

# THE METHOD OF EQUIVALENCE APPLIED TO THREE STATE, TWO INPUT CONTROL SYSTEMS

George R. Wilkens  
 Department of Mathematics  
 University of Hawaii at Manoa  
 Honolulu, HI 96822

## Abstract

The feedback equivalence of three state, two input non-linear control systems is analyzed using Cartan's method of equivalence. The control linear and linear equivalents are classified, including examples of control linear systems which are inequivalent to linear system. Existence of time critical closed loop controls is demonstrated for "fully non-linear" control systems. The existence of natural contact structures and Riemannian structures is demonstrated for certain classes of systems.

## 1. Introduction

The study of geometric control theory took a new direction in 1982 when, for the first time, R. Gardner applied the method of equivalence to it [1]. The method itself, along with many examples, including some applications to control theory, was skillfully presented by R. Gardner at the 1987 CBMS-NSF conference at Texas Tech [2]. Since 1982, numerous applications of the equivalence method have been made to geometric control, and important results have been obtained, including:

- (i) the existence of time optimal closed loop feedbacks [3]
- (ii) the classification of control linear and linear systems [4,5]
- (iii) an optimal algorithm for putting non-linear, linearizable systems into Brunowski normal form [6,7,8].

We will apply the method of equivalence to three state, two input systems. We will completely classify the control linear and linear systems. We will find special geometries attached to the nonlinearizable systems. A natural variational problem is attached to all systems, and we will see an application of it to give time critical closed loop controls.

## 2. Control Systems and Feedback Equivalence

The solution curves of the control system

$$\frac{dx}{dt} = f(x, u) \quad x \in \mathbf{R}^3, u \in \mathbf{R}^2 \quad (1)$$

are equivalent to the integral curves of the Pfaffian system  $\{dx^i - f^i(x, u) dt\}_{i=1}^3$ . (We use the notation  $\{\omega^i\}_{i=1}^n$  to denote the module generated by the linear

span of  $\omega^1, \dots, \omega^n$ .) A feedback transformation is a diffeomorphism  $(\bar{t}, \bar{x}, \bar{u}) = \Phi(t, x, u)$  such that  $\bar{t} = t$ ,  $\bar{x} = \phi(x)$  and  $\bar{u} = \psi(x, u)$ . The feedback transformations act on the control system producing a new system

$$\frac{d\bar{x}}{d\bar{t}} = \bar{f}(\bar{x}, \bar{u}). \quad (2)$$

Systems (1) and (2) are considered to be feedback equivalent, since they are related by a feedback transformation. This notion naturally suggests the inverse problem: given systems (1) and (2), determine if there is a feedback transformation which relates them. Thus we must find a diffeomorphism  $\Phi$  such that

(S') integral curves of (1) map to integral curves of (2)

(F')  $\Phi$  is a feedback transformation.

These two conditions are equivalent to

(S)  $\Phi^*\{d\bar{x}^i - \bar{f}^i d\bar{t}\}_{i=1}^3 = \{dx^i - f^i dt\}_{i=1}^3$

(F)  $\bar{t} = t$ ,  $\Phi^*\{d\bar{x}^i\}_{i=1}^3 = \{dx^i\}_{i=1}^3$  and  $\Phi^*\{d\bar{u}^s\}_{s=1}^2 \equiv 0 \pmod{\{dx^i, du^s\}}$

Conditions (S) and (F) together imply

(I)  $\Phi^*\{d\bar{x}^i - \bar{f}^i d\bar{t}\} \cap \{d\bar{x}^i\} = \{dx^i - f^i dt\} \cap \{dx^i\}$ .

