GEOMETRY OF LINEAR DIFFERENTIAL SYSTEMS

–TOWARDS–
CONTACT GEOMETRY OF SECOND ORDER

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1. Geometry of Jet Spaces.

1.1. **Spaces of Contact Elements (Grassmann Bundle).** The notion of contact manifolds originates from the following space $J(M,n)$ of contact elements: Let M be a (real or complex) manifold of dimension $m+n$. Fixing the number n, we consider the space of n-dimensional **contact elements** to M, i.e., the **Grassmannian bundle** over M consisting of all n-dimensional contact elements to M;

$$J(M,n) = \bigcup_{x \in M} J_x \xrightarrow{\pi} M,$$

where $J_x = \mathrm{Gr}(T_x(M), n)$ is the Grassmann manifold of all n-dimensional subspaces of the tangent space $T_x(M)$ to M at x. Each element $u \in J(M,n)$ is a linear subspace of $T_x(M)$ of codimension m, where $x = \pi(u)$. Hence we have a differential system C of codimension m on $J(M,n)$ by putting:

$$C(u) = \pi_*^{-1}(u) \subset T_u(J(M,n)) \xrightarrow{\pi_*} T_x(M).$$

for each $u \in J(M,n)$. C is called the **Canonical System** on $J(M,n)$. We can introduce the *inhomogeneous Grassmann coordinate* of $J(M,n)$ around $u_o \in J(M,n)$ as folllows; Take a coordinate system $U'; (x_1, \cdots, x_n, z^1, \cdots, z^m)$ of M around $x_o = \pi(u_o)$ such that $dx_1 \wedge \cdots \wedge dx_n \mid_{u_o} \neq 0$. Then we have the coordinate system $(x_1, \cdots, x_n, z^1, \cdots, z^m, p_1^1, \cdots, p_n^m)$ on the neighborhood

$$U = \{u \in \pi^{-1}(U') \mid \pi(u) = x \in U' \quad \text{and} \quad dx_1 \wedge \cdots \wedge dx_n \mid_u \neq 0\};$$

of u_o by defining functions $p_i^\alpha(u)$ on U as follows;

$$dz^\alpha \mid_u = \sum_{i=1}^n p_i^\alpha(u)\ dx_i \mid_u .$$

On a canonical coordinate system$(x_1, \cdots, x_n, z^1, \cdots, z^m, p_1^1, \cdots, p_n^m)$, C is clearly defined by;

$$C = \{\varpi^1 = \cdots = \varpi^m = 0\},$$

where

$$\varpi^\alpha = dz^\alpha - \sum_{i=1}^n p_i^\alpha\, dx_i, \qquad (\alpha = 1, \cdots, m).$$

$(J(M,n), C)$ is the (geometric) 1-jet space and especially, in case $m = 1$, is the so-called contact manifold. Let M, $\hat{M}$ be manifolds (of dimension $m+n$) and $\varphi : M \to \hat{M}$ be a diffeomorphism between them. Then φ induces the **isomorphism** $\varphi_* : (J(M,n), C) \to (J(\hat{M},n), \hat{C})$, i.e., the differential map $\varphi_* : J(M,n) \to J(\hat{M},n)$ is a diffeomorphism sending C onto $\hat{C}$. The reason why the case $m = 1$ is special is explained by the following theorem of Bäcklund (cf. Theorem 1.4 [Y3]).

Theorem 1.1 (Bäcklund). *Let M and $\hat{M}$ be manifolds of dimension $m+n$. Assume $m \geqq 2$. Then, for an isomorphism $\Phi : (J(M,n), C) \to (J(\hat{M},n), \hat{C})$, there exists a diffeomorphism $\varphi : M \to \hat{M}$ such that $\Phi = \varphi_*$.*

We will give a proof of this theorem in §2.4. as an application of the notion of the symbol algebra of $(J(M,n), C)$, which will be introduced in §2.1.

1.2. **Contact Manifolds.** Let J be a manifold and C be a (linear) differential system on J of codimension 1. Namely C is a subbundle of $T(J)$ of codimension 1. Thus, locally at each point u of J, there exists a 1-form ϖ defined around $u \in J$ such that

$$C = \{\varpi = 0\}.$$

Then (J, C) is called a **contact manifold** if $\varpi \wedge (d\,\varpi)^n$ forms a volume element of J. This condition is equivalent to the following conditions (1) or (2);

(1) The restriction $d\varpi\,|_C$ of $d\varpi$ to $C(u)$ is non-degenerate at each point $u \in J$.

