Lecture Notes On Advanced Calculus II

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Contents

CHAPTER 1

Sequences of Real Numbers

1. Sequences

A **sequence** is an ordered list of numbers. For example,

$$1, 2, 3, 4, 5, 6$$

The order of the sequence is important. For example,

$$2, 1, 4, 3, 6, 5$$

is different from above sequence. An **infinite** sequence is a list which does not end. For example,

$$1, 1/2, 1/3, 1/4, 1/5, \cdots$$

We are going to study infinite sequences. We denote by $\{a_n\}$ the sequence

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

EXAMPLE 1.1. *Here are some examples of infinite sequences.*

(1). $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$

(2). $\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}, \cdots$

(3). $1, -2, 3, -4, 5, \cdots$

Can you find a formula for each of the above sequences?
Answer: (1). $a_n = 1/n$. (2). $a_n = 1/3^n$. (3). $(-1)^{n-1}n$.

2. Limits of Sequences

DEFINITION 2.1. The **limit** of $\{a_n\}$ is A, and is written as

$$\lim_{n\to\infty} a_n = A,$$

if for any $\epsilon > 0$, there is a natural number N such that for every $n > N$, we have

$$|a_n - A| < \epsilon.$$

Remark. 1. Some sequences do not satisfy the above. We call such sequences **divergent**.
2. Sequences which satisfy the above definition, i.e. A exists and is finite, are called **convergent** sequences.

EXAMPLE 2.2. *Prove the following limits by using $\epsilon - N$ definition*

1) $\lim_{n\to\infty} \frac{1}{n} = 0$.

2) $\lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+1}} = 1.$

3) $\lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0.$

SOLUTION. (1). Given any $\epsilon > 0$, we want to find N such that $\left|\frac{1}{n} - 0\right| < \epsilon$ for $n > N$, i.e., $n > \frac{1}{\epsilon}$ for $n > N$. Choose N to be the smallest integer such that $N \geq \frac{1}{\epsilon}$. ($N$ is found now!) When $n > N$, then $n > N \geq \frac{1}{\epsilon}$ or $\left|\frac{1}{n} - 0\right| < \epsilon$. Thus $\lim_{n\to\infty} \frac{1}{n} = 0$.

(2). Given any $\epsilon > 0$, we want to find N such that $\left|\sqrt{\frac{n^2}{n^2+1}} - 1\right| < \epsilon$ for $n > N$. Now

$$\left|\sqrt{\frac{n^2}{n^2+1}} - 1\right| < \epsilon \Leftrightarrow \left|\frac{n}{\sqrt{n^2+1}} - 1\right| < \epsilon \Leftrightarrow \left|\frac{n - \sqrt{n^2+1}}{\sqrt{n^2+1}}\right| < \epsilon$$

$$\Leftrightarrow \left|\frac{n^2 - (n^2+1)}{\sqrt{n^2+1}(n + \sqrt{n^2+1})}\right| < \epsilon \Leftrightarrow \frac{1}{\sqrt{n^2+1}(n + \sqrt{n^2+1})} < \epsilon$$

$$\Leftrightarrow \sqrt{n^2+1}(n + \sqrt{n^2+1}) > \frac{1}{\epsilon}$$

Observe that

$$\sqrt{n^2+1}(n + \sqrt{n^2+1}) > n$$

for $n \geq 1$. Choose N to be the smallest integer such that $N \geq \frac{1}{\epsilon}$. Then, for $n > N$,

$$\sqrt{n^2+1}(n + \sqrt{n^2+1}) > \sqrt{N^2+1}(N + \sqrt{N^2+1}) > N \geq \frac{1}{\epsilon}$$

or $\left|\sqrt{\frac{n^2}{n^2+1}} - 1\right| < \epsilon$. Thus N is found and hence the result.

(3). Given any $\epsilon > 0$, we want to find N such that $\left|\left(\frac{3}{4}\right)^n - 0\right| < \epsilon$ for $n > N$. Observe that

$$\left(\frac{3}{4}\right)^n < \epsilon \Leftrightarrow n \ln\left(\frac{3}{4}\right) < \ln(\epsilon) \Leftrightarrow n > \frac{\ln(\epsilon)}{\ln(3/4)}$$

(**Note.** $\ln(3/4) < 0$!!) Choose N to be the smallest positive integer such that $N \geq \frac{\ln(\epsilon)}{\ln(3/4)}$. When $n > N$, then

$$n > N \geq \frac{\ln(\epsilon)}{\ln(3/4)}$$

or $\left|\left(\frac{3}{4}\right)^n - 0\right| < \epsilon$. The proof is finished. □

THEOREM 2.3. *If $\{a_n\}$ has a limit, then the limit is unique.*

PROOF. Let A and B be limits of $\{a_n\}$. Suppose that $A \neq B$. Choose $\epsilon = \frac{|A-B|}{2}$. Then $\epsilon > 0$ because $A \neq B$. By definition, there exists N_1 and N_2 such that $|a_n - A| < \epsilon$ for $n > N_1$ and $|a_n - B| < \epsilon$ for $n > N_2$. For $n > \max\{N_1, N_2\}$, we have

$$|A-B| = |(A-a_n)+(a_n-B)| \leq |A-a_n| + |a_n-B| < 2\epsilon = 2\frac{|A-B|}{2} = |A-B|,$$

which is a contradiction. Thus $A = B$. □

THEOREM 2.4 (Squeeze or Sandwich Theorem). *Given 3 sequences*

$$\{a_n\}, \{b_n\}, \{c_n\}$$

such that

(i) $a_n \leq b_n \leq c_n$ *for every n and*
(ii) $\lim_{n\to\infty} a_n = A = \lim_{n\to\infty} c_n$,

then $\lim_{n\to\infty} b_n = A$.

PROOF. For any $\epsilon > 0$, there exists N_1 and N_2 such that $|c_n - A| < \epsilon$ for $n > N_1$ and $|a_n - A| < \epsilon$ for $n > N_2$. Let $N = \max\{N_1, N_2\}$. For $n > N$, we have

$$-\epsilon < c_n - A < \epsilon \quad \text{and} \quad -\epsilon < a_n - A < \epsilon$$

$$A - \epsilon < c_n < A + \epsilon \quad \text{and} \quad A - \epsilon < a_n < A + \epsilon.$$

Thus

$$A - \epsilon < a_n \leq b_n \leq c_n < A + \epsilon \quad \text{or} \quad |b_n - A| < \epsilon.$$

By definition, we have $\lim_{n\to\infty} b_n = A$ and hence the result. □

Remark. The above theorem is still applicable if the inequality

$$a_n \leq b_n \leq c_n$$

is true **eventually**.

EXAMPLE 2.5. *Find limits*

1) $\lim_{n\to\infty} \frac{1+\sin n}{n}$.
2) $\left(\frac{3n-1}{4n+1}\right)^n$.

SOLUTION. (1). Since

$$0 \leq \frac{1+\sin n}{n} \leq \frac{2}{n}$$

and $\lim_{n\to\infty} \frac{2}{n} = \lim_{n\to\infty} 0 = 0$, we have

$$\lim_{n\to\infty} \frac{1+\sin n}{n} = 0.$$

(2). Since

$$0 \leq \left(\frac{3n-1}{4n+1}\right)^n \leq \left(\frac{3}{4}\right)^n$$

and $\lim_{n\to\infty} \left(\frac{3}{4}\right)^n = \lim_{n\to\infty} 0 = 0$, we have

$$\lim_{n\to\infty} \left(\frac{3n-1}{4n+1}\right)^n = 0.$$

□

3. Sequences which tend to ∞

DEFINITION 3.1. $\{a_n\}$ *tends to* $+\infty$ *if for each positive number* k*, there is an* N *such that*

$$a_n > k \quad \textit{for all } n > N.$$

Remark. For such sequences, we write as $a_n \to +\infty$ as $n \to \infty$ or

$$\lim_{n\to\infty} a_n = +\infty.$$

EXAMPLE 3.2. The following sequences tend to $+\infty$

1) $a_n = \sqrt{\ln n}$.
2) $a_n = (3/2)^n$.

The sequences $-\ln n$, $-n^2$ and etc then tend to $-\infty$.

THEOREM 3.3 (Reciprocal Rule). *Consider a sequence* $\{a_n\}$.

(i) *If* $a_n > 0$ *for all* n *and* $\lim_{n\to\infty} \frac{1}{a_n} = 0$*, then*

$$\lim_{n\to\infty} a_n = +\infty.$$

(ii) *If* $\lim_{n\to\infty} a_n = \pm\infty$*, then* $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

PROOF. We only prove (i). For each positive integer k, there exists N such that

$$\left|\frac{1}{a_n} - 0\right| < \frac{1}{k}$$

for $n > N$ because $\lim_{n\to\infty} \frac{1}{a_n} = 0$. Then, for $n > N$, $a_n > k$ because $a_n > 0$. This finishes the proof. □

EXAMPLE 3.4. Since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, we have $\lim_{n\to\infty} \sqrt{n} = \infty$. Similarly, since $\lim_{n\to\infty} \sqrt{n} = +\infty$, we have $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

4. Techniques For Computing Limits

THEOREM 4.1. *Let f be a continuous function. Then*

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n).$$

IDEA OF PROOF. By the definition of continuity, when $x \to x_0$, $f(x) \to f(x_0)$. Now $\lim_{n\to\infty} a_n = A$ means that $a_n \to A$ when $n \to \infty$. Thus $f(a_n) \to f(A)$ when $n \to \infty$, that is,

$$\lim_{n\to\infty} f(a_n) = f(A) = f(\lim_{n\to\infty} a_n).$$

□

EXAMPLE 4.2.

$$\lim_{n\to\infty} \sin\left(\frac{n\pi}{2n+1}\right) = \lim_{n\to\infty}\left(\frac{\pi}{2+1/n}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

THEOREM 4.3 (L'Hopital's Rule). *Suppose $a_n = f(n)$, $b_n = g(n)$. If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is of the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$, then*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

History Remark. Although the theorem is named after Marquis de l'Hospital (1661-1704), it should be called Bernoulli's rule. The story is that in 1691, l'Hopital asked Johann Bernoulli (1667-1748) to provide, for a fee, lectures on the new subject of calculus. L'Hopital subsequently incorporated these lectures into the first calculus text, *L'Analyse des infiniment petis (Analysis of infinitely small quantities)*, published in 1696. The initial version of what is now known as l'Hopital's rule first appeared in this text.

EXAMPLE 4.4. *Show that* $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$.

PROOF.

$$\lim_{n\to\infty} \ln\left[\left(1+\frac{x}{n}\right)^n\right] = \lim_{n\to\infty} n \ln\left(1+\frac{x}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right)}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{x}{n}}\cdot\left(-\frac{x}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{x}{1+\frac{x}{n}} = x.$$

Thus $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$. □

THEOREM 4.5. *If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, then*

(1). $\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$,

(2). $\lim_{n\to\infty} ka_n = k\lim_{n\to\infty} a_n$,

(3). $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$,

(4). $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided $b_n \neq 0$ and $\lim_{n\to\infty} b_n \neq 0$.

PROOF. omitted. □

EXAMPLE 4.6. *Find the limit of* $\ln\left(\dfrac{n^2+3n+2}{2+4n+2n^2}\right)+\cos\left(\dfrac{1}{\sqrt{n}}\right)$.

SOLUTION.

$$\begin{aligned}
&\lim_{n\to\infty}\left[\ln\left(\frac{n^2+3n+2}{2+4n+2n^2}\right)+\cos\left(\frac{1}{\sqrt{n}}\right)\right]\\
&=\lim_{n\to\infty}\left[\ln\left(\frac{(n^2+3n+2)/n^2}{(2+4n+2n^2)/n^2}\right)+\cos\left(\frac{1}{\sqrt{n}}\right)\right]\\
&=\lim_{n\to\infty}\left[\ln\left(\frac{1+3/n+2/n^2}{2/n^2+4/n+2}\right)+\cos\left(\frac{1}{\sqrt{n}}\right)\right]\\
&=\ln\left(\frac{1+0+0}{0+0+2}\right)+\cos 0=\ln\left(\frac{1}{2}\right)+1=1-\ln 2.
\end{aligned}$$

□

THEOREM 4.7 (Some Standard Limits). *Some standard limits are given as follows.*

1. $\lim_{n\to\infty}\dfrac{1}{n^p}=0$ *for any fixed* $p>0$.
2. $\lim_{n\to\infty}c^n=0$ *for any fixed* c *where* $|c|<1$.
3. $\lim_{n\to\infty}c^{\frac{1}{n}}=1$ *for any fixed* $c>0$.
4. $\lim_{n\to\infty}\sqrt[n]{n}=1$.
5. $\lim_{n\to\infty}\dfrac{n^p}{c^n}=0$ *for any fixed* p *and* $c>1$.
6. $\lim_{n\to\infty}\dfrac{c^n}{n!}=0$ *for any fixed* c.
7. $\lim_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n=e^x$ *for any fixed* x.
8. $\lim_{n\to\infty}\dfrac{(\ln n)^p}{n^k}=0$ *for any fixed* $k>0$.

PROOF. Assertion 7 was proved in Example 4.4.

1.

$$\lim_{n\to\infty}\frac{1}{n^p}=\left(\lim_{n\to\infty}\frac{1}{n}\right)^p=0^p=0.$$

2. Case 1: When $c=0$, the statement is obvious.
Case 2: When $c>0$, we have

$$\ln\left(\lim_{n\to\infty}c^n\right)=\lim_{n\to\infty}\ln c^n=\lim_{n\to\infty}n\ln c=-\infty.$$

Thus, $\lim_{n\to\infty}c^n=0$.
Case 3: When $c<0$, we have $-|c|^n\le c^n\le|c|^n$ for all n. By Case 2, we have $\lim_{n\to\infty}(-|c|^n)=0=\lim_{n\to\infty}|c|^n$. Hence by Squeeze theorem, we also have $\lim_{n\to\infty}c^n=0$.

3. $\lim_{n\to\infty}c^{\frac{1}{n}}=c^{\lim_{n\to\infty}\frac{1}{n}}=c^0=1$.

4.

$$\ln\left(\lim_{n\to\infty}\sqrt[n]{n}\right)=\lim_{n\to\infty}\ln\sqrt[n]{n}=\lim_{n\to\infty}\frac{\ln n}{n}=0 \text{ (by L'Hopital's rule).}$$

Thus, $\lim_{n\to\infty} \sqrt[n]{n} = e^0 = 1$.

5. Let k be a fixed positive integer such that $p - k < 0$. Then

$$\lim_{n\to\infty} \frac{n^p}{c^n} = \lim_{n\to\infty} \frac{pn^{p-1}}{c^n \ln c} = \lim \frac{p(p-1)n^{p-2}}{c^n(\ln c)^2} = \cdots$$

$$= \lim_{n\to\infty} \frac{p(p-1)\cdots(p-k+1)n^{p-k}}{c^n(\ln c)^k} = \lim_{n\to\infty} \frac{p(p-1)\cdots(p-k+1)}{c^n n^{k-p}(\ln c)^k} = 0$$

by L'Hopital's rule.

6. Let $a_n = \frac{c^n}{n!} = \frac{c \cdot c \cdot \cdots \cdot c}{n(n-1)\cdot \cdots \cdot 1}$. Now fix an integer $M > c$. Then for any $n > M$,

$$0 < a_n = \frac{c \cdot c \cdot \cdots \cdot c}{n(n-1)\cdot \cdots \cdot (M+1)} a_M < \frac{c}{n} a_M.$$

Note that a_M is a fixed number because M is fixed. Since $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{c}{n} a_M$, by the Squeeze theorem, $\lim_{n\to\infty} a_n = 0$.

8. Let $m = \ln n$. Then $n = e^m$. By (5),

$$\lim_{n\to\infty} \frac{(\ln n)^p}{n^k} = \lim_{m\to\infty} \frac{m^p}{e^{km}} = \lim_{m\to\infty} \frac{m^p}{(e^k)^m} = 0,$$

where $e^k > 1$ because $k > 0$. □

Strategy: One can find the limits of many sequences from those of the standard sequences.

EXAMPLE 4.8. *Find the limits*

1) $\lim_{n\to\infty} \frac{8^n + (\ln n)^{10} + n!}{n^6 - n!}$.

2) $\lim_{n\to\infty} \left(1 - \frac{1}{2n+1}\right)^{3n}$.

SOLUTION. (1).

$$\lim_{n\to\infty} \frac{8^n + (\ln n)^{10} + n!}{n^6 - n!} = \lim_{n\to\infty} \frac{8^n/n! + (\ln n)^{10}/n! + 1}{n^6/n! - 1} = \frac{0+0+1}{0-1} = -1.$$

(2).

$$\lim_{n\to\infty} \left(1 - \frac{1}{2n+1}\right)^{3n} = \lim_{n\to\infty} \left[\left(1 + \frac{-1}{2n+1}\right)^{2n+1}\right]^{\frac{3n}{2n+1}}$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{-1}{2n+1}\right)^{2n+1}\right]^{\frac{3}{2+1/n}} = \left(e^{-1}\right)^{\frac{3}{2}} = \frac{1}{e\sqrt{e}}$$

□

5. The Least Upper Bounds and the Completeness Property of $\mathbb{R}$

5.1. From Natural Numbers to Real Numbers. Starting with natural numbers $\mathbb{N} = \{1, 2, 3, \cdots\}$, we obtain real numbers $\mathbb{R}$ by adding more and more *new* numbers in the following steps:

Step 1. By adding 0 and negative numbers, we have integers $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \cdots\}$.

Step 2. Then we have rational numbers $\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{Z},\ q \neq 0 \right\}$.

Step 3. Then, by adding **irrational** numbers, we have all real numbers.

Below we give some examples of **irrational** numbers. Recall that a natural number $p > 1$ is called **prime** if p is NOT divisible by any natural numbers other than p and 1. For instance, 2, 3, 5, 7, 11, $\cdots$ are primes. Every natural number $n > 1$ admits a unique (prime) **factorization**

$$n = p_1 \cdot p_2 \cdots p_k,$$

where each p_i is prime. For instance, $20 = 2 \cdot 2 \cdot 5$ and $66 = 2 \cdot 3 \cdot 11$.

Example 5.1. *If n is a natural number, and there is no natural number whose square is n, then $\sqrt{n}$ is NOT a rational number. In particular, $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}$ are irrational numbers.*

Proof. Suppose that $\sqrt{n}$ is a rational number. We can write $\sqrt{n}$ as $\frac{a}{b}$, where $a, b \in \mathbb{N}$ and $b \neq 0$. Then

$$\sqrt{n} = \frac{a}{b} \qquad \Leftrightarrow \qquad n = \frac{a^2}{b^2} \qquad \Leftrightarrow \qquad a^2 = b^2 n.$$

Any prime occurring in the (unique) factorization of a will occur an even number of times in the factorization of a^2; similarly for b and b^2. By $a^2 = b^2 n$, any prime that occurs in the factorization of n must occur an even number of times, since all primes occurring in the factorization of $b^2 n$ are exactly those occurring in the factorization of a^2.

Thus n can be written as

$$n = (p_1 \cdot p_2 \cdots p_k)^2,$$

where, of course, the p_i's need not be distinct. This, however, is a contradiction to the hypothesis, since n is expressed as the square of a natural number. □

5.2. Bounded Sets.

Definition 5.2. A set of real numbers S is *bounded above* if there exists a finite real number M such that

$$x \leq M \qquad \forall x \in S.$$

M is called an *upper bound* of S.

Definition 5.3. A set of real numbers S is *bounded below* if there exists a finite real number m such that

$$m \leq x \qquad \forall x \in S.$$

m is called a *lower bound* of S.

DEFINITION 5.4. A set which is both bounded above and below is called a *bounded set.*

Remark.
1. Upper bounds and lower bounds are not unique.
2. Some sets only have upper bounds but not lower bounds.
3. Some sets have only lower bounds but not upper bounds.
4. A set which is not bounded is called an *unbounded* set.

EXAMPLE 5.5. *Let* $S = \{r \mid r$ *is a rational number with* $r < \sqrt{2}\}$*. Then* S *is bounded above.*

THEOREM 5.6. *Every convergent sequence is bounded.*

PROOF. Let $\{a_n\}$ be a sequence convergent to A. For $\epsilon = 1$, there exists N such that $|a_n - A| < 1$ or $A - 1 < a_n < A + 1$ for $n > N$. Choose M and m to be the largest and smallest number of the finite numbers

$$a_1, a_2, \ldots, a_N, A + 1, A - 1,$$

respectively. When $n \leq N$, we have $m \leq a_n \leq M$ because M (m) is the largest (smallest) number of the above finite set. When $n > N$, we have

$$m \leq A - 1 < a_n < A + 1 \leq M.$$

Thus, for all n, we have $m \leq a_n \leq M$ and so $\{a_n\}$ is bounded. The proof is finished. ☐

COROLLARY 5.7 (Test for divergence). *If* $\{a_n\}$ *is unbounded, then* $\{a_n\}$ *diverges.*

Remark.
1. The converse may not be true, i.e., divergent sequence need not be unbounded.
2. The inverse may not be true, i.e., a bounded sequence may not be convergent.

Example. The sequence $\{1, -1, 1, -1, \cdots\}$ is bounded but NOT convergent.

5.3. Infimum and Supremum. Recall that any finite set of real numbers has a greatest element (maximum) and a least element (minimum).

EXAMPLE 5.8. $\{-2.5,\ 3.1,\ -4.4,\ 4.5,\ 5\}$

However, this property does not necessarily hold for infinite sets.

EXAMPLE 5.9. $\{1, 2, 3, 4, \cdots, \}$.

DEFINITION 5.10. A real number M $(\neq \pm\infty)$ is called the *least upper bound* or *supremum* of a set E if

(i) M is an upper bound of E, i.e., $x \leq M$ for every $x \in E$, and
(ii) if $M' < M$, then M' is not an upper bound of E (i.e., there is an $x \in E$ such that $M' < x$).

We write $M = \sup E$.

Remark.
(i) $\sup E$ is unique whenever it exists.
(ii) The main difference between $\sup E$ and $\max E$ is that $\sup E$ may not be an element of E, whereas $\max E$ must be an element of E if it does exist).
(iii) If E has a maximum, then $\sup E = \max E$.

EXAMPLE 5.11. 1. Let $E = \{r \in \mathbb{Q} \mid 0 \le r \le \sqrt{2}\}$. Then $\sup E = \sqrt{2}$ but $\max E$ does not exist because $\sqrt{2}$ is not a rational number, that is, $\sup E \notin E$.
2. Let $E = \{1/2, 2/3, 3/4, 4/5, 5/6, \cdots\}$. Then $\sup E = 1$ and $\max E$ does not exist.
3. Let $E = \{1, 1/2, 1/3, 1/4, 1/5, \cdots\}$. Then $\max E = 1 = \sup E$.

DEFINITION 5.12. A real number m ($\neq \pm\infty$) is called the *greatest lower bound* or *infimum* of a set E if

(i) m is a lower bound of E, i.e., $m \le x$ for every $x \in E$, and
(ii) if $m' > m$, then m' is not a lower bound of E (i.e., there exists an $x \in E$ such that $x < m'$).

We write $m = \inf E$.

Remark.
(i) $\inf E$ is unique whenever it exists.
(ii) The main difference between $\inf E$ and $\min E$ is that $\inf E$ may not be an element of E, whereas $\min E$ must be an element of E if it does exist.
(iii) If E has a minimum, then $\inf E = \min E$.

EXAMPLE 5.13. 1. Let $E = \{1, 1/2, 1/3, 1/4, \cdots,\}$. Then $\inf E = 0$ but $\min E$ does not exist.
2. Let $E = \{r \in \mathbb{Q} \mid 0 \le r \le \sqrt{2}\}$. Then $\min E = \inf E = 0$.

5.4. The Completeness of $\mathbb{R}$. Consider the set $E = \{r \in \mathbb{Q} \mid r^2 < 2\}$. Then E is a bounded subset of **rational** numbers, but $\sup E = \sqrt{2}$ is NOT a rational number. For considering sup and inf of bounded subsets of rational numbers, we may obtain **irrational** numbers. For bounded subsets of real numbers, sup and inf are still **real** numbers. This is called **completeness property** of $\mathbb{R}$. In details, we have the following.

THEOREM 5.14 (Completeness Axiom of $\mathbb{R}$). *The following statement hold for subsets of real numbers:*

(i) *If E is bounded above, then* $\sup E$ *exists.*
(ii) *If E is bounded below, then* $\inf E$ *exists.*

Remark. For assertion (i), it just means that if a subset of real numbers E is bounded above, then $\sup E$ exists as a **real** number. Compare with rational case: if a subset of rational numbers E is bounded above, then $\sup E$ exists only as a **real** number, but it need not be a **rational** number.
Remark. Richard Dedekind (a German mathematician, 1831-1916), in 1872, used algebraic techniques to construct real number system $\mathbb{R}$ from $\mathbb{Q}$. His basic ideas are as follows.

Given a rational number r, we can construct two sets $U = \{x \in \mathbb{Q} \mid x \ge r\}$ and $L = \{x \in \mathbb{Q} \mid x < r\}$. (One can also construct $U = \{x \in \mathbb{Q} \mid x > r\}$ and $L = \{x \in \mathbb{Q} \mid x \le r\}$.) The sets U and L have the property that

(1). U and L are subsets of $\mathbb{Q}$;
(2). $U \cup L = \mathbb{Q}$;
(3). $U \neq \emptyset$;
(4). $L \neq \emptyset$;
(5). $U \cap L = \emptyset$; and

(6). every element in U is greater than every element in L.

Such a paring (U, L) is called a **Dedekind cut**. Then we can use $\inf U$ (or $\sup L$) to define a *new* number. This is Dedekind's idea to construct all real numbers by using rational numbers. For instance, let $U = \{x \in \mathbb{Q} \mid x^2 > 2\}$ and $L = \{x \in \mathbb{Q} \mid x^2 > 2\}$. Then $\inf U = \sup L = \sqrt{2}$. Another way to construct real numbers using rational numbers was introduced by Georg Cantor (1845-1917). We will explain Cantor's ideas in the section of Cauchy sequences.

Recall that a set E is bounded if and only if it is bounded above and bounded below. Thus the Completeness Axiom leads to

COROLLARY 5.15. *If E is bounded, then both $\sup E$ and $\inf E$ exist.*

6. Monotone Sequences

DEFINITION 6.1. $\{a_n\}$ is called **monotone increasing (decreasing)** if

$$a_n \leq \ (\geq)\ a_{n+1}$$

for every n, that is,

$$a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots$$

$(a_1 \geq a_2 \geq a_3 \geq \cdots)$.

