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Some constructions of projectively flat Finsler metrics

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Abstract In this paper, we find some solutions to a system of partial differential equations that characterize the projectively flat Finsler metrics. Further, we discover that some of these metrics actually have the zero flag curvature.

Keywords: Randers metric, (α, β)-metric, Finsler metric, projectively flat metric, *S*-curvature.

1 Introduction

It is the Hilbert's Fourth Problem in the smooth case to study and characterize the Finsler metrics on an open domain in $\mathbb{R}^n$ such that geodesics are straight lines. Finsler metrics on an open domain in $\mathbb{R}^n$ with this property are said to be projectively flat. According to ref. [1], a Finsler metric $F = F(x, y)$ on on open subset in $\mathbb{R}^n$ is projectively flat if and only if it satisfies the following partial differential equations:

$$F_{x^k y^i} y^k = F_{x^i}. \tag{1}$$

By the Beltrami theorem, a Riemannian metric on a manifold is locally projectively flat, if and only if it has the constant sectional curvature. Every Riemannian metric of the constant sectional curvature μ is locally isometric to the following projectively flat metric on a ball in $\mathbb{R}^n$:

$$\bar{\alpha} = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}. \tag{2}$$

However, there are lots of projectively flat Finsler metrics which are not of the constant flag curvature (the flag curvature is an analogue of the sectional curvature in Riemannian geometry). In ref. [2], Shen studied the local structure of projectively flat metrics with the constant flag curvature. In ref. [3], Chen *et al.* studied the projectively flat metrics with isotropic S-curvature. There are two notable projectively flat metrics on the unit ball B^n in $\mathbb{R}^n$:

$$\bar{F} = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} + \frac{\langle x, y\rangle}{1 - |x|^2}, \tag{3}$$

and

$$F = \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle)^2}{(1 - |x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}. \tag{4}$$

$\bar{F}$ is the well-known Funk metric and F is the metric introduced by Berwald[4]. $\bar{F}$ has the negative constant flag curvature $\mathbf{K} = -\frac{1}{4}$, and F has the zero flag curvature $\mathbf{K} = 0$. Moreover, $\bar{F}$ has the constant S-curvature, but F even does not have the isotropic S-curvature. What is interesting to us is the following relationship between $\bar{F}$ and F. Let

$$\bar{\alpha} := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2}, \qquad \bar{\beta} := \frac{\langle x, y\rangle}{1 - |x|^2}$$

and $\lambda := \frac{1}{1-|x|^2}$. Then $\bar{F} = \bar{\alpha} + \bar{\beta}$, and

$$F = \frac{(\alpha + \beta)^2}{\alpha} = \alpha + 2\beta + \frac{\beta^2}{\alpha}, \tag{5}$$

where $\alpha := \lambda\bar{\alpha}$ and $\beta = \lambda\bar{\beta} = \frac{1}{2}d\lambda$.

The above examples inspire us to construct some projectively flat Finsler metrics in similar forms and seek for those with the constant flag curvature. Consider a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ on a manifold M. Assume that $\bar{F}$ is locally projectively flat, then $\bar{\alpha}$ is locally projectively flat and $\bar{\beta}$ is closed. Locally, we can express $\bar{\alpha}$ by (2) and we can express $\bar{\beta}$ as a differential of some scalar function $\rho = \rho(x)$, i.e. $\bar{\beta} = \frac{1}{2}d\rho$. We wish to find scalar functions $\lambda = \lambda(x) > 0$ and $\sigma = \sigma(x)$ such that for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}d\sigma$, the Finsler metric in the following form is projectively flat,

$$F := \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha},$$

where ϵ, k are constants with $k \neq 0$. In this paper, we succeed in constructing such metrics.

Theorem 1.1. On an open domain in $\mathbb{R}^n$, let

$$h := \frac{\eta|x|^2}{(1 + \sqrt{1 + \mu|x|^2})\sqrt{1 + \mu|x|^2}} + \frac{d_1 + \langle a, x\rangle}{\sqrt{1 + \mu|x|^2}},$$

where η and d_1 are constants. Let

$$\lambda := d_3 + 2k\eta d_2 h - k\mu d_2 h^2, \tag{6}$$

$$\sigma := \pm 2\int \sqrt{d_2\lambda}\,dh, \tag{7}$$

where d_2, d_3 are constants with $d_2 > 0$ and $d_3 + 2k\eta d_1 d_2 - k\mu d_1^2 d_2 > 0$ so that $\lambda > 0$ on an open neighborhood of the origin. Then for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}d\sigma$, the following Finsler metric,

$$F := \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha},$$

is projectively flat on its domain.

