Introduction to Algebraic Topology

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Chapter 1

Introduction

1.1 Sets

Let X and Y be sets. The notation $Y \subseteq X$ means that Y is a subset of X and $Y \subset X$ means that Y is a proper subset of X, that is $Y \subseteq X$ and $Y \neq X$. Let $X \setminus Y$ denote the set

$$X \setminus Y = \{x | x \in X \quad \text{and} \quad x \notin Y\}.$$

The empty set is denoted by $\emptyset$.

Let X and Y be sets. The Cartesian product is defined by

$$X \times Y = \{(x, y) | x \in X, y \in Y\}.$$

Note: If X and Y are finite sets of m and n elements, respectively, then $X \times Y$ is a finite set of mn elements.

Let $X_1, \cdots, X_n$ be sets. The Cartesian product is defined by

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \cdots, x_n) | x_i \in X_i, 1 \leq i \leq n\}.$$

The infinite Cartesian product is defined similarly. For example, let $\{X_\alpha | \alpha \in I\}$ be a family of sets. Then

$$\prod_{\alpha \in I} X_\alpha = \{(x_\alpha) | x_\alpha \in X_\alpha, \alpha \in I\}.$$

The α-coordinate projection

$$\pi_\alpha \colon \prod_{\alpha' \in I} X_{\alpha'} \to X_\alpha$$

is defined by

$$\pi_\alpha((x_{\alpha'})) = x_\alpha.$$

Theorem 1.1.1 *Let $\{X_\alpha | \alpha \in I\}$ be a family of set. Then the Cartesian product $\prod_{\alpha\in I} X_\alpha$ satisfies the following universal lifting property:*

> *Let X be any set and let $f_\alpha: X \to X_\alpha$ be any function for each $\alpha \in I$. Then there is a unique function*
>
> $$f: X \to \prod_{\alpha\in I} X_\alpha$$
>
> *such that*
>
> $$f_\alpha = \pi_\alpha \circ f$$
>
> *for each α.*

Proof. Let $f: X \to \prod_{\alpha\in I} X_\alpha$ be a function defined by

$$f(x) = (f_\alpha(x))$$

for each $x \in X$. Then f is a function with the property that $f_\alpha(x) = \pi_\alpha \circ f(x)$ for any x and so $f_\alpha = \pi_\alpha \circ f$. This shows the existence of the universal lifting property. Let $g: X \to \prod_{\alpha\in I} X_\alpha$ be any function with the property that $f_\alpha = \pi_\alpha \circ g$ for each α. Then the α-th coordinate of $g(x)$ is $f_\alpha(x)$ for each $x \in X$. Thus $g = f$ defined above. This shows the uniqueness of the universal lifting property.

Let $f: X \to Y$ be a function. Then the image of f is defined by

$$\mathrm{Im}(f) = f(X) = \{y \in Y | y = f(x) \quad \text{for} \quad \text{some} \quad x \in X\}.$$

The identity function on X is denoted by id_X, id or 1. Thus $\mathrm{id}(x) = x$.

Exercise 1.1.1 *Let $f: X \to Y$ be a function. Let $\{X_\alpha | \alpha \in I\}$ be a family of subsets of X. Then*

1) show that

$$f(\bigcup_{\alpha\in I} X_\alpha) = \bigcup_{\alpha\in I} f(X_\alpha);$$

2) show that

$$f(\bigcap_{\alpha\in I} X_\alpha) \subseteq \bigcap_{\alpha\in I} f(X_\alpha);$$

3) show by example that

$$f(\bigcap_{\alpha\in I} X_\alpha) \neq \bigcap_{\alpha\in I} f(X_\alpha)$$

in general.

Let $f: X \to Y$ be a function. Let A be a subset of X. Then the restriction $f|_A: A \to Y$ is the function defined by

$$f|_A(a) = f(a)$$

for $a \in A$. Let B be a subset of Y. The pre-image $f^{-1}(B)$ is defined by

$$f^{-1}(B) = \{x \in X | f(x) \in B\}.$$

Note that $f^{-1}(B)$ could be an empty set.

Exercise 1.1.2 *Let $f: X \to Y$ be a function. Let $\{B_\beta|\beta \in J\}$ be a family of subsets of Y. Then*

1) show that

$$f^{-1}(\bigcup_{\beta\in J} B_\beta) = \bigcup_{\beta\in J} f^{-1}(B_\beta);$$

2) show that

$$f^{-1}(\bigcap_{\beta\in J} B_\beta) = \bigcap_{\beta\in J} f^{-1}(B_\beta);$$

3) show that

$$f^{-1}(Y \setminus B_\beta) = X \setminus f^{-1}(B_\beta).$$

A function $f: X \to Y$ is said to be bijective if it is one-to-one and onto. In this case the inverse is denoted by $f^{-1}: Y \to X$. Note that f^{-1} is also bijective. If there is a bijective function f from X to Y, we call that X is isomorphic to Y as sets.

Exercise 1.1.3 *Let X be a set. Let X_α be a family of sets with indices α in a set I. Suppose that $X_\alpha = X$ for each α. Show that $\prod_{\alpha\in I} X_\alpha$ is isomorphic to the set of functions from I to X.*

A relation on a set X is a subset $\sim$ of $X \times X$. We write $x \sim y$ if $(x, y) \in \sim$. A relation on X is an *equivalence relation* if it satisfies

1) the reflexive condition: $x \sim x$ for all $x \in X$;

2) the symmetric condition: If $x \sim y$, then $y \sim x$;

3) the transitive condition: If $x \sim y$ and $y \sim z$, then $x \sim z$.

The equivalence class of x is the set

$$\{x\} = \{y \in X | x \sim y\}.$$

Exercise 1.1.4 *Let $\sim$ be an equivalence relation on X. Show that each element of X belongs to exactly one equivalence class.*

1.2 Monoids and Groups

A *binary operation (multiplication)* on a set X is a function $\mu\colon X \times X \to X$. We abbreviate $\mu(x, y)$ to xy or $x+y$. A *monoid* M is a set M together with a multiplication $\mu\colon M \times M \to M$ satisfying the following conditions:

1) (identity) there exists an element $1 \in M$ such that

$$1x = x1 = x$$

for any $x \in M$;

2) (associativity) the equation

$$(x_1x_2)x_3 = x_1(x_2x_3)$$

holds for any $x_1, x_2, x_3 \in M$.

A *group* is a monoid G satisfying

3) (inverse) For each $x \in G$, there exists an element $x^{-1} \in G$ such that

$$xx^{-1} = x^{-1}x = 1.$$

In other words, a group is a monoid in which every element is invertible. Note that if x is invertible, then the inverse of x is unique. A group (or monoid) G is said to *abelian* or *commutative* if $xy = yx$ for any $x, y \in G$. Let G and H be monoids (or groups). Then the Cartesian product $G \times H$ is a monoid (or group) under the multiplication defined by

$$(g,h)(g',h') = (gg',hh').$$

In additive case, we write $G \oplus H$ for $G \times H$.

A subset H of a group (monoid) is a *subgroup (submonoid)* of G if H is a group (monoid) under the binary operation of G. Let H be a subgroup (submonoid) of G and let $g \in G$. The left and right cosets of H by g are defined by

$$gH = \{gh | h \in H\} \qquad Hg = \{hg | h \in H\}.$$

Example 1.2.1 Let $\mathbb{Z}^+$ be the set of non-negative integers. Then $\mathbb{Z}^+$ is a monoid under the addition $+$. $\mathbb{Z}^+$ is a submonoid of $\mathbb{Z}$. $\mathbb{Z}$ is often called the group completion of the monoid $\mathbb{Z}^+$, i.e. the "smallest group" that contains $\mathbb{Z}^+$. The set of natural numbers is a monoid under the multiplication. The "group completion" of natural numbers is the set of positive rational numbers with the multiplication.

In general, monoids and the "group completion" of monoids are very complicated and there are many research papers about these topics.

Let G and H be monoids (or groups). A homomorphism $f\colon G \to H$ is a function such that $f(1) = 1$ and

$$f(xy) = f(x)f(y)$$

for any $x, y \in G$.

Exercise 1.2.1 *Let G and Hbe groups and let $f\colon G \to H$ be a function such that $f(xy) = f(x)f(y)$ for any $x, y \in G$. Show that*

1) $f(1) = 1$;

2) $f(x^{-1}) = (f(x))^{-1}$ for any $x \in G$.

Let G and H be monoids (or groups). The *kernel* of a homomorphism $f\colon G \to H$ is the set

$$\mathrm{Ker}(f) = \{x \in G | f(x) = 1\}.$$

Note that a homomorphism f is one-to-one (a monomorphism) if and only if

$$\mathrm{Ker}(f) = \{1\}.$$

A monoid (or group) G is called isomorphic to H if there is a bijective homomorphism $f\colon G \to H$. In this case, we write $G \cong H$ or $f\colon G \cong H$.

A subgroup K of G is normal if $gxg^{-1} \in K$ for all $g \in G$ and $x \in K$. Let G and H be groups. Then the kernel of a homomorphism $f\colon G \to H$ is a normal subgroup of G. The image of f is a subgroup of H which is not normal in general.

Exercise 1.2.2 *Let K be a normal subgroup of a group G. Show that*

1) *$gK = Kg$ for any $g \in G$;*

2) *the set*

$$G/K = \{gK | g \in G\}$$

is a group under the operation

$$(gK)(g'K) = (gg')K.$$

The group G/K is called the quotient group of G by K.

Let G be a group and let $g \in G$. The subgroup generated by g is the subset

$$\langle g \rangle = \{g^n | n \in \mathbb{Z}\}.$$

Proposition 1.2.2 *Let G be a group and let $g \in G$. Then $\langle g \rangle$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ for some n.*

Proof. Let $\phi\colon \mathbb{Z} \to \langle g \rangle$ be the function defined by

$$\phi(n) = g^n.$$

Then ϕ is a homomorphism of groups because

$$\phi(m+n) = g^{m+n} = g^m g^n = \phi(m)\phi(n).$$

Note that ϕ is an epimorphism, that is ϕ is onto. If $g^m \neq 1$ for any positive integer m, then ϕ is an isomorphism. Suppose that $g^m = 1$ for some positive integer m. Let

$$n = \min\{m | g^m = 1, m > 0\}.$$

Then

$$\mathrm{Ker}(\phi) = n\mathbb{Z}$$

and so

$$\langle g \rangle \cong \mathbb{Z}/n\mathbb{Z}.$$

If $G = \langle g \rangle$ for some g, we say that G is a *cyclic* group with generator g. A set of *generators* for a group G is a subset S of G such that each element in G is a product of powers of elements taken from S. A group G is called *finitely generated* if it is generated by a finite subset.

A *free abelian group* of rank n is the direct sum

$$\mathbb{Z}^{\oplus n} = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}.$$

Theorem 1.2.3 (Decomposition Theorem) *Let G be a finitely generated abelian group. Then G is isomorphic to*

$$H_0 \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_m,$$

where H_0 is a free abelian group and H_i is a cyclic group of prime power order for $1 \leq i \leq m$.

The proof can be found in any text book of algebra.

A commutator in a group G is an element

$$[g, h] = ghg^{-1}h^{-1}.$$

for some elements $g, h \in G$. The commutator subgroup $[G, G]$ is the subgroup of G generated by all commutators of G. The commutator subgroup $[G, G]$ is normal. The group $G/[G, G]$ is called the *abelianization* of the group G. Note that a group G is abelian if and only if the commutator subgroup $[G, G]$ is trivial. A group G is called perfect if $[G, G] = G$. An example of perfect groups is the alternating groups A_n for $n > 4$. Non-commutative groups are much more complicated than abelian groups.

1.3 G-sets

Let G be a group. A set X is called a *left G-set* if there is an operation $\mu: G \times X \to X$, $(g, x) \to g \cdot x$, such that

1) $1 \cdot x = x$ for all $x \in X$;

2) $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$.

A set X is called a *right G-set* if there is an operation $\mu: X \times G \to X$, $(g, x) \to x \cdot g$, such that

1) $x \cdot 1 = x$ for all $x \in X$;

2) $x \cdot (gh) = (x \cdot g) \cdot h$ for all $g, h \in G$ and $x \in X$.

Example 1.3.1 Let H be a subgroup of a group G. Then the set of left cosets $\{gH | g \in G\}$ is a left G-set and the set of right cosets $\{Hg | g \in G\}$ is a right G-set.

Theorem 1.3.2 *Let X be a left G-set. For any $g \in G$, the function $\theta_g\colon X \to X$ defined by*

$$x \to g \cdot x$$

is a bijective.

Proof. From the definition, we have that $\theta_g \theta_h = \theta_{gh}$ and $\theta_1 = \mathrm{id}_X$. Thus

$$\theta_g \theta_{g^{-1}} = \mathrm{id}_X = \theta_{g^{-1}} \theta_g$$

and so θ_g is a bijective.

Similarly, if X is a right G-set, then the function $\theta_g\colon X \to X$ defined by $x \to x \cdot g$ is a bijective.

1.4 Categories and Functors

A category may be thought of intuitively as consisting of sets, possibly with additional structure, and functions, possibly preserving additional structure. More precisely, a category $\mathcal{C}$ consists of

1) A class of objects

2) For every ordered pair of objects X and Y, a set $\mathrm{Hom}(X, Y)$ of *morphisms* with *domain* X and *range* Y; if $f \in \mathrm{Hom}(X, Y)$, we write $f\colon X \to Y$ or $X \xrightarrow{f} Y$

3) For every ordered triple of objects X, Y and Z, a function associating to a pair of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ their *composite*

$$g \circ f\colon X \to Z$$

These satisfy the following two axioms:

Associativity. If $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$, then

$$h \circ (g \circ f) = (h \circ g) \circ f\colon X \to W.$$

Identity. For every object Y there is a morphism $\mathrm{id}_Y\colon Y \to Y$ such that if $f\colon X \to Y$, then $\mathrm{id}_Y \circ f = f$, and if $h\colon Y \to Z$, then $h \circ \mathrm{id}_Y = h$.

A category is said to be *small* if the class of objects is a set. The category of sets means the category in which the objects are sets and the morphisms are functions. The category of sets is NOT small. But there are many small categories. For instance, the category of finite sets, that is in which the objects are finite sets and the morphisms are functions between finite sets. We list some examples of categories:

1) The category of sets and functions.

2) The category of pointed sets (A pointed set means a non-empty set X with a *base point* $x_0 \in X$) and pointed functions (that is the functions that preserving the base points).

3) The category of finite ordered sets and monotone functions (that is $f(x) \leq f(y)$ is $x \leq y$). This category is usually denoted by Δ. The objects in Δ are given by $\{0, 1, \cdots, n\}$ for $n \geq 0$ and the morphisms in Δ are given by monotone function from $\{0, 1, \cdots, m\}$ to $\{0, 1, \cdots, n\}$ for any m, n.

4) The category of groups and homomorphisms.

5) The category of monoids and homomorphisms.

6) The category of topological spaces and continuous functions. Topological space is a generalization of the usual spaces such as Euclidian spaces $\mathbb{R}^n$, spheres, *polyhedra*, metric spaces and etc. We will give the definition of topological space in the next chapter.

Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{C}' \subseteq \mathcal{C}$ is a category such that

a) The objects of $\mathcal{C}'$ are also objects of $\mathcal{C}$;

b) For objects X' and Y' of $\mathcal{C}'$, $\mathrm{Hom}_{\mathcal{C}'}(X', Y')$ is a subset of $\mathrm{Hom}_{\mathcal{C}}(X', Y')$ and

c) If $f' \colon X' \to Y'$ and $g' \colon Y' \to Z'$ are morphisms of $\mathcal{C}'$, their composite in $\mathcal{C}'$ equals their composite in $\mathcal{C}$.

$\mathcal{C}'$ is called a full subcategory of $\mathcal{C}$ if $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ and for objects X' and Y' in $\mathcal{C}'$, $\mathrm{Hom}_{\mathcal{C}'}(X', Y') = \mathrm{Hom}_{\mathcal{C}}(X', Y')$. For example, the category of groups and homomorphisms is a subcategory of the category of sets and functions but it is not a full subcategory. The category of finite sets and functions is a full subcategory of the category of sets and functions.

Let $\mathcal{C}$ be a category. A morphism $f \colon X \to Y$ is called an equivalence if there is a morphism $g \colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* (or *contravariant functor*) T from $\mathcal{C}$ to $\mathcal{D}$ consists of an object function which assigns to every object X of $\mathcal{C}$ an object $T(X)$ of $\mathcal{D}$ and a morphism function which assigns to every morphism $f: X \to Y$ of $\mathcal{C}$ a morphism $T(f): T(X) \to T(Y)$ [or $T(f): T(Y) \to T(X)$] of $\mathcal{D}$ such that

a) $T(\mathrm{id}_X) = \mathrm{id}_{T(X)}$ and

b) $T(g \circ f) = T(g) \circ T(f)$ [or $T(g \circ f) = T(f) \circ T(g)$].

Theorem 1.4.1 *Let T be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$. Then T maps equivalences in $\mathcal{C}$ to equivalences in $\mathcal{D}$.*

Proof. Assume that T is covariant (the argument is similar if T is contravariant). Let $f: X \to Y$ be an equivalence and let $f^{-1}: Y \to X$ be its inverse. Since $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$, $T(f^{-1}) \circ T(f) = \mathrm{id}_{T(X)}$ and $T(f) \circ T(f^{-1}) = \mathrm{id}_{T(Y)}$. Thus $T(f)$ is an equivalence.

A topological problem on spaces is to how to classify topological spaces. In other words, roughly speaking, how to know whether a space X is homeomorphic to another space Y or not. Basic ideas in algebraic topology is to introduce various functors from the category of topological spaces to "algebraic" categories such as the category of groups, the category of abelian groups, and the category of modules and etc. Homology, fundamental group and higher homotopy groups are most important functors from the category of spaces to the category of groups.

For example, we will know that the fundamental group of $\mathbb{R} \setminus \{0\}$ is $\mathbb{Z}$ but the fundamental group of $\mathbb{R}^2 \setminus \{0\}$ is $\{0\}$. By Theorem 1.4.1, we have that $\mathbb{R} \setminus \{0\}$ is not homeomorphic to $\mathbb{R}^2 \setminus \{0\}$ and so $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. This is a simple example. Actually we will be able to classify all of (2-dimensional) surfaces in this course using the fundamental group.

Chapter 2

General Topolgy

2.1 Metric spaces

Let X be a set. A *metric* d for X is a function $d\colon X \times X \to \mathbb{R}$ satisfying

1) $d(x, y) = 0$ if and only if $x = y$;

2) (triangle inequality)
$$d(x, y) + d(x, z) \geq d(y, z).$$

In this case X is called a metric space with the metric d.

Proposition 2.1.1 *If d is a metric for X, then $d(x, y) \geq 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$.*

Proof. By the triangle inequality, we have

$$2d(x, y) = d(x, y) + d(x, y) \geq d(y, y) = 0,$$

$$d(x, y) = d(x, y) + d(x, x) \geq d(y, x),$$
$$d(y, x) = d(y, x) + d(y, y) \geq d(x, y).$$

Thus $d(x, y) \geq 0$ and $d(x, y) = d(y, x)$.

Exercise 2.1.1 a) Show that each of the following is a metric for $\mathbb{R}^n$:

$$d(x, y) = (\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2} = ||x - y||; \quad d(x, y) = \begin{cases} 0 & \text{if} \quad x = y, \\ 1 & \text{if} \quad x \neq y; \end{cases}$$

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|; \qquad d(x,y) = \max_{1 \le i \le n} |x_i - y_i|.$$

b) Show that $d(x,y)$ does not define a metric on $\mathbb{R}$.

c) Show that $d(x,y) = \min_{1 \le i \le n} |x_i - y_i|$ does not define a metric on $\mathbb{R}^n$.

d) Let d be a metric. Show that d' defined by

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

is also a metric.

Definition 2.1.2 Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f \colon X \to Y$ is said to be *continuous* at $x \in X$ if for any $\epsilon_x > 0$ there exists $\delta_x > 0$ such that $d_Y(f(x), f(y)) < \epsilon_x$ whenever $d_X(x,y) < \delta_x$. The function f is said to be *continuous* if it is continuous at all points $x \in X$.

Exercise 2.1.2 Let X be a metric space with metric d. Let $y \in X$.Show that the function $f \colon X \to \mathbb{R}$ defined by $f(x) = d(x,y)$ is continuous.

Definition 2.1.3 A subset U of a metric space (X, d) is said to be open if for any $x \in U$ there exists $\epsilon_x > 0$ such that if $y \in X$ and $d(y,x) < \epsilon_x$ then $y \in U$.

In other words U is open if for any $x \in U$ there exists an $\epsilon_x > 0$ such that the open ball

$$B_{\epsilon_x}(x) = \{y \in X | d(y,x) < \epsilon_x\} \subseteq U.$$

Exercise 2.1.3 a) Show that $B_\epsilon(x)$ is always an open set for any x and any $\epsilon > 0$.

b) Which of the following subsets of $\mathbb{R}^2$ (with the usual metric) are open?

$$\{(x,y) | x^2 + y^2 < 1\} \cup \{(1,0)\}, \quad \{(x,y) | x^2 + y^2 \le 1\},$$

$$\{(x,y) || x| < 1\}, \quad \{(x,y) | x + y < 0\},$$

$$\{(x,y) | x + y \ge 0\}, \quad \{(x,y) | x + y = 0\}.$$

Exercise 2.1.4 Show that if $\mathcal{U}$ is the family of open sets arising from a metric space then

i) The empty set $\emptyset$ and the whole set belong to $\mathcal{U}$;

ii) The intersection of two members of $\mathcal{U}$ belongs to $\mathcal{U}$;

iii) The union of any number of members of $\mathcal{U}$ belongs to $\mathcal{U}$.

Theorem 2.1.4 *A function $f\colon X \to Y$ between two metric spaces is continuous if and only if for any open set U in Y the set $f^{-1}(U)$ is open in X.*

Proof. Suppose that f is continuous and U is open in Y. Let $x \in f^{-1}(U)$. Then $f(x) \in U$. Since U is open, there exists $\epsilon > 0$ such that $B_\epsilon(f(x)) \subseteq U$. By the definition, there exists $\delta > 0$ such that $f(y) \in B_\epsilon(f(x))$ whenever $y \in B_\delta(x)$, that is $f(B_\delta(x)) \subseteq B_\epsilon(f(x)$. Thus

$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(x)) \subseteq f^{-1}(U)$$

and so $f^{-1}(U)$ is open.

Conversely let $x \in X$; then $B_\epsilon(f(x))$ is an open subset of Y and so $f^{-1}(B_\epsilon(f(x)))$ is an open subset of X. Thus there exists $\delta > 0$ with

$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x))).$$

In other words $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \delta$, that is f is continuous.

2.2 Topological Spaces

Definition 2.2.1 Let X be a set. A *topology* $\mathcal{U}$ for X is a collection of subsets of X satisfying

i) $\emptyset$ and X are in $\mathcal{U}$;

ii) the intersection of two members of $\mathcal{U}$ is in $\mathcal{U}$;

iii) the union of any number of members of $\mathcal{U}$ is in $\mathcal{U}$.

The set X with $\mathcal{U}$ is called a *topological space*. The members $U \in \mathcal{U}$ are called the *open sets*.

Exercise 2.2.1 Let $\mathcal{U}$ be a topology for X. Show that the intersection of a finite number of members of $\mathcal{U}$ is in $\mathcal{U}$.

Note: The intersection of infinitely many open sets is called a *Borel set* which is not open in general.

Let X be a metric space and let $\mathcal{U}$ be the family of open sets. Then $\mathcal{U}$ is a topology. This topology is called the *metric topology.* Note that two different metrics may give rise to the same topology.

Exercise 2.2.2 *Let X be a metric space with metric d. Let d' be the new metric defined in Exercise 2.1.1. Then (X,d) and (X,d') has the same topology.*

Given a set X there may be different choices of topologies for X.

Exercise 2.2.3 *Let $X = \{a,b\}$. Show that there are four different topologies given as follows:*

$$\mathcal{U}_1 = \{\emptyset, X\}, \mathcal{U}_2 = \{\emptyset, \{a\}, X\}, \mathcal{U}_3 = \{\emptyset, \{b\}, X\}, \mathcal{U}_4 = \{\emptyset, \{a\}, \{b\}, X\}.$$

Exercise 2.2.4 *Let X be a set. Let $\mathcal{U}_1 = \{\emptyset, X\}$, let $\mathcal{U}_2 = \mathcal{S}(X)$ be the set of all subsets of X and let*

$$\mathcal{U}_3 = \{U \subseteq X | X \setminus U \quad \text{is} \quad \text{finite}\}.$$

Show that $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{U}_3$ are topologies for X.

$\mathcal{U}_1$ is called *indiscrete topology,* $\mathcal{U}_2$ is called *discrete topology* and $\mathcal{U}_3$ is called *finite complement topology.*

Let X be a topological space and let A be a subset of X. The largest open set contained in A, this is denoted by $\overset{\circ}{A}$ and is called the *interior* of A. For example, let $X = \mathbb{R}^n$. Then the interior of the closed ball

$$D_r(x) = \{y | d(x,y) \leq r\}$$

is the open ball $B_r(x) = \{y | d(x,y) < r\}$.

Exercise 2.2.5 *Let $X = \mathbb{R}^n$ with the usual topology. Let*

$$I^n = \overbrace{[0,1] \times \cdots \times [0,1]}^{n}.$$

Show that

$$\overset{\circ}{I^n} = \overbrace{(0,1) \times \cdots \times (0,1)}^{n}.$$

Let X be a topological space. A subset $N \subseteq X$ with $x \in N$ is called a *neighborhood* of x if there is an open set U with $x \in U \subseteq N$. For example, if X is a metric space, then the closed ball $D_\epsilon(x)$ and the open ball $B_\epsilon(x)$ are neighborhoods of x.

Exercise 2.2.6 *Let X be a topological space. Prove each of the following statements.*

a) For each point $x \in X$ there is at least one neighborhood of x.

b) If N is a neighborhood of x and $N \subseteq M$ then M is also a neighborhood of x.

c) If M and N are neighborhoods of x then so is $N \cap M$.

d) For each $x \in X$ and each neighborhood N of x there exists a neighborhood U of x such that $U \subseteq N$ and U is a neighborhood of each of its points.

Definition 2.2.2 *A subset C of a topological space X is said to closed if $X \setminus C$ is open.*

Theorem 2.2.3 *i) $\emptyset$ and X are closed;*

ii) the union of any pair of closed sets is closed;

iii) the intersection of any number of closed sets is closed.

Note: The union of infinitely many closed sets is not closed in general.

Exercise 2.2.7 *Let X be a set and let $\mathcal{V}$ be a family of subsets of X satisfying*

i) $\emptyset, X \in \mathcal{V}$;

ii) the union of any pair of members of $\mathcal{V}$ belongs to $\mathcal{V}$;

iii) the intersection of any number of members of $\mathcal{V}$ belongs to $\mathcal{V}$.

Show that $\mathcal{U} = \{X - V | V \in \mathcal{V}\}$ is a topology for X.

Let Y be a subset of a topological space X. The set

$$\bar{Y} = \bigcap \{F | F \supseteq Y \quad F \quad \text{is} \quad \text{closed}\}$$

is called the *closure* of Y. The set $Y' = \bar{Y} \setminus Y$ is called the set of *limit points* of Y.

Proposition 2.2.4 *Let Y be a subset of a topological space X. Then $x \in \bar{Y}$ if and only if for every neighborhood N of x, $N \cap Y \neq \emptyset$.*

Proof. Let $x \in \bar{Y}$ and suppose that N is a neighborhood of $x$with $N \cap Y = \emptyset$. Then there is an open neighborhood U of x with $U \subseteq N$. Thus $X \setminus U$ is a closed set and $Y \subseteq X \setminus U$. It follows that $\bar{Y} \subseteq X \setminus U$ and so $x \notin U$. One gets a contradiction.

Conversely suppose that $x \notin \bar{Y}$. Since $\bar{Y}$ is closed, $X \setminus \bar{Y}$ is an (open) neighborhood of x so that $(X \setminus \bar{Y}) \cap Y = \emptyset$ is a contradiction.

Exercise 2.2.8 *Let $X = \mathbb{R}$ with the usual topology. Find the closure of each of the following subsets of X:*

$$A = \{1, 2, 3, \cdots\}, B = \{x | x \quad \text{is} \quad \text{rational}\}, C = \{x | x \quad \text{is} \quad \text{irrational}\}.$$

Exercise 2.2.9 Prove each of the following statements.

a) If Y is a subset of a topological space X with $Y \subseteq F \subseteq X$ and F is closed then $\bar{Y} \subseteq F$.

b) Y is closed if and only if $Y = \bar{Y}$.

c) $\bar{\bar{Y}} = \bar{Y}$.

d) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

e) $X \setminus \overset{\circ}{Y} = \overline{X \setminus Y}$.

f) $\bar{Y} = Y \cup \partial Y$ where $\partial Y = \bar{Y} \cap \overline{(X \setminus Y)}$ (∂Y is called the boundary of Y).

g) Y is closed if and only if $\partial Y \subseteq Y$.

h) $\partial Y = \emptyset$ if and only if Y is both open and closed.

i) For $a < b \in \mathbb{R}$

$$\partial(a, b) = \partial[a, b] = \{a, b\}.$$

2.3 Continuous Functions

Definition 2.3.1 A function $f\colon X \to Y$ between two topological spaces is said to be *continuous* if for every open set U of Y the preimage $f^{-1}(U)$ is open in X.

A continuous function from a topological space to a topological space is often simply called a map. The *category of topological spaces* is defined as follows: the objects are topological spaces and the morphisms are maps, that is continuous functions.

Theorem 2.3.2 *Let X and Y be topological spaces. A function $f\colon X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed for any closed subset C of Y.*

Proof. Suppose that f is continuous and let C be a closed set in Y. Then $Y \setminus C$ is an open set and so $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$ is an open set. It follows that $f^{-1}(C)$ is a closed set. Now suppose that $f^{-1}(C)$ is closed for any closed set C and let U be an open set. Then $Y \setminus U$ is a closed set and so $X \setminus f^{-1}(U) = f^{-1}(Y \setminus U)$. Thus $f^{-1}(U)$ is an open set.

So far we have two general methods to see whether a function is continuous or not, that is by the definition or by the theorem above. If $f\colon X \to Y$ is a function between metric spaces, then we can also use $\epsilon - \delta$ method to test whether f is continuous or not. As we know in calculus that the compositions of continuous functions is still continuous. This is actually true in general.

Theorem 2.3.3 *Let X, Y and Z be topological spaces. If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions then the composite $g \circ f\colon X \to Z$ is continuous.*

Proof. Let U be any open set in Z. Then $g^{-1}(U)$ is an open set in Y and so $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is an open set in X.

Definition 2.3.4 Let X and Y be topological spaces. We say that X and Y are homeomorphic if there exist continuous functions $f\colon X \to Y, g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. We write $X \cong Y$ and say that f and g are *homeomorphisms* between X and Y.

By the definition, a function $f\colon X \to Y$ is a homeomorphism if and only if

i) f is a bijective;

ii) f is continuous and

iii) f^{-1} is also continuous.

Equivalently f is a homeomorphism if and only if 1) f is a bijective, 2) f is continuous and 3) f is an open map, that is f sends open sets to open sets. Thus a homeomorphism between X and Y is a bijective between the points and the open sets of X and Y.

A very general question in topology is how to classify topological spaces under homeomorphisms. For example, we know (from complex analysis and others) that any simple closed loop is homeomorphic to the unit circle S^1. Roughly speaking topological classification of curves is known. The topological classification of (two-dimensional) surfaces is known as well. However the topological classification of 3-dimensional manifolds (we will learn manifolds later.) is quite open. The famous Poicaré conjecture is related to this problem.

Exercise 2.3.1 Give an example of spaces X, Y and a continuous bijective $f\colon X \to Y$ such that f^{-1} is NOT continuous. (Hint: Give a set X. Look at the discrete topology, the indiscrete topology and the identity function.)

