Lie symmetry analysis and some new exact solutions of the Wu–Zhang equation

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The Lie symmetry analysis and the basic similarity reductions are performed for the Wu–Zhang equation, a $2+1$ dimensional nonlinear dispersive wave equation. Some new exact solutions generated from the similarity transformation are provided. They demonstrate some new three-dimensional features of a single solitary wave and two interacting solitary waves.
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I. INTRODUCTION

The $2+1$ dimensional nonlinear dispersive wave equation

$$
\begin{aligned}
&u_t+uu_x+vu_y+w_x=0,\\
&v_t+uv_x+vv_y+w_y=0,\\
&w_t+(uw)_x+(vw)_y+\tfrac{1}{3}(u_{xxx}+u_{xyy}+v_{xxy}+v_{yyy})=0,
\end{aligned}
\tag{1}
$$

where (u,v) is the horizontal projection of the surface velocity of a water particle, w is the total water depth ($w-1$ being the wave elevation), is regarded as Wu–Zhang (WZ) equation by Ref. 1. The WZ equation is derived in Ref. 2 from the Euler equation with a perturbation scheme under the assumption that the amplitude of wave elevation is small and the wave is long compared with the water depth (scaled to be 1). The WZ equation can be used to model the three dimensional behavior of solitary waves on a uniform layer of water, such as oblique interaction, oblique reflection from a vertical wall and turning in a curved channel.

If the waves propagate in only one dimension, e.g., along y coordinate, then the WZ equation is reduced to the classical Boussinesq equation

$$
\begin{aligned}
&v_t+vv_y+w_y=0,\\
&w_t+(vw)_y+\tfrac{1}{3}v_{yyy}=0,
\end{aligned}
\tag{2}
$$

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which is known to be integrable and equivalent to Broer–Kaup (BK) system[3,4] and a member of Ablowitz–Kaup–Newell–Segur (AKNS) system[5] that has a tri-Hamiltonian structure. Its exact bidirectional N-soliton solution has been provided by Ref. 5.

Reference 1 provides the Painlevé analysis of the WZ equation. It obtains some exact solutions by using the standard Weiss–Tabor–Carnevale Painlevé truncation expansion. However, the Lie symmetry analysis of the WZ equation is not available yet.

Since the WZ equation is a physical extension of the classical Boussinesq equation, it allows bidirectional soliton solution in any direction in the (x,y) plane. It might have an exact solution that can be used to describe obliquely interacting solitons. This paper is one of a series study towards a good understanding of the WZ equation.

We perform the Lie symmetry analysis in Sec. II, present the $1+1$ similarity reductions in Sec. III and provide a few new exact solutions of the WZ equation in Sec. IV. Finally we summarize the paper in Sec. V.

II. LIE POINT SYMMETRIES

In this section we perform Lie symmetry analysis for the $2+1$-dimensional system (1). Let us consider a one-parameter Lie group of infinitesimal transformation[6]

$$
\begin{aligned}
x &\to x+\epsilon X(x,y,t,u,v,w),\\
y &\to y+\epsilon Y(x,y,t,u,v,w),\\
t &\to t+\epsilon T(x,y,t,u,v,w),\\
u &\to u+\epsilon U(x,y,t,u,v,w),\\
v &\to v+\epsilon V(x,y,t,u,v,w),\\
w &\to w+\epsilon W(x,y,t,u,v,w)
\end{aligned}
\tag{3}
$$

with a small parameter $\epsilon \ll 1$. The vector field associated with the above group of transformations can be written as

$$
\underline{u} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} + W\frac{\partial}{\partial w}. \tag{4}
$$

An invariance of system (1) under transformation (3) leads to the expressions for the functions X, Y, T, U, V,W of the form (throughout this paper we use symbolic package MAPLE to perform all calculation)

$$
\begin{aligned}
X &= c_8xt+c_7x+c_6y+c_4t+c_1,\\
Y &= c_8yt+c_7y-c_6x+c_5t+c_2,\\
T &= c_8t^2+2c_7t+c_3,\\
U &= -(c_8ut+c_7u-c_6v-c_8x-c_4),\\
V &= -(c_8vt+c_7v+c_6u-c_8y-c_5),\\
W &= -(2c_8wt+2c_7w),
\end{aligned}
\tag{5}
$$

where c_i, $i=1,\dots,8$ are arbitrary constants. The presence of these arbitrary constants leads to a finite-dimensional Lie algebra of symmetries. A general element of this algebra is written as

$$\underline{v}=\underline{v}_1c_1+\underline{v}_2c_2+\underline{v}_3c_3+\underline{v}_4c_4+\underline{v}_5c_5+\underline{v}_6c_6+\underline{v}_7c_7+\underline{v}_8c_8, \tag{6}$$