Let us take a look at the special case: when $d_1 = d_2 = 1$, $d_3 = 0$, $\eta = 0$, $\mu = -1$, $\epsilon = 2$, $k = 1$,

$$\lambda = \frac{(1+\langle a,x\rangle)^2}{1-|x|^2} = \sigma.$$

For $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}d\sigma$, the Finsler metric $F = (\alpha+\beta)^2/\alpha$ is projectively flat. More important, this metric has the zero flag curvature! Thus we obtain the following

Theorem 1.2. Let $a \in \mathbb{R}^n$ be an arbitrary constant vector with $|a| < 1$. The following Finsler metric is projectively flat the with zero flag curvature $\mathbf{K} = 0$,

$$F := \frac{[(1+\langle a,x\rangle)(\sqrt{|y|^2-(|x|^2|y^2-\langle x,y\rangle^2)}+\langle x,y\rangle)+(1-|x|^2)\langle a,y\rangle]^2}{(1-|x|^2)^2\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}. \quad (8)$$

The metric in (8) is an extension of Berwald's famous example in (5). The curvature property follows from the classification theorem in ref. [2]. See sec. 6 below.

As we know, if a projectively flat Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ has the isotropic S-curvature, $S = \frac{1}{2}(n+1)c(x)\bar{F}$, where $c = c(x)$ is a scalar function, then $\bar{\alpha}$ is given by (2) and $\bar{\beta} = \frac{1}{2}d\rho$ for some scalar function $\rho = \rho(x)$. Both $c = c(x)$ and $\rho = \rho(x)$ can be explicitly determined. For such metrics we have the following

Theorem 1.3. Let $\bar{F} = \bar{\alpha} + \bar{\beta}$ be an n-dimensional projectively flat Randers metric with the isotropic S-curvature, $S = \frac{1}{2}(n+1)c(x)\bar{F}$, on an open domain in $\mathbb{R}^n$. There is a scalar function $\lambda = \lambda(x) > 0$ such that for $\alpha := \lambda\bar{\alpha}$ and $\beta := \lambda\bar{\beta}$, the Finsler metric in the following form is projectively flat,

$$F := \frac{(\alpha+\beta)^2}{\alpha}. \quad (9)$$

We will give a direct proof for Theorem 1.3 in sec. 5 below. In the special case when $\mu = -1$ and $c = \frac{1}{2}$, $\bar{F} = \bar{\alpha} + \bar{\beta}$ is a generalized Funk metric[2,5], where $\bar{\alpha}$ is given in (2) and $\bar{\beta}$ is given by

$$\bar{\beta} = \frac{\langle x,y\rangle}{1-|x|^2} + \frac{\langle a,y\rangle}{1+\langle a,x\rangle}.$$

Then for $\alpha := \lambda\bar{\alpha}$ and $\beta := \lambda\bar{\beta}$, where $\lambda = (1+\langle a,x\rangle)^2/(1-|x|^2)$, the Finsler metric $F = (\alpha+\beta)^2/\alpha$ is just the metric in (8) with the zero flag curvature $\mathbf{K} = 0$.

Recently, Senarath and Thornley have given an equation in local coordinates that characterizes the projectively flat Finsler metrics in the form $F = \alpha + \beta^2/\alpha$. In particular, they show that if $\bar{F} = \bar{\alpha} + \bar{\beta}$ is projectively flat, then $\tilde{F} := \bar{\alpha} + \bar{\beta}^2/\bar{\alpha}$ cannot be locally projectively flat unless it is locally Minkowskian. According to our results, there are scalar functions $\lambda = \lambda(x) > 0$ and $\sigma = \sigma(x)$ such that for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}d\sigma$, $F = \alpha + \beta^2/\alpha$ is projectively flat (Theorem 1.1). If in addition, $\bar{F}$ has the isotropic S-curvature, one can even choose $\beta = \lambda\bar{\beta}$ (Theorem 1.3).