A *pointed space* means a topological space X together with a point $x_0 \in X$. The point x_0 is called the base point of X. We often write $*$ for x_0. Let X and Y be pointed spaces with base points x_0 and y_0, respectively. A map $f\colon X \to Y$ is called a *pointed map* if $f(x_0) = y_0$. The category of pointed topological spaces means a category in which the objects are pointed spaces and the morphisms are pointed maps.

2.4 Induced Topology

Definition 2.4.1 Let X be a topological space and let S be a subset of X. The topology on S *induced* by the topology of X is the family of the sets of the form $U \cap S$ where U is an open set in X. We call that the subset S with induced topology is a subspace of X.

Note: By this definition, an open set V in S means $V = U \cap S$ for some open set U in X. The induced topology is also called the subspace topology.

Exercise 2.4.1 *Let X be a topological space with the topology $\mathcal{U}$ and let S be a subset of X. Show that*

$$\mathcal{U} \cap S = \{U \cap S | U \in \mathcal{U}\}$$

is a topology for S.

Example 2.4.2 Let S^n be the n-sphere, that is,

$$S^n = \{x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\} \subseteq \mathbb{R}^{n+1}$$

with the induced topology. Then S^n is a (closed) subspace of $\mathbb{R}^{n+1}$. Note that $S^n = \mathbb{R}^{n+1} \cap S^n$ is an open set in S^n but S^n is NOT open in $\mathbb{R}^{n+1}$.

Proposition 2.4.3 *Let S be a subspace of a topological space X. Then the inclusion function $i: S \to X$ is continuous.*

Proof. Let U be an open set in X. Then $i^{-1}(U) = U \cap S$ is an open set in S.
Note: One can show that the subspace topology is the smallest topology such that the inclusion is continuous.

Proposition 2.4.4 *Let S be a subspace of a topological space X. Then*

1) If S is open in X, then any open set in the subspace S is open in X;

2) If S is closed in X, then any closed set in the subspace S is closed in X.

Proof. The proofs of 1) and 2) are more or less identical. We only prove assertion 2). Let V be a closed set in S. Then $S \setminus V$ is an open set in S. By the definition, there is an open set U in X such that

$$S \setminus V = U \cap S.$$

Thus $V = (X \setminus U) \cap S$. Since S and $X \setminus U$ are closed, V is closed.

Exercise 2.4.2 Show that

1) the subspace (a, b) of $\mathbb{R}$ is homeomorphic to $\mathbb{R}$. (Hint: Use functions like $x \to \tan(\pi(cx + d))$ for suitable c and d.)

2) the subspaces $(1, \infty), (0, 1)$ of $\mathbb{R}$ are homeomorphic. (Hint: $x \to 1/x$.)

3) $S^n \setminus \{(0, 0, \cdots, 0, 1)\}$ is homeomorphic to $\mathbb{R}^n$ with the usual topology. (Hint: Define $\phi: S^n \setminus \{(0, 0, \cdots, 0, 1)\} \to \mathbb{R}^n$ by

$$\phi(x_1, x_2, \cdots, x_{n+1}) = (\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}} \cdots, \frac{x_n}{1 - x_{n+1}})$$

and $\psi: \mathbb{R}^n \to S^n \setminus \{(0, 0, \cdots, 0, 1)\}$ by

$$\psi(x_1, \cdots, x_n) = \frac{1}{1 + \|x\|^2}(2x_1, 2x_2, \cdots, 2x_n, \|x\|^2 - 1).)$$

A map $f: X \to Y$ is called an *embedding* if f is one-to-one and X is homeomorphic to the image $f(X)$ with the subspace topology. The embedding problem in topology is as follows:

> Given a topological space X. Can we embed X into $\mathbb{R}^n$ for some n? If not, can we embed X into a Hilbert space? If yes, what is the minimal number n such that X can be embedded in $\mathbb{R}^n$? This number is called the embedding number of X.

This question is important (and difficult in general) because a topological space X could be very abstract but the spaces $\mathbb{R}^n$ are much easier to be understood. For instance, the circle S^1 can embed in $\mathbb{R}^2$ but S^1 can not embed in $\mathbb{R}^1$. Thus the embedding number of S^1 is 2. Well sometimes a space X could be very simple but it could have a very complicated embedding in $\mathbb{R}^n$.

A *knot* K is a subspace of $\mathbb{R}^3$ that is homeomorphic to the circle S^1. Two knots K_1 and K_2 are *similar* if there is a homeomorphism $h: \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K_1) = K_2$. The knot theory is to study the classification of knots under this relation.

2.5 Quotient Topology

Definition 2.5.1 Let $f: X \to Y$ be a surjective function from a topological space X to a set Y. The *quotient topology* on Y with respect to f is the family

$$\mathcal{U}_f = \{U | f^{-1}(U) \quad \text{is open in} \quad X\}.$$

Exercise 2.5.1 Show that $\mathcal{U}_f$ above is a topology for Y.

Note: After giving the quotient topology on Y the function $f: X \to Y$ is continuous.

Example 2.5.2 (Projective Spaces) Let set $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are defined as follows:

$$\mathbb{R}P^n = \{l | l \quad \text{is a line in} \quad \mathbb{R}^{n+1} \quad \text{with} \quad 0 \in l\},$$

$$\mathbb{C}P^n = \{l | l \quad \text{is a complex line in} \quad \mathbb{C}^{n+1} \quad \text{with} \quad 0 \in l\}.$$

The topologies in $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are given by the quotient topology under the quotient maps $\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ and $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$, respectively.

By this example, one can see that the quotient space Y could be much more complicated than the original space X. The following theorem gives a general method to see whether a function from Y to another space is continuous or not.

Theorem 2.5.3 *Let X be a topological space and let $f: X \to Y$ be a surjective. Suppose that Y are given the quotient topology with respect to f. Then a function $g: Y \to Z$ from Y to a topological space Z is continuous if and only if the composite $g \circ f$ is continuous.*

Proof. Suppose that g is continuous. Since f is continuous, the composite $g \circ f$ is continuous.

Now suppose that the composite $g \circ f$ is continuous. Let U be any open set in Z. Then

$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$$

is open in X and so $g^{-1}(U)$ is open in Y by the definition of quotient topology.

Exercise 2.5.2 Show that

1) $\mathbb{R}P^1 \cong S^1$;

2) $\mathbb{C}P^1 \cong S^2$.

The famous Hopf fibration is the composite

$$S^3 \hookrightarrow \mathbb{C}^2 \setminus \{0\} \twoheadrightarrow \mathbb{C}P^1 \cong S^2.$$

Let A be a subspace of a space X. The space X/A is the quotient space

$$X/A = X/\sim,$$

where $\sim$ is the equivalence relation generated by

$$a \sim b$$

for any $a, b \in A$. As a set $X/A = (X \setminus A) \cup \{*\}$, where $*$ is the eqivalence class of any particular choice of elements in A. The topology in X/A is given by the quotient topology. Roughly speaking X/A is the quotient space X by pinching out A to be one point.

Exercise 2.5.3 *Show that D^n/S^{n-1} is homeomorphic to S^n.*

The canonical inclusions $\mathbb{R}^n \to \mathbb{R}^{n+1}, \mathbb{C}^n \to \mathbb{C}^{n+1}$ given by $(x_1, \cdots, x_n) \to (x_1, \cdots, x_n, 0)$ induce the maps $\mathbb{R}P^n \to \mathbb{R}P^{n+1}$ and $\mathbb{C}P^n \to \mathbb{C}P^{n+1}$, respectively. Thus $\mathbb{R}P^n$ and $\mathbb{C}P^{n+1}$ can be considered as the subspaces of $\mathbb{R}P^{n+1}$ and $\mathbb{C}P^{n+1}$, respectively, for each n.

Exercise 2.5.4 *Show that* $\mathbb{R}P^{n+1}/\mathbb{R}P^n \cong S^{n+1}$ *and* $\mathbb{C}P^{n+1}/\mathbb{C}P^n \cong S^{2n+1}$.

A *fibrewise topological space* means a map $f\colon X \to Y$. In the case where f is an onto, it often called a *bundle*. For each $y \in Y$, the subspace $f^{-1}(y) \subseteq X$ is called the *fibre* at y. Let $f\colon X \to Y$ be a bundle. Then

$$X = \bigcup_{y \in Y} f^{-1}(y)$$

and so X can be considered as the union of subspaces $f^{-1}(y)$ with indexes in a topological space Y. Fibre bundles and covering spaces are special bundles. We will study covering spaces in the next chapter. The category of fibrewise topological spaces is a category in which the objects are fibrewise topological spaces and the morphisms are given by the commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & X' \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f'} \\ Y & \xrightarrow{\ \psi\ } & Y'. \end{array}$$

In other words, the working objects for fibrewise topology are continuous maps and the "relations" between the working objects are the diagrams above. One finds surprisingly that many results in the homotopy theory of topological spaces also holds for the homotopy theory of fibrewise topological spaces. Well the latter one is much more "abstract".

2.6 Product Spaces, Wedges and Smash Products

Let X and Y be topological spaces with topologies $\mathcal{U}_X$ and $\mathcal{U}_Y$, respectively. Let

$$\mathcal{U}_{X\times Y} = \{\bigcup_{\alpha} U_\alpha \times V_\alpha \subseteq X \times Y | U_\alpha \in \mathcal{U}_X, V_\alpha \in \mathcal{U}_Y\},$$

that is any member in $\mathcal{U}_{X\times Y}$ is the union of Cartesian products of open sets of X and Y.

Exercise 2.6.1 Let X and Y be topological spaces. Show that $\mathcal{U}_{X\times Y}$ is a topology for $X \times Y$.

Definition 2.6.1 Let X and Y be topological spaces. The *(Cartesian) product* $X \times Y$ is the set $X \times Y$ with the topology $\mathcal{U}_{X \times Y}$.

Exercise 2.6.2 Show that $\mathbb{R}^2$ with the usual topology is the Cartesian product $\mathbb{R}^1 \times \mathbb{R}^1$.

Theorem 2.6.2 *Let $X \times Y$ be the Cartesian product of spaces X and Y. Then a set $W \subseteq X \times Y$ is open if and only if for any $w \in W$ there exist U_w and V_w such that U_w is open in X, V_w is open in Y and $w \in U_w \times V_w \subseteq W$.*

Proof. Let W be an open set in $X \times Y$ and let $w \in W$. Then $W = \bigcup_\alpha U_\alpha \times V_\alpha$, where U_α and V_α are open in X and Y, respectively. Thus there exists an index α such that $w \in U_\alpha \times V_\alpha$. Choose $U_w = U_\alpha$ and $V_w = V_\alpha$. Coversely let w run over all elements in W we have

$$W = \bigcup_{w \in W} U_w \times V_w$$

and so W is open.

Let $\pi_X \colon X \times Y \to X$, $(x, y) \to x$, and $\pi_Y \colon X \times Y \to Y$, $(x, y) \to y$, be the coordinate projections. Since $\pi_X^{-1}(U) = U \times Y$ and $\pi_Y^{-1}(V) = X \times V$, the coordinate projections π_X and π_Y are continuous. Let $f \colon Z \to X$ and $g \colon Z \to Y$ be any maps from a space Z to X and Y, respectively. Let $\phi \colon Z \to X \times Y$ be the function defined by $\phi(z) = (f(z), g(z)$. Then ϕ is the unique function such that $\pi_X \circ \phi = f$ and $\pi_Y \circ \phi = g$.

Lemma 2.6.3 *The function ϕ defined above is continuous.*

Proof. Let U and V be open sets in X and Y, respectively. Then $\phi^{-1}(U \times V) = \{z | f(z) \in U, g(z) \in V\} = f^{-1}(U) \cap g^{-1}(V)$ is an open set in Z. Now consider any open set W in $X \times Y$. Let z be any element in $\phi^{-1}(W)$ and let $w = \phi(z)$. There exist open sets U_w and V_w such that $w \in U_w \times V_w \subseteq W$. Thus $z \in \phi^{-1}(U_w \times V_w) \subseteq \phi^{-1}(W)$ and $\phi^{-1}(W)$ is a neighborhood of each of its points. It follows that $\phi^{-1}(W)$ is open.

By using the categorical language, Lemma 2.6.3 shows

Theorem 2.6.4 *Let X and Y be topological spaces. Then Cartesian product $X \times Y$ is the product of X and Y in the category of topological spaces.*

Theorem 2.6.5 *For any $y \in Y$, the subspace $X \times \{y\} \subseteq X \times Y$ is homeomorphic to X.*

Proof. Let $f\colon X \times \{y\} \to X$ be the function defined by $f(x,y) = x$. Since f is the composite

$$f\colon X \times \{y\} \hookrightarrow X \times Y \xrightarrow{\pi_X} X,$$

the function f is continuous. Clearly f is a bijective. It suffices to show that f is an open map, that is f sends open sets to open sets. Suppose that W is an open set in $X \times \{y\}$. Then

$$W = (\bigcup_{\alpha} U_\alpha \times V_\alpha) \cap X \times \{y\}$$

for some open sets U_α and V_α in X and Y, respectively. It follows that

$$f(W) = \bigcup_{\alpha, y \in V_\alpha} U_\alpha$$

is open.

Now we look at "infinite" Cartesian products. Let $\{X_\alpha | \alpha \in J\}$ be a set of topological spaces. Recall that the Cartesian product $\prod_{\alpha \in J} X_\alpha$ of the sets X_α is the set of collections of elements (x_α), one element x_α in each X_α. Now An open set in $\prod_{\alpha \in J} X_\alpha$ is defined to be the any union of the following sets

$$U_{\alpha_1,\cdots,\alpha_n} = \{(x_\alpha) | x_{\alpha_1} \in U_{\alpha_1}, \cdots, x_{\alpha_n} \in U_{\alpha_n}\},$$

where $a_1, \cdots, a_n$ is any finite set of elements of J. This gives the topology on the product $\prod_{\alpha \in J} X_\alpha$.

Proposition 2.6.6 *The product topology on $\prod_{\alpha \in J} X_\alpha$ is the smallest topology such that each coordinate projection*

$$\pi_\alpha\colon \prod_{\alpha' \in J} X_{\alpha'} \to X_\alpha$$

is continuous.

Proof. Let $\mathcal{V}$ be a topology on $\prod_{\alpha' \in J} X_{\alpha'}$ such that each coordinate projection π_α is continuous. Let U_α be an open set in X_α. Then

$$\pi^{-1}(U_\alpha) = \{(x'_\alpha) | x_\alpha \in U_\alpha\} \in \mathcal{V}. \tag{2.1}$$

Since the product topology is given by the any union of any finite intersections of the sets of the forms 2.1, it follows taht the product topology is smaller than $\mathcal{V}$.

Let X and Y be pointed spaces with base points x_0 and y_0, respectively. Then the *wedge* $X \vee Y$ of X and Y is defined to be the quotient space

$$(X \coprod Y)/\{x_0, y_0\}.$$

The topology in $X \vee Y$ is given by the quotient topology under the quotient map $q\colon X \coprod Y \to X \vee Y$. This topology can be described as follows. A subset U in $X \vee Y$ is open if and only if $q^{-1}(U)$ is open. There are two cases. If $* \notin U$, then $q^{-1}(U)$ is either an open set in X that does not contain x_0 or an open set in Y that does not contain y_0. If $* \in U$, then $q^{-1}(U) = U_1 \coprod U_2$ for some open set U_1 in X that contains x_0 and some open set U_2 in Y that contains y_0. Thus

$$\mathcal{U}_{X\vee Y} = \{q(U)|x_0 \notin U \in \mathcal{U}_X\} \cup \{q(V)|y_0 \notin V \in \mathcal{U}_Y\} \cup \{q(U_1 \coprod U_2)|x_0 \in U_1 \in \mathcal{U}_X, y_0 \in U_2 \in \mathcal{U}_Y\}.$$

Proposition 2.6.7 *Let X and Y be pointed spaces with base points x_0 and y_0, respectively. Then $X \vee Y$ is homeomorphic to the subspace $(X \times \{y_0\}) \cup (\{x_0\} \times Y) \subseteq X \times Y$.*

Proof. Let $Z = (X \times \{y_0\}) \cup (\{x_0\} \times Y)$ be the subspace of $X \times Y$. Let $f_X\colon X \to X \times Y$ and $f_Y\colon Y \to X \times Y$ be the maps defined by $f_X(x) = (x, y_0)$ and $f_Y(y) = (x_0, y)$. The there is a unique map $\phi\colon X \vee Y \to Z$ such that $\phi \circ f_X = i_X$ and $\phi \circ f_Y = i_Y$. Clearly ϕ is bijective. It suffices to show that ϕ is an open map. If $U \in \mathcal{U}_X$ with $x_0 \notin U$, then $\phi(U) = Z \cap (U \times Y)$ is open in Z. If $V \in \mathcal{U}_Y$ with $y_0 \notin V$, then $\phi(V) = Z \cap (X \times V)$ is open in Z. If $U = q(U_1 \coprod U_2)$ with $x_0 \in U_1 \in \mathcal{U}_X$ and $y_0 \in U_2 \in \mathcal{U}_Y$, then $\phi(U) = Z \cap (U_1 \times U_2)$. Thus q is an open map.

Let X and Y be pointed spaces. The *smash product* $X \wedge Y$ is defined by

$$(X \times Y)/((X \times \{y_0\}) \cup (\{x_0\} \times Y)).$$

We write $x \wedge y$ for elements in $X \wedge Y$, where $x \in X$ and $y \in Y$.

Theorem 2.6.8 *Given three pointed spaces X, Y and Z, $(X \vee Y) \wedge Z$ is homeomorphic to $(X \wedge Z) \vee (Y \wedge Z)$.*

Proof. The function $f\colon X \times Y \times Z \to X \times Z \times Y \times Z$, defined by $f(x, y, z) = (x, z, y, z)$, is clearly continuous. Let g be the composite

$$g\colon X \times Y \times Z \xrightarrow{f} X \times Z \times Y \times Z \xrightarrow{\text{proj.}} (X \wedge Z) \times (Y \times Z).$$

Then

$$g((X \vee Y) \times Z) \subseteq (X \wedge Z) \vee (Y \wedge Z).$$

Moreover, the map g sends $(X \vee Y) \vee Z$ to the base point, so that g induces a map

$$\tilde{g}\colon (X \vee Y) \wedge Z \to (X \wedge Z) \vee (Y \wedge Z),$$

where $\tilde{g}((x, y_0) \wedge z) = x \wedge z$ in $X \wedge Z$ and $g((x_0, y) \wedge z) = y \wedge z$ in $Y \wedge Z$.

Conversely, let $h\colon (X \wedge Z) \vee (Y \wedge Z) \to (X \vee Y) \wedge Z$ be the map such that $h|_{X \wedge Z}$ and $h|_{Y \wedge Z}$ are the inclusions $X \wedge Z \hookrightarrow (X \wedge Y) \wedge Z$ and $Y \wedge Z \hookrightarrow (X \wedge Y) \wedge Z$, respectively. Then $h(x \wedge z) = (x, y_0) \wedge z$ and $h(y \wedge z) = (x_0, y) \wedge z)$ so that $\tilde{g} \circ h$ and $h \circ \tilde{g}$ are identities, and hence $\tilde{g}$ is a homeomorphism.

Exercise 2.6.3 Show that $S^n \wedge S^m \cong S^{n+m}$ for any n, m.

2.7 Topological Groups and Orbit Spaces

A pointed topological space X is called an H-space of there is a continuous multiplication $\mu\colon X \times X \to X$, $(x, y) \to xy$, such that $x_0 x = x x_0 = x$. The base point x_0 is often denoted as $*$ or 1. Equivalently, a pointed space X is an H-space if and only if there is a map $\mu\colon X \times X \to X$ such that $\mu|_{X \vee X} = \nabla$, where $\nabla\colon X \vee X \to X$ is the fold map defined by $\nabla(x, x_0) = x$ and $\nabla(x_0, x) = x$.

An H-space is called *associative* if diagram

$$\begin{array}{ccc}
X \times X \times X & \xrightarrow{\mu \times \mathrm{id}_X} & X \times X \\
\Big\downarrow{\scriptstyle \mathrm{id}_X \times \mu} & & \Big\downarrow{\scriptstyle \mu} \\
X \times X & \xrightarrow{\mu} & X
\end{array}$$

is commutative. An associative H-space is called a *topological monoid*. In other words, a topological monoid is monoid as a set such that the multiplication is continuous. A *topological group* G means a topological monoid such that there is a map $\nu\colon G \to G$, $x \to x^{-1}$, with $xx^{-1} = 1 = x^{-1}x$, that is the inverse is a continuous function.

Let X be a space and let G be a topological group. We say that G *acts* on X and that X is a G-space if there is map $\mu\colon G \times X \to X$, denoted by $(g, x) \to g \cdot x$, such that

i) $1 \cdot x = x$ for all $x \in X$;

Moreover, the map g sends $(X \vee Y) \vee Z$ to the base point, so that g induces a map

$$\tilde{g}\colon (X \vee Y) \wedge Z \to (X \wedge Z) \vee (Y \wedge Z),$$

where $\tilde{g}((x, y_0) \wedge z) = x \wedge z$ in $X \wedge Z$ and $g((x_0, y) \wedge z) = y \wedge z$ in $Y \wedge Z$.

Conversely, let $h\colon (X \wedge Z) \vee (Y \wedge Z) \to (X \vee Y) \wedge Z$ be the map such that $h|_{X\wedge Z}$ and $h|_{Y\wedge Z}$ are the inclusions $X \wedge Z \hookrightarrow (X \wedge Y) \wedge Z$ and $Y \wedge Z \hookrightarrow (X \wedge Y) \wedge Z$, respectively. Then $h(x \wedge z) = (x, y_0) \wedge z$ and $h(y \wedge z) = (x_0, y) \wedge z)$ so that $\tilde{g} \circ h$ and $h \circ \tilde{g}$ are identities, and hence $\tilde{g}$ is a homeomorphism.

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Let X be a space and let G be a topological group. We say that G *acts* on X and that X is a G-space if there is map $\mu\colon G \times X \to X$, denoted by $(g, x) \to g \cdot x$, such that

i) $1 \cdot x = x$ for all $x \in X$;

ii) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $x \in X$ and $g, h \in G$, that is the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id}_G \times \mu} & G \times X \\ \Big\downarrow{\scriptstyle \mu_G \times \mathrm{id}_X} & & \Big\downarrow{\scriptstyle \mu} \\ G \times X & \xrightarrow{\quad \mu \quad} & X \end{array}$$

commutes.

Theorem 2.7.1 *Suppose that X is a G-space. Then the function $\theta_g\colon X \to X$ given by $x \to g \cdot x$ is a homeomorphism. It follows that there is a homomorphism from G to the group of homeomorphisms of X.*

Proof. The function θ_g is the composite

$$X \cong \{g\} \times X \subseteq G \times X \xrightarrow{\mu} X.$$

Thus θ_g is continuous. From the definition of G-space we see that $\theta_g \circ \theta_h = \theta_{gh}$ and $\theta_1 = \mathrm{id}_X$. Thus $\theta_g \circ \theta_{g^{-1}} = \mathrm{id}_X = \theta_{g^{-1}} \circ \theta_g$ and so θ_g is a homeomorphism. Now the function $g \to \theta_g$ is a homomorphism from G to the group of homeomorphisms of X.

Let X be a G-space. We can define an equivalence relation $\sim$ on X by

$$x \sim y \Leftrightarrow g \cdot x = y \text{ for some } g \in G.$$

The quotient space $X/\sim$, denoted by X/G, with the quotient topology is called the *quotient space* of X by G.

Example 2.7.2 1) Let $G = \mathbb{Z}/2 = \{\pm 1\}$ with discrete topology and let $X = S^n$. The G-action on X is given by $\pm 1 \cdot x = \pm x$. Then $S^n/(\mathbb{Z}/2) \cong \mathbb{R}P^n$.

2) Let $G = \mathbb{Z}$ with the discrete topology and let $X = \mathbb{R}$. The action of G on $\mathbb{R}$ is given by $n \cdot x = n + x$. Then $\mathbb{R}/\mathbb{Z} \cong S^1$.

3) Let $G = S^1 \subseteq \mathbb{C}$. Then G is a topological group under the multiplication. Let $S^{2n-1} \subseteq \mathbb{R}^{2n} = \mathbb{C}^n$ be the unit sphere. Let G act on S^{2n-1} by

$$\alpha \cdot (z_1, z_2, \cdots, z_n) = (\alpha z_1, \alpha z_2, \cdots, \alpha z_n).$$

Then $S^{2n-1}/S^1 \cong \mathbb{C}P^n$.

4) Let M_n be the set of $n \times n$-matrices over $\mathbb{R}$. Then $M_n = \mathbb{R}^{n^2}$ is a topological space. Let
$$\mathrm{GL}(n, \mathbb{R}) = \{A \in M_n | \det(A) \neq 0\} \subseteq M_n$$
with the subspace topology. Then $\mathrm{GL}(n, \mathbb{R})$ is a topological group, which called the *general linear group.*

5) Let $O(n)$ be the group of (real) orthogonal $n \times n$ matrices. $O(n)$ is regarded as a subspace of $\mathbb{R}^{n^2}$ with the subspace topology. For $k \leq n$ $O(k)$ is regarded as the set of matrices of the form
$$\begin{pmatrix} A & 0 \\ 0 & I_{n-k} \end{pmatrix}$$
with A an orthogonal $k \times k$-matrix and I_{n-k} the $(n-k) \times (n-k)$ identity matrix. Then $O(k)$ is a topological subgroup of $O(n)$. In $O(n)$ we also have the subgroup $SO(n)$ of orthogonal matrices with determinant 1, that is $SO(n)$ is the kernel of $\det \colon O(n) \to \mathbb{Z}/2$.

6) Let $U(n)$ denote the group of $n \times n$ unitary matrices regarded as a subspace of $\mathbb{C}^{n^2}$. We have the inclusions
$$U(1) \subseteq U(2) \subseteq U(3) \subseteq \cdots \subseteq U(n) \subseteq \cdots.$$
Thus $U(k)$ is a topological subgroup of $U(n)$ for $k \leq n$. We also have the subgroup $SU(n) \subseteq U(n)$ of $n \times n$ unitary matrices with determinant 1, that is $SU(n)$ is the kernel of $\det \colon U(n) \to S^1$.

Theorem 2.7.3 *Suppose that X is a G-space. Then the canonical projection $\pi \colon X \to X/G$ is an open mapping.*

Proof. Let U be an open set in X. Then
$$\pi^{-1}(\pi(U)) = \{x \in X | \pi(x) \in \pi(U)\}$$
$$= \{x \in X | x = g \cdot y \text{ for some } y \in U \text{ some } g \in G\} = \bigcup_{g \in G} g \cdot U.$$
Since $\theta_g \colon X \to X$ is a homeomorphism for each $g \in G$, $g \cdot U$ is open for each g then so $\pi^{-1}(\pi(U))$ is open and hence $\pi(U)$ is open in X/G.

Exercise 2.7.1 1) Let X be a G-space and define the *stabilizer* of $x \in X$ to be the subspace

$$G_x = \{g \in G | g \cdot x = x\}$$

of G. Show that G_x is a topological subgroup of G.

2) Let X be a G-space and define the *orbit* of $x \in X$ to be the subspace

$$G \cdot x = \{g \cdot x | g \in G\}$$

of X. Prove that $G \cdot x$ and $G \cdot y$ are either disjoint or equal for any $x, y \in X$.

2.8 Compact Spaces, Hausdorff Spaces and Locally Compact Spaces

Let X be a space. A *cover* of a subset S is a collection of subsets $\{U_j | j \in J\}$ of X such that

$$S \subseteq \bigcup_{j \in J} U_j.$$

A cover is called *finite* if the indexing set J is finite. Let $\{U_j | j \in J\}$ and $\{V_k | k \in K\}$ be covers of the subset S of X. $\{U_j | j \in J\}$ is called a *subcover* of $\{V_k | k \in K\}$ if

$$\{U_j | j \in J\} \subseteq \{V_k | k \in K\}.$$

Definition 2.8.1 Let X be a space. A subset S is called to be *compact* if every open cover of S has a finite subcover. In particular, a space X is compact if every open cover of X has a finite subcover.

Exercise 2.8.1 Show that a subset S of a space X is compact if and only if it is compact as a space given the induced topology.

Exercise 2.8.2 Show that $[0, 1] \subseteq \mathbb{R}$ is compact.

The following theorem is useful.

Theorem 2.8.2 *Let $f\colon X \to Y$ be a map. If $S \subseteq X$ is a compact subspace, then $f(S)$ is compact.*

Proof. Suppose that $\{U_j|j \in J\}$ be an open cover of $f(S)$. Then $\{f^{-1}(U_j)|j \in J\}$ is an open cover of S. Since S is compact, there exists a finite subset K of J such that

$$S \subseteq \bigcup\{f^{-1}(U_k)|k \in K\}.$$

But $f(f^{-1}(U_k)) \subseteq U_k$ and so

$$f(S) \subseteq \bigcup\{f(f^{-1}(U_k))|k \in K\} \subseteq \{U_k|k \in K\}$$

which is a finite subcover of $\{U_j|j \in J\}$.

Theorem 2.8.3 *A closed subset of a compact space is compact.*

Proof. Let X be a compact space and let S be a closed subset of X. Let $\{U_j\}$ be an open cover of S. Since $S \subseteq \bigcup\{U_j|j \in J\}$ we see that

$$X \subseteq \bigcup\{U_j|j \in J\} \cup (X \setminus S)$$

and so there is a finite subcover

$$X \subseteq \bigcup\{U_k|k \in K\} \cup (X \setminus S).$$

Thus

$$S \subseteq \bigcup\{U_k|k \in K\}$$

which is a finite subcover of $\{U_j|j \in J\}$.

Theorem 2.8.4 *Let X and Y be spaces. Then X and Y are compact if and only if $X \times Y$ is compact.*

Proof. Suppose that $X \times Y$ is compact. Since $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ are continuous, X and Y are compact. Conversely assume that X and Y are compact. Let $\{W_j|j \in J\}$ be an open cover of $X \times Y$. By definition

$$W_j = \bigcup_{k \in K(j)} (U_{j,k} \times V_{j,k})$$

where $U_{j,k}$ and $V_{j,k}$ are open in X and Y, respectively. Thus

$$X \times Y \subseteq \bigcup_{j \in J, k \in K(j)} U_{j,k} \times V_{j,k}.$$

For each $x \in X$ the subspace $\{x\} \times Y$ is compact and so there is a finite subcover

$$\{x\} \times Y \subseteq \bigcup_{i=1}^{n(x)} U_i(x) \times V_i(x).$$

Let $U'(x) = \bigcap_{i=1}^{n(x)} U_i(x)$. Then $U'(x)$ is an open neighborhood of x and

$$X \subseteq \bigcup_{x \in X} U'(x)$$

Since X is compact, there are finite points $x_1, \cdots, x_m$ such that

$$X \subseteq \bigcup_{j=1}^{m} U'(x_j).$$

It follows that

$$X \times Y \subseteq \bigcup_{1 \le j \le m, 1 \le i \le n(x_j)} U'(x_j) \times V_i(x_j) \subseteq \bigcup_{1 \le j \le m, 1 \le i \le n(x_j)} U_i(x_j) \times V_i(x_j).$$

Since for each $U_i(x_j) \times V_i(x_j)$ there is an index k such that

$$U_i(x_j) \times V_i(x_j) \subseteq W_k,$$

there is a finite subcover of $\{W_j | j \in J\}$ covering $X \times Y$.

A space X is called *Hausdorff* if for every pair of distinct points x and y there are open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$. In other words, X is Hausdorff if for any $x \neq y$ in X there are neighborhood $N(x)$ and $N(y)$ of x and y, respectively such that $N(x) \cap N(y) = \emptyset$. Hausdorff space is also called T_2-space. In a Hausdorff space X, any point x is a closed subset. (This is not true for general topological space. For example, the indiscrete topology.)