EXAMPLE 6.2. 1. The sequence $\{1/n\}$ is monotone decreasing.
2. The sequence $\{1/2, 2/3, 3/4, 4/5, 5/6, \cdots\}$ is monotone increasing.

PROPOSITION 6.3. *A monotone increasing (decreasing) sequence is bounded below (above).*

PROOF. Let $\{a_n\}$ be a monotone increasing sequence, that is,

$$a_1 \leq a_2 \leq a_3 \leq \cdots .$$

Then a_1 is a lower bound for $\{a_n\}$ and hence the result. □

THEOREM 6.4 (Monotone Convergence Theorem). *Let $\{a_n\}$ be a sequence.*

(i) *If $\{a_n\}$ is monotone increasing and bounded above, then $\{a_n\}$ is convergent and*

$$\lim_{n\to\infty} a_n = \sup_n a_n.$$

(ii) *If $\{a_n\}$ is monotone decreasing and bounded below, then $\{a_n\}$ is convergent and*

$$\lim_{n\to\infty} a_n = \inf_n a_n.$$

PROOF. (i). Suppose $\{a_n\}$ is monotone increasing and bounded above. Then by the Completeness Axiom of $\mathbb{R}$, $\sup_n a_n$ exists (finite). Now, given $\epsilon > 0$, since $\sup_n a_n - \epsilon < \sup_n a_n$, it follows that $\sup_n a_n - \epsilon$ is not an upper bound of $\{a_n\}$. In other words, there exists an integer N such that $a_N > \sup_n a_n - \epsilon$. Then for all $n > N$, we have

$$\sup_n a_n - \epsilon < a_N \leq a_n \leq \sup_n a_n < \sup_n a_n + \epsilon \quad (\text{since } n > N).$$

Equivalently, $\left|a_n - \sup_n a_n\right| < \epsilon$ for all $n > N$ and so $\lim_{n\to\infty} a_n = \sup_n a_n$ (exists).

The proof of (ii) is similar. □

Example 6.5. Let $a_n = \dfrac{n}{n+1}$, that is, $\{a_n\} = \{1/2, 2/3, 3/4, \cdots\}$. Then a_n is monotone increasing and bounded above. Thus

$$\sup_n a_n = \lim_{n\to\infty} a_n = 1.$$

Corollary 6.6. *If $\{a_n\}$ is monotone increasing (decreasing), then either*

(i) *$\{a_n\}$ is convergent or*

(ii) $\lim_{n\to\infty} a_n = +\infty(-\infty)$.

Proof. Suppose $\{a_n\}$ is monotone increasing, then either $\{a_n\}$ is bounded above or not bounded above.

Case (a): If $\{a_n\}$ is bounded above, then by the Monotone Convergence Theorem, $\{a_n\}$ converges.

Case (b): If $\{a_n\}$ is not bounded above, then $\{a_n\}$ has no upper bounds. Thus for any given $k > 0$, k is not an upper bound of $\{a_n\}$. In other words, there exists N such that

$$a_N > k.$$

Since $\{a_n\}$ is monotone increasing, it follows that for all $n > N$,

$$a_n \geq a_N > k.$$

Therefore, $\lim_{n\to\infty} a_n = +\infty$.

The proof for the case when $\{a_n\}$ is monotone decreasing is similar. □

7. Subsequences

Example 7.1. The following are the subsequences of $\{a_n\} = \{1, -1, 1, -1, 1, -1, \cdots\}$.

$\{a_{2n-1}\} = \{1, 1, 1, \cdots\}$

$\{a_{2n}\} = \{-1, -1, -1, \cdots\}$.

In general, **subsequences** of $\{a_n\}$ are of the form $\{a_{n_k}\}$, $k = 1, 2, 3, ...$, with

$$n_1 < n_2 < n_3 < \cdots .$$

Note. The rule is that we should choose a_{n_1} first and then a_{n_2} with $n_2 > n_1$ and then a_{n_3} with $n_3 > n_2$, so far and so on (up to infinite). Thus n_1 is at least 1, n_2 is at least 2, n_3 is at least 3, $\cdots$.

Theorem 7.2. *Suppose $\lim_{n\to\infty} a_n = A$. Then every subsequence of $\{a_n\}$ also converges to A, that is,*

$$\lim_{k\to\infty} a_{n_k} = A.$$

Proof. For any given $\epsilon > 0$, since $\lim_{n\to\infty} a_n = A$, there exists N such that

$$|a_n - A| < \epsilon \quad \text{for all} \quad n > N.$$

Then for all $k > N$, we have

$$n_k \geq k > N.$$

Hence

$$|a_{n_k} - A| < \epsilon \quad \text{for all} \quad k > N.$$

Therefore, $\lim\limits_{k\to\infty} a_{n_k} = A$. □

COROLLARY 7.3. *Suppose that $\{a_n\}$ has two subsequences that converge to different limits. Then $\{a_n\}$ is divergent.* □

EXAMPLE 7.4. *The sequence $\{1, -1, 1, -1, \cdots\}$ is divergent because $\{a_{2n-1}\} = \{1, 1, \cdots\}$ converges to 1 and $\{a_{2n}\} = \{-1, -1, \cdots\}$ converges to -1.*

8. The Limit Superior and Inferior of a Sequence

Given a sequence $\{a_n\}$, we can form another sequence $\{b_n\}$ given by

$$b_n = \sup_{k\geq n} a_k = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\}.$$

EXAMPLE 8.1. *Let $\{a_n\} = \{1, -1, 1, -1, \cdots\}$. Then*

$$b_n = \sup_{k\geq n} a_k = \sup\{\pm 1, \mp 1, \pm 1, \mp 1, \cdots\} = 1.$$

PROPOSITION 8.2. *For any sequence $\{a_n\}$, the associated sequence $\{b_n\} = \{\sup\limits_{k\geq n} a_k\}$ is always monotone decreasing.*

PROOF. For each n,

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\} \geq \sup\{a_{n+1}, a_{n+2}, \cdots\} = b_{n+1}.$$

□

DEFINITION 8.3. The **limit superior** of $\{a_n\}$, denoted by $\limsup a_n$ or $\limsup\limits_{n\to\infty} a_n$ or $\overline{\lim\limits_{n\to\infty}}\, a_n$ is defined to be $\lim\limits_{n\to\infty} b_n$, i.e.

$$\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \sup_{k\geq n} a_k.$$

EXAMPLE 8.4. 1. Let $\{a_n\} = \{1, -1, 1, -1, 1, -1, \cdots\}$.

$$\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} 1 = 1.$$

2. Let $\{a_n\} = \{1, 2, 3, \cdots\}$. Then

$$b_n = \sup_{k\geq n} a_k = \sup\{n, n+1, \cdots\} = +\infty$$

and so $\overline{\lim\limits_{n\to\infty}}\, a_n = \lim\limits_{n\to\infty} b_n = +\infty$.

3. Let $\{a_n\} = \{-1, -2, -3, \cdots\}$. Then

$$b_n = \sup_{k\geq n} a_k = \sup\{-n, -n-1, \cdots\} = -n$$

and so $\overline{\lim\limits_{n\to\infty}}\, a_n = \lim\limits_{n\to\infty} b_n = -\infty$.

THEOREM 8.5. *Given any sequence $\{a_n\}$, either*

(1). $\overline{\lim\limits_{n\to\infty}}\, a_n$ *exists (finite), or*

(2). $\overline{\lim_{n\to\infty}} a_n = +\infty$, *or*
(3). $\overline{\lim_{n\to\infty}} a_n = -\infty$.

PROOF. If $\{a_n\}$ is not bounded above, then each b_n is $+\infty$, and thus

$$\overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} b_n = +\infty.$$

If $\{a_n\}$ is bounded above, then each b_n is finite. Since $\{b_n\}$ is monotone decreasing, by Corollary 1.7.3, $\{a_n\}$ converges (to a finite limit), or $\lim_{n\to\infty} b_n = -\infty$. □

Similarly, given any sequence $\{a_n\}$, we can form another sequence $\{c_n\}$ given by

$$c_n = \inf_{k\geq n} a_k = \inf\{a_n, a_{n+1}, a_{n+2}, \cdots\}.$$

DEFINITION 8.6. The **limit inferior** of $\{a_n\}$, denoted by $\liminf a_n$ or $\liminf_{n\to\infty} a_n$ or $\underline{\lim}_{n\to\infty} a_n$ is defined to be $\lim_{n\to\infty} c_n$, i.e.

$$\underline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} \inf_{k\geq n} a_k.$$

EXAMPLE 8.7. 1. Let $\{a_n\} = \{1, -1, 1, -1, 1, -1, \cdots\}$.

$$\underline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} (\inf\{\pm 1, \mp 1, \pm 1, \mp 1, \cdots) = \lim_{n\to\infty} -1 = -1.$$

2. Let $\{a_n\} = \{1, 2, 3, \cdots\}$. Then

$$c_n = \inf_{k\geq n} a_k = \inf\{n, n+1, \cdots\} = n$$

and so $\underline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} c_n = +\infty$.
3. Let $\{a_n\} = \{-1, -2, -3, \cdots\}$. Then

$$c_n = \inf_{k\geq n} a_k = \inf\{-n, -n-1, \cdots\} = -\infty$$

and so $\underline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} c_n = -\infty$.

PROPOSITION 8.8. *(i). As in Proposition 8.2, for any given sequence $\{a_n\}$, the associated sequence $\{c_n\} = \{\inf_{k\geq n} a_k\}$ is always monotone increasing.*
(ii). As in Theorem 8.5, for any given $\{a_n\}$, $\underline{\lim}_{n\to\infty} a_n$ either exists (finite), or $+\infty$, or $-\infty$).

Remark. We always have

$$\underline{\lim}_{n\to\infty} a_n \leq \overline{\lim_{n\to\infty}} a_n$$

because $c_n \leq b_n$.

PROPOSITION 8.9. *(i). If $\overline{\lim_{n\to\infty}} a_n = B$ with $B \neq -\infty$, then given $\epsilon > 0$, there exists N such that $a_n < B + \epsilon$ for all $n > N$.*
(ii). $\underline{\lim}_{n\to\infty} a_n = C$ with $C \neq +\infty$, then given $\epsilon > 0$, there exists N such that $a_n > C - \epsilon$ for all $n > N$.

PROOF. (i). If $B = +\infty$, the assertion is obvious and so we assume that B is finite. Since $\overline{\lim_{n\to\infty}}\, a_n = B$, given any $\epsilon > 0$, there exists N such that for all $n > N$,

$$|b_n - B| < \epsilon \quad \Rightarrow \quad b_n < B + \epsilon \quad \Rightarrow \quad \sup\{a_n, a_{n+1}, \cdots\} < B + \epsilon,$$

i.e. $a_n, a_n + 1, \cdots < B + \epsilon$ for all $n > N$. Proof of (ii) is similar. □

Warning!! Given a sequence $\{a_n\}$, $\overline{\lim_{n\to\infty}}\, a_n$ is a different concept from $\sup_n a_n$. From the definition, we have

$$b_1 = \sup_n a_n = \sup\{a_1, a_2, \cdots\}$$

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\}$$

with $b_1 \geq b_2 \geq b_3 \geq \cdots$ and $\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} b_n$. Thus we have the relation

$$\overline{\lim_{n\to\infty}}\, a_n \leq \sup_n a_n = b_1,$$

but $\overline{\lim_{n\to\infty}}\, a_n$ need not be equal to $\sup_n a_n$ in general. Similarly,

$$\underline{\lim_{n\to\infty}}\, a_n \geq \inf_n a_n = c_1,$$

but need not be equal to in general.

The correct understanding is that $\overline{\lim}$ is the **largest subsequential limit** of convergent subsequences, including those possible subsequences tending to $+\infty$ or $-\infty$. Similarly, $\underline{\lim}$ is the **smallest subsequential limit**. This is described in the following theorem.

THEOREM 8.10. *Let $\{a_n\}$ be any sequence. Let $B = \overline{\lim_{n\to\infty}}\, a_n$ and let $C = \underline{\lim_{n\to\infty}}\, a_n$.*

(i) *Let $\{a_{n_k}\}$ be* **any** *subsequence of $\{a_n\}$ such that $\lim_{k\to\infty} a_{n_k}$ exists, $+\infty$, or $-\infty$. Then*

$$C = \underline{\lim_{n\to\infty}}\, a_n \leq \lim_{k\to\infty} a_{n_k} \leq \overline{\lim_{n\to\infty}}\, a_n = B.$$

(ii) *There exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that*

$$\lim_{k\to\infty} a_{n_k} = B.$$

(iii) *There exists a subsequence $\{a_{m_k}\}$ of $\{a_n\}$ such that*

$$\lim_{k\to\infty} a_{m_k} = C.$$

PROOF. Let $b_n = \sup\{a_n, a_{n+1}, \cdots\}$ and let $c_n = \inf\{a_n, a_{n+1}, \cdots\}$.
(i). Since $n_k \geq k$, we have

$$c_k = \inf\{a_k, a_{k+1}, \cdots\} \leq a_{n_k} \leq b_k = \sup\{a_k, a_{k+1}, \cdots\}$$

and so

$$C = \lim_{k\to\infty} c_k \leq \lim_{k\to\infty} a_{n_k} \leq \lim_{k\to\infty} b_k = B.$$

(ii). We consider three cases $B = +\infty$, $-\infty$ or finite.
Case I. $B = -\infty$.

Since $b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\} \geq a_n$ and $\lim_{n\to\infty} b_n = B = -\infty$, we have $\lim_{n\to\infty} a_n = -\infty = B$. In this case, we can choose $\{a_n\}$ itself as a subsequence with the desired property.

Case II. $B = +\infty$. In this case we are going to construct a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\lim_{k\to\infty} a_{n_k} = B = +\infty$.

Since

$$b_1 \geq b_2 \geq \cdots \geq b_n \geq \cdots \geq B = \lim_{n\to\infty} b_n = +\infty,$$

we have

$$b_1 = b_2 = \cdots = +\infty,$$

that is $b_n = +\infty$ for all n. Since $b_1 = \sup\{a_1, a_2, \cdots\} = +\infty$, there exists n_1 such that $a_{n_1} > 1$ because 1 is NOT an upper bound of $\{a_1, a_2, \cdots\}$. Since

$$b_{n_1+1} = \sup\{a_{n_1+1}, a_{n_1+2}, a_{n_1+3}\cdots\} = +\infty,$$

there exists a_{n_2} such that $n_2 > n_1$ and $a_{n_2} > 2$ because 2 is NOT an upper bound of $\{a_{n_1+1}, a_{n_1+2}, a_{n_1+3}\cdots\}$. Now, by induction, suppose that we have constructed $a_{n_1}, a_{n_2}, \cdots, a_{n_k}$ such that $n_1 < n_2 < \cdots < n_k$ and

$$a_{n_s} > s$$

for $1 \leq s \leq k$. Since

$$b_{n_k+1} = \sup\{a_{n_k+1}, a_{n_k+2}, \cdots\} = +\infty,$$

there exists $a_{n_{k+1}}$ such that $n_{k+1} > n_k$ and

$$a_{n_{k+1}} > k + 1$$

because $k+1$ is NOT an upper bound of $\{a_{n_k+1}, a_{n_k+2}, a_{n_k+3}\cdots\}$. The induction is finished and so we obtain a subsequence $\{a_{n_1}, a_{n_2}, \cdots\}$ with the property that

$$a_{n_k} > k$$

for any k. Since $\lim_{k\to\infty} k = +\infty$, we have

$$\lim_{k\to\infty} a_{n_k} = +\infty = B.$$

Case III. B is finite. We are going to construct a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\lim_{k\to\infty} a_{n_k} = B$.

Since

$$b_1 = \sup\{a_1, a_2, \cdots\},$$

$b_1 - 1$ is not an upper bound of $\{a_1, a_2, \cdots\}$ and so there exists a_{n_1} such that

$$a_{n_1} > b_1 - 1.$$

Since

$$b_{n_1+1} = \sup\{a_{n_1+1}, a_{n_1+2}, \cdots\},$$

$b_{n_1+1} - \frac{1}{2}$ is not an upper bound of $\{a_{n_1+1}, a_{n_1+2}, \cdots\}$ and so there exists a_{n_2} such that $n_2 > n_1$ and

$$a_{n_2} > b_{n_1+1} - \frac{1}{2}.$$

Now, by induction, suppose that we have constructed $a_{n_1}, a_{n_2}, \cdots, a_{n_k}$ such that $n_1 < n_2 < \cdots < n_k$ and

$$a_{n_s} > b_{n_{s-1}+1} - \frac{1}{s}$$

for $1 \leq s \leq k$. Since

$$b_{n_k+1} = \sup\{a_{n_k+1}, a_{n_k+2}, \cdots\},$$

$b_{n_k+1} - \dfrac{1}{k+1}$ is not an upper bound of $\{a_{n_k+1}, a_{n_k+2}, \cdots\}$ and so there exists $a_{n_{k+1}}$ such that $n_{k+1} > n_k$ and

$$a_{n_{k+1}} > b_{n_k+1} - \frac{1}{k+1}.$$

The induction is finished and so we obtain a subsequence $\{a_{n_1}, a_{n_2}, \cdots\}$ with the property that

$$a_{n_k} > b_{n_{k-1}+1} - \frac{1}{k}$$

for any k. Consider the inequality

$$b_{n_{k-1}+1} - \frac{1}{k} < a_{n_k} \leq b_{n_k}.$$

Since $\{b_n\}$ is convergent, we have

$$\lim_{k\to\infty} b_{n_k} = \lim_{k\to\infty} b_{n_{k-1}+1} = \lim_{n\to\infty} b_n = B$$

and

$$\lim_{k\to\infty} (b_{n_{k-1}+1} - \frac{1}{k}) = B - 0 = B.$$

Thus, by the Squeeze theorem, we have

$$\lim_{k\to\infty} a_{n_k} = B = \overline{\lim_{n\to\infty}}\, a_n.$$

(iii). The proof is similar to that of (ii). We leave it to you as a tutorial question. □

Example 8.11. *Find the limit inferior and limit superior of the following sequences*

i) $\left\{\dfrac{1-2(-1)^n n}{3n+2}\right\}$,

ii) $\left\{(1+(-1)^n)\sin\dfrac{n\pi}{4}\right\}$,

iii) $\{[1.5+(-1)^n]^n\}$.

Solution. (i). Note that

$$a_{2k} = \frac{1-2\cdot 2k}{3\cdot 2k+2} \qquad \lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} \frac{1/k-4}{6+2/k} = -\frac{4}{6} = -\frac{2}{3}$$

$$a_{2k-1} = \frac{1+2\cdot(2k-1)}{3\cdot(2k-1)+2} \qquad \lim_{k\to\infty} a_{2k-1} = \lim_{k\to\infty} \frac{1/k+2\cdot(2-1/k)}{3\cdot(2-1/k)+2/k} = \frac{4}{6} = \frac{2}{3}$$

Thus the subsequential limits are $\pm\dfrac{2}{3}$ and so

$$\overline{\lim_{n\to\infty}}\, a_n = \frac{2}{3} \quad \text{and} \quad \underline{\lim_{n\to\infty}}\, a_n = -\frac{2}{3}.$$

(ii). The sequence

$$\left\{(1+(-1)^n)\sin\frac{n\pi}{4}\right\} = \left\{0, 2\sin\frac{2\pi}{4} = 2, 0, 2\sin\frac{4\pi}{4} = 0, 0, 2\sin\frac{6\pi}{4} = -2,\right.$$
$$\left. 0, 2\sin\frac{8\pi}{4} = 0, 0, 2\sin\frac{10\pi}{4} = 2, \cdots \right\}.$$

The subsequential limits are -2, 0 and 2. Thus

$$\overline{\lim_{n\to\infty}}\, a_n = 2 \quad \text{and} \quad \underline{\lim}_{n\to\infty}\, a_n = -2.$$

(iii). Note that

$$a_{2k} = 2.5^{2k} \qquad \lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} \left(2.5^k\right)^2 = +\infty$$

$$a_{2k-1} = 0.5^{2k-1} \qquad \lim_{k\to\infty} a_{2k-1} = \lim_{k\to\infty} \left[\left(\frac{1}{2}\right)^k\right]^{(2k-1)/k} = \lim_{k\to\infty} \left[\left(\frac{1}{2}\right)^k\right]^{2-1/k} = 0^2 = 0.$$

The subsequential limits are 0 and $+\infty$. Thus

$$\overline{\lim_{n\to\infty}}\, a_n = +\infty \quad \text{and} \quad \underline{\lim}_{n\to\infty}\, a_n = 0.$$

□

The following theorem was originally proved by Bernhard Bolzano (1781-1848) and modified slightly by Karl Weierstrass (1815-1897).

COROLLARY 8.12 (Bolzano-Weierstrass). *Every bounded sequence has a convergent subsequence.*

PROOF. Let $\{a_n\}$ be a bounded sequence. Since $\{a_n\}$ is bounded,

$$-\infty < inf\{a_1, a_2, \cdots\} = c_1 \le \underline{\lim}_{n\to\infty}\, a_n \le \overline{\lim_{n\to\infty}}\, a_n \le b_1 = \sup\{a_1, a_2, \cdots\} < +\infty.$$

Thus $\overline{\lim_{n\to\infty}}\, a_n$ is finite. By Part (ii) of Theorem 8.10, there exists a convergent subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$\lim_{k\to\infty} a_{n_k} = \overline{\lim_{n\to\infty}}\, a_n.$$

□

THEOREM 8.13. $\underline{\lim}_{n\to\infty}\, a_n = \overline{\lim_{n\to\infty}}\, a_n$ *(finite, $+\infty$, $-\infty$) if and only if* $\lim_{n\to\infty} a_n$ *exists (finite), $+\infty$, or $-\infty$.*

PROOF. Suppose that $\underline{\lim}_{n\to\infty}\, a_n = \overline{\lim_{n\to\infty}}\, a_n = A$. Let $b_n = \sup\{a_n, a_{n+1}, \cdots\}$ and let $c_n = \inf\{a_n, a_{n+1}, \cdots\}$. Then

$$c_n = \inf\{a_n, a_{n+1}, \cdots\} \le a_n \le b_n = \sup\{a_n, a_{n+1}, \cdots\}.$$

By the assumption, we have

$$\lim_{n\to\infty} c_n = \underline{\lim}_{n\to\infty}\, a_n = \overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} b_n.$$

By the Squeeze theorem, the sequence $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = \underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n.$$

Conversely suppose that $\{a_n\}$ converges, tends to $+\infty$, or tends to $-\infty$. Let $A = \lim_{n\to\infty} a_n$ and let $\{a_{n_k}\}$ be any subsequence of $\{a_n\}$. By Theorem 7.2, we have $\lim_{k\to\infty} a_{n_k} = A$. Thus the only subsequential limit of $\{a_n\}$ is A. By Theorem 8.10, we have

$$\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = A = \lim_{n\to\infty} a_n.$$

□

Remark. This theorem means that

1) If $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n$, then

$$\lim_{n\to\infty} a_n = \underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n.$$

2) If $\lim_{n\to\infty} a_n$ exists, $+\infty$ or $-\infty$, then

$$\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} a_n.$$

EXAMPLE 8.14. *Let $\{a_n\}$ be a sequence. Show that $\overline{\lim}\,|a_n| = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$, if and only if $\lim_{n\to\infty} a_n = 0$*

PROOF. Suppose that $\overline{\lim}\,|a_n| = 0$. From $|a_n| \geq 0$, we have

$$0 = \lim_{n\to\infty} 0 = \underline{\lim}\, 0 \leq \underline{\lim}\,|a_n| \leq \overline{\lim}\,|a_n| = 0.$$

Thus

$$\underline{\lim}\,|a_n| = \overline{\lim}\,|a_n| = 0$$

and so $\lim_{n\to\infty} |a_n|$ exists and

$$\lim_{n\to\infty} |a_n| = \underline{\lim}\,|a_n| = \overline{\lim}\,|a_n| = 0$$

Conversely suppose that $\lim_{n\to\infty} |a_n| = 0$. Then

$$\underline{\lim}\,|a_n| = \overline{\lim}\,|a_n| = \lim_{n\to\infty} |a_n| = 0$$

because $\{a_n\}$ is convergent. Hence $\overline{\lim}\,|a_n| = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$.

Now suppose that $\lim_{n\to\infty} |a_n| = 0$. From

$$-|a_n| \leq a_n \leq |a_n|,$$

we have $\lim_{n\to\infty} a_n = 0$ by the Squeeze theorem.

Conversely suppose that $\lim_{n\to\infty} a_n = 0$. Then

$$\lim_{n\to\infty} |a_n| = |\lim_{n\to\infty} a_n| = |0| = 0$$

because the function $f(x) = |x|$ is continuous. Hence

$$\lim_{n\to\infty} |a_n| = 0 \iff \lim_{n\to\infty} a_n = 0.$$

□

EXAMPLE 8.15. *Let $a_n > 0$ for all n. Prove that* $\overline{\lim}\sqrt[n]{a_n} \le \overline{\lim}\dfrac{a_{n+1}}{a_n}$.

PROOF. Let $B = \overline{\lim}\dfrac{a_{n+1}}{a_n}$. If $B = +\infty$, clearly $\overline{\lim}\sqrt[n]{a_n} \le B = +\infty$. So we may assume that $B < +\infty$. Since $a_n > 0$, we have $\dfrac{a_{n+1}}{a_n} > 0$ and so $B \ge 0$. Thus B is a finite nonnegative number. By Proposition 8.9, given any $\epsilon > 0$, there exists N such that

$$\frac{a_{n+1}}{a_n} < B + \epsilon$$

for $n > N$. Fixed any n with $n > N$, we have

$$0 < \frac{a_{n+1}}{a_n},\ \frac{a_{n+2}}{a_{n+1}},\ \frac{a_{n+3}}{a_{n+2}}, \cdots < B + \epsilon$$

and so, for any $k \ge 1$, we have

$$0 < \frac{a_{n+1}}{a_n} \cdot \frac{a_{n+2}}{a_{n+1}} \cdot \frac{a_{n+3}}{a_{n+2}} \cdots \frac{a_{n+k}}{a_{n+k-1}} = \frac{a_{n+k}}{a_n} \le (B+\epsilon)^k$$

$$\Longrightarrow \quad a_{n+k} \le a_n(B+\epsilon)^k \qquad \Longrightarrow \quad a_{n+k}^{\frac{1}{n+k}} \le a_n^{\frac{1}{n+k}} \cdot (B+\epsilon)^{\frac{k}{n+k}} \quad \text{for any} \quad k \ge 1.$$

Let k tends to ∞. We have

$$\lim_{k\to\infty} a_n^{\frac{1}{n+k}} \cdot (B+\epsilon)^{\frac{k}{n+k}} = \lim_{k\to\infty} a_n^{\frac{1}{n+k}} \cdot \lim_{k\to\infty} (B+\epsilon)^{\frac{1}{n/k+1}} = a_n^0 \cdot (B+\epsilon) = B + \epsilon.$$

Thus

$$\begin{aligned} \overline{\lim_{m\to\infty}}\, a_m^{\frac{1}{m}} &= \overline{\lim_{k\to\infty}}\, a_{n+k}^{\frac{1}{n+k}} \le \overline{\lim_{k\to\infty}}\, a_n^{\frac{1}{n+k}} \cdot (B+\epsilon)^{\frac{k}{n+k}} \\ &= \lim_{k\to\infty} a_n^{\frac{1}{n+k}} \cdot (B+\epsilon)^{\frac{k}{n+k}} = B + \epsilon. \end{aligned}$$

In other words,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le B + \epsilon$$

for any $\epsilon > 0$ and so

$$\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = \lim_{\epsilon\to 0}\left(\overline{\lim_{n\to\infty}} \sqrt[n]{a_n}\right) \le \lim_{\epsilon\to 0}(B+\epsilon) = B,$$

that is,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le \overline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n}.$$

□

EXERCISE 8.1. *Let $a_n > 0$ for all n. Prove that* $\underline{\lim}\dfrac{a_{n+1}}{a_n} \le \underline{\lim}\sqrt[n]{a_n}$.