2 Projectively flat (α,β)-metrics

Let $\alpha := \sqrt{a_{ij}y^iy^j}$ be a Riemannian metric, $\beta := b_iy^i$ a 1-form, and $\phi = \phi(s)$ be a positive C^∞ function defined in a neighborhood of the origin $s = 0$. Let

$$F = \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}. \tag{10}$$

It is known that $F = \alpha\phi(\beta/\alpha)$ is a Finsler metric for any α and β with $\|\beta\|_\alpha < b_o$ if and only if

$$\phi(s) > 0, \quad (\phi(s) - s\phi'(s)) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \leqslant b < b_o). \tag{11}$$

By taking $b = s$, one obtains

$$\phi(s) - s\phi'(s) > 0, \quad (|s| < b_o)$$

(see ref. [6]). A Finsler metric F is called an (α,β)-metric if it is in the above form (10) with ϕ satisfying (11) and β satisfying $\|\beta\|_\alpha < b_o$.

Let $F = \alpha\phi(\beta/\alpha)$ be an (α,β)-metric with $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$. Let

$$s_{ij} := \frac{1}{2}\Big(b_{i|j} - b_{j|i}\Big), \qquad r_{ij} := \frac{1}{2}\Big(b_{i|j} + b_{j|i}\Big),$$
$$s_j := b^k s_{kj}, \qquad b := \sqrt{b^ib_i},$$

where $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α. The spray coefficients G^i of F are given by

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \Psi\Big\{-2\alpha Q s_0 + r_{00}\Big\}\Big\{\chi\frac{y^i}{\alpha} + b^i\Big\},$$

where

$$Q = \frac{\phi'}{\phi - s\phi'},$$
$$\chi = \frac{(\phi - s\phi')\phi'}{\phi\phi''} - s,$$
$$\Psi = \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.$$

The above formula is given in refs. [6,7].

Now we consider the following special function

$$\phi = 1 + \epsilon s + ks^2,$$

where ϵ and k are constants with $k \neq 0$. In virtue of (11), we obtain the following condition on ϵ and k:

$$1 + \epsilon s + ks^2 > 0, \qquad 1 + 2kb^2 - 3ks^2 > 0, \qquad (|s| \leqslant b < 1). \tag{12}$$

Namely, $F = \alpha + \epsilon\beta + k\beta^2/\alpha$ is a Finsler metric for any α and β with $\|\beta\|_\alpha < 1$ if and only if ϵ and k satisfy (12).

From now on, we always assume that ϵ and k satisfy (12). Then

$$Q = \frac{\epsilon + 2ks}{1 - ks^2}, \qquad \Psi = \frac{k}{1 + 2kb^2 - 3ks^2}, \qquad \chi = \frac{\epsilon - 3k\epsilon s^2 - 4k^2s^3}{2k(1 + \epsilon s + ks^2)}.$$

Further, we assume that β is closed, i.e.

$$s^i_0 = 0, \qquad s_0 = 0, \tag{13}$$

and it satisfies

$$r_{00} = \tau\Big\{\Big(\frac{1}{k} + 2b^2\Big)\alpha^2 - 3\beta^2\Big\}, \tag{14}$$

where $\tau = \tau(x)$ is a scalar function. Then

$$G^i = G^i_\alpha + \tau\Big\{\alpha\chi y^i + \alpha^2 b^i\Big\}. \tag{15}$$

If in addition,

$$G^i_\alpha = \tau\Big\{\theta y^i - \alpha^2 b^i\Big\}, \tag{16}$$

where $\theta = p_i y^i$ is a local 1-form, then

$$G^i = \tau\Big\{\alpha\chi + \theta\Big\}y^i.$$

In this case, F is projectively flat.

Lemma 2.1. Let ϵ and k be constants satisfying (12). If α and β satisfies (13), (14) and (16), then $F = \alpha + \epsilon\beta + \frac{k\beta^2}{\alpha}$ is a solution of (1).

Recently Shen and Yildirim[1)] has showed that $F = \alpha + \epsilon\beta + \frac{k\beta^2}{\alpha}$ satisfies (1) if and only if (13), (14) and (16) hold.

Matsumoto[8] has proved that in dimension $n \geqslant 3$, the Finsler metric $F = \alpha + \frac{\beta^2}{\alpha}$ with $b \neq 0$ is a Douglas metric if and only if there is a scalar function $\tau = \tau(x)$ such that β is closed and

$$r_{00} = \tau\Big\{(1 + 2b^2)\alpha^2 - 3\beta^2\Big\}.$$

We conjecture that $F = \alpha + \epsilon\beta + k\beta^2/\alpha$ is a Douglas metric if and only if (13) and (14) hold.

