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Approaches to solving nonlinear ODEs

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Abstract

Approaches for investigating nonlinear differential equations (DEs) are described and the status of corresponding programs is reported. Investigations include the determination of infinitesimal symmetries of single ODEs/PDEs or -systems, and for ODEs the determination of first integrals, of factorizations into DEs of lower order and of equivalent Lagrangians.

1. Introduction

To solve differential equations (DEs) a number of approaches are available for different types of DEs to solve. If possible, one will of course use an algorithm and a corresponding computer algebra program to solve the problem exactly and in full generality or to show that the solution does not belong to a well defined class, e.g. that it is not expressible in terms of elementary functions or possibly integrals of them or in terms of solutions of linear second-order ODEs. DEs for which such algorithms are currently available must be ordinary (ODEs) and linear, or related lower order nonlinear, e.g. of Riccati-type. For other nonlinear first-order ODEs one can try the Prelle–Singer algorithm (see e.g. [11,10]).

If DEs are more complicated or are nonsolvable in the above sense then a further possibility could be to apply 'transformation algorithms'. For a restricted class of DEs like nonlinear second-order ODEs those algorithms can decide whether by a transformation (of a restricted class of transformations e.g. point transformations or a subclass or the full class of contact transformations) the DE can be converted into a solvable type (e.g. linear type) or a known other type like one of the Painlevé ODEs. Algorithmic ways, which help to find them, are, for example, given in [1–3,13].

If such transformation algorithms are not applicable or not successful then not everything is lost. What one always can try is to guess a solution or the structure of a solution and to solve the remaining simpler equations. The strategy is the same as with the transformation algorithms – a property which would be advantageous for the solution-process (like point-equivalence to a linear DE) is expressed as a system of DEs which inevitably is overdetermined in the sense that there are more equations than functions involved.

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What is a more systematic way of guessing the structure of the solution? It is obviously not to make an ansatz like

$$y = \frac{\sum_{i=1}^{l} a_i x^{p_i} \mathrm{e}^{q_i x}}{\sum_{j=1}^{k} b_j x^{r_j} \mathrm{e}^{s_j x}}, \quad a_i, p_i, q_i, b_j, r_j, s_j = \text{const.}$$

for the solution of a nonlinear ODE. The weakness of such an ansatz (if it is not motivated otherwise) is that it covers only a small fraction of the whole set of possible solutions and probably the uninteresting ones.

What is necessary is to try ansätze, i.e. restrictions, which are mathematically helpful by cutting out most of the cases without excluding interesting solutions and without having to know or guess details about the structure of the solution beforehand.

The following analogy has no deeper meaning, it only aims to make the purpose of the strategy clearer.

Take, for example, the situation in optics where a light beam is to be reduced in intensity without losing essential information, i.e. without disturbing the image transported by the beam. If the aperture would be placed close to the screen then obviously parts of the image would be lost completely. The way this is accomplished there, is to place the aperture in the focal plane. A so-placed aperture simply cutting out only rays far from the optical axis reduces the overall brightness and keeps all parts of the image.

The same strategy is pursued with our ansätze. In order to do simple cuts, i.e. simple ansätze without knowing any structure beforehand and still not losing interesting solutions of the DE, we have to transform our problem and place restrictions on the transformed form. Doing a transformation will not simplify the task of solving any given ODE but it will resolve the contradiction between simple ansätze and exclusion of interesting solutions. The problem of finding the general solution of an nth-order ODE means finding one special function of $n+1$ arguments $y = y(x, c_1, \ldots, c_n)$ with constants of integration c_i up to re-parametrizations $\hat{c}_i = \hat{c}_i(c_j)$. The re-parametrizations involve only n functions of n arguments and have a measure of zero compared with the set of functions of $n+1$ arguments. The problem of finding a function of $n+1$ arguments will turn up in all the ansätze we will describe in the different sections.