where

$$\underline{v}_1=\frac{\partial}{\partial x},$$

$$\underline{v}_2=\frac{\partial}{\partial y},$$

$$\underline{v}_3=\frac{\partial}{\partial t},$$

$$\underline{v}_4=\frac{\partial}{\partial u}+t\frac{\partial}{\partial x},$$

$$\underline{v}_5=t\frac{\partial}{\partial y}+\frac{\partial}{\partial v},$$

$$\underline{v}_6=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}-u\frac{\partial}{\partial v}+v\frac{\partial}{\partial u},$$

$$\underline{v}_7=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+2t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v}-2w\frac{\partial}{\partial w},$$

$$\underline{v}_8=xt\frac{\partial}{\partial x}+yt\frac{\partial}{\partial y}+t^2\frac{\partial}{\partial t}+(y-vt)\frac{\partial}{\partial v}+(x-ut)\frac{\partial}{\partial u}-2wt\frac{\partial}{\partial w}, \tag{7}$$

construct a basis of the vector space. The associated Lie algebra among these vector fields becomes

	$\underline{v}_1$	$\underline{v}_2$	$\underline{v}_3$	$\underline{v}_4$	$\underline{v}_5$	$\underline{v}_6$	$\underline{v}_7$	$\underline{v}_8$
$\underline{v}_1$	0	0	0	0	0	$-\underline{v}_2$	$\underline{v}_1$	$\underline{v}_4$
$\underline{v}_2$		0	0	0	0	$\underline{v}_1$	$\underline{v}_2$	$\underline{v}_5$
$\underline{v}_3$			0	$\underline{v}_1$	$\underline{v}_2$	0	$2\underline{v}_3$	$\underline{v}_7$
$\underline{v}_4$				0	0	$-\underline{v}_5$	$-\underline{v}_4$	0
$\underline{v}_5$					0	$\underline{v}_4$	$-\underline{v}_5$	0
$\underline{v}_6$						0	0	0
$\underline{v}_7$							0	$2\underline{v}_8$
$\underline{v}_8$								0

where the entry in jth row and kth column represents the commutator $[v_j,v_k]$, and $\{\underline{v}_1,\underline{v}_2,\underline{v}_3\}$, $\{\underline{v}_4,\underline{v}_5\}$, $\{\underline{v}_7,\underline{v}_8\}$, $\{\underline{v}_1,\underline{v}_2,\underline{v}_4,\underline{v}_5,\underline{v}_6\}$ are some of the subalgebras.

We now consider a point transformation

$$G:(x,y,t,u,v,w)\mapsto(\xi,\eta,\zeta,P,Q,R). \tag{8}$$

From the transformation (1), we have the corresponding one-parameter group of symmetries of the WZ equation

$$
\begin{aligned}
&G_1:(x,y,t,u,v,w)\mapsto(x+\epsilon,y,t,u,v,w),\\
&G_2:(x,y,t,u,v,w)\mapsto(x,y+\epsilon,t,u,v,w),\\
&G_3:(x,y,t,u,v,w)\mapsto(x,y,t+\epsilon,u,v,w),\\
&G_4:(x,y,t,u,v,w)\mapsto(x+t\epsilon,y,t,u+\epsilon,v,w),\\
&G_5:(x,y,t,u,v,w)\mapsto(x,y+t\epsilon,t,u,v+\epsilon,w),\\
&G_6:(x,y,t,u,v,w)\mapsto(x\cos\epsilon+y\sin\epsilon,-x\sin\epsilon+y\cos\epsilon,t,u\cos\epsilon+v\sin\epsilon,-u\sin\epsilon\\
&\qquad +v\cos\epsilon,w),\\
&G_7:(x,y,t,u,v,w)\mapsto(xe^{\epsilon},ye^{\epsilon},te^{2\epsilon},ue^{-\epsilon},ve^{-\epsilon},we^{-2\epsilon}),\\
&G_8:(x,y,t,u,v,w)\mapsto\left(\frac{x}{1-t\epsilon},\frac{y}{1-t\epsilon},\frac{t}{1-t\epsilon},u(1-t\epsilon)+x\epsilon,v(1-t\epsilon)+y\epsilon,w(1-t\epsilon)^2\right).
\end{aligned}
\tag{9}
$$

We observe that G_1 and G_2 are space translations, G_3 is a time translations, G_4 and G_5 are Galilean boost, G_6 is a rotation, G_7 is a scaling for all variables with different ratios. G_8 is a time-dependent scaling. The entire symmetry group is obtained by composing one-dimensional subgroups G_i, $i=1,\ldots,8$. When G is an element of this group, if $u(x,y,t)$, $v(x,y,t)$, $w(x,y,t)$ is a solution of WZ equation, then $P(\xi,\eta,\zeta)$, $Q(\xi,\eta,\zeta)$, $R(\xi,\eta,\zeta)$ is also a solution of WZ equation.