3 Deformation of Randers metrics

First, for a scalar function $f = f(x)$, we use the following notations:

$$f_0 := \frac{\partial f}{\partial x^i}(x)y^i, \qquad f_{00} = \frac{\partial^2 f}{\partial x^i \partial x^j}(x)y^i y^j.$$

Let $\bar{\alpha}$ be given in (2) defined on a ball $B^n(r)$ in $\mathbb{R}^n$ and $\bar{\beta} = \frac{1}{2}d\rho$, where $\rho = \rho(x)$ is a scalar function on $B^n(r)$. We are going to find scalar functions $\lambda = \lambda(x) > 0$ and $\sigma = \sigma(x)$ such that for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}\sigma_0$, the Finsler metric $F = \alpha + \epsilon\beta + \frac{k\beta^2}{\alpha}$ is projectively flat.

Let $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}\sigma_0$ for some scalar functions λ and σ. We have

$$b^2 = \frac{1}{4\lambda^2}|\bar{\nabla}\sigma|^2. \tag{17}$$

1) Shen, Z. M., Yildirim, G. C., On a class of projectively flat metrics with constant flag curvature, preprint.

The spray coefficients of α are given by

$$G^i_\alpha = \bar{G}^i + \frac{\lambda_0}{\lambda} y^i - \frac{1}{2}\bar{a}^{ij}\frac{\lambda_{x^j}}{\lambda}\bar{\alpha}^2 = \Big\{ - \frac{\mu\langle x,y\rangle}{1+\mu|x|^2} + \frac{\lambda_0}{\lambda}\Big\} y^i - \frac{1}{2}a^{ij}\frac{\lambda_{x^j}}{\lambda}\alpha^2.$$

Note that $b^i = \frac{1}{2}\sigma_{x^j}a^{ji}$. Hence eq. (16) is equivalent to

$$\frac{\lambda_{x^i}}{\lambda} = \tau\sigma_{x^i}. \tag{18}$$

Since β is exact, it satisfies (13). Then $r_{00} = b_{0|0}$ is given by

$$r_{00} = \frac{\partial b_i}{\partial x^j} y^i y^j - 2b_m G^m_\alpha = \frac{1}{2}\sigma_{00} - \sigma_0\Big\{ - \frac{\mu\langle x,y\rangle}{1+\mu|x|^2} + \frac{\lambda_0}{\lambda}\Big\} + \frac{1}{2}\sigma_{x^i}\bar{a}^{ij}\frac{\lambda_{x^j}}{\lambda}\bar{\alpha}^2. \tag{19}$$

By (17) and (18), we rewrite (19) as follows:

$$r_{00} = \frac{1}{2}\sigma_{00} - \sigma_0\Big\{ - \frac{\mu\langle x,y\rangle}{1+\mu|x|^2} + \tau\sigma_0\Big\} + 2\tau b^2\alpha^2. \tag{20}$$

By (20), we see that (14) is equivalent to

$$\frac{1}{2}\sigma_{00} - \sigma_0\Big\{ - \frac{\mu\langle x,y\rangle}{1+\mu|x|^2} + \tau\sigma_0\Big\} = \frac{\tau}{k}\alpha^2 - 3\tau\beta^2. \tag{21}$$

By $\beta = \frac{1}{2}\sigma_0$, we can simplify (21) to

$$\sigma_{00} + \frac{2\mu\langle x,y\rangle}{1+\mu|x|^2}\sigma_0 = 2\tau\Big\{\frac{\lambda^2}{k}\bar{\alpha}^2 + \frac{1}{4}\sigma_0^2\Big\}. \tag{22}$$

By Lemma 2.1 and the above arguments, we obtain the following

Lemma 3.1. Let ϵ and k be constants satisfying (12). Let $\bar{\alpha}$ be given in (2). Suppose that there are scalar functions $\lambda = \lambda(x) > 0$, $\sigma = \sigma(x)$ and $\tau = \tau(x)$ such that (18) and (22) hold. Then for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}\sigma_0$, the Finsler metric $F = \alpha + \epsilon\beta + \frac{k\beta^2}{\alpha}$ is projectively flat.

Let

$$\lambda = \lambda(h), \qquad \sigma = \sigma(h),$$

where $h = h(x)$ is a scalar function. Thus both λ and σ are functions of x. In this case, (18) is equivalent to

$$\tau = \frac{\lambda'}{\lambda\sigma'}, \tag{23}$$

and (22) is equivalent to

$$h_{00} + \frac{2\mu\langle x,y\rangle}{1+\mu|x|^2}h_0 = \frac{(\lambda^2)'}{k(\sigma')^2}\bar{\alpha}^2 + \Big(\ln\frac{\sqrt{\lambda}}{|\sigma'|}\Big)' h_0^2. \tag{24}$$

By Lemma 3.1, we obtain the following

Lemma 3.2. Let ϵ and k be constants satisfying (12). Let $\lambda = \lambda(h) > 0$ and $\sigma = \sigma(h)$ such that there is a scalar function $h = h(x)$ satisfying (24). Then for $\alpha := \lambda\bar{\alpha}$, $\beta := \frac{1}{2}\sigma_0$, the Finsler metric $F = \alpha + \epsilon\beta + \frac{k\beta^2}{\alpha}$ is projectively flat.

