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Nonlocal Conservation Laws of the DNLS-Equation

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Abstract

An infinite hierarchy of nonlocal conservation laws for the derivative non-linear Schrödinger (DNLS-) equation is derived by means of the geometrical method introduced by Carnevale and Tenenblat. The two lowest order laws are related to conservation equations which have been found by other methods.

1. Introduction

The derivative nonlinear Schrödinger (DNLS-) equation

$$q_t + \{|q|^2 q\}_x + \mathrm{i}q_{xx} = 0 \tag{1.1}$$

has been found relevant for the study of weakly nonlinear and dispersive Alfven waves [1], drifting filamentations formed in nonlinear electrostatic waves in magnetized plasmas [2], and the evolution of light pulses in optical fibres [3].

The mathematical properties of eq. (1.1) can be summarized as follows: It is a completely integrable Hamiltonian system with an infinite hierarchy of conserved polynomial densities in involution with respect to a symplectic form. The transformation to action-angle variables is given by the inverse scattering transform (IST) [4, 5]. Furthermore, eq. (1.1) possesses soliton solutions which have been classified in [10].

One readily finds that (1.1) itself is a conservation equation with conserved density

$$D_0 = q \tag{1.2}$$

By inspecting the hierarchy of polynomial conserved densities one observes that (1.2) is not included in this hierarchy [4, 5]. The same conclusion holds true for the nonlocal conserved density [6]:

$$\begin{aligned} D_{-1} &= \tfrac{1}{2}\mathrm{i}\,(pq - p^*q^*) \\ p &= \int_{-\infty}^{x} q^*\,\mathrm{d}x \end{aligned} \tag{1.3}$$

which was found by means of Noether's theorem: The Lagrangian of the DNLS-equation (1.1) is invariant under the gauge transformation

$$q \to q\exp(\mathrm{i}\varepsilon) \qquad p \to p\exp(\mathrm{i}\varepsilon)$$

The interpretation of D_{-1} as waveaction density becomes apparent when expressing the corresponding conservation law in terms of the energy density $|\hat{q}_k|^2$ of the spectral component k [7]:

$$\int_{-\infty}^{\infty} D_{-1}\,\mathrm{d}x = -2\pi\, P\int_{-\infty}^{\infty} \frac{|\hat{q}_k|^2}{k}\,\mathrm{d}k.$$

where P denotes the principal value and

$$\hat{q}_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} q(x)\exp(-\mathrm{i}kx)\,\mathrm{d}x$$

One can further show that D_0 and D_{-1} are conserved densities of the extended DNLS-equation [7]

$$\begin{aligned} &q_t + \{q(|q|^2 - \sigma H(|q|^2)\}_x + \mathrm{i}q_{xx} = 0 \\ &H(V) = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{V(x')}{x - x'}\,\mathrm{d}x' \end{aligned} \tag{1.4}$$

where the additional term $-\sigma(qH(|q|^2))_x$ describes the effect of resonant particles on the Alfven wave modulations. In fact, the numerical scheme which has been applied to the initial value problem of (1.4), implies corresponding discrete versions of $\int_{-\infty}^{\infty} D_0\,\mathrm{d}x = \text{const.}$ and $\int_{-\infty}^{\infty} D_{-1}\,\mathrm{d}x = \text{const.}$ [8].

This serves as the motivation for the present study: The objective of this paper is to show that the densities D_0 and D_{-1} are included in an infinite hierarchy of conserved densities of (1.1). All the densities in this hierarchy, except D_0, are nonlocal. The conserved densities are derived by employing the geometrical approach introduced by Cavalcante and Tenenblat [9]. Their method is based on the interpretation of soliton equations as descriptions of pseudospherical surfaces.

The present paper is organized in the following way: In Section 2, a description of the Cavalcante–Tenenblat procedure is presented. Section 3 contains the explicit derivation of the nonlocal conserved densities, while concluding remarks are given in Section 4.

