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Non-classical and potential symmetry analysis of Richard's equation for moisture flow in soil

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Abstract

This paper focuses upon the derivation of the non-classical symmetries of Bluman and Cole as they apply to Richard's equation for water flow in an unsaturated uniform soil. It is shown that the determining equations for the non-classical case lead to four highly non-linear equations which have been solved in five particular cases. In each case the corresponding similarity ansatz has been derived and Richard's equation is reduced to an ordinary differential equation. Explicit solutions are produced when possible. Richard's equation is also expressed as a potential system and in reviewing the classical Lie solutions a new symmetry is derived together with its similarity ansatz. Determining equations are then produced for the potential system using the non-classical algorithm. This results in an under-determined set of equations and an example symmetry that reveals a missing classical case is presented. An example of a classical and a non-classical symmetry reduction applied to the infiltration of moisture in soil is presented. The condition for surface invariance is used to demonstrate the equivalence of a classical Lie and a potential symmetry.

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1. Introduction

The symmetry analysis presented here is motivated by problems associated with water flow in unsaturated soils and the need described by Philip [1] to develop more quasi-analytic and analytic methods to describe the strongly non-linear Fokker–Planck equations which describe such a flow. It is worth reflecting that most of the water in the hydrological cycle is located in the unsaturated soil between the time of rainfall and its return to the atmosphere. It is clear therefore that processes of water movement in unsaturated soil play a central role in problems of, for example, irrigation, plant ecology and solute transport. Perhaps less obvious is that the importance of unsaturated flow is greatest in a dry country, such as Australia where as

Philip [1] states 93% of precipitation enters the soil and only 1% is returned to the atmosphere by evaporation. Moreover whilst much of the theory associated with water flow was developed in the moist environments of Europe the challenge for modellers is in the description of flow for arid and semiarid environments of, for example, Africa and middle-east. It is for these applications where the Fokker–Planck equation, applied to flow of moisture in its vapour phase, requires further analysis.

Water flow in unsaturated soil, under suction because of capillary forces, was first described by Richards [2] through the equation:

$$\frac{\partial u}{\partial t} = \nabla \cdot (K \nabla \Psi) + \frac{\partial K}{\partial x} \tag{1}$$

where u is the volumetric water content, K is the hydraulic conductivity, x is the vertical space coordinate and $\Psi = \Psi(u)$ is the matric potential, negative in unsaturated soils which describes the potential of force interaction between solid soil and water. The analysis here will focus upon non-hysteretic cases where $\Psi(u)$ is a single-valued function and where the diffusivity is defined through

$$D = K \frac{\mathrm{d}\Psi}{\mathrm{d}u}. \tag{2}$$

Only one-dimensional flow will be considered and the soil will be taken to be homogeneous so that both D and K are functions of u alone. Richard's equation therefore will have the form:

$$\Delta(x, t, u, u_t, u_x, u_{xx}) \equiv u_t - (Du_x)_x - K_u u_x = 0 \tag{3}$$

where a suffix indicates a partial derivative.

The use of similarity methods to describe flow in unsaturated soil is not new, though limited in scope and has found practical application in the theory of infiltration. For example, in the case of horizontal absorption Philip [3] describes travelling wave solutions based upon an ansatz using the Boltzmann similarity variable ω:

$$u = u(\omega) \qquad \omega = xt^{-\frac{1}{2}} \tag{4}$$

although the resulting mathematical forms for the diffusivity are not well adapted for fitting empirical data.

In a second example, with application to horizontal flow, with Neuman boundary conditions the infiltration process is described using the ansatz:

$$u(x, t) = \psi(\omega) t^{\frac{1}{n+2}} \qquad \omega = xt^{-\frac{n+1}{n+2}} \qquad D(u) = cu^n. \tag{5}$$

Of course, similarity solutions of Richard's equation as described by (3) are well known. Sposito [4], Edwards [5] and El-labany *et al* [6] have conducted a classical Lie analysis of this equation, and Sophocleous [7] has presented a classical analysis of the equation in its potential form. Gandarias [8] presents potential symmetries for a form of heterogeneous porous media with power law diffusivity and hydraulic conductivity. There is however no detailed analysis of the non-classical approach of Bluman and Cole [9] applied to equation (3) although Gandarias *et al* [10] have presented a non-classical analysis of the equation in a restricted form. It is the aim here to review the current range of similarity solutions of Richard's equation as described by (3) and to extend the range of such solution by undertaking a non-classical symmetry analysis. In doing so new potential symmetries will also be presented. Examples of the applicability of a classical and also a non-classical symmetry reduction to descriptions of water flow, in particular, infiltration in unsaturated soils are introduced in section 6.

2. Classical results for Richard's equation

In the classical Lie group method one-parameter infinitesimal point transformations,with group parameter ε are applied to the dependent and independent variables (x, t, u). In this case, the transformation are

$$\bar{x} = x + \varepsilon\eta_1(x,t,u) + O(\epsilon^2) \qquad \bar{t} = t + \varepsilon\eta_2(x,t,u) + O(\epsilon^2)$$
$$\bar{u} = u + \varepsilon\phi(x,t,u) + O(\epsilon^2) \tag{6}$$

and the Lie method, see, for example, Clarkson and Mansfield [11], requires form invariance of the solution set:

$$\Sigma \equiv \{u(x,t), \Delta = 0\}. \tag{7}$$

This results in a system of overdetermined, linear equations for the infinitesimals η_1, η_2, ϕ. The corresponding Lie algebra of symmetries is the set of vector fields

$$\mathcal{X} = \eta_1(x,t,u)\frac{\partial}{\partial x} + \eta_2(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}. \tag{8}$$

The condition for invariance of (3) is the equation

$$\mathcal{X}_E^{(2)}(\Delta)|_{\Delta=0} = 0 \tag{9}$$

where the second prolongation operator $\mathcal{X}_E^{(2)}$ is written in the form

$$\mathcal{X}_E^{(2)} = \mathcal{X} + \phi^{[t]}\frac{\partial}{\partial u_t} + \phi^{[x]}\frac{\partial}{\partial u_x} + \phi^{[xx]}\frac{\partial}{\partial u_{xx}} \tag{10}$$

where $\phi^{[t]}, \phi^{[x]}$ and $\phi^{[xx]}$ are defined through the transformations of the partial derivatives of u. In particular

$$\bar{u}_{\bar{x}} = u_x + \varepsilon\phi^{[x]}(x,t,u) + O(\epsilon^2)$$
$$\bar{u}_{\bar{t}} = u_t + \varepsilon\phi^{[t]}(x,t,u) + O(\epsilon^2) \tag{11}$$
$$\bar{u}_{\bar{x}\bar{x}} = u_{xx} + \varepsilon\phi^{[xx]}(x,t,u) + O(\epsilon^2).$$

Once the infinitesimals are determined the symmetry variables may be found from condition for invariance of surface $u = u(x,t)$:

$$\Omega = \phi - \eta_1 u_x - \eta_2 u_t = 0. \tag{12}$$

Throughout the following, it has been found convenient to set

$$D = H_u \tag{13}$$

and also the MACSYMA program `symmgrp.max` has been used to calculate the determining equations.