4 Proof of Theorem 1.1

In this section, we are going to use Lemma 3.2 to prove Theorem 1.1. We first construct a scalar function $h = h(x)$. Then we find $\lambda = \lambda(h)$ and $\sigma = \sigma(h)$ such that (24) holds.

Let $\xi = \xi(x)$ and

$$h := \frac{\xi(x)}{\sqrt{1+\mu|x|^2}}.$$

We have

$$h_{00} + \frac{2\mu\langle x, y\rangle}{1+\mu|x|^2} h_0 = \frac{\xi_{00}}{\sqrt{1+\mu|x|^2}} - \mu h \bar{\alpha}^2, \tag{25}$$

where $\bar{\alpha}$ is given in (2). Assume that ξ satisfies the following equation:

$$\xi_{00} = \eta\sqrt{1+\mu|x|^2}\ \bar{\alpha}^2. \tag{26}$$

Then it follows from (25) that

$$h_{00} + \frac{2\mu\langle x, y\rangle}{1+\mu|x|^2} h_0 + (\mu h - \eta)\bar{\alpha}^2 = 0. \tag{27}$$

Solving (26), we obtain

$$\xi := d_1 + \langle a, x\rangle + \frac{\eta|x|^2}{1+\sqrt{1+\mu|x|^2}},$$

where η and d_1 are constants and $a \in \mathbb{R}^n$ is a constant vector. Then h is given by

$$h := \frac{d_1 + \langle a, x\rangle}{\sqrt{1+\mu|x|^2}} + \frac{\eta|x|^2}{(1+\sqrt{1+\mu|x|^2})\sqrt{1+\mu|x|^2}}. \tag{28}$$

In virtue of (27), in order to find $\lambda = \lambda(h)$ and $\sigma = \sigma(h)$ satisfying (24), it suffices to solve the following equations:

$$\Big(\ln\frac{\sqrt{\lambda}}{|\sigma'|}\Big)' = 0, \tag{29}$$

$$\mu h - \eta + \frac{[\lambda^2]'}{k(\sigma')^2} = 0, \tag{30}$$

From (29), we obtain

$$\sigma' = \pm 2\sqrt{d_2\lambda}, \tag{31}$$

where d_2 is a positive constant. Plugging it to (30) yields

$$2k\mu d_2 h - 2k\eta d_2 + \lambda' = 0.$$

We obtain

$$\lambda = d_3 + 2k\eta d_2 h - k\mu d_2 h^2.$$

Plugging it into (31) we obtain the following formula for σ:

$$\sigma = \pm 2\int \sqrt{d_2\lambda}\, dh.$$

This proves the theorem.

5 Proof of Theorem 1.3

In this section, we are going to prove Theorem 1.3. Let $\bar{F} = \bar{\alpha} + \bar{\beta}$ be a locally projectively flat metric on an n-dimensional manifold. We may assume that $\bar{\alpha}$ is given by (2) on an open neighborhood of the origin and $\bar{\beta} = \frac{1}{2}\rho_0$ for some scalar function $\rho = \rho(x)$. Now we assume that $\bar{F}$ has the isotropic S-curvature, $S = \frac{1}{2}(n+1)c\bar{F}$, where $c = c(x)$ is a scalar function. According to ref. [3],

$$\rho = \begin{cases} \ln \frac{(1+\langle a,x\rangle)^2}{1-|x|^2} & \text{if } \mu + 4c^2 = 0, \mu = -1, \\ 2(1+\langle a,x\rangle) & \text{if } \mu + 4c^2 = 0, \mu = 0, \\ -\int \frac{4}{\mu+4c^2} dc & \text{if } \mu + 4c^2 \neq 0. \end{cases}$$

When $\mu + 4c^2 \neq 0$, the scalar function $c = c(x)$ is determined by

$$c_{00} + \frac{2\mu\langle x,y\rangle}{1+\mu|x|^2} c_0 = -c(\mu+4c^2)\bar{\alpha}^2 + \frac{12cc_0^2}{\mu+4c^2}. \tag{32}$$

