Mechanical Systems II.
System of Differential Equations

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Abstract. The course treats a modern approach to classical mechanics. We shall gradually develop Lagrangian mechanics. Many subjects to be covered will be of general nature having application in all types of dynamical systems as optimization and stability. The first lecture will be a general introduction to conservative mechanics. We shall focus on so called second order differential equations. The subject of the second lecture will be the calculus of variations, which is a tool of dynamic optimization. The trajectory of a mechanical system is such that it uses minimum energy to get to a new state. Here the calculus of variation will be used to formulate equations of motion for a conservative system. It will be shown during the third lecture that the trajectories of mechanical systems solve so-called Euler-Lagrange equation. Forth lecture will be dealing with static optimization. I shall introduce Lagrange multipliers for this purpose. Final fifth lecture will be devoted to mechanical systems with constraints. I hope that we will have enough time to take some examples, which will build up your intuition and experience in Lagrangian mechanics.

Classical mechanics provides a huge number of beautiful examples of systems of differential equations. A simple example of a pendulum is used in the lecture to explain flavor and general features of the differential equations describing motion. A notion of a conservative mechanical system will be introduced. The order of a differential equation corresponding to motion can be huge. Here conservation laws might be a help, since they decreases the essential size. The findings will be used to show that Newton's law of gravitation implies Kepler's law of planets' motion.

1 Phase Plane

An important tool in analyzing an autonomous differential equation in $\mathbb{R}^2$ is to make a quick hand sketch of phase plane trajectories. We shall consider system of the form

$$\begin{aligned} \frac{dx}{dt} &= f(x,y), \\ \frac{dy}{dt} &= g(x,y). \end{aligned} \tag{1}$$

Notice that the right hand side of the differential equations does not depend explicitly on t. A differential equation or system of differential equations with this property will be called **autonomous**. The second notation I would like to introduce is a notion of **trajectory**. Trajectory is the solution $(x(t), y(t))$ of (1) for a given initial condition (x_0, y_0).

The key to hand sketching the solutions or trajectories of (1) is to use **isoclines**. An isocline is a locus of points, at which the trajectories have the same slope, in other words where

$$\frac{dy/dt}{dx/dt} = k = \text{const} \tag{2}$$

There are two particularly easy drawn isoclines: those of **horizontal slope** (where $\frac{dy}{dt} = \dot{y} = 0$) and those of **vertical slope** (where $\frac{dx}{dt} = \dot{x} = 0$). The isoclines of horizontal and vertical slopes are sometimes called **nullclines**. Points where the isoclines cross are called **singular points** or **zeros**.

The second key to hand sketching phase plane trajectories is to use the signs of $\dot{x}$ and $\dot{y}$ to tell where trajectories are moving left or right, up or down.

The following steps will lead you through the process.

1. Write the equation as

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \tag{3}$$

   and sketch isoclines of
   (a) **horizontal slope,** $\dot{y} = 0$. Mark the isocline with little horizontal slope marks. Indicate the regions on each side of this isocline with vertical arrows up where $\dot{y} > 0$ or down where $\dot{y} < 0$.
   (b) **vertical slope,** $\dot{x} = 0$. Mark the isocline with little vertical slope marks. Indicate the regions on each side of this isocline with horizontal arrows right where $\dot{x} > 0$ or left where $\dot{x} < 0$.
2. In each region determined by these isoclines, put together the horizontal and vertical arrows. Then sketch the resultant direction field using these components.
3. Trace some sample trajectories through the direction field, following all arrows and slope marks.

Example 1.

$$\begin{aligned} \dot{x} &= y - x - 2 \\ \dot{y} &= x^2 - y \end{aligned} \tag{4}$$

gives

$$\frac{dy}{dx} = \frac{x^2 - y}{y - x - 2} \tag{5}$$

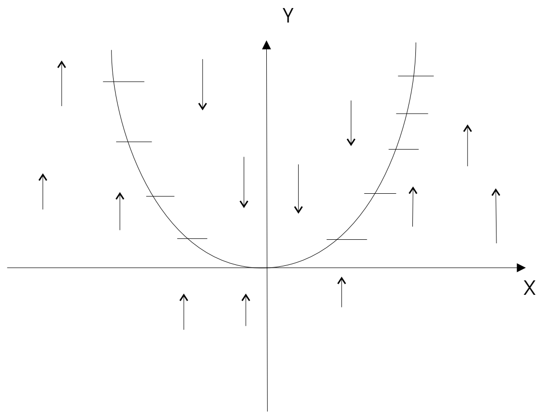

Fig. 1. Step 1a: Isoclines of horizontal slope $\dot{y} = x^2 - y = 0$.

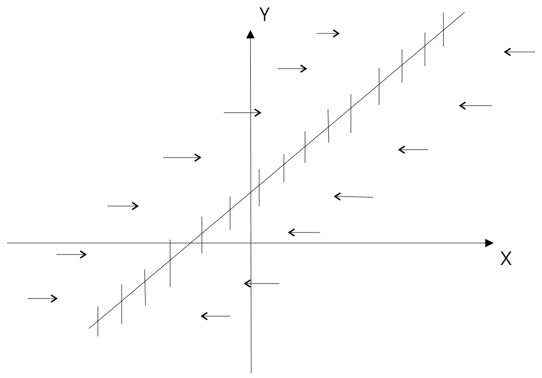

Fig. 2. Step 1b: Isoclines of vertical slope $\dot{x} = y - x - 2 = 0$.

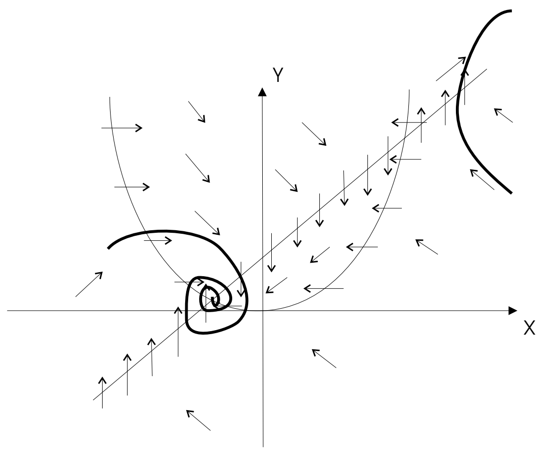

Fig. 3. Step 2a: Put together the horizontal and vertical arrows.

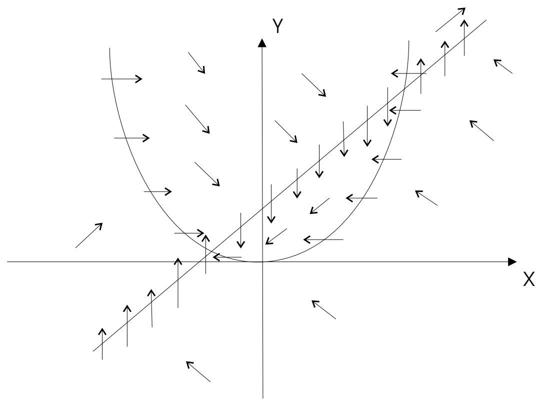

Fig. 4. Step 2b: Sketch the resultant direction field using horizontal and vertical arrows.

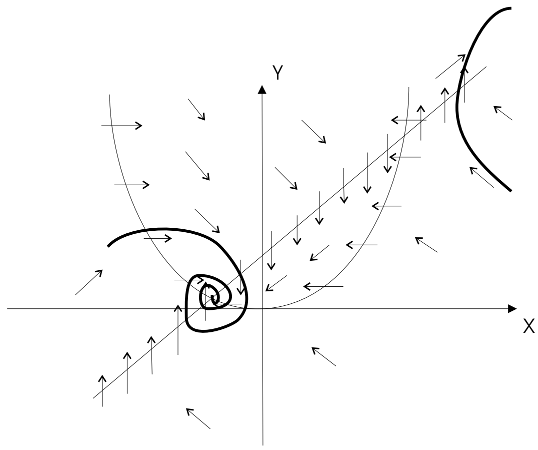

Fig. 5. Step 3: Trace some sample trajectories through the direction field, following all arrows and slope marks.

2 Mechanical Systems with One Degree of Freedom

In this section I shall use the simple pendulum as an example to discuss some aspects of general theory. Consider the pendulum with motion restricted to a plane, with a bob of mass m on a string of length l, as illustrated in Figure 6.

I shall tackle the problem of describing motion of the pendulum using the Newton's Laws. The pendulum sweeps $l\theta$ arc length. The velocity is then $l\dot{\theta}$, and the tangential acceleration is $l\ddot{\theta}$. We have

$$ml\ddot{\theta} = -mg\sin\theta \tag{6}$$

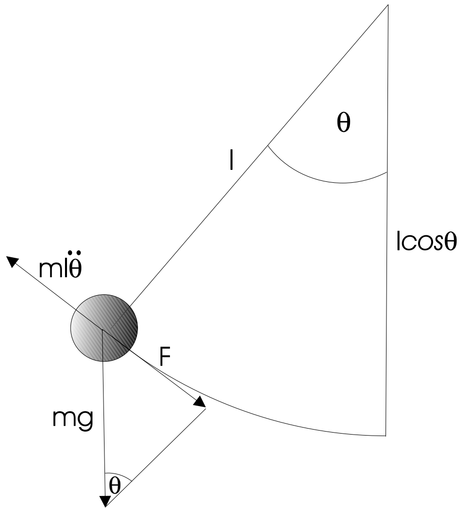

Fig. 6. Pendulum with motion restricted to a plane with a bob of mass m on a string of length l .

Thus

$$\ddot{\theta} = -\frac{g}{l}\sin\theta. \tag{7}$$

Instead of working with a second order differential equation I shall use a system of 1st order differential equations (consisting of two equations).

$$\begin{aligned}\dot{\theta} &= y \\ \dot{y} &= -\frac{g}{l}\sin\theta\end{aligned} \tag{8}$$

The pendulum is an example (a very simple one) of a class of **conservative** mechanical systems. I shall elaborate on this later on. For now on we shall introduce a notion of potential energy for the pendulum

$$U = -m\frac{g}{l}\cos\ddot{\theta}. \tag{9}$$

Then the equation (7) becomes

$$m\ddot{\theta} = -\frac{\mathrm{d}U}{\mathrm{d}\theta} \tag{10}$$

If $U : \mathbb{R}^n \to \mathbb{R}$ is differentiable function on Euclidean space then we shall denote $\frac{\mathrm{d}U}{\mathrm{d}x} = \left(\frac{\partial U}{\partial x_1}, ..., \frac{\partial U}{\partial x_1}\right)$ the gradient of the function U. Note that $n = 1$ in equation (8).