In the case of Richard's equation (3) the nine well-known, for example, Sposito [4], Edwards [5], overdetermined linear determining equations are

$$\eta_{2_u} = 0 \qquad \eta_{2_x} = 0 \tag{14}$$
$$\eta_{1_u} H_{uu} - \eta_{1_{uu}} H_u = 0 \qquad \eta_{2_u} H_{uu} + \eta_{2_{uu}} H_u = 0 \tag{15}$$
$$\eta_{2_x} H_{uu} + \eta_{2_{ux}} H_u + \eta_{1_u} = 0 \qquad \phi_x K_u + \phi_{xx} H_u - \phi_t = 0 \tag{16}$$
$$\eta_{2_x} H_u K_u - \phi H_{uu} + \eta_{2_{xx}} H_u^2 - \eta_{2_t} H_u + 2\eta_{1_x} H_u = 0 \tag{17}$$
$$2\eta_{1_u} H_u K_u + \phi H_u H_{uuu} - \phi H_{uu}^2 + \phi_u H_u H_{uu} + \phi_{uu} H_u^2 - 2\eta_{1_{ux}} H_u^2 = 0 \tag{18}$$
$$\phi H_u K_{uu} - \phi H_{uu} K_u + \eta_{1_x} H_u K_u + 2\phi_x H_u H_{uu} + 2\phi_{ux} H_u^2 - \eta_{1_{xx}} H_u^2 + \eta_{1_t} H_u = 0. \tag{19}$$

As may be seen from table 1 the classical symmetries are given for power and exponential functions of H and K in which the infinitesimal η_1 and η_2 are linear functions of x and t and where ϕ is linear in u. Note that each of these symmetries has been used by Edwards [5] to reduce Richard's equation to an ordinary differential equation.

Table 1. Classical symmetries of Richard's equation (based on Sposito [4] and Edwards [5]).

Entry	Functions H and K	Infinitesimals
1	$H = cu^{\lambda}$	$\eta_1 = (\lambda - \mu)x, \phi = u$
	$K = ku^{\mu}$	$\eta_2 = (\lambda - 2\mu + 1)t$
2	$H = cu^{\lambda}$	$\eta_1 = \lambda x, \phi = u$
	$K = k \ln u$	$\eta_2 = (\lambda + 1)t$
3	$H = cu^{\lambda}$	$\eta_1 = (\lambda - 1)x - kt$
	$K = k(u \ln u - u)$	$\eta_2 = (\lambda - 1)t, \phi = u$
4	$H = c\,\mathrm{e}^{\lambda u}$	$\eta_1 = (\lambda - \mu)x$
	$K = k\,\mathrm{e}^{\mu u}$	$\eta_2 = (\lambda - 2\mu))t, \phi = 1$
5	$H = c\,\mathrm{e}^{\lambda u}, K = ku^2$	$\eta_1 = \lambda x - 2kt, \eta_2 = \lambda t, \phi = 1$
6	$H = cu, K = ku^2$	$\eta_1 = -x, \eta_2 = -2t, \phi = u$
7	$H = cu$	$\eta_1 = -2kxt, \eta_2 = -2kt^2$
	$K = ku^2$	$\phi = x + 2kut$
8	$H = cu, K = ku^2$	$\eta_1 = -2kt, \eta_2 = 0, \phi = 1$

3. Non-classical point symmetry

The non-classical method is a generalization of the classical Lie group approach due to Bluman and Cole [9] that incorporates the invariant surface condition (12) into the condition (9) for form invariance of Richard's equation (3). It follows that non-classical symmetries of Richard's equation may be found by solving the non-linear set of determining equations:

$$\mathcal{X}_E^{(2)}(\Delta)|_{\Delta=0,\Omega=0} = 0. \tag{20}$$

It is important to note that the non-classical method is just one example of a more general conditional symmetry approach described by Ibragimov [13] in which the condition (9) for form invariance of a partial differential equation is supplemented by an additional condition.

To apply (20) two cases must be considered as follows.

3.1. Case A: $\eta_2 = 1, \eta_1 \equiv \eta(x, t, u)$

It is straightforward to show that there are four non-linear determining equations as follows:

$$H_u(\eta_u H_{uu} - \eta_{uu} H_u) = 0 \tag{21}$$

$$\phi_x H_u K_u + \phi^2 H_{uu} + \phi_{xx} H_u^2 - \phi_t H_u - 2\eta_x \phi H_u = 0 \tag{22}$$

$$2\eta_u H_u K_u + \phi H_u H_{uuu} - \phi H_{uu}^2 + \phi_u H_u H_{uu} + \phi_{uu} H_u^2 - 2\eta_{ux} H_u^2 + 2\eta\eta_u H_u = 0 \tag{23}$$

$$\begin{aligned}\phi H_u K_{uu} - \phi H_{uu} K_u + \eta_x H_u K_u + 2\phi_x H_u H_{uu} - \eta\phi H_{uu}\\ + 2\phi_{ux} H_u^2 - \eta_{xx} H_u^2 - 2\eta_u \phi H_u + \eta_t H_u + 2\eta\eta_x H_u = 0.\end{aligned} \tag{24}$$

Solutions of these equations will be generated below.

3.2. Case B: $\eta_2 = 0, \eta_1 \equiv \eta \equiv 1$

In this case the four determining equations are

$$\phi H_{uu} = 0 \tag{25}$$

$$\phi_x K_u + \phi_{xx} H_u - \phi_t = 0 \tag{26}$$

$$\phi H_u H_{uuu} - \phi H_{uu}^2 + \phi_u H_u H_{uu} + \phi_{uu} H_u^2 = 0 \tag{27}$$

$$\phi H_u K_{uu} - \phi H_{uu} K_u + 2\phi_x H_u H_{uu} + 2\phi_{ux} H_u^2 = 0. \tag{28}$$

There is only one solution of these equations as follows, namely the infinite symmetry

$$\eta = 1 \qquad H = cu \qquad K = ku \qquad \phi = \frac{\partial u}{\partial x} = c_0 u + g \tag{29}$$

where $g = g(x, t)$ and satisfies

$$c g_{xx} + k g_x - g_t = 0. \tag{30}$$

Substituting the relations into Richard's equation (3) gives rise to the solution

$$u(x, t) = \mathrm{e}^{c_0 x} \left\{ \int g(x, t)\, \mathrm{d}x + \varphi(t) \right\}. \tag{31}$$

4. Symmetry reductions for case A

Using (21) and (23) the following explicit forms for $\eta(x, t, u)$ and $\phi(x, t, u)$ may be obtained:

$$\eta(x, t, u) = f(x, t) H(u) + g(x, t) \tag{32}$$

$$\phi(x, t, u) = H_u^{-1}\{f_x H^2 - 2f Z(u) + 2fgX(u) + 2f^2 W(u) + HS(x, t) + R(x, t)\} \tag{33}$$

where f, g, R, S depend on x, t and W, X and Z depend only on u. Explicit non-classical symmetries may now be found by considering two sub-cases $f = 0$ and $f \neq 0$ separately.