The general solution of (32) will be given below (see ref. [3]). Let

$$\lambda := \begin{cases} (1+\langle a,x\rangle)^2/(1-|x|^2) & \text{if } \mu + 4c^2 = 0, \mu = -1, \\ 1 & \text{if } \mu + 4c^2 = 0, \mu = 0, \\ 1/|16c^2 \pm 4| & \text{if } \mu + 4c^2 \neq 0, \mu = \pm 1, \\ 4/c^2 & \text{if } \mu + 4c^2 \neq 0, \mu = 0. \end{cases}$$

We are going to show that for $\alpha := \lambda\bar{\alpha}$, $\beta = \lambda\bar{\beta}$, the Finsler metric in the following form is projectively flat,

$$F = \frac{(\alpha+\beta)^2}{\alpha}. \tag{33}$$

(a) Assume that $\mu + 4c^2 = 0$ and $\mu = 0$. In this case, we let $h = h(x)$ be the function in (28) with $k = 1$ and $\mu = 0$, i.e.

$$h = d_1 + \langle a,x\rangle + \frac{\eta}{2}|x|^2.$$

Let

$$\lambda = d_3 + 2\eta d_2 h, \qquad \sigma = \pm 2\int \sqrt{d_2\lambda}\, dh.$$

Then $\lambda = \lambda(h), \sigma = \sigma(h)$ and $h = h(x)$ satisfy (24) with $k = 1$ and $\mu = 0$. By Lemma 3.2, for $\alpha := \lambda\bar{\alpha}$ and $\beta := \frac{1}{2}\sigma_0$, the Finsler metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ is projectively flat.

If $d_1 = 1, d_2 = 1, d_3 = 1$ and $\eta = 0$,

$$h = 1 + \langle a,x\rangle, \qquad \rho = 2h = \sigma, \quad \lambda = 1,$$

then

$$\beta = \frac{1}{2}\sigma_0 = h_0 = \frac{1}{2}\rho_0 = \bar{\beta}.$$

Then

$$F = \frac{(\bar{\alpha}+\bar{\beta})^2}{\bar{\alpha}}$$

is a Minkowski metric.

Assume that $\mu+4c^2=0$, $\mu=-1$. By reversing the metric, if necessary, we can also assume that $c=\frac{1}{2}$. Let $h=h(x)$ be the function in (28) with $k=1$.

$$h=\frac{d_1+\langle a,x\rangle}{\sqrt{1-|x|^2}}+\frac{\eta|x|^2}{(1+\sqrt{1-|x|^2})\sqrt{1-|x|^2}}.$$

Let

$$\lambda=d_3+2\eta d_2 h+d_2h^2,\qquad \sigma=\pm 2\int\sqrt{d_2\lambda}dh.$$

Then $\lambda=\lambda(h),\sigma=\sigma(h)$ and $h=h(x)$ satisfy (24) with $k=1$ and $\mu=-1$. By Lemma 3.2, for $\alpha:=\lambda\bar{\alpha}$ and $\beta:=\frac{1}{2}\sigma_0$, the Finsler metric $F=\frac{(\alpha+\beta)^2}{\alpha}$ is projectively flat.

If $d_1=1$, $d_2=1$, $d_3=0$ and $\eta=0$, then

$$h=\frac{1+\langle a,x\rangle}{\sqrt{1-|x|^2}},\qquad \rho=2\ln h,\qquad \lambda=h^2=\sigma.$$

In this case,

$$\beta=\frac{1}{2}\sigma_0=hh_0=\lambda\frac{h_0}{h}=\frac{1}{2}\lambda\rho_0=\lambda\bar{\beta}.$$

Then

$$F=\frac{(\alpha+\beta)^2}{\alpha}=\lambda\frac{(\bar{\alpha}+\bar{\beta})^2}{\bar{\alpha}}.\tag{34}$$

(b) Assume that $\mu+4c^2\neq 0$. There is a constant η and a function $f=f(c)$ satisfying

$$\frac{\mu f-\eta}{f'}=c(\mu+4c^2),\qquad \frac{f''}{f'}+\frac{12c}{\mu+4c^2}=0.\tag{35}$$

The function f is given by

$$f(c)=\begin{cases}\dfrac{c}{\sqrt{|\mu+4c^2|}} & \text{if } \mu\neq 0\quad (\text{taking } \eta=0),\\ -\dfrac{1}{2c^2} & \text{if } \mu=0\quad (\text{taking } \eta=-4).\end{cases}\tag{36}$$

We use η and f to define a scalar function $c=c(x)$.

$$f(c):=h(x)=\begin{cases}\dfrac{d_1+\langle a,x\rangle}{\sqrt{1+\mu|x|^2}} & \text{if } \mu\neq 0,\\ d_1+\langle a,x\rangle-2|x|^2 & \text{if } \mu=0.\end{cases}$$

By (27) and (35), one can easily verify that $c=c(x)$ satisfies (32). Thus we obtain a general solution of (32) by solving $f(c)=h$ for c. See ref. [3].