2. The Cavalcante–Tenenblat procedure

The goal of this section is to review a systematic procedure for constructing conservation laws

$$\mathscr{D}_{\mathscr{j},\,t} + \mathscr{F}_{\mathscr{j},\,x} = 0 \tag{2.1}$$

for differential equations describing pseudospherical surfaces (p.s.s.) worked out by Cavalcante and Tenenblat. (For details concerning the concept p.s.s., consult Flanders [11].) The original formulation of the concept equations describing p.s.s. presupposes real valued solutions to the equations. We extend the definition of this concept to complex valued functions:

Let M be a 2-dimensional differentiable manifold with coordinates (x, t). A function $q(x, t)$ describes a p.s.s. if it is a necessary and sufficient condition for the existence of differentiable functions f_{ij}, $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant 2$ depending on $q, q_x, \ldots$ such that the 1-forms

$$\omega_i = f_{i1}\,\mathrm{d}x + f_{i2}\,\mathrm{d}t \tag{2.2}$$

satisfy the structure equations of a p.s.s. [11], i.e.,

$$\begin{aligned} \mathrm{d}\omega_1 &= \omega_3 \wedge \omega_2 \\ \mathrm{d}\omega_2 &= \omega_1 \wedge \omega_3 \\ \mathrm{d}\omega_3 &= \omega_1 \wedge \omega_2 \end{aligned} \tag{2.3}$$

As a consequence, each solution of the differential equation provides a metric on M whose Gaussian curvature is constant, equal to -1.

By inserting (2.2) into (2.3) the evolution equations for f_{ij}

$$\begin{aligned} -f_{11,t} + f_{12,x} &= f_{31}f_{22} - f_{21}f_{32} \\ -f_{21,t} + f_{22,x} &= f_{11}f_{32} - f_{12}f_{31} \\ -f_{31,t} + f_{32,x} &= f_{11}f_{22} - f_{12}f_{21} \end{aligned} \tag{2.4}$$

are derived.

Then, according to theorem 2.1 in [9] the following holds true:

(i) The system

$$\begin{aligned} \phi_x &= f_{31} + f_{11}\sin\phi + f_{21}\cos\phi \\ \phi_t &= f_{32} + f_{12}\sin\phi + f_{22}\cos\phi \end{aligned} \tag{2.5}$$

is completely integrable for ϕ.

(ii) For any solution ϕ of 2.5)

$$\omega = (f_{11}\cos\phi - f_{21}\sin\phi)\,\mathrm{d}x + (f_{12}\cos\phi - f_{32}\sin\phi)\,\mathrm{d}t \tag{2.6}$$

is a closed 1-form i.e. $\mathrm{d}\omega = 0$.

(iii) If f_{ij} are analytic functions of a parameter ζ at zero, then the solutions $\phi(x, t, \zeta)$ of (2.5) and the 1-form ω are analytical in ζ at zero, too.

In order to state the main result in the Cavalcante–Tenenblat procedure we need to fix our notation:

We assume that the f_{ij} are analytic at $\zeta = 0$ i.e.,

$$f_{ij}(x, t, \zeta) = \sum_{k=0}^{\infty} f_{ij}^{k}(x, t)\,\zeta^k \tag{2.7}$$

Then the solution of (2.5) and the 1-form ω given by (2.6) can be expressed as

$$\begin{aligned} \phi(x, t, \zeta) &= \sum_{j=0}^{\infty} \phi_j(x, t)\,\zeta^j \\ \omega(x, t, \zeta) &= \sum_{j=0}^{\infty} \omega^j(x, t)\,\zeta^j \end{aligned} \tag{2.8}$$

We consider the following functions of ζ for fixed x and t:

$$\begin{aligned} C(\zeta) \equiv \cos\phi &= \cos\left(\sum_{j=0}^{\infty} \phi_j\zeta^j\right) \\ S(\zeta) \equiv \sin\phi &= \sin\left(\sum_{j=0}^{\infty} \phi_j\zeta^j\right) \end{aligned} \tag{2.9}$$