Theorem 2.8.5 *A compact subset A of a Hausdorff space X is closed.*

Proof. We may assume that $A \neq \emptyset$ and $A \neq X$. Given $x \in X \setminus A$. For each $a \in A$, there are disjoint open sets $U_a(x)$ and $V_a(x)$ such that $a \in U_a(x)$ and $x \in V_a(x)$. Since

$$A \subseteq \bigcup_{a \in A} U_a(x)$$

and A is compact, there are finite points $a_1, \cdots, a_m$ in A such that

$$A \subseteq \bigcup_{i=1}^{m} U_{a_i}(x).$$

Now the set $V(x) = \cap_{i=1}^{m} V_{a_i}(x)$ is an open neighborhood of x with

$$A \cap V(x) \subseteq (\bigcup_{i=1}^{m} U_{a_i}(x)) \cap V(x) = \emptyset$$

and so $V(x) \subseteq X \setminus A$, which means that $X \setminus A$ is open or A is closed.

In particular, if X can be embedded into $\mathbb{R}^n$ then X must be Hausdorff.

Theorem 2.8.6 (Heine-Borel) *A subset S of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Suppose that S is compact. By Theorem 2.8.5, S is closed. Now

$$S \subseteq \bigcup_{x \in S} B_1(x)$$

and so there exist finite points $x_1, \cdots, x_m$ in S such that

$$S \subseteq \bigcup_{i=1}^{m} B_1(x_i).$$

Thus S is bounded. Conversely suppose that S is closed and bounded. There exists positive number $r >> 0$ such that

$$S \subseteq [-r, r]^n.$$

Since $[-r, r]$ is compact, $[-r, r]^n$ is compact. By Theorem 2.8.3, the closed subspace S is compact.

Exercise 2.8.3 Let X and Y be spaces. Then X and Y are Hausdorff if and only if $X \times Y$ is Hausdorff.

Thus the spaces like n-Torus $T^n = S^1 \times S^1 \times \cdots \times S^1$ are (compact) Hausdorff.

Exercise 2.8.4 Let X and Y be topological spaces. Show that

1) If X is Hausdorff, then any subspace of X is Hausdorff;
2) X and Y are Hausdorff if and only if $X \times Y$ is Hausdorff;
3) X is Hausdorff if and only if the diagonal $\Delta(X) = \{(x,x) \in X^2 | x \in X\}$ is a closed subset of X^2;
4) $O(n)$, $SO(n)$, $U(n)$ and $SU(n)$ are compact Hausdorff spaces;
5) Let $f\colon X \to Y$ be a map. Suppose that X is compact Hausdorff and Y is Hausdorff. Then f is a closed map. Reduce that a bijective map from a compact Hausdorff space to a Hausdorff space is a homeomorphism.

A quotient map $f\colon X \to Y$ is also called *identification* map. A quotient space may not be Hausdorff.

Theorem 2.8.7 *Let $f\colon X \to Y$ be an identification map. Suppose that X is Hausdorff. If f is closed and $f^{-1}(y)$ is compact for any $y \in Y$, then Y is Hausdorff.*

Proof. Let y_1 and y_2 be distinct points in Y. For each $x \in f^{-1}(y_1)$ and $a \in f^{-1}(y_2)$, there exist a pair of disjoint open sets $U_{x,a}$ and $V_{x,a}$ with $x \in U_{x,a}$ and $a \in V_{x,a}$. Fixed $x \in f^{-1}(y_1)$ $\{V_{x,a} | a \in f^{-1}(y_2)\}$ is an open cover of f^{y_2}. By the assumption, $f^{-1}(y_2)$ is compact and so there are finite points $a_1(x), \cdots, a_{m(x)}(x)$ such that

$$f^{-1}(y_2) \subseteq \bigcup_{i=1}^{m(x)} V_{x,a_i(x)}.$$

Let $V(x) = \bigcup_{i=1}^{m(x)} V_{x,a_i(x)}$ and let $U(x) = \bigcap_{i=1}^{m(x)} U_{x,a_i(x)}$. Then $U(x)$ is an open neighborhood of x and $V(x)$ is an open neighborhood of $f^{-1}(y_2)$ with $U(x) \cap V(x) = \emptyset$. Since $f^{-1}(y_1) \subseteq \bigcup_{x \in f^{-1}(y_1)} U(x)$ and $f^{-1}(y_1)$ is compact, there are finite points $x_1, \cdots, x_s \in f^{-1}(y_1)$ such that

$$f^{-1}(y_1) \subseteq \bigcup_{j=1}^{s} U(x_j).$$

Let $U = \bigcup_{j=1}^{s} U(x_j)$ and $V = \bigcap_{j=1}^{s} V(x_j)$. Then U and V are disjoint open sets with $f^{-1}(y_1) \subseteq U$ and $f^{-1}(y_2) \subseteq V$. Since f is closed, $f(X \setminus U)$ and $f(X \setminus V)$ are closed subsets in Y and so $W_1 = Y \setminus f(X \setminus U)$ and $W_2 = Y \setminus f(X \setminus V)$ are open subsets in Y with $y_1 \in W_1$ and $y_2 \in W_2$. We show that $W_1 \cap W_2 = \emptyset$. Suppose that $y \in W_1 \cap W_2$. Then $y \notin f(X \setminus U)$ and $y \notin f(X \setminus V)$. Therfore $f^{-1}(y) \cap (X \setminus U) = \emptyset$ and $f^{-1}(y) \cap (X \setminus V) = \emptyset$.It follows that $f^{-1}(y) \subseteq U \cap V = \emptyset$ and hence $W_1 \cap W_2 = \emptyset$.

Corollary 2.8.8 *Let X be a compact Hausdorff space. Then*

i) If G is a finite group and X is a G-space, then X/G is a compact Hausdorff space;

ii) If A is closed subspace of X, then X/A is compact Hausdorff.

Exercise 2.8.5 A space is called *normal* (T_4-space) if every point in X is closed and every pair of disjoint closed sets has disjoint open neighborhood. Let G be a compact topological space and let X be a normal G-space. Show that X/G is Hausdorff.

(Hint: Let $\pi\colon X \to X/G$ be the quotient map. For each $y \in X/G$, $\pi^{-1}(y) = G \cdot x$ for some $x \in X$ with $\pi(x) = y$. Show that the orbit $G \cdot x$ is a quotient of G. Since G is compact, the orbit $G \cdot x$ is compact and so it is closed because T_4-space is Hausdorff. Let $y_1 \neq y_2$ be distinct points in X/G. Then $\pi^{-1}(y_1)$ and $\pi^{-1}(y_2)$ are disjoint closed set and so they have disjoint open neighborhood, say U and V. By Theorem 2.7.3, $\pi(U)$ and $\pi(V)$ are disjoint open neighborhoods of y_1 and y_2, respectively.)

For example, $\mathbb{R}P^n$ is compact Hausdorff space because $\mathbb{R}P^n$ is the quotient of S^n by the action of $\mathbb{Z}/2$. $\mathbb{C}P^n$ is a compact Hausdorff space because it is the quotient of S^{2n-1} by S^1.

A space X is called *locally compact* if every point x in X has a compact neighborhood.

Exercise 2.8.6 Let X be a locally compact Hausdorff space. Given a point $x \in X$ and a neighborhood U of x. Show that there is an open set V such that $x \in V \subseteq \bar{V} \subseteq U$ and $\bar{V}$ is compact. (Hint: Let W be a compact neighborhood of x, that is there is an open set U_1 such that $x \in U_1 \subseteq W$ and W is compact. Let $V_1 = U_1 \cap U$. Then V_1 is an open neighborhood of x and $\bar{V}_1 \setminus V_1$ is compact because it is a closed subset of the compact space W. Let $A = \bar{V}_1 \setminus V_1$. For each $y \in A$, there exist disjoint open sets $U(y)$ and $V(y)$ such that $y \in U(y)$ and $x \in V(y)$ because X is Hausdorff.Since A is compact and $A \subseteq \bigcup_{y \in A} U(y)$, there are finite points $y_1, \cdots, y_n$ such that $A \subseteq \bigcup_{i=1}^n U(y_i)$. Let

$$V = V_1 \cap \bigcap_{i=1}^{n} V(y_i).$$

Then V is an open neighborhood of x with

1) $\bar{V} \cap A = \emptyset$ (because V is disjoint with an open neighbor hood, $\bigcup_{i=1}^n U(y_i)$, of A;

2) $\bar{V} \subseteq \bar{V}_1$ because $V \subseteq V_1$ and

3) $\bar{V}$ is compact because it is a closed subset of $\bar{V}_1$.

By 1) and 2) above, we have that $\bar{V} \subseteq V_1 \subseteq U$.

Theorem 2.8.9 *(a) If $p: X \to Y$ is a quotient map and Z is a locally compact Hausdorff space, then $p \times \mathrm{id}_Z: X \times Z \to Y \times Z$ is a quotient map.*

(b) If A is a compact subspace of a space X and $p: X \to X/A$ is the quotient map, then for any space Z, $p \times \mathrm{id}_Z: X \times Z \to (X/A) \times Z$ is a quotient map.

*Proof.*Let $\pi = p \times \mathrm{id}_Z$.
(a) Let A be a subset of $Y \times Z$ such that $\pi^{-1}(A)$ is open in $X \times Z$. We show that A is open. Let $(y_0, z_0) \in Y \times Z$. Choose $x_0 \in X$ such that $p(x_0) = y_0$.

Since $\pi^{-1}(A)$ is open and Z is locally compact, there are open sets U_1 in X and V in Z such that $\bar{V}$ is compact, $U_1 \times V$ is an open neighborhood of (x_0, z_0) and $U_1 \times \bar{V} \subseteq \pi^{-1}(A)$. The point here is that $p^{-1}(p(U_1))$ is not necessarily open in X but it contains U_1. We do the following construction.

Suppose that U_i is an open neighborhood of x_0 such that $U_i \times \bar{V} \subseteq (p \times \mathrm{id}_Z)^{-1}(A)$. We construct an open set U_{i+1} of X such that

$$p^{-1}(p(U)) \times \bar{V} \subseteq U_{i+1} \times \bar{V} \subseteq \pi^{-1}(A),$$

as follows: For each point $x \in p^{-1}(p(U_i))$ the space $\{x\} \times \bar{V}$ lies in $\pi^{-1}(A)$. Using compactness of $\bar{V}$, we choose a neighborhood W_x of x such that $W_x \times \bar{V} \subseteq \pi^{-1}(A)$. Let U_{i+1} be the union of the open sets W_x; then U_{i+1} is the desired open set of X.

Finally, let U be the union of the open sets $U_1 \subseteq U_2 \subseteq \cdots$. Then $U \times V$ is a neighborhood of (x_0, z_0) and $U \times \bar{V} \subset \pi^{-1}(A)$. Since

$$U \subseteq p^{-1}(p(U)) = p^{-1}(\bigcup_{i=1}^{\infty} p(U_i)) = \bigcup_{i=1}^{infty} p^{-1}(p(U_i)) \subseteq \bigcup_{i=1}^{\infty} U_{i+1} = U,$$

we have $p^{-1}(p(U)) = U$ and so $p(U)$ is open in Y. Thus

$$p(U) \times V = \pi(U \times V) \subseteq A$$

is a neighborhood of (x_0, z_0) lying in A, as desired.
(b) Again it suffices to show that a subset U in $X/A \times Z$ is open if $\pi^{-1}(U)$ is open in $X \times Z$. As in case (a), let $(y_0, z_0) \in U$ and let $x_0 \in X$ such that $p(x_0) = y_0$.

If $x_0 \in A$, then $A \times \{z_0\} \subseteq \pi^{-1}(U)$. Since A is compact, a similar argument to that used in case (a) shows that there exist open sets $V \subseteq X$ and $W \subseteq Z$ such that

$$A \times \{z_0\} \subseteq V \times W \subseteq \pi^{-1}(U).$$

But then $(y_0, z_0) \in p(V) \times W \subseteq U$; $p(V)$ is open since $p^{-1}(p(V)) = V$ (because $A \subseteq V$), and so $p(V) \times W$ is open.

If on the other hand $x \notin A$, there certainly exit open sets $V \subseteq X$ and $W \subseteq Z$ such that $(x_0, z_0) \in V \times W \subseteq \pi^{-1}(U)$ and if $V \cap A = \emptyset$, then $p(V) \times W$ is open. However, if $V \cap A \neq \emptyset$, then $(p(A), z_0) \in U$, and we have already seen that we can then write

$$(p(A), z_0) \in p(\tilde{V}) \times \tilde{W} \subseteq U.$$

But then $(y_0, z_0) \in p(V \cup \tilde{V}) \times (W \cap \tilde{W}) \subseteq U$; $p(V \cup \tilde{V})$ is open since $A \subseteq \tilde{V}$, and so once again (x_0, z_0) is contained in an open subset of U. It follows that U is open. ♠

Corollary 2.8.10 *If $p\colon A \to B$ and $q\colon C \to D$ are quotient maps and if the domain of p and the range of q are locally compact Hausdorff spaces, then*

$$p \times q\colon A \times B \to C \times D$$

is a quotient map.

Proof. We can write $p \times q$ as the composite

$$A \times B \xrightarrow{\mathrm{id}_A \times q} A \times D \xrightarrow{p \times \mathrm{id}_D} C \times D.$$

Since each of these maps is a quotient map, so is the composite $p \times q$. ♠

Theorem 2.8.11 *If X and Y are compact and X is Hausdorff, then $(X \wedge Y) \wedge Z$ is homeomorphic to $X \wedge (Y \wedge Z)$.*

Proof. Write p for the various quotient maps of the form $X \times Y \to X \wedge Y$, and consider the diagram

$$\begin{array}{ccc}
X \times Y \times Z & = & X \times Y \times Z \\
\downarrow{\scriptstyle p \times \mathrm{id}_Z} & & \downarrow{\scriptstyle \mathrm{id}_X \times p} \\
(X \wedge Y) \times Z & & X \times (Y \wedge Z) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
(X \wedge Y) \wedge Z & & X \wedge (Y \wedge Z).
\end{array}$$

Since X and Y are compact, $X \vee Y$ is compact. By Theorem 2.8.9, the map

$$p \times \mathrm{id}_Z \colon X \times Y \times Z \to (X \wedge Y) \times Z$$

is a quotient map. Since X is locally compact and Hausdorff, again by Theorem 2.8.9, the map $\mathrm{id}_X \times p$ is a quotient map. It follows that both $p \circ (p \times \mathrm{id}_Z)$ and $p \circ (\mathrm{id}_X \times p)$ are quotient maps. The identity map $\mathrm{id} \colon X \times Y \times Z \to X \times Y \times Z$ induces maps

$$f \colon (X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z) \text{ and}$$

$$g \colon X \wedge (Y \wedge Z) \to (X \wedge Y) \wedge Z$$

that are clearly homeomorphisms. ♠

Exercise 2.8.7 Show that $X \wedge (Y \wedge Z)$ is homeomorphic to $X \wedge (Y \wedge Z)$ if X and Z are locally compact and Hausdorff.

2.9 Mapping Spaces and Compact-open Topology

Given spaces X and Y, the *mapping space* $\mathrm{Map}(X, Y)$ consists of all (continuous) maps from X to Y. The topology in $\mathrm{Map}(X, Y)$ is given by so-called *compact-open* topology that is defined as follows.

> Let K be a compact set in X and let U be an open set in Y. Let
>
> $$W_{K,U} = \{f \in \mathrm{Map}(X, Y) | f(K) \subseteq U\}.$$
>
> The compact-open topology in $\mathrm{Map}(X, Y)$ is generated by $W_{K,U}$ where K runs over all compact subsets in X and U runs over all open sets in Y. In other words, an open set in $\mathrm{Map}(X, Y)$ is a union of a finite intersection of the subsets with the form $W_{K,U}$.

If X and Y are pointed spaces. Then pointed mapping space, denoted by Y^X or $\mathrm{Map}_*(X, Y)$, is the subspace of $\mathrm{Map}(X, Y)$ consisting of all pointed (continuous) maps, that all of maps $f \colon X \to Y$ with $f(x_0) = y_0$.

Exercise 2.9.1 Let Y be a space and let X be a space with discrete topology. Show that the compact-open topology on

$$\mathrm{Map}(X, Y) = \prod_{x \in X} Y_x,$$

where Y_x is a copy of Y, is the same as the product topology.

Let $f\colon A \to X$ and $g\colon Y \to B$ be maps. Then the function $g^f\colon \mathrm{Map}(X,Y) \to \mathrm{Map}(A,B)$ is defined by

$$g^f(\lambda) = g \circ \lambda \circ f$$

for $\lambda\colon X \to Y$. If f and g are pointed maps, then g^f induces the map $g^f\colon \mathrm{Map}_*(X,Y) \to \mathrm{Map}_*(A,B)$ because if $\lambda \in \mathrm{Map}_*(X,Y)$, that is λ is a pointed map, then $g^f(\lambda)$ is a pointed map.

Proposition 2.9.1 *Let $f\colon A \to X$ and $g\colon Y \to B$ be* [pointed] *maps. Then $g^f\colon \mathrm{Map}(X,Y) \to \mathrm{Map}(A,B)$ [$g^f\colon \mathrm{Map}_*(X,Y) \to \mathrm{Map}_*(A,B)$] is continuous.*

Proof. Take a sub-basic open set $W_{K,U}$ in $\mathrm{Map}(A,B)$, where K is compact in A and U is open in B. Then

$$(g^f)^{-1}(W_{K,U}) = \{\lambda\colon X \to Y | g \circ \lambda f(K) \subseteq U\}$$

$$= \{\lambda\colon X \to Y | \lambda(f(K)) \subseteq g^{-1}(U)\} = W_{f(K),g^{-1}(U)}$$

because $f(K)$ is compact in X and $g^{-1}(U)$ is open in Y. Thus g^f is continuous. ♠

Let X and Y be pointed spaces. Then both $\mathrm{Map}_*(X,Y)$ and $\mathrm{Map}(X,Y)$ are pointed spaces, where the base-point is the constant map $c\colon X \to Y$, $c(x) = y_0$. Let $i\colon \{x_0\} \to X$ be the inclusion, then we have the sequence

$$\mathrm{Map}_*(X,Y) \hookrightarrow \mathrm{Map}(X,Y) \xrightarrow{\mathrm{id}_Y^i} \mathrm{Map}(\{x_0\},Y) \cong Y.$$

This sequence is called the canonical fibration for mapping spaces. Observe that $\lambda \in \mathrm{Map}_*(X,Y)$ is and only if $\mathrm{id}_Y^i(\lambda)$ is the base-point. As sets, one can see that $\mathrm{Map}(X,Y)$ is isomorphic to $\mathrm{Map}_*(X,Y) \times Y$. But as spaces $\mathrm{Map}(X,Y)$ is quite different from $\mathrm{Map}_*(X,Y) \times Y$ in general. The pointed mapping space $\mathrm{Map}_*(S^1,Y)$ is denoted by ΩY, which is called the *loop space* of Y. The mapping space $\mathrm{Map}(S^1,Y)$ is often denoted by ΛY and is call the *free loop* space of Y in many references. We will see that ΩY is actually an H-space, while ΛY is not in general. It was found in physics that some problems related to so-called *n-body* problem in physics are related to the homology of ΛY for certain spaces Y. There are many machines in algebraic topology for computing the homology of ΩY, but the determination of the homology of ΛY for many interesting spaces Y remains as interesting problems and have been studied by people.

Proposition 2.9.2 *(a) If Z is a subspace of Y, then $\mathrm{Map}(X,Z)$ is a subspace of $\mathrm{Map}(X,Y)$;*

(b) If Z is a pointed subspace of Y, then $\mathrm{Map}_*(X, Z)$ *is a subspace of* $\mathrm{Map}_*(X, Y)$.

Proof. The proofs of assertions (a) and (b) are similar. So we only prove assertion (a). We have to show that a set is open in $\mathrm{Map}(X, Z)$ if and only if it is the intersection with $\mathrm{Map}(X, Z)$ of a set that is open in $\mathrm{Map}(X, Y)$. Let $j: Z \to Y$ be the inclusion. Then $j^{\mathrm{id}_X}: \mathrm{Map}(X, Z) \to \mathrm{Map}(X, Y)$ is continuous, so that if $U \subseteq \mathrm{Map}(X, Y)$ is open, $U \cap \mathrm{Map}(X, Z) = (j^{\mathrm{id}_X})^{-1}(U)$ is open in $\mathrm{Map}(X, Z)$. To prove the converse, it sufficient to consider an open set in $\mathrm{Map}(X, Z)$ of the form $W_{K,U}$, where $K \subseteq X$ is compact and $U \subseteq Z$ is open. But $U = V \cap Z$ for some open set V in Y and

$$W_{K,V} \cap \mathrm{Map}(X, Z) = \{f: X \to Y | f(K) \subseteq V \text{ and } f(X) \subseteq Z\}$$

$$= \{f: X \to Z | f(K) \subseteq V \cap Z = U\} = W_{K,U}.$$

That is an open set in $\mathrm{Map}(X, Z)$ is the intersection with $\mathrm{Map}(X, Z)$ of an open set in $\mathrm{Map}(X, Y)$. ♠

Given spaces X and Y, the *evaluation map*

$$e: \mathrm{Map}(X, Y) \times X \to Y$$

is defined by

$$e(\lambda, x) = \lambda(x)$$

for $x \in X$ and $\lambda: X \to Y$. If X and Y are pointed spaces, the restriction of e gives the evaluation map $e: \mathrm{Map}_*(X, Y) \times X \to Y$. If λ is the constant map or x is the base point x_0, then $e(\lambda, x) = y_0$. That is $e(\mathrm{Map}_*(X, Y) \vee X) = y_0$ and so e induces the *evaluation map*

$$e: \mathrm{Map}_*(X, Y) \wedge X \to Y.$$

Theorem 2.9.3 *Let X and Y be pointed spaces. If X is locally compact Hausdorff, then the evaluation maps*

$$e: \mathrm{Map}(X, Y) \times X \to Y \text{ and}$$

$$e: \mathrm{Map}_*(X, Y) \wedge X \to Y$$

are continuous.

Proof. Let U be an open set in Y and that $e(\lambda, x) = \lambda(x) \in U$. Then $x \in \lambda^{-1}(U)$ which is open in X. Since X is locally compact and Hausdorff, there exists an open set V in X such that $x \in V \subseteq \bar{V} \subseteq \lambda^{-1}(U)$, and $\bar{V}$ is compact. Consider

$W_{\bar{V},U} \times V \subseteq \mathrm{Map}(X,Y) \times X$; this contains (λ, x) and if (λ', x') is another point in it, then

$$e(\lambda', x') = \lambda'(x') \in \lambda'(\bar{V}) \subseteq U.$$

Thus $W_{\bar{V},U} \times V \subseteq e^{-1}(U)$ and so $e^{-1}(U)$ is open or $e\colon \mathrm{Map}(X,Y) \times X \to Y$ is continuous. It follows that the restriction

$$e\colon \mathrm{Map}_*(X,Y) \times X \to Y$$

is continuous and so $e\colon \mathrm{Map}_*(X,Y) \wedge X \to Y$ is continuous. ♠

Note: The evaluation $e\colon \mathrm{Map}(X,Y) \times X \to Y$ may NOT be continuous in general. This is somewhat "not-so-good" in the category of topological spaces. Norman Steenrod then introduced "compact generated topological spaces" as a convenient category of topological spaces [4]. We just give the definition of compactly generated space. A space X is called *compactly generated* if X is Hausdorff and each subset A of X with the property that $A \cap C$ is closed for every compact subset C of X is itself closed. A locally compact Hausdorff space is compactly generated.

Theorem 2.9.4 *Let X, Y and Z be pointed spaces. Suppose that X and Y are Hausdorff. Then*

(a) $\mathrm{Map}(X \coprod Y, Z) \cong \mathrm{Map}(X,Z) \times \mathrm{Map}(Y,Z)$;

(b) $\mathrm{Map}_*(X \vee Y, Z) \cong \mathrm{Map}_*(X,Z) \times \mathrm{Map}_*(Y,Z)$.

Proof. We only prove assertion (b). Let x_0 and y_0 are base points of X and Y respectively, and define

$$i_X\colon X \to X \vee Y, \quad i_Y\colon Y \to X \vee Y$$

by $i_X(x) = (x, y_0)$ and $i_Y(y) = (x_0, y)$. Then i_X and i_Y are continuous. Define a function

$$\theta\colon \mathrm{Map}_*(X,Z) \times \mathrm{Map}_*(Y,Z) \to \mathrm{Map}_*(X \vee Y, Z \vee Z)$$

by $\theta(\lambda, \mu) = \lambda \vee \mu$ for $\lambda\colon X \to Z$ and $\mu\colon Y \to Z$. Consider the composites

$$\phi\colon Z^{X\vee Y} \xrightarrow{\Delta} Z^{X\vee Y} \times Z^{X\vee Y} \xrightarrow{\mathrm{id}_Z^{i_X} \times \mathrm{id}_Z^{i_Y}} Z^X \times Z^Y \quad \text{and}$$

$$\psi\colon Z^X \times Z^Y \xrightarrow{\theta} (Z \vee Z)^{X\vee Y} \xrightarrow{\nabla} Z^{X\vee Y},$$

where Δ is the diagonal map and $\nabla Z \vee Z \to Z$ is the fold map, that is $\nabla(z, z_0) = \nabla(z_0, z) = z$ for $z \in Z$. Given $\nu\colon X \vee Y \to Z$, $\phi(\nu = (\nu \circ i_X, \nu \circ i_Y)$ and given

$\lambda: X \to Z$ and $\mu: Y \to Z$, $\psi(\lambda, \mu) = \nabla(\lambda \vee \mu)$. Thus $\phi \circ \psi$ and $\psi \circ \phi$ are identity functions, and the only point that remains in showing ϕ is a homeomorphism is to show that θ is continuous.

To do so, consider the set $W_{K,U}$, where $K \subseteq X \vee Y$ is compact and $U \subseteq Z \vee Z$ is open. Now

$$\theta^{-1}(W_{K,U}) = \{(\lambda, \mu) | (\lambda \vee \mu)(K) \subseteq U\}$$
$$= \{(\lambda, \mu) | \lambda(K \cap X) \subseteq U \cap (Z \times \{z_0\} \text{ and } \mu(K \cap Y) \subseteq U \cap (\{z_0\} \times Z)\}.$$

Clearly $U_1 = U \cap (Z \times \{z_0\})$ and $U_2 = U \cap (\{z_0\} \times Z)$ are open. But since X and Y are Hausdorff, so is $X \times Y$ and hence is $X \vee Y$; thus K, X, Y are closed in $X \vee Y$, so that $K \cap X$ and $K \cap Y$ are closed and hence compact. That is,

$$\theta^{-1}(W_{K,U}) = W_{K \cap X, U_1} \times W_{K \cap Y, U_2}$$

so that θ is continuous and hence ϕ is a homeomorphism. ♠

Let X be a topological space and let $\mathcal{S}$ be a family of subsets of X. $\mathcal{S}$ is called a *sub-base* of open sets if any member in $\mathcal{S}$ is open and any open set in X is a union of finite intersections of members in $\mathcal{S}$. In other words if $\mathcal{S}$ is a sub-base of open sets then the topology on X is generated by $\mathcal{S}$. We are going to give a result involving $(Y \times Z)^X$ and $Y^X \times Z^X$. We need the following lemma.

Lemma 2.9.5 *Let X be a Hausdorff space and let $\mathcal{S}$ be a sub-base of open sets for a space Y. Then the sets of the form $W_{K,U}$ for $K \subseteq X$ compact and $U \in \mathcal{S}$, form a sub-base of open sets for* $\mathrm{Map}(X, Y)$.

Proof. Let $K \subseteq X$ be compact, $V \subseteq Y$ be open and let $\lambda \in W_{K,V}$. Then $V = \bigcup_\alpha V_\alpha$, where V_α is a finite intersection of members in $\mathcal{S}$, and so

$$K \subseteq \bigcup_\alpha \lambda^{-1}(V_\alpha);$$

hence, since K is compact, a finite collection of the sets $\lambda^{-1}(V_\alpha)$, say $\lambda^{-1}(V_1), \cdots, \lambda^{-1}(V_n)$, suffice to cover K. Given $x \in K$, there exists r such that $x \in \lambda^{-1}(V_r)$. Since K is a compact Hausdorff space and $K \cap \lambda^{-1}(V_r)$ is an open neighborhood of x, there exists an open set A_x in K such that

$$x \in A_x \subseteq \bar{A}_x \subseteq K \cap \lambda^{-1}(V_r).$$

Again, a finite collection of the open sets A_x will cover K, and their closures are each contained in just one set of the form $\lambda^{-1}(V_r)$. Thus by taking suitable unions of $\bar{A}_x$'s, we can write $K = \bigcup_{r=1}^n K_r$, where $K_r \subseteq \lambda^{-1}(V_r)$ and K_r is closed and so compact. It follows that

$$\lambda \in \bigcap_{r=1}^{n} W_{K_r,V} \subseteq W_{K,V},$$

since if $\mu(K_r) \subseteq V_r$, for each r, then $\mu(K) \subseteq \bigcup_{r=1}^n V_r \subseteq V$. But if, say, $V_r = \bigcup_{s=1}^m U_s$ for $U_s \in \mathcal{S}$, then $W_{K_r,V_r} = \bigcap_{s=1}^m W_{K_r,U_s}$. Hence λ is contained in a finite intersection of sets of the form W_{K_r,U_s} for $U_s \in \mathcal{S}$ and this intersection is contained in $W_{K,V}$. ♠

Theorem 2.9.6 *Let X, Y and Z be pointed spaces. Suppose that X is Hausdorff. Then*

$$\mathrm{Map}(X, Y \times Z) \cong \mathrm{Map}(X, Y) \times \mathrm{Map}(X, Z) \quad \text{and}$$
$$\mathrm{Map}_*(X, Y \times Z) \cong \mathrm{Map}_*(X, Y) \times \mathrm{Map}_*(X, Z).$$

Proof. We only prove that

$$\mathrm{Map}(X, Y \times Z) \cong \mathrm{Map}(X, Y) \times \mathrm{Map}(X, Z).$$

Let $p_Y: Y \times Z \to Y$ and $p_Z: Y \times Z \to Z$ be coordinate projections. Define a function

$$\theta: \mathrm{Map}(X, Y) \times \mathrm{Map}(X, Z) \to \mathrm{Map}(X \times X, Y \times Z)$$

by $\theta(\lambda, \mu) = \lambda \times \mu$ for $\lambda: X \to Y$ and $\mu: X \to Z$. Consider the composites

$$\phi: \mathrm{Map}(X, Y\times Z) \xrightarrow{\Delta} \mathrm{Map}(X, Y\times Z)\times \mathrm{Map}(X, Y\times Z) \xrightarrow{p_Y^{\mathrm{id}_X} \times p_Z^{\mathrm{id}_X}} \mathrm{Map}(X, Y)\times \mathrm{Map}(X, Z)$$

$$\psi: \mathrm{Map}(X, Y) \times \mathrm{Map}(X, Z) \xrightarrow{\theta} \mathrm{Map}(X \times X, Y \times Z) \xrightarrow{\mathrm{id}^{\Delta}} \mathrm{Map}(X, Y \times Z),$$

where Δ is a diagonal map. If $\nu: X \to Y \times Z$, then $\phi(\nu) = (p_Y \circ \nu, p_Z \circ \nu)$ and if $\lambda: X \to Y$ and $\mu: X \to Z$, then $\psi(\lambda, \mu) = (\lambda \times \mu) \circ \Delta$. Thus $\phi \circ \psi$ and $\psi \circ \phi$ are identity functions, and it remains only to prove that θ is continuous.