4.1. Sub-case $f = 0$

When $f = 0$ equations (32) and (33) become

$$\eta = g \qquad \phi = H_u^{-1}\{HS(x, t) + R(x, t)\} \tag{34}$$

and the determining equations (22) and (24) are now

$$(HH_{uu}K_u - HH_u K_{uu} + gH_{uu})(R + HS) + H_u^3(g_{xx} - 2S_x) - H_u^2(g_x K_u + g_t + 2gg_x) = 0 \tag{35}$$

and

$$H_u^3(R_{xx} + HS_{xx}) + H_{uu}(R + HS)^2 + H_u^2 K_u (R_x + HS_x) - 2g_x H_u^2 (R + HS) - H_u^2 (R_t + HS_t) = 0. \tag{36}$$

The following two solutions of these equations have been found. In the first case:

$$f = 0 \qquad H = \frac{c}{u} \qquad K = \frac{k}{u} \tag{37}$$

where c and k are constants and where the infinitesimals are

$$\eta = c_2\, \mathrm{e}^{-\frac{kx}{c}} \qquad \phi = \frac{c_2 k u}{c}\, \mathrm{e}^{-\frac{kx}{c}}. \tag{38}$$

These may be substituted into the surface invariant condition (12) and the method of characteristics employed to determine the following ansatz for $u(x, t)$:

$$u(x, t) = \psi(\omega)\, \mathrm{e}^{\frac{kx}{c}} \tag{39}$$

where the similarity variable $\omega(x, t)$ is given by

$$\mathrm{e}^{\frac{k\omega}{c}} = \mathrm{e}^{\frac{kx}{c}} - \frac{k c_2 t}{c}. \tag{40}$$

Substitution of these relationships into Richard's equation (3) gives the ordinary differential equation

$$\frac{4c\psi_{\omega\omega}}{\psi^2} - \frac{8c\psi_\omega^2}{\psi^3} - \frac{2k\psi_\omega}{\psi^2} - c_2\,\mathrm{e}^{\frac{k\omega}{c}}\psi_\omega = 0. \tag{41}$$

Clearly equations (40) and (41) together define $\psi(\omega)$ and $\omega(x,t)$ to give the final form of the solution (39).

In the second reduction, the following solutions of (35) and (36) have also been found

$$f = 0 \qquad H = \text{arbitrary} \qquad K = \text{arbitrary} \tag{42}$$

with infinitesimals

$$\eta = \eta(t) \qquad \phi = 0. \tag{43}$$

Substitution of these results into Richard's equation gives rise to standard travelling wave solutions. It is interesting to note that these straightforward solutions do not satisfy the determining equations for classical symmetry (14) to (19).

4.2. The sub-case $f \neq 0$

For this sub-case, substitution of (32) and (33) into the determining equation (23) gives rise to the following condition on f and g:

$$a_0 - ga_2 - fa_4 = 0 \tag{44}$$

where a_i are constants for which the functions Z, X and W satisfy

$$\frac{Z_u}{H_u} - K = a_0u + a_1 \qquad \frac{X_u}{H_u} + u = a_2u + a_3 \tag{45}$$

$$\frac{W_u}{H_u} + \int H\,\mathrm{d}u = a_4u + a_5. \tag{46}$$

The remaining two determining equations (22) and (24) have the lengthy form:

$$\begin{aligned}
&f_x\left(4H_u^2Z_u - H^2H_uK_{uu} + H^2K_uH_{uu} - HH_u^2K_u\right) + 2fZ\left(H_uK_{uu} - K_uH_{uu}\right)\\
&\quad + 2f^2\left(-HH_{uu}Z - 2H_u^2Z - H_uK_{uu}W + K_uH_{uu}W\right)\\
&\quad + 2fg(-H_{uu}Z - H_uK_{uu}X + K_uH_{uu}X)\\
&\quad + f_xg\left(-4H_u^2X_u + H^2H_{uu} - 2HH_u^2\right) + 2fg_x\left(-2H_u^2X_u - HH_u^2\right)\\
&\quad + 2f^2g\left(HH_{uu}X + 2H_u^2X + H_{uu}W\right) + 2fg^2H_{uu}X\\
&\quad + ff_x\left(H^3H_{uu} - 8H_u^2W_u\right) + 2f^3W\left(HH_{uu} + 2H_u^2\right) - 2H_u^3S_x\\
&\quad + (K_uH_{uu} - H_uK_{uu})(R + HS) + f\left(HH_{uu} + 2H_u^2\right)(R + HS)\\
&\quad + gH_{uu}(R + HS) - g_xH_u^2K_u - 3f_{xx}HH_u^3 + g_{xx}H_u^3\\
&\quad - f_tHH_u^2 - g_tH_u^2 - 2gg_xH_u^2 = 0
\end{aligned} \tag{47}$$

$$\begin{aligned}
&4f^2H_{uu}Z^2 + ff_x\left(-4H^2H_{uu}Z + 4HH_u^2Z + 4H_u^2K_uW\right)\\
&\quad + f_{xx}\left(H^2H_u^2K_u - 2H_u^3Z\right) + fg_x\left(4H_u^2Z + 2H_u^2K_uX\right)\\
&\quad - 8f^2gH_{uu}XZ - 8f^3H_{uu}WZ - 4fH_{uu}Z(R + HS)\\
&\quad - 2f_xH_u^2K_uZ + 2f_tH_u^2Z + 4f^2g^2H_{uu}X^2\\
&\quad + ff_xg\left(4H^2H_{uu}X - 4HH_u^2X\right) + f_xg_x\left(4H_u^3X - 2H^2H_u^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+8f^3gH_{uu}WX+4fgH_{uu}X(R+HS)+2f_xgH_u^2K_uX+2fg_{xx}H_u^3X\\
&+2f_{xx}gH_u^3X-2fg_tH_u^2X-4fgg_xH_u^2X-2f_tgH_u^2X+4f^4H_{uu}W^2\\
&+f^2f_x\left(4H^2H_{uu}W-4HH_u^2W\right)+f_x^2\left(4H_u^3W+H^4H_{uu}-2H^3H_u^2\right)\\
&+4f^2H_{uu}W(R+HS)+4ff_{xx}H_u^3W-4f^2g_xH_u^2W\\
&-4ff_tH_u^2W-H_u^2(R_t+HS_t)+H_u^3\left(R_{xx}+HS_{xx}\right)+H_u^2K_u(R_x+HS_x)\\
&+H_{uu}(R+HS)^2+f_x\left(2H^2H_{uu}-2HH_u^2\right)(R+HS)\\
&-2g_xH_u^2(R+HS)+f_{xxx}H^2H_u^3-f_{xt}H^2H_u^2=0.
\end{aligned}
\tag{48}
$$

The following symmetry reductions are possible from these equations.

