



## The Affine Group of a Lie Group

Joseph A. Wolf

*Proceedings of the American Mathematical Society*, Vol. 14, No. 2. (Apr., 1963), pp. 352-353.

Stable URL:

<http://links.jstor.org/sici?sici=0002-9939%28196304%2914%3A2%3C352%3ATAGOAL%3E2.0.CO%3B2-Z>

*Proceedings of the American Mathematical Society* is currently published by American Mathematical Society.

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/about/terms.html>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/journals/ams.html>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

---

The JSTOR Archive is a trusted digital repository providing for long-term preservation and access to leading academic journals and scholarly literature from around the world. The Archive is supported by libraries, scholarly societies, publishers, and foundations. It is an initiative of JSTOR, a not-for-profit organization with a mission to help the scholarly community take advantage of advances in technology. For more information regarding JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

# THE AFFINE GROUP OF A LIE GROUP

JOSEPH A. WOLF<sup>1</sup>

1. If  $G$  is a Lie group, then the group  $\mathbf{Aut}(G)$  of all continuous automorphisms of  $G$  has a natural Lie group structure. This gives the semi-direct product  $\mathbf{A}(G) = G \cdot \mathbf{Aut}(G)$  the structure of a Lie group. When  $G$  is a vector group  $\mathbf{R}^n$ ,  $\mathbf{A}(G)$  is the ordinary affine group  $\mathbf{A}(n)$ . Following L. Auslander [1] we will refer to  $\mathbf{A}(G)$  as the *affine group of  $G$* , and regard it as a transformation group on  $G$  by  $(g, \alpha): h \rightarrow g \cdot \alpha(h)$  where  $g, h \in G$  and  $\alpha \in \mathbf{Aut}(G)$ ; in the case of a vector group, this is the usual action on  $\mathbf{A}(n)$  on  $\mathbf{R}^n$ .

If  $B$  is a compact subgroup of  $\mathbf{A}(n)$ , then it is well known that  $B$  has a fixed point on  $\mathbf{R}^n$ , i.e., that there is a point  $x \in \mathbf{R}^n$  such that  $b(x) = x$  for every  $b \in B$ . For  $\mathbf{A}(n)$  is contained in the general linear group  $\mathbf{GL}(n+1, \mathbf{R})$  in the usual fashion, and  $B$  (being compact) must be conjugate to a subgroup of the orthogonal group  $\mathbf{O}(n+1)$ . This conjugation can be done leaving fixed the  $(n+1, n+1)$ -place matrix entries, and is thus possible by an element of  $\mathbf{A}(n)$ . This done, the translation-parts of elements of  $B$  must be zero, proving the assertion.

L. Auslander [1] has extended this theorem to compact abelian subgroups of  $\mathbf{A}(G)$  when  $G$  is connected, simply connected and nilpotent. We will give a further extension.

**THEOREM.** *Let  $G$  be a connected Lie group and let  $S$  be the identity component of the radical of  $G$ . Then the following conditions are equivalent:*

1. *If  $B$  is a compact subgroup of the affine group  $\mathbf{A}(G)$ , then  $G$  has an element  $x$  such that  $b(x) = x$  for every  $b \in B$ .*
2. *Every compact subgroup of  $\mathbf{A}(G)$  is conjugate to a subgroup of  $\mathbf{Aut}(G)$ .*
3.  *$G$  has no nontrivial compact subgroup.*
4.  *$G$  is homeomorphic to Euclidean space.*
5.  *$S$  is simply connected and  $G/S$  is a direct product of copies of the universal covering group of the real special linear group  $\mathbf{SL}(2, \mathbf{R})$ .*
6.  *$G$  is simply connected, and every simple analytic subgroup of  $G$  is a 3-dimensional noncompact group.*

Equivalence of (3) and (4) is contained in the Cartan-Iwasawa theorem [3, Theorem 13]. Equivalence of (4) and (5) follows from

---

Received by the editors March 12, 1962.

<sup>1</sup> National Science Foundation Fellow.

Chevalley's theorem on the topology of solvable groups [2], the fact that the universal covering of  $\mathbf{SL}(2, \mathbf{R})$  is the only simple Lie group homeomorphic to Euclidean space, and the global Levi-Whitehead decomposition of  $G$ . It is not difficult to see that (5) is equivalent to (6) and it is clear that (1) is equivalent to (2). Finally, a nontrivial compact subgroup of  $G$  is a compact subgroup of  $\mathbf{A}(G)$  which is not conjugate to a subgroup of  $\mathbf{Aut}(G)$ ; thus (2) implies (3). The proof of the Theorem is now reduced to the proof that (3) implies (2).