III. 1+1 SIMILARITY REDUCTIONS

After determining the infinitesimal generators, the similarity variables can be found by solving the characteristic equations[6]

$$
\frac{\mathrm{d}x}{X}=\frac{\mathrm{d}y}{Y}=\frac{\mathrm{d}t}{T}=\frac{\mathrm{d}u}{U}=\frac{\mathrm{d}v}{V}=\frac{\mathrm{d}w}{W}. \tag{10}
$$

It is easy to know that the generator $\underline{v}_1$ has an invariance

$$
\xi=y,\quad \eta=t,\quad P=u,\quad Q=v,\quad R=w.
$$

Under this transformation, WZ equation is reduced to a system of PDE with two independent variables ξ and η and three dependent variables P, Q, and R. The reduced equation is WZ equation but with u, v, and w independent of x, i.e., $u_x=v_x=w_x=0$. In fact the set of two equations on v and w is identical to the system (2) and the other one is a linear equation on u

$$
u_t+vu_y=0, \tag{11}
$$

which can be solved with a method of characteristic line. Therefore one can obtain a solution of WZ equation (1) from a solution of the classical Boussinesq equation (2).

For the generator $\underline{v}_2$, we have a similar result except now the solution is independent of y.

For the generator $\underline{v}_3$, WZ equation is reduced to its steady case.

For the generator of Galilean transformation $\underline{v}_4=(\partial/\partial u)+t(\partial/\partial x)$, we have the following similarity variables:

$$
\xi=y,\quad \eta=t,\quad P=ut-x,\quad Q=v,\quad R=w. \tag{12}
$$

The reduced PDE becomes

$$P_\eta + QP_\xi = 0,$$

$$Q_\eta + QQ_\xi + R_\xi = 0,$$

$$R_\eta + \frac{R}{\eta} + Q_\xi R + QR_\xi + \frac{1}{3}Q_{\xi\xi\xi} = 0. \tag{13}$$

One may notice that the second and third equations are very closely related to the classical Boussinesq equation (2) except the extra term R/η.

The generator $\underline{v}_5$ has a similar result as $\underline{v}_4$.

For the rotation transformation $\underline{v}_6$, the similarity variables are

$$\xi = x^2 + y^2, \quad \eta = t, \quad P = -xv + yu, \quad Q = yv + xu, \quad R = w. \tag{14}$$

Then WZ equation is reduced to

$$P_\eta + 2QP_\xi = 0,$$

$$\xi Q_\eta + 2\xi^2 R_\xi + 2\xi QQ_\xi - (P^2 + Q^2) = 0,$$

$$R_\eta + 2QR_\xi + 2RQ_\xi + \tfrac{8}{3}Q_{\xi\xi} + \tfrac{8}{3}\xi Q_{\xi\xi\xi} = 0. \tag{15}$$

Of course one may choose another set of invariants to be the similarity variables and obtain a different reduced system. For example, if we take

$$\xi = x^2 + y^2, \quad \eta = t, \quad P = \frac{-xv + yu}{x^2 + y^2}, \quad Q = \frac{yv + xu}{x^2 + y^2}, \quad R = w, \tag{16}$$

then the reduced system reads

$$P_\eta - 2QP_\xi \xi - 2QP = 0,$$

$$Q_\eta + Q^2 + 2QQ_\xi \xi - P^2 + 2R_\xi = 0, \tag{17}$$

$$2RQ + 2QR_\xi \xi + 2RQ_\xi \xi + R_\eta + \tfrac{16}{3}Q_\xi + \tfrac{8}{3}Q_{\xi\xi\xi}\xi^2 + \tfrac{32}{3}Q_{\xi\xi}\xi = 0,$$

which is equivalent to the system (15).

For the scaling transformation generated by $\underline{v}_7$, the similarity variables are

$$\xi = \frac{x}{\sqrt{t}}, \quad \eta = \frac{y}{\sqrt{t}}, \quad P = u\sqrt{t}, \quad Q = v\sqrt{t}, \quad R = wt. \tag{18}$$

The WZ equation is reduced to a system with two independent variables but in a more complicated form

$$(\xi - 2P)P_\xi + (\eta - 2Q)P_\eta + P - 2R_\xi = 0,$$

$$(\xi - 2P)Q_\xi + (\eta - 2Q)Q_\eta + Q - 2R_\eta = 0, \tag{19}$$

$$6R(Q_\eta + P_\xi - 1) - 3(\eta - 2Q)R_\eta - 3(\xi - 2P)R_\xi + 2(P_{\xi\xi\xi} + P_{\eta\eta\xi} + Q_{\eta\eta\eta} + Q_{\eta\xi\xi}) = 0.$$

The similarity variables corresponding to $\underline{v}_8$ are

$$\xi=\frac{x}{t}, \quad \eta=\frac{y}{t}, \quad P=ut-\xi t, \quad Q=vt-\eta t, \quad R=wt^2. \tag{20}$$

The WZ equation is also reduced to its steady case.

We would like to point out that in this section we have only reported the $1+1$ similarity reduction generated by the single but different basic infinitesimal generators $\underline{v}_j$. More $1+1$ reduced systems can be obtained by considering a proper linear combination of different basic generators. We may also implement the symmetry analysis and similarity reduction upon a $1+1$ reduced system and obtain a corresponding ODE system.