4.2.1. First reduction. In this sub-case,

$$f=c_0 \qquad K_u=-c_0H \tag{49}$$

where c_0 is constant and the corresponding infinitesimals are

$$\eta=c_0H \qquad \phi=0. \tag{50}$$

These results enable closed form solutions of Richard's equation to be found in following way. Using the surface invariant condition (12) and method of characteristics, it may be shown that

$$u(x,t)=\psi(\omega) \qquad \omega=x-c_0Ht \tag{51}$$

from which the second equation gives

$$(\omega_{uu}+c_0H_{uu}t)u_x^2+(\omega_u+c_0H_ut)u_{xx}=0. \tag{52}$$

Thus with Richard's equation

$$H_{uu}u_x^2+H_uu_{xx}=0 \qquad H=a\omega+b \tag{53}$$

and so finally

$$H(u)=\frac{ax+b}{1+ac_0t}. \tag{54}$$

4.2.2. Second reduction. Consider now the sub-case

$$f=\frac{k}{c} \qquad H=cu \qquad K=ku^2 \tag{55}$$

with infinitesimals

$$\eta=k(u+h) \qquad \phi=-\frac{k^2u^2(u+h)}{c} \tag{56}$$

where $h=h(x,t)$ such that

$$h_t=ch_{xx}+2khh_x. \tag{57}$$

When $h=0$, the method of characteristics applied to the surface invariant condition gives the ansatz:

$$u(x,t)=\left(\frac{2k^2t}{c}+\psi(\omega)\right)^{-\frac{1}{2}} \tag{58}$$

where ψ and ω are such that

$$x=\frac{c}{k}\left(\frac{2k^2t}{c}+\psi(\omega)\right)^{\frac{1}{2}}-\frac{c}{k}\psi^{\frac{1}{2}}+\omega. \tag{59}$$

Table 2. Non-classical symmetries of Richard's equation.

Entry	Functions H and K	Infinitesimals $\phi = \eta u_x + u_t$
1	$H = \frac{c}{u}$	$\eta = c_2\,\mathrm{e}^{-\frac{kx}{c}}$
	$K = \frac{k}{u}$	$\phi = \frac{c_2 k u}{c}\,\mathrm{e}^{-\frac{kx}{c}}$
2	H, K arbitrary	$\eta = \eta(t)$
		$\phi = 0$
3	$H = H(u)$	$\eta = c_0 H$
	$K_u = -c_0 H$	$\phi = 0$
4	$H = cu$	$\eta = k(u+h)$
	$K = ku^2$	$\phi = -\frac{k^2u^2(u+h)}{c}$
		$h_t = ch_{xx} + 2khh_x$
5	$H = c\,\mathrm{e}^{\lambda u}$	$\eta = -\frac{cc_1\lambda\,\mathrm{e}^{\lambda u}}{c_0+c_1x+c_2t}$
	$K = k\,\mathrm{e}^{\lambda u}$	$\phi = \frac{kc_1\lambda\,\mathrm{e}^{\lambda u}-c_2}{\lambda(c_0+c_1x+c_2t)}$

Substitution of these relationships into Richard's equation gives rise to the ordinary differential equation for $\psi(\omega)$:

$$\psi\psi_{\omega\omega} - \frac{3\psi_\omega^2}{2} + \frac{3k\sqrt{\psi}\psi_\omega}{c} - \frac{2k^2\psi}{c^2} = 0. \tag{60}$$

4.2.3. Third reduction. In this sub-case,

$$f = -\frac{c_1\lambda}{c_0 + c_1x + c_2t} \qquad H = c\,\mathrm{e}^{\lambda u} \qquad K = k\,\mathrm{e}^{\lambda u} \tag{61}$$

and the infinitesimals are

$$\eta = -\frac{cc_1\lambda\,\mathrm{e}^{\lambda u}}{c_0 + c_1x + c_2t} \qquad \phi = \frac{kc_1\lambda\,\mathrm{e}^{\lambda u} - c_2}{\lambda(c_0 + c_1x + c_2t)}. \tag{62}$$

In the case when $c_2 = 0$ it may be shown that surface invariance with characteristics gives

$$\mathrm{e}^{\lambda u} = \psi(\omega)\,\mathrm{e}^{\frac{k}{c}(\omega - x)} \tag{63}$$

where

$$c_1k\lambda\psi\,\mathrm{e}^{\frac{k\omega}{c}}t = \left(c_0 + c_1\omega - \frac{cc_1}{k}\right)\mathrm{e}^{\frac{k\omega}{c}} - \left(c_0 + c_1x - \frac{cc_1}{k}\right)\mathrm{e}^{\frac{kx}{c}}. \tag{64}$$

The ordinary differential equation for $\psi(\omega)$ is obtained by direct substitution into (3) and is found to be

$$c(c_0 + c_1\omega)\psi_{\omega\omega} + [k(c_0 + c_1\omega) - cc_1]\psi_\omega - kc_1\psi = 0. \tag{65}$$

The non-classical symmetries found above are summarized in table 2.

5. Potential symmetry—classical algorithm

Richard's equation may also be written as the potential system $\boldsymbol{\Delta} \equiv (\Delta_1, \Delta_2) = 0$ where

$$\Delta_1 = v_x - u = 0 \qquad \Delta_2 = v_t - H_u u_x - K = 0. \tag{66}$$

In this case, the classical Lie analysis is based upon the infinitesimal transformations:

$$\begin{aligned} \bar{x} &= x + \varepsilon\eta_1(x,t,u,v) + O(\epsilon^2) \qquad & \bar{t} &= t + \varepsilon\eta_2(x,t,u,v) + O(\epsilon^2) \\ \bar{u} &= u + \varepsilon\phi_1(x,t,u,v) + O(\epsilon^2) \qquad & \bar{v} &= v + \varepsilon\phi_2(x,t,u,v) + O(\epsilon^2). \end{aligned} \tag{67}$$

Note that since $u = v_x$ these define contact transformations for v. The associated generator

$$\mathcal{X} = \eta_1 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial t} + \phi_1 \frac{\partial}{\partial u} + \phi_2 \frac{\partial}{\partial v} \tag{68}$$

and the condition for form invariance of (66) are found by applying the first prolongation so that now

$$\mathcal{X}_E^{(1)}(\mathbf{\Delta})|_{\Delta_1=0,\Delta_2=0} = 0. \tag{69}$$

It may be shown that the seven linear determining equations are

$$\eta_{2_u} = 0 \tag{70}$$

$$\eta_{2_v} H_u - \eta_{1_u} = 0 \tag{71}$$

$$u\eta_{2_v} + \eta_{2_x} = 0 \tag{72}$$

$$\phi_{2_u} - u\eta_{1_u} = 0 \tag{73}$$

$$2\eta_{1_u} K + \phi_1 H_{uu} - \phi_{2_v} H_u + \phi_{1_u} H_u + \eta_{2_t} H_u - \eta_{1_x} H_u = 0 \tag{74}$$

$$\phi_{2_u} K - u\eta_{1_u} K - u\phi_{2_v} H_u - \phi_{2_x} H_u + \phi H_u + u^2 \eta_{1_v} H_u + u\eta_{1_x} H_u = 0 \tag{75}$$

$$\begin{aligned}\phi_1 H_u K_u - \eta_{1_u} K^2 - \phi_1 H_{uu} K - \phi_1 H_u K + u\eta_{1_v} H_u K \\ + \eta_{1_x} H_u K + u\phi_{1_v} H_u^2 + \phi_{1_x} H_u^2 - \phi_{2_t} H_u + u\eta_{1_t} H_u = 0.\end{aligned} \tag{76}$$

There are two main cases to be considered. In the first $\eta_{1_v} \neq 0$ whilst in the second, considered in detail by Sophocleous [7], $\eta_{1_v} = 0$.

