ action, we can obtain some weaker results without assuming that the $k(p)$ are all odd. As long as $n \geqq 1$, the compactness of Σ implies that f must have at least one critical circle. The circles of maxima and minima of f cannot coincide unless f is constant, so there are at least two critical circles if $n \geqq 3$. Finally, if there are exactly two critical circles, the complement of a tubular neighborhood of the circle of maxima can be deformed along the gradient lines of f into a tubular neighborhood of the circle of minima. If $n \geqq 5$, this is impossible because the complement of a circle in S^n is simply connected, so there must be at least three critical circles. Unfortunately, this line of argument stops at $n \geqq 5$. In fact, some preliminary work suggests that there exists a function on S^7 whose critical point set consists of three circles. Further progress on this problem must depend, therefore, on special properties of the function f.

The preceding results on spheres of periodic orbits can be applied to hamiltonian systems near an equilibrium point. Let (P, Ω, H) be a hamiltonian system and let $p \in P$ be a critical point of H. We may assume that $H(p) = 0$. Suppose that there exist canonical coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ around p such that

$$H = \frac{1}{2} \sum_{i=1}^{l} \lambda_i (x_i^2 + y_i^2) + H_2 + H_* \ ,$$

where H_2 is a quadratic form in $(x_{l+1}, \cdots, x_n, y_{l+1}, \cdots, y_n)$, and H_* vanishes at the origin together with all its partial derivatives of first and second order. If we write $(z_1, \cdots, z_{2(n-l)})$ for $(x_{l+1}, \cdots, x_n, y_{l+1}, \cdots, y_n)$, the hamiltonian vector field ξ_H is

$$\sum_{i=1}^{l} \lambda_i\left(y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i}\right)$$
$$+ \sum_{i,j=1}^{2(n-l)} a_{ij}z_i\frac{\partial}{\partial z_j} + \sum_{i=1}^{n}\left(b_i\frac{\partial}{\partial x_i} + c_i\frac{\partial}{\partial y_i}\right),$$

where $A = (a_{ij})$ is a matrix of constants and the b_i's and c_j's are functions which vanish at the origin together with their partial derivatives of first order.

Now we make the further assumption that each λ_i is of the form $k_i\lambda$, where λ is a positive real number, each k_i is a positive integer, $k_1 = 1$, and no eigenvalue of A is an integral multiple of $\sqrt{-1}\cdot\lambda$. It is easy to verify, now, that the canonical boundary value problem $(P, \Omega, H - H_*$, energy $=$ $1, \Delta_p)$ has as non-degenerate solution manifold the $2l - 1$ dimensional ellipsoid

$$\Sigma = \left\{(x, y, t) \,\middle|\, \frac{1}{2}\sum_{i=1}^{l} \lambda_i(x_i^2 + y_i^2) = 1,\ x_{l+1} = \cdots = x_n \right.$$
$$\left. = y_{l+1} = \cdots = y_n = 0\,, \qquad t = \frac{2\pi}{\lambda}\right\}.$$

Now define the functions $\mathcal{K}_\varepsilon$, for $\varepsilon \neq 0$, by $\mathcal{K}_\varepsilon(x, y) = (1/\varepsilon^2)H(\varepsilon x, \varepsilon y)$. The quadratic homogeneity of the first two terms of the decomposition of H implies that

$$\mathcal{K}_\varepsilon(x, y) = (H - H_*)(x, y) + \frac{1}{\varepsilon^2}H_*(\varepsilon x, \varepsilon y)\ .$$

Since H_* vanishes to third order in x and y, $(1/\varepsilon^2)H_*(\varepsilon x, \varepsilon y)$ and $\mathcal{K}_\varepsilon(x, y)$, as functions of ε, x, and y, extend smoothly over the hypersurface $\varepsilon = 0$ in such a way that $\mathcal{K}_0(x, y) = (H - H_*)(x, y)$.

By the previous discussion on spherical periodic manifolds, we conclude that, for ε sufficiently close to zero, the boundary value problem $(P, \Omega, \mathcal{K}_\varepsilon$, energy $= 1, \Delta_P)$ has at least $(1/2)(2l - 1 + 1) = l$ circles of solutions, provided that all the integers k_i are odd. Since the transformation $(x, y) \mapsto (\varepsilon x, \varepsilon y)$ multiplies Ω by ε^2, it is easy to see that these l circles of solutions give l circles of solutions to the problem $(P, \Omega, H$, energy $= \varepsilon^2, \Delta_P)$. In other words, there are at least l periodic orbits for (P, H, Ω) on each level surface of H near the equilibrium point p.

If the k_i are not all odd, we must content ourselves with the weaker result that there is at least 1 periodic orbit on each level surface if $l \geqq 1$, at least 2 if $l \geqq 2$, and at least 3 if $l \geqq 3$.