Since X is Hausdorff, by Lemma 2.9.5, it sufficient to consider sets of the form $W_{K,U\times V}$, where $K \subseteq X \times X$ is compact and $U \subseteq Y$, $V \subseteq Z$ are open. Then

$$\theta^{-1}(W_{K,U\times V}) = \{(\lambda, \mu) | (\lambda \times \mu)(K) \subseteq U \times V\} = \{(\lambda, \mu) | K \subseteq \lambda^{-1}(U) \times \mu^{-1}(V)\}.$$

But if $p_1, p_2: X \times X \to X$ be the first and the second coordinate projections, then $p_1(K)$ and $p_2(K)$ are compact, and $K \subseteq \lambda^{-1}(U) \times \mu^{-1}(V)$ if and only if $p_1(K) \times p_2(K) \subseteq \lambda^{-1}(U) \times \mu^{-1}(V)$. Hence

$$\theta^{-1}(W_{K,U\times V}) = W_{p_1(K),V} \times W_{p_2(K),V}$$

and so θ is continuous. ♠

At this point we posses rules for manipulating mapping spaces analogous to the index laws $a^{b+c} = a^b \cdot a^c$ and $(a \cdot b)^c = a^c \cdot b^c$ for real numbers, and it remains to investigate what rule, if any, corresponds to the index law $a^{b \cdot c} = (a^b)^c$. Now we define the 'association map'.

Given spaces X, Y and Z, the *(unreduced) association map* is the function $\alpha\colon \mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$ defined by

$$[\alpha(\lambda)(x)](y) = \lambda(x, y)$$

for $x \in X$, $y \in Y$ and $\lambda\colon X \times Y \to Z$.

To justify this definition, we have to show that $\alpha(\lambda)$ is an element in $\mathrm{Map}(X, \mathrm{Map}(Y, Z))$. For a fixed x, the function $\alpha(\lambda)(x)\colon Y \to Z$ is continuous because it is the composite

$$Y \cong \{x\} \times Y \subseteq X \times Y \xrightarrow{\lambda} Z.$$

Thus at least $\alpha(\lambda)$ is a function from X to $\mathrm{Map}(Y, Z)$.

Proposition 2.9.7 *The function $\alpha(\lambda)\colon X \to \mathrm{Map}(Y, Z)$ is continuous.*

Proof. Consider $W_{K,U}$, where $K \subseteq Y$ is compact and $U \subseteq Z$ is open. If $x \in X$ is a point such that $\alpha(\lambda)(x) \in W_{K,U}$, then $\lambda(\{x\} \times K) \subseteq U$ or $\{x\} \times K \subseteq (\lambda)^{-1}(U)$. Since $\lambda^{-1}(U)$ is open and K is compact, there is an open set V in X such that

$$\{x\} \times K \subseteq V \times K \subseteq \lambda^{-1}(U).$$

That is

$$x \in V \subseteq (\alpha(\lambda))^{-1}(W_{K,U})$$

and so $\alpha(\lambda)$ is continuous. ♠

Thus the function $\alpha\colon \mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$ is well-defined. Now we consider the pointed case. Let X, Y and Z be pointed spaces. Let $p\colon X \times Y \to X \wedge Y$ be the quotient map. Then we have the map

$$\mathrm{id}_Z^p\colon \mathrm{Map}_*(X \wedge Y, Z) \to \mathrm{Map}_*(X \times Y, Z) \subseteq \mathrm{Map}(X \times Y, Z).$$

Clearly α maps the image of id_Z^p into the subspace

$$\mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z)) \subseteq \mathrm{Map}(X, \mathrm{Map}_*(Y, Z)) \subseteq \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

because if $\lambda: X \wedge Y \to Z$, then $\lambda \circ p: X \times Y \to Z$ has the property that

$$\lambda \circ p|_{X \vee Y}: X \vee Y \to Z$$

is the constant map and so $\alpha(\lambda)(x_0)(y) = \alpha(x)(y_0) = z_0$ for any x, y. Thus the association map $\alpha: \mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$ induces the *reduced association map*

$$\bar{\alpha}: \mathrm{Map}_*(X \wedge Y, Z) \to \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z))$$

with

$$[\bar{\alpha}(\lambda)(x)](y) = \lambda(x \wedge y)$$

for $x \in X$, $y \in Y$ and $\lambda: X \wedge Y \to Z$.

Proposition 2.9.8 *If X is Hausdorff, then the association map*

$$\alpha: \mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

is continuous and therefore the reduced association map

$$\bar{\alpha}: Z^{X \wedge Y} \to (Z^Y)^X$$

is continuous.

Proof. By Lemma 2.9.5, it suffices to consider $\alpha^{-1}(W_{K,U})$, where $K \subseteq X$ is compact and $U \subseteq \mathrm{Map}(Y, Z)$ is of the form $W_{L,V}$ for $L \subseteq Y$ compact and $V \subseteq Z$ open. Now

$$\alpha^{-1}(W_{K,U} = \{\lambda|(\alpha(\lambda)(K) \subseteq W_{L,V}\} = \{\lambda|\lambda(K \times L) \subseteq V\} = W_{K \times L,V}.$$

Thus α is continuous. ♠

Theorem 2.9.9 *(a) For all spaces X, Y and Z, the functions*

$$\alpha: \mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \quad \text{and}$$

$$\bar{\alpha}: Z^{X \wedge Y} \to (Z^Y)^X$$

are one-to-one.

(b) If Y is locally compact Hausdorff, then both α and $\bar{\alpha}$ are onto.

(c) If both X and Y are locally compact Hausdorff, then α is a homeomorphism.

(d) If both X and Y are compact and Hausdorff, then $\bar{\alpha}$ is homeomorphism.

Proof. (a) We only show that $\bar{\alpha}$ is one-to-one. Let $\lambda, \mu\colon X \wedge Y \to Z$ such that $\alpha(\lambda) = \alpha(\mu)$. Then for any $x \in X$ and $y \in Y$, we have

$$\lambda(x \wedge y) = [\alpha(\lambda)(x)](y) = [\alpha(\mu)(x)](y) = \mu(x \wedge y),$$

so that $\lambda = \mu$.
(b) Let $\lambda\colon X \to \mathrm{Map}(Y, Z)$ be a map. Let $\mu\colon X \times Y \to Z$ be the composite

$$X \times Y \xrightarrow{\lambda \times \mathrm{id}_Y} \mathrm{Map}(Y, Z) \times Y \xrightarrow{e} Z,$$

where e is the evaluation map. By Theorem 2.9.3, the evaluation e is continuous and so is μ. Clearly $\alpha(\mu) = \lambda$ and so α is onto. Now given a pointed map $\lambda'\colon X \to \mathrm{Map}_*(Y, Z)$, let $\mu'\colon X \wedge Y \to Z$ be the composite

$$X \wedge Y \xrightarrow{\lambda' \wedge \mathrm{id}_Y} \mathrm{Map}_*(Y, Z) \wedge Y \xrightarrow{e} Z,$$

where e is the evaluation. Again by Theorem 2.9.3 e is continuous and so is μ'. Clearly $\bar{(}\mu') = \lambda'$ and so $\bar{\alpha}$ is onto.
(c) Certainly α is continuous, one-to-one and onto, so we have only to show that the inverse to α is continuous. Let θ be the composite

$$\theta\colon \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \times X \times Y \xrightarrow{e \times \mathrm{id}_Y} \mathrm{Map}(Y, Z) \times Y \xrightarrow{e} Z,$$

where e are evaluations. By Theorem 2.9.3, θ is continuous. By Proposition 2.9.7, the function

$$\alpha(\theta)\colon \mathrm{Map}(X, \mathrm{Map}(Y, Z) \to \mathrm{Map}(X \times Y, Z)$$

is continuous. Clearly $\alpha(\theta)$ is the inverse of the association map α.
(d) By Theorem 2.8.11, there is a homeomorphism

$$(Z^Y)^X \wedge (X \wedge Y) \cong ((Z^Y)^X \wedge X) \wedge Y.$$

Let ψ be the composite

$$(Z^Y)^X \wedge (X \wedge Y) \cong ((Z^Y)^X \wedge X) \wedge Y \xrightarrow{e \wedge \mathrm{id}_Y} Z^Y \wedge Y \xrightarrow{e} Z.$$

Then ψ is continuous and

$$\bar{\alpha}(\psi)\colon (Z^Y)^X \to Z^{X \wedge Y}$$

is the inverse to the reduced association $\bar{\alpha}$. ♠

Let X be a pointed space. The *n-fold loop space* $\Omega^n(X)$ of X is defined by

$$\Omega^n(X) = \mathrm{Map}_*(S^n, X).$$

Exercise 2.9.2 Let X and Y be pointed spaces. Show that $\Omega^n(X \times Y) \cong \Omega^n(X) \times \Omega^n(Y)$ and $\Omega^{n+m}(X) \cong \Omega^m(\Omega^n(X))$.

2.10 Manifolds and Configuration Spaces

A Hausdorff space M is called an *n-manifold* if each point of M has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$.

For example, $\mathbb{R}^n$ and the n-sphere S^n is an n-manifold. A 2-dimensional manifold is called a *surface*. The objects traditionally called 'surfaces in 3-space' can be made into manifolds in a standard way. The compact surfaces have been classified as spheres or projective planes with various numbers of handles attached.

Exercise 2.10.1 Show that the real projective space $\mathbb{R}P^n$ is an n-manifold and the complex projective space $\mathbb{C}P^n$ is a $2n$-manifold.

By the definition of manifold, the closed n-disk D^n is not an n-manifold because it has the 'boundary' S^{n-1}. D^n is an example of 'manifolds with boundary'. We give the definition of manifold with boundary as follows.

A Hausdorff space M is called an *n-manifold with boundary* ($n \geq 1$) if each point in M has a neighborhood homeomorphic to an open set in the half space

$$\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_n \geq 0\}.$$

Manifold is one of models that we can do calculus 'locally'. By means of calculus, we need local coordinate systems. Let $x \in M$. By the definition, there is a an open neighborhood $U(x)$ of x and a homeomorphism ϕ_x from $U(x)$ onto an open set in $\mathbb{R}^n_+$. The collection $\{(U(x), \phi_x) | x \in M\}$ has the property that 1) $\{U(x) | x \in M\}$ is an open cover and 2) ϕ_x is a homeomorphism from $U(x)$ onto an open set in $\mathbb{R}^n_+$. The subspace $\phi_x(U_x)$ in $\mathbb{R}^n_+$ plays a role as a local coordinate system. The collection $\{(U(x), \phi_x) | x \in M\}$ is somewhat too large and we may like less local coordinate systems. This can be done as follows.

Let M be a space. A *chart* of M is a pair (U, ϕ) such that 1) U is an open set in M and 2) ϕ is a homeomorphism from U onto an open set in $\mathbb{R}^n_+$. An *atlas* for M means a collection of charts $\{(U_\alpha, \phi_\alpha) | \alpha \in J\}$ such that $\{U_\alpha | \alpha \in J\}$ is an open cover of M.

Proposition 2.10.1 *A Hausdorff space M is a manifold (with boundary) if and only if M has an atlas.*

Proof. Suppose that M is a manifold. Then the collection $\{(U(x), \phi_x) | x \in M\}$ is an atlas. Conversely suppose that M has an atlas. For any $x \in M$ there exists α such that $x \in U_\alpha$ and so U_α is an open neighborhood of x that is homeomorphic to an open set in $\mathbb{R}^n_+$. Thus M is a manifold. ♠

We define a subset ∂M as follows: $x \in \partial M$ if there is a chart (U_α, ϕ_α) such that $x \in U_\alpha$ and $\phi_\alpha(x) \in \mathbb{R}^{n-1} = \{x \in \mathbb{R}^n | x_n = 0\}$. ∂M is called the boundary of M. For example the boundary of D^n is S^{n-1}.

Proposition 2.10.2 *Let M be a n-manifold with boundary. Then ∂M is an $(n-1)$-manifold without boundary.*

Proof. Let $\{(U_\alpha, \phi_\alpha)|\alpha \in J\}$ be an atlas for M. Let $J' \subseteq J$ be the set of indices such that $U_\alpha \cap \partial M \neq \emptyset$ if $\alpha \in J'$. Then Clearly

$$\{(U_\alpha \cap \partial M, \phi_\alpha|_{U_\alpha \cap \partial M}|\alpha \in J'\}$$

can be made into an atlas for ∂M. ♠

Definition 2.10.3 A Hausdorff space M is called a *differential manifold of class C^k* if there is an atlas of M

$$\{(U_\alpha, \phi_\alpha|\alpha \in J\}$$

such that

> For any $\alpha, \beta \in J$, the composites
>
> $$\phi_\alpha \circ \phi_\beta^{-1} \colon \phi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n_+$$
>
> is differentiable of class C^k.

The atlas $\{(U_\alpha, \phi_\alpha|\alpha \in J\}$ is called a *differential atlas of class C^k* on M.

Two differential atlases of class C^k $\{(U_\alpha, \phi_\alpha)|\alpha \in I\}$ and $\{(V_\beta, \psi_\beta)|\beta \in J\}$ are called *equivalent* if

$$\{(U_\alpha, \phi_\alpha)|\alpha \in I\} \cup \{(V_\beta, \psi_\beta)|\beta \in J\}$$

is again a differential atlas of class C^k (this is an equivalence relation). A *differential structure of class C^k* on M is an equivalence class of differential atlases of class C^k on M. Thus a differential manifold of class C^k means a manifold with a differential structure of class C^k. A *smooth* manifold means a differential manifold of class C^∞. **Note:** A general manifold is also called *topological manifold*. Kervaire and Milnor [2] have shown that the topological sphere S^7 has 28 distinct oriented smooth structures.

Let M be a smooth manifold and let $\{(U_\alpha, \phi_\alpha)|\alpha \in J\}$ be a C^∞-atlas for M. For $\alpha, \beta \in J$, the function

$$\phi_\alpha \circ \phi_\beta^{-1}\colon \phi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n_+$$

is a smooth map from an open set in $\mathbb{R}^n_+$ to an open set in $\mathbb{R}^n_+$. The Jacobian matrix

$$M_{\alpha\beta}(x) = \left(\frac{\partial(\phi_\alpha \circ \phi_\beta^{-1})_i}{\partial x_j}\bigg|_{\phi_\beta(x)} \right)$$

is invertible for any $x \in U_\alpha \cap U_\beta$. A smooth manifold M is called *orientable* if there is an C^∞-atlas $\{(U_\alpha \phi_\alpha|\alpha \in J\}$ for M such that the determinant of the Jacobian

$$\det(M_{\alpha\beta}(x)) > 0$$

for any $\alpha, \beta \in J$ and $x \in U_\alpha \cap U_\beta$. For example $\mathbb{R}P^n$ is orientable if and only if n is odd. On the other hand $\mathbb{C}P^n$ is orientable for any n.

Definition 2.10.4 let M and N be smooth manifolds of dimensions m and n respectively. A map $f\colon M \to N$ is called *smooth* if for some smooth atlases $\{(U_\alpha, \phi_\alpha|\alpha \in I\}$ for M and $\{(V_\beta, \psi_\beta)\beta \in J\}$ for N the functions

$$\psi_\beta \circ f \circ \phi_\alpha^{-1}|_{\phi_\alpha(f^{-1}(V_\beta)\cap U_\alpha)}\colon \phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha) \to \mathbb{R}^n_+$$

are of class C^∞.

Proposition 2.10.5 *If $f\colon M \to N$ is smooth with respect to atlases*

$$\{(U_\alpha, \phi_\alpha|\alpha \in I\}, \quad \{(V_\beta, \phi_\beta|\beta \in J\}$$

for M, N then it is smooth with respect to equivalent atlases

$$\{(U'_\delta, \theta_\delta|\alpha \in I'\}, \quad \{(V'_\gamma, \eta_\gamma|\beta \in J'\}$$

Proof. Since f is smooth with respect with the atlases

$$\{(U_\alpha, \phi_\alpha|\alpha \in I\},\ \{(V_\beta, \phi_\beta|\beta \in J\},$$

f is smooth with respect to the smooth atlases

$$\{(U_\alpha, \phi_\alpha|\alpha \in I\} \cup \{(U'_\delta, \theta_\delta|\alpha \in I'\}, \quad \{(V_\beta, \phi_\beta|\beta \in J\} \cup \{(V'_\gamma, \eta_\gamma|\beta \in J'\}$$

by look at the local coordinate systems. Thus f is smooth with respect to the atlases

$$\{(U'_\delta, \theta_\delta | \alpha \in I'\},\ \{(V'_\gamma, \eta_\gamma | \beta \in J'\}. \spadesuit$$

Thus the definition of smooth maps between two smooth manifolds is independent of choice of atlas.

Let M be a m-manifold. The *(ordered) configuration space* $F(M,n)$ is defined by

$$F(M,n) = \{(x_1, \cdots, x_n) \in M^n | x_i \neq x_j \text{ for } i \neq j\}.$$

In other words, the configuration space $F(M,n)$ is the subspace of the Cartesian product M^n by removing the 'flat' diagonals. The symmetric group Σ_n acts on $F(M,n)$ by permuting coordinates. The *(unordered) configuration space* $B(M,n)$ is the quotient of $F(M,n)$ by Σ_n, that is

$$B(M,n) = F(M,n)/\Sigma_n.$$

Clearly both $F(M,n)$ and $B(M,n)$ are mn-manifolds. configuration spaces are arisen from many areas in mathematics and physics. In geometry and physics, the diagonals play as singularities in many cases and so we have to remove them, then this gives the configuration space. In combinatorics, the homology of configuration spaces is related to 'subspace arrangements'. The determination of the homology of $F(M,n)$ and $B(M,n)$ still remains open for general manifold M though it is known for many cases. The fundamental groups of configuration spaces are interesting as well. A typical example is that the fundamental group of $F(\mathbb{R}^2, n)$ is the pure braid group K_n and the fundamental group of $B(\mathbb{R}^2, n)$ is the Artin braid group B_n. The braid groups are important in group theory, low dimensional topology and mathematical physics. In homotopy theory, configuration spaces are used to construct various combinatorial models for mapping spaces. (As we have seen that mapping spaces are quite complicated, the construction means that we construct certain 'simpler spaces' that has the same homotopy groups and homology groups of a mapping space. So if one needs to know the homotopy groups and homology groups of a complicated mapping space, one may look at these simpler spaces.)

Chapter 3

Homotopy and The Fundamental Groups

3.1 Homotopy

3.1.1 Homotopy Relative to a Subspace

The problem of classifying topological spaces and continuous maps up to topological equivalence (homeomorphism) does not seem to be amenable to attack directly by computable algebraic functors. Many of the computable functors, because they are computable, are invariant under continuous deformation. Therefore they cannot distinguish between spaces (or maps) that can be continuously deformed from one to the other; the most that can be hoped for from such functors is that they characterize the space (or map) up to continuous deformation.

The intituive concept of a continuous deformation will be made precise in this section in the concept of homotopy. This leads to the homotopy category which is fundamental for algebraic topology. Its objects are topological spaces and its morphisms are equivalence classes of continuous maps (two maps being equivalent of one can be continuously deformed into the other).

Roughly speaking two continuous maps $f_0, f_1 \colon X \to Y$ are said to be homotopic if there is an intermediate family of maps $f_t \colon X \to Y$ for $0 \leq t \leq 1$ which vary continuously with respect to t. Let $I = [0, 1]$.

Definition 3.1.1 Let $f, g \colon X \to Y$ be two maps. We say that f is homotopic to g if there is a continuous map $F \colon X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$

for any $x \in X$. The map F is called a *homotopy* between f and g. We write $f \simeq g$ or $F: f \simeq g$.

For each $0 \leq t \leq 1$, we denote $F(x,t)$ by $F_t(x)$. So gives a family of maps $F_t: X \to Y$. Just keep in mind that F_t is continuous in t as a map from I to $\mathrm{Map}(X, Y)$. A map $f: X \to Y$ is called *null homotopic* if f is homotopic to a constant map.

Definition 3.1.2 Suppose that A is a subset of X and that $f, g: X \to Y$ are maps. We say that f is *homotopic to g relative* to A, denoted $f \simeq g \operatorname{rel} A$ or $F: f \simeq g \operatorname{rel} A$, if there is a homotopy $F: X \times I \to Y$ such that

1) $F(x, 0) = f(x)$ for any $x \in X$;

2) $F(x, 1) = g(x)$ for any $x \in X$ and

3) $F(a, t) = f(a)$ for any $a \in A$ and $t \in I$.

We say that f is *null homotopic relative to A* if f is homotopic to a constant map relative to A.

Note that $f(a) = g(a)$ for all $a \in A$ if $f \simeq g \operatorname{rel} A$. When $A = \emptyset$, $f \simeq g \operatorname{rel} A$ is equivalent to $f \simeq g: X \to Y$. Given two maps $f, g: X \to Y$ such that $f(a) = g(a)$ for $a \in A$. The question whether f is homotopic to g relative to A is in fact an 'extension question' by the following diagram

$$\begin{array}{ccc} X \times \{0\} \cup X \times \{1\} \cup A \times I & \hookrightarrow & X \times I \\ \| & & \downarrow F \\ X \times \{0\} \cup X \times \{1\} \cup A \times I & \xrightarrow{\phi} & Y, \end{array}$$

where $\phi|_{X \times \{0\}} = f$, $\phi|_{X \times \{1\}} = g$ and $\phi(a, t) = f(a) = g(a)$ for $a \in A$ and $t \in I$. In other words such an extension F exists if and only if f is homotopic to g relative to A.

Theorem 3.1.3 *Homotopy relative to A is an equivalence relation in the set of maps from X to Y.*

Proof. *Reflexivity.* For $f: X \to Y$, define $F: X \times I \to Y$ by $F(x,t) = f(x)$. Thus $f \simeq a \operatorname{rel} A$.

Symmetry. Given $F: f \simeq g \operatorname{rel} A$, define $F': g \simeq f \operatorname{rel} A$ by

$$F'(x,t) = F(x, 1-t).$$

Transitivity. Given $F: f \simeq g \operatorname{rel} A$ and $G: g \simeq h \operatorname{rel} A$, define $H: f \simeq h \operatorname{rel} A$ by

$$H(x,t) = \begin{cases} F(x,2t) & 0 \leq t \leq 1/2 \\ G(x,2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

Note that H is continuous because its restriction to each of closed sets $X \times [0,1/2]$ and $X \times [1/2,1]$ is continuous.♠

It follows that the set of maps from X to Y is partitioned into disjoint equivalence classes by the relation of homotopy relative to A. These equivalence classes are called *homotopy classes relative to* A. We use the notation $[X,Y]_A$ to denote this set of homotopy classes. Given $f: X \to Y$, we use $[f]_A$ to denote the element in $[X,Y]_A$ determined by f. For (unpointed) spaces X and Y, the notation $[X,Y]$ usually means $[X,Y]_\emptyset$.

Theorem 3.1.4 *Let A and B be subspaces of X and Y respectively. Let $f_0, f_1: X \to Y$ be homotopic relative to A and $g_0, g_1: Y \to Z$ be homotopic relative to B such that $f_1(A) \subseteq B$. Then $g_0 \circ f_0 \simeq g_1 \circ f_1 \operatorname{rel} A$.*

Proof. Let $F: f_0 \simeq f_1 \operatorname{rel} A$ and $G: g_0 \simeq g_1 \operatorname{rel} B$. Then the composite

$$X \times I \xrightarrow{F} Y \xrightarrow{g_0} Z$$

is a homotopy relative to A from $g_0 \circ f_0$ to $g_0 \circ f_1$, and the composite

$$X \times I \xrightarrow{f_1 \times \mathrm{id}_I} Y \times I \xrightarrow{G} Z$$

is a homotopy relative to $f_1^{-1}(B)$ from $g_0 \circ f_1$ to $g_1 \circ f_1$. Since $A \subseteq f_1^{-1}(B)$, we have shown that$g_0 \circ f_0 \simeq g_0 \circ f_1 \operatorname{rel} A$ and $g_0 \circ f_1 \simeq g_1 \circ f_1 \operatorname{rel} A$. The result follows from Theorem 3.1.3. ♠

3.1.2 Pointed Homotopy

Let X and Y be pointed spaces and let $f, g: X \to Y$ be pointed maps. f is called (pointed) *homotopic* to g is $f \simeq g \operatorname{rel} x_0$, where x_0 is the base point. If there is no confusion, we simply denote $f \simeq g$ for $f \simeq g \operatorname{rel} x_0$ (in pointed case). For pointed spaces, the notation $[X, Y]$ means the set of equivalence classes of pointed maps from X to Y by the relation of homotopy relative to the base point x_0. For any pointed map f, $[f]$ means the homotopy class determined by f. The homotopy category of pointed spaces means the category in which objects are pointed spaces and morphisms are homotopy classes $[f]$. The composition in the homotopy category of pointed spaces is defined by $[f] \circ [g] = [f \circ g]$. Theorem 3.1.4 shows that this is a well-defined composition operation.

Definition 3.1.5 Let X be a pointed space. The *n-homotopy group* $\pi_n(X)$ is defined by

$$\pi_n(X) = [S^n, X]$$

for $n \geq 0$.

Note: $\pi_0(X)$ is NOT a group in general. $\pi_1(X)$ is also called the *fundamental group* of X. We will show that the fundamental group $\pi_1(X)$ is a group for any X (but non-commutative in general). We will also show that $\pi_n(X)$ is an abelian group for $n \geq 2$.

Theorem 3.1.6 *A pointed map $f\colon Y_1 \to Y_2$ gives rise to a function*

$$f_*\colon [X, Y_1] \to [X, Y_2]$$

for any pointed space X with the following properties:

1) *If $f'\colon Y_1 \to Y_2$ is another map, and $f' \simeq f$, then $f_* = f'_*$;*

2) *If $\mathrm{id}\colon Y \to Y$ is the identity map, then $\mathrm{id}_*\colon [X, Y] \to [X, Y]$ is the identity function;*

3) *If $g\colon Y_2 \to Y_3$ is another map, then*

$$(g \circ f)_* = g_* \circ f_*.$$

Proof. Let $[\lambda] \in [X, Y_1]$ be the homotopy class of a map $\lambda\colon X \to Y$. Define

$$f_*([\lambda]) = [f \circ \lambda] \in [X, Y_2].$$

The function f_* is well-defined because if $\lambda'\colon X \to Y_1$ is another map with $[\lambda'] = [\lambda]$, that is $\lambda' \simeq \lambda$, then $[f \circ \lambda'] = [f \circ \lambda]$ by Theorem 3.1.4. Properties 1 to 3 follow immediately from the definition and Theorem 3.1.4. ♠

Let $f\colon X_1 \to X_2$ be any pointed map. Define

$$f^*\colon [X_2, Y] \to [X_1, Y]$$

by

$$f^*([\lambda]) = [\lambda \circ f]$$

for any pointed map $\lambda\colon X_2 \to Y$. By the similar arguments, we have

Theorem 3.1.7 *A pointed map $f\colon X_1 \to X_2$ gives rise to a function*

$$f^*\colon [X_2, Y] \to [X_1, Y]$$

for any pointed space Y with the following properties:

1) If $f'\colon X_1 \to X_2$ is another map, and $f' \simeq f$, then $f^* = f'^*$;

2) If $\mathrm{id}\colon X \to X$ is the identity map, then $\mathrm{id}^*\colon [X, Y] \to [X, Y]$ is the identity function;

3) If $g\colon X_2 \to X_3$ is another map, then

$$(g \circ f)^* = f^* \circ g^*.$$

Theorem 3.1.8 *Let X, Y and Z be pointed spaces. Then*

1) $[X \vee Y, Z] \cong [X, Z] \times [Y, Z]$ and

2) $[X, Y \times Z] \cong [X, Y] \times [X, Z]$.

Proof. 1) Let

$$\theta\colon [X \vee Y, Z] \to [X, Z] \times [Y, Z]$$

be defined by

$$\theta([\lambda]) = (i_X^*([\lambda], i_Y^*([\lambda]))$$

for any $[\lambda] \in [X \vee Y \to Z]$, where $i_X\colon X \to \vee Y$ and $i_Y\colon Y \to X \vee Y$ be the inclusions. We first show that θ is onto. For any $([\lambda_1], [\lambda_2]) \in [X, Z] \times [Y, Z]$, where $\lambda_1\colon X \to Z$ and $\lambda_2\colon Y \to Z$ are pointed maps. Then there is a unique pointed map $\lambda\colon X \vee Y \to Z$ such that $\lambda|_X = \lambda_1$ and $\lambda|_Y = \lambda_2$ and so

$$\theta([\lambda]) = ([\lambda_1], [\lambda_2])$$

or θ is onto. Now we show that θ is one-to-one. Let $\lambda, \lambda' : X \vee Y \to Z$ such that $\theta([\lambda]) = \theta([\lambda'])$. Then $i_X^*([\lambda]) = i_X^*([\lambda'])$ and $i_Y^*([\lambda]) = i_Y^*([\lambda'])$ that is there are pointed homotopies

$$F : \lambda|_X \simeq \lambda'_X \quad \text{and}$$

$$G_: \lambda|_Y \simeq \lambda'|_Y$$

and so the map $H : (X \vee Y) \times I \to Z$ defined by

$$H(x,t) = \begin{cases} F(x,t) & \text{for} \quad x \in X \\ G(x,t) & \text{for} \quad x \in Y \end{cases}$$

is a homotopy from λ to λ'.

2). Let

$$\theta : [X, Y \times Z] \to [X, Y] \times [X, Z]$$

be the function defined by

$$\theta([\lambda]) = (p_{Y*}([\lambda]), p_{Z*}(\lambda]))$$

for any $\lambda : X \to Y \times Z$. Similar arguments show that θ is one-to-one and onto. ♠

Corollary 3.1.9 *Let X and Y be a pointed space. Then*

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$$

for each $n \geq 0$.

Theorem 3.1.10 *Let X, Y and Z be pointed spaces. If Y is locally compact and Hausdorff, then the association map*

$$\bar{\alpha} : \mathrm{Map}_*(X \wedge Y, Z) \to \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z))$$

induces and one-to-one correspondence

$$\bar{\alpha}_* : [X \wedge Y, Z] \xrightarrow{\cong} [X, \mathrm{Map}_*(Y, Z)].$$

Proof. Let $p : X \times Y \to X \wedge Y$ be the quotient map. Suppose that $F : (X \wedge Y) \times I$ be a pointed homotopy between maps $f, g : X \wedge Y \to Z$. Then the map

$$F \circ (p \times \mathrm{id}_I) : X \times Y \times I \to Z$$

sends $X\times\{y_0\}\times I$ and $\{x_0\}\times Y\times I$ to z_0 and so induces a map $F'\colon (X\times I)\wedge Y\to Z$. Then $\alpha(F')\colon X\times I\to Z^Y$ sends $x_0\times I$ to the base-point and is clearly a homotopy between $\bar\alpha_*(f)$ and $\bar\alpha_*(g)$. Thus $\bar\alpha_*\colon[X\wedge Y,Z]\to[X,Z^Y]$ is a well-defined function (though $\bar\alpha$ may not be continuous in general).

By assertion (b) of Theorem 2.9.9, the function

$$\bar\alpha\colon Z^{X\wedge Y}\to(Z^Y)^X$$

is onto and so $\bar\alpha_*\colon[X\wedge Y,Z]\to[X,Z^Y]$ is onto.

Now we show that $\bar\alpha_*$ is one-to-one. Let $f,g\colon X\wedge Y\to Z$ such that $\bar\alpha(f)\simeq\bar\alpha(g)$, that is there is a pointed homotopy $F\colon X\times I\to Z^Y$ such that $F_0=f$ and $F_1=g$. By assertion (b) of Theorem 2.9.9, there is a map $F'\colon(X\times I)\wedge Y\to Z$ such that $\bar\alpha(F')=F$. Let $q\colon X\times Y\times I\to(X\times I)\wedge Y$ be the quotient map defined by $q(x,y,t)=(x,t)\wedge y$. Then $F'\circ q$ sends $X\times\{y_0\}\times I$ and $\{x_0\}\times Y\times I$ to z_0. BY Theorem 2.8.9, the map $(X\times Y)\times I\to(X\wedge Y)\times I$ is a quotient map because I is locally compact and Hausdorff. Thus $F\circ q$ induces a map $F''\colon(X\wedge Y)\times I\to Z$, which is clearly a pointed homotopy between f and g. ♠

Corollary 3.1.11 *Let X be a pointed space. Then*

$$\pi_n(X)\cong\pi_0(\Omega^nX)\cong\pi_1(\Omega^{n-1}X)$$

for any $n\geq 1$.