5.1. *The case* $\eta_{1_v} \neq 0$

This case does not appear to have been considered in the literature and the corresponding determining equations give rise to the following solution

$$H = c\left(\frac{u}{c_0 + uc_1}\right)^{\frac{1}{c_0}-1} \qquad K = ku\left(\frac{u}{c_0 + uc_1}\right)^{\frac{1}{c_0}-1} \qquad \text{when} \quad c_0 \neq 0, 1 \tag{77}$$

$$H = c\,\mathrm{e}^{-\frac{1}{c_1 u}} \qquad K = ku\,\mathrm{e}^{-\frac{1}{c_1 u}} \qquad \text{when} \quad c_0 = 0 \tag{78}$$

$$H = c\ln\left(\frac{u}{1 + c_1 u}\right) \qquad K = ku \qquad \text{when} \quad c_0 = 1 \tag{79}$$

with infinitesimals

$$\begin{aligned}\eta_1 &= c_0 x + c_1 v \qquad & \eta_2 &= c_3 + t \\ \phi_1 &= -u(c_0 + c_1 u) \qquad & \phi_2 &= 0.\end{aligned} \tag{80}$$

In this case there are two invariant surface conditions for u, v namely

$$\Omega_1 = \phi_1 - \eta_1 u_x - \eta_2 u_t = 0 \tag{81}$$

$$\Omega_2 = \phi_2 - \eta_1 v_x - \eta_2 v_t = 0 \tag{82}$$

and so from (80)

$$-u(c_0 + c_1 u) = (c_0 x + c_1 v)u_x + (c_3 + t)u_t \tag{83}$$

and

$$0 = (c_0 x + c_1 v)v_x + (c_3 + t)v_t. \tag{84}$$

However, since (84) is a differential consequence of (83) it is necessary only to solve (84) with the result that

$$\omega = \omega(v) \tag{85}$$

where

$$\omega = \begin{cases} \frac{1}{c_0}\left[\left(\frac{c_3}{t+c_3}\right)^{c_0}(c_0x + c_1v) - c_1v\right] & c_0 \neq 0 \\ x - c_1v\ln\left(\frac{t+c_3}{c_3}\right) & c_0 = 0. \end{cases} \tag{86}$$

It follows from (86) that

$$v_x = u = \begin{cases} \dfrac{\left(\frac{c_3}{t+c_3}\right)^{c_0}}{\omega_v - \frac{c_1}{c_0}\left(\left(\frac{c_3}{t+c_3}\right)^{c_0} - 1\right)} & c_0 \neq 0 \\ \dfrac{1}{\omega_v + c_1\ln\left(\frac{c_3+t}{c_3}\right)} & c_0 = 0 \end{cases} \tag{87}$$

and so with

$$F(v) = c_0\omega + c_1v \tag{88}$$

then Richard's equation reduces to the ordinary differential equations:

$$\frac{c(1-c_0)}{c_0}F_{vv} - \frac{FF_v^{\frac{1}{c_0}}}{c_3} - kF_v = 0 \qquad \text{when} \quad c_0 \neq 0, 1 \tag{89}$$

$$cF_{vv} - \frac{FF_v}{c_3} - kF_v = 0 \qquad \text{when} \quad c_0 = 1 \tag{90}$$

and

$$\frac{c}{c_1}\omega_{vv} - \frac{c_1v}{c_3}\,\mathrm{e}^{\frac{1}{c_1\omega_v}} - k = 0 \qquad c_0 = 0. \tag{91}$$

Note, for example, that in the case $c_0 = 1/2$, $c_3 = 1$ the explicit solution for v is

$$\operatorname{erf}\left(\frac{\frac{\omega}{2} + c_1v}{\sqrt{2c}}\right) = a_0 + a_1\,\mathrm{e}^{\frac{kv}{c}}. \tag{92}$$

5.2. *The case* $\eta_{1_v} = 0$

In this case it follows that $\eta_2 = \eta_2(t)$ and the corresponding results have been studied in detail by Sophocleous [7] and table 3 is based upon his results.

5.3. *Potential symmetry—non-classical algorithm*

A natural extension of the analysis of the previous section is the extension to a non-classical consideration of the system (66). In this case the determining equations are found by incorporating the surface conditions (81) and (82) so form invariance is found by applying the first prolongation:

$$\mathcal{X}_E^{(1)}(\boldsymbol{\Delta})|_{\Delta_1=0,\Delta_2=0,\Omega_1=0,\Omega_2=0} = 0. \tag{93}$$

It may be shown that the resulting two determining equations are

$$\begin{aligned} &\phi_{2_u}K - u\eta_uK - u\phi_{2_v}H_u - \phi_{2_x}H_u + \phi_1H_u + u^2\eta_vH_u \\ &\quad + u\eta_xH_u - \phi_2\phi_{2_u} + u\eta\phi_{2_u} + u\eta_u\phi_2 - u^2\eta\eta_u = 0 \end{aligned} \tag{94}$$

Table 3. Potential symmetries of Richard's equation (based upon Sophocleous [7]).

Entry	Functions H and K	Infinitesimals
1	$H = cu$	$\eta_1 = (1-\mu)(x-k_0t)$, $\phi_1 = u$
	$K = ku^{\mu} + k_0u$	$\eta_2 = 2(1-\mu)t$, $\phi_2 = (2-\mu)v$
2	$H = cu$	$\eta_1 = \mu(k_0t - x)$, $\eta_2 = -2\mu t$
	$K = k\,e^{\lambda u} + k_0u$	$\phi_1 = 1$, $\phi_2 = -\mu v + x + k_0t$
3	$H = cu$	$\eta_1 = -kt$, $\eta_2 = 2t$
	$K = k\ln u + k_0u$	$\phi_1 = u$, $\phi_2 = 2v + kt$
4	$H = cu$	$\eta_1 = x - k_0t$, $\eta_2 = 0$
	$K = ku\ln u + k_0u$	$\phi_1 = u$, $\phi_2 = v$
5	$H = cu$	$\eta_1 = -2kt$, $\eta_2 = 0$
	$K = ku^2$	$\phi_1 = 1$, $\phi_2 = x$
6	$H = cu$	$\eta_1 = -2kxt$, $\eta_2 = -2kt^2$
	$K = ku^2$	$\phi_1 = x + 2kut$, $\phi_2 = ct + \frac{t^2}{2}$
7	$H = c\,e^{\lambda u}$	$\eta_1 = \lambda x - 2kt$, $\eta_2 = \lambda t$
	$K = ku^2$	$\phi_1 = 1$, $\phi_2 = x + \lambda v$
8	$H = c\,e^{\lambda u}$	$\eta_1 = (\lambda-\mu)x - k_0\mu t$, $\eta_2 = (\lambda - 2\mu)t$
	$K = k\,e^{\mu u} + k_0u$	$\phi_1 = 1$, $\phi_2 = x + k_0t + (\lambda-\mu)v$
9	$H = cu^{\lambda}$	$\eta_1 = (\lambda-\mu)x + k_0(\mu-1)t$, $\phi_1 = u$
	$K = ku^{\mu} + k_0u$	$\eta_2 = (\lambda - 2\mu + 1)t$, $\phi_2 = (\lambda - \mu + 1)v$
11	$H = cu^{\lambda}$	$\eta_1 = \frac{(\lambda-1)}{2}(x+kt)$, $\eta_2 = 0$
	$K = ku$	$\phi_1 = u$, $\phi_2 = \frac{(\lambda+1)}{2}v$
12	$H = cu^{\lambda}$	$\eta_1 = (\lambda-1)x - kt$, $\phi_1 = u$
	$K = ku\ln(u) + k_0u$	$\eta_2 = (\lambda-1)t$, $\phi_2 = \lambda v$
13	$H = cu^{\lambda}$	$\eta_1 = \lambda x - k_0t$, $\phi_1 = u$
	$K = k\ln(u) + k_0u$	$\eta_2 = (\lambda+1)t$, $\phi_2 = (\lambda+1)v + kt$