Let

$$\lambda:=d_3+2\eta d_2h-\mu d_2h^2,$$
$$\sigma:=\pm 2\int\sqrt{d_2\lambda}dh,$$

where $\eta=0$ if $\mu\neq 0$ and $\eta=-4$ if $\mu=0$. Then $\lambda=\lambda(h),\sigma=\sigma(h)$ and $h=h(x)$ satisfy (24) with $k=1$. By Lemma 3.2, for $\alpha:=\lambda\bar{\alpha}$ and $\beta:=\frac{1}{2}\sigma_0$, the Finsler metric $F=\frac{(\alpha+\beta)^2}{\alpha}$ is projectively flat.

We can express β as

$$\beta=\frac{1}{2}\sigma_0=\pm\sqrt{d_2\lambda}h_0=\pm\sqrt{d_2\lambda}f'(c)c_0=\delta\bar{\beta},$$

where

$$\delta := \mp\frac{1}{2}(\mu + 4c^2)\sqrt{d_2\lambda} f'(c).$$

We can choose d_3 and the sign of σ such that $\delta = \lambda$.

Observe that

$$\lambda = \begin{cases} \dfrac{d_3|\mu + 4c^2| - d_2\mu c^2}{|\mu + 4c^2|} & \text{if } \mu \neq 0, \\[2ex] \dfrac{d_3c^2 + 4d_2}{c^2} & \text{if } \mu = 0; \end{cases}$$

$$\delta = \mp \begin{cases} \dfrac{\mu\sqrt{d_2[d_3|\mu + 4c^2| - d_2\mu c^2]}}{2|\mu + 4c^2|} & \text{if } \mu \neq 0, \\[2ex] \dfrac{2\sqrt{d_2(d_3c^2 + 4d_2)}}{c|c|} & \text{if } \mu = 0. \end{cases}$$

Take

$$d_3 = \begin{cases} \dfrac{d_2\mu}{4}\mathrm{sign}(\mu + 4c^2) & \text{if } \mu \neq 0, \\[2ex] 0 & \text{if } \mu = 0. \end{cases}$$

Then

$$\lambda = \begin{cases} \dfrac{d_2\mu^2}{4|\mu + 4c^2|} & \text{if } \mu \neq 0, \\[2ex] \dfrac{4d_2}{c^2} & \text{if } \mu = 0; \end{cases}$$

$$\delta = \mp \begin{cases} \dfrac{d_2\mu|\mu|}{4|\mu + 4c^2|} & \text{if } \mu \neq 0, \\[2ex] \dfrac{4d_2}{c|c|} & \text{if } \mu = 0. \end{cases}$$

Clearly, we can choose the sign of σ such that $\delta = \lambda$. This proves Theorem 1.3.

6 Zero flag curvature

In this section, we are going to show that the Finsler metric F in (34) has the zero flag curvature. This fact follows from Theorem 1.3 in ref. [2].

According to Theorem 1.3 in ref. [2], for any Minkowski norm $\psi = \psi(y)$ on $\mathbb{R}^n$ and any positively homogeneous function of degree one, $\varphi = \varphi(y)$, on $\mathbb{R}^n$, the following function F is a projectively flat Finsler metric with the zero flag curvature on an open neighborhood of the origin in $\mathbb{R}^n$,

$$F := \psi(y + Px)\Big\{1 + P_{y^m}x^m\Big\}, \tag{37}$$

where $P = P(x, y)$ is defined by

$$P = \varphi(y + Px). \tag{38}$$

The proof is based on the following important equation satisfied by P,

$$P_{x^k} = PP_{y^k}. \tag{39}$$

When φ is a Minkowski norm on $\mathbb{R}^n$, the function P is a Finsler metric on the open domain

$$\Omega_\varphi := \Big\{ y \in \mathbb{R}^n \mid \varphi(y) < 1 \Big\}.$$

This metric is called the Funk metric of φ. Let

$$\psi := \varphi(y)\phi\Big(\frac{\langle a, y\rangle}{\varphi(y)}\Big),$$

where $\phi = \phi(s)$ is a positive C^∞ function of s and $a \in \mathbb{R}^n$ is a constant vector such that ψ is a Minkowski norm on $\mathbb{R}^n$. By (38), we have

$$\frac{\langle a, y + P\, x\rangle}{\varphi(y + P\, x)} = \frac{\langle a, y\rangle + P\langle a, x\rangle}{P} = \langle a, x\rangle + \frac{\langle a, y\rangle}{P}.$$