3.1.3 Path Connected Components

Let X be a topological space. A *path* in X means a continuous map $\lambda\colon I\to X$. $\lambda(0)$ is called the *initial point* and $\lambda(1)$ is called the it final or it terminal point. Clearly, a path in X is a homotopy from one point space to X. Given a space X, define an equivalence relation by $x\simeq y$ if there is a path in X joining x and y. Let x be any point in X. The *path-connected component* of X that contains x is defined to be the subspace

$$\{y\in X|y\simeq x\}\subseteq X.$$

A space X is called *path-connected* if X has only one path-connected component. In other words, X is path-connected if for any two points x,y in X there is a path joining x and y. By Theorem 3.1.3, $\simeq$ is an equivalence relation on X and so X is a disjoint union of its path-connected components. Let $X/\simeq$ be the set of equivalence classes of X by $\simeq$.

Exercise 3.1.1 Let $f: X \to Y$ be a map. If X is path-connected, then the image $f(X)$ is path-connected.

Exercise 3.1.2 Let X be a non-empty space and let x_0 be any point in X which is regarded as the base point. Then

$$\pi_0(X) \cong (X/\simeq).$$

In particular, X is path-connected if and only if $\pi_0(X)$ is the one-point set $\{0\}$.

Exercise 3.1.3 Let X and Y be topological spaces. Then X and Y are path-connected if and only if $X \times Y$ is path-connected. (Hint: $\pi_0(X \times Y) \cong \pi_0(X) \times \pi_0(Y)$.)

3.1.4 Homotopy Equivalences and Contractible Spaces

A map $f: X \to Y$ is called an *homotopy equivalence* if there is a map $g: Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. The map g is called a homotopy inverse of f. A space X is called *homotopy equivalent* to Y if there is a homotopy equivalence between X and Y. In this case, we call that X has the same homotopy type of Y, denoted by $X \simeq Y$.

A space X is called *contractible* if the identity map is homotopic to some constant map from X to itself.

Proposition 3.1.12 *Any two maps of an arbitrary space to a contractible space are homotopic.*

Proof. Let Y be a contractible space and suppose that $\mathrm{id}_Y \simeq c$, where $c: Y \to Y$ is a constant map. Let $f_0, f_1: X \to Y$ be any two maps. Then

$$f_0 = \mathrm{id}_Y \circ f_0 \simeq c \circ f_0 = c \circ f_1 \simeq \mathrm{id}_Y \circ f_1 = f_1$$

and so $f_0 \simeq f_1$. ♠

Corollary 3.1.13 *If Y is a contractible space, then any two constant maps of Y to itself are homotopic, and the identity map is homotopic to any constant map of Y to itself.*

Exercise 3.1.4 Show that any vector space V over $\mathbb{R}$ is contractible. (Hint: Check that the map $F: V \times I \to V$, $(x, t) \to (1 - t)x$, is a homotopy between the identity map and a constant map.)

Theorem 3.1.14 *A space is contractible if and only if it has the same homotopy type as a one-point space.*

Proof. Assume that X is contractible and let F be a homotopy between the identity map and a constant map $c: X \to X$, $x \to x_0$. Let P be the one-point space $\{x_0\}$ and let $f: X \twoheadrightarrow P$ and $j: P \subseteq X$. Then $f \circ j = \mathrm{id}_P$ and $F: \mathrm{id}_X \simeq j \circ f$. Thus f is a homotopy equivalence from X to P.

Conversely, if X has the same homotopy type as a one-point space P, let $f: X \to P$ be a homotopy equivalence with homotopy inverse $g: P \to X$. Then $\mathrm{id}_X \simeq g \circ f$. Because $g \circ f$ is a constant map, X is contractible. ♠

Corollary 3.1.15 *Any two contractible spaces have the same homotopy type, and any continuous map between contractible spaces is a homotopy equivalence.*

Proof. Let X and Y be two contractible spaces. Let P be a one-point space. Then $X \simeq P \simeq Y$ and so $X \simeq Y$. The second part follows from Proposition 3.1.12. ♠

Theorem 3.1.16 *Let p_0 be any point of S^n and let $f: S^n \to Y$. The following are equivalent:*

(a) f is null homotopic;

(b) f can be continuously extended over D^{n+1};

(c) f is null homotopic relative to p_0.

Proof. (*a*) ⇒ (*b*). Let $F: f \simeq c$, where c is the constant map of S^n to $y_0 \in Y$. Define an extension $\tilde{f}$ of f over E^{n+1} by

$$\tilde{f}(x) = \begin{cases} y_0 & 0 \leq \|x\| \leq 1/2 \\ F(x/\|x\|, 2 - 2\|x\|) & 1/2 \leq \|x\| \leq 1. \end{cases}$$

Since $F(x,1) = y_0$ for all $x \in S^n$, the map $\tilde{f}$ is well-defined. $\tilde{f}$ is continuous because its restriction to each of the closed sets $\{x \in E^{n+1} | 0 \leq \|x\| \leq 1/2\}$ and $\{x \in E^{n+1} | 1/2 \leq \|x\| \leq 1\}$ is continuous. Since $F(x,0) = f(x)$ for $x \in S^n$, $\tilde{f}|_{S^n} = f$ and $\tilde{f}$ is a continuous extension of f to D^{n+1}.

(*b*) ⇒ (*c*). If f has the continuous extension $\tilde{f}: E^{n+1} \to Y$, define $F: S^n \times I \to Y$ by

$$F(x,t) = \tilde{f}((1-t)x + tp_0).$$

Then $F(x,0) = \tilde{f}(x) = f(x)$ and $F(x,1) = \tilde{f}(p_0)$ for $x \in S^n$. Since $F(p_0,t) = \tilde{f}(p_0)$ for $t \in I$, F is a homotopy relative to $\{p_0\}$ from f to a constant map.

(*c*) ⇒ (*a*). This is obvious. ♠

Exercise 3.1.5 Show that any continuous map from S^n to a contractible space has a continuous extension over E^{n+1}.

Exercise 3.1.6 The *comb space* Y is defined by

$$Y = \{(x, y) \in \mathbb{R}^2 | 0 \leq y \leq 1, x = 0, 1/n \text{ or } y = 0, 0 \leq x \leq 1\}.$$

Show that the identity map of Y is homotopic to the constant map to $(0, 1) \in Y$. (Hint: By Proposition 3.1.12, it suffices to show that Y is contractible. Let $F\colon Y \times I \to Y$ be defined by $F((x, y), t) = (x, (1-t)y)$. Then F is a homotopy from id_Y to the projection of Y to the x-axis. Since the latter map is homotopic to a constant map, Y is contractible.)

3.2 Retraction and Deformation

This section is concerned mainly with inclusion maps. We consider whether such a map has a left inverse, a right inverse and a two-sided inverse in either the category of spaces or the homotopy category.

A subspace A of X is called a *retract* of X if the inclusion $i\colon A \to X$ has a left inverse, that is, there is a map $r\colon X \to A$ such that $r \circ i = \mathrm{id}_A$. A subspace A is called a *weak retract* of X if $i\colon A \subseteq X$ has a left homotopy inverse, that is there is a map $r\colon X \to A$ such that $r \circ i \simeq \mathrm{id}_A$.

Example 3.2.1 Let A be the comb space of exercise 3.1.6 and let $X = I^2$. Then A is a weak retract of X because both A and X are contractible and so the inclusion $i\colon A \subseteq X$ is a homotopy equivalence (in particular i has a left inverse. We show that A is not a retract of X. Suppose that there were a retraction $r\colon X \to A$. Let $x_0 = (0, 1) \in A$. Then $r(x_0) = x_0$. Let $U = \{y | \|y - x_0\| < 1/2\} \cap A = B1/2(x_0) \cap A$ be the open neighborhood of x_0. There is an open neighborhood V of x_0 in I^2 such that $r(V) \subseteq U$. Let ϵ be a small positive number such that $B_\epsilon(x_0) \cap I^2 \subseteq V$. Since $B_\epsilon(x_0) \cap I^2$ is path-connected, the image $r(B_\epsilon(x_0) \cap I^2) \subseteq U$ is path connected in U. Let $m \neq n$ be positive integers such that $1/m, 1/n < \epsilon$. Then $(1/m, 1), (1/n, 1) \in r(B_\epsilon(x_0) \cap I^2)$ because r is a retraction and so there is a path λ in $r(B_\epsilon(x_0) \cap I^2) \subseteq U$ joining them. This contracts to that $(1/m, 1)$ and $(1/n, 1)$ lie in different path-connected components of U.

Exercise 3.2.1 Show that a subspace $i\colon A \subseteq X$ is a weak retract if and only if $i^*\colon [X, A]_\emptyset \to [A, A]_\emptyset$ is onto.

Despite the fact that, in general, a weak retract need not be a retract, these concepts do coincide when A is a suitable subspace of X. This occurs frequently enough to warrant special consideration and will prove of use later. Let (X, A) be a pair of spaces (that is A is a subspace of X) and Y be a space. (X, A) is said to have the *homotopy extension property with respect to* Y if, given maps $g: X \to Y$ and $G: A \times I \to Y$ such that $g(a) = G(a, 0)$ for $x \in A$, there is a map $F: X \times I \to Y$ such that $F(x, 0) = g(x)$ for $x \in X$ and $F|_{A\times I} = G$. In other words, the following commutative diagram holds

$$\begin{array}{ccc} X \times 0 \cup A \times I & \xrightarrow{g \cup G} & Y \\ \big\downarrow & & \big\| \\ X \times I & \overset{F}{\dashrightarrow} & Y, \end{array}$$

for any $g: X \to Y$ and $G: A \times I \to Y$ such that $g|_A = G_{A\times 0}$.

Proposition 3.2.2 *Suppose that (X, A) has the homotopy extension property with respect to Y and $f_0, f_1: A \to Y$ are homotopic. If f_0 has an extension to X, then so is f_1.*

Proof. Let $G: A \times I \to Y$ be the homotopy from f_0 to f_1 and let $g: X \to Y$ be the extension of f_0. By the definition, there is a map $F: X \times I \to Y$ such that $F|_{X\times 0\cup A\times I} = g \cup G$. Then $F_1: X \to Y$ is an extension of f_1. ♠

Of particular importance is the case when (X, A) has the homotopy extension property with respect to any space. More generally, a map $f: X' \to X$ is called a *cofibration* if for any space Y and any given maps $g: X \to Y$ and $G: X' \times I \to Y$ such that

$$G(x', 0) = g(f(x'))$$

for any $x' \in X'$, there exists a map $F: X \times I \to Y$ such that $F(x, 0) = g(x)$ and $F(f(x'), t) = G(x', t)$ for any $x \in X$, $x' \in X'$ and $t \in I$. The existence of F is equivalent to the existence of a map represented by the dotted arrow which makes the following diagram commutative:

$$\begin{array}{ccccc} X' \times I & \xrightarrow{f \times \mathrm{id}_I} & X \times I & \hookleftarrow & X \times 0 \\ \downarrow G & & \downarrow & & \downarrow g \\ Y & = & Y & = & Y. \end{array}$$

Thus an inclusion $i: A \subseteq X$ is a cofibration if and only if (X, A) has the homotopy extension property with respect to any space Y.

Proposition 3.2.3 *An inclusion $i: A \subseteq X$ is a cofibration if and only if $X \times 0 \cup A \times I$ is a retract of $X \times I$.*

Proof. Suppose that $i: A \subseteq X$ is a cofibration. Then the identity map of $X \times 0 \cup A \times I$ can be extended to $X \times I$ and so $X \times 0 \cup A \times I$ is a retract of $X \times I$. Conversely, let

$$r: X \times I \to X \times 0 \cup A \times I$$

be a retraction. Let Y be any space and let $g: X \to Y$ and $G: A \times I \to Y$ be maps such that $G|_{A \times 0} = g|_A$. Then the composite

$$X \times I \xrightarrow{r} X \times 0 \cup A \times I \xrightarrow{g \cup G} Y$$

is an extension of $g \cup G$. ♠

Exercise 3.2.2 Let $A \subseteq B \subseteq X$ be subspaces. Suppose that $A \subseteq B$ and $B \subseteq X$ are co-fibrations. Show that $A \subseteq X$ is a cofibration.

Exercise 3.2.3 Show that $S^n \subseteq D^{n+1}$ is a cofibration.

Theorem 3.2.4 *If (X, A) has the homotopy extension property with respect to A, then A is a weak retract of X if and only if A is a retract of X.*

Proof. We show that any weak retraction $r: X \to A$ is, in fact, homotopic to a retraction. Let $G: A \times I \to A$ be the homotopy from $r \circ i$ to id_A. Because (X, A) has the homotopy extension property with respect to A, there is an extension $F: X \times I \to A$ such that $F(x, 0) = r(x)$ and $F(a, t) = G(a, t)$. Then $F_1: X \to A$ is a retraction. ♠

Given $X' \subseteq X$, a *deformation* D of X' in X is a homotopy $D: X' \times I \to X$ such that $D(x', 0) = x'$ for $x' \in X'$. If, moreover, $D(X' \times 1)$ is contained in a subspace A

of X, D is said to be a*deformation of X' into A* and X' is said to be *deformable in X into A*. A space X is said to be *deformable* into a subspace A if it is deformable in itself into A. Thus a space X is contractible if and only if it is deformable into one of its points.

Exercise 3.2.4 Show that a space X is deformable into a subspace A if and only if the inclusion $i: A \subseteq X$ has a right homotopy inverse.

Note that an inclusion $i: A \subseteq X$ never has a right inverse in the category of topological spaces except the trivial case $A = X$.

A subspace $A \subseteq X$ is called a *weak deformation retract* of X if the inclusion $i: A \subseteq X$ is a homotopy equivalence.

Exercise 3.2.5 Show that A is a weak deformation retract of X if and only if A is a weak retract of X and X is deformable into A.

A is called a *deformation retract* if there is a retraction r of X to A such that $r \circ i \simeq \mathrm{id}_X$. A is called a *strong deformation retract* of X if there is a retraction r of X to A such that $r \circ i \simeq \mathrm{id}_X \ \mathrm{rel}\, A$.

Exercise 3.2.6 Suppose that X is deformable into a retract A. Show that A is a deformation retraction of X.

Theorem 3.2.5 *If (X, A) has the homotopy extension property with respect to A, then A is a weak deformation retract of X if and only if A is a deformation retract of X.*

Proof. Since (X, A) has the extension property with respect to A and A is a weak retract of X, A is a retract of X. Let $r: X \to A$ be a retraction. Since $i: A \subseteq X$ is a homotopy equivalence, i has a right homotopy inverse and so r is a right homotopy inverse of i. Thus A is a deformation retract of X.

Theorem 3.2.6 *If $(X \times I, (X \times 0) \cup (A \times I) \cup (X \times 1))$ has the homotopy extension property with respect to X and A is closed in X, then A is a deformation retract of X if and only if A is a strong deformation retract of X.*

Proof. $\Leftarrow$ is obvious by definition. $\Rightarrow$. Let $r: X \to A$ be a retract and let $F: X \times I \to X$ be a homotopy from id_X to $r \circ i$, where $i: A \subseteq X$. A homotopy

$$G: ((X \times 0) \cup (A \times I) \cup (X \times 1)) \times I \to X$$

is defined by the equations

$$\begin{array}{ll} G((x,0),t') = x & x \in X, t' \in I \\ G((a,t),t') = F(a,(1-t')t) & a \in A, t, t' \in I \\ G((x,1),t') = F(r(x),1-t') & x \in X, t' \in I. \end{array}$$

G is well-defined, because for $a \in A$

$$G((a,0)t') = a = F(a,0)$$

by the first two equations and

$$G((a,1),t') = F(a,1-t') = F(r(a),1-t')$$

by the last two equations. G is continuous because its restriction to each of the closed sets $X \times o \times I$, $A \times I \times I$ and $X \times 1 \times I$ is continuous. Furthermore

$$G|_{((X\times 0)\cup(A\times I)\cup(X\times 1))\times 0} = F_{(X\times 0)\cup(A\times I)\cup(X\times 1)}$$

[because $F(x,0) = x$ and since r is a retraction, $F(r(x),1) = ir(r(x)) = F(x,1)$.] Thus G restrict to $((X\times 0)\cup(A\times I)\cup(X\times 1))\times 0$ can be extended to $(X\times I)\times 0$. From the homotopy extension property in the hypothesis, G restrict to $((X\times 0)\cup(A\times I)\cup(X\times 1))\times 1$ can be extended to $(X\times I)\times 1$. Let $G'\colon (X\times I)\times 1 \to X$ be such an extension, and define $H\colon X\times I \to X$ by $H(x,t) = G'((x,t),1)$. Then we have

$$\begin{array}{ll} H(x,0) = G'((x,0),1) = G((x,0),1) = x & x \in X \\ H(x,1) = G((x,1),1) = F(r(x),0) = r(x) & x \in X \\ H(xa,t) = G((a,t),1) = F(a,0) = a & a \in A, t \in I \end{array}$$

and so H is a homotopy relative to A from id_X to $i \circ r$, or A is a strong deformation retract of X. ♠

3.3 H-spaces and Co-H-spaces

In this section, a space X means a pointed space. The notation $[X,Y]$ means the set of pointed homotopy classes of pointed maps from X to Y.

3.3.1 H-spaces

An H-*space* consists of a pointed space P together with a continuous multiplication $\mu: P \times P \to P$ for which the (unique) constant map $c: P \to P$ is a *homotopy identity*, that is, the following diagram

$$\begin{array}{ccccc} P & \xrightarrow{(c,\,\mathrm{id}_P)} & P \times P & \xleftarrow{(\mathrm{id}_P,\,c)} & P \\ \Big\| & & \Big\downarrow{\scriptstyle \mu} & & \Big\| \\ P & = & P & = & P \end{array}$$

commutes up to homotopy.

Exercise 3.3.1 Let P be a pointed space and let $\mu: P \times P \to P$ be a map. Then μ has a homotopy identity if and only if there is a homotopy commutative diagram

$$\begin{array}{ccc} P \times P & \xrightarrow{\mu} & P \\ \Big\uparrow & & \Big\| \\ P \vee P & \xrightarrow{\nabla} & P, \end{array}$$

where ∇ is the fold map defined by $\nabla(x, x_0) = x$ and $\nabla(x_0, y) = y$.

An H-space P is called *homotopy associative* if the diagram

$$\begin{array}{ccc} P \times P \times P & \xrightarrow{\mu \times \mathrm{id}_P} & P \times P \\ \Big\downarrow{\scriptstyle \mathrm{id}_P \times \mu} & & \Big\downarrow{\scriptstyle \mu} \\ P \times P & \xrightarrow{\mu} & P \end{array}$$

commutes up to homotopy. An H-space P is called *homotopy commutative* if the diagram

$$\begin{array}{ccc} P\times P & \xrightarrow{T} & P\times P \\ \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\mu} \\ P & = & P \end{array}$$

commutes up to homotopy, where $T(x,y)=(y,x)$. A map $\nu\colon P\to P$ is called a *homotopy inverse* if the diagram

$$\begin{array}{ccccc} P\times P & \xrightarrow{\mu} & P & \xleftarrow{\mu} & P\times P \\ \Big\uparrow{\scriptstyle(\nu,\mathrm{id}_P)} & & \Big\| & & \Big\uparrow{\scriptstyle(\mathrm{id}_P,\nu)} \\ P & \xrightarrow{c} & P & \xleftarrow{c} & P \end{array}$$

commutes up to homotopy.

An H-space P is called an H-group if μ is homotopy associative with a homotopy inverse.

Note: We have been to call a space X is an H-space if there is a multiplication $\mu\colon X\times X\to X$ such that μ has a strict identity. In general, a multiplication $\mu\colon X\times X\to X$ that has a homotopy identity may not have a strict identity. But under certain conditions, homotopy identity $\Rightarrow$ strict identity.

Proposition 3.3.1 *Let X be a pointed space with a base point x_0. Let $\mu\colon X\times X\to X$ be a multiplication with a homotopy identity, that is, X is an H-space. Suppose that $X\vee X\subseteq X\times X$ is a cofibration. Then there is a multiplication $\mu'\colon X\times X\to X$ such that μ' has a strict identity.*

Proof. Let $\nabla\colon X\vee X\to X$ be the fold map. Since $\mu|_{X\vee X}\colon X\vee X\to X$ is homotopic to ∇ and $X\vee X\hookrightarrow X\times X$ has homotopy extension property with respect to X, ∇ has an extension $\mu'\colon X\times X\to X$. ♠

Note: It is known that if $\{x_0\}\to X$ is a cofibration, then $X\vee X\to X\times X$ is a cofibration. A base-point x_0 of X is called *non-degenerate* if the inclusion $\{x_0\}\to X$ is a cofibration.

Note: In homotopy theory, there are (were) many questions about H-spaces. We list few of them:

1) Suppose that P is a homotopy associative H-space. Do there exist a space Q and a multiplication μ' on Q such that Q is a topological monoid under μ' and $Q \simeq P$? Suppose that P is path-connected. The answer of this question is: Yes if and only if P is homotopy equivalent to a loop space ΩX for some X. James Stasheff studied this question in 1960's and produced a method to test whether a space is homotopy equivalent to a loop space. His methods has been applied to Quantum Groups in 1980's.

2) Since one knows that S^1, S^3 and S^7 are H-spaces, people asked for which n S^n is an H-space? The answer was given by Adams in 1950's that S^n is an H-space if and only if $n = 1, 3, 7$.

3) We will show that the double loop spaces are homotopy associative and homotopy commutative H-spaces. One has been to ask whether a double loop space is homotopy equivalent to a (strict) commutative topological group. The answer, was given by Milnor in 1950's, is that if a path connected space X is homotopy equivalent to a commutative topological space if and only if X is a product of the spaces Y with the property that Y has at most one possible nontrivial commutative homotopy group, that is there is an integer n such that $\pi_i(Y) = 0$ for $i \neq n$ and $\pi_n(Y)$ is commutative.

Let P and Q be H-spaces. A (pointed) map $f\colon P \to Q$ is called an H-map if the diagram

$$\begin{array}{ccc} P \times P & \xrightarrow{\mu_P} & P \\ \Big\downarrow{\scriptstyle f \times f} & & \Big\downarrow{\scriptstyle f} \\ Q \times Q & \xrightarrow{\mu_Q} & Q \end{array}$$

commutes up to homotopy. If this diagram commutes strictly, we call f is a homomorphism. Clearly a homomorphism is an H-map. On the other hand, an H-map may not be a homomorphism in general.

Problem 3.3.2 *Let X and Y be H-spaces and let $f\colon X \to Y$ be an H-map. Under what conditions on X, Y and f such that there exists a homomorphism $g\colon X \to Y$ with $g \simeq f$?*

This problem has not been studied much and is related to a problem, so-called Freyd conjecture, in homotopy theory.

Now we give some basic properties of H-spaces.

Proposition 3.3.3 *Let P be an H-space. Let Q be a space and let $f\colon Q \to P$ be a pointed map. Suppose that f has a left pointed homotopy inverse. Then Q is an H-space.*

Proof. Let $r\colon Q \to P$ be a left pointed homotopy inverse of f, that is $r \circ f \simeq \mathrm{id}_Q$. Define a multiplication $\mu_Q\colon Q \times Q \to Q$ by the composite

$$Q \times Q \xrightarrow{f\times f} P \times P \xrightarrow{\mu_P} P \xrightarrow{r} Q.$$

Since there is a homotopy commutative diagram

$$\begin{array}{ccccc}
Q \times Q & \xrightarrow{f \times f} & P \times P & \xrightarrow{\mu} & P \\
\uparrow & & \uparrow & & \| \\
Q \vee Q & \xrightarrow{f \vee f} & P \vee P & \xrightarrow{\nabla} & P \\
\| & & \downarrow{\scriptstyle r \vee r} & & \downarrow{\scriptstyle r} \\
Q \vee Q & = & Q \vee Q & \xrightarrow{\nabla} & Q,
\end{array}$$

μ_Q has a homotopy identity and so Q is an H-space. ♠

Theorem 3.3.4 *If P is a homotopy associative H-space (H-group), then $[X, P]$ is a monoid (group) for any X. Furthermore if P is homotopy commutative, then $[X, P]$ is commutative.*

Proof. The multiplication $\mu\colon P \times P \to P$ induces a function

$$\mu_*\colon [X, P] \times [X, P] \cong [X, P \times P] \to [X, P]$$

for any X. This makes $[X, P]$ to be an H-set. Since μ is homotopy associative, μ_* is associative and so $[X, P]$ is a monoid. Furthermore if μ has a homotopy inverse, then μ_* has in inverse and so $[X, P]$ is a group. ♠

Lemma 3.3.5 *Let $f_0, f_1\colon A \to X$ and $g_0, g_1\colon Y \to B$ be pointed maps. Suppose that $f_0 \simeq f_1$ and $g_0 \simeq g_1$ under pointed homotopies. Assume that A is Hausdorff. Then $g_0^{f_0} \simeq g_1^{f_1}\colon \mathrm{Map}_*(X, Y) \to \mathrm{Map}_*(A, B)$.*

Proof. First we show that
$$\mathrm{id}_Y^{f_0} \simeq \mathrm{id}_Y^{f_1}\colon Y^X \to Y^A$$
Let $F\colon A \times I \to X$ be a homotopy from f_0 to f_1. Then F induces a map
$$\phi\colon \mathrm{Map}(X, Y) \xrightarrow{\mathrm{id}_Y^F} \mathrm{Map}(A \times I, Y) \cong \mathrm{Map}(I \times A, Y) \xrightarrow{\alpha} \mathrm{Map}(I, \mathrm{Map}(A, Y)),$$
where the association α is continuous because I is Hausdorff. Since I is locally compact Hausdorff, the association map
$$\alpha\colon \mathrm{Map}(\mathrm{Map}(X, Y) \times I, \mathrm{Map}(A, Y)) \to \mathrm{Map}(\mathrm{Map}(X, Y), \mathrm{Map}(I, \mathrm{Map}(A, Y)))$$
is onto-to-onto and onto. Thus $\alpha^{-1}(\phi)$ defines a map
$$F' = \alpha^{-1}(\phi)\colon \mathrm{Map}(X, Y) \times I \to \mathrm{Map}(A, Y).$$
The map F' is given as follows:
$$F'(\lambda, t)(a) = \lambda \circ F(a, t)$$
for $\lambda\colon X \to Y$ and $t \in I$. Clearly, F' maps $\mathrm{Map}_*(X, Y) \times I$ into $\mathrm{Map}_*(A, Y)$ with $F'_0 = \mathrm{id}_Y^{f_0}$, $F'_1 = \mathrm{id}_Y^{f_1}$ and $F'(*, t) = *$. Thus $\mathrm{id}_Y^{f_0} \simeq \mathrm{id}_Y^{f_1}$.

Now we show that $g_0^{\mathrm{id}_A} \simeq g_1^{\mathrm{id}_A}\colon Y^A \to B^A$. Let $G\colon g_0 \simeq g_1$ be a pointed homotopy. Consider the map
$$G^{\mathrm{id}_A}\colon \mathrm{Map}(A, Y) \times \mathrm{Map}(A, I) \cong \mathrm{Map}(A, Y \times I) \to \mathrm{Map}(A, B).$$
Let a_0 be the base-point of A. The constant map $A \to \{a_0\}$ induces a map
$$\theta\colon I = \mathrm{Map}(\{a_0\}, I) \to \mathrm{Map}(A, I).$$
Note that $\theta(t)$ is just the constant map from A to $t \in I$ for each t. Let G' be the composite
$$G'\colon \mathrm{Map}(A, Y) \times I \xrightarrow{\mathrm{id} \times \theta} \mathrm{Map}(A, Y) \times \mathrm{Map}(A, I) \xrightarrow{G^{\mathrm{id}_A}} \mathrm{Map}(A, B).$$
Then
$$G'(\lambda, t)(a) = G(a, t).$$

Clearly G' maps $\mathrm{Map}_*(A,Y)\times I$ into $\mathrm{Map}_*(A,B)$, $G'_0 = g_0^{\mathrm{id}_A}$, $G'_1 = g_1^{\mathrm{id}_A}$ and $G'(*,t) = *$. Thus $g_0^{\mathrm{id}_A} \simeq g_1^{\mathrm{id}_A}$ and so

$$g_0^{f_0} = g_0^{\mathrm{id}_A} \circ \mathrm{id}_Y^{f_0} \simeq g_1^{\mathrm{id}_A} \circ \mathrm{id}_Y^{f_1} = g_1^{f_1}$$

and therefore we have the result. ♠

Theorem 3.3.6 *Let P be an H-space (H-group) and let X be a pointed Hausdorff space. Then* $\mathrm{Map}_*(X,P)$ *is an H-space (H-group). In particular, $\Omega^n P$ is an H-space for each $n \geq 0$.*

Proof. The multiplication on $\mathrm{Map}_*(X,P)$ is defined by

$$\mu = \mu_P^{\mathrm{id}_X} \colon P^X \times P^X \cong (P\times P)^X \to P^X.$$

The assertion follows from Lemma 3.3.5

3.3.2 co-H-space

A pointed space X is called a *co-H-space* if there is a comultiplication $\mu' \colon X \to X\vee X$ such that μ' has a homotopy co-identity, that is there is a homotopy commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{(\mathrm{id}_X, c)} & X\vee X & \xrightarrow{(c,\mathrm{id}_X)} & X \\ \| & & \uparrow{\scriptstyle \mu'} & & \| \\ X & = & X & = & X, \end{array}$$

where c is the constant map. μ' is called *homotopy coassociative* if there is a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu'} & X\vee X \\ \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu'\vee \mathrm{id}_X} \\ X\vee X & \xrightarrow{\mathrm{id}_X \vee \mu'} & X\vee X\vee X. \end{array}$$

A *homotopy inverse* is a map $\nu\colon X \to X$ such that the diagram

$$\begin{CD} X \vee X @<{\mu'}<< X @>{\mu'}>> X \vee X \\ @V{(\mathrm{id}, \nu)}VV @| @VV{(\nu, \mathrm{id})}V \\ X @<{c}<< X @>{c}>> X. \end{CD}$$

μ' is called *homotopy cocommutative* if there is a homotopy commutative diagram

$$\begin{CD} X @= X \\ @V{\mu'}VV @VV{\mu'}V \\ X \vee X @>{T}>> X \vee X, \end{CD}$$

where $T(x, y) = (y, x)$. An co-H-space X is called an *co-H-group* if μ is homotopy coassociative with a homotopy inverse.

Let X and Y be co-H-spaces. A map $f\colon X \to Y$ is called a *co-H-map* if there is a homotopy commutative diagram

$$\begin{CD} X @>{\mu'_X}>> X \vee X \\ @V{f}VV @VV{f \vee f}V \\ Y @>{\mu'_Y}>> Y \vee Y. \end{CD}$$

Theorem 3.3.7 *Let X be a pointed Hausdorff space. Suppose that X is a co-H-space (co-H-group) with a comultiplication $\mu'\colon X \to X \vee X$. Then $\mathrm{Map}_*(X, Y)$ is an H-space (H-group) for any Y. In particular, $[X, Y] = \pi_0(\mathrm{Map}_*(X, Y))$ is a monoid (group).*

Proof. The multiplication on Y^X is defined by the composite

$$\mu = \mathrm{id}_Y^{\mu'_X}\colon Y^X \times Y^X \cong Y^{X \vee X} \to Y^X.$$

The assertion follows from Lemma 3.3.5.