By Example 8.15 and Exercise 8.1, we have

$$\underline{\lim}\frac{a_{n+1}}{a_n} \le \underline{\lim}\sqrt[n]{a_n} \le \overline{\lim}\sqrt[n]{a_n} \le \overline{\lim}\frac{a_{n+1}}{a_n}.$$

EXERCISE 8.2. *Let $a_n > 0$ for all n. Suppose that the limit $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}$ exists or $+\infty$. Prove that $\lim\limits_{n\to\infty}\sqrt[n]{a_n}$ exists and $\lim\limits_{n\to\infty}\sqrt[n]{a_n} = \lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}$.*

For instance,

$$\lim_{n\to\infty} \sqrt[n]{n!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = +\infty.$$

9. Cauchy Sequences and the Completeness of $\mathbb{R}$

9.1. Cauchy Sequences.

Definition 9.1. $\{a_n\}$ is called a *Cauchy sequence* if given any $\epsilon > 0$, there exists a natural number N such that for all m, $n > N$, we have

$$|a_n - a_m| < \epsilon.$$

Remark. Roughly speaking, a sequence is Cauchy if the width of its tail $\to$ 0 as $n \to \infty$.

Proposition 9.2. *Every Cauchy sequence is bounded.*

Proof. Let $\{a_n\}$ be a Cauchy sequence. Choose $\epsilon = 1$. There exists N such that $|a_n - a_m| < 1$ for $n, m > N$. In particular, $|a_n - a_{N+1}| < 1$ or

$$a_{N+1} - 1 < a_n < a_{N+1} + 1$$

for $n > N$. Let

$$M = \max\{a_1, a_2, \cdots, a_N, a_{N+1} + 1\}$$
$$m = \min\{a_1, a_2, \cdots, a_N, a_{N+1} - 1\}.$$

For $n \leq N$, we have $m \leq a_n \leq M$, and, for $n > N$, we have

$$m \leq a_{N+1} - 1 < a_n < a_{N+1} + 1 \leq M.$$

Thus, for all n, we have $m \leq a_n \leq M$ and so $\{a_n\}$ is bounded. □

9.2. Completeness of $\mathbb{R}$. The following Criterion was formulated by Augustin-Louis Cauchy (1789-1857).

Theorem 9.3 (Cauchy's criterion). *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

Proof. $\Longrightarrow$, i.e., every convergent sequence is Cauchy.

Given that $\{a_n\}$ is convergent, say $\lim\limits_{n\to\infty} a_n = A$. Then for any given $\epsilon > 0$, there exists N such that

$$|a_n - A| < \frac{\epsilon}{2}$$

for all $n > N$. Now for any $m, n > N$,

$$|a_n - a_m| = |(a_n - A) - (a_m - A)| \leq |a_n - A| + |a_m - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

since both $m, n > N$. Therefore, $\{a_n\}$ is a Cauchy sequence.

$\Longleftarrow$, i.e., every Cauchy sequence is convergent.

Given that $\{a_n\}$ is Cauchy. By Proposition 9.2, $\{a_n\}$ is bounded. By the Bolzano-Weierstrass Theorem (Corollary 8.12), there exists a convergent subsequence $\{a_{n_k}\}$ of $\{a_n\}$. Let $A = \lim_{k\to\infty} a_{n_k}$. Given any $\epsilon > 0$, since $\{a_n\}$ is Cauchy, there exists N_1 such that

$$|a_n - a_m| < \frac{\epsilon}{2} \qquad \text{for all } n, m > N_1.$$

Since $\{a_{n_k}\}$ converges to A, there exists K such that

$$|a_{n_k} - A| < \frac{\epsilon}{2} \qquad \text{for all} \quad k > K.$$

Let $N = \max\{K, N_1\}$. Choose an n_k such that $k > N$, for instance, choose n_k to be n_{N+1}. When $n > N$, by triangular inequality,

$$|a_n - A| = |(a_n - a_{n_k} + (a_{n_k} - A)| \le |a_n - a_{n_k}| + |a_{n_k} - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

because $n > N \ge N_1$, $n_k \ge k > N \ge N_1$ and $k > N \ge K$. Therefore $\{a_n\}$ converges to A by the definition. □

Remark. The statement that *every Cauchy sequence in* $\mathbb{R}$ *converges* is often expressed by saying that $\mathbb{R}$ is complete. Note that our proof of the Cauchy Criterion used the Bolzano-Weierstrass Theorem and the proof of that one used the completeness axiom of $\mathbb{R}$ (Theorem 5.14). We did not prove Theorem 5.14, namely we treated Theorem 5.14 as an axiom. (Dedekind and Cantor proved the completeness axiom in 1872 independently.) Conversely, we can treat the Cauchy Criterion as an axiom, and prove Theorem 5.14. In this sense, both Theorem 5.14 and the Cauchy Criterion are regarded as the completeness of $\mathbb{R}$.

Cantor constructed real numbers from rational numbers by using Cauchy sequences. His ideas are as follows. Consider all of the Cauchy sequences $\{a_n\}$ with $a_n \in \mathbb{Q}$. By the Cauchy criterion, $\{a_n\}$ converges to a **real** number. Then he proved that all real numbers can be obtained as the limits of all of the Cauchy sequences $\{a_n\}$ with $a_n \in \mathbb{Q}$. One can think that, by using our standard base 10 number system, any real number admits a (decimal) expansion. For instance, $\sqrt{2} = 1.4142\cdots$, we can define a sequence of rational numbers

$$a_1 = 1, a_2 = 1.4, a_3 = 1.41, a_4 = 1.414, a_5 = 1.4142, \cdots \qquad \text{with} \quad \lim_{n\to\infty} a_n = \sqrt{2}.$$

EXAMPLE 9.4. *Let* $s_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$. *Show that* $\{s_n\}$ *is convergent.*

PROOF. For each $k \ge 1$, we have

$$\begin{aligned}
|s_{n+k} - s_n| &= \left|\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+k)^2}\right) - \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right)\right| \\
&= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+k)^2} \\
&\le \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{(n+k-1)(n+k)} \\
&= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{n+k-1} - \frac{1}{n+k}\right) \\
&= \frac{1}{n} - \frac{1}{n+k} < \frac{1}{n}.
\end{aligned}$$

Given any $\epsilon > 0$, choose N such that $\frac{1}{N} < \epsilon$, that is, $N > \frac{1}{\epsilon}$. When $m > n > N$, from the above,

$$|s_m - s_n| < \frac{1}{n} < \frac{1}{N} < \epsilon.$$

Thus $\{s_n\}$ is a Cauchy sequence and hence $\{s_n\}$ converges by the Cauchy Criterion. □

9.3. Contractive Sequences. In this subsection, we give an application of the Cauchy Criterion.

A sequence $\{a_n\}$ is called **contractive** if there exists b, $0 < b < 1$, such that

$$|a_{n+1} - a_n| \leq b|a_n - a_{n-1}| \tag{1}$$

for all $n \geq 2$.

THEOREM 9.5 (Contractive Theorem). *Every contractive sequence converges. Furthermore, if $\{a_n\}$ is contractive and $A = \lim_{n\to\infty} a_n$, then*

(a). $|A - a_n| \leq \dfrac{b^{n-1}}{1-b}|a_2 - a_1|$, *and*

(b). $|A - a_n| \leq \dfrac{b}{1-b}|a_n - a_{n-1}|$,

where b is the constant in above definition.

PROOF. From Inequality (1), we have

$$|a_{n+1} - a_n| \leq b|a_n - a_{n-1}| \leq b^2|a_{n-1} - a_{n-2}| \leq b^3|a_{n-2} - a_{n-3}| \leq \cdots \leq b^{n-1}|a_2 - a_1|$$

For each $k \geq 1$, we have

$$\begin{aligned} |a_{n+k} - a_n| &= |(a_{n+k} - a_{n+k-1}) + (a_{n+k-1} - a_{n+k-2}) + \cdots + (a_{n+1} - a_n)| \\ &\leq |a_{n+k} - a_{n+k-1}| + |a_{n+k-1} - a_{n+k-2}| + \cdots + |a_{n+1} - a_n| \\ &\leq b^{n+k-1}|a_2 - a_1| + b^{n+k-2}|a_2 - a_1| + \cdots + b^{n-1}|a_2 - a_1| \\ &= (b^{n+k-1} + b^{n+k-2} + \cdots + b^{n-1})|a_2 - a_1| \end{aligned}$$

$$= b^{n-1}(1 + b + b^2 + \cdots + b^k)|a_2 - a_1| = \frac{b^{n-1}\cdot(1 - b^{k+1})}{1-b}|a_2 - a_1| < \frac{b^{n-1}}{1-b}|a_2 - a_1|.$$

Since $0 < b < 1$, we have $\lim_{n\to\infty} b^{n-1} = 0$ by the Standard limits and so

$$\lim_{n\to\infty} \frac{b^{n-1}}{1-b}|a_2 - a_1| = 0.$$

Therefore, given any $\epsilon > 0$, there exists N such that

$$\frac{b^{n-1}}{1-b}|a_2 - a_1| < \epsilon$$

for $n > N$ and so, for all $n > N$ and $k \geq 1$,

$$|a_{n+k} - a_n| < \frac{b^{n-1}}{1-b}|a_2 - a_1| < \epsilon.$$

It follows that $\{a_n\}$ is Cauchy and so it is convergent.

Let $A = \lim_{n\to\infty} a_n$. From the inequality

$$|a_{n+k} - a_n| < \frac{b^{n-1}}{1-b}|a_2 - a_1|,$$

we have

$$|A - a_n| = \left|\lim_{k\to\infty} a_{n+k} - a_n\right| = \lim_{k\to\infty} |a_{n+k} - a_n| \leq \frac{b^{n-1}}{1-b}|a_2 - a_1|.$$

This is Inequality (a).

For each $k \geq 1$, we have

$$|a_{n+k} - a_n| = |(a_{n+k} - a_{n+k-1}) + (a_{n+k-1} - a_{n+k-2}) + \cdots + (a_{n+1} - a_n)|$$
$$\leq |a_{n+k} - a_{n+k-1}| + |a_{n+k-1} - a_{n+k-2}| + \cdots + |a_{n+1} - a_n|$$
$$= |a_{n+1} - a_n| + |a_{n+2} - a_{n+1}| + \cdots + |a_{n+k} - a_{n+k-1}|$$
$$\leq b|a_n - a_{n-1}| + b^2|a_n - a_{n-1}| + \cdots + b^k|a_n - a_{n-1}|$$
$$= (b + b^2 + \cdots + b^k)|a_n - a_{n-1}|$$
$$= b(1 + b + b^2 + \cdots + b^{k-1})|a_n - a_{n-1}| = \frac{b \cdot (1 - b^k)}{1 - b}|a_n - a_{n-1}| < \frac{b}{1-b}|a_n - a_{n-1}|.$$

Thus

$$|A - a_n| = \left|\lim_{k\to\infty} a_{n+k} - a_n\right| = \lim_{k\to\infty} |a_{n+k} - a_n| \leq \frac{b}{1-b}|a_n - a_{n-1}|.$$

This is Inequality (b). □

Example 9.6. *Let $c_1 \in (0,1)$ be arbitrary, and for $n \geq 1$ set $c_{n+1} = \frac{1}{3}(c_n^2 + 1)$.*

1) *Prove that the sequence $\{c_n\}$ is contractive.*
2) *Show that if $c = \lim_{n\to\infty} c_n$, then c is a root of the polynomial $x^2 - 3x + 1$.*
3) *Let $c_1 = \frac{1}{2}$. Determine a value of n such that $|c - c_n| < 10^{-3}$.*

Proof. (1). First we show that $0 < c_n < 1$ for all n by induction. By the assumption $0 < c_1 < 1$. Suppose that $0 < c_n < 1$. Then

$$0 < c_{n+1} = \frac{1}{3}(c_n^2 + 1) < \frac{1}{3}(1 + 1) = \frac{2}{3} < 1.$$

The induction is finished and so $0 < c_n < 1$ for all n. Now from

$$|c_{n+1} - c_n| = \left|\frac{1}{3}(c_n^2 + 1) - \frac{1}{3}(c_{n-1}^2 + 1)\right| = \frac{1}{3}|c_n^2 - c_{n-1}^2|$$
$$= \frac{1}{3}|c_n - c_{n-1}||c_n + c_{n-1}| \leq \frac{1}{3}|c_n - c_{n-1}|(1 + 1) = \frac{2}{3}|c_n - c_{n-1}|,$$

the sequence $\{c_n\}$ is contractive with $b = \frac{2}{3}$.

(2). From

$$c_{n+1} = \frac{1}{3}(c_n^2 + 1),$$
$$c = \lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{3}(c_n^2 + 1) = \frac{1}{3}\left(\lim_{n\to\infty} c_n^2 + 1\right) = \frac{1}{3}(c^2 + 1).$$

Thus $c^2 - 3c + 1 = 0$, that is, c is a root of $x^2 - 3x + 1$.

(3). We use Inequality (a) of the Contractive Theorem.

Note that

$$c_2 = \frac{1}{3}(c_1^2 + 1) = \frac{1}{3}\left(\frac{1}{2^2} + 1\right) = \frac{5}{12}.$$

$$\frac{b^{n-1}}{1-b}|c_2 - c_1| = \frac{\left(\frac{2}{3}\right)^{n-1}}{1 - \frac{2}{3}}\left|\frac{5}{12} - \frac{1}{2}\right| < 10^{-3}$$

$$\implies \quad 3\left(\frac{2}{3}\right)^{n-1} \cdot \frac{1}{12} < \frac{1}{10^3} \quad \implies \left(\frac{2}{3}\right)^{n-1} < \frac{4}{10^3}$$

$$\implies \quad (n-1) \cdot \ln\left(\frac{2}{3}\right) < \ln 4 - 3\ln 10$$

$$\implies \quad (n-1) > \frac{3\ln 10 - \ln 4}{\ln 3 - \ln 2} = 13.62.$$

Thus when $n = 15$, we have

$$|c - c_{15}| < \frac{b^{n-1}}{1-b}|c_2 - c_1| < 10^{-3}.$$

□

Note. From $x^2 - 3x + 1 = 0$, we have

$$x = \frac{3 \pm \sqrt{3^2 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

Since $0 \le c \le 1$, $c = \dfrac{3 - \sqrt{5}}{2}$.

CHAPTER 2

Series of Real Numbers

1. Series

The expression

$$a_1 + a_2 + a_3 + \cdots$$

written alternatively as $\displaystyle\sum_{k=1}^{\infty} a_k$ is called an **infinite series**.

EXAMPLE 1.1. (1). $1 + 2 + 3 + 4 + \cdots$.
(2). $1 + 1/2 + 1/3 + 1/4 + \cdots$
(3). $1 + 1/2^2 + 1/3^2 + 1/4^2 + \cdots$.
(4). $1 + 0 + 1 + 0 + 1 + 0 + \cdots$.

DEFINITION 1.2. Given a series $\displaystyle\sum_{k=1}^{\infty} a_k$, its n^{th} **partial sum** S_n is given by

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The sequence $\{S_n\}$ is called the **sequence of partial sums** of the series $\displaystyle\sum_{k=1}^{\infty} a_k$.

EXAMPLE 1.3. Consider the series $1-1+1-1+1-1+1-1+\cdots$. The $S_{2n-1} = 1$ and $S_{2n} = 0$.

DEFINITION 1.4. Consider the sequence of partial sums $\{S_n\}$ of the series $\displaystyle\sum_{k=1}^{\infty} a_k$. If this sequence converges to a number S, we say that the series $\displaystyle\sum_{k=1}^{\infty} a_k$ **converges** to S and write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = S.$$

If $\{S_n\}$ diverges, then we say $\displaystyle\sum_{k=1}^{\infty} a_n$ **diverges**.

EXAMPLE 1.5 (Geometric Series). Let $a \neq 0$. Consider the series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots .$$

Then the partial sum

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a(1 + r + \cdots + r^{n-1}) = \begin{cases} a\dfrac{1-r^n}{1-r} & r \neq 1 \\ a(n+1) & r = 1 \end{cases}$$

When $-1 < r < 1$, $S_n \to \dfrac{a}{1-r}$ as $n \to \infty$.

When $r > 1$, S_n diverges because $r^n \to +\infty$ as $n \to \infty$.

When $r = 1$, $S_n = a(n+1)$ diverges.

When $r = -1$, $S_n = \dfrac{a[1-(-1)^n]}{2}$ diverges.

When $r < -1$, S_n diverges because $r^n \to \pm\infty$.

Thus **the geometric series** $\sum_{n=0}^{\infty} ar^n$ **converges if and only if** $-1 < r < 1$**, and,**

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

for $-1 < r < 1$**.**

Remark. If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converges, then one always has

(i) $\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$.

(ii) $\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k$.

EXAMPLE 1.6.

$$\begin{aligned}
\sum_{n=1}^{\infty}\left[\left(\frac{1}{4}\right)^n + \left(\frac{1}{5}\right)^n\right] &= \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n + \sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n \\
&= \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right] + \left[\frac{1}{5} + \left(\frac{1}{5}\right)^2 + \cdots\right] \\
&= \frac{1}{4}\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right] + \frac{1}{5}\left[1 + \frac{1}{5} + \left(\frac{1}{5}\right)^2 + \cdots\right] \\
&= \frac{1}{4}\frac{1}{1-\frac{1}{4}} + \frac{1}{5}\frac{1}{1-\frac{1}{5}} \\
&= \frac{1}{4\cdot\frac{3}{4}} + \frac{1}{5\cdot\frac{4}{5}} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.
\end{aligned}$$

THEOREM 1.7. *If* $\sum_{k=1}^{\infty} a_k$ *converges, then* $\lim_{k\to\infty} a_k = 0$.

Proof. Recall that partial sum $S_k = a_1 + a_2 + \cdots + a_k$. We have

$$S_k - S_{k-1} = (a_1 + a_2 + \cdots + a_{k-1} + a_k) - (a_1 + a_2 + \cdots + a_{k-1}) = a_k.$$

Since the series $\sum_{k=1}^{\infty} a_k$ converges, the sequence $\{S_k\}$ converges. Let $S = \lim_{k\to\infty} S_k$. Then

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} (S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = S - S = 0.$$

□

Corollary 1.8 (Divergence Test). *If* $\lim_{n\to\infty} a_n \neq 0$ *(or does not exist), then* $\sum_{n=1}^{\infty} a_n$ *diverges.* □

Example 1.9. (1). The series $\sum_{n=1}^{\infty} (-1)^n$ is divergent because the limit of the n-th term $(-1)^n$ does not exist.

(2). The series $\sum_{n=1}^{\infty} \frac{n!}{n^2}$ is divergent because

$$\lim_{n\to\infty} \frac{n!}{n^2} = \frac{1}{\lim_{n\to\infty} \frac{n^2}{n!}} = \frac{1}{0} = +\infty \neq 0.$$

(3). The series $\sum_{n=1}^{\infty} \frac{2n+1}{3n+2}$ is divergent because

$$\lim_{n\to\infty} \frac{2n+1}{3n+2} = \lim_{n\to\infty} \frac{2+1/n}{3+2/n} = \frac{2}{3} \neq 0.$$

Remark. The divergence test is a "one-way" test, i.e., $\lim_{n\to\infty} a_n = 0$ does NOT imply $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 1.10 (Cauchy Criterion). *The series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if given any* $\epsilon > 0$*, there exists* N *such that*

$$\left| \sum_{k=n+1}^{m} a_k \right| < \epsilon$$

for all $m > n > N$.

Proof. The series $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence of its partial sums $\{S_n\}$ converges, (by definition), if and only if $\{S_n\}$ is Cauchy. The result follows from

$$|S_m - S_n| = |(a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots + a_m) - (a_1 + a_2 + \cdots + a_n)|$$

$$= |a_n + a_{n+1} + \cdots + a_m| = \left| \sum_{k=n+1}^{m} a_k \right|.$$

□

EXAMPLE 1.11 (Harmonic Series). *Show that the series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

PROOF. Note that

$$|S_{2n} - S_n| = \left| \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \right|$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = n \cdot \frac{1}{2n} = \frac{1}{2}.$$

Suppose that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converges. Then $\{S_n\}$ is Cauchy. Given $\epsilon = \frac{1}{2}$. There exists N such that $|S_m - S_n| < \dfrac{1}{2}$ for all $m > n > N$. This contradicts to the above fact that $|S_{2n} - S_n| \geq \dfrac{1}{2}$, where m is chosen to be $2n$. □

Note. The divergence of the harmonic series appears to have been established by Nicole Oresme (1323?-1382) by showing that the sequence $\left\{ \displaystyle\sum_{k=1}^{n} \frac{1}{k} \right\}$ is NOT bounded.

THEOREM 1.12. *Suppose that* **eventually** $a_k \geq 0$. *Then* $\displaystyle\sum_{k=1}^{\infty} a_k$ *converges if and only if* $\{S_n\}$ *is bounded above.*

PROOF. We may assume that $a_k \geq 0$ for all k. Since

$$S_{n+1} - S_n = a_{n+1} \geq 0,$$

the sequence $\{S_n\}$ is monotone increasing. Thus $\displaystyle\sum_{k=1}^{\infty} a_k$ converges if and only if $\{S_n\}$ converges, if and only if $\{S_n\}$ is bounded above (by the Monotone Convergence Theorem). □

Note. Suppose that **eventually** $a_k \geq 0$. This theorem means that the following.

1) If $\{S_n\}$ is bounded above, then $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.

2) If $\{S_n\}$ is NOT bounded above, then $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges.

2. Tests for Positive Series

A series $\displaystyle\sum_{k=1}^{\infty} a_k$ is called a (**eventually**) **positive series** if every term a_k is (eventually) positive.

2.1. Comparison Test.

THEOREM 2.1 (Comparison Test). *Consider 2 positive series* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$. *Suppose that eventually* $0 \leq a_k \leq b_k$.

(i) *If* $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

(ii) *If* $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

PROOF. Assertion (ii) follows immediately from (i). Let $A_n = \sum_{k=1}^{n} a_k$, and $B_n = \sum_{k=1}^{n} b_k$. Then $A_n \leq B_n$ for all n. Suppose that $\sum_{k=1}^{\infty} b_k$ converges, that is, $\sum_{k=1}^{\infty} b_k$ is a (finite) number. Then

$$A_n \leq B_n \leq \sum_{k=1}^{\infty} b_k$$

for all n and so A_n is bounded above. By Theorem 1.12, $\sum_{k=1}^{\infty} a_k$ is convergent. □

EXAMPLE 2.2. The series $\sum_{k=1}^{\infty} \left(\frac{2k-1}{3k+2}\right)^k$ converges because

$$\left(\frac{2k-1}{3k+2}\right)^k \leq \left(\frac{2}{3}\right)^k$$

and the geometric series $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$ converges.

Remark. 1. Suppose $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges. Then NO conclusion can be drawn.

2. Similarly, suppose $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} a_n$ converges. Then NO conclusion can be drawn.

EXAMPLE 2.3.

$$0 \leq \left(\frac{1}{2}\right)^n \leq 2^n \leq 3^n.$$

The series $\sum_{n=1}^{\infty} 3^n$ diverges. Now $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges and $\sum_{n=1}^{\infty} 2^n$ diverges.

COROLLARY 2.4 (Limit Comparison Test). *Suppose that* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *are (eventually) positive series.*

(a). *If* $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ *with* $0 < L < \infty$, *then* $\sum_{k=1}^{\infty} a_k$ *converges* **if and only if** $\sum_{k=1}^{\infty} b_k$ *converges. (So* $\sum_{k=1}^{\infty} a_k$ *diverges if and only if* $\sum_{k=1}^{\infty} b_k$ *diverges.)*

(b). *If* $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$, *(that is,* $a_k << b_k$*), and* $\sum_{k=1}^{\infty} b_k$ **converges**, *then* $\sum_{k=1}^{\infty} a_k$ **converges**.

(c). *If* $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$, *(that is,* $a_k << b_k$*), and* $\sum_{k=1}^{\infty} a_k$ **diverges**, *then* $\sum_{k=1}^{\infty} b_k$ **diverges**.

Remark. If $\lim_{n\to\infty} \frac{a_n}{b_n} = +\infty$, interchange a_n and b_n, and then apply assertions (b) and (c).

PROOF. We may assume that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are positive series.
(a). Since

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L\ (\neq 0, \neq \infty),$$

$\left\{\frac{a_k}{b_k}\right\}$ is bounded above, say, by M. Thus

$$0 \le a_k \le M b_k$$

for all k. Similarly, since $\lim_{k\to\infty} \frac{b_k}{a_k} = \frac{1}{L}\ (\neq 0, \neq \infty)$, $\left\{\frac{b_k}{a_k}\right\}$ is bounded above, say, by M'. Thus $0 \le b_k \le M' a_k$ for all k.

If $\sum_{k=1}^{\infty} a_k$ is convergent, then $\sum_{k=1}^{\infty} M' a_k = M' \sum_{k=1}^{\infty} a_k$ is also convergent. By the comparison test, it follows that $\sum_{k=1}^{\infty} b_k$ is also convergent because $b_k \le M' a_k$.

If $\sum_{k=1}^{\infty} b_k$ is convergent, then $\sum_{k=1}^{\infty} M b_k = M \sum_{k=1}^{\infty} b_k$ is also convergent. By the comparison test, it follows that $\sum_{k=1}^{\infty} a_k$ is also convergent because $a_k \le M b_k$. Hence $\sum_{k=1}^{\infty} a_k$ is convergent if and only if $\sum_{k=1}^{\infty} b_k$ is convergent. Hence $\sum_{k=1}^{\infty} a_k$ is divergent if and only if $\sum_{k=1}^{\infty} b_k$ is divergent.