$$
\begin{aligned}
&\phi_1 H_u K_u - \eta_u K^2 - \phi_1 H_{uu} K - \phi_{1_u} H_u K + u\eta_v H_u K + \eta_x H_u K \\
&\quad - \eta\phi_{2_u} K + 2\eta_u\phi_2 K - u\eta\eta_u K + \phi_1\phi_2 H_{uu} - u\eta\phi_1 H_{uu} \\
&\quad + u\phi_v H_u^2 + \phi_{1_x} H_u^2 - \phi_{2_t} H_u - \phi_2\phi_{2_v} H_u + u\eta\phi_{2_v} H_u - \phi_1\phi_{2_u} H_u \\
&\quad + \phi_{1_u}\phi_2 H_u - \eta_x\phi_2 H_u - u\eta\phi_{1_u} H_u + u\eta_u\phi_1 H_u + u\eta_t H_u \\
&\quad + u\eta\eta_x H_u + \eta\phi_2\phi_{2_u} - u\eta^2\phi_{2_u} - \eta_u\phi_2^2 + u\eta\eta_u\phi_2 = 0.
\end{aligned}
\tag{95}
$$

Note that equation (94) incorporating equations (66) is the condition for the differential consistency of the two surface conditions (81) and (82). This expression may be used to find ϕ_1 with the result that

$$
\phi_1 = \frac{u\eta_u K}{H_u} - \frac{\phi_{2_u} K}{H_u} + \frac{u^2\eta\eta_u}{H_u} - \frac{u_1\phi_2\eta_u}{H_u} - \frac{u\phi_{2_u}\eta}{H_u} + \frac{\phi_2\phi_{2_u}}{H_u} - u^2\eta_v - u\eta_x + u\phi_{2_v} + \phi_{2_x}. \tag{96}
$$

It follows that (95) becomes

$$
\begin{aligned}
&-u^2\eta_v H_u K_u - u\eta_x H_u K_u + u\phi_{2_v} H_u K_u + \phi_{2_x} H_u K_u - u\eta_{uu} K^2 - 2\eta_u K^2 \\
&\quad + \phi_{2_{uu}} K^2 + u^2\eta_v H_{uu} K + u\eta_x H_{uu} K - u\phi_{2_v} H_{uu} K - \phi_{2_x} H_{uu} K \\
&\quad + 3u\eta_v H_u K + 2\eta_x H_u K + 2u^2\eta_{uv} H_u K + 2u\eta_{ux} H_u K - \phi_{2_v} H_u K \\
&\quad - 2u\phi_{2_{uv}} H_u K - 2\phi_{2_{ux}} H_u K - 2u^2\eta\eta_{uu} K + 2u\varphi_2\eta_{uu} K - 4u\eta\eta_u K \\
&\quad + 4\phi_2\eta_u K + 2u\phi_{2_{uu}}\eta K - 2\phi_2\phi_{2_{uu}} K + u^3\eta\eta_v H_{uu} - u^2\phi_2\eta_v H_{uu} \\
&\quad + u^2\eta\eta_x H_{uu} - u\phi_2\eta_x H_{uu} - u^2\phi_{2_v}\eta H_{uu} - u\phi_{2_x}\eta H_{uu} + u\phi_2\phi_{2_v} H_{uu}
\end{aligned}
$$

$$+\phi_2\phi_{2_x}H_{uu} - u^3\eta_{vv}H_u^2 - u\eta_{xx}H_u^2 - 2u^2\eta_{xv}H_u^2 + u^2\phi_{2_{vv}}H_u^2 + \phi_{2_{xx}}H_u^2$$
$$+2u\phi_{2_{xv}}H_u^2 + u\eta_t H_u + 2u^2\eta\eta_v H_u - 2u\phi_2\eta_v H_u + 2u\eta\eta_x H_u$$
$$-2\phi_2\eta_x H_u + 2u^3\eta\eta_{uv}H_u - 2u^2\phi_2\eta_{uv}H_u + 2u^2\eta\eta_{ux}H_u - 2u\phi_2\eta_{ux}H_u$$
$$-2u^2\phi_{2_{uv}}\eta H_u - 2u\phi_{2_{ux}}\eta H_u - \phi_{2_t}H_u + 2u\phi_2\phi_{2_{uv}}H_u + 2\phi_2\phi_{2_{ux}}H_u$$
$$-u^3\eta^2\eta_{uu} + 2u^2\phi_2\eta\eta_{uu} - u_1\phi_2^2\eta_{uu} - 2u^2\eta^2\eta_u + 4u\phi_2\eta\eta_u - 2\phi_2^2\eta_u$$
$$+u^2\phi_{2_{uu}}\eta^2 - 2u_1\phi_2\phi_{2_{uu}}\eta + \phi_2^2\phi_{2_{uu}} = 0. \tag{97}$$

This equation is, of course, under-determined and so in principle there are infinity of symmetries. By way of a particular example however consider the case

$$\phi_2 = 0 \tag{98}$$

with

$$H = cu^\lambda \qquad K = ku^\lambda \tag{99}$$

and $\lambda \neq 0$. Here the determining equations are satisfied by

$$\eta \equiv \eta_1 = \frac{c_0 + v}{u(c_2 + t)(\lambda - 1)} \tag{100}$$

$$\eta_2 = 1 \tag{101}$$

and

$$\phi_1 = \frac{u^{\lambda+1}(c_2+t)c\lambda(1-\lambda) + u^\lambda(c_0+v)(c_2+t)k(1-\lambda) - (c_0+v)^2}{u^\lambda(c_0+v)^2c\lambda(1-\lambda)^2}. \tag{102}$$

This example has been presented because, as may be seen from table 3, the case (99) does not produce classical potential symmetries, that is the non-classical approach uncovers this missing case.