IV. SOME NEW EXACT SOLUTIONS

In this section, we present some new exact particular solutions of WZ system (1) obtained from the three kinds of reduction transformation studied in the last section.

A. Solutions from $\underline{v}_3$ reduction

The system generated by $\underline{v}_3$ is the steady WZ system. Here we are looking for a particular steady solution with a similarity variable $z=k_1x+k_2y$, where the two constants k_1 and k_2 are assumed to satisfy $k_1^2+k_2^2=1$ without a loss of generality. The velocity field (u,v) and the total wave depth w are assumed to be functions of z only. The WZ equation becomes a system of ODE

$$k_1uu'+k_2vu'+k_1w'=0, \tag{21}$$

$$k_1uv'+k_2vv'+k_2w'=0, \tag{22}$$

$$(k_1u+k_2v)w'+(k_1u'+k_2v')w+\tfrac{1}{3}(k_1u'''+k_2v''')=0. \tag{23}$$

Integrating the first two equations gives

$$v=\frac{k_2}{k_1}u+d_1, \quad w=-\frac{1}{2k_1^2}u^2-\frac{k_2}{k_1}d_1u+d_2+k_2^2d_1^2.$$

Substituting into Eq. (23) and integrating it twice yields a single equation for u,

$$u'^2=\frac{3}{4k_1^2}u^4+3\frac{k_2}{k_1}d_1u^3-3d_2u^2+d_3u+d_4,$$

where the four integration constants d_i, $i=1,2,3,4$ are determined by the boundary condition at infinity. For a set of d_i's with four real parameters, $\lambda_1>\lambda_2>\lambda_3>\lambda_4$,

$$d_1=-\frac{1}{4k_2}\sum_{i=1}^{4}\lambda_i, \quad d_2=-\frac{1}{4}\sum_{i,j=1,i<j}^{4}\lambda_i\lambda_j, \quad d_3=-\frac{3}{4}k_1\sum_{j=1}^{4}\frac{1}{\lambda_j}\lambda_1\lambda_2\lambda_3\lambda_4,$$

$$d_4=\tfrac{3}{4}k_1^2\lambda_1\lambda_2\lambda_3\lambda_4.$$

the ODE is written as

$$u'^2=\frac{3}{4k_1^2}(u-k_1\lambda_1)(u-k_1\lambda_2)(u-k_1\lambda_3)(u-k_1\lambda_4),$$

and the solution can be written in terms of a Jacobi elliptic function

$$u=k_1\left(\lambda_4+\frac{\Delta_{24}\Delta_{34}}{\Delta_{24}-\Delta_{23}\,\mathrm{sn}^2(s,m)}\right),\quad \text{or}\quad u=k_1\left(\lambda_2+\frac{\Delta_{24}\Delta_{12}}{\Delta_{24}-\Delta_{14}\,\mathrm{sn}^2(s,m)}\right),$$

$$v=\frac{k_2}{k_1}u-\frac{1}{4k_2}\sum_{i=1}^{4}\lambda_i,\tag{24}$$

$$w=-\frac{1}{2k_1^2}u^2+\frac{1}{4k_1}\sum_{i=1}^{4}\lambda_i u-\frac{1}{4}\sum_{i,j=1,i<j}^{4}\lambda_i\lambda_j+\frac{1}{16}\left(\sum_{i=1}^{4}\lambda_i\right)^2,$$

where

$$s=\frac{1}{4}\sqrt{3\Delta_{13}\Delta_{24}}(z-z_0),\quad m=\sqrt{\frac{\Delta_{14}\Delta_{23}}{\Delta_{13}\Delta_{24}}},\quad \Delta_{ij}=\lambda_i-\lambda_j.$$

For a particular set of integration constants with one parameter λ,

$$d_1=0,\quad d_2=1+\tfrac{1}{2}\lambda^2,\quad d_3=6k_1\lambda,\quad d_4=\tfrac{3}{4}k_1^2\lambda^4-3k_1^2\lambda^2,\tag{25}$$

the solution given by

$$u=k_1\left(\lambda-\frac{2(\lambda^2-1)}{\lambda+\cosh\sqrt{3(\lambda^2-1)}(z-z_0)}\right),\quad v=\frac{k_2}{k_1}u,\tag{26}$$

$$w=1+\frac{2(\lambda^2-1)(1+\lambda\cosh\sqrt{3(\lambda^2-1)}(z-z_0))}{(\lambda+\cosh\sqrt{3(\lambda^2-1)}(z-z_0))^2},\tag{27}$$

describes a steady solitary wave on a uniform layer of water. For the same set of constants as (25), we have another solution

$$u=k_1\left(\lambda+\frac{2(1-\lambda^2)}{\lambda\pm\sin\sqrt{3(1-\lambda^2)}(z-z_0)}\right),\quad v=\frac{k_2}{k_1}u,$$

$$w=1-\frac{2(1-\lambda^2)(1\pm\lambda\sin\sqrt{3(1-\lambda^2)}(z-z_0))}{(\lambda\pm\sin\sqrt{3(1-\lambda^2)}(z-z_0))^2},\tag{28}$$

which has a singularity in a finite domain.