The actual problem-solving is performed when the conditions are solved which the different ansätze in the different approaches provide. Because it always involves the solution of an overdetermined system of PDEs, we tried to write one computer algebra program `CRACK` which does all those tasks. In this talk only examples of applications are given; for more details on `CRACK` see [16].

In Section 2 the method of calculating infinitesimal symmetries of DEs is very shortly reviewed because it is already the most widely known and used method to solve nonlinear DEs. Other methods which are not commonly used, are described in the later sections [14]. What would be of interest mathematically is to relate the different methods. Remarks and examples are made in the last section.

2. Infinitesimal symmetries

Since the symmetry approach is extensively described in the literature and well known, we therefore in this section will only give a short introduction, concentrating on qualitative features. For more details on the subject see the literature survey [5] with 171 references. A description of the program `LIEPDE` which applies the program `CRACK` for automatically determining Lie symmetries is given in [15,16].

The method of computing and applying infinitesimal symmetries of DEs

$$0 = H_A(x, y, y', \ldots, y^{(n)}) \tag{1}$$

is the most widely used method to investigate and solve nonlinear DEs. The intrinsic property of DEs which are exploited in this approach is to be form-invariant against infinitesimal transformations

$$\tilde{x} = x + \epsilon\xi(x, y, \ldots), \qquad \tilde{y} = y + \epsilon\eta(x, y, \ldots),$$

where x is the independent variable, y the dependent variable and ξ, η are the generators of the symmetry. If symmetries of systems of DEs are investigated then y and η obtain an index and if (1) are partial DEs then x and ξ obtain an index. Here form-invariance means that after performing transformations (2) of a DE (-system) (1), these DEs take the same form in coordinates $\tilde{x}, \tilde{y}$ as in coordinates x, y up to higher order corrections in ϵ,

$$H_A(x, y, y', \ldots, y^{(n)}) = H_A(\tilde{x}, \tilde{y}, \tilde{y}', \ldots, \tilde{y}^{(n)}) + \mathrm{O}(\epsilon^2). \tag{2}$$

Since one is normally interested only in the solutions of (1), one therefore is also only interested in symmetries of the solutions of (1), i.e. for finding symmetries which satisfy (2) identically in $x, y, y', \ldots$, the DEs (1) can be used to substitute e.g. $y^{(n)}$ and lower the number of independent variables $x, y, y', \ldots$ of this symmetry-determining problem.

As the theory of Sophus Lie shows [7,8], it is possible to lower the order of a nth-order ODE by one if an infinitesimal symmetry is known and if in addition one can solve a linear first-order PDE, which normally is not a difficult problem, given the simple structure that symmetries mostly have. To find a symmetry, one has to find symmetry generators ξ and η which satisfy a PDE which results from setting to zero the coefficient of ϵ in (2).

How difficult is it to find symmetries, i.e. how many functions of how many variables have to be determined? If we have one unknown function y then the generators ξ, η can be expressed without loss of generality through a single symmetry potential $\Omega = \Omega(x, y, \ldots)$ through $\xi := \Omega_{,y'}$, $\eta := y'\Omega_{,y'} - \Omega$. The usual way to place restrictions in this approach is to simply cut the dependencies of ξ, η or equivalently the dependence of Ω. The general case is represented by $\Omega(x, y, y', \ldots, y^{(n-1)})$, i.e. a function of $n + 1$ arguments to be found.

Back to our analogy. A simple restriction is to let ξ, η or Ω only depend on fewer variables, e.g. on x, y or x, y, y'. The amazing feature of the symmetry approach is that such a crude restriction does not only work for uninteresting DEs. If, for example, a point symmetry, i.e. a symmetry with $\xi = \xi(x, y), \eta = \eta(x, y)$, exists then this would mean that there is a point transformation $\tilde{x} = \tilde{x}(x, y)$, such that $\tilde{x}$ is a symmetry variable, $\partial_{\tilde{x}}$ a symmetry generator and $\tilde{x} \to \tilde{x} + c$ is a symmetry transformation. Then the general solution of the nth order ODE can be written in the form $y = y(\tilde{x} + c_1, c_2, \ldots, c_n)$ with constants of integration $c_1, \ldots, c_n$. Tests show that about $\frac{3}{4}$ of the nonlinear second-order ODE in the book of Kamke [6] which are solved there have at least one point symmetry.