Finally, we define the functions

$$\begin{aligned} H_k^{ij} &\equiv f_{1k}^{i}\frac{\mathrm{d}^{j-i}C}{\mathrm{d}\zeta^{j-1}}(0) - f_{2k}^{i}\frac{\mathrm{d}^{j-i}S}{\mathrm{d}\zeta^{j-i}}(0) \\ L_k^{ij} &\equiv f_{1k}^{i}\frac{\mathrm{d}^{j-i}S}{\mathrm{d}\zeta^{j-i}}(0) + f_{2k}^{i}\frac{\mathrm{d}^{j-i}C}{\mathrm{d}\zeta^{j-i}}(0) \\ F_{1k} &\equiv f_{3k}^{1} + L_k^{11} \\ F_{lk} &\equiv f_{3k}^{l} + \sum_{r=1}^{l-1}\frac{l-r}{r!}H_k^{0r}\phi_{l-r} + \sum_{r=1}^{l}\frac{1}{(1-r)!}L_k^{rl} \end{aligned} \tag{2.10}$$

where i, j, l are nonnegative integers such that $j \geq i$, $l \geq 2$ and $k = 1, 2$.

As an immediate consequence of (i)–(iii), we get the following corollary (corollary 2.2 in [9]):

Let $f_{ij}(x, t, \zeta)$, $1 \leq i \leq 3$, $1 \leq j \leq 2$ be differentiable functions of x and t, analytic at $\zeta = 0$, that satisfy (2.4). Then, with the above notation, the following statements hold:

(a) The solutions ϕ of (2.5) are analytic at $\zeta = 0$; ϕ_0 is determined by

$$\begin{aligned} \phi_{0,x} &= f_{31}^{0} + L_1^{00} \\ \phi_{0,t} &= f_{32}^{0} + L_2^{00} \end{aligned} \tag{2.11}$$

and for $j \geq 1$, ϕ_j are recursively determined by the system

$$\begin{aligned} \phi_{j,x} &= H_1^{00}\phi_j + F_{j1} \\ \phi_{j,t} &= H_2^{00}\phi_j + F_{j2} \end{aligned} \tag{2.12}$$

(b) For any such solution ϕ and any integer $j \geq 0$, ω^j is given by

$$\omega^j = \sum_{i=0}^{j}\frac{1}{(j-i)!}(H_1^{ij}\,\mathrm{d}x + H_2^{ij}\,\mathrm{d}t) \tag{2.13}$$

The closed 1-forms ω^j provide a sequence of conservation laws (2.1) for the equation governing the evolution of $q(x, t)$, with conserved densities $\mathscr{D}_j$ and corresponding fluxes $\mathscr{F}_j$ given by ($j \geq 0$)

$$\mathscr{D}_j = \sum_{i=0}^{j}\frac{1}{(j-i)!}H_1^{ij}$$

and

$$\mathscr{F}_j = -\sum_{i=0}^{j}\frac{1}{(j-i)!}H_2^{ij} \tag{2.14}$$

respectively.

3. The DNLS-equation

In this section we apply the formalism of the preceding section to construct nonlocal conservation equations on the form (2.1) for the DNLS-equation (1.1). The main results are given by eqs. (3.6)–(3.17).

The equation (1.1) is the integrability condition for [4]

$$\begin{aligned} v_{1,x} &= -\mathrm{i}\zeta^2 v_1 + \zeta q v_2 \\ v_{2,x} &= \mathrm{i}\zeta^2 v_2 + \zeta q^* v_1 \\ v_{1,t} &= Av_1 + Bv_2 \\ v_{2,t} &= Cv_1 - Av_2 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} A &= 2\mathrm{i}\zeta^4 + \mathrm{i}|q|^2\zeta^2 \\ B &= -2q\zeta^3 - (\mathrm{i}q_x + |q|^2 q) \\ C &= -2q^*\zeta^3 + (\mathrm{i}q_x^* - |q|^2 q^*) \end{aligned} \tag{3.2}$$

It is possible to rewrite (1.1) on the form

$$\begin{aligned} A_x &= \zeta(qC - q^*B) \\ B_x + 2\mathrm{i}\zeta^2 B &= \zeta(q_t - 2qA) \\ C_x - 2\mathrm{i}\zeta^2 C &= \zeta(q^*t + 2q^*A) \end{aligned} \tag{3.3}$$

by means of (3.2).