With some other choices of the constants, we have more steady solutions listed below.

Case 1:

$$d_1=-\frac{1}{2k_2}(\lambda+\mu),\quad d_2=-\frac{1}{4}(\lambda^2+4\lambda\mu+\mu^2),$$

$$d_3=-\tfrac{3}{2}k_1\lambda\mu(\lambda+\mu),\quad d_4=\tfrac{3}{4}k_1^2\lambda^2\mu^2,$$

$$u=k_1\frac{\lambda-\mu\exp\left(\pm\frac{\sqrt{3}}{2}(\lambda-\mu)(z-z_0)\right)}{1-\exp\left(\pm\frac{\sqrt{3}}{2}(\lambda-\mu)(z-z_0)\right)},\quad v=\frac{k_2}{k_1}u-\frac{1}{2k_2}(\lambda+\mu),\tag{29}$$

$$w=-\frac{1}{8}(\lambda-\mu)^2\,\mathrm{csch}^2\frac{\sqrt{3}}{4}(\lambda-\mu)(z-z_0),$$

or

$$u=k_1\frac{\mu+\lambda\exp\left(\pm\frac{\sqrt{3}}{2}(\lambda-\mu)(z-z_0)\right)}{1+\exp\left(\pm\frac{\sqrt{3}}{2}(\lambda-\mu)(z-z_0)\right)},\quad v=\frac{k_2}{k_1}u-\frac{1}{2k_2}(\lambda+\mu),$$

$$w=\frac{1}{8}(\lambda-\mu)^2\operatorname{sech}^2\frac{\sqrt{3}}{4}(\lambda-\mu)(z-z_0). \tag{30}$$

Case 2:

$$d_1=-\frac{1}{k_2}(\lambda\mp1),\quad d_2=-\frac{3}{2}\lambda(\lambda\mp2),$$

$$d_3=-3k_1\lambda^2(\lambda\mp3),\quad d_4=\tfrac{3}{4}k_1^2\lambda^3(\lambda\mp4),$$

$$u=k_1\left(\lambda\pm\frac{4}{3(z-z_0)^2-1}\right),\quad v=\frac{k_2}{k_1}u-\frac{1}{k_2}(\lambda\mp1),$$

$$w=1-\frac{4(3(z-z_0)^2+1)}{(3(z-z_0)^2-1)^2}. \tag{31}$$

Case 3:

$$d_1=-\frac{\lambda}{k_2},\quad d_2=-\frac{3}{2}\lambda^2,\quad d_3=-3k_1\lambda^3,\quad d_4=\frac{3}{4}k_1^2\lambda^4,$$

$$u=k_1\left(\lambda\mp\frac{2}{\sqrt{3}(z-z_0)}\right),\quad v=\frac{k_2}{k_1}u-\frac{\lambda}{k_2},\quad w=\frac{-2}{3(z-z_0)^2}. \tag{32}$$

Case 4:

$$d_1=0,\quad d_2=\tfrac{1}{4}(\lambda^2-\mu^2),\quad d_3=0,\quad d_4=-\tfrac{3}{2}k_1^2\lambda^2\mu^2,$$

$$u=\frac{k_1\lambda}{\sqrt{1-\operatorname{sn}^2(s,m)}},\quad v=\frac{k_2}{k_1}u,\quad w=\frac{1}{4}(\lambda^2-\mu^2)-\frac{\lambda^2}{2(1-\operatorname{sn}^2(s,m))}, \tag{33}$$

$$s=\tfrac{1}{2}\sqrt{3(\lambda^2+\mu^2)}(z-z_0),\quad m^2=\frac{\mu^2}{\lambda^2+\mu^2}.$$

Case 5:

$$d_1=-\frac{1}{k_2}\mu, \quad d_2=-\frac{1}{4}(6\mu^2+\lambda^2),$$
$$d_3=-\tfrac{3}{2}k_1\mu(2\mu^2+\lambda^2), \quad d_4=\tfrac{3}{4}k_1^2\mu^2(\mu^2+\lambda^2),$$
$$u=k_1\left(\mu\mp\frac{\lambda}{\sinh\frac{\sqrt{3}}{2}\lambda(z-z_0)}\right), \quad v=\frac{k_2}{k_1}u-\frac{\mu}{k_2}, \tag{34}$$
$$w=-\frac{\lambda^2}{4}\left(1+\frac{2}{\sinh^2\frac{\sqrt{3}}{2}\lambda(z-z_0)}\right).$$

Case 6:

$$d_1=0, \quad d_2=-\tfrac{1}{4}(\lambda^2+\mu^2), \quad d_3=0, \quad d_4=\tfrac{3}{4}k_1^2\lambda^2\mu^2,$$
$$u=k_1\lambda\frac{\mathrm{sn}(s,m)}{\mathrm{cn}(s,m)}, \quad v=\frac{k_2}{k_1}u, \quad w=\frac{1}{4}(\lambda^2-\mu^2)-\frac{\lambda^2}{2\,\mathrm{cn}^2(s,m)}, \tag{35}$$
$$s=\frac{\sqrt{3}}{2}\mu(z-z_0), \quad m^2=1-\frac{\lambda^2}{\mu^2} \quad (\lambda^2<\mu^2).$$