In the case $l = 1$, our result is due essentially to Liapounov [15] and Horn [14]; see also §§ 13–15 of [30] and Appendix C of [1]. For results in the case $l > 1$, not altogether encompassed by those in the present paper, see [5], [9], [19], [21], [27], [28], [29].

Added December 21, 1972. The author [37] has recently succeeded in eliminating the hypothesis above that all the k_i be odd. Thus, there are always l periodic orbits on each level surface near an equilibrium at which the hamiltonian is positive definite. The situation in the indefinite case is more complicated. See [27] for some theorems and examples.

11. On the method of averaging

Consider a family $\mathcal{P}_\varepsilon = (P, \Omega, H_\varepsilon, C, R)$ of canonical boundary problems where ε ranges over an interval J containing 0. If Σ is a non-degenerate solution manifold for $\mathcal{P}_0$, we have, by Theorem 9.4, 1-forms $\Gamma(\mathcal{P}_\varepsilon) \in Z^1(\Sigma)$ whose zeros correspond to solutions of $\mathcal{P}_\varepsilon$. $\Gamma(\mathcal{P}_0)$ is the zero form, and $(d/d\varepsilon)\Gamma(\mathcal{P}_\varepsilon)|_{\varepsilon=0}$ is given by Theorem 6.4. The importance of this derivative is shown by Theorem 6.5.

According to Theorem 6.4 and the proof of Theorem 8.4, $(d/d\varepsilon)\Gamma(\mathcal{P}_\varepsilon)|_{\varepsilon=0}$ is equal to $-i_\Sigma^* D_0 \mathrm{L}$, where $L_\varepsilon = \widetilde{F}^C_{H_\varepsilon}(\mathfrak{D}^C_{H_\varepsilon})$. Given $(p, t) \in \Sigma$, to compute $(d/d\varepsilon)\Gamma(\mathcal{P}_\varepsilon)|_{\varepsilon=0}$ at (p, t) we must find a curve $(p_\varepsilon, t_\varepsilon) \in \mathfrak{D}^C_{H_\varepsilon}$, find a tangent vector

$$\eta = \frac{d}{d\varepsilon}\big(\widetilde{F}_{H_\varepsilon}(p_\varepsilon, t_\varepsilon)\big)\big|_{\varepsilon=0} ,$$

and take $F^*_{H_0}(\eta \lrcorner \Omega_x)$.

The vector η can be broken into three parts coming from the derivatives of $\widetilde{F}_{H_\varepsilon}$, p_ε, and t_ε with respect to ε. The contribution from $dp_\varepsilon/d\varepsilon$ is zero, because $\Omega_x = \pi_1^*\Omega - \pi_2^*\Omega$, and the two terms cancel. If we write $\theta = dt_\varepsilon/d\varepsilon|_{\varepsilon=0}$, a function on Σ, then the contribution from θ is equal to $\theta \cdot d(H\circ\pi)$. This contribution will be zero if C is a curve of constant energy or constant time. Finally, the contribution from the variation in $\widetilde{F}_{H_\varepsilon}$ is, as in Moser [21], dA, where

$$A(p, t) = \int_0^t \frac{dH_\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0}\big(F_{H_0}(p, s)\big)ds .$$

Applying Theorem 6.5, we have the following result, which generalizes work of Reeb [26] and Moser [21].

THEOREM 11.1. *Let $\mathcal{P}_\varepsilon = (P, \Omega, H_\varepsilon, C, R)$ be a family of canonical boundary value problems, where C is a curve of constant time or constant energy. Let Σ be a non-degenerate solution manifold of $\mathcal{P}_0$. Let Σ_0 be a compact, weakly stable component of the zero set of dA, where $A: \Sigma \to \mathcal{R}$ is defined by*

$$A(p, t) = \int_0^t \frac{dH_\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0}(F_{H_0}(p, s))ds .$$

Then, given any neighborhood U of Σ_0 in $P \times R$, the problem $\mathcal{P}_\varepsilon$ has a solution in U for all ε sufficiently near 0.

Remarks. Σ_0 is a weakly stable component of the zero set of dA if the type numbers [20] of Σ_0 as a critical set of A are not all zero. In particular, if Σ_0 is a non-degenerate critical manifold of A, this condition is satisfied. If $R = \Delta_p$ and Σ is simply fibred by periodic orbits, the function A is constant on each orbit, so it comes from a function $\bar{A}$ on the orbit manifold. If an orbit is a non-degenerate critical point for $\bar{A}$, it is a non-degenerate critical manifold for A.

The importance of Theorem 11.1 lies in the fact that the computation of A requires only the integration of $dH_\varepsilon/d\varepsilon$ along orbits of (P, Ω, H_0), while the computation of $\Gamma(\mathcal{P}_\varepsilon)$ requires one to integrate the vector field ξ_{H_ε}. The first task is often a relatively simple one (see [21]), while the second may be hopeless.

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