We can find generators for these intersections by choosing  $GL(3)$  valued functions  $\bar{A}_0$  and  $A_0$  such that

$$\bar{A}_0(\bar{x}, \bar{u}) \bar{f}(\bar{x}, \bar{u}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A_0(x, u) f(x, u).$$

Typically,  $A_0$  will be defined on an open set  $U \subset \mathbf{R}^3 \times \mathbf{R}^2$  and  $\bar{A}_0$  will be defined on another open set  $V \subset \mathbf{R}^3 \times \mathbf{R}^2$ . Letting  $\eta_U = A_0 dx$  and  $\bar{\eta}_V = \bar{A}_0 d\bar{x}$  we have  $\{dx^i - f^i dt\} \cap \{dx^i\} = \{\eta_U^1 - dt, \eta_U^2, \eta_U^3\} \cap \{\eta_U^1, \eta_U^2, \eta_U^3\} = \{\eta_U^2, \eta_U^3\}$ . Similarly,  $\{d\bar{x}^i - \bar{f}^i d\bar{t}\} \cap \{d\bar{x}^i\} = \{\bar{\eta}_V^2, \bar{\eta}_V^3\}$ . Finally, condition (S) together with the fact that  $\Phi^*(d\bar{t}) = dt$  implies that  $\Phi^*(\bar{\eta}_V^1) \equiv \eta_U^1 \pmod{\{\eta_U^2, \eta_U^3\}}$ . Together, these observations imply that the control system (1) is feedback equivalent to system (2) if and only if there is a diffeomorphism  $\Phi$  such that  $\bar{t} = t$  and

$$\Phi^* \begin{pmatrix} d\bar{t} \\ \bar{\eta}_V \\ d\bar{u} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & B & C \end{pmatrix} \begin{pmatrix} dt \\ \eta_U \\ du \end{pmatrix}, \quad (3)$$

where  $A^t(1, 0, 0) = {}^t(1, 0, 0)$ . Since  $\bar{\eta}_V$  is independent of  $\bar{t}$  and  $\eta_U$  is independent of  $t$ , we see that (3) is

equivalent to

$$\Phi^* \begin{pmatrix} \bar{\eta}_V \\ d\bar{u} \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} \eta_U \\ du \end{pmatrix}.$$

The condition of feedback equivalence of two systems is therefore reduced to the condition that the derivative of a diffeomorphism  $\Phi$  belongs to the group

$$G = \left\{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \in GL(5) \mid A^t(1,0,0) = {}^t(1,0,0) \right\}.$$

The problem of feedback equivalence is now adapted to the method of equivalence, and we may proceed with its application. We introduce on the set  $U \times G$  the column vector of 1-forms

$$\begin{pmatrix} \eta \\ \mu \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} \eta_U \\ du \end{pmatrix},$$

which form a feedback-invariant, independent set of 1-forms. We use the method of equivalence to derive additional feedback invariants, which we then use to classify the control systems.

### 3. Summary of Results

Given a control system (1) and a suitably chosen  $GL(3)$  valued function  $A_0(x, u)$  with  $A_0 f = {}^t(1, 0, 0)$ , the first major invariant we will uncover is the rank of the bottom two rows of the matrix  $A_0 \partial f / \partial u$ . This  $2 \times 2$  submatrix will either have rank 1 or rank 2. (We are assuming that the rank of  $\partial f / \partial u = 2$ , since otherwise we could eliminate one of the controls by a feedback transformation.) These two cases have a nice geometric interpretation. If we view the control system as defining a two dimensional surface in each of the tangent spaces to  $\mathbf{R}^3$ , then the rank 1 case is equivalent to saying that in each tangent space, the surface determined by  $f$  is ruled by the rays emanating from the origin. For each of these major cases, we get a classification theorem.

It is easy to check that, by construction, the integral curves of the control system coincide with the integral curves of the Pfaffian system  $\{\eta^2, \eta^3\}$  for which  $\eta^1 \neq 0$ , and that the integral  $\int \eta^1$  over a solution curve of the control system gives the time taken to traverse the curve. We see that the variational problem of finding time optimal curves among all solutions curves to (1) is naturally contained in this problem. The Euler-Lagrange equations for this variational problem will appear in a natural way and we will be able to show that for ‘‘generic’’ control systems, there exist time critical closed loop controls. We summarize these results in the following theorems.