Thus

$$\begin{aligned}\psi(y + P\, x) &= \varphi(y + P\, x)\phi\Big(\frac{\langle a, y + P\, x\rangle}{\varphi(y + P\, x)}\Big)\\ &= P\phi\Big(\langle a, x\rangle + \frac{\langle a, y\rangle}{P}\Big).\end{aligned}$$

Then the Finsler metric in (37) can be expressed by

$$F = P\phi\Big(\langle a, x\rangle + \frac{\langle a, y\rangle}{P}\Big)\Big\{1 + P_{y^m}x^m\Big\}.$$

By (39), we have

$$P\Big\{1 + P_{y^m}x^m\Big\} = P + P_{x^m}x^m.$$

Then we obtain the following version of Theorem 1.3 in ref. [2].

Theorem 6.1. Let $\varphi = \varphi(y)$ be a Minkowski norm on $\mathbb{R}^n$ and $P = P(x, y)$ be the Funk metric of φ on a strongly convex domain Ω_φ. Let $\phi = \phi(s)$ be an arbitrary positive C^∞ function and $a \in \mathbb{R}^n$ is a constant vector such that $\psi := \varphi(y)\phi\Big(\langle a, y\rangle/\varphi(y)\Big)$ is a Minkowski norm on $\mathbb{R}^n$. Let

$$F := \phi\Big(\langle a, x\rangle + \frac{\langle a, y\rangle}{P(x, y)}\Big)\Big\{P(x, y) + P_{x^m}(x, y)x^m\Big\}.$$

Then F is projectively flat with a projective factor P and a flag curvature $\mathbf{K} = 0$.

When $\varphi = |y|$, we have

$$P = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2}.$$

Then

$$P + P_{x^m}x^m = \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle)^2}{(1 - |x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}.$$

In this case,

$$\begin{aligned}F =&\phi\Big(\langle a, x\rangle + \frac{(1 - |x|^2)\langle a, y\rangle}{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}\Big)\\ &\times \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle)^2}{(1 - |x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}.\end{aligned}$$

Further, if $\phi(s) = (1+s)^2$, then

$$F := \frac{[(1+\langle a,x\rangle)(\sqrt{|y|^2-(|x|^2|y^2-\langle x,y\rangle^2)}+\langle x,y\rangle)+(1-|x|^2)\langle a,y\rangle]^2}{(1-|x|^2)^2\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}. \tag{40}$$

By Theorem 6.1, F is projectively flat with $\mathbf{K}=0$. Note that the Finsler metric in (40) is the same metric we obtained in (34).

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References

1. Hamel, G., Über die Geometrieen in denen die Geraden die Kürzesten sind, Math. Ann., 1903, 57: 231–264.
2. Shen, Z., Projectively flat Finsler metrics of constant flag curvature, Trans. of Amer. Math. Soc., 2003, 355(4): 1713–1728.
3. Chen, X., Mo, X., Shen, Z., On the flag curvature of Finsler metrics of scalar curvature, J. London Math. Soc., 2003, 68(2): 762–780.
4. Berwald, L., Über die n-dimensionalen Geometrien konstanter Krümmung, in denen die Geraden die kürzesten sind, Math. Z., 1929, 30: 449–469.
5. Shen, Z., Projectively flat Randers metrics of constant curvature, Math. Ann., 2003, 325: 19–30.
6. Shen, Z., Landsberg curvature, S-curvature and Riemann curvature, in A Sampler of Riemann-Finsler Geometry, MSRI Series, Vol. 50, Cambridge: Cambridge University Press, 2004.
7. Kitayama, M., Azuma, M., Matsumoto, M., On Finsler spaces with (α,β)-metric, Regularity, geodesics and main scalars, J. Hokkaido Univ. Education (Section II A), 1995, 46(1): 1–10.
8. Matsumoto, M., Finsler spaces with (α,β)-metric of Douglas type, Tensor, N. S., 1998, 60: 123–134.