Now we prove assertions (b) and (c). $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ implies for every $\epsilon > 0$, there is an N such that

$$\left| \frac{a_k}{b_k} - 0 \right| < \epsilon \quad \forall k > N.$$

We choose $\epsilon = 1$. Then the above inequality is

$$a_k < b_k \quad \forall k > N.$$

We get the result by applying the comparison test. □

Standard series used in comparison and limit comparison tests.

1. The Geometric Series:

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \text{converges} & \text{if } |r| < 1, \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

2. The p-series: for a fixed p,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges} & \text{if } p > 1, \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

To be proved in the subsection on Integral Test.

EXAMPLE 2.5. *Determine the convergence or divergence:*

1) $\displaystyle\sum_{n=1}^{\infty} \frac{1 + \cos n}{n^2}$

2) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n + n^3 + 8}{n^4 - 2n + 3}$

SOLUTION. (1). It is convergent, by the comparison test, because

$$0 \leq \frac{1 + \cos n}{n^2} \leq \frac{2}{n^2}$$

and the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(2). It is divergent, by the limit comparison test, because

$$\lim_{n\to\infty} \frac{\dfrac{\ln n + n^3 + 8}{n^4 - 2n + 3}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n \ln n + n^4 + 8n}{n^4 - 2n + 3} = \lim_{n\to\infty} \frac{\dfrac{\ln n}{n^3} + 1 + \dfrac{1}{n^3}}{1 - \dfrac{2}{n^3} + \dfrac{3}{n^4}} = \frac{0 + 1 + 0}{1 - 0 + 0} = 1$$

and the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. □

EXAMPLE 2.6. *Determine convergence or divergence of* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^k}$, *where* k *is a constant.*

SOLUTION. Let $b_n = \dfrac{1}{(\ln n)^k}$ and let $a_n = \dfrac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{(\ln n)^k}} = \lim_{n\to\infty} \frac{(\ln n)^k}{n} = 0.$$

Since the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent, the series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^k}$ is divergent for any k. □

2.2. Integral Test. Let $f(x)$ be a real-valued function on $[a, +\infty)$ such that $f(x)$ is Riemann integrable, that is, the integral $\int_a^b f(x)\,dx$ exists for every $b > a$. The following theorem is useful.

THEOREM 2.7. *Let $f(x)$ be a real-valued function on $[a, b]$.*

(a). *If $f(x)$ is continuous on $[a, b]$, then $f(x)$ is Riemann integrable on $[a, b]$.*
(b). *If $f(x)$ is monotone on $[a, b]$, then $f(x)$ is Riemann integrable on $[a, b]$.*

The improper integral is defined by

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx = \text{area under } f(x) \text{ over } [a, \infty).$$

Here we say that $\int_a^{\infty} f(x)\,dx$ **converges** if the limit $\lim_{b\to\infty} \int_a^b f(x)\,dx$ exists (finite), i.e., the area under $f(x)$ over $[a, \infty)$ is finite.

We also say that $\int_a^{\infty} f(x)\,dx$ **diverges** if the limit $\lim_{b\to\infty} \int_a^b f(x)\,dx$ does not exist.

THEOREM 2.8 (Integral Test). *Let $f(x)$ be an (***eventually***) **positive monotone decreasing** function on $[1, +\infty)$. Suppose we have a series $\sum_{k=1}^{\infty} a_k$ such that $a_k = f(k)$, then the series $\sum_{k=1}^{\infty} a_k$ and the integral $\int_1^{\infty} f(x)dx$ either both* **converge** *or both* **diverge**.

PROOF. We may assume that $f(x)$ is positive monotone decreasing on $[1, +\infty)$. Let $a_n = f(n)$ for all n.

From the graph, we see that

$$\text{area of the rectangles } \le \text{area under } f(x) \text{ over } [1, n], \text{ i.e.,}$$

$$\sum_{k=2}^{n} f(k) \le \int_1^n f(x)dx \le \int_1^\infty f(x)dx.$$

Thus, if $\int_1^\infty f(x)\,dx < \infty$, then

$$\sum_{k=2}^{n} a_k = \sum_{k=2}^{n} f(k) \le \int_1^\infty f(x)\,dx < \infty,$$

i.e., for all n, $\sum_{k=2}^{n} a_k$ is bounded above by the finite number $\int_1^\infty f(x)\,dx$. Since we also have $a_k \ge 0$, it follows from Theorem 1.12 that $\sum_{k=2}^{\infty} a_k$ converges, and thus $\sum_{k=1}^{\infty} a_k$ also converges.

Next we consider the following graph:

From the graph, it is easy to see that

$$\text{area under the rectangles} \ge \text{area under } f(x) \text{ over } [1, n], \text{ i.e.,} \sum_{k=1}^{n-1} f(k) \ge \int_1^n f(x)\,dx.$$

Thus, if $\sum_{k=1}^{\infty} a_k < \infty$, then

$$\infty > \sum_{k=1}^{\infty} a_k \ge \sum_{k=1}^{n-1} a_k = \sum_{k=1}^{n-1} f(k) \ge \int_1^n f(x)\,dx,$$

i.e., $\int_1^n f(x)\,dx$ is bounded above by the finite number $\sum_{k=1}^{\infty} a_k < \infty$. Letting $n \to \infty$, it follows that we have

$$\int_1^\infty f(x)\,dx \le \sum_{k=1}^{\infty} a_k < \infty.$$

In conclusion, we have $\sum_{k=1}^{\infty} a_k$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges, which also means that $\sum_{k=1}^{\infty} a_k$ diverges if and only if $\int_1^{\infty} f(x)\,dx$ diverges. $\square$

EXAMPLE 2.9. *Show that*

1) *the series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if and only if* $p > 1$.
2) *the series* $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^k}$ *converges if and only if* $k > 1$.

(1). If $p \le 0$, then $\frac{1}{n^p}$ does not tend to 0 and so, by divergence test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges. Assume that $p > 0$. Let $f(x) = \frac{1}{x^p}$ on $[1, +\infty)$. Then $f(x)$ is positive monotone decreasing. Now

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} x^{-p}\,dx = \begin{cases} \left.\frac{1}{-p+1} x^{-p+1}\right|_1^{+\infty} & p \neq 1 \\ \ln(+\infty) - \ln 1 & p = 1 \end{cases}$$

Thus $\int_1^{\infty} f(x)\,dx$ converges if and only if $p > 1$ and so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

(2). Let $f(x) = \frac{1}{x(\ln x)^k}$ on $[2, +\infty)$. Then $f(x)$ is positive. We check that $f(x)$ is *eventually* monotone decreasing. From

$$f'(x) = \left(x^{-1}(\ln x)^{-k}\right)' = -x^{-2}(\ln x)^{-k} - kx^{-1}(\ln x)^{-k-1}\frac{1}{x} = -x^{-2}(\ln x)^{-k-1}(\ln x + k),$$

we have $f'(x) \le 0$ when $\ln x > -k$. Thus $f(x)$ is monotone decreasing when $\ln x > -k$ and so $f(x)$ is eventually monotone decreasing. Now

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x(\ln x)^k}\,dx \xlongequal[dy = \frac{1}{x}dx]{y = \ln x} \int_{\ln 2}^{\infty} \frac{1}{y^k} dy = \begin{cases} \left.\frac{1}{-k+1} y^{-k}\right|_{\ln 2}^{\infty} & k \neq 1 \\ \ln(+\infty) - \ln(\ln 2) & k = 1 \end{cases}$$

Thus $\int_2^{\infty} f(x)\,dx$ converges if and only if $k > 1$ and so the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^k}$ converges if and only if $k > 1$.

2.3. Ratio Test.

THEOREM 2.10 (Ratio Test). *Consider the positive series* $\sum_{n=1}^{\infty} a_n$. *Suppose*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell. \tag{2}$$

(i) *If* $0 \le \ell < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(ii) *If* $1 < \ell \le \infty$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*
(iii) *If* $\ell = 1$, *then the test is inconclusive.*

PROOF. We will prove (i) and (ii). Given any $\epsilon > 0$, it follows from (1) that there exists N such that for all $n > N$,

$$\left|\frac{a_{n+1}}{a_n} - \ell\right| < \epsilon \quad \text{or} \quad \ell - \epsilon < \frac{a_{n+1}}{a_n} < \ell + \epsilon.$$

By repeating using the above inequalities, it follows that for all $m > 0$,

$$a_{N+1}(\ell - \epsilon)^m < a_{N+1+m} < a_{N+1}(\ell + \epsilon)^m. \tag{3}$$

(i). If $\ell < 1$, choose $\epsilon > 0$ such that $\ell + \epsilon < 1$, then $\sum_{m=1}^{\infty} a_{N+1}(\ell+\epsilon)^m$ converges (since it is a geometric series with common ratio satisfying $|r| = \ell + \epsilon < 1$). Together with the right-hand-side of (3), it follows from the comparison test that $\sum_{m=1}^{\infty} a_{N+1+m}$ converges, and thus $\sum_{n=1}^{\infty} a_n$ converges.
(ii). If $\ell > 1$, choose $\epsilon > 0$ such that $\ell - \epsilon > 1$, then by the left-hand-side of (3), we have, for all $m > 0$,

$$a_{N+1+m} \ge a_{N+1}(\ell - \epsilon)^m > a_{N+1} > 0.$$

In particular, $\lim_{n\to\infty} a_n \ne 0$ or does not exist. By the divergence test, $\sum_{n=1}^{\infty} a_n$ diverges.

□

EXAMPLE 2.11. *Determine convergence or divergence.*

1) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.
2) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$.

(1).

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\dfrac{(n+1)!}{(n+1)^{n+1}}}{\dfrac{n!}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!}{n!\cdot\dfrac{(n+1)^n}{n^n}\cdot(n+1)}=\lim_{n\to\infty}\frac{1}{\left(1+\dfrac{1}{n}\right)^n}=\frac{1}{e}<1.$$

Thus the series converges.

(2).

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\dfrac{[(n+1)!]^2}{(2n+2)!}}{\dfrac{(n!)^2}{(2n)!}}=\lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$=\lim_{n\to\infty}\frac{(n+1)^2/n^2}{(2n+2)(2n+1)/n^2}=\lim_{n\to\infty}\frac{1+2/n+1/n^2}{(2+2/n)(2+1/n)}=\frac{1+0+0}{(2+0)\cdot(2+0)}=\frac{1}{4}<1.$$

Thus the series converges.

2.4. Root Test.

THEOREM 2.12. *Consider the series* $\sum_{n=1}^{\infty} a_n$ *with each* $a_n \geq 0$, *and let*

$$\ell=\overline{\lim_{n\to\infty}}\sqrt[n]{a_n}. \tag{4}$$

(i) *If* $0\leq \ell<1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) *If* $1<\ell\leq\infty$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

(iii) *If* $\ell=1$, *then the test is inconclusive.*

PROOF. We will prove (i) and (ii).

(i) Suppose that $\ell<1$. Then for all given $\epsilon>0$, it follows from (4) and Proposition 8.9 of chapter 1, that there exists an N such that $\sqrt[n]{a_n}<\ell+\epsilon$ for all $n>N$. Now choose $\epsilon>0$ s.t. $\ell+\epsilon<1$. Then

$$0\leq a_n<(\ell+\epsilon)^n \quad \text{for all } n>N. \tag{5}$$

Since $\sum_{n=1}^{\infty}(\ell+\epsilon)^n$ converges (as it is a geometric series with common ratio satisfying $|r|=\ell+\epsilon<1$), it follows from (5) and the comparison test that $\sum_{n=1}^{\infty} a_n$ converges.

(ii). We are going to prove (ii) by contradiction. Given that $\ell>1$. Suppose that $\sum_{n=1}^{\infty} a_n$ converges. Then by Theorem 1.7, we have $\lim_{n\to\infty} a_n=0$. In particular, there exists N such that $0 \leq a_n < 1$ for all $n > N$. Hence $\sqrt[n]{a_n} < 1$ for all $n > N$, and it follows that we must have $\ell \leq 1$, which is a contradiction. Hence $\sum_{n=1}^{\infty} a_n$ diverges. $\square$

COROLLARY 2.13 (Simplified Root Test). *Consider the series* $\sum_{n=1}^{\infty} a_n$ *with each* $a_n \geq 0$. *Suppose that* $\lim_{n\to\infty} \sqrt[n]{a_n} = \ell$.

(i) *If* $0 \leq \ell < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) *If* $1 < \ell \leq \infty$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

(iii) *If* $\ell = 1$, *then the test is inconclusive.*

PROOF. We will prove (i) and (ii). Recall from Theorem 8.13 of chapter 1 that if $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists, then $\lim_{n\to\infty} \sqrt[n]{a_n} = \overline{\lim_{n\to\infty}} \sqrt[n]{a_n}$. Then the Corollary follows from Theorem 2.12. $\square$

EXAMPLE 2.14. *Determine convergence or divergence of the series*

$$\sum_{n=1}^{\infty} 2^n \left(1 - \frac{1}{n}\right)^{n^2}.$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left[2^n \left(1 - \frac{1}{n}\right)^{n^2}\right]^{\frac{1}{n}} = \lim_{n\to\infty} 2\left(1 - \frac{1}{n}\right)^n = \frac{2}{e} < 1.$$

Thus the series converges.

EXAMPLE 2.15. *Determine convergence or divergence of the series*

$$\sum_{n=1}^{\infty} (3 + \sin n)^n \left(1 - \frac{2}{n}\right)^{n^2}.$$

$$\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = \overline{\lim_{n\to\infty}} \left[(3 + \sin n)^n \left(1 - \frac{2}{n}\right)^{n^2}\right]^{\frac{1}{n}}$$

$$= \overline{\lim_{n\to\infty}} (3 + \sin n)\left(1 - \frac{2}{n}\right)^n \leq \overline{\lim_{n\to\infty}} 4\left(1 - \frac{2}{n}\right)^n = \frac{4}{e^2} < 1.$$

Thus the series converges.

3. The Dirichlet Test and Alternating Series

3.1. The Dirichlet Test. The following theorem is due to Neils Abel (1802-1829).

THEOREM 3.1 (Abel Partial Summation Formula). *Let $\{a_k\}$ and $\{b_k\}$ be sequences, and let $\{A_k\}_{k\geq 0}$ be a sequence such that*

$$A_k - A_{k-1} = a_k$$

for each $k \geq 1$. Then if $1 \leq p \leq q$,

$$\sum_{k=p}^{q} a_k b_k = A_q b_q - A_{p-1} b_p + \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}).$$

PROOF. Since $a_k = A_k - A_{k-1}$,

$$a_p b_p + a_{p+1} b_{p+1} + \cdots + a_q b_q$$

$$= (A_p - A_{p-1}) b_p + (A_{p+1} - A_p) b_{p+1} + (A_{p+2} - A_{p+1}) b_{p+2} + \cdots + (A_q - A_{q-1}) b_q$$

$$= -A_{p-1} b_p + A_p (b_p - b_{p+1}) + A_{p+1} (b_{p+1} - b_{p+2}) + \cdots + A_{q-1} (b_{q-1} - b_q) + A_q b_q$$

$$= -A_{p-1} b_p + A_q b_q + \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}).$$

□

Remark. There are two canonical choices of $\{A_n\}$

(1). $A_0 = 0$, $A_n = \sum_{k=1}^{n} a_k$. Then $A_n - A_{n-1} = a_n$ for all n.

(2). Suppose that $\sum_{k=1}^{\infty} a_k$ converges. Then we can also choose $\{A_n\}$ by letting $A_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{\infty} a_k = -\sum_{k=n+1}^{\infty} a_k$, $n \geq 1$, $A_0 = -\sum_{k=1}^{\infty} a_k$. In this case, we also have

$$A_n - A_{n-1} = \left(-\sum_{k=n+1}^{\infty} a_k \right) - \left(-\sum_{k=n}^{\infty} a_k \right) = a_n.$$

An application is to give the following theorem of Peter Lejeune Dirichlet (1805-1859).

THEOREM 3.2 (Dirichlet Test). *Suppose that $\{a_k\}$ and $\{b_k\}$ are sequences of real numbers satisfying the following:*

(i). *the sequence of the partial sums $A_n = \sum_{k=1}^{n} a_k$ is bounded,*

(ii). $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$, *and*

(iii). $\lim_{k\to\infty} b_k = 0$.

Then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.

PROOF. Since $\{A_n\}$ is bounded, there exists a positive number $M > 0$ such that $|A_n| < M$ for all n. Also, since $\lim_{n\to\infty} b_n = 0$, given $\epsilon > 0$, there exists N such that $b_n = |b_n - 0| < \frac{\epsilon}{2M}$ for all $n > N$. Now, for $m > n > N$, by Abel partial summation formula,

$$\left|\sum_{k=n+1}^{m} a_k b_k\right| = \left|A_m b_m - A_n b_{n+1} + \sum_{k=n+1}^{m-1} A_k (b_k - b_{k+1})\right|$$

$$\le |A_m| \cdot |b_m| + |A_n| \cdot |b_{n+1}| + \sum_{k=n+1}^{m-1} |A_k| \cdot |b_k - b_{k+1}|$$

$$\le M \cdot b_m + M \cdot b_{n+1} + \sum_{k=n+1}^{m-1} M \cdot (b_k - b_{k+1}) \qquad \text{(because} \quad b_k \ge b_{k+1} \ge 0, \quad |A_k| \le M)$$

$$= M \cdot [b_m + b_{n+1} + (b_{n+1} - b_{n+2}) + (b_{n+2} - b_{n+3}) + \cdots + (b_{m-2} - b_{m-1}) + (b_{m-1} - b_m)]$$

$$= 2M b_{n+1} < 2M \cdot \frac{\epsilon}{2M} = \epsilon.$$

Hence by the Cauchy Criterion, the series $\sum_{k=1}^{\infty} a_k b_k$ converges. □

3.2. Alternating Series Test. An **alternating series** is of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots, \quad \text{or}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$$

with each $a_n > 0$.

EXAMPLE 3.3.

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$-1 + 2 - 3 + 4 - 5 + 6 - \cdots$$

THEOREM 3.4 (The Alternating Series test). *If $\{b_n\}$ is a sequence satisfying*

(i) $b_1 \ge b_2 \ge b_3 \ge \cdots \ge 0$, *and*

(ii) $\lim_{n\to\infty} b_n = 0$,

then $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ *(and* $\sum_{n=1}^{\infty} (-1)^n b_n$*)* **converge**.

PROOF. Let $a_k = (-1)^{k+1}$ and let $A_n = \sum_{k=1}^{n} a_k$. Then

$$|A_n| = \left|1 - 1 + 1 - 1 + \cdots + (-1)^n + (-1)^{n+1}\right| = \begin{cases} 0 & \text{when } n \text{ even} \\ 1 & \text{when } n \text{ odd} \end{cases}$$

Thus $|A_n| \le 1$ for all n, and the Dirichlet test applies. □

EXAMPLE 3.5. *Show that convergence or divergence of the series*

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^p} = \begin{cases} \textit{convergence} & p > 0 \\ \textit{divergence} & p \leq 0. \end{cases}$$

PROOF. If $p \leq 0$, then the n-th term $(-1)^n\frac{1}{n^p}$ does not tend to 0. Thus the series diverges in this case by the divergence test.

Assume that $p > 0$. Let $a_n = \frac{1}{n^p}$. Then $a_n > 0$, monotone decreasing and $\lim_{n\to\infty} a_n = 0$. Thus the alternating series converges in this case.

In conclusion, we have that the series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^p}$ converges when $p > 0$ and diverges when $p \leq 0$. □

Now we are going to give a theorem providing an estimate on the sum of (certain) series. For instance, let $S = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$. By taking the partial sum, we have

$$S \approx S_1 = 1,\ S \approx S_2 = 1 - \frac{1}{2^2} = 0.75,\ S \approx S_3 = 1 - \frac{1}{2^2} + \frac{1}{3^2} \approx 0.861, \ldots.$$

By using computer program, we are able to compute much more, say $S_{1000000}$. A mathematical problem is then what is the 'error' for estimating S by using the partial sum S_n. In other words, how to estimate the remainder

$$R_n = |S - S_n| = |a_{n+1} + a_{n+2} + \cdots|.$$

THEOREM 3.6 (Alternating Series Estimation). *Let $\{b_n\}$ be a sequence satisfying*

(i) $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$, *and*
(ii) $\lim_{n\to\infty} b_n = 0$.

Let

$$S_n = \sum_{k=1}^{n}(-1)^{k+1}b_k \qquad \text{and} \qquad S = \sum_{k=1}^{\infty}(-1)^{k+1}b_k$$

Then the remainder $R_n = |S - S_n| \leq b_{n+1}$ for all n.

PROOF.

$$\begin{aligned} R_n &= \left|(-1)^{n+2}b_{n+1} + (-1)^{n+3}b_{n+2} + \cdots\right| \\ &= \left|b_{n+1} - b_{n+2} + b_{n+3} - b_{n+4} + \cdots\right|. \end{aligned}$$

Since

$$\begin{aligned} &b_{n+1} - b_{n+2} + b_{n+3} - b_{n+4} + \cdots \\ = {}& b_{n+1} - (b_{n+2} - b_{n+3}) - (b_{n+4} - b_{n+5}) - (b_{n+6} - b_{n+7}) - \cdots \leq b_{n+1} \end{aligned}$$

and

$$b_{n+1} - b_{n+2} + b_{n+3} - b_{n+4} + \cdots = (b_{n+1} - b_{n+2}) + (b_{n+3} - b_{n+4}) + \cdots \geq 0,$$

we have $R_n \leq a_{n+1}$. □

EXAMPLE 3.7. *Estimate* $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$ *with error within* 0.001.

SOLUTION. From $\frac{1}{(n+1)^4} \leq 10^{-3}$, we have $n+1 \geq 6$ or $n \geq 5$. Thus

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4} \approx 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4}$$

with error within 0.001. □

3.3. Trigonometric Series. Another application of Dirichlet test is to convergence of trigonometric series. These series will be studied in much detail in the course on Fourier series with a lot of applications in physics and other sciences. (Joseph Fourier, 1768-1830.) We require the following two identities.

LEMMA 3.8. *For* $t \neq 2p\pi$, $p \in \mathbb{Z}$,

$$\sum_{k=1}^{n} \sin kt = \frac{\cos\frac{1}{2}t - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}$$

$$\sum_{k=1}^{n} \cos kt = \frac{\sin\left(n+\frac{1}{2}\right)t - \sin\frac{1}{2}t}{2\sin\frac{1}{2}t}$$

PROOF. We prove the second identity. A proof of the first identity can be found from text book [**2**, pp.297].

Using the trigonometric identity

$$\sin x \cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right),$$

we obtain

$$\sin\frac{1}{2}t\sum_{k=1}^{n}\cos kt = \sum_{k=1}^{n}\sin\frac{1}{2}t\cos kt = \frac{1}{2}\sum_{k=1}^{n}\left[\sin\left(\frac{t}{2}+kt\right) + \sin\left(\frac{t}{2}-kt\right)\right]$$

$$= \frac{1}{2}\sum_{k=1}^{n}\left[\sin\left(k+\frac{1}{2}\right)t - \sin\left(k-\frac{1}{2}\right)t\right]$$

$$= \frac{1}{2}\left\{\left[\sin\left(1+\frac{1}{2}\right)t - \sin\left(1-\frac{1}{2}\right)t\right] + \left[\sin\left(2+\frac{1}{2}\right)t - \sin\left(2-\frac{1}{2}\right)t\right]\right.$$

$$\left. + \left[\sin\left(3+\frac{1}{2}\right)t - \sin\left(3-\frac{1}{2}\right)t\right] + \cdots + \left[\sin\left(n+\frac{1}{2}\right)t - \sin\left(n-\frac{1}{2}\right)t\right]\right\}$$

$$= \frac{1}{2}\left[\sin\left(n+\frac{1}{2}\right)t - \sin\frac{1}{2}t\right].$$

□

THEOREM 3.9 (Trigonometric Series Test). *Let* $\{b_n\}$ *be a sequence satisfying*

(i) $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$, *and*
(ii) $\lim_{n\to\infty} b_n = 0$.

Then

(a). *The series* $\sum_{k=1}^{\infty} b_k \sin kt$ *converges for all* $t \in \mathbb{R}$, *and*

(b). *The series* $\sum_{k=1}^{\infty} b_k \cos kt$ *converges for all* $t \in \mathbb{R}$, *except perhaps* $t = 2p\pi$, $p \in \mathbb{Z}$.

PROOF. (a). Let $a_k = \sin kt$ and let $A_n = \sum_{k=1}^{n} a_k$. If $t = 2p\pi$, then $a_k = \sin 2kp\pi = 0$ and so $A_n = 0$ for $t = 2p\pi$ with $p \in \mathbb{Z}$. If $t \neq 2p\pi$, then

$$|A_n| = \left|\sum_{k=1}^{n} \sin kt\right| = \left|\frac{\cos \frac{1}{2}t - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}\right|$$

$$\leq \frac{\left|\cos\frac{1}{2}t\right| + \left|\cos\left(n+\frac{1}{2}\right)t\right|}{2\left|\sin\frac{1}{2}t\right|} \leq \frac{1+1}{2\left|\sin\frac{1}{2}t\right|} = \frac{1}{\left|\sin\frac{1}{2}t\right|}.$$

Thus $\{|A_n|\}$ is bounded above, and the Dirichlet test applies.

(b). Let $a_k = \cos kt$ and let $A_n = \sum_{k=1}^{n} a_k$. The Dirichlet test applies because

$$|A_n| = \left|\sum_{k=1}^{n} \cos kt\right| = \left|\frac{\sin\left(n+\frac{1}{2}\right)t - \sin\frac{1}{2}t}{2\sin\frac{1}{2}t}\right|$$

$$\leq \frac{\left|\sin\left(n+\frac{1}{2}\right)t\right| + \left|\sin\frac{1}{2}t\right|}{2\left|\sin\frac{1}{2}t\right|} \leq \frac{1+1}{2\left|\sin\frac{1}{2}t\right|} = \frac{1}{\left|\sin\frac{1}{2}t\right|}.$$

□

EXAMPLE 3.10. *Determine convergence or divergence of the series* $\sum_{k=1}^{\infty} \frac{\cos kt}{k^q}$, $t \in \mathbb{R}$, $q > 0$.

SOLUTION. When $t = 2p\pi$ with $p \in \mathbb{Z}$, the series

$$\sum_{k=1}^{\infty} \frac{\cos kt}{k^q} = \sum_{k=1}^{\infty} \frac{1}{k^q}$$

and so it converges for $q > 1$ and diverges for $q \leq 1$.