**Theorem 1.** *Given a control system (1) and a  $GL(3)$  valued matrix  $A_0(x, u)$  such that  $A_0 f = {}^t(1, 0, 0)$ , then the rank of the bottom 2 rows of the  $3 \times 2$  matrix*

*$A_0 \partial f / \partial u$  is a feedback invariant. If the rank of the last 2 rows of the matrix is 1 then we can construct an invariant set of independent 1-forms on a higher space such that the structure of the 2-form*

$$d\eta^3 = \alpha_3^3 \wedge \eta^3 + P \eta^1 \wedge \eta^2 + Q \mu^2 \wedge \eta^2$$

*determines 3 cases. They are:*

- I. *If  $P = Q = 0$  then the system is locally equivalent to the linear normal form*

$$\frac{dx}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}.$$

- II. *If  $P \neq 0, Q = 0$  then the system is locally equivalent to the control linear normal form*

$$\frac{dx}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x^1 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}.$$

- III. *If  $Q \neq 0$  then there is an invariant basis of 1-forms on a higher space, and the structure equations of this basis determine the equivalence class.*

It is worth mentioning that case II in the above theorem is a control linear system. We will check that no linear control system satisfies the conditions of case II; thus we have an example of a control linear system that is not feedback linearizable. The next theorem covers the remaining case.

**Theorem 2.** *Let (1) be a control system as in the statement of theorem 1 except that we now assume that the rank of the bottom 2 rows of the  $3 \times 2$  matrix  $A_0 \partial f / \partial u$  is 2. Then we can construct an invariant set of independent 1-forms on a higher space such that the structure of the 2-form*

$$d\eta^1 = P \eta^1 \wedge \eta^2 + Q \eta^2 \wedge \eta^3 + R \eta^1 \wedge \eta^3 + E \mu^1 \wedge \eta^2 + F \mu^2 \wedge \eta^2 + F \mu^1 \wedge \eta^3 + G \mu^2 \wedge \eta^3$$

*determines 4 cases.*

- I. *If  $d\eta^1 = 0$  then the system is locally equivalent to the linear non-controllable normal form*

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}.$$

- II. *If  $d\eta^1 \neq 0$  and  $d\eta^1 \wedge \eta^1 = 0$  then the system is locally equivalent to the linear controllable normal form*

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}.$$

III. If  $d\eta^1 \wedge \eta^1 \neq 0$ ,  $(d\eta^1)^2 \wedge \eta^1 = 0$  and  $E = F = G = 0$  then the system is locally equivalent to the control linear normal form

$$\frac{dx}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x^2 & 0 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

IV. If  $(d\eta^1)^2 \wedge \eta^1 \neq 0$  we can construct an invariant basis of 1-forms on a 6 dimensional space and the structure equations of this basis determine the equivalence class.

The control linear form in case III is also not feedback linearizable. From the coefficients of  $d\eta^1$ , we form the symmetric bilinear form

$$E(\eta^2)^2 + 2F\eta^2\eta^3 + G(\eta^3)^2.$$

This bilinear form drops to  $U$ , even though none of the individual terms  $E$ ,  $F$ ,  $G$ ,  $\eta^2$  or  $\eta^3$  drop to  $U$ . The bilinear form is related to the Hessian of the Lagrangian in the variational problem  $\int \eta^1$ , and non-degeneracy of the Hessian is equivalent to nondegeneracy of the bilinear form, occurring only in case IV of theorem 2. If the bilinear form is nondegenerate, then we can normalize it to the form

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix},$$

where  $\epsilon_i = \pm 1$ ,  $i = 1, 2$ , depending on the index of the bilinear form. Notice that nondegeneracy is an open condition, since it can be expressed in terms of the non-vanishing of a determinant, and the condition that the bottom 2 rows of  $A_0 \partial f / \partial u$  be linearly independent is also open. Thus the ‘‘generic’’ control system (1) will satisfy the conditions of case IV in theorem 2. The final theorem applies to this nondegenerate case.

**Theorem 3.** *For every control system (1) that satisfies the conditions of case IV in theorem 2, there exist functions  $h(x)$  such that the integral curves of the differential equation*

$$\frac{dx}{dt} = f(x, h(x))$$

*are critical curves for the variational problem  $\int \eta^1$ . In other words, there exist time critical closed loop feedback functions. Moreover, there is a subclass of these systems for which the invariant quadratic form  $(\eta^1)^2 - \epsilon_1(\eta^2)^2 - \epsilon_2(\eta^3)^2$ , which is defined on the space of states and controls,  $U$ , drops to the 3 dimensional space of states. Thus there is a (possibly pseudo) Riemannian metric on the space of states, and the time critical closed loop feedback curves are geodesics for this metric.*

We see that the ‘‘generic’’ control system (1) has an associated time optimal variational problem and

time critical closed loop feedback functions. We also see a familiar geometry, namely Riemannian geometry, contained in this case. There are non-standard geometries in this case as well, and they can be studied using Cartan’s theory of generalized geometries.