We shall use the following definition.

Definition 1. A system with one degree of freedom is a system described by one differential equaation

$$m\ddot{x} = f(x),\ x \in \mathbb{R}. \tag{11}$$

The kinetic energy is the quadratic form

$$T(\dot{x}) = \frac{1}{2}m\dot{x}^2. \tag{12}$$

The potential energy is a function

$$U(x) = -\int_{x_0}^{x} f(\zeta)d\zeta \tag{13}$$

■

Notice that the potential energy determines f, therefore to specify system (11) it is enough to specify the potential energy.

The total energy is the sum

$$E(x, \dot{x}) = U(x) + T(\dot{x}). \tag{14}$$

We shall now state the low of conservation of energy

Theorem 1 (Law of Conservation Energy). *The total energy E of points mowing according to the differential equation*

$$m\ddot{x} = -\frac{\mathrm{d}U}{\mathrm{d}x} \tag{15}$$

is conserved i.e. constant along all the trajectories.

Proof.

$$\frac{d}{dt}(U(x) + T(\dot{x})) = \frac{\mathrm{d}U}{\mathrm{d}x}\dot{x} + \dot{x}\ddot{x} = 0. \tag{16}$$

■

Particularly the total energy of the pendulum $E = \frac{1}{2}my^2 - \frac{mg}{l}\cos\theta$ is conserved. In the following we shall analyze behavior of the pendulum with respect to the energy it has. We shall analyze the phase plane of the system (8) in Figure 7.

Notice also that the potential energy $-\frac{mg}{l}$ is minimal, and it is reached for $\theta = 0$ and for $\theta = \pi$ the potential energy is maximal ($\frac{mg}{l}$). We have the following conclusions:

1. If $E = -\frac{mg}{l}$ the pendulum is at the rest at the bottom of its motion. This is a stable equilibrium (Actually it is Lyapunov stable. You will learn more about different notions of stability during the next semester course Control of Nonlinear Systems).

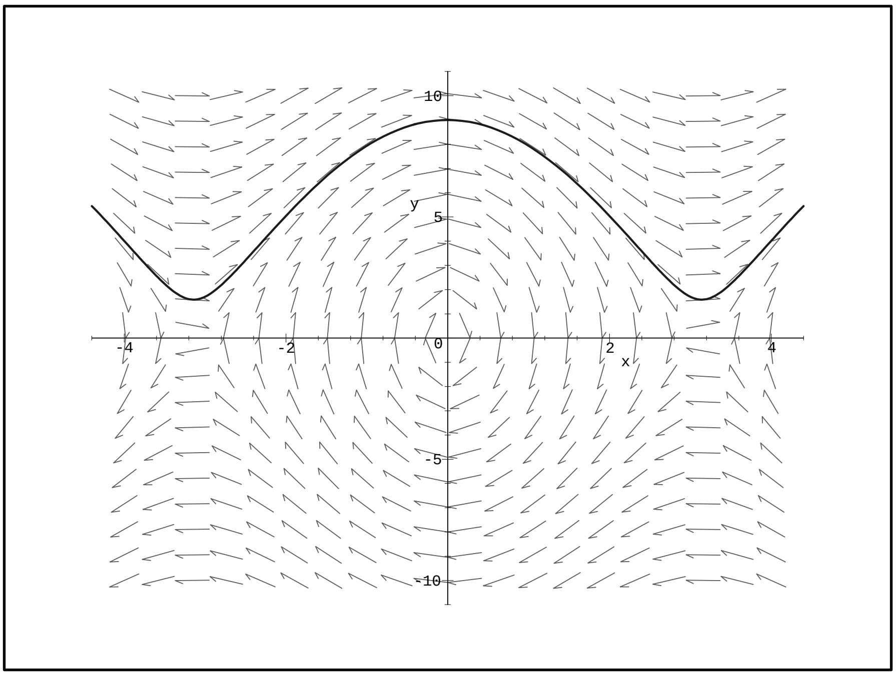

Fig. 7. Phase plane of the pendulum in equation (8). Remember to draw horizontal slope $\dot{y} = 0$, and mark vertical arrows up ($\dot{y} > 0$) and down ($\dot{y} < 0$). Then draw the vertical slope $\dot{x} = 0$, and the horizontal arrows right ($\dot{x} > 0$) or left ($\dot{x} < 0$).

2. If $E < -\frac{mg}{l}$ corresponds to forth and back oscillations.
3. If $E > -\frac{mg}{l}$ the pendulum has enough energy to go full circle.
4. If $E = -\frac{mg}{l}$ the level curves are joining the saddles. For $t \to \infty$ and $t \to -\infty$ the pendulum tends to unstable equilibrium of the very top of the swing. Note that the time required to go from one saddle to the next is infinite.

The solutions coming from or going to saddles are called **separatrices**. They separate the regions of generic from each other.

2.1 Effects of Adding Friction

I shall add to our pendulum equations a perturbation term. This will represent friction, a physically more realistic situation.

The motion of the pendulum bob is now

$$\begin{aligned} \dot{\theta} &= y \\ \dot{y} &= -k \sin\theta - \epsilon y, \end{aligned} \tag{17}$$

where $k = \frac{g}{l}$, and ϵ is the friction coefficient, i.e. friction is assumed to be proportional to the velocity.

The total energy is no longer constant on the trajectories, but decreases. The friction dissipates the energy

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}my(t)^2 - km\cos\theta(t)\right) = my\dot{y} + (km\sin\theta)\dot{\theta} \\ &= my(-k\sin\theta - \epsilon y) + ykm\sin\theta = -m\epsilon y^2 \leq 0. \end{aligned} \tag{18}$$

Notice that the regions $\{(\theta, y) : E(\theta, y) \leq c\}$ for some positive reals c are traps. Why ?. See the phase plane of the pendulum with friction in Figure 8.

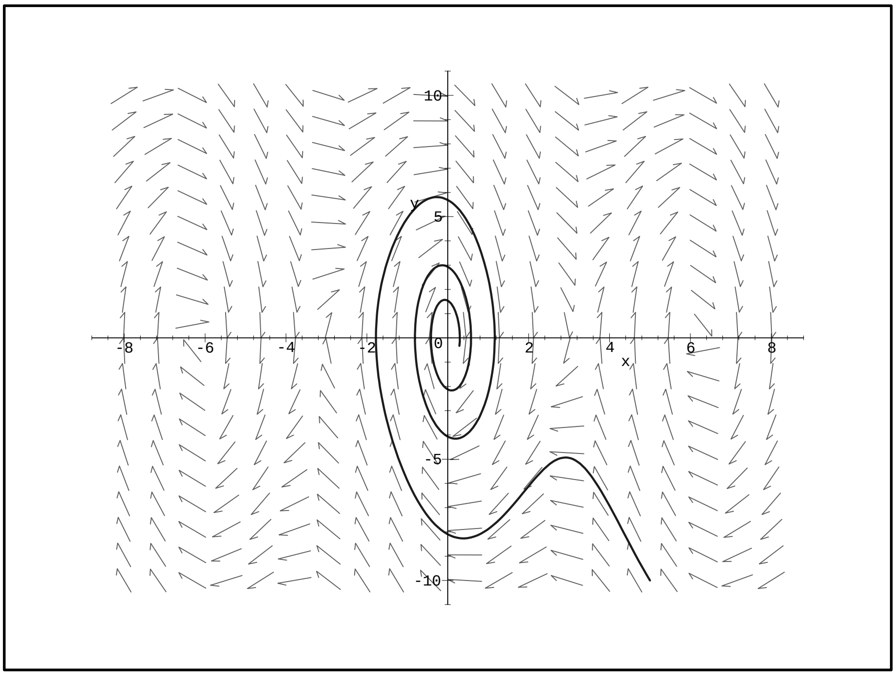

Fig. 8. Phase plane of the pendulum with friction, equation (17).

2.2 Conservation Law

At this point I shall introduce a notion of an **essential size**.

Definition 2. The essential size is the smallest number n such that the differential equation can be transformed into an autonomous first order equations in $\mathbb{R}^n$.■

Example 2. The system

$$\begin{aligned} \ddot{x} &= x^2 - y \\ \ddot{w} &= y - x \end{aligned} \tag{19}$$

has essential size 4, since

$$\begin{aligned} \dot{x} &= w \\ \dot{w} &= x^2 - y \\ \dot{y} &= v \\ \dot{v} &= y - x. \end{aligned} \tag{20}$$

Example 3.

$$\dot{x} = f(x, y) \tag{21}$$

is non-autonomous but can be written as autonomous

$$\begin{aligned} \dot{x} &= f(t, x) \\ \dot{t} &= 1, \end{aligned} \tag{22}$$

thus the essential size is two.

We shall spend some time discussing **conservation laws**. The following definition is in place.

Definition 3 (Conservation Law). If a differential equation in $\mathbb{R}^n$ implies for some function $F : \mathbb{R}^n \to \mathbb{R}$, that $\dot{F}((x(t)) = 0$ for each solution $x(t)$ then $F(x(t)) = c$, where c is a constant along the trajectories of the solutions. This is called a conservation law.■

Conservation laws are important, since they tell how to reduce the essential size of the system.

Example 4. Consider a mechanical system with one degree of freedom:

$$m\ddot{x} = -\frac{\mathrm{d}U}{\mathrm{d}x}. \tag{23}$$

Using the conservation law of the total energy $E = \frac{1}{2}my^2 + U(x)$ on the trajectories of the solutions we get

$$y^2 = 2(E - U(x))/m. \tag{24}$$

For $y \geq 0$ it has the solution $y = \sqrt{2(E - U(x))/m}$, thus $\dot{x} = y$ gives

$$\dot{x} = \sqrt{2(E - U(x))/m} \tag{25}$$

We have reduced the original system to the first order equation in $\mathbb{R}$.

The following theorem points out that if there is a conservation law then the reduction of the essential size is always possible.