When $t \neq 2p\pi$, then the series converges by the trigonometric series test because, for $q > 0$, the sequence $\dfrac{1}{k^q}$ is positive, monotone decreasing and $\lim_{k\to\infty} \dfrac{1}{k^q} = 0$. In conclusion, the series

$$\sum_{k=1}^{\infty} \frac{\cos kt}{k^q} \quad \text{is} \quad \begin{cases} \text{convergent} & q > 1, t \in \mathbb{R} \\ \text{convergent} & 0 < q \leq 1, t \in \mathbb{R}, t \neq 2p\pi, \quad \text{for any} \quad p \in \mathbb{Z} \\ \text{divergent} & 0 < q \leq 1, t = 2p\pi \quad \text{for some} \quad p \in \mathbb{Z} \end{cases}$$

□

4. Absolute and Conditional Convergence

Definition 4.1. *A series* $\sum_{n=1}^{\infty} a_n$ *is called* **absolutely convergent** *if* $\sum_{n=1}^{\infty} |a_n|$ *converges.*

Theorem 4.2. *Every absolutely convergent series is convergent.*

Proof. Suppose that $\sum_{n=1}^{\infty} a_n$ converges absolutely, that is, $\sum_{n=1}^{\infty} |a_n|$ converges by the definition. Let $T_n = \sum_{k=1}^{n} |a_k|$, $S_n = \sum_{k=1}^{n} a_k$. Since $\{T_n\}$ converges, $\{T_n\}$ is Cauchy. Thus, for any $\epsilon > 0$, there is a N such that $|T_n - T_m| < \epsilon$ for all $n, m > N$. For any $n, m > N$, we may assume that $m \geq n$, say $m = n + p$ (as one of them should be greater than another). Then

$$|S_n - S_m| = |S_n - (S_n + a_{n+1} + a_{n+2} + \cdots + a_{n+p})| = |a_{n+1} + a_{n+2} + \cdots + a_{n+p}|$$

$$\leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}| = T_m - T_n = |T_n - T_m| < \epsilon.$$

Thus $\{S_n\}$ is a Cauchy sequence and so it converges. Thus the series $\sum_{n=1}^{\infty} a_n$ converges and hence the result. □

Example 4.3. *Determine convergence or divergence of the series*

$$\sum_{n=2}^{\infty} \frac{\sin n + \frac{1}{2}}{n(\ln n)^2}.$$

Solution. Since

$$\left| \frac{\sin n + \frac{1}{2}}{n(\ln n)^2} \right| \leq \frac{|\sin n| + \frac{1}{2}}{n(\ln n)^2} \leq \frac{1 + \frac{1}{2}}{n(\ln n)^2}$$

and $\displaystyle\sum_{n=2}^{\infty}\frac{1+\frac{1}{2}}{n(\ln n)^2}=\frac{3}{2}\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges by Example 2.9, the series $\displaystyle\sum_{n=2}^{\infty}\left|\frac{\sin n+\frac{1}{2}}{n(\ln n)^2}\right|$ converges. Thus the series $\displaystyle\sum_{n=2}^{\infty}\frac{\sin n+\frac{1}{2}}{n(\ln n)^2}$ converges. □

Remark. If you are testing for absolute convergence, all the techniques for the positive series are applicable.

Q: Is the converse of the Corollary true? I.e., if a series is convergent, will it be absolutely convergent?
A: No, it is not necessarily true.

Example 4.4. The series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ converges by Example 3.5, but it is NOT absolutely convergent by the p-series.

Definition 4.5. *A series* $\displaystyle\sum_{n=1}^{\infty}a_n$ *is said to be* **conditionally convergent** *if* $\displaystyle\sum_{n=1}^{\infty}a_n$ *converges but* $\displaystyle\sum_{n=1}^{\infty}|a_n|$ *diverges.*

Example 4.6. The series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ is conditionally convergent.

Remark 4.7. *Every series is either absolutely convergent, conditionally convergent or divergent.* □

Example 4.8. *The series*

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^p}=\begin{cases}\textit{absolutely convergence} & p>1\\ \textit{conditionally convergence} & 0<p<1\\ \textit{divergence} & p\le 0\end{cases}$$

5. Remarks on the various tests for convergence/divergence of series

1. **n-th term test for divergence:**
   - a test for divergence ONLY, and it works for series with positive and negative terms, e.g. $\displaystyle\sum_{n=1}^{\infty}(-1)^n$.
2. **Comparison test/Limit Comparison test:**
   - when applying these tests, one usually compares the given series with a geometric series or a p-series.
   - generally works for series which look like the geometric series or the p-series,

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{4^n}, \quad \sum_{n=1}^{\infty} \frac{2^{\frac{1}{n}}}{n^2}$.

- when an oscillating factor/term appears, e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{3^n}$, try the Comparison test rather than the Limit Comparison test.

3. **Integral test:**

e.g. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

4. **Ratio test:**

- generally works for series which look like the geometric series, series with $n!$, and certain series defined recursively,

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}, \quad \sum_{n=1}^{\infty} \frac{(2n)!}{4^n \cdot n!}$,

$\displaystyle\sum_{n=1}^{\infty} a_n$, where $a_1 = 1$, $a_n = (\frac{1}{2} + \frac{1}{n})a_{n-1}$, $n = 2, 3, \cdots$.

5. **(Simplified) Root test:**

- generally works for series where a_n involves a high power such as the n-th power,

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}, \sum_{n=1}^{\infty} 2^n \left(1 - \frac{1}{n}\right)^{n^2}$.

6. **Alternating Series test:** - works for alternating series only,

e.g. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$.

Remark. In general, Tests 2 - 5 works only for $\displaystyle\sum_{n=1}^{\infty} a_n$, where $a_n \geq 0$.

CHAPTER 3

Sequences and Series of Functions

1. Pointwise Convergence

1.1. Sequences of Functions. Let I be a (nonempty) subset of $\mathbb{R}$, e.g. $(-1,1)$, $[0,1]$, etc. For each $n \in \mathbb{N}$, let $F_n : I \to \mathbb{R}$ be a function. Then we say $\{F_n\}$ forms a **sequence of functions** on I.

EXAMPLE 1.1. 1. $F_n(x) = x^n, \qquad 0 < x < 1$. Then $\{F_n\}$ forms a sequence of functions on (0,1).
2. $\left\{\left(1+\dfrac{x}{n}\right)^n\right\}$ forms a sequence of functions on $(-\infty, \infty)$.
Write out some terms:

$$F_1(x) = 1 + x \quad F_2(x) = \left(1+\frac{x}{2}\right)^2 \quad F_3(x) = \left(1+\frac{x}{3}\right)^3$$

If we fix the x, and let $n \to \infty$,

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

So for each x, we can define

$$F(x) = \lim_{n\to\infty} F_n(x).$$

DEFINITION 1.2. A sequence $\{F_n\}$ is said to **converge pointwise** to a function F on I if

$$\lim_{n\to\infty} F_n(x) = F(x) \qquad \text{for each } x \in I,$$

i.e., for each $x \in I$ and given any $\epsilon > 0$, there exists an N (which depends on x and ϵ) such that

$$|F_n(x) - F(x)| < \epsilon \qquad \forall n > N.$$

The function F is called the **limiting function** of $\{F_n\}$.

Remark. The limiting function is necessarily unique.

EXAMPLE 1.3. The sequence $\{x^n\}$ converges pointwise on $I = (0,1)$ because the limit $F(x) = \lim_{n\to\infty} x^n = 0$ exists for any $0 < x < 1$.

The sequence $\{x^n\}$ does NOT converge on $[-1,1]$ because the limit $\lim_{n\to\infty} x^n$ does not exist when $x = -1 \in [-1,1]$.

1.2. Series of Functions. A **series of functions** on a set I is of the form

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + \cdots ,$$

where each f_n is a function on I.

EXAMPLE 1.4. Below are some examples.

1. $\displaystyle\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \cdots$
2. $\displaystyle\sum_{k=1}^{\infty} \frac{\sin kx}{k+x} = \frac{\sin x}{1+x} + \frac{\sin 2x}{2+x} + \frac{\sin 3x}{3+x} + \cdots, \qquad 0 \le x \le 1.$

As in chapter 2, we may form the **partial sums**

$$S_n(x) = \sum_{k=1}^{n} f_k(x) = f_1(x) + f_2(x) + \cdots + f_n(x).$$

Then $\{S_n\}$ forms a sequence of functions on I.

DEFINITION 1.5. The series $\displaystyle\sum_{n=1}^{\infty} f_n$ is said to **converge pointwise** (to a function S) on I if $\{S_n\}$ converges pointwise (to S) on I, (i.e. $\displaystyle\lim_{n\to\infty} S_n(x) = S(x)$ for each $x \in I$.)

EXAMPLE 1.6. *What is the pointwise limit of* $\displaystyle\sum_{n=1}^{\infty} x^{n-1}$*, where* $x \in (-1, 1)$*? Does* $\displaystyle\sum_{n=1}^{\infty} x^{n-1}$ *converge pointwise on* $[-1, 1)$

SOLUTION. Consider the partial sum

$$S_n(x) = \sum_{i=1}^{n} x^{i-1} = 1 + x + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$$

for $-1 < x < 1$. Thus

$$\sum_{n=1}^{\infty} x^{n-1} = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1}{1-x}$$

for $x \in (-1, 1)$.

Since the series $\displaystyle\sum_{n=1}^{\infty} x^{n-1}$ diverges when $x = -1$, the series of functions $\displaystyle\sum_{n=1}^{\infty} x^{n-1}$ does NOT converge pointwise on $[-1, 1)$. □

1.3. Some Questions on Pointwise Convergence. Suppose a sequence of functions $\{F_n\}$ converges pointwise to a function F on the interval $[a, b]$. Also suppose a series of functions $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to a function $S(x)$ on $[a, b]$. Among the questions we want to consider the following. Some of these questions were incorrectly believed to be true by many mathematicians prior to nineteenth century, including the famous Cauchy. Cauchy in his text *Cours d'Analyse* "proved" a theorem to the effect that the limit of a convergent sequence of continuous functions was again continuous. As we will see, this result is false!

Question (a). If each F_n is continuous at $p \in [a,b]$, is F necessarily continuous at p? Recall that F is continuous at p if and only if

$$\lim_{t\to p} F(t) = F(p).$$

Since $F(x) = \lim_{n\to\infty} F_n(x)$ for every $x \in [a,b]$, what we really asking is does

$$\lim_{t\to p}\left(\lim_{n\to\infty} F_n(t)\right) = \lim_{n\to\infty}\left(\lim_{t\to p} F_n(t)\right) \ ?$$

Question (a'). For series of functions, we can ask similar question. If each $f_n(x)$ is continuous at p, is $S(x) = \sum_{n=1}^{\infty} f_n(x)$ necessarily continuous at p? Again what we really asking is does

$$\lim_{t\to p}\sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty}\lim_{t\to p} f_n(t) \ ?$$

Question (b). If each F_n is differentiable on $[a,b]$, is F necessarily differentiable on $[a,b]$? If so, does

$$\frac{d}{dx}\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty}\frac{d}{dx}F_n(x) \ ?$$

Question (b'). If each f_n is differentiable on $[a,b]$, is $S(x) = \sum_{n=1}^{\infty} f_n(x)$ necessarily differentiable on $[a,b]$? If so, does

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty}\frac{d}{dx}f_n(x) \ ?$$

Question (c). If each F_n is Riemann integrable on $[a,b]$, is F necessarily Riemann integrable on $[a,b]$? If so, does

$$\int_a^b \lim_{n\to\infty} F_n(x)\ dx = \lim_{n\to\infty}\int_a^b F_n(x)\ dx \ ?$$

Question (c'). If each f_n is Riemann integrable on $[a,b]$, is $S(x) = \sum_{n=1}^{\infty} f_n(x)$ necessarily Riemann integrable on $[a,b]$? If so, does

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\ dx = \sum_{n=1}^{\infty}\int_a^b f_n(x)\ dx \ ?$$

Without **additional hypothesis** the answer to all of these questions is generally **no**. This additional hypothesis is so-called **uniform convergence**, introduced by Weierstrass in 1850. For his many contributions to the subject area, Weierstrass is often referred to as the father of modern analysis. The subsequent study on the subject together with questions from geometry and physics also lead to a new area called **topology** in the end of 19th century. Poincaré is often referred to as the father of topology. Topology together with Riemann geometry provide the mathematics foundation for Einstein's relativity theory. You may learn some basic knowledge of topology and modern geometry in 4000 and 5000 modules.

Below we only give counter-examples to Questions (a) and (c). You may read the text book [**2**] for more examples.

EXAMPLE 1.7 (Counter-example to Question (a)). Consider the functions

$$F_n(x) = x^n, \quad x \in [0,1].$$

For each fixed $x \in [0,1)$, we have

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} x^n = 0 \quad (\text{since } |x| < 1).$$

At $x = 1$, we have

$$\lim_{n\to\infty} F_n(1) = \lim_{n\to\infty} 1^n = 1.$$

Thus $\{F_n\}$ converges pointwise to the function F on the interval $[0,1]$ given by

$$F(x) = \begin{cases} 0, & \text{for } x \in [0,1), \\ 1, & x = 1. \end{cases}$$

Each F_n is continuous on the whole interval $[0,1]$, but F is not continuous at $x = 1$.

This simple example gives a counter example to Cauchy's (false) statement that the limit of continuous functions is continuous, namely the limit of continuous functions need not be continuous.

EXAMPLE 1.8 (Counter-example to Question (c)). Consider the functions

$$F_n(x) = \begin{cases} n^2 x, & 0 < x < \frac{1}{n}, \\ 2n - n^2 x, & \frac{1}{n} \le x < \frac{2}{n}, \\ 0, & \frac{2}{n} \le x < 1. \end{cases}$$

For each fixed $x \in (0,1]$, one sees that $F_n(x) = 0$ whenever $n \ge \frac{2}{x}$, and hence

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} 0 = 0.$$

Also, at $x = 0$, we have

$$\lim_{n\to\infty} F_n(0) = \lim_{n\to\infty} n^2 \cdot 0 = 0.$$

Thus, $\{F_n\}$ converges pointwise to the zero function $F(x) \equiv 0$ on the interval $[0,1]$. For each $n \ge 1$, we have

$$\begin{aligned} \int_0^1 F_n(x)\,dx &= \int_0^{1/n} n^2 x\,dx + \int_{1/n}^{2/n} (2n - n^2 x)\,dx + \int_{2/n}^1 0\,dx \\ &= \frac{n^2 x^2}{2}\bigg|_0^{\frac{1}{n}} + \left(2nx - \frac{n^2 x^2}{2}\right)\bigg|_{\frac{1}{n}}^{\frac{2}{n}} + 0 \\ &= \frac{1}{2} + (4-2) - (2 - \frac{1}{2}) + 0 = 1. \end{aligned}$$

Thus we have

$$\lim_{n\to\infty} \int_0^1 F_n(x)\,dx = \lim_{n\to\infty} 1 = 1 \ne 0 = \int_0^1 F(x)\,dx, \quad \text{i.e.}$$

$$\lim_{n\to\infty}\int_0^1 F_n(x)\,dx \neq \int_0^1 \Big(\lim_{n\to\infty} F_n(x)\Big)\,dx.$$

Reason: At different x, $F_n(x)$ converges to $F(x)$ at different pace (more specifically, in the definition of pointwise convergence, the choice of N depends on both ϵ and x).

2. Uniform Convergence

We define a slightly different concept of convergence.

2.1. Uniform Convergence of Sequences of Functions.

DEFINITION 2.1. $\{F_n\}$ is said to **converge uniformly** to a function F on a set I if for every $\epsilon > 0$, there exists an N (which depends only on ϵ) such that

$$|F_n(x) - F(x)| < \epsilon$$

for ALL $x \in I$ whenever $n > N$.

REMARK 2.2. *If $\{F_n\}$ converges uniformly to F on I, then $\{F_n\}$ converges pointwise to F on I. Conversely, if $\{F_n\}$ converges pointwise to F on I, then $\{F_n\}$ need not converge uniformly to F on I.*

The inequality in the definition can be expressed as

$$F(x) - \epsilon < F_n(x) < F(x) + \epsilon$$

for all $x \in I$ and $n > N$. The geometric interpretation is that for $n > N$ the graph of $y = F_n(x)$ lies in the band spanned by the curves $y = F(x) - \epsilon$ and $y = F(x) + \epsilon$, that is, given any $\epsilon > 0$ the graph of $y = F_n(x)$ **eventually** lies in the band spanned by the curves $y = F(x) - \epsilon$ and $y = F(x) + \epsilon$.

EXAMPLE 2.3. The sequence of function $F_n = x^n$ converges pointwise to 0 on $[0, 1)$. But from the graph we can see that the graph of $F_n = x^n$ DOES NOT eventually lie in the band spanned by $y = -\epsilon$ and $y = +\epsilon$. The geometric reason also tells us that $\{x^n\}$ does not converge uniformly on $[0, 1)$.

2.2. Two Criteria for Uniform Convergence of $\{F_n\}$.

The following theorem is useful (computationally) in determining whether a sequence of functions converges uniformly or not.

THEOREM 2.4 (T-test). *Suppose $\{F_n\}$ is a sequence of functions converging pointwise to a function F on a set I, and let*

$$T_n = \sup_{x\in I} |F_n(x) - F(x)|.$$

Then $\{F_n\}$ converges uniformly to F on I if and only if $\lim_{n\to 0} T_n = 0$.

PROOF. First we prove the 'only if' part. Suppose that $\{F_n\}$ converges uniformly to F on I. Then for any given $\epsilon > 0$, there exists N such that

$$|F_n(x) - F(x)| < \frac{\epsilon}{2} \quad \text{for all } n > N \text{ and } x \in I$$

$$\Rightarrow \; T_n = \sup_{x\in I} |F_n(x) - F(x)| \le \frac{\epsilon}{2} < \epsilon \quad \text{for all } n > N$$

$$\Rightarrow \; |T_n - 0| = T_n < \epsilon \quad \text{for all } n > N.$$

Hence we have $\lim_{n\to 0} T_n = 0$.

Next we prove the 'if' part. Suppose that $\lim_{n\to 0} T_n = 0$. Then for any given $\epsilon > 0$, there exists N such that

$$|T_n - 0| = T_n < \epsilon \quad \text{for all } n > N$$

$$\Rightarrow \; \sup_{x\in I} |F_n(x) - F(x)| < \epsilon \quad \text{for all } n > N$$

$$\Rightarrow \; |F_n(x) - F(x)| < \epsilon \quad \text{for all } n > N \text{ and } x \in I.$$

Hence $\{F_n\}$ converges uniformly to F on I. This finishes the proof of the theorem. □

THEOREM 2.5 (Cauchy's Criterion). *A sequence of functions $\{F_n\}$ converges uniformly on a set I if and only if given any $\epsilon > 0$, there exists a natural number N such that*

$$|F_n(x) - F_m(x)| < \epsilon \quad \textit{for all } x \in I \textit{ and all } m, n > N. \tag{6}$$

Remark: Here N does not depend on x.

PROOF. First we prove the 'only if' part. Suppose that $\{F_n\}$ converges uniformly to the function F on I. Then given any $\epsilon > 0$, there exists N such that

$$|F_n(x) - F(x)| < \frac{\epsilon}{2} \quad \text{for all } x \in I \text{ and all } n > N.$$

Then for all $x \in I$ and $m, n > N$,

$$|F_n(x) - F_m(x)| = |(F_n(x) - F(x)) - (F_m(x) - F(x))|$$

$$\leq |F_n(x) - F(x)| + |F_m(x) - F(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This finishes the proof of the 'only if' part.

Next we prove the 'if' part. Suppose that equation 6 holds. Then for each fixed point $x \in I$, $\{F_n(x)\}$ is a Cauchy sequence of real numbers, and thus by Cauchy's criterion for sequences, the sequence of real numbers $\{F_n(x)\}$ converges. For each $x \in I$, we denote the limit by $F(x) = \lim_{n\to\infty} F_n(x)$. Then $\{F(x)\}_{x\in I}$ forms a function on I, which we denote by F. Given any $\epsilon > 0$, by equation 6, there exists N such that

$$|F_n(x) - F_m(x)| < \frac{\epsilon}{2} \quad \text{for all } x \in I \text{ and all } m, n > N.$$

Then for each fixed $x \in I$ and $n > N$, we have

$$|F_n(x) - F(x)| = |F_n(x) - \lim_{m\to\infty} F_m(x)|$$

$$= \lim_{m\to\infty} |F_n(x) - F_m(x)| \leq \lim_{m\to\infty} \frac{\epsilon}{2} = \frac{\epsilon}{2} < \epsilon.$$

Thus $\{F_n\}$ converges uniformly to F, and this finishes the proof of the 'if' part. □

2.3. Examples.

EXAMPLE 2.6. *Show that* $F_n(x) = \dfrac{\sin^2 x}{n}$, $\quad x \in (-\infty, +\infty)$, *converges uniformly.*

PROOF. The limiting function $F(x)$ is

$$F(x) = \lim_{n\to\infty} \frac{\sin^2 x}{n} = 0$$

for all $x \in (-\infty, +\infty)$. Since

$$T_n = \sup_{x\in(-\infty,+\infty)} |F_n(x) - F(x)| = \sup_{x\in(-\infty,+\infty)} \left|\frac{\sin^2 x}{n}\right| \le \frac{1}{n} \to 0$$

as $n \to \infty$, thus the sequence of functions $\{F_n(x\}$ converges uniformly on $(-\infty, +\infty)$. □

EXAMPLE 2.7. *Determine whether the following sequences of functions converge uniformly on the indicated interval.*

(a) $f_n(x) = \dfrac{n^2 \ln x}{x^n}$, $x \in [1, +\infty)$;

(b) $f_n(x) = \dfrac{n^2 \ln x}{x^n}$, $x \in [2, +\infty)$.

SOLUTION. Let $f(x) = \lim_{n\to\infty} f_n(x) = 0$ for $x \ge 1$.

(a). $T_n = \sup_{x\ge 1} |f_n(x) - 0| = \sup_{x\ge 1} \dfrac{n^2 \ln x}{x^n} = \sup_{x\ge 1} f_n(x)$. From

$$f_n'(x) = n^2 \frac{1}{x} \cdot x^{-n} - n^3 \ln x \cdot x^{-n-1} = \frac{n^2 - n^3 \ln x}{x^{n+1}} = 0,$$

we have $n^2 - n^3 \ln x = 0$ or $x = e^{\frac{1}{n}}$. Observe that $f_n(x)$ is monotone increasing for $1 \le x \le e^{\frac{1}{n}}$ and monotone decreasing for $x \ge e^{\frac{1}{n}}$. Thus

$$T_n = \max_{x\ge 1} f_n(x) = f_n(e^{\frac{1}{n}}) = \frac{n^2 \cdot \frac{1}{n}}{\left(e^{\frac{1}{n}}\right)^n} = \frac{n}{e} \not\to 0$$

as $n \to \infty$ and so $\{f_n(x)\}$ does NOT converge uniformly.

(b). Since $e^{\frac{1}{n}} \le 2$ for $n \ge 2$, the function $f_n(x)$ is monotone decreasing on $[2, +\infty)$ for $n \ge 2$ and so $T_n = \sup_{x\ge 2} |f_n(x) - f(x)| = f_n(2) = \dfrac{n^2 \ln 2}{2^n}$ for $n \ge 2$. Since $\lim_{n\to\infty} T_n = 0$, $\{f_n\}$ converges uniformly on $[2, +\infty)$. □

EXAMPLE 2.8. *Show that* $F_n(x) = \dfrac{n^2 \ln x \sin nx}{x^n}$ *converges uniformly on* $[2, +\infty)$.

SOLUTION. $F(x) = \lim_{n\to\infty} F_n(x) = 0$ for $x \ge 2$. Observe

$$T_n = \sup_{x\ge 2} |F_n(x) - F(x)| = \sup_{x\ge 2} \frac{n^2 \ln x |\sin nx|}{x^n} \le \frac{n^2 \ln 2}{2^n}$$

for $n \ge 2$. Since $\lim_{n\to\infty} T_n = 0$, $\{F_n\}$ converges uniformly. □

Remark. Let $\{F_n\}$ be a sequence of functions on an interval I. To see whether $\{F_n\}$ is uniformly convergent, we may try to do by the following steps.

(1). Determine the limiting function $F(x) = \lim_{n\to\infty} F_n(x)$.

(2). Determine $T_n = \sup_{x\in I} |F_n(x) - F(x)|$.

(3). Check whether $\lim_{n\to\infty} T_n = 0$.

If T_n is difficult to be determined, then we may try to estimate an upper bound of T_n (a lower bound of T_n if we guess that the sequence of functions might not be uniformly convergent).

2.4. Uniform Convergence of Series of Functions.

DEFINITION 2.9. $\sum_{n=1}^{\infty} f_n$ is said to **converge uniformly** (to S) on I if the sequence of its partial sums $\{S_n\}$ converges uniformly (to S) on I.

THEOREM 2.10 (T-test for Series of Functions). *Suppose* $\sum_{n=1}^{\infty} f_n(x)$ *is a series of functions converging pointwise on a set* I*, and let*

$$T_n = \sup_{x\in I} \left| \sum_{k=n+1}^{\infty} f_k(x) \right| .$$

Then $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly on* I *if and only if* $\lim_{n\to\infty} T_n = 0$.

PROOF. Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$ and $S(x) = \sum_{k=1}^{\infty} f_k(x)$. Then

$$|S_n(x) - S(x)| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right|$$

and so the T-test, Theorem 2.4, applies. □

THEOREM 2.11 (Cauchy Criterion). *A series of functions* $\sum_{n=1}^{\infty} f_n$ *converges uniformly on a set* I *if and only if given any* $\epsilon > 0$, *there exists a natural number* N *such that*

$$\left| \sum_{k=n+1}^{m} f_k(x) \right| < \epsilon \quad \text{for all } x \in I \text{ and all } m > n > N.$$

Remark: Here N does not depend on x.

PROOF. The proof follows by applying the Cauchy Criterion to the partial sums $S_n(x) = \sum_{k=1}^{n} f_k(x)$. □

The following test is very useful in verifying that certain series of functions converge uniformly to some functions on an interval.

THEOREM 2.12 (Weierstrass M-test). *Consider a series of functions* $\sum_{k=1}^{\infty} f_k$ *on a set* I. *Suppose that*

(i) $|f_k(x)| \le M_k$ *for all* $x \in I$, $k = 1, 2, \cdots$, *and*

(ii) $\sum_{k=1}^{\infty} M_k$ *converges.*

Then $\sum_{k=1}^{\infty} f_k$ *converges uniformly (to some function) on* I.