Now define the 1-forms ω_1, ω_2 and ω_3 as

$$\begin{aligned} \omega_1 &\equiv \zeta(q + q^*)\,\mathrm{d}x + (C + B)\,\mathrm{d}t \\ \omega_2 &\equiv -2\mathrm{i}\zeta^2\,\mathrm{d}x + 2A\,\mathrm{d}t \\ \omega_3 &\equiv \zeta(q^* - q)\,\mathrm{d}x + (C - B)\,\mathrm{d}t \end{aligned} \tag{3.4}$$

A simple calculation shows that ω_1, ω_2 and ω_3 are the appropriate 1-forms which formally satisfy the structure equations (2.3). Thus the DNLS-equation can be said to describe a p.s.s. The coefficient functions f_{ij} of ω_1, ω_2 and ω_3 are given by

$$\begin{aligned} f_{11} &= (q + q^*)\zeta \\ f_{12} &= (\mathrm{i}q_x^* - \mathrm{i}q_x - |q|^2(q + q^*))\zeta - 2(q + q^*)\zeta^3 \\ f_{21} &= -2\mathrm{i}\zeta^2 \\ f_{22} &= 4\mathrm{i}\zeta^4 + 2\mathrm{i}\zeta^2|q|^2 \\ f_{31} &= (q^* - q)\zeta \\ f_{32} &= (\mathrm{i}q_x^* + \mathrm{i}q_x + |q|^2(q - q^*))\zeta + 2(q - q^*)\zeta^3 \end{aligned} \tag{3.5}$$

The next step consists of determining the sequence of functions ϕ_j (eqs. (2.11)–(2.13)). It follows from (2.10) and (3.5) that

$$\begin{aligned} \phi_{0,x} &= 0 \\ \phi_{0,t} &= 0, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \phi_{1,x} &= q^* - q + (q^* + q)\sin\phi_0 \\ \phi_{1,t} &= \mathrm{i}(q_x^* + q_x) + |q|^2(q - q^*) \\ &\quad + [\mathrm{i}(q_x^* - q_x) - |q|^2(q^* + q)]\sin\phi_0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \phi_{2,x} &= (q + q^*)\,\phi_1\cos\phi_0 - 2\mathrm{i}\cos\phi_0 \\ \phi_{2,t} &= [\mathrm{i}q_x^* - \mathrm{i}q_x - |q|^2(q + q^*)]\phi_1\cos\phi_0 + 2\mathrm{i}|q|^2\cos\phi_0, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \phi_{3,x} &= 2\mathrm{i}\phi_1\sin\phi_0 + \tfrac{1}{2}(q + q^*)(2\phi_2\cos\phi_0 - \phi_1^2\sin\phi_0) \\ \phi_{3,t} &= -2(q + q^*)\sin\phi_0 - 2\mathrm{i}|q|^2\phi_1\sin\phi_0 \\ &\quad + \tfrac{1}{2}(\mathrm{i}q_x^* - \mathrm{i}q_x - |q|^2(q + q^*))(2\phi_2\cos\phi_0 - \phi_1^2\sin\phi_0) \end{aligned} \tag{3.9}$$

For $j \geqslant 4$ the ϕ_j-equations are given by

$$\begin{aligned} \phi_{j,x} &= \frac{1}{(j-1)!}(q + q^*)\frac{\mathrm{d}^{j-1}S}{\mathrm{d}\zeta^{j-1}}(0) - 2\mathrm{i}\frac{1}{(j-2)!}\frac{\mathrm{d}^{j-2}C}{\mathrm{d}\zeta^{j-2}}(0) \\ \phi_{j,t} &= \frac{1}{(j-1)!}(\mathrm{i}q_x^* - \mathrm{i}q_x - |q|^2(q + q^*))\frac{\mathrm{d}^{j-1}S}{\mathrm{d}\zeta^{j-1}}(0) \\ &\quad + \frac{1}{(j-2)!}2\mathrm{i}|q|^2\frac{\mathrm{d}^{j-2}C}{\mathrm{d}\zeta^{j-2}}(0) \\ &\quad + \frac{1}{(j-3)!}2(q + q^*)\frac{\mathrm{d}^{j-3}S}{\mathrm{d}\zeta^{j-3}}(0) \\ &\quad + \frac{1}{(j-4)!}4\mathrm{i}\frac{\mathrm{d}^{j-4}C}{\mathrm{d}\zeta^{j-4}}(0) \end{aligned} \tag{3.10}$$