To have a simple measure of how large the class of ODEs of a given order with point symmetries is, let us notice that for finding point symmetries we have to find two functions ξ, η of two arguments x, y each and for contact symmetries we have to find one function Ω of three variables. This will give us some hint in comparing the chances of success in comparison with other methods by comparing how many free functions are left to be determined.

3. Polynomial first integrals

It is well known that if the Lagrangian of a variational principle has symmetries then corresponding conservation laws can be derived. But this does not mean that there cannot be first integrals with simple structure which do not result from an infinitesimal symmetry. For example the equation

$$0 = y'' + y'(2y + F(x)) + F(x)y^2 - H(x) \tag{3}$$

for a function $y(x)$ has no point symmetry for arbitrary $F(x)$, $H(x)$. But it has a first integral

$$\text{const.} = \left(\frac{\mathrm{d}y}{\mathrm{d}x} + y^2\right) \mathrm{e}^{\int F\,\mathrm{d}x} - \int H \mathrm{e}^{\int F\,\mathrm{d}x}\,\mathrm{d}x, \tag{4}$$

which is linear in y'. Linearity in y' is not an exotic property, as conservation laws of momentum are normally linear and energy conservation laws are only quadratic in the first derivative. A check of the second-order nonlinear ODEs in the book of Kamke shows that not a few ODEs solved there have a first integral at most quadratic in y'.

The corresponding program `FIRINT` starts with the ansatz

$$\text{const.} = h_2(x, y)y'^2 + h_1(x, y)y' + h_0(x, y) \tag{5}$$

and formulates the determining system by total differentiation of (5), identification of y'' with y'' in the given ODE and direct separation of y', to produce in the above example (3)

$$0 = h_{2,y}, \qquad 0 = (H - Fy^2)h_1 + h_{0,x},$$
$$0 = h_{1,x} + h_{2,x} - (2F + 4y)h_2, \qquad 0 = 2(H - Fy^2)h_2 - (F - 2y)h_1 + h_{0,y} + h_{1,x}.$$

It then calls the program `CRACK` which provides the solution (4) for h_0, h_1, h_2.

The program allows for the specification of the order and degree of the polynomial first integral. For an nth-order ODE the ansatz

$$\text{const.} = \sum h_i(x, y, y', \ldots, y^{(n-2)})(y^{(n-1)})^i$$

would already provide an overdetermined system for the functions h_i after a separation w.r.t. the only explicitly occurring variable $y^{(n-1)}$. Naturally a higher degree and/or higher-order ansatz takes more effort to solve. One could also test any other ansatz by the same procedure in which at least one of the variables $x, y, y', \ldots, y^{(n-1)}$ occurs only explicitly but that would require insight into the nature of the problem which we assume we do not have.

Compared with the symmetry approach, the restrictions in the case of a second-order ODE are even less severe, with three free functions of two variables to be determined compared with two functions of two variables there. Also here the resulting overdetermined PDE-system is linear in the h_i. The disadvantage is that a polynomial ansatz for a first integral is not invariant against variable transformations, e.g. point transformations.