6. Examples applied to the problem of infiltration

6.1. A classical Lie/potential symmetry reduction

As a particular practical illustration of the potential symmetry reduction of Richard's equation consider the Dirichlet problem, of the vertical infiltration of water from a soil surface

$$u(0,t) = f(t) \qquad t \geqslant 0 \tag{103}$$

where $f(t)$ is a given function of t, into a uniform soil mass for which

$$u(x,0) = g(x) \qquad x > 0 \tag{104}$$

where x is depth measured positively from the soil surface. Similar problems have been discussed by many authors, for example, Philip [3], Hillel [14] and Miyazaki [15]. They discuss many appropriate mathematical forms for diffusivity and hydraulic conductivity including exponential and power law relationships. In this example it will be assumed that

$$D(u) = H_u \qquad H(u) = cu^\lambda \qquad K(u) = ku^\mu \tag{105}$$

which corresponds to the particular potential symmetry given by table 3 (entry 9). It follows that the two appropriate surface invariant conditions are therefore

$$\Omega_1 = u - (\lambda - \mu)xu_x - (\lambda - 2\mu + 1)tu_t = 0 \tag{106}$$

$$\Omega_2 = (\lambda - \mu + 1)v - (\lambda - \mu)xv_x - (\lambda - 2\mu + 1)tv_t = 0. \tag{107}$$

However, these are differentially related using the argument given by equations (81) to (84) and the potential symmetry is entirely equivalent to the classical symmetry presented in table 1 (entry 1). Integration of equation (106) defines the similarity transformation:

$$u(x,t) = \psi(\omega)(1+t)^{\frac{1}{\lambda-2\mu+1}} \qquad \omega(x,t) = x(1+t)^{-\frac{\lambda-\mu}{\lambda-2\mu+1}} \tag{108}$$

and so from equations (103) and (104):

$$u(0,t) = f(t) = \psi(0)(1+t)^{\frac{1}{\lambda-2\mu+1}} \tag{109}$$

$$u(x,0) = g(x) = \psi(\omega)|_{\omega=x}. \tag{110}$$

The similarity transformation (108) may be used to reduce Richard's equation (3) to the ordinary differential equation:

$$c\lambda\psi^{\lambda-2}\{\psi\psi_{\omega\omega} + (\lambda-1)\psi_\omega^2\} - k\mu\psi^{\mu-1}\psi_\omega + \frac{\omega(\lambda-\mu)\psi_\omega - \psi}{(\lambda-2\mu+1)} = 0 \tag{111}$$

where the minus sign attached to the hydraulic conductivity term indicates that x has a positive value in the downward direction.

Consider now the particular problem based upon data provided by Philip [3], Hillel [14] and Miyazaki [15] where

$$D(u) = 0.75u^4 \Longrightarrow H(u) = 0.15u^5 \text{ cm}^2 \text{ s}^{-1} \qquad K(u) = 0.003u^8 \text{ cm s}^{-1}. \tag{112}$$

Note that the physical units chosen are those commonly used in soil physics. Thus equation (108) is

$$u(x,t) = \psi(\omega)(1+t)^{-\frac{1}{10}} \qquad \omega(x,t) = x(1+t)^{-\frac{3}{10}} \tag{113}$$

and (111) is specifically

$$0.75\psi^4\psi_{\omega\omega} + 3\psi^3\psi_\omega^2 - 0.024\psi^7\psi_\omega + \frac{3\omega\psi_\omega + \psi}{10} = 0. \tag{114}$$

Initially it is assumed that the surface water content is 0.5 $\text{cm}^3\ \text{cm}^{-3}$ and that at a depth of 15 cm the soil is much dryer with volumetric water content of 0.1 $\text{cm}^3\ \text{cm}^{-3}$ then the conditions (109) and (110) give rise to

$$\psi(0) = 0.5 \qquad \psi(15) = 0.1. \tag{115}$$

Equations (114) and (115) together specify a standard boundary value problem whose solution was found numerically using MAPLE software. The result is presented graphically in figure 1 in terms of the curve $u(x,0)$ versus x, entirely equivalent to $\psi(\omega)$ versus ω.

The corresponding infiltration curves are calculated using equation (113) and the results are presented in figure 2 which show the penetration of water into the soil after 200 and 2000 s.

Although this example is somewhat idealized it does demonstrate that the symmetry method offers a straightforward solution approach involving the numerical solution of an ordinary rather than a partial differential equation. Moreover the approach has not been reliant of the use of the Boltzmann transformation (4) which is often seen in the many textbooks on infiltration.

6.2. *A non-classical symmetry reduction*

Consider the physical interpretation of the non-classical symmetry given in table 2 (entry 5) for which

$$D = H_u \qquad H(u) = c\,e^{\lambda u} \qquad K(u) = k\,e^{\lambda u}. \tag{116}$$

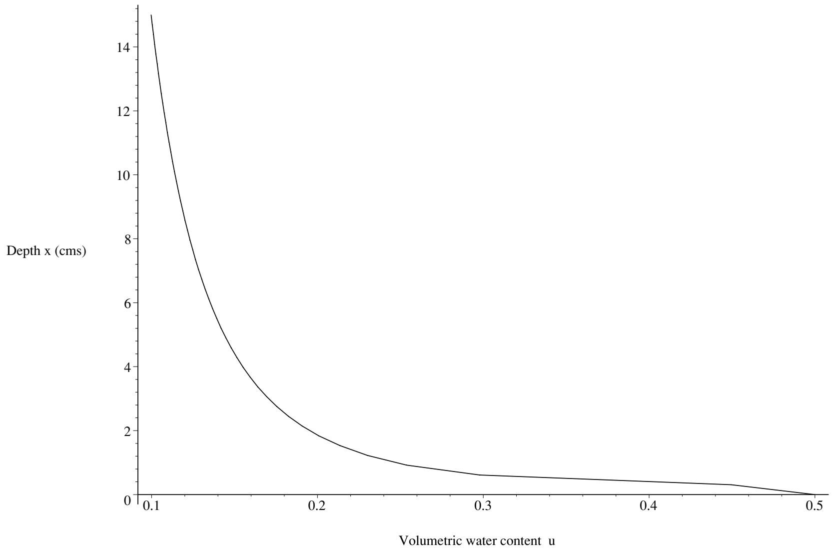

Figure 1. Initial profile of volumetric water content versus depth.

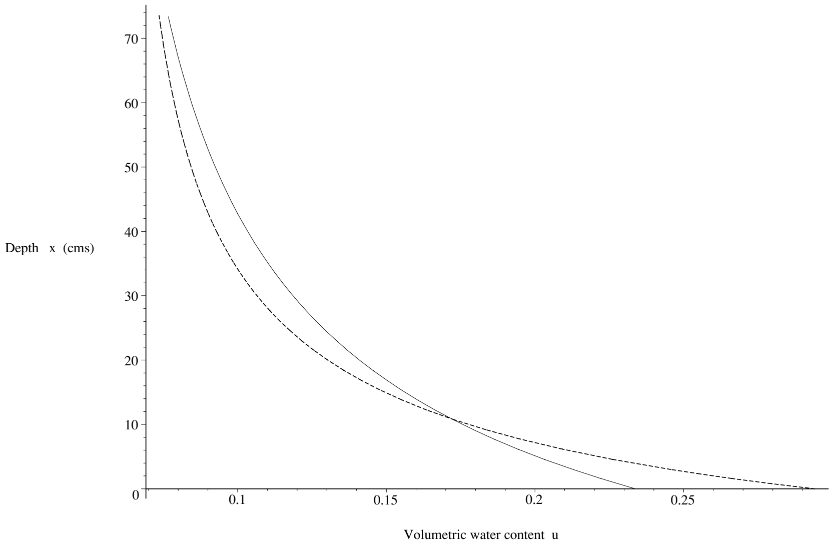

Figure 2. Redistribution of moisture content in a soil, drying at the surface, after 200 and 2000 s.