Theorem 2 (Implicit Function Theorem). *Let $V \subset \mathbb{R}^n$ be open and $p \in V$. Let $f : V \to \mathbb{R}$ be smooth with $f(p) = a$. If $\frac{\partial f}{\partial y_k}(p) \neq 0$ for some coordinate y_k. Then there exists an open neighborhood W of p in V, such that the set of solutions to the equation $f(y) = a$ is the graph of smooth functions $y_k = g(y_1, ..., y_{k-1}, y_{k+1}, ..., y_n)$.*

2.3 System of Three Degrees of Freedom

Consider a particle in a central force $F = f(R)\frac{R}{|R|}$, where $R, F \in \mathbb{R}^3$, and $|\cdot|$ is the Euclidean distance, see Figure 9.

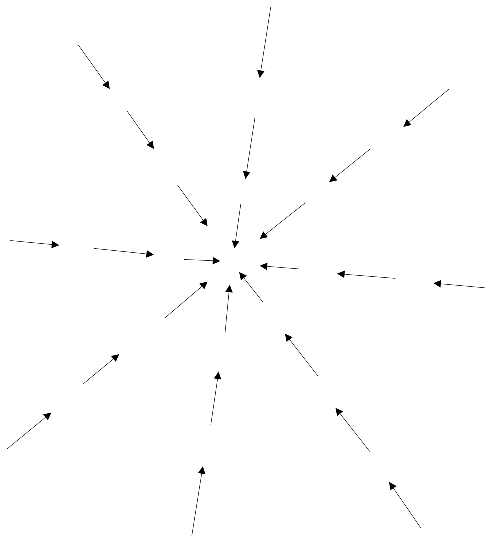

Fig. 9. Central force field.

From the Newton's law $m\ddot{R} = f(R)\frac{R}{|R|}$. Notice that the essential size is 6, since there are two equations for x, y, z coordinates.

Definition 4. The angular momentum of a material point of unit mass relative to the point 0 (center of force) is the vector

$$M = R \times \dot{R} \tag{26}$$

■

Theorem 3 (The Law of Conservation of Angular Momentum). *Under motion in a central force field , the angular momentum M relative to the center of the field 0 does not change with time.*

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}M = \dot{R} \times \dot{R} + R \times \ddot{R} = R \times \ddot{R} = R \times f(R)\frac{R}{|R|} = 0 \tag{27}$$

■

Conservation of the total energy and the angular velocity together imply that the essential size of the equation of motion is 6-3-1=2. Thus the problem of motion in the central force field can be reduced to two dimensional problem. The law of conservation of angular momentum was first discovered by Kepler (1571-1630) through observation of the motion of Mars. We shall spend the rest of this lecture analyzing the **two body problem**, which is an example of motion in a central force.

The constant direction of M ensures that R will never leave the plane spanned by $R(t_0)$ and $\dot{R}(t_0)$ for particular time t_0. This means that the central force field can be assumed to operate only in $\mathbb{R}^2$. The two body problem will be now much easier to deal with if the polar coordinates R, ϕ are introduced in this plane:

$$R = re_r + \phi e_\phi, \tag{28}$$

where e_r and e_ϕ are basis for polar coordinate system (e_r directed along R and e_ϕ perpendicular to it and in the direction of increasing ϕ).

Lemma 1. *We have the relation*

$$\dot{R} = \dot{r}e_r + r\dot{\phi}e_\phi \tag{29}$$

Proof. Notice first that $\dot{e}_r = \dot{\phi}e_\phi$ and $e_\phi = -\dot{\phi}\dot{e}_r$. Then

$$\dot{R} = \frac{\mathrm{d}}{\mathrm{d}t}(re_r) = \dot{r}e_r + r\dot{e}_r = \dot{r}e_r + r\dot{\phi}e_\phi. \tag{30}$$

■

Consequently the angular momentum is

$$M = R\times\dot{R} = R\times\dot{r}e_r + R\times r\dot{\phi}e_\phi = re_r\times\dot{r}e_r + re_r\times r\dot{\phi}e_\phi = r^2\dot{\phi}e_r\times e_\phi. \tag{31}$$

Hence the quantity $r^2\dot{\phi}$ is conserved. This has a simple geometric meaning. Define the area ΔS swept out by the radius vector during Δt time, see Figure 10.

Kepler's Second Law says: *In equal times the radius vector sweeps out equal areas, hence the sectorial velocity* $\frac{\mathrm{d}S}{dt}$ *is constant.* Notice however that

$$\Delta S = S(t + \Delta t) - S(t) = \frac{1}{2}r^2\dot{\phi}\Delta t + o(\Delta t) \tag{32}$$

and $\frac{\mathrm{d}S}{dt} = \frac{1}{2}r^2\dot{\phi} = \frac{1}{2}M$. We conclude that the Kepler's Second Law is equivalent to the angular momentum conservation. Our results can be converted to two body problem.

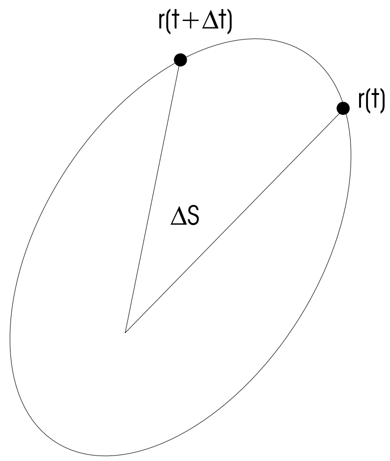

Fig. 10. Definition of the are $\varDelta S$.

Example 5. Assuming two bodies with masses m_1, m_2, and positions x_1, x_2 the **Newton's Universal Law of Gravity** is

$$\ddot{x}_1 = = \frac{Gm_2(x_2 - x_1)}{|x_2 - x_1|^3}$$
$$\ddot{x}_2 = = \frac{Gm_1(x_1 - x_2)}{|x_1 - x_2|^3}. \tag{33}$$

This is a priori a differential equation in $\mathbb{R}^{12}$. The reduction of the dimensions goes in the following way. The **center of mass**

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{34}$$

satisfies $\ddot{X} = 0$. Thus having the position of the first body it is possible to compute the position of the second one. Now I shall use $R = x_1 - X$ to describe position of the first body. Then the motion is given by

$$\ddot{R} = -\frac{Gm_2^3}{(m_1 + m_2)^2} \frac{R}{|R|^3}, \tag{35}$$

and it corresponds to the motion in a central field:

$$F(R) = f(R)\frac{R}{|R|} = f(R)e_r, \text{ where } f(R) = -\frac{Gm_2^3}{(m_1 + m_2)^2} \frac{1}{|R|^2}. \tag{36}$$

In order to give a coordinate-wise description of motion (35) we need to determine $\ddot{R}$

$$\ddot{R} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}e_r + r\dot{\phi}e_\phi) = \ddot{r}e_r + \dot{r}\dot{\phi}e_\phi + \dot{r}\dot{\phi}e_\phi + r\ddot{\phi}e_\phi + r\dot{\phi}^2 e_r$$
$$= \left(\ddot{r} - r\dot{\phi}^2\right) e_r + \left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right) e_\phi. \tag{37}$$

Hence $\ddot{r} - r\dot{\phi}^2 = f(r)$ and $2\dot{r}\dot{\phi} + r\ddot{\phi} = 0$.

By the law of conservation of angular momentum $\dot{\phi} = \frac{M}{r^2}$ thus

$$\ddot{r} - \frac{M^2}{r^3} = f(r). \tag{38}$$

This yields

$$\ddot{r} = -\frac{k}{r} + \frac{M^2}{r^3}, \text{ where } k = \frac{Gm_2^3}{(m_1 + m_2)^2}. \tag{39}$$

It is worth mentioning that the system (39) is conservative for potential energy $U(r) = -\frac{k}{r} + \frac{M}{2r^2}$. I believe you will also find interesting the phase plane drawing for the two body problem, see Figure 11.

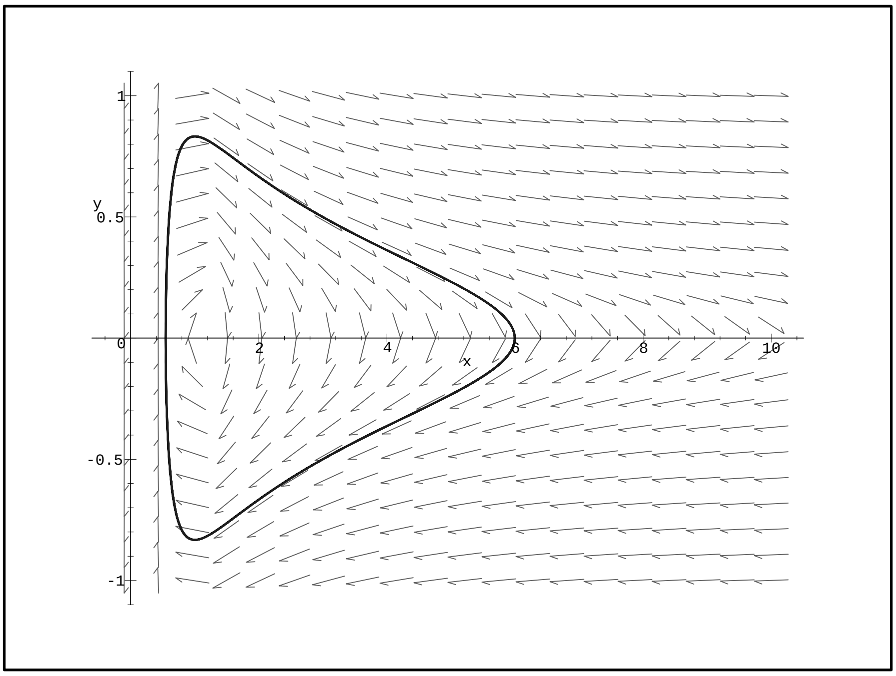

Fig. 11. Phase Plane for Two Body Problem

Mechanical Systems II.
Introduction to Calculus of Variations

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Abstract. The subject of the second lecture is the calculus of variations. A motivating example of the Fermat principle is addressed first. I shall then generalize this example to an arbitrary mechanical system. Finding a trajectory of a mechanical system is reduced to a solution of minimum/maximum problem. The main result of the lecture is the Euler-Lagrange equation, which gives a necessary condition for a solution for such an optimal problem. The calculus of variations is a relatively simple piece of mathematics, which you will find useful in many other aspects of engineering. At the end of the lecture I shall provide a number of examples, which we will solve using the Euler equation.