4. A factorization of ODEs

Inspecting the solved or at least partially integrated nonlinear second-order ODEs in the book of Kamke [6] then half of those that have no point symmetry have been ‘factorized’ into lower-order DEs where one

DE is one of the known first-order standard ODEs (linear, Bernoulli, separable, Riccati) [9]. How can this be formalized and how can ODEs systematically be investigated with respect to such a property? Our aim will be to split the nth-order ODE into a first-order ODE and an $n-1$-order DE such that one can be solved after the other. The method is to substitute in

$$0 = \omega(x, y, y', \ldots, y^{(n)}) \tag{6}$$

derivatives of y:

$$\begin{aligned} &y' \quad \text{by} \quad z(x, y), \\ &y'' \quad \text{by} \quad \frac{\mathrm{d}z(x, y)}{\mathrm{d}x} = \partial_x z + z\partial_y z, \\ &\vdots \end{aligned}$$

which provides an $n-1$-order nonlinear PDE

$$0 = \omega(x, y, z, \partial_x z + z * \partial_y z, \ldots) \tag{7}$$

in $z(x, y)$. If a solution with $n-1$ independent constants of integration can be found then all that remains is to solve the first-order ODE

$$y' = z(x, y). \tag{8}$$

We up to now did not make any restrictions, therefore to solve (7) in general is not easier than to solve (6). According to our philosophy the restrictions should be simple and not require inside knowledge of (6). They will be chosen to enable or simplify the solution of (7) *and* (8) to make optimal use of them.

In four individual investigations z is given a structure by one of the following four ansätze:

$$z = a(x)\,b(y), \tag{9}$$
$$z = a(x)\,y + b(x), \tag{10}$$
$$z = a(x)\,y^n + b(x)\,y, \tag{11}$$
$$z = a(x)\,y^2 + b(x)\,y + c(x), \tag{12}$$

with free functions a, b, c. In the first three cases (9)–(10) this enables the solution of (8) and in case (12) the Riccati equation is equivalent to a second-order linear ODE. Substituting z in (7) gives in cases (10)–(12) an equation which is separable with respect to y giving an overdetermined ODE-system for the remaining functions of x. In case (9) matters are not that simple but still the resulting PDE after substitution is a PDE in two independent variables x, y and only functions of one variable. An algorithm to handle such a type of equation is included in the program CRACK [16] such that in all cases Eq. (7) after substitution of z needs only to be passed to CRACK and will be automatically investigated with good chances for solution.

Compared with the previous methods a disadvantage here is that the resulting overdetermined system is nonlinear in the remaining functions. A further disadvantage is that only functions of one variable are left to determine in comparison with functions of two variables in the other method, i.e. the chances that this factorization will be successful are on average smaller than to find symmetries. An ansatz which has not been implemented but which is equally possible for higher-order ODEs is to substitute y'' by z which involves free functions of two variables, but then the solution of the remaining second-order ODE will probably be more difficult.

An example where an ODE has neither point symmetries nor simple first integrals is the ODE (6.122) in [6]

$$0 = yy'' - y'^2 + F(x)yy' + Q(x)y^2. \tag{13}$$

The overdetermined system resulting from case (10) is

$$0 = b^2, \qquad 0 = -F(x)b - b' + ab, \qquad 0 = -F(x)a - Q(x) - a',$$

with the result

$$a(x) = \left(c_1 - \int Q e^{\int F\,\mathrm{d}x}\,\mathrm{d}x\right)\mathrm{e}^{-\int F\,\mathrm{d}x}, \qquad b(x) = 0.$$

Since this contains one constant of integration, this factorization ansatz does not restrict the solution space of (13) and after solving (10) the general solution is

$$y = c_2 \exp \int \left[\left(c_1 - \int Q\mathrm{e}^{\int F\,\mathrm{d}x}\,\mathrm{d}x\right)\mathrm{e}^{-\int F\,\mathrm{d}x}\right]\mathrm{d}x.$$

5. A Lagrangian for second order ODEs

If a DE cannot be solved analytically then perhaps a re-formulation is useful which is more compact and offers better physical insight. This would be the case if a variational principle would be available which is equivalent to the ODE. In that case a Hamilton Jacobi equation could be formulated and separation of variables could be tried. Also, if a Lagrangian is known then the determination of cyclic variables, i.e. Noether symmetries leads to conservation laws.