Exercise 3.3.2 Let S^1 be identified with $I/\partial I = [0,1]/\{0,1\}$. Show that S^1 is a co-H-group under the comultiplication μ' defined by

$$\mu'(t) = \begin{cases} (2t, *) & 0 \le t \le 1/2 \\ (*, 2t-1) & 1/2 \le t \le 1. \end{cases}$$

and a homotopy inverse ν defined by $\nu(t) = 1 - t$.

By this exercise, we have the following important theorem.

Theorem 3.3.8 *Any loop space ΩX is an H-group. In particular,*

$$\pi_n(X) = \pi_0(\Omega^n X) = \pi_0(\Omega(\Omega^{n-1}(X))$$

is a group for $n \ge 1$.

By the definition, the multiplication on ΩX is induced by the comultiplication $\mu' \colon S^1 \to S^1 \vee S^1$. In other words, $\mu \colon \Omega X \times \Omega X \to \Omega X$ is given by

$$\mu(\lambda, \lambda')(t) = \begin{cases} \lambda(2t) & 0 \le t \le 1/2 \\ \lambda'(2t-1) & 1/2 \le t \le 1. \end{cases}$$

Lemma 3.3.9 *Let $f_0, f_1 \colon X \to A$ and $g_0, g_1 \colon Y \to B$ be pointed maps. Suppose that $f_0 \simeq f_1$ and $g_0 \simeq g_1$ under pointed homotopies. Then*

$$f_0 \wedge g_0 \simeq f_1 \wedge g_1 \colon X \wedge Y \to A \wedge B.$$

Proof. Let $F \colon X \times I \to A$ be a pointed homotopy from f_0 to f_1. Then we have the map

$$F' \colon X \times Y \times I \cong (X \times I) \times Y \xrightarrow{F \times \mathrm{id}_Y} A \times Y \xrightarrow{p} A \wedge Y,$$

where p is the quotient map. Clearly F' factors through $(X \wedge Y) \times I$. Since I is locally compact Hausdorff, the map $p \times \mathrm{id}_I \colon X \times Y \times I \to (X \wedge Y) \times I$ is a quotient map and so F' induces a (pointed) homotopy

$$F'' \colon (X \wedge Y) \times I \to A \wedge Y$$

with $F''_0 = f_0 \wedge \mathrm{id}_Y$ and $F''_1 = f_1 \wedge \mathrm{id}_Y$. Thus $f_0 \wedge \mathrm{id}_Y \simeq f_1 \wedge \mathrm{id}_Y$. Similarly, $\mathrm{id}_A \wedge g_0 \simeq \mathrm{id}_A \wedge g_1$. Thus

$$f_0 \wedge g_0 = (\mathrm{id}_A \wedge g_0) \circ (f_0 \wedge \mathrm{id}_Y) \simeq (\mathrm{id}_A \wedge g_1) \circ (f_1 \wedge \mathrm{id}_Y) = f_1 \wedge g_1$$

and hence the result. ♠

Theorem 3.3.10 *Let X and Y be pointed spaces. Suppose that X is a co-H-space (co-H-group). Then so is $X \wedge Y$.*

Proof. The comultiplication μ' is defined by

$$\mu' \colon X \wedge Y \xrightarrow{\mu'_X \wedge \mathrm{id}_Y} (X \vee X) \wedge Y \cong (X \wedge Y) \vee (X \wedge Y).$$

By Lemma 3.3.9, $X \wedge Y$ is a co-H-space (co-H-group if X is). ♠

Let X be a pointed space. The *n-fold suspension* of X is defined by

$$\Sigma^n X := S^n \wedge X.$$

Note that

$$\Sigma^n X = (S^1 \wedge S^1 \wedge \cdots \wedge S^1) \wedge X = S^1 \wedge \Sigma^{n-1} X$$

if $n \geq 1$ by Theorem 2.8.11. Thus we have

Theorem 3.3.11 *Let X be a pointed space. Then $\Sigma^n X$ is a co-H-group for each $n \geq 1$.*

Now we want to show that $\pi_n(X)$ is abelian for $n \geq 2$.

Lemma 3.3.12 *Let S be an H-set. Suppose that there is a function*

$$\phi \colon S \times S \to S$$

such that

1) *$\phi(x, 1) = x = \phi(1, x)$ for any $x \in S$ and*

2) *$\phi(x_1 x_2, y_1 y_2) = \phi(x_1, y_1)\phi(x_2, y_2)$ for any $x_1, x_2, y_1, y_2 \in S$.*

Then S is a commutative monoid and $\phi(x, y) = xy$ for any $x, y \in S$.

Proof. Let $x * y$ denote $\phi(x, y)$. Since

$$xy = (x * 1)(1 * y) = (x1) * (1y) = x * y,$$

we have $xy = \phi(x, y)$ for any x, y. Since

$$xy = x * y = (1x) * (y1) = (1 * y)(x * 1) = yx,$$

S is commutative. Since

$$x(yz) = x(y * z) = (x * 1)(y * z) = (xy) * (1z) = (xy)z$$

S is associative. Thus S is a commutative monoid. ♠

Suppose that X is a co-H-space and Y is an H-space. Then there are two multiplications on $[X, Y]$, one is induced by the comultiplication $X \to X \vee X$ and another is induced by the multiplication $Y \times Y \to Y$.

Theorem 3.3.13 *Let X be a co-H-space and let Y be an H-space. Then the two multiplication on $[X, Y]$ induced by μ'_X and μ_Y agree and are both associative and commutative.*

Corollary 3.3.14 *Suppose that Y is an H-space. Let X be any pointed space. Then $[\Sigma X, Y]$ is an abelian group. In particular,*

1) $\pi_1(Y)$ is abelian;

2) $\pi_n(Z) = [S^1, \Omega^{n-1}Z]$ is an abelian group for any pointed space Z.

Note. One of differential geometers through the internet has been to asked whether $S^1 \vee S^1$ is homotopy equivalent to a topological group. We will see that $\pi_1(S^1 \vee S^1)$ is a free group of rank 2, that is, two generators. In particular, $\pi_1(S^1 \vee S^1)$ is not abelian and so $S^1 \vee S^1$ is not an H-space or $S^1 \vee S^1$ is not homotopy equivalent to a topological group.

Exercise 3.3.3 Let μ_1 and μ_2 be two multiplications on Y such that Y is an H-space under μ_1 and μ_2. Show that

$$\Omega\mu_1 \simeq \Omega\mu_2 \colon \Omega(Y \times Y) \to \Omega Y.$$

Proof of Theorem 3.3.13. The multiplication on $[X, Y]$ induced by μ'_X is given as follows: For $[f], [g] \in [X, Y]$, $[f][g]$ is the homotopy class represented by the composite

$$X \xrightarrow{\mu'} X \vee X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\nabla} Y,$$

where ∇ is the fold map. The multiplication $[X, Y]$ induced by μ_Y is given as follows: For $[f], [g] \in [X, Y]$, $[f] * [g]$ is the homotopy class represented by the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{\mu} Y.$$

By Lemma 3.3.12, it suffices to show that

$$([f][f']) * ([g][g']) = ([f] * [g])([f'] * [g'])$$

for any f, f', g, g'. This follows from the following homotopy commutative diagram

$$\begin{array}{ccccc}
X & = & X & = & X \\
\downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \Delta} \\
X \vee X & = & X \vee X & \hookrightarrow & X \times X \\
\downarrow{\scriptstyle \Delta \vee \Delta} & & \downarrow{\scriptstyle \mu' \vee \mu'} & & \downarrow{\scriptstyle \mu' \times \mu'} \\
(X \times X) \vee (X \times X) & \xleftarrow{\phi} & X \vee X \vee X \vee X & \hookrightarrow & (X \vee X) \times (X \vee X) \\
\downarrow{\scriptstyle (f \times g) \vee (f' \vee g')} & & \downarrow{\scriptstyle f \vee f' \vee g \vee g'} & & \downarrow{\scriptstyle (f \vee f') \times (g \vee g')} \\
(Y \times Y) \vee (Y \times Y) & \xleftarrow{\phi} & Y \vee Y \vee Y \vee Y & \hookrightarrow & (Y \vee Y) \times (Y \vee Y) \\
\downarrow{\scriptstyle \mu \vee \mu} & & \downarrow{\scriptstyle \nabla \vee \nabla} & & \downarrow{\scriptstyle \nabla \times \nabla} \\
Y \vee Y & & Y \vee Y & \hookrightarrow & Y \times Y \\
\downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \mu} \\
Y & = & Y & = & Y,
\end{array}$$

where

$$\phi\colon Z \vee Z \vee Z \vee Z \to (Z \times Z) \vee (Z \times Z)$$

is the composite

$$Z \vee Z \vee Z \vee Z \xrightarrow{\mathrm{id}_Z \vee T \vee \mathrm{id}_Z} Z \vee Z \vee Z \vee Z \hookrightarrow (Z \times Z) \vee (Z \times Z)$$

with $T(x, y) = (y, x)$. ♠

3.3.3 The James Construction

Let X be a pointed (Hausdorff) space with the base-point $*$. The *James Construction* $J_n(X)$ is defined by

$$J_n(X) = X^n/\sim,$$

where $\sim$ is the equivalence relation generated by

$$(x_1, \cdots, x_{i-1}, *, x_{i+1}, \cdots, x_n) \sim (x_1, \cdots, x_{j-1}, *, x_{j+1}, \cdots, x_n)$$

for any $1 \leq i, j \leq n$ and any $x_s \in X$. The elements in $J_n(X)$ is written as a word

$$w = x_1 x_2 \cdots x_n,$$

where we just keep in mind that, for example,

$$*x_1x_2 = x_1 * x_2 = x_1x_2*$$

in $J_3(X)$.

Let $q_n\colon X^n \to J_n(X)$ be the quotient map. The inclusion $X^{n-1} \hookrightarrow X^n$, $(x_1, \cdots, x_{n-1}) \to (x_1, \cdots, x_{n-1}, *)$, induces a map $i_n\colon J_{n-1}(X) \hookrightarrow J_n(X)$ such that the diagram

$$\begin{array}{ccc} X^{n-1} & \hookrightarrow & X^n \\ \downarrow{\scriptstyle q_{n-1}} & & \downarrow{\scriptstyle q_n} \\ J_{n-1}X & \overset{i_n}{\hookrightarrow} & J_n(X) \end{array}$$

commutes. We show that i_n is a closed map. Let C be a closed set in $J_{n-1}(X)$. Then $q_{n-1}^{-1}(C)$ is a closed set in X^{n-1} and so

$$E_i = \{(x_1, \cdots, x_{i-1}, *, x_{i+1}, \cdots, x_n) \in X^n | (x_1, \cdots, x_{i_1}, x_{i+1}, \cdots, x_n) \in q_{n-1}^{-1}(C)\}$$

is a closed set in X^n because E^i is the intersection of $X^{i-1} \times * \times X^{n-i}$ and $\pi_i^{-1}(q_{n-1}^{-1}(C))$, where

$$\pi_i(x_1, \cdots, x_n) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$

is the coordinate projection. Since

$$q_n^{-1}(i_n(C)) = \bigcup_{i=1}^{n} E_i,$$

$q_n^{-1}(C)$ is a closed set in X^n and so $i_n(C)$ is a closed set in $J_n(X)$. Thus i_n maps $J_{n-1}(X)$ homeomorphically onto the closed subspace $i_n(J_{n-1}(X))$ in $J_n(X)$ and so we may identify $J_{n-1}(X)$ as a closed subspace of $J_n(X)$. This gives a tower of closed spaces

$$J_1(X) \subseteq J_2(X) \subseteq J_3(X) \subseteq \cdots.$$

Define

$$J(X) = \bigcup_{n=1}^{\infty} J_n(X)$$

with so-called *weak topology*, that is, C is a closed set in $J(X)$ if and only if $C \cap J_n(X)$ is closed in $J_n(X)$ for each n. This makes $J(X)$ to be a topological spaces and each $J_n(X)$ is a closed subspace of $J(X)$.

An exact definition of weak topology is as follows. Let X be a space and let $\{A_\alpha\}$ be a family of closed set in whose union is X. We say X has the *weak topology* with respect to $\{A_\alpha\}$ if it satisfies the following condition: a subset C of X, whose intersection with each of A_α is closed, is itself closed. Let X be a set, and let $\{A_\alpha\}$ be a family of topological spaces, each a subset of X. We shall say that $\{A_\alpha\}$ is a *coherent family* (of topological spaces) on X if

1) $X = \bigcup_\alpha A_\alpha$;

2) $A_\alpha \cap A_\beta$ is a closed set of A_α for each α, β;

3) for every α, β, the topologies induced on $A_\alpha \cap A_\beta$ by A_α and A_β coincide.

Let A_α be a coherent family on X. Define a subset C of X to be closed if $C \cap A_\alpha$ is closed for each α. Then

1) X is a topological space (that is the completements of the closes sets form a topology on X);

2) Each A_α is a closed subspace of X;

3) X has the weak topology with respect to $\{A_\alpha\}$.

Lemma 3.3.15 *Suppose X has the weak topology with respect to $\{A_\alpha\}$. Let U be a subset in X. Then U is open if and only if $U \cap A_\alpha$ is open for each α.*

Proof. If U is open, clearly $U \cap A_\alpha$ is open because A_α is a subspace. Conversely, assume that $U \cap A_\alpha$ is open for each α. Then

$$(X \setminus U) \cap A_\alpha = A_\alpha \setminus (U \cap A_\alpha)$$

is closed in A_α for each α and so $X \setminus U$ is closed or U is open. ♠

Lemma 3.3.16 *Suppose X has the weak topology with respect to $\{A_\alpha | \alpha \in I\}$ and Y has the weak topology with respect to $\{B_\beta | \beta \in J\}$. Then $X \times Y$ has the weak topology with respect to*

$$\{A_\alpha \times B_\beta | \alpha \in I, \beta \in J\}.$$

Proof. Let C be a subset in $X \times Y$ such that $C \cap (A_\alpha \times B_\beta)$ is closed for any α, β. Let $U = X \times Y \setminus C$. We show that U is open. Since

$$U \cap A_\alpha \times B_\beta = A_\alpha \times B_\beta \setminus (C \cap A_\alpha \times B_\beta),$$

$U \cap A_\alpha \times B_\beta$ is open for any α and β. Let $\phi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the coordinate projections. Given any $\alpha \in I$, then

$$\pi_X(U) \cap A_\alpha = \bigcup_{\beta \in J} \pi_X(U \cap A_\alpha \times B_\beta).$$

Since $U \cap A_\alpha \times B_\beta$ is open in $A_\alpha \times B_\beta$ and $\pi_X|_{A_\alpha \times B_\beta}$ is the first coordinate projection,

$$\pi_X(U \cap A_\alpha \times B_\beta)$$

is open in A_α for each β and so the union

$$\pi_X(U) \cap A_\alpha$$

is open in A_α for any given α. It follows that $\pi_X(U)$ is open in X because X has the weak topology. Similarly, $\pi_Y(U)$ is open in Y. Thus U is open in $X \times Y$ and hence the result. ♠.

Theorem 3.3.17 *Let X be a pointed locally compact Hausdorff space. Then $J(X)$ is a topological monoid.*

Proof. The composite

$$X^n \times X^m = X^{n+m} \xrightarrow{q_{n+m}} J_{n+m}(X)$$

factors through $J_n(X) \times J_m(X)$, that is there is a map $\mu_{n,m}: J_n(X) \times J_m(X) \to J_{n+m}(X)$ such that the diagram

$$\begin{array}{ccc} X^n \times X^m & = & X^{n+m} \\ \downarrow{\scriptstyle q_n \times q_m} & & \downarrow{\scriptstyle q_{n+m}} \\ J_n(X) \times J_m(X) & \xrightarrow{\mu_{n,m}} & J_{n+m}(X), \end{array}$$

where $q_m \times q_n$ is a quotient map because X is locally compact Hausdorff. By writing down the elements, we have

$$\mu_{n,m}(x_1x_2\cdots x_n, y_1y_2\cdots y_m) = x_1x_2\cdots x_ny_1y_2\cdots y_m$$

and so $\mu_{m,n}$ induces a unique function $\mu: J(X)\times J(X) \to J(X)$ such that the diagram

$$\begin{array}{ccc} J_n(X) \times J_m(X) & \xrightarrow{\mu_{n,m}} & J_{n+m}(X) \\ \cap\downarrow & & \cap\downarrow \\ J(X) \times J(X) & \xrightarrow{\mu} & J(X) \end{array}$$

commutes for any n, m. We show that μ is continuous. Let C be any closed set in $J(X)$. For any n, m, $C \cap J_{n+m}(X)$ is closed and so

$$\mu^{-1}(C) \cap (J_n(X) \times J_m(X)) = \mu^{-1}_{n,m}(C \cap J_{n+m}(X))$$

is closed. By Lemma 3.3.16, $J(X) \times J(X)$ has the weak topology with respect to $\{J_m(X) \times J_n(X)\}$. Thus C is closed and hence μ is continuous. Clearly μ has the identity $* = 1$ and is associative. Thus $J(X)$ is a topological monoid. ♠

We write $X^{(n)}$ for the n-fold self smash of X.

Theorem 3.3.18 *There is an homeomorphism*

$$J_n(X)/J_{n-1}(X) \cong X^{(n)}$$

for each n.

Proof. Let $q_n\colon X^n \to J_n(X)$ and $p_n\colon X^n \to X^{(n)}$ be the quotient maps. Then p_n factors $J_n(X)$, that is, there is a function $p'_n\colon J_n(X) \to X^{(n)}$ such that $p_n = p'_n \circ q_n$. It follows that p'_n is quotient map. Since $p'_n(J_{n-1}(X)) = *$, p'_n induces a quotient map

$$p''_n\colon J_n(X)/J_{n-1}(X) \to X^{(n)}.$$

The map p''_n is a homeomorphism because it is one-to-one, onto and a quotient map. ♠

One of applications of the James construction to H-spaces is as follows.

Theorem 3.3.19 *Let X be a pointed space. Then X is an H-space with a strict identity if and only if X is a (pointed) retract of a topological monoid.*

Proof. Suppose that X is a retract of a topological monoid M. Let $j\colon X \to M$ be the inclusion with $j(*) = 1$ and let $r\colon M \to X$ be a retraction with $r(1) = *$. Define a multiplication on X by

$$X \times X \xhookrightarrow{j\times j} M \times M \xrightarrow{\mu} M \xrightarrow{r} X.$$

Then $*x = x* = x$ for $x \in X$. Conversely, suppose that there is a multiplication $\mu\colon X \times X \to X$ with a strict identity. We write $x \cdot y$ for $\mu(x, y)$. Define a map

$$\phi_n\colon X^n \to X$$

by

$$\phi_n(x_1, x_2, \cdots, x_n) = ((\cdots((x_1 \cdot x_2) \cdot x_3) \cdots) \cdot x_n).$$

Since $* = 1$ is a strict identity for μ, the map ϕ_n factors through the quotient $J_n(X)$, that is there is a map $\phi'_n\colon J_n(X) \to X$ such that the diagram

$$\begin{array}{ccc} X^n & \xrightarrow{\phi_n} & X \\ \downarrow{\scriptstyle q_n} & & \| \\ J_n(X) & \xrightarrow{\phi'_n} & X \end{array}$$

commutes. Clearly

$$\phi'_n|_{J_{n-1}(X)} = \phi'_{n-1}\colon J_{n-1}(X) \to X$$

and so ϕ'_n induces a unique function $\phi'\colon J(X) \to X$ such that

$$\phi'_n = \phi'|_{J_n(X)}.$$

We show that ϕ' is continuous. Let C be a closed set in X. Then

$$\phi'^{-1}(C) \cap J_n(X) = \phi_n'^{-1}(C)$$

is a closed set in $J_n(X)$ for each n. Thus $\phi'(C)$ is closed because $J(X)$ has the weak topology with respect to $\{J_n(X)\}$. Since

$$\phi'|_{J_1(X)} = \phi_1' \colon J_1(X) = X \to X$$

is the identity map, the map ϕ' is a retraction and hence the result because $J(X)$ is a topological monoid. ♠

Note. If $*$ is non-degenerate, that is $* \to X$ is a cofibration, then X is an H-space with a homotopy identity if and only if X is an H-space with a strict identity. (See Proposition 3.3.1) Thus suppose that $*$ is non-degenerate, then X is an H-space if and only if X is a retract of a topological monoid.

Note. It is known that $J(X) \simeq \Omega\Sigma X$ if X is a path-connected CW-complex. (For this reason, $J(X)$ is known as a 'combinatorial model' for loop suspensions. For instance, $J(S^1) \simeq \Omega S^2$.) Thus suppose that X is a path-connected CW-complex with a non-degenerate base-point, then X is an H-space is and only if X is a retract of a loop space.

3.4 The fundamental Group

3.4.1 The fundamental Groupoid

Let λ and μ be two paths in X with $\lambda(1) = \mu(0)$. Then the product $\lambda * \mu$ is defined by

$$(\lambda * \mu)(t) = \begin{cases} \lambda(2t) & 0 \le t \le 1/2 \\ \mu(2t-1) & 1/2 \le t \le 1. \end{cases}$$

Two paths λ and λ' are briefly said to be *homotopic*, denoted by $\lambda \simeq \lambda'$, if they are homotopic relative to $\partial I = \{0, 1\}$. Note that if $\lambda \simeq \lambda'$, then $\lambda(0) = \lambda'(0)$ and $\lambda(1) = \lambda'(1)$.

Lemma 3.4.1 *Let $\lambda_0, \lambda_1, \mu_0, \mu_1$ are paths in X with $\lambda_0(1) = \mu_0(0)$ and $\lambda_1(1) = \mu_1(0)$. If $\lambda_0 \simeq \lambda_1$ and $\mu_0 \simeq \mu_1$, then $\lambda_0 * \mu_0 \simeq \lambda_1 * \mu_1$.*

Proof. Let $F \colon lambda_0 \simeq \lambda_1$ and $G \colon \mu_0 \simeq \mu_1$ be the homotopies relative to ∂I. Then $H \colon I \times I \to X$ defined by

$$H(t, s) = \begin{cases} F(2t, s) & 0 \le t \le 1/2 \\ G(2t-1, s) & 1/2 \le t \le 1 \end{cases}$$

We show that ϕ' is continuous. Let C be a closed set in X. Then

$$\phi'^{-1}(C) \cap J_n(X) = \phi_n'^{-1}(C)$$

is a closed set in $J_n(X)$ for each n. Thus $\phi'(C)$ is closed because $J(X)$ has the weak topology with respect to $\{J_n(X)\}$. Since

$$\phi'|_{J_1(X)} = \phi_1' \colon J_1(X) = X \to X$$

is the identity map, the map ϕ' is a retraction and hence the result because $J(X)$ is a topological monoid. ♠

Note. If $*$ is non-degenerate, that is $* \to X$ is a cofibration, then X is an H-space with a homotopy identity if and only if X is an H-space with a strict identity. (See Proposition 3.3.1) Thus suppose that $*$ is non-degenerate, then X is an H-space if and only if X is a retract of a topological monoid.

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3.4 The fundamental Group

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Two paths λ and λ' are briefly said to be *homotopic*, denoted by $\lambda \simeq \lambda'$, if they are homotopic relative to $\partial I = \{0, 1\}$. Note that if $\lambda \simeq \lambda'$, then $\lambda(0) = \lambda'(0)$ and $\lambda(1) = \lambda'(1)$.

Lemma 3.4.1 *Let $\lambda_0, \lambda_1, \mu_0, \mu_1$ are paths in X with $\lambda_0(1) = \mu_0(0)$ and $\lambda_1(1) = \mu_1(0)$. If $\lambda_0 \simeq \lambda_1$ and $\mu_0 \simeq \mu_1$, then $\lambda_0 * \mu_0 \simeq \lambda_1 * \mu_1$.*

Proof. Let $F \colon lambda_0 \simeq \lambda_1$ and $G \colon \mu_0 \simeq \mu_1$ be the homotopies relative to ∂I. Then $H \colon I \times I \to X$ defined by

$$H(t, s) = \begin{cases} F(2t, s) & 0 \le t \le 1/2 \\ G(2t-1, s) & 1/2 \le t \le 1 \end{cases}$$

is a homotopy relative to ∂I between $\lambda_0 * \mu_0$ and $\lambda_1 * \mu_1$. ♠

Lemma 3.4.2 *Suppose that $\lambda_0, \lambda_1, \lambda_2$ are paths in X with $\lambda_0(1) = \lambda_1(0)$ and $\lambda_1(1) = \lambda_2(0)$. Then $(\lambda_0 * \lambda_1) * \lambda_2 \simeq \lambda_0 * (\lambda_1 * \lambda_2)$.*

Proof. The map $F: I \times I \to X$ defined by

$$F(t,s) = \begin{cases} \lambda_0((4t)/(1+s)) & 0 \leq t \leq (s+1)/4, \\ \lambda_1(4t-s-1) & (s+1)/4 \leq t \leq (s+2)/4, \\ \lambda_2((4t-s-2)/(2-s)) & (s+2)/4 \leq t \leq 1; \end{cases}$$

is a homotopy relative to ∂I between $(\lambda_0 * \lambda_1) * \lambda_2$ and $\lambda_0 * (\lambda_1 * \lambda_2)$.

For each $x \in X$, we define $\epsilon_x: I \to X$ as the constant path with $\epsilon_x(t) = x$ for any t.

Lemma 3.4.3 *Let λ be in path in X with $\lambda(0) = x$ and $\lambda(1) = y$. Then $\epsilon_x * \lambda \simeq \lambda$ and $\lambda * \epsilon_y \simeq \lambda$.*

Proof. The map $F: I \times I \to X$ defined by

$$F(t,s) = \begin{cases} x & 0 \leq t \leq (1-s)/t, \\ \lambda((2t-1+s)/(1+s)) & (1-s)/2 \leq t \leq 1; \end{cases}$$

is a homotopy relative to ∂I between $\epsilon_x * \lambda$ and λ. The map $G: I \times I \to X$ defined by

$$G(t,s) = \begin{cases} \lambda(\frac{2}{1+s}t) & 0 \leq t \leq \frac{1+s}{2} \\ y & \frac{1+s}{2} \leq t \leq 1. \end{cases}$$

is a homotopy relative to ∂I between $\lambda * \epsilon_y$ and λ. ♠

Given a path λ in X, the inverse λ^{-1} is defined by $\lambda^{-1}(t) = \lambda(1-t)$.

Lemma 3.4.4 *Let λ be a path in X with $\lambda(0) = x$ and $\lambda(1) = y$. Then $\lambda * \lambda^{-1} \simeq \epsilon_x$ and $\lambda^{-1} * \lambda \simeq \epsilon_y$.*

Proof. The map $F: I \times I \to X$ defined by

$$F(t,s) = \begin{cases} \lambda(2t(1-s)) & 0 \leq t \leq 1/2, \\ \lambda((2-2t)(1-s)) & 1/2 \leq t \leq 1; \end{cases}$$

is a homotopy relative to ∂I between $\lambda * \lambda^{-1}$ and ϵ_x. Similarly $\lambda^{-1} * \lambda \simeq \epsilon_y$. ♠

A category is called *small* if the class of objects is a set. A *groupoid* is a small category in which every morphism is an equivalence. Let X be a space. Let category $\mathcal{P}(X)$ is defined by:

the objects in $\mathcal{P}(X)$ are points in X and morphisms from x to y are path classes from x to y. The composite operation is defined by $[\mu] \circ [\lambda] = [\lambda * \mu]$ for a path λ from x to y and a path μ from y to z.

By the lemmas above, we have

Theorem 3.4.5 *Let X be a space. Then $\mathcal{P}(X)$ is a groupoid.*

3.4.2 Change of Base

Let X be a space with $x \in X$. Consider x is the basepoint of X. Then $\pi_1(X, x) = \pi_1(X)$ is called the *fundamental group* of X with base point x. Recall that $\pi_1(X, x)$ is a group, where the multiplication is given by the path multiplication. Note that the fundamental group depends on the choice of the base point x.

Theorem 3.4.6 *Let $x, y \in X$. If there is a path in X from x to y, then the groups $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic.*

Proof. Let λ be a path from x to y, that is $\lambda(0) = x$ and $\lambda(1) = y$. Define a function

$$\chi_\lambda \colon \pi_1(X, x) \to \pi_1(X, y)$$

by

$$\chi_\lambda([\mu]) = [\lambda^{-1} * \mu * \lambda].$$

This is a homomorphism of groups because

$$\chi_\lambda([\mu][\mu']) = [\lambda^{-1} * \mu * \mu' * \lambda] = [\lambda^{-1} * \mu * \lambda * \lambda^{-1} * \mu' * \lambda]$$
$$= [\lambda^{-1} * \mu\lambda][\lambda^{-1} * \mu * \lambda] = \chi_\lambda([\mu])\chi_\lambda([\mu']).$$

λ^{-1} is path from y to x and so

$$\chi_{\lambda^{-1}} \colon \pi_1(X, y) \to \pi_1(X, x).$$

For $\mu \in \pi_1(X, x)$, we have

$$\chi_{\lambda^{-1}} \circ \chi_\lambda([\mu]) == [\lambda * \lambda^{-1}\mu * \lambda * \lambda^{-1}] = [\mu]$$

and so $\chi_{\lambda^{-1}} \circ \chi_\lambda = \mathrm{id}$. Similarly $\chi_\lambda \circ \chi_{\lambda^{-1}} = \mathrm{id}$. Thus χ_λ is an isomorphism of groups. ♠

Let $f: X \to Y$ be a map. Then f induces a homomorphism of groups

$$f_*: \pi_1(X, x) = [S^1, X] \to \pi_1(Y, f(x))[S^1, Y].$$

If $f \simeq g \operatorname{rel} x$, then

$$f_* = g_*: \pi_1(X, x) \to \pi_1(Y, y),$$

where $y = f(x) = g(x)$. If $X \simeq Y$ relative the base-point, then $\pi_1(X) \cong \pi_1(Y)$.

Exercise 3.4.1 *Prove that if there is a path in X from x_0 to x_1, then $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic.*

3.4.3 The fundamental Group of a Circle

The map $\mathrm{e}: \mathbb{R} \to S^1$ is defined by

$$\mathrm{e}(t) = \exp^{2\pi i t}.$$

Then e is continuous, $\mathrm{e}(t_1 + t_2) = \mathrm{e}(t_1)\,\mathrm{e}(t_2)$ and $\mathrm{e}(t_1) = \mathrm{e}(t_2)$ if and only if $t_1 - t_2$ is an integer. It follows that $\mathrm{e}\,|_{(-1/2,1/2)}$ is a homeomorphism of the open interval $(-1/2, 1/2)$ onto $S^1 \setminus \{\exp(\pi i)\}$. Let

$$\log: S^1 \setminus \{\exp(\pi i)\} \to (-1/2, 1/2)$$

be the inverse of $\mathrm{e}\,|_{(-1/2,1/2)}$.

A subset $X \subseteq \mathbb{R}^n$ is called *starlike* from a point x_0 if, whenever $x \in X$, the closed segment $[x_0, x]$ from x_0 to x lies in X.