In theorems 1 and 2, there are 3 linear systems and 2 control linear systems. It can be shown that every control linear system (and hence every linear system) must be equivalent to one of these 5 systems. Thus we have a complete classification of all the linear and control linear systems with 3 state and 2 control variables.

#### 4. Proofs of Theorems

We begin the method of equivalence by computing the Lie algebra of the group  $G$  defined in section 2. Letting  $S: G \rightarrow GL(5)$  be the inclusion map, we see that the right invariant Maurer-Cartan form is

$$dSS^{-1} = \begin{pmatrix} \tilde{\alpha} & 0 \\ \tilde{\beta} & \tilde{\gamma} \end{pmatrix},$$

where  $\tilde{\alpha}$ ,  $\tilde{\beta}$  and  $\tilde{\gamma}$  are matrices of right invariant 1-forms on  $G$ . The condition  $A^t(1, 0, 0) = {}^t(1, 0, 0)$  implies that the first column of the matrix  $\tilde{\alpha}$  must be 0. The non-zero entries of  $dSS^{-1}$  form a basis for the right invariant 1-forms on  $G$ . Knowing the shape of  $dSS^{-1}$ , we know that we can write

$$d \begin{pmatrix} \eta \\ \mu \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \wedge \begin{pmatrix} \eta \\ \mu \end{pmatrix} + \begin{pmatrix} T_1 \\ T_2 \end{pmatrix},$$

where the first column of  $\alpha$  is 0, and  $T_1$  and  $T_2$  are column vectors of 2-forms that are quadratic in the  $\eta$ ’s and the  $\mu$ ’s. We can modify the  $\beta$ ’s and the  $\gamma$ ’s so as to make  $T_2 = 0$ , and we can modify the  $\alpha$ ’s so that  $T_1$  has no terms that are linear in  $\{\eta^2, \eta^3\}$ . Thus the first structure equation is

$$d \begin{pmatrix} \eta \\ \mu \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \wedge \begin{pmatrix} \eta \\ \mu \end{pmatrix} + \begin{pmatrix} M\mu \wedge \eta^1 \\ 0 \end{pmatrix}. \quad (4)$$

Equation (4) determines the  $3 \times 2$  matrix of functions  $M$  uniquely. We can get a formula for  $M$  by computing  $d\eta \pmod{\eta^2, \eta^3}$ . Performing this calculation will show that

$$M = -A A_0 \frac{\partial f}{\partial u} C^{-1}.$$

The first obvious invariant is the rank of  $M$ , which is also the rank of  $\partial f / \partial u$ . We will assume that the rank of  $M = 2$ , since otherwise we could eliminate one of the controls by a feedback transformation.

From the explicit formula for  $M$  and because the matrix  $A$  must satisfy the condition  $A^t(1, 0, 0) = {}^t(1, 0, 0)$ , we see that the rank of the last 2 rows of  $M$

is also invariant. Since the rank of  $M = 2$ , we have 2 possibilities.

[1] The rank of the last 2 rows equals 1.

[2] The rank of the last 2 rows equals 2.

We compute the infinitesimal action on  $M$  by using the identity  $d^2\eta = 0$  and equation (4). We get that

$$0 \equiv (dM - \alpha M + M\gamma) \wedge \mu - M\mu \wedge (M^1\mu) \pmod{\eta^1, \eta^2, \eta^3}, \quad (5)$$

where  $M^1$  is the first row of the matrix  $M$ .