Case 7:

$$d_1=\frac{\lambda}{2k_2}, \quad d_2=-\tfrac{1}{8}(3\lambda^2+b^2), \quad d_3=\tfrac{3}{8}k_1\lambda(\lambda^2+b^2), \quad d_4=\tfrac{3}{64}k_1^2(\lambda^2+b^2)^2,$$
$$u=-\frac{k_1}{2}\left(\lambda\mp b\tan\frac{\sqrt{3}}{4}b(z-z_0)\right), \quad v=\frac{k_2}{k_1}u+\frac{\lambda}{2k_2}, \tag{36}$$
$$w=-\frac{1}{8}b^2\left(1+\tan^2\frac{\sqrt{3}}{4}b(z-z_0)\right).$$

These solutions are of mathematical interests, even though some of them are not physically meaningful for the water wave because the total water depth w either goes to zero at infinity or has a singularity in a finite domain.

B. Solutions from $\underline{v_4}$ reduction

The last two equations in the reduced system (13) from the generator v_4 are closely related to the classical Boussinesq equation except the extra term R/η. Their relation is very similar to that of KdV and cKdV equations. In fact, with the following transformation

$$\bar{\xi}=\frac{\xi+d_1}{d_1\eta}+d_2, \quad \bar{\eta}=-\frac{1}{d_1^2\eta}+d_3,$$
$$\bar{P}=d_4P+d_5, \quad \bar{Q}=\left(Q-\frac{\xi+d_1}{\eta}\right)d_1\eta, \quad \bar{R}=Rd_1^2\eta^2 \tag{37}$$

the system (13) is converted to

$$\overline{P}_{\overline{\eta}}+\overline{Q}\,\overline{P}_{\overline{\xi}}=0,$$

$$\overline{Q}_{\overline{\eta}}+\overline{Q}\,\overline{Q}_{\overline{\xi}}+\overline{R}_{\overline{\xi}}=0, \tag{38}$$

$$\overline{R}_{\overline{\eta}}+\overline{Q}_{\overline{\xi}}\overline{R}+\overline{Q}\,\overline{R}_{\overline{\xi}}+\tfrac{1}{3}\overline{Q}_{\overline{\xi}\overline{\xi}\overline{\xi}}=0,$$

where the last two equations are the classical Boussinesq equation. We can make use of the property to construct new solutions of WZ equation. For example, starting with a single soliton solution of the classical Boussinesq equation

$$\overline{Q}=\frac{2\,(\lambda^2-1)}{\lambda+\cosh(\sqrt{3\,\lambda^2-3}\,(\overline{\xi}-\lambda\,\overline{\eta}))}, \tag{39}$$

$$\overline{R}=\frac{2\,(\lambda^2-1)[1+\lambda\,\cosh(\sqrt{3\,\lambda^2-3}(\overline{\xi}-\lambda\,\overline{\eta}))]}{[\lambda+\cosh(\sqrt{3\,\lambda^2-3}(\overline{\xi}-\lambda\,\overline{\eta}))]^2}+1, \tag{40}$$

we are able to obtain a particular solution of WZ equation

$$u=\frac{1}{d_4 t}(d_4 x+\overline{P}-d_5), \tag{41}$$

$$v=\frac{2(\lambda^2-1)}{d_1 t[\lambda+\cosh(\sqrt{3\,\lambda^2-3}\;\;s)]}+\frac{y+d_1}{t}, \tag{42}$$

$$w=\frac{2(\lambda^2-1)[1+\lambda\,\cosh(\sqrt{3\,\lambda^2-3}\;\;s)]}{d_1^2 t^2[\lambda+\cosh(\sqrt{3\,\lambda^2-3}\;\;s)]^2}+\frac{1}{d_1^2 t^2}, \tag{43}$$

where the phase function s is given by

$$s=\overline{\xi}-\lambda\,\overline{\eta}=\frac{1}{d_1 t}\left[y-d_1(\lambda d_3-d_2)t+d_1+\frac{\lambda}{d_1}\right]$$

and $\overline{P}$ is a solution of linear equation

$$\overline{P}_{\overline{\eta}}+\overline{Q}\,\overline{P}_{\overline{\xi}}=0$$

with $\overline{Q}$ given by (39) and d_j, $j=1,\ldots,5$ are arbitrary constants but $d_1 d_4\neq 0$. The solution describes a single solitary wave that is uniform along x direction and travels along y direction. The wave travels with a speed $d_1(\lambda d_3-d_2)$. Since d_2 and d_3 are two free parameters, the solitary wave can be made still by choosing $d_2=\lambda d_3$. It is double-peaked when $\lambda>2$ just like the solution of the classical Boussinesq equation. Its amplitude of the total water depth for the case of $\lambda<2$ is $(2\lambda-1)/(d_1 t)^2$, which is singular at time $t=0$ and decreases as time goes on from 0 to $+\infty$, and its wavelength increases like $d_1 t$. For $t>0$, the mass loss under the solitary wave is due to the sinks at infinity. One may notice that the velocity $u\rightarrow\pm\infty$ as $x\rightarrow\pm\infty$ and $v\rightarrow\pm\infty$ as $y\rightarrow\pm\infty$. The solitary wave solution of the WZ equation has a three-dimensional feature that has not been presented before for other $2+1$ dimensional nonlinear dispersive wave equations.