The conditions for a function $L(x, y_i, y_j')$ to be an equivalent Lagrangian for a given system of ODEs are known to be the Helmholz conditions [4,12]. Only for a single ODE the existence of such an L is always guaranteed which makes it roughly as difficult to find L as to solve the ODE. As in the first two approaches a function of $n + 1 = 3$ variables is to be found and an ansatz will include functions of two variables. A simple first guess which is motivated from classical mechanics, to take L to be quadratic in y', is already rather successful as a test of the nonlinear second-order ODEs in [6] shows. The ansatz is

$$L = u(x, y)(y')^2 + v(x, y), \tag{14}$$

with functions $u(x, y)$ and $v(x, y)$ to be determined. A term linear in y' corresponds to a total derivative of an x, y-dependent expression and can be discarded. Furthermore an additive purely x-dependent function and an overall constant factor are dropped from the result. The corresponding overdetermined PDE-system which is determined by the program LAGRAN for a given second-order ODE and solved by CRACK, results by

- formulating the Euler–Lagrange equation for (14),
- substituting y'' through the ODE and
- regarding x, y, y' as independent variables in the resulting equation.

Since y' occurs only explicitly, a separation is possible which, for example, for the ODE (6.53) in [6]

$$y'' = -F(x)y' - Q(x)G(y) - y'^2 G(y) \tag{15}$$

gives the condition

$$0 = -2F(x)y'u - 2Q(x)G(y)u + 2u,_x y' + u,_y y'^2 - v,_y - 2y'^2G(y)u,$$

which has to be satisfied identically in x, y, y'. To determine a Lagrangian or to show that none exists is usually much simpler than the approaches of previous sections because the system for u, v is linear and of first order only. On the other hand it is normally also less useful for solving the system exactly. In the example (15) the result for arbitrary parametric functions $F(x)$, $G(y)$, $Q(x)$ is

$$L = \mathrm{e}^{\int F\,\mathrm{d}x}\left(2Q\int \mathrm{e}^{2\int G\,\mathrm{d}y}G\,\mathrm{d}y - \mathrm{e}^{2\int G\,\mathrm{d}y}y'^2\right).$$

6. Comparison of the different approaches

To conclude this talk the main features of the different approaches will be summarized.

If one favorable of the described methods is to be chosen, e.g. to be run first in a computer program for solving ODEs, then it would be the infinitesimal symmetry approach. Reasons are:

- The resulting overdetermined PDE-system for ξ, η or Ω is linear and even homogeneous. This is possible because the problem is investigated infinitesimally at first. The price is to have to do integrations afterwards to find the similarity variable(s). So what is achieved is to split the whole problem into two steps (1. finding the symmetry, 2. integration of first-order PDEs) which are to be done *sequentially*.
- The existence of symmetries represented by ξ, η as functions of x, y is an intrinsic property of the problem. Replacing the DEs (1) by an algebraic combination of them which leaves the ideal invariant does not change symmetries of them. Also the performance of point transformations of (1) does not change the symmetries, only their coordinate representation $\xi(x, y)$, $\eta(x, y)$.

For comparison, the approach of determining integrating factors and by that first integrals are linear problems as well but the simplicity of the structure of first integrals is not an intrinsic property of the problem and is e.g. lost in point transformations. Nevertheless, mostly the ODE is given in appropriate variables and if a 'momentum'- or 'energy'-conservation law is behind it then it will most probably be found.

A factorization of an nth-order ODE into a standard first-order ODE plus a PDE of order $n - 1$ is less likely to succeed and furthermore leads to a nonlinear overdetermined DE-system but still it can be successful if other methods fail as many examples in [6] show.

An interesting approach is to determine a quadratic Lagrangian and to investigate this instead of the ODE because it is the most compact way to formulate the problem. The determining equations for L are linear and of first order and therefore mostly simple to solve. From all the nonlinear second-order ODEs in [6] over 60% have such a simple Lagrangian, many of them have e.g. no point symmetries.

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