1 Introduction

If the function f attains a maximum or minimum value at a point x in the interiod of its domain, then the derivative is zero at that point

$$\dot{f}(x) = 0. \tag{1}$$

This equation may be satisfied even if f does not attain an extreme value at x.

A point which satisfies this equation is called a **critical point** for the function f. What if the domain is n-dimensional ? We again suppose that the function f attains an extreme value at a point x in $\mathbb{R}^n$ and in the interior of its domain. We introduce a vector η (also in $\mathbb{R}^n$) which serves as a direction for analysis of the function. Then the point $x + \epsilon\eta$, where ϵ is a real number, represents a point on the straight line through x in the direction h. Thus the function of ϵ given by the expression $f(x + \epsilon\eta)$ attains an extreme value at $t\epsilon = 0$. So now

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(x + \epsilon\eta)\right|_{\epsilon=0} = 0 \tag{2}$$

The quantity on the left side of this equation is often called the **directional derivative** of f in the direction h, and it is denoted $D_\eta f(x)$ (you will meet this notion frequently in the next semester course Nonlinear Control). The chain rule of multivariable calculus enable us to write this equation in the form

$$\sum_{j=1}^{n} \frac{\partial f(x)}{\partial x_j} \eta_j = 0. \tag{3}$$

Since (3) must hold for every choice of the direction η, we may use the n coordinate vectors $h = (0, ..., 0, 1, 0, ..., 0)$ to conclude that (3) is equivalent to

$$\frac{\partial f(x)}{\partial x_j} = 0 \text{ for all } 1 \leq j \leq n. \tag{4}$$

I shall stress the following observations. A point x satisfying (4) is said to be a **critical point** for f. Any extreme point for f is a critical point, but not conversely. The point $x + \epsilon\eta$ in the analysis is said to be a **variation** of the fixed point x. The idea is that ϵ is small in the absolute value and it is only limiting behavior of $f(x + \epsilon\eta)$ as $\epsilon \to 0$ that interests us. We proceed from (3) to (4) by judicious choices of ϵ.

2 Fermat's Principle

Fermat (1601-1665) put forward the following hypothesis. The light beam from a point P to a point Q moves such that the time necessary to cover the distance from P to Q is smallest possible. The velocity of light in a homogeneous matter is $\frac{c}{n}$, where c is the velocity of light in the vacuum, and n is a scalar value, so called refraction index. We shall treat the refraction index as a function in $\mathbb{R}^3$, then the time of covering the distance from P to Q is

$$\int_P^Q dt = \int_p^Q \frac{n(x, y, z)}{c} ds, \tag{5}$$

where x, y, and z are parameterized by the arc length s that is $x = x(s),\ y = y(s),\ z = z(s)$. In a homogeneous matter the fastest distance is also the shortest one, i.e. a straight line.

Consider the following intellectual experiment. The ray passes through a stack of homogeneous plates with different refraction indexes n_i as in Figure 1.

In each plate the ray is a straight light. The only places where the ray refracts are the boundary between the plates. Therefore the ray forms a polygon, and the time for passing the distance form P to Q is

$$t = \sum_{i=1}^{p} \frac{n_i}{c} s_i = \frac{1}{c} \sum_{i=1}^{p} n_i \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} \tag{6}$$

The task is now to determine the polygon giving the minimum time for passing from P to Q. We may analyze the problem for each plate separately. Since the plate i has the width determined by x_i and x_{i+1}, and the ray has initial position (x_i, y_i) the minimum passing time is determined by

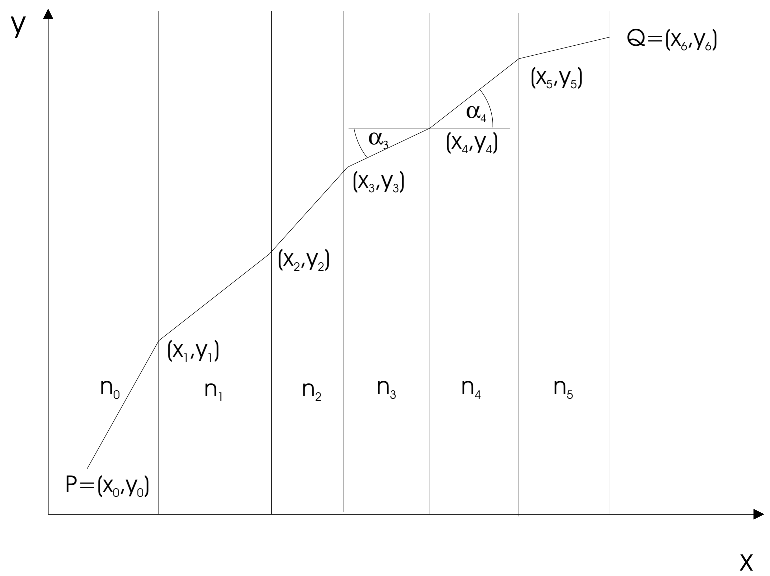

Fig. 1. The ray passes through a stack of homogeneous planes with different refraction indexes.

$$c\frac{\partial t}{\partial y_i} = \frac{n_{i-1}(y_i - y_{i-1})}{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}} - \frac{n_i(y_{i+1} - y_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} = 0. \tag{7}$$

From Figure 1

$$\sin\alpha_{i-1} = \frac{n_{i-1}(y_i - y_{i-1})}{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}}, \tag{8}$$

thus $n_{i-1}\sin\alpha_{i-1} - n_i \sin\alpha_i = 0$ and

$$\frac{n_{i-1}}{n_i} = \frac{\sin\alpha_i}{\sin\alpha_{i-1}} \tag{9}$$

which we recognize as the **law of refraction**. What if our medium changes the refraction index continuously. The problem changes completely its nature. We shall study all curves $\xi : [x_P,\ x_Q] \to [y_P,\ y_Q]$ going from $P = (x_P, y_P)$ to $Q = (x_Q, y_Q)$ and determine which of those minimizes the integral

$$I(\xi) = \int_{x_P}^{x_Q} \frac{1}{c} n(x) ds. \tag{10}$$

Substituting $ds = \sqrt{dx^2 + d\xi^2} = dx\sqrt{1 + \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^2}$ into equation (10) one gets

$$I(\xi) = \int_{x_P}^{x_Q} \frac{1}{c} n(x)\sqrt{1 + \dot{\xi}^2} dx. \tag{11}$$

The moral is that a problem of computing a trajectory of a mechanical system between two points can be reduced to computing the trajectory minimizing a certain functional defined by an integral equation.

Notice that I called I in equation (11) a **functional**. By this I mean a map defined on a set of curves with real values.

2.1 Euler-Lagrange Equation

I the future we shall study the problem of finding the curve ξ minimizing

$$I(\xi) = \int_a^b F(x, \xi, \dot{\xi}) dx, \tag{12}$$

where a and b are some real numbers, and $F(x, u, v)$ is a function of three variable ($F : \mathbb{R}^3 \to \mathbb{R}$). In the above formulation I was imprecise, since I did not specify what is the domain of the functional I. In the sequel the domain of I, in other words the allowed curves, are those continuously differentiable curves $\tilde{\xi} : [a,\ b] \to \mathbb{R}$ that start at a and end at b.

Assume that ξ is the minimum. Then all other curves joining a and b are obtained by adding a **variation** $\delta\xi \in C^1$ to $\hat{\xi}$, see Figure 2.

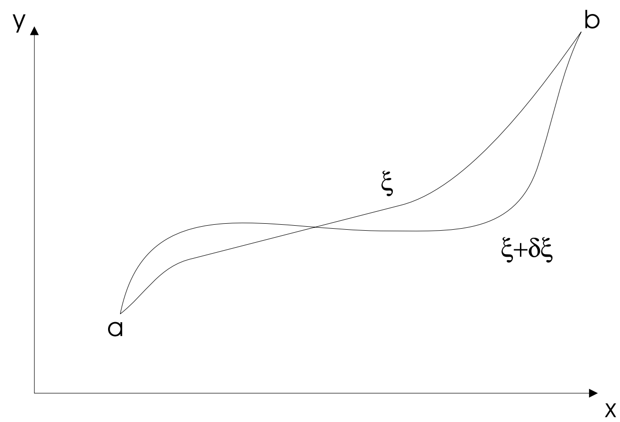

Fig. 2. The ray passes through a stack of homogeneous planes with different refraction indexes.

$$\tilde{\xi} = \xi + \delta\xi. \tag{13}$$

Since $\tilde{\xi}(a) = \xi(a)$ and $\tilde{\xi}(b) = \xi(b)$ it follows that $\delta\xi(a) = \delta\xi(b) = 0$.

In order to reduce the problem of finding a minimum of the functional I to the problem of finding a minimum of a real function of one variable I shall introduce the following trick. I represent $\delta\xi(x) = \epsilon\eta(x)$, where ϵ is a real variable and ϵ is unknown but fixed curve from a to b. Isn't this the same idea as in Section 1? We call functions η **test functions**. I shall use a notation $I(\epsilon) := I(\xi + \epsilon\eta)$, then $I(0) \leq I(\epsilon)$ for all ϵ. Thus $\frac{\mathrm{d}I(\epsilon)}{\mathrm{d}\epsilon}|_{\epsilon=0} = 0$! Or using the notion of the directional derivative

$$D_\eta I = 0. \tag{14}$$

Let us compute this expression. I shall use the following notation $F_z(x, y, z) = \frac{\partial F(x,y,z)}{\partial z}$

$$\begin{aligned} D_\eta I &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} I(\xi + \epsilon\eta) = \int_a^b \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big(F(x, \xi + \epsilon\eta, \dot{\xi} + \epsilon\dot{\eta})\Big)\,dx \\ &= \int_a^b \Big(F_\xi(x, \xi, \dot{\xi})\eta + F_{\dot{\xi}}(x, \xi, \dot{\xi})\dot{\eta}\Big)\,dx \end{aligned} \tag{15}$$

What about applying the integration by parts to the last term of (15). Does this help to simplify the directional derivative? We try

$$\int_a^b F_{\dot{\xi}}(x, \xi, \dot{\xi})\dot{\eta}dx = F_{\dot{\xi}}(x, \xi, \dot{\xi})\eta\Big|_a^b - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x} F_{\dot{\xi}}(x, \xi, \dot{\xi})\eta dx. \tag{16}$$

Notice that $\eta(a) = \eta(b) = 0$ hence the first summand disappears. We are left with the equation

$$D_\eta I = \int_a^b \left(F_\xi(x, \xi, \dot{\xi}) - \frac{\mathrm{d}}{\mathrm{d}x} F_{\dot{\xi}}(x, \xi, \dot{\xi})\right)\eta dx = 0 \tag{17}$$

which has to be true not only for one particular η but for all curves η joining a and b.