Lemma 3.4.7 *Let X be compact and starlike from $x_0 \in X$. Given any continuous map $f: X \to S^1$ and any $t_0 \in \mathbb{R}$ such that $\mathrm{e}(t_0) = f(x_0)$, there exists a continuous map $\tilde{f}: X \to \mathbb{R}$ such that $\tilde{f}(x_0) = t_0$ and $\mathrm{e} \circ \tilde{f}(x) = f(x)$ for all $x \in X$.*

Proof. Clearly we can translate X so that it is starlike from the origin; hence there is no loss of generality in assuming $x_0 = 0$. Since X is compact, f is uniformly continuous and there exists $\epsilon > 0$ such that if $\|x - x'\| < \epsilon$, then $\|f(x) - f(x')\| < 2$ [that is, $f(x)$ and $f(x')$ are not antipodes in S^1]. Since X is bounded, there exists a positive integer n such that $\|x\|/n < \epsilon$ for all $x \in X$. Then for each $0 \le j < n$ and all $x \in X$

$$\|\frac{(j+1)x}{n} - \frac{jx}{n}\| = \frac{\|x\|}{n} < \epsilon$$

and so

$$\|f\left(\frac{(j+1)x}{n}\right) - f\left(\frac{jx}{n}\right)\| < 2.$$

It follows that the quotient $f((j+1)x/n)/f(jx/n)$ is a point of $S^1 \setminus \{\exp(\pi i)\}$. Let $g_j: X \to S^1 \setminus \{\exp(\pi i)\}$ for $0 \leq j < n$ be the map defined by

$$g_j(x) = \frac{f((j+1)x/n)}{f(jx/n)}.$$

Then for all $x \in X$, we see that

$$f(x) = f(0)g_0(x)g_1(x)\cdots g_{n-1}(x).$$

We define $\tilde{f}: X \to \mathbb{R}$ by

$$\tilde{f}(x) = t_0 + \log(g_0(x)) + \log(g_1(x)) + \cdots + \log(g_{n-1}(x)).$$

Since f' is the sum of $n+1$ continuous functions from X to $\mathbb{R}$, it is continuous. Clearly $\tilde{f}(0) = t_0$ and $\mathrm{e} \circ \tilde{f} = f$. ♠

Lemma 3.4.8 *Let X be a connected space and let $\tilde{f}, \tilde{g}: X \to \mathbb{R}$ be maps such that $\mathrm{e} \circ \tilde{f} = \mathrm{e} \circ \tilde{g}$ and $\tilde{f}(x_0) = \tilde{g}(x_0)$ for some $x_0 \in X$. Then $\tilde{f} = \tilde{g}$.*

Proof. Let $h = \tilde{f} - \tilde{g}: X \to \mathbb{R}$. Since $\mathrm{e} \circ \tilde{f} = \mathrm{e} \circ \tilde{g}$, $\mathrm{e} \circ h$ is the constant map of X to $1 \in S^1$. Thus h is a continuous map from X to $\mathbb{R}$, taking only integral values. Because X is connected, h is constant, and since $h(x_0) = 0$, $h(x) = 0$ for all $x \in X$. ♠

Let $\alpha: I \to S^1$ be a closed path at 1. Because I is starlike from 0 and $\alpha(0) = 1 = \mathrm{e}(0)$, it follows from Lemmas 3.4.7 and 3.4.8 there exists a unique lifting $\tilde{\alpha}: I \to \mathbb{R}$ such that $\tilde{\alpha}(0) = 0$ and $\mathrm{e} \circ \tilde{\alpha} = \alpha$. Because $\mathrm{e}(\tilde{\alpha}(1)) = \alpha(1) = 1$, it follows that $\tilde{\alpha}(1)$ is an integer. We define the degree of α by

$$\deg(\alpha) = \tilde{\alpha}(1).$$

Lemma 3.4.9 *Let α and β be homotopic closed paths in S^1 at 1. Then* $\deg(\alpha) = \deg(\beta)$.

Proof. Let $F: I \times I \to S^1$ be a homotopy relative to ∂I from α to β. Because $I \times I$ is a starlike set of $\mathbb{R}^2$ from $(0,0)$, it follows that there is a (unique) lifting $\tilde{F}: I \times I \to \mathbb{R}$ such that $\tilde{F}(0,0) = 0$ and $\mathrm{e} \circ \tilde{F} = F$. Since F is a homotopy relative to ∂I, $F(0,t) = F(1,t) = 1$ for all $t \in I$. Thus $\tilde{F}(0,t)$ and $\tilde{F}(1,t)$ take on only integral values for all $t \in I$. It follows that $\tilde{F}(0,t)$ must be constant and $\tilde{F}(1,t)$ must be constant. Because $\tilde{F}(0,0) = 0$, $\tilde{F}(0,t) = 0$ for all t. Let $\tilde{\alpha}, \tilde{\beta}: I \to \mathbb{R}$ be the maps defined by $\tilde{\alpha}(t) = \tilde{F}(t,0)$ and $\tilde{\beta}(t) = \tilde{F}(t,1)$. Then $\tilde{\alpha}(0) = \tilde{\beta}(0) = 0$, $\mathrm{e} \circ \tilde{\alpha} = \alpha$ and $\mathrm{e} \circ \tilde{\beta} = \beta$. Thus

$$\deg(\alpha) = \tilde{\alpha}(1) = \tilde{F}(1,0) = \tilde{F}(1,t) = \tilde{F}(1,1) = \tilde{\beta}(1) = \deg(\beta). \spadesuit$$

It follows that there is a well-defined function deg from $\pi_1(S^1, 1)$ to $\mathbb{Z}$ defined by

$$\deg([\alpha]) = \deg(\alpha).$$

Theorem 3.4.10 *The function* deg *is an isomorphism of groups*

$$\deg: \pi_1(S^1, 1) \cong \mathbb{Z}.$$

Proof. To prove that deg is a homomorphism, let α and β be two closed paths in S^1 at 1 and let $\alpha\beta$ be the closed path which is their pointwise product in the group multiplication of S^1. We know from Theorem 3.3.13 that $[\alpha] * [\beta] = [\alpha\beta]$. Let $\tilde{\alpha}, \tilde{\beta}: I \to \mathbb{R}$ be such that $\tilde{\alpha}(0) = \tilde{\beta}(0) = 0$, $\mathrm{e} \circ \tilde{\alpha} = \alpha$ and $\mathrm{e} \circ \tilde{\beta} = \beta$. Let $\tilde{\gamma} = \tilde{\alpha} + \tilde{\beta}: I \to \mathbb{R}$. Then $\tilde{\gamma}(0) = 0$ and $\mathrm{e}(\tilde{\gamma}) = \alpha\beta$. Thus

$$\deg([\alpha] * [\beta]) = \deg([\alpha\beta]) = \tilde{\gamma}(1) = \tilde{\alpha}(1) + \tilde{\beta}(1) = \deg([\alpha]) + \deg([\beta]).$$

The map deg is an epimorphism: For any integer n, let $\tilde{\alpha}: I \to \mathbb{R}$ be the path defined by $\tilde{\alpha}(t) = tn$ and let $\alpha = \mathrm{e} \circ \tilde{\alpha}: I \to S^1$. Then clearly $\deg([\alpha]) = n$.

The map deg is a monomorphism: If $\deg([\alpha]) = 0$, then there is a path $\tilde{\alpha}: I \to \mathbb{R}$ with $\tilde{\alpha}(0) = \tilde{\alpha}(1) = 0$ and $\mathrm{e} \circ \tilde{\alpha} = \alpha$. Since $\mathbb{R}$ is contractible, $\tilde{\alpha} \simeq \epsilon_0$ and

$$\alpha = \mathrm{e} \circ \tilde{\alpha} \simeq \mathrm{e} \circ \epsilon_0 = \epsilon_1. \spadesuit$$

Exercise 3.4.2 *Show that the map $f: S^1 \to S^1$, $z \to z^n$ is of degree n.*

Corollary 3.4.11 *The fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$.*

Theorem 3.4.12 (The Fundamental Theorem of Algebra) *Every non-constant complex polynomial has a root.*

Proof. We may assume without loss of generality that our polynomial has the form

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

with $n \geq 1$. Assume that p has no zero.

Let S_r be the circle $|z| = r$ of radius r. Choose $r >> 0$ such that

$$r^n > |a_1|r^{n-1} + |a_2|r^{n-2}| + \cdots + |a_{n-1}|r + |a_n|.$$

Let $F\colon S_r \times I \to \mathbb{C}$ be the map defined by

$$F(z,t) = z^n + t(a_1 z^{n-1} + \cdots + a_n).$$

Since

$$|F(z,t)| \geq |z|^n - t(|a_1||z|^{n-1} + \cdots + |a_n|) > 0$$

for $|z| = r$ and $0 \leq t \leq 1$, the image of F lies in $\mathbb{C} \setminus \{0\}$. Let

$$G\colon S^1 \times I \to S^1$$

be the composite

$$S^1 \xrightarrow{rz} S_r \xrightarrow{\frac{F(z,t)}{F(r,t)}} \mathbb{C} \setminus \{0\} \xrightarrow{\frac{z}{|z|}} S^1.$$

Then $G(1,t) = (F(r,t)/F(r,t))/|F(r,t)/F(r,t)| = 1$ for $t \in I$, $G(z,0) = z^n$ and

$$G(z,1) = \frac{p(rz)}{p(r)} \frac{|p(r)|}{|p(rz)|}.$$

Thus $f(z) = (|p(r)|/(p(r)|p(rz)|))p(rz) \simeq z^n$ is of degree n.

Let $H\colon S^1 \times I \to S^1$ be the map defined by

$$H(z,t) = \frac{p(rzt}{p(rt)} \frac{|p(rt)|}{|p(rzt)|},$$

where H is well-defined (and so it is continuous) because $p(z)$ is never zero. Then $H(1,t) = 1$ for all t, $H(z,0) = 1$ and $H(z,1) = f(z)$. It follows that $f(z)$ is of degree 0, which is a contradiction (unless $n = 0$).

Theorem 3.4.13 (Brouwer Fixed Point Theorem) *Any continuous map $f\colon D^2 \to D^2$ has a fixed point, that is a point x such that $f(x) = x$.*

Proof. Suppose that $x \neq f(x)$ for all $x \in D^2$. Then we may define a map $\phi\colon D^2 \to S^1$ by setting $\phi(x)$ to be the point on S^1 obtained from the intersection of the line segment from $f(x)$ to x extended to meet S^1. Let $i\colon S^1 \to D^2$ be the inclusion. Then $\phi \circ i = \mathrm{id}_{S^1}$. Thus there is a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z} = \pi_1(S^1) & = & \mathbb{Z} = \pi_1(S^1) \\
\downarrow i_* & & \uparrow \phi_* \\
0 = \pi_1(D^2) & = & 0 = \pi_1(D^2),
\end{array}$$

which is impossible. This contradiction proves the result. ♠

Exercise 3.4.3 Show that $\pi_n(S^1) = 0$ for $n > 1$. (Hint: Let $q\colon I^n \to S^n = I^n/\partial I^n$ the pinch map. Let $f\colon S^n \to S^1$ be any map. Consider $f \circ q\colon I^n \to S^1$. Since I^n is starlike, there is a unique lifting $\alpha\colon I^n \to \mathbb{R}$ such that $\alpha(0) = 0$ and $\mathrm{e} \circ \alpha = f \circ q$. Since

$$\mathrm{e} \circ \alpha(x) = f \circ q(x) = f(*) = 1$$

for $x \in \partial I^n$, $\mathrm{e} \circ \alpha|_{\partial I^n}$ is the constant map and so $\alpha|_{\partial I^n}$ is a continuous map from ∂I^n to integers. It follows that $\alpha|_{\partial I^n}$ is a constant map because $\partial I^n \cong S^{n-1}$ is path-connected when $n > 1$. Since $\alpha(0) = 0$, $\alpha(x) = 0$ for $x \in \partial I^n$ and so α induces a map $\bar{\alpha}\colon S^n = I^n/\partial I^n \to \mathbb{R}$. Since $\mathrm{e} \circ \alpha = f \circ q$, we have $\mathrm{e} \circ \bar{\alpha} = f$. Since $\mathbb{R}$ is contractible, $\bar{\alpha} \simeq \epsilon_0$ and so

$$f = \mathrm{e} \circ \alpha \simeq \mathrm{e} \circ \epsilon_0 = \epsilon.$$

This show that any map from S^n to S^1 is null homotopic and so $\pi_n(S^1) = 0$.)

3.4.4 Simply Connected Spaces

A space X is said to be *n-connected* for $n \geq 0$ if every continuous map $f\colon S^k \to X$ for $k \leq n$ has a continuous extension over E^{k+1}. A 1-connected space is also said to *simply connected*. Note that if $0 \leq m \leq n$, an n-connected space is m-connected. It follows from Theorem 3.1.16 that a space X is n-connected if and only if it is path-connected and $\pi_k(X, x)$ is trivial for every base point $x \in X$ and $1 \leq k \leq n$. By Exercise 3.4.1, X is n-connected if and only if it is path-connected and $\pi_k(X, x_0) = 0$ for $1 \leq k \leq n$ and any particular choice of base point x_0. Note that X is 0-connected if and only if X is path-connected. By Exercise 3.1.5, we have

Lemma 3.4.14 *A contractible space is n-connected for every $n \geq 0$.*

Exercise 3.4.4 Let λ and μ be paths in X from x to y. Suppose that X is simply connected. Then $\lambda \simeq \mu$.

Lemma 3.4.15 *Suppose that $X = U \cup V$ with U, V open and simply connected and $U \cap V$ non-empty and path connected. Then X is simply connected.*

Proof. Let f be any path in X. Then $f^{-1}(U)$ is an open set of I and so $f^{-1}(U)$ is a disjoint union of open intervals. Let

$$f^{-1}(U) = \bigcup_{\alpha} (a_\alpha, b_\alpha)$$

be a disjoint union of open intervals (a_α, b_α). Since $f^{-1}(V)$ is open in I,

$$f^{-1}(V) = \bigcup_{\beta} (c_\beta, d_\beta).$$

Since

$$I = \bigcup_{\alpha,\beta} (a_\alpha, b_\alpha) \cup (c_\beta, d_\beta)$$

and I is compact, there exists a finite subcover

$$I = \bigcup_{i=1}^{m} (a_i, b_i) \cup \bigcup_{j=1}^{n} (c_j, d_j).$$

It follows that there are finite numbers

$$t_1 = 0 < t_2 < \cdots < t_q = 1$$

such that $[t_s, t_{s+1}]$ is either contained in (a_i, b_i) for some i or in (c_j, d_j) for some j. Let

$$f_s(t) = f(t_s + t(t_{s+1} - t_s)).$$

Then f_s is a path that starts with $f(t_s)$ and ends with $f(t_{s+1})$. If $[t_s, t_{s+1}] \subseteq (a_i, b_i)$ for some i, then $f_s(I) = f([t_s, t_{s+1}]) \subseteq U$, that is f_s is a path in U. Otherwise, $[t_s, t_{s+1}] \subseteq (c_j, d_j)$ for some j and f_s is a path in V. It follows that

$$f = f_1 * f_2 * \cdots * f_q,$$

where f_s is either in U or V and so

$$[f] = [f_1][f_2] \cdots [f_q].$$

We show by induction that

If f is a loop with $f(0) = f(1) \in U \cap V$ such that $[f] = [f_1][f_2]\cdots[f_q]$ with f_j is either a path in U or a path in V, then $[f] = 0$.

The assertion will follow from this statement.

If $q = 1$, then $[f] = [f_1]$ and so f_1 must be a loop. If f_1 is a loop in U, then $[f_1] = 0$ because U is simply connected and so $[f] = 0$. Otherwise f_1 is a loop in V and $[f] = [f_1] = 0$ because V is simply connected. Assume that the statement holds for $< q$. Let $[f] = [f_1]\cdots[f_q]$. We may assume that f_1 is a path in U without loss of generality. Let $i \geq 1$ be the largest integer such that f_j is a path in U for $j \leq i$. Then $f_1 * f_2 * \cdots * f_i$ is a path in U and f_{i+1} is a path in V. It follows that

$$f_i(1) = f_{i+1}(0) \in U \cap V.$$

Since $U \cap V$ is path connected, there is a path λ in $U \cap V$ from $f(0)$ to $f_i(1)$. Since U is simply connected, and $f_1 * \cdots * f_i$ and λ are paths in U from $f(0)$ to $f_i(1)$, we have

$$[f_1]\cdots[f_i] = [\lambda]$$

and so

$$[f] = [\lambda][f_{i+1}][f_{i+2}]\cdots[f_q] = [\lambda * f_{i+1}][f_{i+2}]\cdots[f_q] = 0$$

by induction, where $\lambda * f_{i+1}$ is a path in V. By induction. ♠

Corollary 3.4.16 *S^n is simply connected for $n \geq 2$.*

Exercise 3.4.5 Let X be a space. The *unreduced suspension* $\Sigma^u X$ is the quotient space of $I \times X$ obtained by identifying $0 \times X$ to a point and $1 \times X$ to a (different) point. Suppose that X is path-connected. Show that $\Sigma^u X$ is simply connected.

Note: $\Sigma X = \Sigma^u X / I \times *$. If $I \times * \to \Sigma^u X$ is a cofibration (this is true if $* \to X$ is a cofibration), then $\Sigma^u X \simeq \Sigma X$. Thus if $* \to X$ is a cofibration and X is path-connected, then ΣX is simply connected.

3.5 The Seifert-Van Kampen Theorem

In this section, we provide a useful theorem for calculations of fundamental groups.

3.5.1 Free Groups and Free Products of groups

Let X be a set. The *free group* $F(X)$ generated by X is a group that satisfies the following universal property:

1) $X \subseteq F(X)$ is a subset.

2) Let G be any group and let $f\colon X \to G$ be any function. There exists a unique homomorphism of groups $\tilde{f}\colon F(X) \to G$ such that $\tilde{f}|_X$.

It is known that for any X $F(X)$ exists and unique up to isomorphism. There is an explicit construction of the free group $F(X)$ in terms of words:

$$w = x_1^{\epsilon_1} \cdots x_k^{\epsilon_k},$$

where $x_j \in X$ and $\epsilon_j = \pm 1$. For instance, if $X = \{x_1, \cdots, x_n\}$, the words on X are given by

$$x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k}$$

for $k \leq 0$, $\epsilon_j = \pm 1$ and $1 \leq i_j \leq n$. A word w is called *reduced* if for each $1 \leq j \leq k$ $x_j \neq x_{j+1}$ or $x_j = x_{j+1}$ with $\epsilon_j \neq -\epsilon_{j+1}$. As a set $F(X)$ is given by all reduced words and the multiplication on $F(X)$ is given by the formal product of words, where we use the rule:

$$x_i^{-1} x_i = x_i x_i^{-1} = 1$$

for each i. For example,

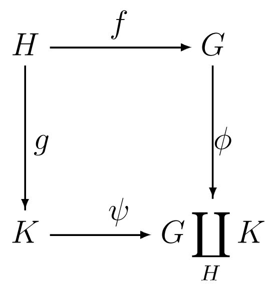
$$(x_1 x_2^{-1} x_3) \cdot (x_3^{-1} x_1) = x_1 x_2^{-1} x_1.$$

Clearly $F(X)$ is NOT a commutative group if X has more than one element because $x_1 x_2$ and $x_2 x_1$ are different words in $F(X)$ for $x_1, x_2 \in X$.

Definition 3.5.1 Let $f\colon H \to G$ and $g\colon H \to K$ be homomorphisms of groups. The *push-out* $G \coprod_H K$ is a group that satisfies the following universal properties:

1) There are homomorphisms of groups $\phi\colon G \to G \coprod_H K$ and $\psi\colon K \to G \coprod_H K$

such that $\phi \circ f = \psi \circ g$, that is the diagram



,

commutes;

2) Let Γ be any group and let $\phi' \colon G \to \Gamma$ and $\psi' \colon K \to \Gamma$ be homomorphisms with $\phi' \circ f = \psi' \circ g$. Then there is a unique homomorphism $\theta \colon G \coprod_H K \to \Gamma$ such that $\phi' = \theta \circ \phi$ and $\psi' = \theta \circ \psi$.

When H is the trivial group, $G \coprod K = G \coprod_{\{1\}} K$ is called the *free product* of G and K. It is known in group theory that the push-out (so-called free product with amalgamation in group theory) always exists. The universal property show that $G \coprod_H K$ must be unique up to isomorphism if it exists. The combinatorial construction $G \coprod_H K$ can be given as follows:

First we construct the free product $G \coprod K$ can be given by the words

$$w = \alpha_1 \cdots \alpha_k,$$

where $\alpha_j \in G$ or K for each j. w is reduced if each $\alpha_j \neq 1$ and α_j and α_{j+1} are not lie in the same group. The product of two reduced words is the reduced words obtained from the formal product of them. For instance, let $\alpha_1, \alpha_2 \in G$ and $\beta_1 \in K$. Then

$$(\alpha_1 \beta_1 \alpha_2)(\alpha_2^{-1} \beta_1^{-1} \alpha_2) = \alpha_1 \alpha_2 \in G.$$

The push-out $G \coprod_H K$ is the quotient group of $G \coprod K$ by the normal subgroup generated by

$$f(h) g(h)^{-1}$$

for $h \in H$. We can check that this construction satisfies the universal property: The homomorphisms $\phi \colon G \to G \coprod_H K$ and $\psi \colon K \to G \coprod_H K$ are canonical map given by $\phi(g)$ is word represented by g and $\psi(k)$ is the word represented by k for $g \in G$ and $k \in K$. By the relation above $\phi \circ f = \psi \circ g$ in the group $\coprod_H K$ (NOT $G \coprod K$).

Assume that $\phi' \colon G \to \Gamma$ and $\psi' \colon K \to \Gamma$ be homomorphisms with $\phi' \circ f = \psi' \circ g$. First there is a unique homomorphism $\theta' \colon G \coprod K \to \Gamma$ such that $\theta'(g) = \phi'(g)$ and $\theta'(k) = \psi'(k)$. Since $\phi' \circ f = \psi' \circ g$, we have that

$$\theta'(f(h)g(h)^{-1}) = 1$$

for $h \in H$ and so θ' induces a unique homomorphism $\theta \colon G \coprod_H K \to \Gamma$ with the desired property.

Example 3.5.2 If K is the trivial group, then $G \coprod K = G$ and so $G \coprod_H K$ is the quotient group of G by the normal subgroup generated by the image of $f \colon H \to G$.

$\mathbb{Z} \coprod \mathbb{Z} = F(x_1, x_2)$ is a free group generated by two generators. In general, the n-fold free product of $\mathbb{Z}$ is a free group of rank n, that is n free generators.

$\mathbb{Z}/m \coprod \mathbb{Z}/n$ is the quotient group of $F(x_1, x_2)$ by the relations:

$$x_1^m = 1, x_2^n = 1.$$

3.5.2 The Seifert-Van Kampen Theorem

Theorem 3.5.3 *Let X be a space. Suppose that $X = U_1 \cup U_2$ such that U_1, U_2 are open and $U_1 \cap U_2$ is non-empty and path connected. Let $x_0 \in U_1 \cap U_2$ be a base-point of X. Then*

$$\pi_1(X, x_0) = \pi_1(U_1, x_0) \coprod_{\pi_1(U_1 \cap U_2, x_0)} \pi_1(U_2, x_0).$$

Sketch of Proof. Let $j^1 \colon U_1 \to X$, $j^2 \colon U_2 \to X$, $i^1 \colon U_1 \cap U_2 \to U_1$ and $i^2 \colon U_1 \cap U_2 \to U_2$ be inclusions. Since $j^1 \circ i^1 = j^2 \circ i^2$, the homomorphisms $j^1_* \colon \pi_1(U_1) \to \pi_1(X)$ and $j^2_* \colon \pi_1(U_2) \to \pi_1(X)$ induces a homomorphism

$$\theta \colon \pi_1(U_1) \coprod_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \to \pi_1(X).$$

Let $\lambda \colon S^1 \to X$ be a loop in X. By the proof of Lemma 3.4.15, we have

$$[\lambda] = [\lambda_1][\lambda_2] \cdots [\lambda_k],$$

where λ_j is a path either in U_1 or U_2. We may assume that λ_1 is a path in U_1. Let i be the largest number such that $\lambda_1, \cdots, \lambda_i$ are paths in U_1. Then $\lambda_i(1) = \lambda_{i+1}(0) \in$

$U_1 \cap U_2$. Since $U_1 \cap U_2$ is path connected, there is a path μ in $U_1 \cap U_2$ from $\lambda(0)$ to $\lambda_i(1) = \lambda_{i+1}(0)$. Then

$$[\lambda] = [\lambda_1 * \cdots * \lambda_i * \mu^{-1}][\mu\lambda_{i+1}][\lambda_{i+2}] \cdots [\lambda_k].$$

Now $\lambda_1 * \cdots * \lambda_i * \mu^{-1}$ is a loop in U_1 and $\mu * \lambda_{i+1}$ is a path in U_2. By repeating this step (one can do this by induction), finally one can written down $[\lambda]$ as a product of elements from $\pi_1(U_1)$ or $\pi_1(U_2)$ and so θ is an epimorphism.

It is more complicated to show that θ is a monomorphism. So we omit this part of proof. ♠

3.5.3 Calculations of the fundamental Group

By Using the Seifert-van Kampen theorem, we can compute the fundamental groups of a lot of spaces.

Example 3.5.4 $\pi_1(S^1 \vee S^1) = F(x_1, x_2)$. *In general,* $\pi_1(\vee^n S^1) = F(x_1, \cdots, x_n)$.

Proof. Let x be an element in S^1 different from the base point. Let $U = S^1 \vee (S^1 \setminus \{x\})$ and $V = (S^1 \setminus \{x\}) \vee S^1$. Then U and V are open sets in $S^1 \vee S^1$. Since $U \simeq S^1$ and $V \simeq S^1$, we have $\pi_1(U) = \mathbb{Z}$ and $\pi_1(V) = \mathbb{Z}$. Now $U \cap V = (S^1 \setminus \{x\}) \vee (S^1 \setminus \{x\})$ is contractible, $\pi_1(U \cap V) = \{1\}$. By the Seifert-van Kampen theorem, we have

$$\pi_1(S^1 \vee S^1) = \mathbb{Z} \coprod_{\{1\}} \mathbb{Z} = F(x_1, x_2).$$

By induction, one can show that $\pi_1(\vee^n S^1) = F(x_1, \cdots, x_n)$. ♠
Note: By this example, we know that $\vee^n S^1$ is NOT an H-space if $n > 1$.

Example 3.5.5 $\pi_1(\mathbb{R}P^1) = \mathbb{Z}$ *and* $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$ *for* $n \geq 2$.

Proof. Clearly $\mathbb{R}P^1 \cong S^1$ and so $\pi_1 \mathbb{R}P^1 = \mathbb{Z}$. Now we compute $\pi_1 \mathbb{R}P^2$.

Recall that $\mathbb{R}P^2/\mathbb{R}P^1 \cong S^2$. Let $x \in \mathbb{R}P^2 \setminus \mathbb{R}P^1$ and let $U = \mathbb{R}P^2 \setminus \{x\}$. Then U is homotopy equivalent to $\mathbb{R}P^1$ and so $\pi_1(U) = \mathbb{Z}$. Let V be an open neighborhood of x that is homeomorphic to the open disk B^2 and is disjoint from $\mathbb{R}P^1$. Then $\pi_1 V = 0$. Clearly $U \cap V \simeq S^1$, $\pi_1(U \cap V) = \mathbb{Z}$. Let $j: U \cap V \to U$ be the inclusion. Then $j_*: \pi_1(U \cap V) \to \pi_1(U)$ is multiple by 2. Thus by the Seifert-van Kampen theorem $\pi_1(\mathbb{R}P^2) = \pi_1(U \cup V) = \mathbb{Z}/2\mathbb{Z}$.

Now we show that $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$ by induction. Assume that $\pi_1(\mathbb{R}P^{n-1}) = \mathbb{Z}/2$ with $n \geq 3$. Let $x \in \mathbb{R}P^n \setminus \mathbb{R}P^{n-1}$. Let $U = \mathbb{R}p^n \setminus \{x\}$. Then $U \simeq \mathbb{R}P^{n-1}$ and

$\pi_1(U) = \mathbb{Z}/2$. Let V be a small neighborhood of x with $V \cong B^n$. Then $\pi_1(V) = 0$. Clearly $U \cap V \simeq S^{n-1}$. Since $n \geq 3$, S^{n-1} is simply connected and so $\pi_1(U \cap V) = 0$. It follows that $\pi_1(\mathbb{R}P^n) = \pi_1(U \cup V) = \mathbb{Z}/2$. ♠

Exercise 3.5.1 Show that $\mathbb{C}P^n$ is simply connected for each $n \geq 1$.

Note: By looking at fundamental groups, we already know that any $\mathbb{R}P^m$ is NOT homeomorphic to $\mathbb{C}P^n$.

Exercise 3.5.2 Let $T_g = T\#T\#\cdots\#T$ be the g-fold connected sum of the torus T. Show that $\pi_1(T_g)$ is the quotient group of the free group $F(c_1, d_1, c_2, d_2, \cdots, c_g, d_g)$ by the one relation:

$$c_1 d_1 c_1^{-1} d_1^{-1} c_2 d_2 c_2^{-1} d_2^{-1} \cdots c_g d_g c_g^{-1} d_g^{-1} = 1.$$

3.5.4 Groups and Spaces

Let X be a space. The *unreduced cone* $CX = I \times X / 1 \times X$. Clearly the cone CX is contractible for any X. There is a relation between groups and so-called 2-complexes.

Lemma 3.5.6 *Let $\phi: F(x_1, \cdots, x_m) \to F(y_1, \cdots, y_n)$ be a homomorphism. Then there is a (continuous) map $f: \vee^m S^1 \to \vee^n S^1$ such that*

$$f_* = \phi: \pi_1(\bigvee^m S^1) = F(x_1, \cdots, x_m) \to \pi_1(\bigvee^n S^1) = F(y_1, \cdots y_n).$$

Proof. The homomorphism ϕ is uniquely determined by the elements $\phi(x_1), \cdots, \phi(x_m)$ in $F(y_1, \cdots, y_n)$. Since $\pi_1(\vee^n S^1) = F(y_1, \cdots, y_n)$, there are maps

$$f_1, \cdots, f_m: S^1 \to \bigvee^n S^1$$

such that $[f_j] = (f_j)_*([\mathrm{id}]) = \phi(x_j)$. Let $f: \vee^m S^1 \to \vee^n S^1$ be the map induced by $f_1, \cdots, f_m$. The $f_* = \phi$. ♠

Let G be a group with generators $x_1, \cdots, x_k$ and relations $R_1, \cdots, R_q$, where each R_j is a word in the free group $F(x_1, \cdots, x_k)$. The group G is the quotient group of $F(x_1, \cdots, x_k)$ by the normal subgroup generated by $R_1, \cdots, R_q$. Now we can construct a space $X = X(G)$ such that $\pi_1(X) = G$ as follows:

First we choose the wedges of circles, $X_1 = \vee^k S^1$ and $X_2 = \vee^q S^1$. Now we define a map $f: \vee^q S^1 \to \vee^k S^1$ such that f restricted to the j-th copy of S^1 is a representative of the element $R_j \in \pi_1(\vee^k S^1) = F(x_1, \cdots, x_k)$. Define

$$X = X_1 \coprod CX_2 / \sim,$$

where $\sim$ is the equivalence relation generated by

$$(0, x) \sim f(x)$$

for $x \in \vee^q S^1$. We show that $\pi_1(X) = G$. Let $x = 1 \times X_2$ be the element in $CX_2 = I \times X_2 / 1 \times X_2$, where $X_2 = \vee^q S^1$. Let

$$U = X \setminus x = X_1 \coprod (CX_2 \setminus \{x\}) / \sim .$$

Then $U \simeq X_1 = \vee^k S^1$ and so $\pi_1(U) = F(x_1, \cdots, x_k)$. Let V be image of $(2/1, 1] \times X_2$ in CX_2. Then V is an open neighborhood of x with $\pi_1(V) = 0$ (V is contractible). Clearly that $U \cap V \simeq X_2 = \vee^q S^1$. Thus $\pi_1(X)$ is the quotient group of $F(x_1, \cdots x_k)$ by the normal subgroup generated by

$$\mathrm{Im}(\pi_1(U \cap V) \to \pi_1(U)) = \mathrm{Im}(f_* \colon \pi_1(\vee^q S^1) \to \pi_1(\vee^k S^1)),$$

which is the normal subgroup generated by $R_1, \cdots R_q$. Thus $\pi_1(X) = G$.