Remark. Weierstrass M-test **only** states that if a series of functions satisfies conditions (i) and (ii), it converges uniformly on I. If a series of functions **does not** satisfy these two conditions, the test fails, namely, **no conclusion** that you can claim from this test.

PROOF. Since $\sum_{k=1}^{\infty} M_k$ converges (by (ii)), by the Cauchy Criterion, given any $\epsilon > 0$, there exists N such that

$$\sum_{k=n+1}^{m} M_k = \left| \sum_{k=n+1}^{m} M_k \right| < \epsilon \quad \text{for all } m > n > N$$

because $M_k \ge 0$. Then for all $x \in I$, we have

$$\left| \sum_{k=n+1}^{m} f_k(x) \right| \le \sum_{k=n+1}^{m} |f_k(x)| \le \sum_{k=n+1}^{m} M_k < \epsilon.$$

Thus, by the Cauchy criterion, $\sum_{k=1}^{\infty} f_k$ converges uniformly on I. □

2.5. Examples.

EXAMPLE 2.13. *Show that* $\sum_{n=1}^{\infty} \frac{\cos^n x}{n^2 + x}$ *converges uniformly on* $(0, \infty)$.

PROOF. Since

$$\left| \frac{\cos^n x}{n^2 + x} \right| \le \frac{1}{n^2}$$

for all $x \in (0, \infty)$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p-series, the series of functions $\sum_{n=1}^{\infty} \frac{\cos^n x}{n^2 + x}$ converges uniformly by the Weierstrass M-test and hence the result. □

EXAMPLE 2.14. *Does the series of functions*

$$\sum_{n=1}^{\infty} n^2 x^n \sin nx$$

converge uniformly on the interval $[0, \frac{1}{2}]$*? Justify your answer.*

SOLUTION. Note that

$$\left| n^2 x^n \sin nx \right| \leq \frac{n^2}{2^n}$$

for $x \in [0, \frac{1}{2}]$. Since

$$\lim_{n\to\infty} \frac{\dfrac{(n+1)^2}{2^{n+1}}}{\dfrac{n^2}{2^n}} = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2} = \frac{1}{2} < 1,$$

the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges and so the series of functions $\sum_{n=1}^{\infty} n^2 x^n \sin nx$ converges uniformly on $[0, \frac{1}{2}]$ by the Weierstrass M-test. □

EXAMPLE 2.15. *Show that the series* $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ *converges uniformly on* $[0, 1]$.

Note. The M-test fails for this example because $\sup_{x\in[0,1]} \left| (-1)^{n+1} \frac{x^n}{n} \right| = \frac{1}{n}$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. In this case, we use T-test or Cauchy Criterion.

PROOF. Let $a_n(x) = \frac{x^n}{n}$. Then, for $0 \leq x \leq 1$, we have

$$a_1(x) \geq a_2(x) \geq \cdots \geq 0 \qquad \text{and} \qquad \lim_{n\to\infty} a_n(x) = 0.$$

Thus the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ converges pointwise on $[0, 1]$ by the Alternating Series Test. By the Alternating Series Estimation,

$$T_n = \sup_{0\leq x\leq 1} \left| \sum_{k=n+1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \right| \leq \sup_{0\leq x\leq 1} \frac{x^{n+1}}{n+1} = \frac{1}{n+1}.$$

Since $\lim_{n\to\infty} \frac{1}{n+1} = 0$, the series $(-1)^{n+1} \frac{x^n}{n}$ converges uniformly by the T-test. □

EXAMPLE 2.16. *Show that the series* $\sum_{n=1}^{\infty} \frac{x^n}{n}$ *does not converge uniformly on* $[0, 1)$.

PROOF. Observe that

$$T_n = \sup_{0\leq x<1} |S_n(x) - S(x)| = \sup_{0\leq x<1} \left| \sum_{k=n+1}^{\infty} \frac{x^k}{k} \right| = \sup_{0<x<1} \sum_{k=n+1}^{\infty} \frac{x^k}{k}.$$

$$= \sup_{0<x<1}\left(\frac{x^{n+1}}{n+1}+\frac{x^{n+2}}{n+2}+\cdots\right) \geq \sup_{0<x<1}\left(\frac{x^{n+1}}{n+1}+\frac{x^{n+2}}{n+2}+\cdots+\frac{x^{2n}}{2n}\right)$$

$$= \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} \qquad \text{because } \frac{x^k}{k} \text{ monotone increasing on } (0,1)$$

$$\geq \frac{1}{2n}+\frac{1}{2n}+\cdots+\frac{1}{2n} = n\cdot\frac{1}{2n}=\frac{1}{2}.$$

Thus the sequence $\{T_n\}$ does not tend to 0 and so the series of functions $\displaystyle\sum_{k=1}^{\infty}\frac{x^k}{k}$ does NOT converge uniformly on $[0,1)$ by the T-test. □

3. Uniform Convergence of $\{F_n\}$ and Continuity

In this section we will prove that the limit of uniformly convergent sequence of continuous functions is again continuous.

THEOREM 3.1. *Let $\{F_n\}$ be a sequence of continuous functions on an interval I. Suppose that $\{F_n\}$ converges uniformly to a function F on I. Then F is continuous on I.*

PROOF. Fix any point $x_0 \in I$. We are going to show that F is continuous at the point x_0. Given any $\epsilon > 0$, since $\{F_n\}$ converges uniformly to F on I, there exists N such that

$$|F_n(x) - F(x)| < \frac{\epsilon}{3} \quad \text{for all } x \in I \text{ and all } n > N.$$

Next we fix an $n > N$ (say, $n = [N] + 1$). Since F_n is continuous at x_0, there exists $\delta > 0$ (here δ depends on x_0 and ϵ) such that for all x satisfying $|x-x_0| < \delta$, we have

$$|F_n(x) - F_n(x_0)| < \frac{\epsilon}{3}.$$

Then for all x satisfying $|x - x_0| < \delta$, we have

$$\begin{aligned}|F(x) - F(x_0)| &= |F(x) - F_n(x) + F_n(x) - F_n(x_0) + F_n(x_0) - F(x_0)|\\ &\leq |F(x) - F_n(x)| + |F_n(x) - F_n(x_0)| + |F_n(x_0) - F(x_0)|\\ &< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Thus F is continuous at x_0. Since x_0 is arbitrary, it follows that F is continuous on I. This finishes the proof of the theorem. □

EXAMPLE 3.2. *Find the pointwise limit F of the sequence*

$$F_n(x) = \frac{x^{2n}}{1+x^{2n}}, \qquad x \in [0,1].$$

Show using Theorem 3.1 that the convergence is not uniform.

SOLUTION. If $0 \leq x < 1$, we have

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty}\frac{x^{2n}}{1+x^{2n}} = \frac{0}{1+0} = 0.$$

If $x = 1$, then $F_n(1) = \dfrac{1}{2}$. Thus the limiting function $F(x)$ is

$$F(x) = \begin{cases} 0 & 0 \le x < 1 \\ \\ \frac{1}{2} & \quad x = 1 \end{cases}$$

Because each $F_n(x)$ is continuous but $F(x)$ is not, the sequence of functions $\{F_n(x)\}$ does not converge uniformly on $[0, 1]$ by Theorem 3.1. □

It is possible that each F_n and $F(x) = \lim\limits_{n\to\infty} F_n(x)$ are continuous, but $\{F_n\}$ does not converge uniformly to $F(x)$.

Example 3.3. *Let $F_n(x) = nxe^{-nx^2}$. Show that*
1) *Each $F_n(x)$ is continuous on $I = [0, 1]$.*
2) *$\{F_n(x)\}$ converges pointwise to a continuous function on I.*
3) *$\{F_n(x)\}$ does not converge uniformly on I.*

Proof. Each $F_n(x)$ is continuous because it is a well-defined elementary function on $[0, 1]$. Let $F(x) = \lim\limits_{n\to\infty} F_n(x)$ on $[0, 1]$. When $x = 0$, $F_n(0) = 0$ for each n and $F(0) = \lim_{n\to\infty} 0 = 0$, when $x \neq 0$ (fixed),

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} nxe^{-nx^2} = \lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{n\to\infty} \frac{x}{e^{nx^2} \cdot x^2} = \lim_{n\to\infty} \frac{1}{xe^{nx^2}} = 0.$$

Thus $F(x) = 0$ is continuous on $[0, 1]$. Now $T_n = \sup\limits_{0\le x\le 1} |F_n(x) - F(x)| = \sup\limits_{0\le x\le 1} F_n(x)$. From

$$F_n'(x) = ne^{-nx^2} + nxe^{-nx^2} \cdot (-2nx) = ne^{-nx^2}(1 - 2nx^2) = 0,$$

we have $x = \pm\sqrt{\dfrac{1}{2n}}$. Since $F_n(0) = 0$ and $F_n(1) = ne^{-n}$,

$$\sup_{0\le x\le 1} F_n(x) = \max_{0\le x\le 1} F_n(x) = \max\left\{0,\ \frac{n}{e^n},\ n \cdot \frac{1}{\sqrt{2n}} \cdot e^{-n\cdot\frac{1}{2n}}\right\} = \sqrt{\frac{n}{2e}}.$$

Thus $T_n = \sqrt{\dfrac{n}{2e}}$. Since $\lim\limits_{n\to\infty} T_n = +\infty$, the sequence $\{F_n\}$ does not converge uniformly to $F(x)$ on $[0, 1]$. □

Corollary 3.4. *Suppose that $\sum\limits_{k=1}^{\infty} f_k$ converges uniformly to a function S on an interval I. Suppose that each f_k is continuous on I. Then S is also continuous on I.*

Proof. Consider the sequence of partial sums $\{S_n\}$ on I, where we have $S_n = \sum\limits_{k=1}^{n} f_k$. Then $\{S_n\}$ converges uniformly to S on I. If each f_k is continuous on I, then each S_n is also continuous on I. Then by Theorem 3.1, S is also continuous on I. □

Example 3.5. *Is $\sum\limits_{n=1}^{\infty} \dfrac{x}{n^2 e^{nx}}$, $\quad x \in (0, \infty)$, a continuous function?*

SOLUTION. Let $f_n(x) = \dfrac{x}{n^2 e^{nx}}$. We find an upper bound of $f_n(x)$. Observe that

$$f_n'(x) = \left(\frac{x}{n^2 e^{nx}}\right)' = \left(\frac{xe^{-nx}}{n^2}\right)' = \frac{e^{-nx} - nxe^{-nx}}{n^2} = \frac{e^{-nx}\left(\dfrac{1}{n} - x\right)}{n}.$$

We obtain that $f_n'(x) > 0$ for $0 < x < \frac{1}{n}$ and $f_n'(x) < 0$ for $x > \frac{1}{n}$. It follows that $f_n(x)$ is monotone increasing on $(0, \frac{1}{n}]$ and monotone decreasing on $[\frac{1}{n}, +\infty)$. Thus

$$\sup_{0<x<+\infty} f_n(x) = \max_{0<x<+\infty} f_n(x) = f_n\left(\frac{1}{n}\right) = \frac{\frac{1}{n}}{n^2 e^{n\cdot\frac{1}{n}}} = \frac{1}{en^3}.$$

Let $M_n = \dfrac{1}{en^3}$. Then $|f_n(x)| \leq M_n$ for $x \in (0, \infty)$. Since $\displaystyle\sum_{n=1}^{\infty} M_n$ converges by the p-series, the series of functions $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n^2 e^{nx}}$ converges uniformly by Weierstrass M-test on $(0, \infty)$. According to Corollary 3.4, the function $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n^2 e^{nx}}$ is continuous on $(0, \infty)$. □

It is possible that each f_n and $\displaystyle\sum_{n=1}^{\infty} f_n$ are continuous, but $\displaystyle\sum_{n=1}^{\infty} f_n$ does not converge uniformly.

EXAMPLE 3.6. Consider the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

for $-1 < x < 1$. In this case, each x^n and $\displaystyle\sum_{n=0}^{\infty} x^n$ are continuous on $I = (0, 1)$. We show that $\displaystyle\sum_{n=0}^{\infty} x^n$ does not converge uniformly by T-test. Since

$$\begin{aligned} T_n &= \sup_{-1<x<1} \left|x^{n+1} + x^{n+2} + \cdots\right| \geq \sup_{0\leq x<1} \left|x^{n+1} + x^{n+2} + \cdots\right| \\ &= \sup_{0\leq x<1} \left(x^{n+1} + x^{n+2} + \cdots\right) \geq \sup_{0\leq x<1} x^{n+1} = 1, \end{aligned}$$

the sequence $\{T_n\}$ does not tend to 0 and so the series $\displaystyle\sum_{n=0}^{\infty} x^n$ does not converge uniformly on $(-1, 1)$ by the T-test. □

4. Uniform Convergence and Integration

Before we go on, we first recall some facts about Riemann integrals (Reference: Our Text Book [**2**, Chapter 6, pp. 208-216]).

4.1. Review of Riemann Integration. Let f be a bounded function on a finite interval $[a, b]$. A **partition** P of $[a, b]$ is a set of points $\{x_0, x_1, \cdots, x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

For such a partition P and $i = 1, 2, \cdots n$, we denote

$$M_i(f) = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

$$m_i(f) = \inf_{x \in [x_{i-1}, x_i]} f(x).$$

The **upper (Riemann) sum** of f with respect to the partition P is defined to be

$$U(P, f) := \sum_{i=1}^{n} M_i(f) \Delta x_i.$$

Here $\Delta x_i = x_i - x_{i-1}$. Similarly the **lower (Riemann) sum** of f with respect to P is defined to be

$$L(P, f) := \sum_{i=1}^{n} m_i(f) \Delta x_i.$$

The **upper** and **lower integrals** of f, denoted by $\overline{\int_a^b} f(x), dx$ and $\underline{\int_a^b} f(x), dx$, respectively, are defined by

$$\overline{\int_a^b} f(x)\, dx = \inf\{U(P, f) \mid P \text{ is a partition of } [a, b]\},$$

$$\underline{\int_a^b} f(x)\, dx = \sup\{L(P, f) \mid P \text{ is a partition of } [a, b]\}.$$

Remark. If f is bounded, then

$$\underline{\int_a^b} f(x)\, dx \leq \overline{\int_a^b} f(x)\, dx,$$

see [**2**, Theorem 6.1.4, p.211].

A function f is said to be **Riemann integrable** on $[a, b]$ if

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx.$$

The common value is called the **Riemann integral** of f over $[a, b]$, and it is denoted by $\int_a^b f(x)\, dx = \underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx$.

Remark. In the following, (1) and (2) are from [**2**, Theorem 6.1.8].

1) Any continuous function on a finite interval $[a, b]$ is Riemann integrable.
2) Any monotone function on a finite interval $[a, b]$ is Riemann integrable.
3) Any elementary function is continuous on its domain.

The **elementary** functions are the polynomials, rational functions, power functions (x^a), exponential functions (a^x), logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by five operations of addition, subtraction, multiplication, division, and composition.

4.2. The Theorems.

THEOREM 4.1. *Let $\{F_n\}$ be a sequence of Riemann integrable functions on a finite interval $[a,b]$. Suppose that $\{F_n\}$ converges uniformly to a function F on $[a,b]$. Then F is Riemann integrable on $[a,b]$, and*

$$\lim_{n\to\infty}\int_a^b F_n(x)\,dx = \int_a^b F(x)\,dx \quad i.e. \quad \lim_{n\to\infty}\int_a^b F_n(x)\,dx = \int_a^b \Big(\lim_{n\to\infty} F_n(x)\Big)\,dx.$$

PROOF. By definition, given any $\epsilon > 0$, there exists N such that

$$|F_n(x) - F(x)| < \frac{\epsilon}{2(b-a)}$$

for all $n > N$ and $x \in [a,b]$. Thus

$$F_n(x) - \frac{\epsilon}{2(b-a)} < F(x) < F_n(x) + \frac{\epsilon}{2(b-a)}$$

for all $x \in [a,b]$ and $n > N$. It follows that

(7)
$$\underline{\int_a^b}\left(F_n(x) - \frac{\epsilon}{2(b-a)}\right)dx \le \underline{\int_a^b} F(x)\,dx \le \overline{\int_a^b} F(x)\,dx \le \overline{\int_a^b}\left(F_n(x) + \frac{\epsilon}{2(b-a)}\right)dx$$

for $n > N$. Since $F_n(x)$ is Riemann integrable, we have

$$\underline{\int_a^b}\left(F_n(x) - \frac{\epsilon}{2(b-a)}\right)dx = \int_a^b \left(F_n(x) - \frac{\epsilon}{2(b-a)}\right)dx$$

$$= \int_a^b F_n(x)\,dx - \frac{\epsilon}{2(b-a)}\cdot(b-a) = \int_a^b F_n(x)\,dx - \frac{\epsilon}{2},$$

$$\overline{\int_a^b}\left(F_n(x) + \frac{\epsilon}{2(b-a)}\right)dx = \int_a^b \left(F_n(x) + \frac{\epsilon}{2(b-a)}\right)dx$$

$$= \int_a^b F_n(x)\,dx + \frac{\epsilon}{2(b-a)}\cdot(b-a) = \int_a^b F_n(x) + \frac{\epsilon}{2}.$$

Together with Inequality (7), we obtain

$$\int_a^b F_n(x)\,dx - \frac{\epsilon}{2} \le \underline{\int_a^b} F(x)\,dx \le \overline{\int_a^b} F(x)\,dx \le \int_a^b F_n(x)\,dx + \frac{\epsilon}{2} \tag{8}$$

for $n > N$. It follows that

$$0 \le \overline{\int_a^b} F(x)\,dx - \underline{\int_a^b} F(x)\,dx \le \epsilon.$$

Let $\epsilon \to 0$, we have

$$\overline{\int_a^b} F(x)\,dx - \underline{\int_a^b} F(x)\,dx = 0$$

and so $F(x)$ is Riemann integrable. From Inequality (8), we have

$$\int_a^b F_n(x)\,dx - \frac{\epsilon}{2} \le \int_a^b F(x)\,dx \le \int_a^b F_n(x)\,dx + \frac{\epsilon}{2}, \quad \text{that is,}$$

$$\left| \int_a^b F_n(x)\,dx - \int_a^b F(x)\,dx \right| \le \frac{\epsilon}{2} < \epsilon$$

for $n > N$ and hence

$$\lim_{n\to\infty} \int_a^b F_n(x)\,dx = \int_a^b F(x)\,dx.$$

□

Example 4.2. *Compute, justifying your answer,*

$$\lim_{n\to\infty} \int_0^1 \frac{\sin nx}{n + x^2}\,dx.$$

Solution. Let $F_n(x) = \dfrac{\sin nx}{n + x^2}$. Then the limiting function

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{\sin nx}{n + x^2} = 0$$

for any given $0 \le x \le 1$, by the Squeeze theorem, because

$$-\frac{1}{n} \le \frac{\sin nx}{n + x^2} \le \frac{1}{n}$$

and $\lim\limits_{n\to\infty} \dfrac{1}{n} = -\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. Since

$$T_n = \sup_{0\le x\le 1} |F_n(x) - F(x)| = \sup_{0\le x\le 1} \left| \frac{\sin nx}{n + x^2} \right| \le \frac{1}{n},$$

$\lim\limits_{n\to\infty} T_n = 0$ by the Squeeze Theorem and so the sequence of functions $\{F_n\}$ converges uniformly to $F(x)$. Thus

$$\lim_{n\to\infty} \int_0^1 \frac{\sin nx}{n + x^2}\,dx = \int_0^1 \lim_{n\to\infty} \frac{\sin nx}{n + x^2}\,dx = \int_0^1 0\,dx = 0.$$

□

Corollary 4.3. *Suppose that* $\sum\limits_{k=1}^{\infty} f_k$ *converges uniformly to a function* S *on an interval* $[a, b]$*. Suppose that each* f_k *is a Riemann integrable (bounded) function on* $[a, b]$*. Then* S *is also Riemann integrable on* $[a, b]$*, and*

$$\int_a^b S(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx, \quad \textit{i.e.} \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx.$$

PROOF. Consider the sequence of partial sums $\{S_n\}$ on $[a, b]$, where $S_n = \sum_{k=1}^{n} f_k$. Then $\{S_n\}$ converges uniformly to S on $[a, b]$. If each f_k is Riemann integrable on $[a, b]$, then each S_n is also Riemann integrable on $[a, b]$. Then by Theorem 4.1, S is also Riemann integrable on $[a, b]$, and

$$\int_a^b S(x)\,dx = \lim_{n\to\infty} \int_a^b S_n(x)\,dx = \lim_{n\to\infty} \int_a^b \sum_{k=1}^{n} f_k(x)\,dx$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b f_k(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx.$$

□

By using this theorem, we have the following amazing formula.

EXAMPLE 4.4. *Show that*

$$\ln 2 = \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} = \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots$$

PROOF. Since $\left|x^{n-1}\right| \le \left(\frac{1}{2}\right)^{n-1}$ for $0 \le x \le \frac{1}{2}$ and the geometric series $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ converges, the series $\sum_{n=1}^{\infty} x^{n-1}$ converges uniformly to $\frac{1}{1-x}$ on $[0, \frac{1}{2}]$ by the Weierstrass M-test. Thus we have

$$\int_0^{\frac{1}{2}} \frac{1}{1-x} = \int_0^{\frac{1}{2}} \sum_{n=1}^{\infty} x^{n-1}\,dx$$
$$= \sum_{n=1}^{\infty} \int_0^{\frac{1}{2}} x^{n-1}\,dx = \sum_{n=1}^{\infty} \frac{x^n}{n}\bigg|_0^{1/2} = \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}.$$

Since

$$\int_0^{\frac{1}{2}} \frac{1}{1-x} = -\ln(1-x)\Big|_0^{\frac{1}{2}} = -\ln\left(1 - \frac{1}{2}\right) = -\ln\left(\frac{1}{2}\right) = \ln 2,$$

we obtain the formula

$$\ln 2 = \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} = \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots.$$

□

REMARK 4.5. Example 4.4 gives a way to estimate the number $\ln 2$ because the remainder

$$R_n = \sum_{k=n+1}^{\infty} \frac{1}{k \cdot 2^k} = \frac{1}{(n+1)2^{n+1}} + \frac{1}{(n+2)2^{n+2}} + \cdots$$
$$< \frac{1}{(n+1)2^{n+1}} + \frac{1}{(n+1)2^{n+2}} + \frac{1}{(n+1)2^{n+3}} + \cdots$$

$$= \frac{1}{(n+1)2^{n+1}} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) = \frac{1}{(n+1)2^{n+1}} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{(n+1)2^n}.$$

For instance,

$$\ln 2 \approx \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots + \frac{1}{10 \cdot 2^{10}}$$

with error less than $\dfrac{1}{11 \cdot 2^{10}} = \dfrac{1}{11264}$.

4.3. Remarks on Theorem 4.1. For completeness we include the following result, which does not require *uniform convergence*. The proof requires new theory called **Lebesgue integration**. There are applications in the area of probability and statistics.

THEOREM 4.6 (Bounded Convergence Theorem). *Let $\{F_n\}$ be a sequence of Riemann integrable functions converging pointwise to $F(x)$ on $[a,b]$. Suppose that*

(i) *$F(x)$ is Riemann integrable, and*
(ii) *there exists a positive constant M such that*

$$|F_n(x)| \le M$$

for all $x \in [a,b]$ and $n \in \mathbb{N}$.

Then

$$\lim_{n\to\infty} \int_a^b F_n(x)\,dx = \int_a^b F(x)\,dx \qquad = \int_a^b \lim_{n\to\infty} F_n(x)\,dx$$

5. Uniform Convergence and Differentiation

THEOREM 5.1. *Let $\{F_n\}$ be a sequence of functions on $[a,b]$ such that*

(i) *each F_n' exists and is continuous on $[a,b]$,*
(ii) *$\{F_n\}$ converges pointwise to a function F on $[a,b]$, and*
(iii) *$\{F_n'\}$ converges uniformly on $[a,b]$.*

Then F is differentiable on $[a,b]$, and for all $x \in [a,b]$,

$$F'(x) = \lim_{n\to\infty} F_n'(x), \text{ i.e. } \frac{d}{dx}\Big(\lim_{n\to\infty} F_n(x)\Big) = \lim_{n\to\infty}\left(\frac{d}{dx} F_n(x)\right).$$

Remark. Here the differentiability and continuity at the endpoints a and b refer to the one sided derivatives and limits respectively.

PROOF. By (iii), there exists a function g such that $\{F_n'\}$ converges uniformly to g on $[a,b]$. In particular, $\lim\limits_{n\to\infty} F_n'(x) = g(x)$ for all $x \in [a,b]$. By (i), since each F_n' is continuous on $[a,b]$, F_n' is also Riemann integrable on $[a,b]$, and by the fundamental theorem of calculus,

$$\int_a^x F_n'(t)\,dt = F_n(x) - F_n(a) \quad \text{for all } x \in [a,b].$$

Letting $n \to \infty$, we have, for all $x \in [a,b]$,

$$\lim_{n\to\infty} \int_a^x F_n'(t)\,dt = \lim_{n\to\infty} \big(F_n(x) - F_n(a)\big) = F(x) - F(a). \tag{9}$$

On the other hand, since $\{F_n'\}$ converges uniformly to g on $[a, b]$, it follows from Theorem 4.1 that

$$\lim_{n\to\infty} \int_a^x F_n'(t)\,dt = \int_a^x \big(\lim_{n\to\infty} F_n'(t)\big)dt = \int_a^x g(t)\,dt.$$

Together with equation 9, it follows that

$$F(x) - F(a) = \int_a^x g(t)\,dt \quad \text{for all } x \in [a, b].$$

By (i) and Theorem 3.1, g is continuous on $[a, b]$. Then by the fundamental theorem of calculus, we have

$$\frac{d}{dx}\int_a^x g(t)\,dt = g(x).$$

Together with equation 9, it follows that F is also differentiable on $[a, b]$, and for all $x \in [a, b]$,

$$\frac{d}{dx}\big(F(x) - F(a)\big) = \frac{d}{dx}\int_a^x g(t)\,dt = g(x), \quad \text{i.e.} F'(x) = g(x), \quad \text{i.e.}$$

$$\frac{d}{dx}\big(\lim_{n\to\infty} F_n(x)\big) = \lim_{n\to\infty}\big(\frac{d}{dx}F_n(x)\big).$$

The proof is finished. □

Remark 5.2. *By inspecting the proof, Theorem 5.1 still holds when the closed interval $[a, b]$ is replaced by (a, b), $(a, b]$ or $[a, b)$.*

This theorem can be generalized as follows.