The next trick is to apply the **Fundamental Lemma of the Calculus of Variations**. We shall use the following notation: $C^0[a, b]$ are all continuous functions defined on the closed interval $[a, b]$, similarly $C^1[a, b]$ means continuously differentiable functions on $[a, b]$.

Lemma 1. *If $f \in C^0[a, b]$ and*

$$\int_a^b f(x)\eta(x)dx = 0 \tag{18}$$

for each function $\eta \in C^0[a, b]$ with $\eta(a) = \eta(b) = 0$ if and only if $f(x) = 0\ \forall x \in [a, b]$.

Before I will proof this very useful lema (definitely worth of remembering) I shall apply it to our minimization problem in (17) and get

$$F_\xi(x,\xi,\dot{\xi}) - \frac{\partial}{\partial x} F_{\dot{\xi}}(x,\xi,\dot{\xi}) = 0 \tag{19}$$

This formula bears names of Euler (1707-1783) and Lagrange (1736-1813), the **Euler-Lagrange equation**.

Proof. I shall prove the lemma by a contradiction. Assume that there is a function f such that $f \neq 0$ and it satisfies (18). It means that there is a point $x_0 \in (a,b)$ for which $f(x_0) \neq 0$. Assume without loss of generality that $f(x_0) > 0$. But f is continuous thus there exist a positive number c and an interval (x_1, x_2) around x_0 such that $f(x) > 0$ for $x \in (x_1, x_2)$. Pick η as a bump function in Figure 3. Then

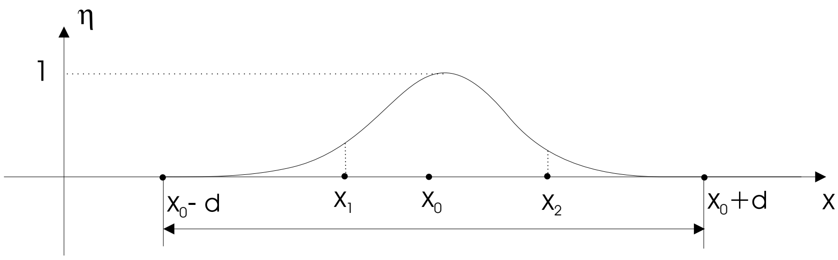

Fig. 3. The graph of bump function with $f(x) \neq 0$ for $x \in (x_0 - d, x_0 + d)$ and the maximum value equal 1.

$$\int_a^b f(x)\eta(x)dx = \int_{x_0-d}^{x_0+d} f(x)\eta(x)dx \geq \int_{x_1}^{x_2} c \cdot 1 \; dx = c(x_2 - x_1) > 0 \tag{20}$$

which contradicts the assumption. ■

Notice that in the proof we did not make any use of a particular coordinate system (rectangular, polar) therefore the results are valid in each coordinate system.

Before making any generalization to $\mathbb{R}^n$ with arbitrary dimension n I owe you a simple example.

Example 1. I would like to make sure that the shortest distance between two points in the Euclidean space $\mathbb{R}^2$ is the straight line.

We have given points a and b and shall compute the curve ξ with shortest distance between them

$$\int_a^b ds = \sqrt{1 + \dot{\xi}^2} dx. \tag{21}$$

Apply the Euler-Lagrange equation for $F(x, \dot{\xi}) = \sqrt{1+\dot{\xi}^2}$ and get

$$F_{\xi}(x, \dot{\xi}) - \frac{\mathrm{d}}{\mathrm{d}x} F_{\dot{\xi}}(x, \dot{\xi}) = 0 + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\dot{\xi}}{\sqrt{1+\dot{\xi}^2}}\right) = 0 \tag{22}$$

Thus

$$\frac{\dot{\xi}}{1+\dot{\xi}^2} = C^2, \tag{23}$$

where C is a constant. This give $\dot{\xi}^2 = \frac{C^2}{1-C^2}$ and finally we get a line

$$\xi = Dx + E, \tag{24}$$

for some constants D and E.■

We make the following observation. If $F = F(x, \dot{\xi})$, then $\frac{\partial}{\partial x} F_{\dot{\xi}} = 0$ hence $F^{'}_{\dot{\xi}} = const$.

Example 2. Imagine a soap film spanned between two circular rings, see Figure 4. Our task is to compute what form the film will take on. We know that to generate a film of are dA we shall use work corresponding to CdA, where C is the tension constant of the surface. The soap film takes the form such that its area is smallest possible. Translating this into integral equation we get

$$I(r) = \int_{-a}^{a} dA = 2\pi \int_{-a}^{a} r(x) ds = 2\pi \int_{-a}^{a} r\sqrt{1+\dot{r}^2} dx, \tag{25}$$

where $r(x)$ is the radius of the film at x and s is the arc length. The boundary conditions are $r(a) = r(-a) = b$

We shall apply Euler-Lagrange equation to $F(x, r, \dot{r}) = r\sqrt{1+\dot{r}^2}$. Notice that in fact F does not depend implicitly on x.

If $F(x, r, \dot{r}) = F(r, \dot{r})$ with $\dot{r} \neq 0$ one can use the following trick

$$\int_a^b F(r, \dot{r}) dx = \int_a^b F\left(r, \frac{1}{\dot{x}}\right) \dot{x} dr, \tag{26}$$

since $r(x(r)) = id$ at least in the small neighborhood of r. Hence $\frac{\mathrm{d}r(x)}{\mathrm{d}x}\frac{\mathrm{d}x(r)}{\mathrm{d}r} = 1$ and $x_y = y_x$. Define

$$G(r, \dot{x}) = F(r, 1/\dot{x})\dot{x}, \tag{27}$$

and apply the Euler-Lagrange equation to G. This gives

$$G_{\dot{x}} = C, \text{ where } C \text{ is a constant.} \tag{28}$$

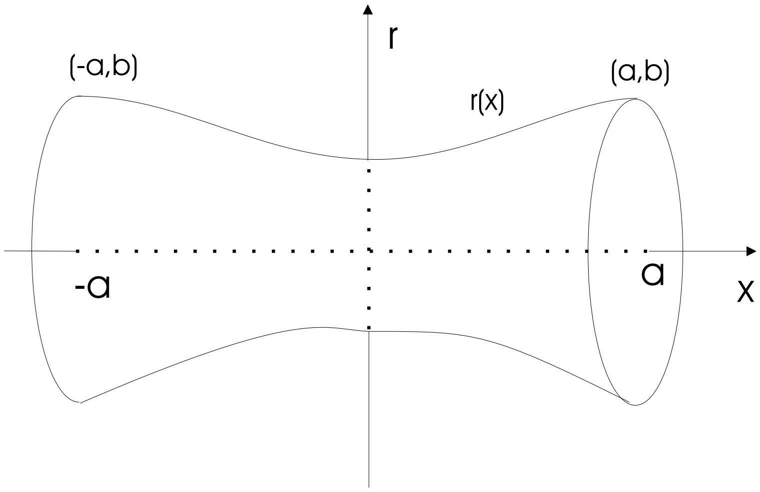

Fig. 4. A soap film spanned between two circular rings.

Substitute F with G yields

$$-F_{1/\dot{x}}(r, 1/\dot{x})\dot{x}^{-2}\dot{x} + F(r, 1/\dot{x}) = C, \tag{29}$$

hence

$$-F_{\dot{r}}(r, \dot{r})\dot{r} + F(r, \dot{r}) = C. \tag{30}$$

Equipped with (30) we can proceed we the example. Recall that $F(r, \dot{r}) = r\sqrt{1 + \dot{r}^2}$ then Euler-Lagrange equation is

$$C = -F_{\dot{r}}\dot{r} + F = -\frac{r\dot{r}}{\sqrt{1+\dot{r}^2}}\dot{r} + r\sqrt{1+\dot{r}^2}. \tag{31}$$

We conclude that

$$\frac{r}{\sqrt{1+\dot{r}^2}} = C \tag{32}$$

which leads to

$$\dot{r} = \pm\sqrt{\frac{r^2}{C^2} - 1} \tag{33}$$

I will not bother you with solving this ordinary differential equation at this stage. I will only state that it has a nice solution, see (1)

$$r = \frac{a}{\alpha}\cosh\left(\alpha\frac{x}{a}\right), \tag{34}$$

where the constant α is a solution of

$$\frac{b}{a}\alpha = \cosh \alpha. \tag{35}$$

Try to calculate (35) using your computer. Ups! you will find out that for small ratios b/a there are no solutions, simply the two rings are too far away apart. Conversely for some small values of b/a there are two solutions corresponding to the minimal and maximal area. ■

The Euler-Lagrange equation is only a necessary condition for the functional I to attain extremum value at ξ.

3 Euler-Lagrange Equation Generalized to $\mathbb{R}^n$

Without being too specific about hypotheses, suppose that D is a reasonable bounded open set in $\mathbb{R}^n$ with closure $\bar{D}$, and suppose that F is a smooth real-valued function defined on $\bar{D} \times \mathbb{R}^{n+1}$. Then for any C^1 function $u : \bar{D} \to \mathbb{R}$ we can define the integral

$$I(u) = \int_D F\left(x, u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right) dx. \tag{36}$$

Usually u will have to satisfy same restrictions on the boundary ∂D of D. These restrictions are called **boundary conditions**. A common example is the so-called **Dirichlet** (1805-1859) **condition**, in which the restriction of u to ∂D is a given function defined on ∂D.