We now start with the solution of two-soliton head-on collision of the classical Boussinesq equation[5]

$$\overline{Q}=\frac{2(\lambda_1+\lambda_2)(\lambda_2^2-\lambda_1^2-k_2^2\tanh^2\xi_2+k_1^2\tanh^2\xi_1)}{(k_2\tanh\xi_2-k_1\tanh\xi_1)^2-(\lambda_1+\lambda_2)^2}, \tag{44}$$

$$\overline{R}=\frac{1}{\sqrt{3}}\overline{Q}_{\overline{\xi}}+\frac{k_2\tanh\xi_2-k_1\tanh\xi_1-\lambda_1-\lambda_2}{k_2\tanh\xi_2-k_1\tanh\xi_1+\lambda_1+\lambda_2}\times\left[1+2\frac{(\lambda_1+\lambda_2)(k_1\tanh\xi_1-\lambda_1)(k_2\tanh\xi_2+\lambda_2)}{k_2\tanh\xi_2-k_1\tanh\xi_1+\lambda_1+\lambda_2}\right], \tag{45}$$

$$\xi_1=\frac{\sqrt{3}}{2}k_1(\overline{\xi}-\lambda_1\overline{\eta}),\quad \xi_2=\frac{\sqrt{3}}{2}k_2(\overline{\xi}+\lambda_2\overline{\eta}),\quad k_i=\sqrt{\lambda_i^2-1},\quad \lambda_i>1,\quad i=1,2,$$

and construct a new solution of the WZ equation

$$u=\frac{1}{d_4t}(d_4x+\overline{P}-d_5),\quad v=\frac{\overline{Q}}{d_1t}+\frac{y+d_1}{t},\quad w=\frac{\overline{R}}{d_1^2t^2}, \tag{46}$$

where $\overline{P}$ solves $\overline{P}_{\overline{\eta}}+\overline{QP}_{\overline{\xi}}=0$, $\overline{Q}$ and $\overline{R}$ are given by (44) and (45) with

$$\overline{\xi}=\frac{y+d_1}{d_1t}+d_2,\quad \overline{\eta}=-\frac{1}{d_1^2t}+d_3.$$

The two phase functions are

$$s_1=\overline{\xi}-\lambda_1\overline{\eta}=\frac{1}{d_1t}\left[y-d_1(\lambda_1d_3-d_2)t+d_1+\frac{\lambda_1}{d_1}\right],$$

$$s_2=\overline{\xi}+\lambda_2\overline{\eta}=\frac{1}{d_1t}\left[y-d_1(-\lambda_2d_3-d_2)t+d_1-\frac{\lambda_2}{d_1}\right].$$

The two wave speeds are

$$c_1=d_1(\lambda_1d_3-d_2),\quad c_2=d_1(-\lambda_2d_3-d_2).$$

If we pick $d_2=-\lambda_2d_3$, then $c_1=d_1d_3(\lambda_1+\lambda_2)$ and $c_2=0$, solitary wave 2 will stand still and solitary wave 1 will pass through solitary wave 2. Like the single solitary wave solution, the amplitude of the total water depth of both solitary waves decreases like $1/(d_1t)^2$, the wave length increases like d_1t for $t>0$. The mass loss under the two solitary waves is due to the sinks at infinity of both x and y directions.

Figure 1 shows the total water depth $w(y,t)$ from (46) for the interaction of two single-peak solitary waves. The parameters are chosen to ensure that the solitary wave 2 with higher amplitude stands still near the origin and solitary wave 1 with lower amplitude passes through the solitary wave 2 as time goes on. After the elastic collision, each one experiences a backward phase shift. The phase shift of solitary wave 2 is visible in Fig. 1 by comparing the lowerest dashed and solid lines. Figure 2 shows the same solution with λ_1 and λ_2 larger than 2, so that the two solitary waves are double peaked. The phase shift of solitary wave 2 is more visible because the two solitary waves have larger amplitude.