**Proof of Theorem 1.** Theorem 1 is about case [1]. In this case we can reduce to the subbundle of  $U \times G$  defined by the equation

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and equation (5) becomes the 2 equations

$$\begin{aligned} 0 &\equiv \left( \gamma - \begin{pmatrix} 0 & \alpha_2^1 \\ 0 & \alpha_2^2 \end{pmatrix} \right) \wedge \mu - \begin{pmatrix} 0 \\ \mu^2 \wedge \mu^1 \end{pmatrix} \\ 0 &\equiv \alpha_2^3 \wedge \mu^2. \end{aligned} \pmod{\eta^1, \eta^2, \eta^3}$$

The structure equations (4) can now be written in the form

$$d \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} = \begin{pmatrix} \alpha_2^1 & \alpha_3^1 \\ \alpha_2^2 & \alpha_3^2 \end{pmatrix} \wedge \begin{pmatrix} \eta^2 \\ \eta^3 \end{pmatrix} + \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix} \wedge \eta^1 \quad (6)$$

$$d\eta^3 = \alpha_3^3 \wedge \eta^3 + P\eta^1 \wedge \eta^2 + Q\mu^2 \wedge \eta^2 \quad (7)$$

$$d\mu = \beta \wedge \eta + \begin{pmatrix} 0 & \alpha_2^1 \\ 0 & \alpha_2^2 \end{pmatrix} \wedge \mu + \begin{pmatrix} 0 \\ \mu^2 \wedge \mu^1 \end{pmatrix} \quad (8)$$

By differentiating equation (7), we compute that the infinitesimal action on the functions  $P$  and  $Q$  is

$$\begin{aligned} dP - P\alpha_3^3 + P\alpha_2^2 + Q\beta_1^2 &\equiv 0 \\ dQ - Q\alpha_3^3 + 2Q\alpha_2^2 &\equiv 0, \end{aligned} \pmod{\eta^1, \eta^2, \eta^3, \mu^1, \mu^2}$$

and we see that  $Q$  is acted on by multiplication, and  $P$  is acted on by multiplication and, if  $Q \neq 0$ , by translation. There are 3 possible cases:

I.  $P = Q = 0 \iff d\eta^3 \wedge \eta^3 = 0$ .

II.  $P \neq 0$  and  $Q = 0 \iff d\eta^3 \wedge \eta^3 \neq 0$  and  $d\eta^3 \wedge \eta^3 \wedge \eta^1 = 0$ .

III.  $Q \neq 0 \iff d\eta^3 \wedge \eta^3 \wedge \eta^1 \neq 0$ .

These are the cases that are listed in theorem 1. We begin with case I.

If  $P = Q = 0$ , then equation (7) is simply  $d\eta^3 = \alpha_3^3 \wedge \eta^3$  and the structure equations, (6), (7) and (8) form an involutive system with constant torsion. Thus, all of the control systems in this case are

equivalent, and any one of them is a normal form. It is easy to show that the control system

$$\frac{dx}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$$

is in this class and is therefore a normal form. This proves case I of theorem 1.

In case II, a calculation finally yields the structure equations

$$\begin{aligned} d \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} &= \begin{pmatrix} \alpha_2^1 & \alpha_3^1 \\ \alpha_2^2 & \alpha_3^2 \end{pmatrix} \wedge \begin{pmatrix} \eta^2 \\ \eta^3 \end{pmatrix} + \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix} \wedge \eta^1 \\ d\eta^3 &= \alpha_2^2 \wedge \eta^3 + \eta^1 \wedge \eta^2 + \mu^1 \wedge \eta^3 \\ d\mu &= \begin{pmatrix} \beta_1^1 & \beta_2^1 & \beta_3^1 \\ \beta_1^2 & -2\alpha_3^2 - \beta_1^1 & \beta_3^2 \end{pmatrix} \wedge \eta + \\ &\begin{pmatrix} 0 & \alpha_2^1 \\ 0 & \alpha_2^2 \end{pmatrix} \wedge \mu + \begin{pmatrix} 0 \\ \mu^2 \wedge \mu^1 \end{pmatrix}. \end{aligned}$$

The reduced equations form an involutive system with constant torsion. Thus, every control system in this case is equivalent. It is easy to check that the control system

$$\frac{dx}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x^1 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$$

is in this case and is therefore a normal form for this case. This proves case II of theorem 1.

In case III, a lengthy calculation gives structure equations that uniquely determine all of the 1-forms in the equations, and we have an invariant basis of 1-forms, or identity structure, on a 9 dimensional space. This completes the proof of case III of theorem 1.