We are then going to investigate possible critical points (among functions u) for I. We shall again apply for this purpose variations $u + \epsilon\eta$ of u, where the test function η are smooth and zero on ∂D. For each such test function we can calculate the directional derivative

$$\begin{aligned} D_\eta I(u) &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} I(u+\epsilon\eta)\Big|_{\epsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_D F(x, u+\epsilon\eta, u_{x_1}+\epsilon\eta_{x_1,...})\Big|_{\epsilon=0} \qquad (37) \\ &= \int_D \frac{\mathrm{d}}{\mathrm{d}\epsilon} F(x, u+\epsilon\eta, u_{x_1}+\epsilon\eta_{x_1,...})\Big|_{\epsilon=0} = \int_D \left(\frac{\partial F}{\partial u}\eta + \frac{\partial F}{\partial u_{x_1}}\eta_{x_1} + ...\right) dx. \end{aligned}$$

Mimicking the scalar case from the last section we integrate by parts to get rid of all term $\frac{\partial g}{\partial u_{x_i}}$. Note also that no integration over ∂D is required, thanks to the fact that η is zero on δD. The result is

$$D_\eta I(u) = \int_D \left(\frac{\partial F}{\partial u} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\frac{\partial F}{\partial u_{x_1}}\right)\eta dx. \tag{38}$$

As you surely guessed we apply now the Fundamental Lemma of the Calculus of Variations to conclude that if u is a critical point of I, then the

Euler-Lagrange equation is satisfied

$$\left(\frac{\partial F}{\partial u} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\frac{\partial F}{\partial u_{x_1}}\right) = 0. \tag{39}$$

References

[1] Erik Hansen. *Variations-Regning*. Plyteknisk Forlag, 1976.

Mechanical Systems II
Lagrange Mechanics

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Abstract. The third lecture deals with application of the calculus of variations to classical mechanics. Hamilton's principle says that the motion of a mechanical system form time a to b is such that the integral of the difference of kinetic and potential energies is minimal for the correct path of the motion. Necessary conditions for the minimum of such a functional is the Euler-Lagrange equation. We shall spend some time in the lecture discussing a notion of the generalized forces, dissipation, and constraints in the mechanical systems.

1 Lagrange Mechanics

During the last lecture we have discussed the Fermat's principle, which says that the light between two points a and b takes the path such that the passing time defined by the integral

$$\int_a^b \frac{l}{c} n(x,y,z)\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}dt \tag{1}$$

is smallest possible. In the above equation t is a parameter. I formulated the principle so that the light ray is a curve of the minimum passing time. However, this is correct only if the points a and b are sufficiently near each other. If they are very far apart it could happen that the light ray actually gives the maximum time! This is true for many applications of the calculus of variations in physics. Therefore the correct formulation of the Fermat's Principle is that the functional (1) is critical for the correct path. The word "critical" is often exchanged with "**stationary**" in the literature on classical mechanics, see (1).

The instantaneous configuration of a mechanical system with n degrees of freedom is described by the values of the n **generalized coordinates** $q_1, ..., q_n$, and corresponds to a particular point in a Cartesian hyperspace where the q's form the n coordinate axes. This n-dimensional space is known as configuration apace. I must emphasize that configuration space has no necessary connection with the physical three-dimensional space (of a particle or a robot), just as the generalized coordinate coordinates are not necessary position coordinates.

Definition 1 (Hamilton's Principle). The motion of of a mechanical system from time a to b is such the integral

$$I(t, q, \dot{q}) = \int_a^b L(t, q, \dot{q})dt, \tag{2}$$

where $L = T - U$, has a stationary value (is critical) for correct path of the motion. The function L is called the **Lagrangian**. ■

Using our experience from Lecture 2 we conclude that the necessary condition for a trajectory to be a critical point of (2) is that the Euler-Lagrange equation

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0, \quad \text{where } \dot{q}_i = \frac{\mathrm{d}q_i}{\mathrm{d}t} \tag{3}$$

is satisfied along the trajectory. We could write (3) more compactly in the vector notation

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = 0, \tag{4}$$

where $q = (q_1, ..., q_n)$ and correspondingly $\dot{q} = (\dot{q}_1, ..., \dot{q}_n)$.

You should think about Hamilton's principle as one replacing/generalizing Newton's laws. This is illustrated by the following example.

Example 1. In Figure 1 I depicted the Atwood's machine with two masses M_1 and M_2 and an inextensible rope of the length l. Consider that all motions are without slipping. The objective is to find a differential equation describing motion of the machine.

In order to use the Hamilton's principle we have to derived expressions for kinetic and potential energies. The Atwood's machine has one degree of freedom, and we may treat the distance x as the generalized coordinate. Kinetic energy is then

$$T = \frac{1}{2}M_1\dot{x}^2 + \frac{1}{2}M_2\dot{y}^2 = \frac{1}{2}(M_1 + M_2)\dot{x}^2. \tag{5}$$

Potential energy U is

$$\begin{aligned} U &= M_1 x' g + M_2 y' g = M_1(h - x)g + M_2(h - y)g \\ &= (M_1 + M_2)hg - M_1 xg - M_2 yg = -M_1 xg + M_2 xg + const, \end{aligned} \tag{6}$$

where g is the gravitational constant.

Now the Lagrangian becomes

$$L = T - U = (M_1 + M_2)\dot{x}^2 + (M_1 - M_2)gx + const. \tag{7}$$

It follows that

$$\frac{\partial L}{\partial x} = (M_1 - M_2)g \tag{8}$$

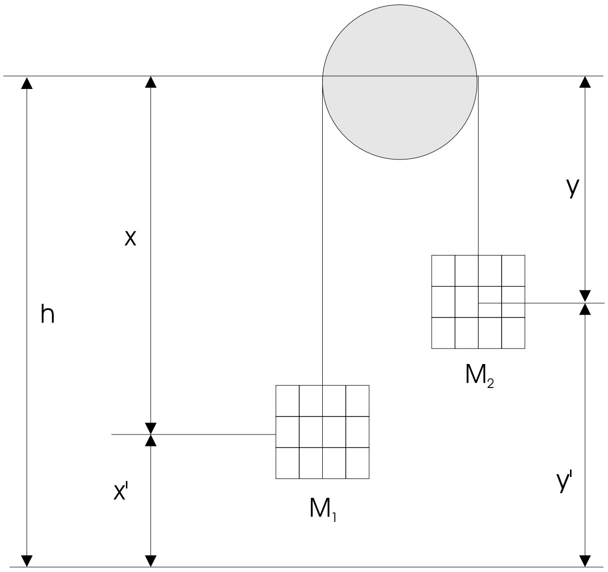

Fig. 1. The Atwood's machine consists of two masses M_1, M_2 and an inextensible rope of the length l. The distance y is defined as $y := l - x$

and

$$\frac{\partial L}{\partial \dot{x}} = (M_1 + M_2)\dot{x}. \tag{9}$$

Thus the equation of motion is

$$\ddot{x} = \frac{M_1 - M_2}{M_1 + M_2} g. \tag{10}$$

■

So far we have establish tools to deal with mechanical systems for which potential energy and kinetic energy is known. The nest step is to examine ways of including external forces.

2 Generalized Forces

There is a type of the external forces, which is straightforward to include. This is a conservative force Q. Recall from the first lecture that the conservative force is the differential of a certain potential energy $U : \mathbb{R}^n \to \mathbb{R}$

$$Q = -\frac{\mathrm{d}U(q)}{\mathrm{d}q} = -\left(\frac{\partial U}{\partial q_1}, ..., \frac{\partial U}{\partial q_n}\right). \tag{11}$$

Consider the Lagrangian $L(q,\dot{q}) = T(q,\dot{q}) - U(q)$ then the Euler Lagrange equation (4) gives

$$-\frac{\partial T}{\partial q} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}} = Q \tag{12}$$

Lagrange's equations can be made more general by including a term of an external force as in (12). Once again we shall ask the question about the trajectory of a mechanical system from time a to b. But now the system is influenced by the force field $Q(q,\dot{q})$. By a **force field** I mean a force with possibly different values for different different values of the generalized coordinates q and their velocities $\dot{q}$.

We derived the Euler-Lagrange equation during Lecture 2 by considering a family $\tilde{q}(\epsilon) = q + \epsilon\eta$, where q was the stationary solution, ϵ was a parameter and $\eta(a) = \eta(b) = 0$. In the notation above I have replaced ξ by q in order to relate the calculus of variations from the last lecture to the classical mechanics.

Notice that the test function is

$$\eta = \frac{\mathrm{d}q}{\mathrm{d}\epsilon}. \tag{13}$$

In physical literature η is called the **virtual displacement**, since it can be interpreted as a virtual displacement from a point on the actual path to some point on the neighboring varied path. Now we are ready to state integral Lagrange-d'Alembert Principle.

Definition 2 (Lagrange-d'Alembert Principle). The trajectory q of a mechanical system system influenced by a force field Q is such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\int_a^b L(q+\epsilon\eta, \dot{q}+\epsilon\dot{\eta})dt + \int_a^b Q(q,\dot{q})\eta dt = 0 \tag{14}$$

for each virtual displacement η ■

Remark 1. The second summand is an expression for work done by the force Q to displace the mechanical system along η.

Using the findings from the last lecture (think) The Lagrange-d'Alembert Principle in (14) can be rewritten as

$$\int_a^b \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} + Q\right)\eta dt = 0. \tag{15}$$

By the Fundamental Lemma of the Calculus of Variations we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q. \tag{16}$$

I will be a bit sloppy and refer in the sequel to (16) as the Lagrange-d'Alembert Principle.

2.1 Coordinate Transformations

The expression "generalized coordinates" means that coordinates are not necessary related to standard Euclidean basis. Thus for example if we try to model a robot with n links. The angle of each joint may serve as the generalized coordinates. I have also introduced the notion of a generalized force or a force field. They were parameterized by q and $\dot{q}$. At this point I would like to discuss a problem of changing coordinates from given q to some r; $r(q)$ may be treated as a function of q.