Theorem 5.3. *Let $\{F_n\}$ be a sequence of* **differentiable** *functions on $[a, b]$ such that*

(a) *$\{F_n(x_0)\}$ converges for some $x_0 \in [a, b]$, and*
(b) *$\{F_n'\}$ converges* **uniformly** *on $[a, b]$.*

Then $\{F_n\}$ converges uniformly to a function $F(x)$ on $[a, b]$ with,

$$F'(x) = \lim_{n\to\infty} F_n'(x), \ \textit{i.e.}\ \frac{d}{dx}\big(\lim_{n\to\infty} F_n(x)\big) = \lim_{n\to\infty}\big(\frac{d}{dx}F_n(x)\big).$$

The proof of this theorem is omitted, see [**2**, Theorem 8.5.1, pp.340-341].

Corollary 5.4. *Let $\sum_{k=1}^{\infty} f_k$ be a series of* **differentiable** *functions on $[a, b]$ such that*

(a) *$\sum_{k=1}^{\infty} f_k(x_0)$ converges for some $x_0 \in [a, b]$, and*
(b) *$\sum_{k=1}^{\infty} f_k'$ converges* **uniformly** *on $[a, b]$.*

Then $\sum_{k=1}^{\infty} f_k$ *converges uniformly to a function* $S(x)$ *on* $[a, b]$, *and for all* $x \in [a, b]$,

$$S'(x) = \sum_{k=1}^{\infty} f_k'(x), \quad \textit{i.e.} \ \frac{d}{dx}\Big(\sum_{k=1}^{\infty} f_k(x)\Big) = \sum_{k=1}^{\infty} \frac{d}{dx} f_k(x).$$

PROOF. Consider the partial sums $S_n(x) = \sum_{k=1}^{n} f_k(x)$ on $[a, b]$. By (a), $\{S_n(x_0)\}$ converges, and, by (b), $\{S_n'(x)\}$ converges uniformly on $[a, b]$. By Theorem 5.3, $\{S_n(x)\}$ converges uniformly to a function $S(x)$ on $[a, b]$ and so $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly to $S(x)$ on $[a, b]$ by the definition. Furthermore, by Theorem 5.3, for all $x \in [a, b]$, $S'(x) = \lim_{n\to\infty} S_n'(x)$, that is,

$$\frac{d}{dx}\Big(\sum_{k=1}^{\infty} f_k(x)\Big) = \lim_{n\to\infty} \Big(\sum_{k=1}^{n} f_k(x)\Big)' = \lim_{n\to\infty} \sum_{k=1}^{n} f_k'(x) = \sum_{k=1}^{\infty} \frac{d}{dx} f_k(x).$$

□

As an application, we give a proof of binomial series. Let a be any real number. The binomial number, a chooses n, is defined by

$$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}$$

for positive integers n. For instance, $\binom{\frac{1}{2}}{1} = \frac{1}{2}$, $\binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)}{2!} = -\frac{1}{8}$,

$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} = \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{6} = \frac{1}{16}.$$

We also use the convention that $\binom{a}{0} = 1$ for any a.

THEOREM 5.5 (Binomial Series). *Let* a *be any real constant. Then*

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \cdots = 1 + \sum_{n=1}^{\infty} \binom{a}{n} x^n = \sum_{n=0}^{\infty} \binom{a}{n} x^n$$

for $|x| < 1$.

PROOF. Let ρ be any fixed positive number with $0 < \rho < 1$ and let $I = [-\rho, \rho]$. Consider the series of functions

$$1 + \sum_{n=1}^{\infty} \binom{a}{n} x^n$$

converges when $x = 0 \in [-\rho, +\rho]$. So it satisfies condition (a) of Corollary 5.4. We check condition (b) of Corollary 5.4, namely, the series of functions

$$\sum_{n=1}^{\infty} \binom{a}{n} n x^{n-1}$$

converges uniformly on $[-\rho, +\rho]$. Note that

$$\left|\binom{a}{n} n x^{n-1}\right| \leq \left|\binom{a}{n}\right| n\rho^{n-1}$$

for $x \in [-\rho, +\rho]$. Let $M_n = \left|\binom{a}{n}\right| n\rho^{n-1}$. Then

$$\lim_{n\to\infty} \frac{M_{n+1}}{M_n} = \lim_{n\to\infty} \frac{\left|\binom{a}{n+1}\right|(n+1)\rho^n}{\left|\binom{a}{n}\right| n\rho^{n-1}} = \lim_{n\to\infty} \frac{\dfrac{|a|\cdot|a-1|\cdots|a-n|}{(n+1)!}\cdot(n+1)\cdot\rho^n}{\dfrac{|a|\cdot|a-1|\cdots|a-n+1|}{n!}\cdot n\cdot\rho^{n-1}}$$

$$= \lim_{n\to\infty} \frac{|a|\cdot|a-1|\cdots|a-n|\cdot n!\cdot(n+1)\cdot\rho}{(n+1)!\cdot|a|\cdot|a-1|\cdots|a-n+1|\cdot n}$$

$$= \lim_{n\to\infty} \frac{|a-n|\cdot\rho}{n} = \lim_{n\to\infty} |a/n - 1|\cdot\rho = \rho < 1.$$

Thus the series $\sum_{n=1}^{\infty} M_n$ converges and so the series of functions $\sum_{n=1}^{\infty} \binom{a}{n} n x^{n-1}$ converges uniformly on $[-\rho, \rho]$, by the Weierstrass M-test.

By Corollary 5.4, the series $1 + \sum_{n=1}^{\infty} \binom{a}{n} x^n$ converges uniformly to a function $f(x)$ on $[-\rho, \rho]$ with

$$f'(x) = \sum_{n=1}^{\infty} \binom{a}{n} n x^{n-1}.$$

Next we are going to set up a differential equation that

$$(1+x)f'(x) = af(x).$$

(**Note.** Since the goal is to show that $f(x) = (1+x)^a$, this equation is observed from that, if $y = (1+x)^a$, then $y' = a(1+x)^{a-1}$ and so $(1+x)y' = (1+x)^a = y$.)

Now

$$f'(x) = \sum_{n=1}^{\infty} \binom{a}{n} n x^{n-1} = \sum_{n=1}^{\infty} \frac{a(a-1)\cdots(a-n+1)\cdot n}{n!} x^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{a(a-1)\cdots(a-n+1)}{(n-1)!} x^{n-1} = a\sum_{n=1}^{\infty} \binom{a-1}{n-1} x^{n-1} \quad \text{and}$$

$$(1+x)f'(x) = f'(x) + xf'(x) = a\sum_{n=1}^{\infty} \binom{a-1}{n-1} x^{n-1} + a\sum_{n=1}^{\infty} \binom{a-1}{n-1} x^n$$

$$= a\sum_{n=0}^{\infty} \binom{a-1}{n} x^n + a\sum_{n=1}^{\infty} \binom{a-1}{n-1} x^n$$

$$= a\left\{1 + \sum_{n=1}^{\infty}\left[\binom{a-1}{n} + \binom{a-1}{n-1}\right] x^n\right\} = a\left[1 + \sum_{n=1}^{\infty} \binom{a}{n} x^n\right] = af(x),$$

where

$$\begin{aligned}
&\binom{a-1}{n}+\binom{a-1}{n-1}\\
&=\frac{(a-1)(a-2)\cdots(a-1-n+1)}{n!}+\frac{(a-1)(a-2)\cdots(a-1-n+2)}{(n-1)!}\\
&=\frac{(a-1)(a-2)\cdots(a-1-n+2)}{n!}(a-1-n+1+n)\\
&=\frac{a(a-1)\cdots(a-n+1)}{n!}=\binom{a}{n}.
\end{aligned}$$

Let $y=f(x)$. Then we obtain the differential equation

$$\begin{aligned}
(1+x)\frac{dy}{dx}&=ay \qquad \frac{dy}{y}=\frac{a\,dx}{1+x}\\
\Longrightarrow \int\frac{dy}{y}&=\int\frac{a\,dx}{1+x}\\
\Longrightarrow \ln|y|&=a\ln|1+x|+A=\ln|1+x|^a+A\\
\Longrightarrow |y|&=e^{\ln|y|}=e^A|1+x|^a\\
\Longrightarrow y&=C|1+x|^a,
\end{aligned}$$

where $C=\pm e^A$ is a constant. By putting $x=0$,

$$C=(1+0)^a=y(0)=1+\sum_{n=1}^{\infty}\binom{a}{n}0^n=1.$$

Thus $y=|1+x|^a$ or

$$(1+x)^a=1+\sum_{n=1}^{\infty}\binom{a}{n}x^n$$

for $|x|\le\rho$ because $1+x>0$ when $|x|\le\rho<1$. Since ρ is any number with $0<\rho<1$, the formula

$$(1+x)^a=1+\sum_{n=1}^{\infty}\binom{a}{n}x^n$$

holds for all $x\in(-1,1)$. □

For instance,

$$\sqrt{1+x}=(1+x)^{\frac12}=\sum_{k=0}^{\infty}\binom{\frac12}{k}x^k=1+\frac12x+\frac{\left(\frac12\right)\left(-\frac12\right)}{2!}x^2+\cdots$$

$$\sqrt{1-x^3}=(1-x^3)^{\frac12}=\sum_{k=0}^{\infty}\binom{\frac12}{k}\left(-x^3\right)^k=1-\frac12x^3+\frac{\left(\frac12\right)\left(-\frac12\right)}{2!}x^6+\cdots$$

$$\sqrt{4.1}=\left(4+\frac{1}{10}\right)^{\frac12}=2\left(1+\frac{1}{40}\right)^{\frac12}=2\sum_{k=0}^{\infty}\binom{\frac12}{k}\left(\frac{1}{40}\right)^k.$$

Example 5.6. *Evaluate* $\sqrt{4.1}$ *with error less than* 0.001.

SOLUTION.

$$\sqrt{4.1} = \left(4 + \frac{1}{10}\right)^{\frac{1}{2}} = 2\left(1 + \frac{1}{40}\right)^{\frac{1}{2}} = 2\sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(\frac{1}{40}\right)^k.$$

$$= 2 + 2\sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} \left(\frac{1}{40}\right)^k.$$

Now

$$\binom{\frac{1}{2}}{k} = \frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-k+1\right)}{k!} = (-1)^{k+1}\frac{\frac{1}{2}\cdot\left(1-\frac{1}{2}\right)\cdots\left(k-\frac{1}{2}-1\right)}{k!}$$

for $k \geq 2$. Let

$$b_k = \frac{\frac{1}{2}\cdot\left(1-\frac{1}{2}\right)\cdots\left(k-\frac{1}{2}-1\right)}{k!}\left(\frac{1}{40}\right)^k.$$

Then, for $k \geq 2$, we have $b_k \geq 0$,

$$\frac{b_{k+1}}{b_k} = \frac{\dfrac{\frac{1}{2}\cdot\left(1-\frac{1}{2}\right)\cdots\left(k-\frac{1}{2}-1\right)\cdot\left(k-\frac{1}{2}\right)}{(k+1)!}\left(\dfrac{1}{40}\right)^{k+1}}{\dfrac{\frac{1}{2}\cdot\left(1-\frac{1}{2}\right)\cdots\left(k-\frac{1}{2}-1\right)}{k!}\left(\dfrac{1}{40}\right)^k}$$

$$= \frac{k-\frac{1}{2}}{40(k+1)} \leq 1,$$

that is, $b_2 \geq b_3 \geq \cdots \geq 0$, and $\lim_{k\to\infty} b_k = 0$ by the Squeeze Theorem because

$$0 \leq b_k \leq \frac{\frac{1}{2}\cdot 1\cdot 2\cdots(k-1)}{k!}\left(\frac{1}{40}\right)^k = \frac{1}{2k\cdot 40^k}$$

for $k \geq 2$ and $\lim_{k\to\infty} \frac{1}{2k\cdot 40^k} = 0$. By the alternating series estimation, from

$$2\cdot\left|\binom{\frac{1}{2}}{k+1}\left(\frac{1}{40}\right)^{k+1}\right| < 0.001,$$

we have $k \geq 1$, because $2b_2 = \frac{1}{6400} < 0.001$, and so

$$\sqrt{4.1} \approx 2 + 2\binom{\frac{1}{2}}{1}\frac{1}{40} = 2.025$$

with error less than 0.001. □

6. Power Series

6.1. Power Series.

DEFINITION 6.1. A **power series** in x is of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

EXAMPLE 6.2. Below are some examples

1. $\sum_{n=0}^{\infty}(n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$
2. $\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

DEFINITION 6.3. A *power series* in $x - x_0$ is of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

EXAMPLE 6.4. Here are some examples.

1. $\sum_{n=0}^{\infty}(x-1)^n = 1 + (x-1) + (x-1)^2 + \cdots$.
2. $\sum_{n=1}^{\infty} n^2(x+2)^n = (x+2) + 2^2(x+2)^2 + 3^2(x+2)^3 + \cdots$.

Warning. Don't expand out the terms $a_n(x-x_0)^n$ in the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ because, when you rearrange terms in an (infinite) series, you may get different values. (For partial sums, you can expand out, if it is necessary, because there are only finitely many terms.)

Question: Given a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, when does it converge and when does it diverge? In other words, what is the domain of the function $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. We are going to answer this question.

6.2. Radius of Convergence.

DEFINITION 6.5. *Given a power series* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, *the* **radius of convergence** R *is defined by*

$$R = \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}}.$$

If $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \infty$, *we take* $R = 0$, *and if* $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = 0$, *we set* $R = \infty$. *If* $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}$ *exists,* R *is also given by*

$$R = \frac{1}{\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}}$$

Remark. Recall that

$$\varliminf \frac{|a_{n+1}|}{|a_n|} \le \varliminf \sqrt[n]{|a_n|} \le \varlimsup \sqrt[n]{|a_n|} \le \varlimsup \frac{|a_{n+1}|}{|a_n|}.$$

If $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|}$ exists, then

$$\varliminf \frac{|a_{n+1}|}{|a_n|} = \varlimsup \frac{|a_{n+1}|}{|a_n|}$$

and so $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ exists and

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

EXAMPLE 6.6. *What is the radius of convergence for the series*

$$1 + \frac{x}{3} + \frac{x^2}{4^2} + \frac{x^3}{3^3} + \frac{x^4}{4^4} + \frac{x^5}{3^5} + \frac{x^6}{4^6} + \cdots$$

SOLUTION. Since

$$a_n = \begin{cases} \dfrac{1}{4^{2k}} & n = 2k \\[2ex] \dfrac{1}{3^{2k-1}} & n = 2k-1, \end{cases}$$

we have

$$\sqrt[n]{|a_n|} = \begin{cases} \dfrac{1}{4} & n = 2k \\[2ex] \dfrac{1}{3} & n = 2k-1. \end{cases}$$

Thus $\varlimsup\limits_{n\to\infty} \sqrt[n]{|a_n|} = \dfrac{1}{3}$ and so the radius of convergence

$$R = \frac{1}{\varlimsup\limits_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{\frac{1}{3}} = 3.$$

□

EXAMPLE 6.7. *Find the radius of convergence of the power series* $\displaystyle\sum_{n=0}^{\infty} \frac{(4x+3)^n}{n^3}$

SOLUTION. Observe that

$$\sum_{n=1}^{\infty} \frac{(4x+3)^n}{n^3} = \sum_{n=1}^{\infty} \frac{4^n}{n^3} \cdot \left(x + \frac{3}{4}\right)^n.$$

Thus

$$R = \frac{1}{\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \dfrac{4^{n+1}\cdot n^3}{(n+1)^3 \cdot 4^n}} = \frac{1}{\lim\limits_{n\to\infty} \dfrac{4}{\left(1+\frac{1}{n}\right)^3}} = \frac{1}{4}.$$

□

THEOREM 6.8. *Given any power series* $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ *with radius of convergence* R, $0 \le R \le \infty$, *then the series* $\sum_{k=0}^{\infty} a_k(x-x_0)^k$

(i) *converges absolutely for all* x *with* $|x-x_0| < R$, *and*
(ii) *diverges for all* x *with* $|x-x_0| > R$.

PROOF. By definition, the radius of convergence $R = \dfrac{1}{\overline{\lim}\,|a_k|^{\frac{1}{k}}}$. Assertion (i) follows from the root test because, from

$$\overline{\lim}\left|a_k(x-x_0)^k\right|^{\frac{1}{k}} = \overline{\lim}\,|a_k|^{\frac{1}{k}}\cdot|x-x_0| = |x-x_0|\cdot\overline{\lim}\,|a_k|^{\frac{1}{k}} = |x-x_0|\cdot\frac{1}{R} < R\cdot\frac{1}{R} < 1,$$

the series $\sum_{k=0}^{\infty}\left|a_k(x-x_0)^k\right|$ converges.

Next we are going to prove (ii) by contradiction. Suppose that $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ converges at a point x with $|x-x_0| > R$. Then by Theorem 1.7, we have

$$\lim_{k\to\infty} a_k(x-x_0)^k = 0.$$

Let $\epsilon = 1$. Then there exists N such that

$$|a_k(x-x_0)^k - 0| < 1 \quad \text{for all } k > N \Rightarrow \quad |a_k(x-x_0)^k|^{\frac{1}{k}} < 1 \quad \text{for all } k > N$$

$$\Rightarrow \quad |a_k|^{\frac{1}{k}} < \frac{1}{|x-x_0|} \quad \text{for all } k > N \Rightarrow \quad \sup_{n\ge k}|a_k|^{\frac{1}{k}} \le \frac{1}{|x-x_0|} \quad \text{for all } n > N$$

$$\Rightarrow \quad \overline{\lim}\,|a_k|^{\frac{1}{k}} \le \frac{1}{|x-x_0|} \Rightarrow \quad \frac{1}{R} \le \frac{1}{|x-x_0|} < \frac{1}{R},$$

which is a contradiction. Hence we must have $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ diverges at each x satisfying $|x-x_0| > R$. □

6.3. Interval of convergence. In view of Theorem 6.8, for a power series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ with radius of convergence R, the set of points at which $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is convergent form an interval called the **interval of convergence**, which must be either

$$(x_0-R, x_0+R), \qquad (x_0-R, x_0+R],$$
$$[x_0-R, x_0+R) \quad \text{or} \quad [x_0-R, x_0+R].$$

EXAMPLE 6.9. *Find the interval of convergence of the power series.*

$$\text{(i)} \sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2} \qquad \text{(ii)} \sum_{n=1}^{\infty}\frac{(x-2)^n}{n} \qquad \text{(iii)} \sum_{n=1}^{\infty} n(x-2)^n$$

SOLUTION. (i). First we find the radius of convergence

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{n^2}{(n+1)^2}} = \frac{1}{\lim\limits_{n\to\infty} \frac{1}{(1+\frac{1}{n})^2}} = 1.$$

Next we check the ending-points $x_0 \pm R = 2 \pm 1 = 1, 3$. When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which is convergent by Example 3.5. When $x = 3$, the series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is convergent by the p-series. Thus the interval of convergence is $[1, 3]$.

(ii). The radius of convergence is

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{n}{(n+1)}} = \frac{1}{\lim\limits_{n\to\infty} \frac{1}{(1+\frac{1}{n})}} = 1.$$

Now we check the ending-points $x_0 \pm R = 1, 3$. When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which is convergent by Example 3.5. When $x = 3$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$, which is divergent by the p-series. Thus the interval of convergence is $[1, 3)$.

(iii). The radius of convergence is

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{n+1}{n}} = \frac{1}{\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)} = 1.$$

Now we check the ending-points $x_0 \pm R = 1, 3$. When $x = 1$, the series is $\sum_{n=1}^{\infty} n(-1)^n$, and when $x = 3$, the series is $\sum_{n=1}^{\infty} n$. Both of these series are divergent by the divergence test. Thus the interval of convergence is $(1, 3)$. □

6.4. Uniform Convergence of Power Series.

THEOREM 6.10 (Abel Theorem). *Let* $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ *be a power series, and let* $R \geq 0$.

(1). *If* $\sum_{n=0}^{\infty} a_n R^n$ *converges, then the series of functions* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *converges uniformly on* $[x_0, x_0 + R]$.

(2). *If* $\sum_{n=0}^{\infty} a_n(-R)^n$ *converges, then the series of functions* $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ *converges uniformly on* $[x_0 - R, x_0]$.

PROOF. We only prove assertion (1). The proof of assertion (2) is similar to.

We may assume that $R > 0$. Let $t = \dfrac{x - x_0}{R}$. Then

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n R^n t^n.$$

We are going to show that this series converges uniformly on $0 \le t \le 1$, that is, $x_0 \le x \le x_0 + R$. Write $\bar{a}_n$ for $a_n R^n$. By the assumption, the series $\sum_{n=0}^{\infty} \bar{a}_n = \sum_{n=0}^{\infty} a_n R^n$ converges. Let $A_n = \sum_{k=0}^{n} \bar{a}_k - \sum_{k=0}^{\infty} \bar{a}_k = -\sum_{k=n+1}^{\infty} \bar{a}_k$. Then

$$A_n - A_{n-1} = \left(-\sum_{k=n+1}^{\infty} \bar{a}_k \right) - \left(-\sum_{k=n}^{\infty} \bar{a}_k \right) = \bar{a}_n.$$

By Abel Partial Summation Formula, for $0 \le n < m$, $0 \le t \le 1$,

$$\left| \sum_{k=n+1}^{m} \bar{a}_k t^k \right| = \left| A_m t^m - A_n t^{n+1} + \sum_{k=n+1}^{m-1} A_k (t^k - t^{k+1}) \right|$$

$$\le |A_m| t^m + |A_n| t^{n+1} + \sum_{k=n+1}^{m-1} |A_k| \cdot t^k (1 - t).$$

Since $\sum_{n=0}^{\infty} \bar{a}_n$ converges, the remainders $\sum_{k=n+1}^{\infty} \bar{a}_k = -A_n$ tends to 0 and so $\lim_{n\to\infty} A_n = 0$. Given $\epsilon > 0$, there exists N such that $|A_n| < \dfrac{\epsilon}{2}$ for $n > N$. Now, for $m > n > N$ and $0 \le t \le 1$,

$$\left| \sum_{k=n+1}^{m} \bar{a}_k t^k \right| \le\le |A_m| t^m + |A_n| t^{n+1} + \sum_{k=n+1}^{m-1} |A_k| \cdot t^k (1 - t)$$

$$< \frac{\epsilon}{2} t^m + \frac{\epsilon}{2} t^{n+1} + \sum_{k=n+1}^{m-1} \frac{\epsilon}{2} \cdot t^k \cdot (1 - t)$$

$$= \frac{\epsilon}{2} \left(t^m + t^{n+1} + (1 - t)(t^{n+1} + t^{n+2} + \cdots + t^{m-1}) \right)$$

$$= \frac{\epsilon}{2} \left(t^m + t^{n+1} + (t^{n+1} + t^{n+2} + \cdots + t^{m-1}) - (t^{n+2} + t^{n+3} + \cdots + t^m) \right) = \frac{\epsilon}{2} \cdot 2t^{n+1} \le \epsilon.$$

By the Cauchy Criterion, the series of functions $\sum_{n=0}^{\infty} \bar{a}_n t^n = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges uniformly on $0 \le t \le 1$, or on $x_0 \le x \le x_0 + R$. □

Theorem 6.11 (Uniform Convergence Theorem). *Let $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ be a power series of radius of convergence $R > 0$. Let I be the interval of convergence. Then* *the series of functions* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *converges uniformly on any* **closed interval** $[c,d] \subseteq I$.

Remark. (1). $I = [x_0-R, x_0+R]$, $[x_0-R, x_0+R)$, $(x_0-R, x_0+R]$, or (x_0-R, x_0+R). If $I = [x_0-R, x_0+R]$, then the power series converges on I. In other cases, since I is not closed, the theorem says that the power series converges on any **closed** sub-interval of I. For instance, if $x_0 = 0$, $R = 1$, and $I = [-1, 1)$, then the power series converges uniformly on $[-1, 0.9]$, $[0, 0.9]$ and etc, but it need not converge uniformly on $[-1, 1)$ or $[0, 1)$.
(2). In any of the four cases, the power series converges uniformly on **any closed sub-interval** of (x_0-R, x_0+R).

PROOF. There are three cases: (i). $c \le x_0 \le d$, (ii). $x_0 < c \le d$, or (iii). $c \le d < x_0$.

Case (i). $c \le x_0 \le d$. Since $\sum_{n=0}^{\infty} a_n(d-x_0)^n$ and $\sum_{n=0}^{\infty} a_n(c-x_0)^n$ (because $c, d \in I$), where $d-x_0 \ge 0$ and $c-x_0 \le 0$, the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges uniformly on $c \le x \le x_0]$ and $x_0 \le x \le d$ by the Abel theorem and so on the union $[c,d] = [c,x_0] \cup [x_0,d]$.

Case (ii). $x_0 < c \le d$. Since $\sum_{n=0}^{\infty} a_n(d-x_0)^n$ converges, the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges uniformly on $[x_0,d]$ by the Abel Theorem and so on the sub-interval $[c,d] \subseteq [x_0,d]$.

Case (iii). $c \le d < x_0$. Since $\sum_{n=0}^{\infty} a_n(c-x_0)^n$ converges, the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges uniformly on $[c,x_0]$ and so on $[c,d] \subseteq [c,x_0]$. □

COROLLARY 6.12. *Let* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *be a power series of radius of convergence* $R > 0$, *and let* I *be the interval of convergence. Suppose that* $[c,d]$ *is a closed sub-interval of* I. *Then*

$$\int_c^d \sum_{n=0}^{\infty} a_n(x-x_0)^n \, dx = \sum_{n=0}^{\infty} a_n \frac{(d-x_0)^{n+1} - (c-x_0)^{n+1}}{n+1}.$$

PROOF. Since $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges uniformly on $[c,d]$,

$$\int_c^d \sum_{n=0}^{\infty} a_n(x-x_0)^n \, dx = \sum_{n=0}^{\infty} \int_c^d a_n(x-x_0)^n \, dx = \sum_{n=0}^{\infty} a_n \frac{(d-x_0)^{n+1} - (c-x_0)^{n+1}}{n+1}.$$

□

COROLLARY 6.13 (Abel). *Let* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *be a power series of radius of convergence* $R>0$.