The conservation laws are easily deduced by using (2.14). One readily obtains

$$\begin{aligned} \mathscr{D}_0 &= 0 \\ \mathscr{F}_0 &= 0, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \mathscr{D}_1 &= (q + q^*)\cos\phi_0 \\ \mathscr{F}_1 &= (\mathrm{i}q_x - \mathrm{i}q_x^* + |q|^2(q + q^*))\cos\phi_0, \end{aligned} \tag{3.12}$$

$$\begin{aligned} \mathscr{D}_2 &= (-(q + q^*)\phi_1 + 2\mathrm{i})\sin\phi_0 \\ \mathscr{F}_2 &= ((\mathrm{i}q_x^* - \mathrm{i}q_x - |q|^2(q + q^*))\phi_1 + 2\mathrm{i}|q|^2)\sin\phi_0, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \mathscr{D}_3 &= -\tfrac{1}{2}(q^* + q)(2\phi_2\sin\phi_0 + \phi_1^2\cos\phi_0) + 2\mathrm{i}\phi_1\cos\phi_0 \\ \mathscr{F}_3 &= \tfrac{1}{2}(\mathrm{i}q_x^* - \mathrm{i}q_x - |q|^2(q + q^*))(2\phi_2\sin\phi_0 + \phi_1^2\cos\phi_0) \\ &\quad + 2\mathrm{i}|q|^2\,\phi_1\cos\phi_0 + 2(q + q^*)\cos\phi_0 \end{aligned} \tag{3.14}$$

and for $j \geqslant 4$:

$$\begin{aligned} \mathscr{D}_j &= \frac{1}{(j-1)!}(q + q^*)\frac{\mathrm{d}^{j-1}C}{\mathrm{d}\zeta^{j-1}}(0) + \frac{1}{(j-2)!}2\mathrm{i}\frac{\mathrm{d}^{j-2}S}{\mathrm{d}\zeta^{j-2}}(0) \\ \mathscr{F}_j &= \frac{1}{(j-1)!}(\mathrm{i}q_x - \mathrm{i}q_x^* + |q|^2(q + q^*))\frac{\mathrm{d}^{j-1}C}{\mathrm{d}\zeta^{j-1}}(0) \\ &\quad + \frac{1}{(j-2)!}2\mathrm{i}|q|^2\frac{\mathrm{d}^{j-2}S}{\mathrm{d}\zeta^{j-2}}(0) \\ &\quad + \frac{1}{(j-3)!}2(q + q^*)\frac{\mathrm{d}^{j-3}C}{\mathrm{d}\zeta^{j-3}}(0) \\ &\quad + \frac{1}{(j-4)!}4\mathrm{i}\frac{\mathrm{d}^{j-4}S}{\mathrm{d}\zeta^{j-4}}(0) \end{aligned}$$

for the densities $\mathscr{D}_j$ and the fluxes $\mathscr{F}_j$.

One immediately notices that $\mathscr{D}_1$ can be expressed as

$$\mathscr{D}_1 = (D_0 + D_0^*)\cos\phi_0 \tag{3.16}$$

Thus the conserved density D_0 is recovered from the first nontrivial density in our hierarchy.

By using (1.3), (3.7) and the assumption $\phi_1 \to 0$ as $x \to -\infty$ one finds that

$$\phi_1 = p - p^* + (p + p^*)\sin\phi_0$$

Thus $\mathscr{D}_2$ can be expressed as

$$\mathscr{D}_2 = (-(q^* + q)(p - p^* + (p + p^*)\sin\phi_0) + 2\mathrm{i})\sin\phi_0 \tag{3.17}$$

Moreover, it is easy to check that p^*q is a conserved density of the DNLS-equation. Hence D_{-1} can be recovered from our hierarchy of conserved densities, too.