Denote the virtual displacement of r by η_r and the virtual displacement of q by η_q then

$$\eta_{r\,i} = \frac{\mathrm{d}r_i(q(\epsilon))}{\mathrm{d}\epsilon} = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \eta_{q\,j} \tag{17}$$

What about the generalized force, is it influenced by the coordinate change? In order to answer this question we shall examine the work term in (14). The work is independent of the coordinates. The work you did by going to the university today is the same independent on we count the distance in meters or in yards. If we denote the force field in coordinates q by Q, and the work in the coordinates r by F we get

$$\sum_i F_i \eta_{ri} = \sum_j Q_j \eta_{qj}, \tag{18}$$

thus

$$\sum_i F_i \eta_{ri} = \sum_i F_i \sum_j \frac{\partial r_i}{\partial q_j} \eta_{q\,j} = \sum_j \left(\sum_i F_i \frac{\partial r_i}{\partial q_j} \right) \eta_{q\,j}. \tag{19}$$

It follows that

$$Q_j = \sum_i F_i \frac{\partial r_i}{\partial q_j}, \tag{20}$$

or

$$Q = \left(\frac{\mathrm{d}r}{\mathrm{d}q} \right)^T F, \tag{21}$$

where $\frac{\mathrm{d}r}{\mathrm{d}q}$ is the Jacobian.

Example 2. We shall consider motion of a particle in $\mathbb{R}^2$ with canonical coordinates (x,y), which is influenced by a constant external force F. Our task is to describe motion of the particle in the polar coordinates (r, θ). We use the relations

$$\begin{aligned} x(t) &= r(t) \cos \theta(t) \\ y(t) &= r(t) \sin \theta(t) \end{aligned} \tag{22}$$

and

$$\begin{aligned}\dot{x} &= -r\dot{\theta}\sin\theta + \dot{r}\cos\theta \\ \dot{y} &= \quad r\dot{\theta}\cos\theta + \dot{r}\sin\theta\end{aligned} \tag{23}$$

Kinetic energy is then

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\theta})^2\right). \tag{24}$$

The generalized force is

$$Q = \begin{bmatrix} -r\sin\theta & r\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} F := \begin{bmatrix} rF_\theta \\ F_r \end{bmatrix}. \tag{25}$$

We apply the Lagrange-d'Alembert Principle (16) coordinate-wise. The equations

$$\frac{\partial T}{\partial \theta} = 0, \; \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{26}$$

yield

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = rF_\theta. \tag{27}$$

Correspondingly

$$\frac{\partial T}{\partial r} = mr\dot{\theta}^2 \text{ and } \frac{\partial T}{\partial \dot{r}} = m\dot{r} \tag{28}$$

give

$$m\ddot{r} - mr\dot{\theta}^2 = F_r. \tag{29}$$

■

2.2 Dissipation

We shall address a situation when frictional forces are acting on a mechanical system. It frequently happens that the frictional force is proportional to the velocity of the system

$$F_i = -k_i v_i, \text{ where } v_i = \dot{r}_i. \tag{30}$$

Similarly as for the conservative force one can define a function $\mathcal{F}(v)$ in the following way

$$F_i = -\frac{\partial \mathcal{F}}{\partial v_i}. \tag{31}$$

The function $\mathcal{F}$ is called the **Rayleigh's dissipation function**. For

$$\mathcal{F} = \frac{1}{2}\sum_i v_i^2 k_i \tag{32}$$

we get exactly (30).

Using the results for the last section the generalized force is

$$\begin{aligned} Q_j &= \sum_i F_i \frac{\partial r_i}{\partial q_j} = -\sum_i \frac{\partial \mathcal{F}}{\partial v_i}\frac{\partial r_i}{\partial q_j} = -\sum_i \frac{\partial \mathcal{F}}{\partial v_i}\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = -\sum_i \frac{\partial \mathcal{F}}{\partial v_i}\frac{\partial v}{\partial \dot{q}_j} \\ &= -\sum_i \frac{\partial \mathcal{F}}{\partial \dot{q}_i}. \end{aligned} \tag{33}$$

Hence

$$Q = -\frac{\mathrm{d}\mathcal{F}}{d\dot{q}} \tag{34}$$

In (33) we used the following observation. If

$$\dot{r}_i = \frac{\mathrm{d}r_i(q_1, ..., q_n)}{\mathrm{d}t} = \frac{\partial r_i}{\partial q_1}\dot{q}_1 + ... + \frac{\partial r_i}{\partial q_i}\dot{q}_i + ... + \frac{\partial r_i}{\partial q_n}\dot{q}_n \tag{35}$$

then

$$\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j} \tag{36}$$

In conclusion the Euler-Lagrange equations is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = -\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\dot{q}} \tag{37}$$

2.3 Mechanical Systems with Constraints

Before I proceed with the theoretical part of this section I will consider the following example. A hoop of mass M is rolling without a slipping down a hill in Figure 2. We choose the distance x and the angle θ as the generalized coordinates. However, we observe that there is a relation between the two coordinates

$$rd\theta = dx. \tag{38}$$

The task in the following is to derive the Euler-Lagrange equations for a mechanical system with a family of m constraints of the form

$$f_l(\tilde{q}_1, ..., \tilde{q}_n) = 0, \;\; l = 1, ..., m. \tag{39}$$

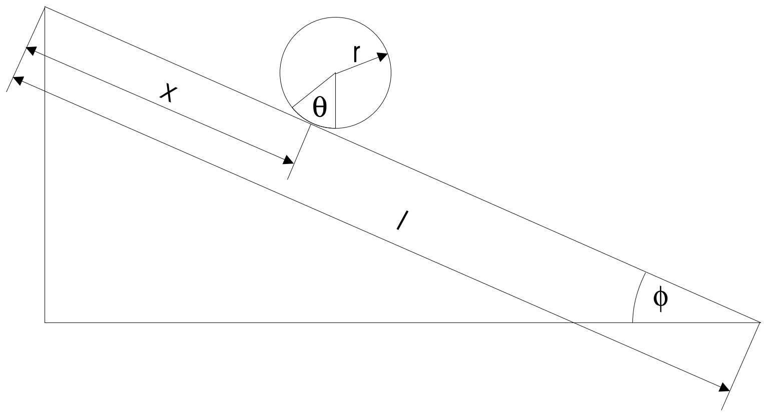

Fig. 2. The hoop of the radius r and the mass M is rolling without slipping down the hill with a slope ϕ.

We shall call such constraints **nonholonomic**. In order to incorporate this new ingredient into the calculus of variations we consider the directional derivative of the constraint

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} f_l(q_1 + \epsilon\eta_1, ..., q_n + \epsilon\eta_n) = \sum_{j=1}^{n} \frac{\partial f_l}{\partial q_j}\eta_j = 0. \tag{40}$$

For short I shall write

$$\sum_{j=1}^{n} a_{lj}\eta_j = 0. \tag{41}$$

Here I will skip the rigor (I will discuss the details of the theory of Lagrange Multipliers the next time), and I will only mention that multiplication (41) by some undetermined function λ_l does not change it. If (41) holds it is true then

$$\lambda_l \sum_{j=1}^{n} a_{lj}\eta_j = 0. \tag{42}$$

The functions λ_l are called the **Lagrange multipliers**.

We want to reduce the number of virtual displacements in Hamilton's Principle

$$\int_a^b dt \sum_j \left(\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} \right) \eta_j = 0 \tag{43}$$

to independent ones. We can combine (43) with m equations of constraint on the virtual displacement η_j by summing (42) over l and integrating the results with respect to time from a to b:

$$\int_a^b dt \sum_{l=1}^{m} \lambda_l \sum_{j=1}^{n} a_{lj}\eta_j = 0. \tag{44}$$

The sum of (43) and (44) is then the relation

$$\int_a^b dt \sum_j \left(\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} + \sum_{l=1}^{n} \lambda_l a_{lj} \right) \eta_j = 0 \tag{45}$$

Now it is time to use the Fundamental Lemma of the Calculus of Variations to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_{l=1}^{n} \lambda_l a_{lj}. \tag{46}$$

The term $Q_j = \sum_{l=1}^{n} \lambda_l a_{lj}$ in (46) is often called the force of constraints.

We shall proceed now with the example in Figure 2 from the beginning of this section.

Example 3. Kinetic energy is the sum of two contributions due to the translational and rotational motion:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}Mr^2\dot{\theta}^2. \tag{47}$$

Potential energy is

$$U = Mg(l - x)\sin\phi. \tag{48}$$

The coefficients of the equation of constraint (38), $rd\theta - dx = 0$, are $a_\theta = r$ and $a_x = -1$. We apply the equation (46) with $L = T - U$ to θ

$$Mr^2\ddot{\theta} = \lambda r \tag{49}$$

and to x

$$M\ddot{x} - Mg\sin\theta = -\lambda. \tag{50}$$

Differentiating (38) one gets one more equation

$$r\dot{\theta}dt = \dot{x}dt \text{ or } r\dot{\theta} = \dot{x}. \tag{51}$$

Differentiating it one more we get

$$r\ddot{\theta} = \ddot{x} \tag{52}$$

Equations (49) and (52) yield

$$M\ddot{x} = \lambda. \tag{53}$$

But from (53) and (50) we have

$$\frac{1}{2}Mg\sin\theta = \lambda. \tag{54}$$

Finally the equations (49) and (50) yield

$$\begin{aligned}\ddot{x} &= \frac{1}{2}g\sin\theta \\ \ddot{\theta} &= \frac{1}{2r}g\sin\theta.\end{aligned} \tag{55}$$

■

The last remark in this section is that the work of the force of constraints is zero, since

$$\sum_j Q_j\eta_j = \sum_j\sum_l \lambda_l a_{lj}\eta_j = \sum_l \lambda_l \sum_j a_{lj}\eta_j = 0 \tag{56}$$

References

[1] Herbert Goldstein. *Classical Mechanics*. Addison-Wesley, 1980.

Mechanical Systems II.
Method of Lagrange Multipliers

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Abstract. So far our approach to classical mechanics was limited to finding a critical point of a certain functional. In this lecture I shall present a first-order necessary condition for an extremum problem with additional equality constraints. This is so called a method of Lagrange multipliers.

1 Lagrange Conditions

We shall deal with a first-order necessary condition for an extremum problem with equality constraints. The main result presented in this chapter is the Lagrange Multiplier Theorem. To better understand the idea of the underlying theorem we shall limit ourself to functions of two variables and one equality constraint. I shall not focus on exact properties of the functions used below, I will merely assume that all considered functions are continuously differentiable.