2. Suppose that  $G$  has no nontrivial compact subgroup. Then  $G$  is simply connected, and it follows that  $\mathbf{Aut}(G)$  has only finitely many connected components because  $\mathbf{Aut}(G)$  is isomorphic to the group  $\mathbf{Aut}(\mathfrak{G})$  of automorphisms of the Lie algebra  $\mathfrak{G}$  of  $G$ , and  $\mathbf{Aut}(\mathfrak{G})$  is a real algebraic matrix group. Thus  $\mathbf{A}(G)$  has only finitely many connected components. The Cartan-Iwasawa theorem [3, Theorem 13] is valid for Lie groups with only finitely many components; thus  $\mathbf{A}(G)$  has maximal compact subgroups and, if  $K$  is one of them, every compact subgroup of  $\mathbf{A}(G)$  is conjugate to a subgroup of  $K$ . The proof that (3) implies (2) is now reduced to the proof that  $\mathbf{Aut}(G)$  contains a maximal compact subgroup of  $\mathbf{A}(G)$ .

Let  $K \subset \mathbf{Aut}(G) \subset \mathbf{A}(G)$  be a maximal compact subgroup of  $\mathbf{Aut}(G)$ ; we will prove that  $K$  is a maximal compact subgroup of  $\mathbf{A}(G)$ . Let  $K'$  be a maximal compact subgroup of  $\mathbf{A}(G)$  with  $K \subset K'$ ; we must prove  $K = K'$ . It is easily seen that  $K$  meets every component of  $\mathbf{A}(G)$ ; it follows that we need only prove that  $K$  and  $K'$  have the same identity component. Again because  $K \subset K'$ , it suffices to show that  $\dim K = \dim K'$ . Let  $f: \mathbf{A}(G) \rightarrow \mathbf{Aut}(G)$  be the canonical homomorphism  $(g, \alpha) \rightarrow \alpha$  with kernel  $G$ .  $K \cap G$  and  $K' \cap G$  are compact subgroups of  $G$  and thus are trivial by hypothesis. Furthermore  $K = f(K) = f(K')$  because  $K$  is a maximal compact subgroup of  $\mathbf{Aut}(G)$ , and because  $f(K)$  is contained in the compact subgroup  $f(K')$ . This gives  $\dim K = \dim f(K) = \dim f(K') = \dim K'$  which proves the Theorem.

3. It is worth remarking that the main part of the Theorem—the equivalence of (1), (2) and (3)—can be proved in the same way when  $G$  is assumed to have only finitely many connected components.

#### REFERENCES

1. L. Auslander, *A fixed point theorem for nilpotent Lie groups*, Proc. Amer. Math. Soc. **9** (1958), 822–823.
2. C. Chevalley, *On the topological structure of solvable groups*, Ann. of Math. (2) **42** (1941), 668–675.
3. K. Iwasawa, *On some types of topological groups*, Ann. of Math. (2) **50** (1949), 507–558.

## LINKED CITATIONS

- Page 1 of 1 -



*You have printed the following article:*

### **The Affine Group of a Lie Group**

Joseph A. Wolf

*Proceedings of the American Mathematical Society*, Vol. 14, No. 2. (Apr., 1963), pp. 352-353.

Stable URL:

<http://links.jstor.org/sici?sici=0002-9939%28196304%2914%3A2%3C352%3ATAGOAL%3E2.0.CO%3B2-Z>

---

*This article references the following linked citations. If you are trying to access articles from an off-campus location, you may be required to first logon via your library web site to access JSTOR. Please visit your library's website or contact a librarian to learn about options for remote access to JSTOR.*

## **References**

### <sup>1</sup> **A Fixed Point Theorem for Nilpotent Lie Groups**

Louis Auslander

*Proceedings of the American Mathematical Society*, Vol. 9, No. 5. (Oct., 1958), pp. 822-823.

Stable URL:

<http://links.jstor.org/sici?sici=0002-9939%28195810%299%3A5%3C822%3AAFPTFN%3E2.0.CO%3B2-E>

**NOTE:** *The reference numbering from the original has been maintained in this citation list.*