Now let $\phi \colon G \to H$ be a homomorphism. Suppose that G has generators $x_1, \cdots x_k$ with relations $R_1, \cdots R_q$ and H has generators $y_1, \cdots y_s$ with relations $S_1, \cdots S_t$. Then there is a homomorphism $\tilde{\phi} \colon F(x_1, \cdots, x_k) \to F(y_1, \cdots, y_t)$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \uparrow & & \uparrow \\ F(x_1, \cdots, x_k) & \xrightarrow{\tilde{\phi}} & F(y_1, \cdots, y_s) \end{array}$$

commutes. Thus there is a map $f \colon X_1(G) \to X_1(H)$ such that

$$f_* = \tilde{\phi} \colon \pi_1(X_1(G)) \to \pi_1(X_1(H)).$$

Let $j \colon X_1(H) \to X(H)$ be the inclusion. Then the composite

$$\theta \colon X_2(G) \longrightarrow X_1(G) \xrightarrow{f} X_1(H) \hookrightarrow X(H)$$

is null homotopic because its restriction to each copy of S^1 induces the trivial element in the fundamental group of $X(H)$. It follows that there is a map $\tilde{\theta} \colon CX_2(G) \to X(H)$ such that $\tilde{\theta}|_{X_2(G)} = \theta$. Now the map $j \circ f$ and $\tilde{\theta}$ defines a map

$$\bar{f} \colon X(G) = X_1(G) \coprod CX_2(G) / \sim \to X(H).$$

Clearly $\bar{f}_* = \phi \colon \pi_1(X(G)) \to \pi_1(X(H))$. Thus we have the following theorem.

Theorem 3.5.7 *For any group G, there is a group $X(G)$ such that $\pi_1(X(G)) = G$. If $\phi: G \to H$ is a homomorphism, there is a map $f: X(G) \to X(H)$ such that*

$$f_* = \phi: \pi_1(X(G) = G \to \pi_1(X(H)) = H.$$

Note: The space $X(G)$ is not unique (even up to homotopy) because a group G can be written down in terms of different generator-relation systems.

Example 3.5.8 If a group G has only one relation (such group are called *one relator groups*, the construction of $X(G)$ is quite simple which can be described as follows:

Let $x_1, \cdots, x_k$ be generators for G and let $R = x_{i_1}^{\epsilon_1} \cdots x_{i_t}^{\epsilon_t}$ be the only relation for G. We may assume that R is an unreduced word such that all $x_1, \cdots, x_k$ occur in R.

Let Y be a t-sided polygonal region with counter-clockwise orientation. The sides in Y are labeled by $x_{i_1}, \cdots, x_{i_t}$. The j-th side is chosen to be in a positive direction [negative direction] if $\epsilon_j = 1$ [if $\epsilon_j = -1$].

Let X be the quotient space of Y by identifying 1) all vertices to be one point and 2) all oriented sides labeled by the same letter.

We can show that $\pi_1(X) = G$. Let x be an inner point in Y. Let $U = X \setminus \{x\}$. Then $U \simeq \vee^k S^1$. Let V be an open ϵ-neighborhood of x in Y (and so in X). Then $\pi_1(V) = 0$. Clearly $U \cap V \simeq S^1$ and so $\pi_1(U \cap V) = \mathbb{Z}$. Let $j: U \cap V \to U$ be the inclusion and let $\alpha = [\mathrm{id}_S^1]$ be the generator for $\pi_1(U \cap V)$. Then

$$j_*(\alpha) = x_{i_1}^{\epsilon_1} \cdots x_{i_t}^{\epsilon_t}.$$

Thus $\pi_1(X) = G$. ♠

Chapter 4

Covering Spaces

4.1 Covering Spaces

Definition 4.1.1 A map $p\colon \tilde{X} \to X$ is a *covering projection* and $\tilde{X}$ (or $(\tilde{X}, p)$ is a *covering space* of X if

1) p is onto, and

2) for any $x \in X$ there is an open neighbourhood U (called an *elementary neighbourhood*) of x such that
$$p^{-1}(U) = \coprod_{\alpha \in J} U_\alpha$$
is a topological disjoint union of open sets (called *sheets*), each U_α is mapped homeomorphically onto U by p. (So $p^{-1}(U) \cong U \times$ (discrete space.)

Roughly speaking covering space just means that 'locally' the preimage $p^{-1}(U)$ is disjoint union of copies of U.

Example 4.1.2 (1). Any homeomorphism $p\colon \tilde{X} \to X$ is a one-sheeted covering projection.

(2). Let F be a discrete space and $\tilde{X} = X \times F$. Then the coordinate projection $p\colon \tilde{X} \to X$ is a covering projection.

(3). The projection $p\colon S^n \to \mathbb{R}P^n$ is a two-sheeted covering projection.

(4). $p\colon S^1 \to S^1$, $z \mapsto z^n$, is an n-sheeted covering.

(5). The exponential map $\mathrm{e}\colon \mathbb{R} \to S^1$ is a covering with infinite sheets.

Exercise 4.1.1 Let $p\colon \tilde{X} \to X$ and $q\colon \tilde{Y} \to Y$ be covering projections. Show that $p \times q\colon \tilde{X} \times \tilde{Y} \to X \times Y$ is also a covering projection.

Let G be a group and let Y be a G-space. For $g \in G$ and a subset $S \subseteq Y$, let $g \cdot S$ denote the set $\{g \cdot x | x \in S\}$.

Definition 4.1.3 Let G be a (discrete) group and let Y be a G-space. A G-action on Y is called *properly discontinuous* if

> for any $y \in Y$ there exists a neighbourhood W_y such that
>
> $$g_1 \neq g_2 \quad \Rightarrow \quad g_1 \cdot W_y \cap g_2 \cdot W_y = \emptyset$$
>
> (or, equivalently, $g \neq 1 \quad \Rightarrow \quad g \cdot W_y \cap W_y = \emptyset$).

Theorem 4.1.4 *Let X be a G-space. If the G-action on X is properly discontinuous, then $X \to X/G$ is a covering.*

Proof. Let $p\colon X \to X/G$ be the quotient map. By Theorem **??**, p is an open map. For any $x \in X$, let W be an open neighbourhood satisfying the condition of proper discontinuity. Then $p(U)$ is an open neighbourhood of $p(x)$ and

$$p^{-1}(W) = \coprod_{g \in G} g \cdot W$$

is a disjoint union of open subsets of X. Furthermore $p|_{g \cdot W}\colon g \cdot W \to p(W)$ is a continuous open bijective map and hence a homeomorphism. ♠

Exercise 4.1.2 Let X be a G-space. Suppose that $X \to X/G$ is a covering. Show that the G-action on X is properly discontinuous.

Now the next question is how can we know a group-action is properly discontinuous. Recall that a group G acts freely on X if $g \cdot x \neq x$ for all $x \in X$ and $g \in G$ with $g \neq 1$.

Exercise 4.1.3 Let X be a G-space. Suppose that the G-action on X is properly discontinuous. Then G acts freely on X.

Theorem 4.1.5 *Let G be a finite group and let X be a Hausdorff G-space. Then the G-action on X is properly discontinuous if and only if G acts freely on X.*

Proof. $\Rightarrow$ is obvious (see Exercise 4.1.3). $\Leftarrow$ Let $G = \{g_0 = 1, g_1, \cdots, g_n\}$. Since X is Hausdorff, there exist open neighbourhoods $U_0, \cdots U_n$ of $g_0 \cdot x, \cdots, g_n \cdot x$, respectively such that $U_0 \cap U_j = \emptyset$ for $1 \leq j \leq n$. Let $U = \cap_{j=0}^{n} g_j^{-1} \cdot U_j$. Then U is an open neighbourhood of x with $g_j \cdot U \cap U = \emptyset$ for each $1 \leq j \leq n$ because

$$g_j \cdot U = g_j \cdot \bigcap_{i=0}^{n} g_i^{-1} U_i = \bigcap_{i=0}^{n} g_j (g_i^{-1} \cdot U_i)$$

$$= \bigcap_{i=0}^{n} (g_j g_i^{-1}) \cdot U_i \subseteq (g_j g_j^{-1}) \cdot U_j = U_j.$$

Thus the G-action on X is properly discontinuous. ♠

Note: If G has infinite elements, a free G-action may or may not be properly discontinuous. In other words, the quotient $X \to X/G$ may or may not be a covering even if G acts freely on X and X is Hausdorff.

Now we have more examples of covering spaces.

Example 4.1.6 1) Let $\mathbb{Z}$ act on $\mathbb{R}$ by $x \mapsto x+n$. Then this action is discontinuous and so $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ is a covering.

2) Let $\mathbb{Z}^n = \mathbb{Z}^{\oplus n}$ act on $\mathbb{R}^n$ by $(x_1, \cdots, x_n) \mapsto (x_1 + l_1, \cdots, x_n + l_n)$ for $x_j \in \mathbb{R}$ and $l_j \in \mathbb{Z}$. Then this action is discontinuous and so $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = S^1 \times \cdots \times S^1$ is a covering. In particular, when $n = 2$, we have the covering projection $: \mathbb{R}^2 \to T = S^1 \times S^1$.

3) Let p be a prime integer and let $q_1, \cdots q_n$ be integers prime to p. We define a $\mathbb{Z}/p$-action on

$$S^{2n+1} = \{(z_0, \cdots, z_n) \in \mathbb{C}^n \,|\, |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1\}$$

by

$$l \cdot (z_0, \cdots, z_n) = (\mathrm{e}^{2\pi i l/p} z_0, \mathrm{e}^{2\pi i l q_1/p} z_1, \cdots, \mathrm{e}^{2\pi i l q_n/p} z_n).$$

We show that this action is free. Suppose that

$$l \cdot (z_0, \cdots, z_n) = (z_0, \cdots, z_n).$$

Then

$$e^{2\pi i l q_j/p} z_j = z_j$$

for each $0 \leq j \leq n$, where $q_0 = 1$. Since $(z_0, \cdots z_n) \in S^{2n+1}$, there exists $z_{j_0} \neq 0$ for some j_0. It follows that

$$e^{2\pi i l q_{j_0}/p} = 1$$

and so $lq_{j_0} \equiv 0 \bmod p$. Since $q_{j_0} \not\equiv 0 \bmod p$ and p is a prime, $l \equiv 0 \bmod p$, that is l is the identity in $\mathbb{Z}/p$. Thus this action is free.

Since S^{2n+1} is Hausdorff, $S^{2n+1} \to S^{2n+1}/(\mathbb{Z}/p)$ is a covering. The quotient $S^{2n+1}/(\mathbb{Z}/p)$, denoted by $L^n(p, q_1, \cdots, q_n)$, is called a *lens space*. Note that $L^n(2) = \mathbb{R}P^{2n+1}$.

4) Let p be any non-zero integer. We define a $\mathbb{Z}/p$-action on

$$S^{2n+1} = \{(z_0, \cdots, z_n) \in \mathbb{C}^n | |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1\}$$

by

$$l \cdot (z_0, \cdots, z_n) = (e^{2\pi i l/p} z_0, e^{2\pi i l/p} z_1, \cdots, e^{2\pi i l/p} z_n).$$

The argument above show that this action is free. (**Note:** in this case, we do not need to assume that p is a prime.) The quotient $S^{2n+1}/(\mathbb{Z}/p)$ is denoted by $L^n(p)$. Again we have a covering projection $S^{2n+1} \to L^n(p)$.

5) Let M be a manifold and let

$$F(M, n) = \{(x_1, \cdots, x_n) \in M^n | x_i \neq x_j \text{ for } i \neq j\}$$

be a ordered configuration space. Let the symmetric group Σ_n act on $F(M, n)$ by permuting positions. Then $F(M, n) \to F(M, n)/\Sigma_n$ is a covering. The quotient $F(M, n)/\Sigma_n$, denoted by $B(M, n)$, is called the space of *unordered configurations.*

6) Let G be a (Hausdorff) topological group and let H be a finite subgroup of G. Let G/H be the set of left cosets with quotient topology. Then $G \to G/H$ is a covering. (**Note:** One can directly show that $G \to G/H$ is a covering if H is a discrete subgroup of G (without assuming that H is finite).

4.2 The Lifting Theorem For Covering Spaces

If $p\colon \tilde{X} \to X$ is a covering and $f\colon Y \to X$ is a map, then a *lifting* of f is a continuous map $\tilde{f}\colon Y \to \tilde{X}$ such that $f = p \circ \tilde{f}$.

The lifting problem is: Given a map $f\colon Y \to X$.

i) When does there exist a lifting of f?

ii) Must such a lifting be unique?

The 'uniqueness' can be answered as follows.

Lemma 4.2.1 *Let $p\colon \tilde{X} \to X$ be a covering and let $\tilde{f}, \bar{f}\colon Y \to \tilde{X}$ be two lifting of $f\colon Y \to X$. Suppose that Y is connected and $\tilde{f}(y_0) = \bar{f}(y_0)$ for some $y_0 \in Y$. Then $\tilde{f} = \bar{f}$.*

Proof. Let $Y' = \{y \in Y | \tilde{f}(y) = \bar{f}(y)\}$. Then $y_0 \in Z$. It suffices to show that Z is open and closed. (**Note:** A space Y is connected if and only if Y and $\emptyset$ are only open and closed subsets of Y (or, equivalently, Y is not disjoint union of two open subsets). A path-connected space is connected, but a connected space may not be path connected in general.)

First we show that Y' is an open subset of Y. Let $y \in Y'$ and let U be an elementary neighbourhood of $f(y)$ in X. There is a (unique) sheet U_α of $p^{-1}(U)$ such that $\tilde{f}(y) = \bar{f}(y) \in U_\alpha$. Then $\tilde{f}^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\alpha)$ is an open neighbourhood of y. Since $p|_{U_\alpha}\colon U_\alpha \to U$ is a homeomorphism,

$$\tilde{f}|_{\tilde{f}^{-1}(U_\alpha)\cap\bar{f}^{-1}(U_\alpha)} = \bar{f}|_{\tilde{f}^{-1}(U_\alpha)\cap\bar{f}^{-1}(U_\alpha)}.$$

Thus

$$\tilde{f}^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\alpha) \subseteq Y'$$

and so Y' is open.

Now we show that $Y \setminus Y'$ is open. Let $y \in Y \setminus Y'$ and let U be an elementary neighbourhood of $f(y)$ in X. Since $\tilde{f}(y) \neq \bar{f}(y)$, there are two different sheets U_α and U_β of $p^{-1}(U)$ such that $\tilde{f}(y) \in U_\alpha$ and $\bar{f}(y) \in U_\beta$. ($\alpha \neq \beta$ because p restricted to each sheet is a homeomorphism.) Now $\tilde{f}^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\beta)$ is an open neighbourhood of y. Since $U_\alpha \cap U_\beta = \emptyset$, $\tilde{f}(z) \neq \bar{f}(z)$ for any $z \in f^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\beta)$ and so

$$\tilde{f}^{-1}(U_\alpha \cap \bar{f}^{-1}(U_\beta) \subseteq Y \setminus Y'.$$

Thus $Y \setminus Y'$ is open and hence the result. ♠

Corollary 4.2.2 *Suppose that $\tilde{X}$ is connected and $\phi: \tilde{X} \to \tilde{X}$ is a map such that $p \circ \phi = p$. If $\phi(x_1) = x_1$ for some $x_i \in \tilde{X}$, then ϕ is the identity map.*

Proof. Both ϕ and the identity map $\mathrm{id}_{\tilde{X}}$ are liftings of the map $p: \tilde{X} \to X$. Since $\phi(x_1) = \mathrm{id}_{\tilde{X}}(x_1)$, the assertion follows from Lemma 4.2.1. ♠

Let X be a pointed space with a base-point x_0 and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$.

Theorem 4.2.3 (Path-lifting Theorem) *Let $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering. Then*

i) Every path $\lambda: (I, 0) \to (X, x_0)$ has a unique lifting $\tilde{\lambda}: (I, 0) \to (\tilde{X}, \tilde{x}_0)$.

ii) Every map $F: (I \times I, (0,0)) \to (X, x_0)$ has a unique lifting $\tilde{F}: (I \times I, (0,0)) \to (\tilde{X}, \tilde{x}_0)$.

Proof. We already prove the uniqueness of a lifting. So we only need to prove the existence.

i) There exist $0 = t_0 < t_1 < \cdots t_m = 1$ such that $\lambda([t_i, t_{i+1}])$ is contained in some elementary neighborhood of each i. We show that there is a lifting $\tilde{\lambda}_i: [0, t_i] \to \tilde{X}$ of $\lambda|_{[0,t_i]}$ by induction on i. When $i = 0$, $\tilde{\lambda}_0: [0, 0] \to \tilde{X}$ is given by $\tilde{\lambda}(0) = \tilde{x}_0$. Suppose that there is a lifting $\tilde{\lambda}_i: [0, t_i] \to \tilde{X}$. Since $\lambda([t_i, t_{i+1}])$ lies in an elementary neighbourhood. There is a unique lifting $\mu: [t_i, t_{i+1}] \to \tilde{X}$ of $\lambda|_{[t_i,t_{i+1}]}$ such that $\mu(t_i) = \tilde{\lambda}_i(t_i)$ (The map μ is obtained by composing $\lambda|_{[t_i,t_{i+1}]}$ with the inverse homeomorphism to p-restricted-to-the-sheet-containing-$\tilde{\lambda}_i(t_i)$. Let

$$\tilde{\lambda}_{i+1} = \tilde{\lambda}_i \cup \mu: [0, t_{i+1}] \to \tilde{X}.$$

Then $\tilde{\lambda}_{i+1}$ is a lifting of $\lambda|_{[0,t_{i+1}]}$. This gives a construction of $\tilde{\lambda}$ by induction.

ii) The proof essentially follows from the same idea, that is there are sequence $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that F maps each small rectangle $R_{i,j} = [s_i, s_{i+1}] \times [t_j, t_{j+1}]$ into an elementary neighbourhood and then defined $\tilde{F}$ inductively over the rectangles

$$R_{0,0}, R_{0,1}, \cdots, R_{0,n}, R_{1,0}, \cdots. \spadesuit$$

Corollary 4.2.4 (Monodromy Lemma) *Let $\tilde{\lambda}_0, \tilde{\lambda}_1: (I, 0) \to (\tilde{X}, \tilde{x}_0)$ be paths with $p \circ \tilde{\lambda}_0 \simeq p \circ \tilde{\lambda}_1$. Then $\tilde{\lambda}_0 \simeq \tilde{\lambda}_1$. In particular, $\tilde{\lambda}_0(1) = \tilde{\lambda}_1(1)$.*

Proof. Let $\lambda_0 = p \circ \tilde{\lambda}_0$ and $\lambda_1 = p \circ \lambda_1$. Let $F: I \times I \to X$ be a homotopy relative to $\{0, 1\}$ from λ_0 to λ_1. Then there is a unique lifting $\tilde{F}: I \times I \to \tilde{X}$ of F with $\tilde{F}(0, 0) = \tilde{\lambda}_0(0) = \tilde{\lambda}_1(0)$. Then

1) $\tilde{F}(t,0) = \tilde{\lambda}_0(t)$ for any t because both of them are lifting of λ_0 with $\tilde{F}(0,0) = \tilde{\lambda}_0(0)$. And $\tilde{F}(1,0) = \tilde{\lambda}_0(1)$.

2) $\tilde{F}(0,s) = \epsilon_{\tilde{\lambda}_0(0)}$ because both of them are liftings of $F(0,s) = \epsilon_{\lambda(0)}$ with $\tilde{F}(0,0) = \lambda_0(0)$. And $\tilde{F}(0,1) = \tilde{\lambda}_0(0) = \tilde{\lambda}_1(0)$.

3) $\tilde{F}(t,1) = \tilde{\lambda}_1(t)$ because $\tilde{F}(0,1) = \tilde{\lambda}_1(0)$ and both of them are liftings of λ_1. In particular, $\tilde{F}(1,1) = \tilde{\lambda}_1(1)$.

4) $\tilde{F}(1,s) = \epsilon_{\tilde{\lambda}_0(1)}$ because $\tilde{F}(1,0) = \tilde{\lambda}_0(1)$ and both of them are liftings of $\epsilon_{\lambda_0(1)}$.

This show that $\tilde{F}$ is a path homotopy from $\tilde{\lambda}_0$ to $\tilde{\lambda}_1$. ♠

If in Corollary 4.2.4 we consider only loops, then we immediately have

Theorem 4.2.5 *If $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering, then $p_*\colon \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is a monomorphism.*

Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering projection. The function $\psi\colon \pi_1(X, x_0) \to p^{-1}(x_0)$ is defined by $[\alpha] \mapsto \tilde{\alpha}(1)$, where $\tilde{\alpha}\colon (I, 0, 1) \to (\tilde{X}, \tilde{x}_0, \tilde{\alpha}(1))$ is the unique lifting of α as in Theorem 4.2.3. The function ψ is well-defined by the Monodromy Lemma (Corollary 4.2.4).

Exercise 4.2.1 Suppose that $\tilde{X}$ is path-connected. Show that the function $\psi\colon \pi_1(X, x_0) \to p^{-1}(x_0)$ is onto.

Hint: Let $y \in p^{-1}(x_0)$. There is a path β from $\tilde{x}_0$ to y. Let $\alpha = p \circ \beta$. Then $\beta = \tilde{\alpha}$ by the uniqueness of the lifting and so $\psi([\alpha]) = \tilde{\alpha}(1) = \beta(1) = y$.

Theorem 4.2.6 *If $\tilde{X}$ is simply connected, then ψ is a bijection.*

Proof. By Exercise 4.2.1, it suffices to show that ψ is one-to-one.

Suppose that $[\alpha], [\beta] \in \pi_1(X, x_0)$ with $\psi([\alpha]) = \psi([\beta]) = y \in p^{-1}(x_0)$, that is $\tilde{\alpha}(1) = \tilde{\beta}(1) = y$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are the liftings of $[\alpha]$ and β, respectively. Since $\tilde{X}$ is simply connected, $[\tilde{\alpha} * \tilde{\beta}^{-1}] = 1 \in \pi_1(\tilde{X}, \tilde{x}_0)$. Thus

$$[\alpha][\beta]^{-1} = [(p \circ \tilde{\alpha}) * (p \circ \tilde{\beta}^{-1}] = [p \circ (\tilde{\alpha} * \tilde{\beta}^{-1})] = p_*([\tilde{\alpha} * \tilde{\beta}^{-1}]) = p_*(1) = 1.$$

Hence $[\alpha] = [\beta] \in \pi_1(X, x_0)$. ♠

Now suppose that the quotient $p\colon \tilde{X} \to \tilde{X}/G$, $\tilde{x} \mapsto [\tilde{x}]$, is a covering space arising from a properly discontinuous group action. Here we can do much better.

Since $p^{-1}([\tilde{x}_0]) = G \cdot \tilde{x} = \{g \cdot tildex_0 | g \in G\}$, we can identify $p^{-1}([\tilde{x}_0])$ with G by $g \cdot \tilde{x}_0 \leftrightarrow g$. (Recall: $g \cdot \tilde{x}_0 = g' \cdot \tilde{x}_0 \Rightarrow g = g'$ by the properly discontinuous property.)

Theorem 4.2.7 *If $\tilde{X}$ is path-connected, then $\psi: \pi_1(\tilde{X}/G, [\tilde{x}_0]) \to G$ is a group epimorphism with kernel $p_*\pi_1(\tilde{X}, \tilde{x}_0)$.*

Proof. (i) By Exercise 4.2.1, the function ψ is onto.
(ii) To see that it's a homomorphism, recall that the lifting $\tilde{\alpha}: (I, 0, 1) \to (\tilde{X}, \tilde{x}_0, \tilde{\alpha}(1))$ of a loop α representing $[\alpha] \in \pi_1(\tilde{X}/G, [\tilde{x}_0])$ has $\alpha(1) = g_\alpha \cdot \tilde{x}_0$ for some unique $g_\alpha \in G$ (independent of choice of $\alpha \in [\alpha]$.)

Given $[\alpha], [\beta] \in \pi_1(\tilde{X}/G, [\tilde{x}])$, with α, β lifting to $\tilde{\alpha}: (I, 0, 1) \to (\tilde{X}, \tilde{x}_0, g_\alpha \cdot \tilde{x}_0)$, $\tilde{\beta}: (I, 0, 1) \to (\tilde{X}, \tilde{x}_0, g_\beta \cdot \tilde{x}_0)$, note that in general $\tilde{\alpha} * \tilde{\beta}$ is not defined (since $g_\alpha \cdot \tilde{x}_0 \neq \tilde{x}_0$). However the map $g_\alpha \cdot: \tilde{X} \to \tilde{X}$ composes with $\tilde{\beta}$ to give

$$g_\alpha \cdot \tilde{\beta}: (I, 0, 1) \to (\tilde{X}, g_\alpha \cdot \tilde{x}_0, g_\alpha \cdot (g_\beta \cdot \tilde{x}_0))$$

which lifts β (Note $g_\alpha \cdot \tilde{\beta}$ is from $g_\alpha \cdot \tilde{x}_0$ to $g_\alpha \cdot (g_\beta \cdot \tilde{x}_0)$). Thus

$$\tilde{\alpha} * (g_\alpha \cdot \tilde{\beta}): (I, 0, 1) \to (\tilde{X}, \tilde{x}_0, g_\alpha g_\beta \cdot \tilde{x}_0)$$

is well-defined and lifts $\alpha * \beta$. Since this lifting of $\alpha * \beta$ has final point $g_\alpha g_\beta \cdot \tilde{x}_0$, we have $\psi([\alpha * \beta]) = g_\alpha g_\beta$ and hence

$$\psi([\alpha][\beta]) = \psi([\alpha * \beta]) = g_\alpha g_\beta = \psi([\alpha])\psi([\beta]).$$

(iii) If $\psi([\alpha]) = e \in G$, then $\tilde{\alpha}(1) = e \cdot \tilde{x}_0 = \tilde{x}_0$, making $\tilde{\alpha}$ a loop. Hence

$$[\alpha] = [p \circ \tilde{\alpha}] = p_*([\tilde{\alpha}]) \in p_*\pi_1(\tilde{X}, \tilde{x}_0).$$

Conversely, for any $\tilde{\alpha}: (I, \partial I) \to (\tilde{X}, \tilde{x}_0)$, $p \circ \tilde{\alpha}$ has lifting $\tilde{\alpha}$ with $\tilde{\alpha}(1) = e \cdot \tilde{x}_0$, and so $\psi(p_*([\tilde{\alpha}])) = \psi([p \circ \tilde{\alpha}]) = e \in G$. ♠

Corollary 4.2.8 *Suppose that $\tilde{X}$ is path-connected space on which the group G acts properly discontinuously. Then*

$$\psi: \pi_1(\tilde{X}/G, \tilde{x}_0) \longrightarrow G$$

is an isomorphism if and only if $\tilde{X}$ is simply-connected.

Example 4.2.9 1) Since S^n is simply connected for $n \geq 2$, we have

$$\pi_1(\mathbb{R}P^n) = \pi_1(S^n/\mathbb{Z}/2) = \mathbb{Z}/2$$

for $n \geq 2$.

2) $\pi_1(L^n(p)) = \pi_1(S^{2n+1}/\mathbb{Z}/p) = \mathbb{Z}/p$ $(n \geq 1)$.

3) $\pi_1(S^1) = \pi_1(\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$.

A space X is said to be *locally path-connected* if, for each $x \in X$ and any open neighbourhood U of x, there is an open neighbourhood V of x such that $x \in V \subseteq U$ and any two points in V can be connected by a path in U.

Theorem 4.2.10 (Lifting Theorem) *Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering space. Let $f\colon (Y, y_0) \to (X, x_0)$ be a map. Suppose that Y is path-connected and locally path-connected. Then $f\colon (Y, y_0) \to (X, x_0)$ admits a unique lifting $\tilde{f}\colon (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ if and only if*

$$f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0).$$

Sketch of Proof. $\Rightarrow$ is obvious. $\Leftarrow$ By Lemma 4.2.1, if f admits a lifting, then the lifting is unique. Thus it suffices to prove the existence of the lifting. The construction of $\tilde{f}$ is as follows:

> For each $y \in Y$, since Y is path-connected, there is a path $\alpha\colon (I, 0, 1) \to (Y, y_0, y)$. So lift $f \circ \alpha\colon (I, 0) \to (X, x_0)$ uniquely (by Theorem 4.2.3) to $\overline{f \circ \alpha}\colon (I, 0) \to (\tilde{X}, \tilde{x}_0)$. Let
>
> $$\tilde{f}(y) = \overline{f \circ \alpha}(1).$$
>
> Then $p \circ \tilde{f} = f$.

We must prove that

i) $\tilde{(}y)$ is independent of choice of $\alpha\colon (I, 0, 1) \to (Y, y_0, y)$, that is $\tilde{f}$ is well-defined as a function, and

ii) $\tilde{f}$ is continuous.

We omit this part of the proof. ♠

Corollary 4.2.11 *Any maps from a simply-connected locally path-connected (Y, y_0) lifts (uniquely).*

Corollary 4.2.12 *Any map from $(S^n, (1, 0, \cdots, 0))$ lifts uniquely ($n \geq 2$).*

Corollary 4.2.13 *For $n \geq 2$, $p_*\colon \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism.*

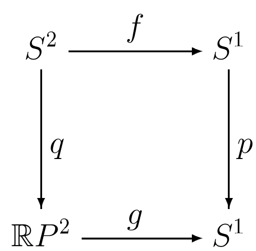
Proof. By Corollary 4.2.12, p_* is onto. By Corollary 4.2.11, p_* is one-to-one because $S^n \times I$ is simply connected for $n \geq 2$.

Theorem 4.2.14 (Borsuk-Ulam) *There exists no map $f\colon S^2 \to S^1$ such that $f(-x) = -f(x)$ for any x.*

Proof. Let $q\colon S^2 \to \mathbb{R}P^2$ be the covering projection, and suppose that for all $x \in S^2$

$$f(-x) = -f(x).$$

Then we can define $g\colon \mathbb{R}P^2 \to S^1$ by $g(\pm x) = (f(x))^2$, making $g \circ q = p \circ f$, where $p\colon S^1 \to S^1$ is defined by $z \mapsto z^2$.



Since $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2$, $g_*\pi_1(\mathbb{R}P^2)$ is a torsion subgroup of $\pi_1(S^1) = \mathbb{Z}$ and hence $g_*\pi_1(\mathbb{R}P^2) = 0$. Thus, by Theorem **??**, there is a lifting $\tilde{g}\colon \mathbb{R}P^2 \to S^1$ such that $g = p \circ \tilde{g}$. (Note the map p is a covering.) Since $\tilde{g} \circ q$ and f are two liftings of $g \circ q$, we have

$$\tilde{g} \circ q = f.$$

It follows that

$$f(x) = \tilde{g} \circ q(x) = \tilde{g} \circ q(-x) = f(-x) = -f(x),$$

a contradiction. ♠

Corollary 4.2.15 *If $g\colon S^2 \to \mathbb{R}^2$ is an antipode-preserving map, that is $g(-x) = -g(x)$, then some $x \in S^2$ has $g(x) = 0$.*

Proof. Otherwise $f\colon S^2 \to S^1 \quad x \to \frac{g(x)}{\|g(x\|}$ contradicts Theorem 4.2.14.

Corollary 4.2.16 *If $h\colon S^2 \to \mathbb{R}^2$, then some $x \in S^2$ has $h(x) = h(-x)$; so h is not injective.*

Proof. If this were not the case, then $g\colon S^2 \to \mathbb{R}^2 \quad x \mapsto h(x) - h(-x)$ would contradicts Corollary 4.2.15. ♠

Corollary 4.2.17 *No subspace of* $\mathbb{R}^2$ *is homeomorphic to* S^2.

Example 4.2.18 Regard the Earth as S^2 and the functions

$P\colon S^2 \to \mathbb{R}$, $x \mapsto$ barometric pressure at x,

$T\colon S^2 \to \mathbb{R}$, $x \mapsto$ temperature at x

as continuous. Then Corollary 4.2.16 says that

$$h\colon S^2 \to \mathbb{R}^2 \quad h(x) = (P(x), T(x))$$

has $h(-x) = h(x)$ for some $x \in S^2$, in other words, there are always two antipodal places on Earth with the same temperature and pressure. ♠