Similarly we can obtain a new exact solution of the WZ equation by using the multisoliton solution of the classical Boussinesq equation.[5]

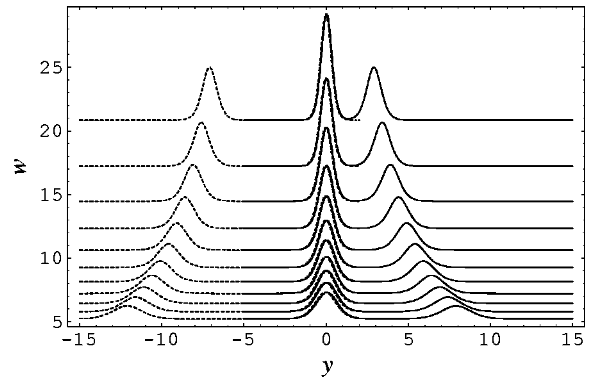

FIG. 1. The interaction of two single-peak solitary waves, $\lambda_1=1.1$, $\lambda_2=1.2$, $d_1=\sqrt{\lambda_2}$, $d_3=10$, $d_2=-\lambda_2 d_3$. The dashed lines are for the total water depth w as a function of space y for the time instances of $t=-0.4$, -0.38, -0.36, -0.34, -0.32, -0.3, -0.28, -0.26, -0.24, -0.22, and -0.2 bottom up. The solid lines are for the time instances of $t=0.2$, 0.22, 0.24, 0.26, 0.28, 0.3, 0.32, 0.34, 0.36, 0.38, 0.4 top down.

C. Solution from v_8 reduction

The system generated by v_8 is also the steady WZ system. With the steady solution (26) and (27) and the similarity variables (20), we obtain a new solution of the WZ equation

$$u=\frac{x}{t}+\frac{k_1}{t}\left(\lambda-\frac{2(\lambda^2-1)}{\lambda+\cosh\sqrt{3(\lambda^2-1)}\ \dfrac{k_1x+k_2y-z_0t}{t}}\right),\quad k_1^2+k_2^2=1,$$

$$v=\frac{y}{t}+\frac{k_2}{t}\left(\lambda-\frac{2(\lambda^2-1)}{\lambda+\cosh\sqrt{3(\lambda^2-1)}\ \dfrac{k_1x+k_2y-z_0t}{t}}\right),$$

$$w=\frac{1}{t^2}+\frac{2(\lambda^2-1)\left(1+\lambda\cosh\sqrt{3(\lambda^2-1)}\ \dfrac{k_1x+k_2y-z_0t}{t}\right)}{t^2\left(\lambda+\cosh\sqrt{3(\lambda^2-1)}\ \dfrac{k_1x+k_2y-z_0t}{t}\right)^2}.$$

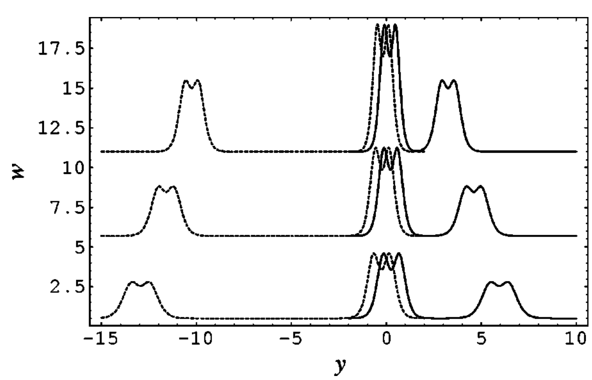

FIG. 2. The interaction of two double-peak solitary waves, $\lambda_1=3$, $\lambda_2=4$, $d_1=\sqrt{\lambda_2}$, $d_3=1$, $d_2=-\lambda_2 d_3$. The dashed lines are for the total water depth w as a function of space y for the time instances of $t=-0.7$, -0.6, and -0.5 bottom up. The solid lines are for the time instances of $t=0.5$, 0.6, and 0.7 top down.

The solution describes a solitary wave with an amplitude of the total water depth decreasing like $1/t^2$ and wavelength increasing like t. The wave speed of the solitary wave is z_0, which can be an arbitrary number.

V. SUMMARY

We have performed Lie symmetry analysis for the Wu–Zhang (WZ) equation and found its algebraic structure. The WZ equation is shown to have a finite dimension of Lie algebra, which means that the equation is less integrable than other integrable $2+1$ dimensional system, such as Kadomtsev–Petviashvili (KP) equation, Davey–Stewartson (DS) equation, Nizhnik–Novikov–Veselov (NNV) equation and $2+1$ dimensional sine-Gorden (sG) system,[7–14] which have infinite dimension of Lie algebra. The result agrees with that from Painlevé analysis.[1]

We have also obtained some new exact solutions of WZ equation by using the similarity transformation approach. They are of mathematical interest even though most of them are not physically meaningful. The solution demonstrate that a solitary wave could travel with arbitrary speed, its amplitude decreases and wave-length increases with time, and solitary waves with any kind of amplitudes could take over each other. These new features are due to the velocity sinks at infinity in both x and y directions. The three-dimensional feature of solitary waves seems to be a new phenomenon to us.

Since the WZ equation has a rotation symmetry, it admits a solitary wave solution along any direction in (x,y) plane. Two such solitary waves on two different directions could have an oblique interaction. The WZ equation can be used to model the process, but the question remains open whether the obliquely interacting solitary wave solution can be written in a closed-form. This is left for further research.

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