**Proof of Theorem 2.** We now proceed to the second major case, [2], where the rank of the last 2 rows of the matrix  $M$  is 2. Theorem 2 covers this case. We can normalize the matrix  $M$  to

$$M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and equations (4) and (5) imply that the structure equations have the form

$$\begin{aligned} d\eta^1 &= P\eta^1 \wedge \eta^2 + Q\eta^2 \wedge \eta^3 + R\eta^1 \wedge \eta^3 + \\ &E\mu^1 \wedge \eta^2 + F\mu^2 \wedge \eta^2 + F\mu^1 \wedge \eta^3 + \\ &G\mu^2 \wedge \eta^3 \end{aligned}$$

$$d \begin{pmatrix} \eta^2 \\ \eta^3 \end{pmatrix} = \begin{pmatrix} \alpha_2^2 & \alpha_3^2 \\ \alpha_2^3 & \alpha_3^3 \end{pmatrix} \wedge \begin{pmatrix} \eta^2 \\ \eta^3 \end{pmatrix} + \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix} \wedge \eta^1$$

$$d\mu = \beta \wedge \eta + \begin{pmatrix} \alpha_2^2 & \alpha_3^2 \\ \alpha_2^3 & \alpha_3^3 \end{pmatrix} \wedge \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix}.$$

The equation for  $d\eta^1$  shows that  $\eta^1$  drops to the state-control space,  $U$ . Thus  $\eta^1$  is an invariant 1-form on  $U$ . For notational convenience we will write

$$\gamma = \begin{pmatrix} \alpha_2^2 & \alpha_3^2 \\ \alpha_2^3 & \alpha_3^3 \end{pmatrix}.$$

All of the torsion is in the equation for  $d\eta^1$ , and differentiating  $d\eta^1$  gives the infinitesimal action. Part of the action is expressed by the equation

$$d \begin{pmatrix} E & F \\ F & G \end{pmatrix} + {}^t\gamma \begin{pmatrix} E & F \\ F & G \end{pmatrix} + \begin{pmatrix} E & F \\ F & G \end{pmatrix} \gamma \equiv 0 \quad (9) \\ (\text{mod } \eta, \mu).$$

The symmetric matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is acted on as a quadratic form. In fact we can show that the invariant quadratic form

$$E(\eta^2)^2 + 2F\eta^2\eta^3 + G(\eta^3)^2 \quad (10)$$

drops to the space  $U$  by showing that the Lie derivative of (10) along every vector field tangent to the fibers of  $U \times G$  over  $U$  is equal to 0. This is easy to do since these vector fields are precisely the ones which annihilate the  $\eta$ 's and the  $\mu$ 's, and since the Lie derivative is a derivation. The verification follows easily using the structure equations and equation (9).

The first case to consider is the case where all of the invariants vanish, i.e.,  $d\eta^1 = 0$ . In this case we have an involutive system with constant torsion, so all systems in this case are equivalent. It is easy to check that the control system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$$

falls in this case and is therefore a normal form for this case. This proves I of theorem 2.

The next case to consider is the case where  $d\eta^1 \neq 0$  and  $d\eta^1 \wedge \eta^1 = 0$ . This implies that  $E = F = G = Q = 0$  and that the row vector  $(P, R) \neq 0$ . A calculation gives us the structure equations

$$d\eta^1 = \eta^1 \wedge \eta^2 \\ d\eta^2 = \mu^1 \wedge \eta^1 \\ d\eta^3 = \alpha_2^3 \wedge \eta^2 + \alpha_3^3 \wedge \eta^3 + \mu^2 \wedge \eta^1 \\ d\mu = \begin{pmatrix} \beta_1^1 & 0 & 0 \\ \beta_2^1 & \beta_2^2 & \beta_3^2 \end{pmatrix} \wedge \eta + \begin{pmatrix} -\mu^1 \wedge \eta^2 \\ \alpha_2^3 \wedge \mu^1 + \alpha_3^3 \wedge \mu^2 \end{pmatrix}.$$

Again, these form an involutive system with constant torsion. It is easy to check that the control system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$$

falls in this case and is thus a normal form. This proves case II of theorem 2.