For this case equations (63) to (65) apply and without loss of generality it will be assumed that $c_0 = cc_1/k$. This has the advantage of rendering equation (65) integrable with the result that

$$\psi(\omega) = A\left(2 - \frac{k\omega}{c}\right)\mathrm{e}^{\frac{k\omega}{c}} + \omega + B \tag{117}$$

where A and B are constants. In addition (63) and (64) may be written in the form

$$x = \left(\frac{\omega}{\psi(\omega)} + k\lambda t\right)\mathrm{e}^{\lambda u} \tag{118}$$

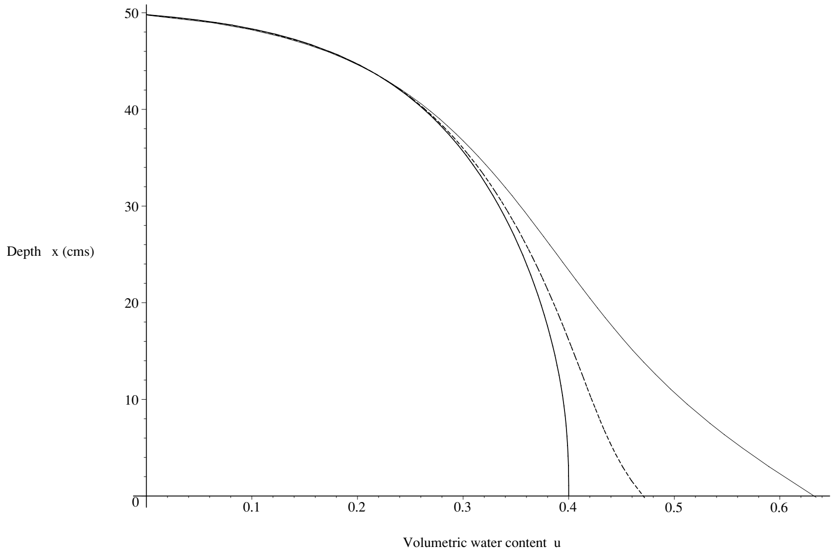

Figure 3. Redistribution of moisture at times $t = 0$, 1000 and 4000 s. Note that at each depth the volumetric moisture content increases as time passes.

$$e^{\lambda u} = \psi(\omega)\, e^{-\frac{k}{c}(\omega - x)} \tag{119}$$

where $x = \omega$ when $t = 0$ and where the x variable will once again be measured positively in the downwards direction

From (119) this initial condition means that

$$e^{\lambda u(x,0)} = \psi(x). \tag{120}$$

If in addition $u\,(0, 0) = u_0$ then equations (120) and (117) together imply that

$$\psi(\omega) = A\left(2 - \frac{k\omega}{c}\right) e^{\frac{k\omega}{c}} + \omega - 2A + e^{\lambda u_0}. \tag{121}$$

The constant A may be found by additionally imposing the condition:

$$u(x_L, 0) = u_L \tag{122}$$

to equations (120) and (121).

It follows that in this way $\psi(\omega)$ may be fully specified and so equations (118) and (119) can be used parametrically to produce curves of u versus x for particular values of t.

As a specific example consider a soil of thickness 50 cm which initially has volumetric moisture content $u_0 = 0.4$ at the surface and which is initially dry $u_L = 0$ at $x_L = 50$ cm. In addition suppose the diffusivity and hydraulic conductivity are given by

$$D = H_u \qquad H(u) = 0.0001\, e^{9u}\ \text{cm}^2\ \text{s}^{-1} \qquad K(u) = 0.000\,014\, e^{9u}\ \text{cm s}^{-1}. \tag{123}$$

These conditions together with equations (120) to (122) imply that $A = -7.133$ and so $\psi(\omega)$ is fully defined. It follows that equations (118) and (119) can now be used parametrically to produce curves which describe the redistribution of volumetric moisture content u versus depth x for particular values of time t. The results are given in figure 3.

7. Conclusion

In this paper the main aim has focused upon Richard's equation in its one-dimensional form as it applies to the flow of moisture in an unsaturated soil or other unsaturated porous media. In particular, the classical and non-classical symmetry reduction methods have been applied to the equation of flow in a uniform soil with unspecified hydraulic conductivity and moisture diffusivity. The analysis has centred upon Richard's equation expressed both in its standard second-order form, and also as a potential system of two first-order partial differential equations.

A review of the classical symmetries has been presented in tables 1 and 3 and as expected there are apparently many more symmetries for the potential system than for the standard second-order system form of Richard's equation. However, in section 6 an example with $H = cu^{\lambda}$, $K = ku^{\mu}$ is given that demonstrates the equivalence of a potential and a classical symmetry reduction. This is achieved using two surface invariant conditions which are differentially dependent. It is to be expected that this method will demonstrate the equivalence of many of the entries in tables 1 and 3. Nonetheless, it is also clear that the potential method does give rise to new symmetries and one apparently not previously published is presented at equations (77) to (79). The corresponding reduction to a system of first-order ordinary differential equations has been presented.

A non-classical symmetry reduction of Richard's equation is presented in sections 3 and 4, and the determining equation are shown to be over-determined, heavily non-linear and very lengthy. Nonetheless, it has been possible to determine solutions in five cases although the nature of $H(u)$ and $K(u)$ lacks the generality of the classical case. However, in each case the reduction of Richard's equation to an ordinary differential equation is presented and in the particular instance when $H = H(u)$, $K_u = -c_0 H$ an exact analytic solution is presented. A further analytic solution is presented in section 6. The non-classical method is also discussed in the context of the potential system in section 5 and this results in two non-linear undetermined equations for the three infinitesimals η, ϕ_1 and ϕ_2. However, it is also shown that one of these equations merely expresses the differential dependence of two surface conditions and so only one determining equation remains for the two functions η and ϕ_2. It follows that this equation can be closed only by the introduction of a further condition and there is need for research to determine the nature of such additional conditions. In this paper the assumption $\phi_2 = 0$ is introduced to enable a non-classical potential symmetry to be found.

In section 6 a case of classical (Lie/potential) symmetry and also a non-classical symmetry is applied to a Dirichlet problem of the vertical infiltration of moisture from the surface into a dry soil. Richard's equation is reduced to an ordinary boundary value problem which is solved numerically in the classical case whilst an analytic solution is presented for the non-classical example. The results are presented in figures 2 and 3. Whilst this method illustrates the usefulness of the symmetry method underspecialized boundary conditions there is no doubt that further research is necessary to establish its applicability under a broader range of physical circumstances and also the possible range of initial and boundary problems to which they may correspond.

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