4. Concluding remarks

In the present paper we have shown a straightforward extension of the Cavalcante–Tenenblat procedure for constructing nonlocal conservation laws of an evolution equation for a complex valued field, the DNLS-equation (1.1). Furthermore, it has been pointed out that the two lowest order laws are related to conservation equations which have been derived by other methods.

An alternative approach to the problem of deriving conservation laws has been suggested by Sasaki [12]. One proceeds as follows:

Let $\mathbf{\Omega}$ be the traceless 2 × 2-matrix of 1-forms defined by

$$\mathbf{\Omega} = \frac{1}{2}\begin{bmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{bmatrix}$$

where ω_1, ω_2 and ω_3 are given by (3.4) and (3.5). We associate a pair of completely integrable Pfaffian equations

$$\mathrm{d}\boldsymbol{v} = \mathbf{\Omega}\boldsymbol{v}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{4.1}$$

with the nonlinear evolution equation under consideration. (In the DNLS-case these Pfaffians are equivalent with the eqs. (3.1)–(3.2)). The evolution equation is now equivalent to the integrability condition

$$\mathrm{d}\mathbf{\Omega} - \mathbf{\Omega} \wedge \mathbf{\Omega} = 0 \tag{4.2}$$

also called *the zero-curvature* equation. Then, by employing the projective transformations $\Gamma = v_2/v_1$ and $\Theta = v_1/v_2$, one readily obtains the Ricatti-equations

$$\begin{aligned} d\Gamma - \sigma_3 + 2\Gamma\sigma_1 + \Gamma^2\sigma_2 \equiv \gamma_1 &= 0 \\ d\Theta - \sigma_2 - 2\Theta\sigma_1 + \Theta^2\sigma_3 \equiv \gamma_2 &= 0 \end{aligned} \tag{4.3}$$

where $\omega_2 = 2\sigma_1$, $\omega_1 - \omega_3 = 2\sigma_2$ and $\omega_1 + \omega_3 = 2\sigma_3$ (ω_1, ω_2 and ω_3 satisfy the structure equations (2.3)).

By differentiating (4.3) and using (4.3) and (2.3) we get

$$\begin{aligned} d\gamma_1 &= 2\gamma_1 \wedge (\sigma_1 + \Gamma\sigma_2) \\ d\gamma_2 &= -2\gamma_2 \wedge (\sigma_2 - \Theta\sigma_3) \end{aligned} \tag{4.3}$$

which are necessary and sufficient conditions for the γ_i's to be completely integrable (Frobenius theorem on complete integrability; see Flanders [11] for more details).

The equations (4.3) show that the 1-forms

$$\begin{aligned} \varepsilon_1 &\equiv \sigma_1 + \Gamma\sigma_2 \\ \varepsilon_2 &\equiv \sigma_1 - \Theta\sigma_3 \end{aligned} \tag{4.4}$$

are closed 1-forms i.e.,

$$\begin{aligned} d\varepsilon_1 &= 0 \\ d\varepsilon_2 &= 0 \end{aligned} \tag{4.5}$$

The equations (4.5) are the desired conservation laws. According to Sasaki [5] the infinite hierarchy of *polynomial conserved densities* are obtained by inserting *expansions in inverse powers of* ζ *for* ε_i into (4.5). On the other hand, the nonlocal conserved densities are derived by assuming *expansions in power series of* ζ in the Cavalcante–Tenenblat procedure. We conjecture that our hierarchy of conservation laws can be derived by means of Sasakis method by assuming power series expansions in ζ.

Finally, we point out that the integrability of the system (1.1) follows from the existence of an infinite set of independent polynomial conservation laws which are in involution with respect to given weak symplectic form [4, 5]. No *a priori* knowledge of the nonlocal laws is required to predict this property. Thus the polynomial laws form a complete set. This observation serves as a motivation for the following conjecture: We believe that the nonlocal laws presented in Section 3 are expressible in terms of the local laws. The proof of this conjecture should be carried out by making the appropriate identifications of the nonlocal conserved densities in scattering data space.

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