The next case we consider is the case where  $d\eta^1 \wedge \eta^1 \neq 0$  and  $(d\eta^1)^2 \wedge \eta^1 = 0$  which implies that

$$\det \begin{pmatrix} E & F \\ F & G \end{pmatrix} = 0.$$

We will only consider the rank 0 case, i.e.,  $E = F = G = 0$ . Another calculation gives the structure equations

$$d\eta^1 = \eta^1 \wedge \eta^2 + \eta^2 \wedge \eta^3 \\ d\eta^2 = \mu^1 \wedge \eta^1 + \mu^2 \wedge \eta^2 - \mu^1 \wedge \eta^3 \\ d\eta^3 = \alpha_2^3 \wedge \eta^2 + \mu^2 \wedge \eta^1 - \mu^2 \wedge \eta^3 \\ d\mu = \begin{pmatrix} \beta_1^1 & \beta_2^1 & -\beta_1^1 \\ \beta_2^1 & \beta_2^2 & -\beta_2^1 \end{pmatrix} \wedge \eta + \\ \left( \begin{array}{c} 2\mu^2 \wedge \mu^1 \\ \alpha_2^3 \wedge \mu^1 + \mu^1 \wedge \eta^1 - \mu^1 \wedge \eta^3 \end{array} \right).$$

This is another involutive system with constant torsion, and a normal form for this case is

$$\frac{dx}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x^2 & 0 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is in this class and is therefore a normal form for this class. This proves case III of theorem 2.

We consider the final case  $(d\eta^1)^2 \wedge \eta^1 \neq 0$  which implies that

$$\det \begin{pmatrix} E & F \\ F & G \end{pmatrix} \neq 0.$$

Notice that since  $\eta^1$  drops to the 5 dimensional space of states and controls,  $U$ , it defines a contact structure on the space of states and controls. The infinitesimal action on the above matrix shows that it can be normalized to  $\text{diag}(\epsilon_1, \epsilon_2)$ , where  $\epsilon_i = \pm 1$ , for  $i = 1, 2$ , depending on the index of the symmetric matrix. Continuing the calculation, we eventually get an identity structure, with partial structure equations

$$d\eta^1 = \epsilon_1 \mu^1 \wedge \eta^2 + \epsilon_2 \mu^2 \wedge \eta^3 \\ d \begin{pmatrix} \eta^2 \\ \eta^3 \end{pmatrix} \equiv \begin{pmatrix} 0 & \alpha_3^2 \\ -\epsilon_1 \epsilon_2 \alpha_3^2 & 0 \end{pmatrix} \wedge \begin{pmatrix} \eta^2 \\ \eta^3 \end{pmatrix} + \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix} \wedge \eta^1 \\ (\text{mod } \bigwedge_2(\eta, \mu)).$$

This completes the proof of case IV of theorem 2.

In case IV of theorem 2, we can form the invariant quadratic form  $(\eta^1)^2 - \epsilon_1(\eta^2)^2 - \epsilon_2(\eta^3)^2$  on the

space of states and controls,  $U$ . This quadratic form only involves differentials of the state variables, and we ask when will it drop from the space of states and controls to the space of states. A simple Lie derivative argument shows that the form drops if and only if the the congruence in the last equation is an equality. If this occurs then we can write  $d\eta = \theta \wedge \eta$  where

$$\theta = \begin{pmatrix} 0 & \epsilon_1 \mu^1 & \epsilon_2 \mu^2 \\ \mu^1 & 0 & \alpha_3^2 \\ \mu^2 & -\epsilon_1 \epsilon_2 \alpha_3^2 & 0 \end{pmatrix}$$

and  $\theta$  satisfies the equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\epsilon_1 & 0 \\ 0 & 0 & -\epsilon_2 \end{pmatrix} \theta + {}^t\theta \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\epsilon_1 & 0 \\ 0 & 0 & -\epsilon_2 \end{pmatrix} = 0,$$

thus  $\theta$  is the ‘‘Levi-Civita’’ connection for the pseudo-Riemannian metric. We see that the structure equations in this case are the familiar structure equations of Riemannian geometry.