(a). *If* $\sum_{n=0}^{\infty} a_n R^n$ *converges, then*

$$\lim_{x\to(x_0+R)^-}\sum_{n=0}^{\infty} a_n(x-x_0)^n=\sum_{n=0}^{\infty} a_n R^n.$$

(b). *If* $\sum_{n=0}^{\infty} a_n(-R)^n$ *converges, then*

$$\lim_{x\to(x_0-R)^+}\sum_{n=0}^{\infty} a_n(x-x_0)^n=\sum_{n=0}^{\infty} a_n(-R)^n.$$

PROOF. (a). By the Abel theorem, the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges uniformly on $[x_0,x_0+R]$ and so the function $f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$ continuous on $[x_0,x_0+R]$. Hence

$$\sum_{n=0}^{\infty} a_n R^n=f(x_0+R)=\lim_{x\to(x_0+R)^-} f(x)=\lim_{x\to(x_0+R)^-}\sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

The proof of (b) is similar to that of (a). □

EXAMPLE 6.14. From the geometric series $\sum_{n=0}^{\infty} x^n=\frac{1}{1-x}$ for $|x|<1$, we have

$$\frac{1}{1+t}=\sum_{n=0}^{\infty}(-1)^n t^n$$

by letting $x=-t$. For any $x\in(-1,1)$, we have

$$\ln(1+x)=\int_0^x\frac{1}{1+t}\,dt=\sum_{n=0}^{\infty}(-1)^n\int_0^x t^n\,dt=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}.$$

Since $\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$ converges when $x=1$, we have

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}=\lim_{x\to1^-}\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}=\lim_{x\to1^-}\ln(1+x)=\ln 2.$$

In other words,

$$\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots.$$

EXAMPLE 6.15. From the geometric series $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$ for $|x| < 1$, we have

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

by letting $x = -t^2$. For any $x \in (-1, 1)$, we have

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Since $\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ converges when $x = \pm 1$, it converges uniformly on $[-1, 1]$ and so

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{for all} \quad |x| \le 1.$$

In particular,

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

EXAMPLE 6.16. From

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{for all} \quad |x| \le 1,$$

we have

$$\arctan x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2(2n+1)}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

for $|x| \le 1$ and so

$$\int_0^1 \arctan x^2\,dx = \sum_{n=0}^{\infty} (-1)^n \int_0^1 \frac{x^{4n+2}}{2n+1}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)(4n+3)}.$$

7. Differentiation of Power Series

LEMMA 7.1. *Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $a_n \ge 0$, $\lim_{n\to\infty} b_n$ exists with $\lim_{n\to\infty} b_n \ne 0$. Then $\overline{\lim}\, a_n b_n = \overline{\lim}\, a_n \cdot \lim_{n\to\infty} b_n$.*

PROOF. Let $B = \lim_{n\to\infty} b_n$.Given any $\epsilon > 0$, there exists N such that $|b_n - B| < \epsilon$ for $n > N$, that is,

$$B - \epsilon < b_n < B + \epsilon \qquad \text{for} \quad n > N.$$

Thus, since $a_n \ge 0$,

$$a_n(B - \epsilon) < a_n b_n < a_n(B + \epsilon) \qquad \text{for} \quad n > N$$

and so

$$(B - \epsilon)\,\overline{\lim}\, a_n = \overline{\lim}\, a_n(B - \epsilon) \le \overline{\lim}\, a_n b_n \le \overline{\lim}\, a_n(B + \epsilon) = (B + \epsilon)\,\overline{\lim}\, a_n.$$

Now, by letting ϵ tend to 0, we have

$$B \cdot \overline{\lim}\, a_n \leq \overline{\lim}\, a_n b_n \leq B \cdot \overline{\lim}\, a_n.$$

Thus

$$\overline{\lim}\, a_n b_n = B \cdot \overline{\lim}\, a_n = \overline{\lim}\, a_n \cdot \lim_{n\to\infty} b_n.$$

□

THEOREM 7.2. *Suppose that* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *has radius of convergence* $R > 0$, *and*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n, \qquad |x-x_0| < R.$$

Then

(a). *The power series* $\sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$ *has radius of convergence* R, *and*

(b). $f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$ *for* $|x-x_0| < R$.

PROOF. (a). By Lemma 7.1,

$$\overline{\lim_{n\to\infty}}\, |na_n|^{\frac{1}{n}} = \overline{\lim_{n\to\infty}}\, |a_n|^{\frac{1}{n}} \cdot n^{\frac{1}{n}} = \overline{\lim_{n\to\infty}}\, |a_n|^{\frac{1}{n}} \cdot \lim_{n\to\infty} \sqrt[n]{n}$$

$$\overline{\lim_{n\to\infty}}\, |a_n|^{\frac{1}{n}} \cdot 1 = \overline{\lim_{n\to\infty}}\, |a_n|^{\frac{1}{n}} = \frac{1}{R}.$$

Thus the power series

$$\sum_{n=0}^{\infty} n a_n (x-x_0)^n = \sum_{n=1}^{\infty} n a_n (x-x_0)^n = (x-x_0) \cdot \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

has radius of convergence R and so has $\sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$.

(b). For any ρ with $0 < \rho < R$, the series of functions $\sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$ converges uniformly on $|x-x_0| \leq \rho$ by the Uniform Convergence Theorem because the closed interval $[x_0-\rho, x_0+\rho] \subseteq (x_0-R, x_0+R)$. The result follows from Theorem 5.3. □

Remark. The formula $f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$ need not hold at the end points $x = x_0 \pm R$ in general even if the interval of convergence of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $[x_0-R, x_0+R]$.

EXAMPLE 7.3. From Example 6.15,

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{for all} \quad |x| \leq 1.$$

But

$$\frac{1}{1+x^2} = (\arctan x)' = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

only holds for $|x| < 1$ because when $x = \pm 1$, the right hand side diverges (and the left hand side $= \frac{1}{2}$).

COROLLARY 7.4. *Suppose that* $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ *has radius of convergence* $R > 0$ *with pointwise limiting function* $f(x)$ *on* $|x-x_0| < R$ *(i.e.* $f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ *on* $|x-x_0| < R$*), then* $f(x)$ *has derivatives of all orders on* $|x-x_0| < R$*, and for each* n*,*

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)(k-2)\cdots(k-n+1)a_k(x-x_0)^{k-n}. \tag{10}$$

In particular,

$$a_k = \frac{f^{(k)}(x_0)}{k!} \quad \text{for all } k. \tag{11}$$

(i.e. we have $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$*.)*

PROOF. The result is obtained by successively applying the previous theorem to f, f', f'', and etc. Equation follows by setting $x = x_0$ in Equation 10, that is,

$$f^{(n)}(x) = n!a_n + (n+1)n\cdots 2a_{n+1}(x-x_0) + (n+2)(n+1)\cdots 3a_{n+2}(x-x_0)^2 + \cdots .$$

□

EXAMPLE 7.5. From the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1,$$

we have

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

for $|x| < 1$, and so

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + 4x^4 + \cdots \qquad |x| < 1.$$

By letting $x = \frac{1}{2}$, we have

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$$

EXAMPLE 7.6. Consider the function

$$y = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

The radius of convergence

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{n!}{(n+1)!}} = \frac{1}{\lim\limits_{n\to\infty} \frac{1}{n+1}} = \frac{1}{0} = \infty.$$

Thus for any $x \in (-\infty, +\infty)$,

$$y' = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = y$$

$$\Longrightarrow \frac{dy}{dx} = y \qquad \Longrightarrow \frac{dy}{y} = dx \qquad \Longrightarrow \int \frac{dy}{y} = \int dx \qquad \Longrightarrow \ln|y| = x + A$$

$$\Longrightarrow |y| = e^{x+A} \qquad \Longrightarrow y = Ce^x, \quad C = \pm e^A \quad \text{constant.}$$

Let $x = 0$.

$$C = Ce^0 = y(0) = 1 + 0 + 0 + \cdots = 1.$$

Hence we obtain the formula

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

DEFINITION 7.7. *A real-valued function f defined on an open interval I is said to be* **infinitely differentiable** *on I if all (higher) derivatives $f^{(n)}(x)$, $n \geq 1$, exist. The set of infinitely differentiable functions on I is denoted by $C^\infty(I)$.*

As a consequence, the functions $f(x) = \sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ are infinitely differentiable on $(x_0 - R, x_0 + R)$ if $R > 0$.

COROLLARY 7.8 (Uniqueness Theorem). *Suppose that $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ and $\sum\limits_{n=0}^{\infty} b_n(x-x_0)^n$ are two power series which converge for $|x - x_0| < R$ with $R > 0$. Then*

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} b_n(x - x_0)^n \qquad \text{for} \quad |x - x_0| < R$$

if and only if $a_k = b_k$ for all $k = 0, 1, 2, 3, \cdots$

PROOF. Suppose that $a_k = b_k$ for all $k = 0, 1, 2, 3 \cdots$. Then $\sum\limits_{n=0}^{\infty} a_n(x - x_0)^n = \sum\limits_{n=0}^{\infty} b_n(x - x_0)^n$.

Conversely, suppose that $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n = \sum\limits_{n=0}^{\infty} b_n(x-x_0)^n = f(x)$ for $|x-x_0| < R$. Then $a_n = \dfrac{f^{(n)}(x_0)}{n!}$ and $b_n = \dfrac{f^{(n)}(x_0)}{n!}$. Thus $a_n = b_n$ for all $n = 0, 1, 2, 3, \cdots$. $\square$

8. Taylor Series

8.1. History Remarks. The study of sequences and series of functions has its origins in the study of power series representation of functions. The power series of $\ln(1+x)$ was known to Nicolaus Mercator (1620-1687) by 1668, and the power series of many other functions such as $\arctan x$, $\arcsin x$, and etc, were discovered around 1670 by James Gregory (1625-1683). All these series were obtained without any reference to calculus. The first discoveries of Issac Newton (1642-1727), dating back to the early months of 1665, resulted from his ability to express functions in terms of power series. His treatise on calculus, published in 1737, was appropriately entitled *A treatise of the methods of fluxions and infinite series.* Among his many accomplishments, Newton derived the power series expansion of $(1+x)^{m/n}$ using algebraic techniques. This series and the geometric series were crucial in many of his computations. Newton also displayed the power of his calculus by deriving the power series expansion of $\ln(1+x)$ using term-by-term integration of the expansion of $1/(1+x)$. Colin Maclaurin (1698-1746) and Brooks Taylor (1685-1731) were among the first mathematicians to use Newton's calculus in determining the coefficients in the power series expansion of a function. Both realized that if a function $f(x)$ had a power series expansion $\displaystyle\sum_{n=0}^{\infty} a_n(x-x_0)^n$, then the coefficients a_n had to be given by $\dfrac{f^{(n)}(x_0)}{n!}$.

8.2. Taylor Polynomials and Taylor Series.

Definition 8.1. *Let $f(x)$ be a function defined on an open interval I, and let $x_0 \in I$ and $n \geq 1$. Suppose that $f^{(n)}(x)$ exists for all $x \in I$. The polynomial*

$$T_n(f, x_0)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

is called the **Taylor polynomial** *of* **order** *n of f at the point x_0. If f is infinitely differentiable on I, the power series*

$$T(f, x_0)(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is called the **Taylor series** *of f and x_0.*

For the special case $x_0 = 0$, the Taylor series of a function f is often referred to as **Maclaurin series**. The first few Taylor polynomials are as follows:

$$T_0(f, x_0)(x) = f(x_0),$$
$$T_1(f, x_0)(x) = f(x_0) + f'(x_0)(x-x_0),$$
$$T_2(f, x_0)(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2,$$
$$T_3(f, x_0)(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3.$$

The Taylor polynomial $T_1(f, x_0)$ is the **linear approximation** of f at x_0, that is **the tangent line** passing through $(x_0, f(x_0))$ with slope $f'(x_0)$.

In general, the Taylor polynomial T_n of f is a polynomial of degree less than or equal to n that satisfies the conditions

$$T_n^{(k)}(f, x_0)(x_0) = f^{(k)}(x_0)$$

for $0 \le k \le n$. Since $f^{(n)}(x_0)$ might be zero, T_n could very well be a polynomial of degree strictly less than n.

EXAMPLE 8.2. *Find the Maclaurin series of e^x.*

SOLUTION. Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$. Thus $f^{(n)}(0) = 1$ and so the Maclaurin series of e^x is

$$1 + x + \frac{x^2}{2!} + \cdots .$$

□

EXAMPLE 8.3. *Find the Taylor series of $f(x) = \sin x$ at $x_0 = \pi$.*

SOLUTION.

$$f(x) = \sin x \qquad f'(x) = \cos x \qquad f''(x) = -\sin x \qquad f'''(x) = -\cos x \qquad \cdots$$

$$f(\pi) = 0 \qquad f'(\pi) = -1 \qquad f''(\pi) = 0 \qquad f'''(\pi) = 1, \qquad \cdots$$

$$T_3(f, \pi)(x) = 0 - (x - \pi) + 0 + \frac{1}{3!}(x - \pi)^3.$$

$$T(f, \pi)(x) = -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \cdots = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{(2k+1)!}(x-\pi)^{2k+1}.$$

□

EXAMPLE 8.4. *Find the Taylor series of $f(x) = \dfrac{1}{x}$ at $x_0 = 3$.*

PROOF.

$$\frac{1}{x} = \frac{1}{3 - (3 - x)} = \frac{1}{3} \cdot \frac{1}{1 - \frac{3-x}{3}}$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{3 - x}{3}\right)^n = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n}(x - 3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}(x - 3)^n.$$

Thus the Taylor series of $\dfrac{1}{x}$ at $x_0 = 3$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}(x - 3)^n.$$

□

8.3. Taylor Theorem. In view of Theorem 7.4, we may ask the following question:

Question: Given a infinitely differentiable function $f(x)$, does the equality

$$f(x) = T(f, x_0)(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \tag{12}$$

hold for $|x - x_0| < R$?
(Here R is the radius of the convergence of the Taylor series.)
It turns out that in general, the answer is NO. (See Example 8.5 for an example of a function such that equation 12 does not hold.)

However, as we have seen, the above equality does hold for some elementary functions such as $e^x, \sin x$, $\cos x$, $\ln(1+x)$, $(1+x)^a$, $\arctan x$, and etc. We are going to give certain hypothesis such that the above equality holds for some functions.

Example 8.5 (Counter-example to the Question). Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Then we will show that $f(x) \neq$ its Taylor series at $x_0 = 0$.

Proof. First we compute $f'(0)$. By substituting $y = \dfrac{1}{x^2}$

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}} - 0}{x} = \lim_{x\to 0} \frac{1}{xe^{\frac{1}{x^2}}} = \lim_{x\to 0} \frac{1}{x^2 e^{\frac{1}{x^2}}} \cdot x \\ &= \lim_{y\to\infty} \frac{y}{e^y} \cdot \lim_{x\to 0} x = 0 \cdot 0 \quad \text{(by L'Hopital's rule)} \quad = 0. \end{aligned}$$

For $x \neq 0$,

$$f'(x) = \frac{d}{dx}\left(e^{-\frac{1}{x^2}}\right) = 2x^{-3}e^{-\frac{1}{x^2}}.$$

Thus,

$$f'(x) = \begin{cases} 2x^{-3}e^{-\frac{1}{x^2}} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Next we compute $f''(x)$. By substituting $y = \dfrac{1}{x^2}$,

$$\begin{aligned} f''(0) &= \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{2x^{-3}e^{-\frac{1}{x^2}} - 0}{x} = 2\lim_{x\to 0} \frac{1}{x^4 e^{\frac{1}{x^2}}} \\ &= 2\lim_{y\to\infty} \frac{y^2}{e^y} = 0 \quad \text{(by L'Hopital's rule).} \end{aligned}$$

Again, for $x \neq 0$,

$$f''(x) = \frac{d}{dx}\left(2x^{-3}e^{-\frac{1}{x^2}}\right) = (-6x^{-4} + 4x^{-6})e^{-\frac{1}{x^2}}.$$

Similar calculations will lead to

$$f(0) = f'(0) = f''(0) = f^{(3)}(0) = f^{(4)}(0) = \cdots = 0.$$

Thus we have

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{0}{k!} x^k = 0 + 0x + 0x^2 + \cdots = 0.$$

Clearly, at any $x \neq 0$, $f(x) = e^{-\frac{1}{x^2}} \neq 0$. Therefore, $f(x) \neq \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$. □

The **remainder** or **error function** between $f(x)$ and $T_n(f, x_0)$ is defined by

$$R_n(f, x_0)(x) = f(x) - T_n(f, x_0)(x).$$

Clearly

$$f(x) = \lim_{n\to\infty} T_n(f, x_0)(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

if and only if

$$\lim_{n\to\infty} R_n(f, x_0)(x) = 0.$$

To emphasize this fact, we state it as a theorem

THEOREM 8.6. *Suppose that f is an infinitely differentiable function on an open interval I and $x_0 \in I$. Then, for $x \in I$,*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

if and only if $\lim_{n\to\infty} R_n(f, x_0)(x) = 0$. □

The remainder $R_n(f, x_0)$ has been studied much and there are various forms of $R_n(f, x_0)$. We only provide one result called **Lagrange Form of the Remainder**, attributed by Joseph Lagrange (1736-1813). But this result sometimes also referred to as Taylor's theorem.

THEOREM 8.7 (Taylor Theorem). *Let f be a function on an open interval I, $x_0 \in I$ and $n \in \mathbb{N}$. If $f^{(n+1)}(t)$ exists for every $t \in I$, then for any $x \in I$, there exists a ξ between x_0 and x such that*

$$R_n(f, x_0)(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}. \tag{13}$$

Thus

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1},$$

for some ξ between x and x_0.

PROOF. Recall that

$$R_n(f, x_0) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Fixed $x \in I$, let M be defined by $R_n(f, x_0) = M(x - x_0)^{n+1}$, that is,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + M(x - x_0)^{n+1}.$$

(**Note.** M depends on x.) Our goal is to show that $M = \dfrac{f^{n+1}(\xi)}{(n+1)!}$ for some ξ between x_0 and x.

We construct a function

$$g(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(t - x_0)^k - M(t - x_0)^{n+1}$$

$$= f(t) - \left(f(x_0) + f'(x_0)(t - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(t - x_0)^n \right) - M(t - x_0)^{n+1}.$$

By taking derivatives, we have

$$g(x_0) = g'(x_0) = g''(x_0) = \cdots = g^{(n)}(x_0) = 0.$$

and

$$g^{(n+1)}(t) = f^{(n+1)}(t) - (n+1)!M. \tag{14}$$

For convenience, let's assume $x > x_0$. By the choice of M, $g(x) = 0$. By applying the mean value theorem to g on the interval $[x_0, x]$, there exists c_1, $x_0 < c_1 < x$, such that

$$0 = g(x) - g(x_0) = g'(c_1)(x - x_0) \qquad \Longrightarrow g'(c_1) = 0.$$

Since $g'(x_0) = g'(c_1) = 0$, by applying the mean value theorem to g' on the interval $[x_0, c_1]$, there exists c_2, $x_0 < c_2 < c_1$, such that

$$0 = g'(c_1) - g'(x_0) = g''(c_2)(c_1 - x_0) \qquad \Longrightarrow g''(c_2) = 0.$$

Continuing this manner, we obtain points $c_1, c_2, \cdots, c_n$, $x_0 < c_n < c_{n-1} < \cdots < c_2 < c_1 < x$, such that $g'(c_1) = 0$, $g''(c_2) = 0$, $g'''(c_3) = 0$, $\cdots$, $g^{(n)}(c_n) = 0$. By applying the mean value theorem once more to $g^{(n)}$ on $[x_0, c_n]$, there exists ξ, $x_0 < \xi < c_n$, such that

$$0 = g^{(n)}(c_n) - g^{(n)}(x_0) = g^{(n+1)}(\xi)(c_n - x_0) \qquad \Longrightarrow g^{(n+1)}(\xi) = 0.$$

From Equation (14),

$$0 = g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - (n+1)!M,$$

that is, $M = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}$ for some ξ between x_0 and x (because $x_0 < \xi < c_n < x$). □

Example 8.8. *Show that*

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (|x| < \infty).$$

PROOF. First the right hand side is the Maclaurin series of $f(x) = \sin x$ because

$$f(x) = \sin x \qquad f'(x) = \cos x \qquad f''(x) = -\sin x \qquad f'''(x) = -\cos x \qquad \cdots$$
$$f(0) = 0 \qquad f'(0) = 1 \qquad f''(0) = 0 \qquad f'''(0) = -1, \qquad \cdots$$

Next let $a_n = \left|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right|$. By the ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}\right|}{\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right|}$$
$$= \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)} = 0 < 1$$

for all x. Thus $R = +\infty$.

In the last step we show that the remainder tends to 0. Since $f(x) = \sin x$, the higher derivatives $f^{(n+1)}(x)$ is either $\pm \sin x$ or $\pm \cos x$. Thus $|f^{(n+1)}(x)| \le 1$ for all x and all n. By the Taylor Theorem,

$$|R_n(f,0)(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)} x^{n+1}\right| \le \frac{|x|^{n+1}}{(n+1)!}.$$

Since $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, we have $\lim_{n\to\infty} |R_n(f,0)(x)| = 0$, by the Squeeze Theorem, or $\lim_{n\to\infty} R_n(f,0)(x) = 0$ for all x. Hence

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (|x| < \infty).$$

□

8.4. Some Standard Power Series. Below is a list of Maclaurin series of some elementary functions.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (|x| < \infty)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (|x| < \infty).$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (|x| < \infty).$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad (-1 < x \le 1).$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (|x| < 1).$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots \qquad (|x| < 1).$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \qquad (-1 \le x \le 1)$$

$$(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n = 1 + \binom{a}{1} x + \binom{a}{2} x^2 + \binom{a}{3} x^3 + \cdots \qquad (|x| < 1),$$

where

$$\binom{a}{k} = \frac{a \cdot (a-1) \cdot (a-2) \cdots (a-k+1)}{k!}$$

for any real number a and integers $k \ge 1$, and $\binom{a}{0} = 1$

Remark. From these power series, we can obtain Maclaurin series of various more complicated functions by using operations such as substitution, addition, subtraction, multiplication, division, integrals, derivatives and etc. For the Maclaurin series of $\sin x^2$ can be obtained by replacing x by x^2 in the Maclaurin series of $\sin x$. By using multiplication, we can obtain the Maclaurin series of $e^x \cdot \sin x$. By using long division, we can obtain the Maclaurin series of $\tan x = \dfrac{\sin x}{\cos x}$. By taking integral, we can obtain the Maclaurin series of non-elementary functions, for instance $f(x) = \int_0^x \sin t^2 \, dt$.

As an application, we are going to compute number π.

Computation of π

Step 1. Find the Maclaurin series of $\arcsin x$ for $|x| < 1$.

For $|x| < 1$,

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt = \int_0^x \left(1 + (-t^2)\right)^{-\frac{1}{2}} dt = \int_0^x \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-t^2)^k dt$$

$$= \sum_{k=0}^{\infty} \int_0^x \binom{-\frac{1}{2}}{k} (-1)^k t^{2k} dt = \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \frac{x^{2k+1}}{2k+1} = x + \sum_{k=1}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \frac{x^{2k+1}}{2k+1}$$

Note that

$$\binom{-\frac{1}{2}}{k} = \frac{\left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{2} - 1\right) \cdots \left(-\frac{1}{2} - k + 1\right)}{k!} = \frac{\left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdots \left(-\frac{2k-1}{2}\right)}{k!}$$

$$= (-1)^k \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{k! \cdot 2^k}$$

for $k \ge 1$. Thus

(15)

$$\arcsin x = x + \sum_{k=1}^{\infty} (-1)^k (-1)^k \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{2^k \cdot k! \cdot (2k+1)} x^{2k+1} = x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{2^k \cdot k! \cdot (2k+1)} x^{2k+1}$$

for $|x| < 1$.

Step 2. Find a series expansion of $\dfrac{\pi}{6}$ using $\dfrac{\pi}{6} = \arcsin \dfrac{1}{2}$.

From Equation (15), we obtain the following formula.

$$\frac{\pi}{6} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot \dots \cdot (2k-1)}{k! \cdot (2k+1) \cdot 2^{3k+1}}. \tag{16}$$

Step 3. Estimate the remainder.

The remainder of the formula 16 can be estimated as follows.

$$R_n = |S - S_n| = \sum_{k=n+1}^{\infty} \frac{1 \cdot 3 \cdot \dots \cdot (2k-1)}{2^k \cdot k! \cdot (2k+1) \cdot 2^{2k+1}} < \sum_{k=n+1}^{\infty} \frac{1}{(2k+1)2^{2k+1}}$$

$$< \sum_{k=n+1}^{\infty} \frac{1}{(2n+3)2^{2k+1}} = \frac{1}{(2n+3)2^{2n+3}} \left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right)$$

$$= \frac{1}{(2n+3)2^{2n+3}\left(1 - \frac{1}{4}\right)} = \frac{1}{3(2n+3)2^{2n+1}}$$

For instance, let $n = 10$, we have

$$\pi \approx 6\left(\frac{1}{2} + \sum_{k=1}^{10} \frac{(1 \cdot 3 \cdot \dots \cdot (2k-1)}{k! \cdot (2k+1) \cdot 2^{3k+1}}\right)$$

with error less than

$$6 \cdot \frac{1}{3 \cdot 23 \cdot 2^{21}} = \frac{1}{23 \cdot 2^{20}} = \frac{1}{24117248}.$$

If we choose $n = 20$, we have

$$\pi \approx 6\left(\frac{1}{2} + \sum_{k=1}^{20} \frac{(2k-1)!!}{k! \cdot (2k+1) \cdot 2^{3k+1}}\right)$$

with error less than

$$6 \cdot \frac{1}{3 \cdot 43 \cdot 2^{41}} = \frac{1}{43 \cdot 2^{40}} = \frac{1}{47278999994368} < 10^{-13}.$$

If $n = 40$, the error is less than $\dfrac{1}{1003408430280142215006126 08} < 10^{-26}$.

Remark. There are several other methods for computing π. For instance, we can also use

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

but one needs a huge number of terms to get enough accuracy. (So this method is no good for computational purpose!) Another method is to use the formula of John Machin (1680-1751):

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4},$$

see our text book [**1**, Problem 7, p.813] for details. Machin used his method in 1706 to find π correct to 100 decimal places. In 1995 Jonathan and Peter Borwein of Simon Fraster University and Yasumasa Kanada of the University of Tokyo calculated the value of π to $4, 294, 967, 286$ decimal places!

Another story on computing π is a Chinese mathematician and astronomer Tsu Ch'ung Chi (430-501). He gave the rational approximation $\frac{355}{113}$ to π which is correct to 6 decimal places. He also proved that

$$3.1415926 < \pi < 3.1415927$$

a remarkable result (**Note.** He was a person lived 1500 years ago!), on which it would be nice to have more details but Tsu Ch'ung Chi's book, written with his son, is lost. (His method is to cut off the circle by equal pieces to get his approximation to π.) Tsu's astronomical achievements include the making of a new calendar in 463 which never came into use. (According to the article of J J O'Connor and E F Robertson in http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Tsu.html.)

Remark. Those, who are interested in more applications of Taylor series, can try to finish the applied project, *Radiation from the Stars*, in our text book [**1**, pp.808-809].

Bibliography

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