We shall regard a problem of finding the minimum $x^* = (x_1^*, x_2^*)$ of a function $f : \mathbb{R}^2 \to \mathbb{R}$ subject to a constraint equation

$$h(x) = 0,\ h : \mathbb{R}^2 \to \mathbb{R}. \tag{1}$$

It will be vital for further analysis to introduce a notion of a level set. The **level set** through a point x^* is the set $\{x \in \mathbb{R}^2 : h(x) = h(x^*)\}$.

Let $h(x^*) = 0$ (i.e. x^* satisfies the constraint equation) and the differential

$$\mathrm{d}h(x^*) = \left(\frac{\partial h(x_1,x_2)}{\partial x_1}, \frac{\partial h(x_1,x_2)}{\partial x_1}\right)\bigg|_{x^*} \neq 0 \tag{2}$$

we then can parameterize the level set $\{x : h(x) = 0\}$ in a neighborhood of x^* by a curve $x : (a,b) \to \mathbb{R}^2$ (Why is it possible? Think about Implicit Function Theorem from the first lecture):

$$x(t) = (x_1(t), x_2(t))\,,\ t \in (a,b),\ x(t^*) = x^*,\ \dot{x}(t^*) \neq 0. \tag{3}$$

Lemma 1. $\mathrm{d}h(x^*)$ *is orthogonal to* $\dot{x}(t^*)$

Proof. Since h is zero on the curve $x(t)$, we have that for all $t \in (a,b)$,

$$h(x(t)) = 0. \tag{4}$$

Differentiating with respect to t and applying the chain rule yield

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} h(x(t)) = \langle dh(x(t)), \dot{x}(t) \rangle, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ means the scalar product in $\mathbb{R}^2$. Therefore $dh(x^*)$ is orthogonal to $\dot{x}(t^*)$. ■

Now suppose x^* is a minimizer of $f : \mathbb{R}^2 \to \mathbb{R}$ on the set $\{x \in \mathbb{R}^2 : h(x) = 0\}$.

Lemma 2. $\mathrm{d}f(x^*)$ *is orthogonal to* $\dot{x}(t^*)$.

Proof. Observe that the composite function

$$\phi(t) = f(x(t)) \tag{6}$$

achieves the minimum at t^*. Consequently, the necessary condition for the unconstrained extremum problem implies

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}(t^*) = 0. \tag{7}$$

Applying the chain rule gives

$$0 = \frac{\mathrm{d}\phi}{\mathrm{d}t}(t^*) = \langle df(x(t^*)), \dot{x}(t^*) \rangle = \langle df(x^*), \dot{x}(t^*) \rangle. \tag{8}$$

Thus $df(x^*)$ is orthogonal to $\dot{x}(t^*)$. ■

We shall now combine Lemmas 1 and 2. That is, both $df(x^*)$ and $dh(x^*)$ are orthogonal to $\dot{x}(t^*)$. Therefore, the vectors $df(x^*)$ and $dh(x^*)$ are parallel. In other words $df(x^*)$ is a scalar multiple of $dh(x^*)$. The above observation allows us to formulate the **Lagrange theorem** for functions of two variables with one constraint.

Theorem 1 (Lagrange's Theorem). *Let the point* x^* *be a minimizer (maximizer) of* $f : \mathbb{R}^2 \to \mathbb{R}$ *subject to the constraint* $h(x) = 0,\ h : \mathbb{R}^2 \to \mathbb{R}$. *Then* $df(x^*)$ *and* $dh(x^*)$ *are parallel. Thus, if* $\mathrm{d}h(x^*) \neq 0$, *then there exists a scalar* λ *such that*

$$df(x^*) + \lambda^* \mathrm{d}h(x^*) = 0. \tag{9}$$

In the above theorem, we refer to λ^* as the **Lagrange multiplier**.

Note that the Lagrange condition is only necessary but not sufficient. We shall now generalize the Lagrange's theorem for the case when $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^m,\ m \leq n$. We shall need a notion of a regular point . A point x^* is a **regular point** of a map $g : \mathbb{R}^n \to \mathbb{R}^m$ if

$$\text{rank } Dg(x^*) = \text{rank } \frac{\mathrm{d}g}{\mathrm{d}x}(x^*) = m \tag{10}$$

Theorem 2 (Lagrange Multiplier Theorem). *Let x^* be a local minimizer (or maximizer) of $f : \mathbb{R}^n \to \mathbb{R}$, subject to $h(x) = 0,\ h : \mathbb{R}^n \to \mathbb{R}^m,\ m \leq n$. Assume that x^* is a regular point of h. Then there exists $\lambda^* \in \mathbb{R}^m$ such that*

$$Df(x^*) + {\lambda^*}^{\mathrm{T}} Dh(x^*) = 0. \tag{11}$$

I shall not prove the Lagrange Multiplier Theorem in this lecture, merely state that the proof uses similar reasoning as given above for the case $n = 2,\ m = 1$ and refer you to (1), pp. 335-343.

It is often convenient to introduce the so-called **Lagrangian function** $l : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

$$l(x, \lambda) \equiv f(x) + \lambda^{\mathrm{T}} h(x). \tag{12}$$

The necessary condition (11) in the Lagrange Multiplier Theorem for a local minimizer x^* can be reformulated as

$$Dl(x^*, \lambda^*) = 0. \tag{13}$$

In other words, the necessary condition in the Lagrange Multiplier Theorem is equivalent to the first-order necessary condition for unconstrained optimization applied to the Lagrange function in (12). This is seen by differentiating l

$$0 = Dl(x, \lambda) = \left[\frac{\partial l(x, \lambda)}{\partial x},\ \frac{\partial l(x, \lambda)}{\partial \lambda}\right], \tag{14}$$

but $0 = \frac{\partial l(x,\lambda)}{\partial x} = Df(x) + \lambda^{\mathrm{T}} Dh(x)$ gives the condition (11) and $0 = \frac{\partial l(x,\lambda)}{\partial \lambda} = h(x)$, which is the equation of constrains.

Example 1. Consider the problem of finding the extrema of the function

$$f:\ \mathbb{R}^2 \to \mathbb{R},\ f(x_1, x_2) = x_1^2 + x_2^2 \tag{15}$$

on the ellipse

$$\{(x_1, x_2) \in \mathbb{R}^2 : h(x_1, x_2) = x_1^2 - 2x_2^2 - 1 = 0\}. \tag{16}$$

The lagrange function is

$$l(x_1, x_2, \lambda) = x_1^2 + x_2^2 + \lambda(x_1^2 + 2x_2^2 - 1). \tag{17}$$

The necessary conditions for l to reach extremum at $(x_1^*, x_2^*, \lambda^*)$ are

$$0 = \left.\frac{\partial l(x_1, x_2, \lambda)}{\partial x_1}\right|_{(x_1^*, x_2^*, \lambda^*)} = 2x_1^* + 2\lambda^* x_1^* \tag{18}$$

$$0 = \left.\frac{\partial l(x_1, x_2, \lambda)}{\partial x_2}\right|_{(x_1^*, x_2^*, \lambda^*)} = 2x_2^* + 4\lambda^* x_2^* \tag{19}$$

$$0 = \left.\frac{\partial l(x_1, x_2, \lambda)}{\partial \lambda}\right|_{(x_1^*, x_2^*, \lambda^*)} = (x_1^*)^2 + 2(x_2^*)^2 - 1. \tag{20}$$

From (18) $x_1 = 0$ or $\lambda = -1$. If $x_1 = 0$, then $x_2 = \pm\frac{\sqrt{2}}{2}$ and $\lambda = \frac{1}{2}$. If $\lambda = -1$ then $x_2 = 0$ and $x_1 = \pm 1$. In conclusion we have the following solutions

$$x^{(1)} = \left(0, -\frac{\sqrt{2}}{2}\right),\ x^{(2)} = \left(0, \frac{\sqrt{2}}{2}\right),\ x^{(3)} = (-1, 0)\,,\ x^{(4)} = (1, 0) \quad (21)$$

and

$$f(x^{(1)}) = f(x^{(2)}) = \frac{1}{2} \quad (22)$$

$$f(x^{(3)}) = f(x^{(4)}) = 1. \quad (23)$$

Thus $x^{(1)}$ and $x^{(2)}$ correspond to minimum, whereas the function f reaches maximum at $x^{(3)}$ and $x^{(4)}$. ■

Example 2. We shall consider an example from linear algebra.

Maximize the quadratic form $f : \mathbb{R}^n \to \mathbb{R},\ x \mapsto x^{\mathrm{T}}Qx$, where $Q^{\mathrm{T}} = Q$, subject to $x^{\mathrm{T}}Px = 1$, where $P^{\mathrm{T}} = P$ and P is nonsingular. The function h in the Lagrange Multiplier Theorem is

$$h : \mathbb{R}^n \to \mathbb{R},\ x \mapsto 1 - x^{\mathrm{T}}Px. \quad (24)$$

The Lagrange function is now

$$l(x, \lambda) = x^{\mathrm{T}}Qx + \lambda\left(1 - x^{\mathrm{T}}Px\right). \quad (25)$$

As in the previous example we compute partial derivatives of l

$$\frac{\partial l}{\partial x}(x^*, \lambda^*) = 2\,(x^*)^{\mathrm{T}}\,Q - 2\lambda^*(x^*)^{\mathrm{T}}P \quad (26)$$

$$\frac{\partial l}{\partial \lambda}(x^*, \lambda^*) = 1 - (x^*)^{\mathrm{T}}Px^*. \quad (27)$$

Thus the necessary conditions are

$$Qx^* = \lambda Px^* \quad (28)$$

$$(x^*)^{\mathrm{T}}Px^* = 1. \quad (29)$$

But since P is nonsingular the equation (28) is equivalent to

$$P^{-1}Qx^* = \lambda^* x^*, \quad (30)$$

which is the equation for the eigenvalues λs of the matrix $P^{-1}Q$. On the other hand multiplying (30) by xP yields

$$(x^*)^{\mathrm{T}}Qx^* = \lambda^*(x^*)^{\mathrm{T}}Px^* = \lambda^* \quad (31)$$

We conclude that the maximizer of $x^{\mathrm{T}}Qx$ subject to $x^{\mathrm{T}}Px = 1$ is the eigenvector of the matrix $P^{-1}Q$ corresponding to the maximal eigenvalue. ■

References

[1] S. Zak E. K.P. Chong. *An Introduction to Optimization*. John Wiley, 1976.