**Proof of Theorem 3.** Assume we are in case IV of theorem 2. It is clear from the definition of  $\eta^1$  that integrating  $\eta^1$  along solution curves of (1) gives the time taken to traverse the curve from the initial endpoint to the final endpoint. In [1] we see that the Euler-Lagrange equations for the variational problem  $\int \eta^1$  are computed from  $d\eta^1$  and, in this case, are given by the Pfaffian equations  $\eta^2 = \eta^3 = \mu^1 = \mu^2 = 0$ . From the structure equations we have that the Pfaffian system  $\{\eta^2, \eta^3, \mu^1, \mu^2\}$  is completely integrable; thus we can find a non-singular  $2 \times 2$  matrix  $T$  and functions  $g_1, g_2$  such that

$$\begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix} \equiv T \begin{pmatrix} dg_1 \\ dg_2 \end{pmatrix} \pmod{\eta^2, \eta^3}.$$

Since  $dg_i \wedge \eta^2 \wedge \eta^3 \wedge \mu^1 \wedge \mu^2 = 0$ , we see that  $g_i$  is a function on the space of states and controls,  $U$ , for  $i = 1, 2$ . Now  $0 \neq \mu^1 \wedge \mu^2 \wedge \eta^1 \wedge \eta^2 \wedge \eta^3 = \det T \det(\partial g/\partial u) du^1 \wedge du^2 \wedge \eta^1 \wedge \eta^2 \wedge \eta^3$  which implies that  $\det(\partial g/\partial u) \neq 0$ . If we pick 2 constants,  $c_1$  and  $c_2$ , then by the implicit function theorem we can solve for  $u^1$  and  $u^2$  as functions of  $(x^1, x^2, x^3)$  from the equation  ${}^t(g_1, g_2) = {}^t(c_1, c_2)$ . Let  $u = h(x)$  be the solution and substitute  $h$  into equation (1). Then the solution curves of the equation  $dx/dt = f(x, h(x))$  necessarily satisfy the Pfaffian equations  $\eta^2 = \eta^3 = 0$ . Since  $g$  is constant along the solution curves, it also satisfies the equations  $dg_1 = dg_2 = 0$  and therefore  $\mu^1 = \mu^2 = 0$ . Hence, these solution curves satisfy the Euler-Lagrange equations for  $\int \eta^1$ , and we have time critical closed loop feedback functions. This proves theorem 3.

## 5. Closing Remarks

A simple analysis verifies that every linear system is equivalent to one of the three linear forms in this paper. A similar analysis shows that every control linear is equivalent to either a linear system or one of the two control linear systems in this paper. We may reasonably call the systems that are not equivalent to control linear or linear systems the ‘‘truly non-linear’’ systems. It is interesting to note that the truly non-linear systems all have identity structures and therefore constitute a generalized geometry in the sense of Cartan. We even saw a familiar geometry, namely Riemannian, in one of the cases. This suggests that Cartan’s approach will continue to be of significant help in the study of the truly non-linear control systems.

## References

- [1] R.B. Gardner, *Differential geometric methods in interfacing control theory*, in **Differential Geometric Control Theory**, R. Brockett, R. Millman, H. Sussman eds. Birkhuser Progress in Mathematics **27** (1983), Boston, pp. 117-180.
- [2] R.B. Gardner, **The Method of Equivalence and Its Applications**, CBMS–NSF Regional Conference Series in Applied Mathematics 58, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1989.
- [3] R.B. Gardner, W.F. Shadwick, G.R. Wilkens, *A geometric isomorphism with applications to closed loop controls*, **SIAM J. Control and Optimization** **27**, No. 6, (1989) pp. 1361–1368.
- [4] R.B. Gardner, W.F. Shadwick, *Feedback equivalence of control systems*, **Systems and Control Lett.** **8** (1987) pp. 463–465.
- [5] R.B. Gardner, W.F. Shadwick, *Overdetermined equivalence problems with an application to feedback equivalence*, in **Differential Geometry: The Interface Between Pure and Applied Mathematics**, M. Lucksic, C. Martin and W. Shadwick (Eds.) AMS, Providence (1987) pp 111–119.
- [6] R.B. Gardner, W.F. Shadwick, G.R. Wilkens, *Feedback equivalence and symmetries of Brunowski normal forms*, **Contemporary Mathematics** **97** (1989), pp115–130.
- [7] R.B. Gardner, W.F. Shadwick, *Symmetry and the implementation of feedback linearization*, preprint (1989), to appear in Systems and Control Letters.
- [8] R.B. Gardner, W.F. Shadwick, *The GS-algorithm for exact linearization to Brunowski normal form*, preprint (1990).