(2) There exists a coframe $\{\,\varpi, \omega_1, \ldots, \omega_n, \pi_1, \ldots, \pi_n\}$ defined around $u \in J$ such that the following holds;

$$d\varpi \equiv \omega_1 \wedge \pi_1 + \cdots + \omega_n \wedge \pi_n \qquad (\text{mod} \quad \varpi)$$

A contact manifold (J, C) of dimension $2n+1$ can be regarded locally as a space of 1-jets for one unknown function by the following theorem of Darboux.

Theorem 1.2 (Darboux). *At each point of a* **contact manifold** *J, there exists a canonical coordinate system $(x_1, \ldots, x_n, z, p_1, \ldots, p_n)$ such that*

$$C = \{dz - \sum_{i=1}^{n} p_i\, dx_i = 0\}$$

We will give a proof of this theorem in §1.4.

Starting from a contact manifold (J, C), we can construct the geometric second order jet space $(L(J), E)$ as follows: We consider the **Lagrange-Grassmann bundle** $L(J)$ over J consisting of all n-dimensional integral elements of (J, C);

$$L(J) = \bigcup_{u \in J} L_u,$$

where L_u is the Grassmann manifolds of all lagrangian (or **legendrian**) subspaces of the symplectic vector space $(C(u), d\varpi)$. Here ϖ is a local contact form on J. Let π be the projection of $L(J)$ onto J. Then the canonical system E on $L(J)$ is defined by

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \xrightarrow{\pi_*} T_u(J), \qquad \text{for} \quad v \in L(J).$$

Let us fix a point $v_o \in L(J)$. Starting from a canonical coordinate system $(x_1, \cdots, x_n, z, p_1, \cdots, p_n)$ defined on a neiborhood U' of the contact manifold (J, C) around $u_o = \pi(v_o)$ such that $dx_1 \wedge \cdots \wedge dx_n\,|_{v_o} \neq 0$, we can introduce a coordinate system (x_i, z, p_i, p_{ij}) $(1 \leqq i \leqq j \leqq n)$ on

$$U = \{v \in \pi^{-1}(U') \mid \pi(v) = u \in U' \quad \text{and} \quad dx_1 \wedge \cdots \wedge dx_n\,|_v \neq 0\} \subset L(J)$$

by defining functions $p_{ij}(v)$ on U as follows;

$$dp_i \mid_v = \sum_{i-!}^{n} p_{ij}(v) dx_j \mid_v .$$

Then, since $v \in C(u)$, we have $dz \mid_v = \sum_{i-1}^{n} p_i(u) dx \mid_v$ and , since $d\varpi \mid_v = 0$, we get $p_{ij} = p_{ji}$ from

$$d\varpi \mid_v = \sum_{i=1}^{n} dx_i \mid_v \wedge dp_i \mid_v = \sum_{i,j=1}^{n} p_{ij}(v) dx_i \mid_v \wedge dx_j \mid_v = 0.$$

Thus E is defined on this canonical coordinate system by

$$E = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\},$$

where

$$\varpi = dz - \sum_{i=1}^{n} p_i \, dx_i, \qquad \text{and} \qquad \varpi_i = dp_i - \sum_{j=1}^{n} p_{ij} \, dx_j \qquad \text{for} \quad i = 1, \cdots, n.$$

Let (J, C), $(\hat{J}, \hat{C})$ be contact manifolds of dimension $2n + 1$ and $\varphi : (J, C) \to (\hat{J}, \hat{C})$ be a contact diffeomorphism between them. Then φ induces an isomorphism $\varphi_* : (L(J), E) \to (L(\hat{J}), \hat{E})$. Conversely we have (cf. Theorem 3.2 [Y1])

Theorem 1.3. *Let (J, C) and $(\hat{J}, \hat{C})$ be contact manifolds of dimension $2n + 1$. Then, for an isomorphism $\Phi : (L(J), E) \to (L(\hat{J}), \hat{E})$,there exists a contact diffeomorphism $\varphi : (J, C) \to (\hat{J}, \hat{C})$ such that $\Phi = \varphi_*$.*

Our first aim is to formulate the submanifold theory for $(L(J), E)$, which will be given in §3.

1.3. **Derived Sytems and Cauchy Characteristic Systems.** Now we prepare basic notions for **linear differential systems** (or Pfaffian systems). By a (linear) differential system (M, D), we mean a subbundle D of the tangent bundle $T(M)$ of a manifold M of dimension d. Locally D is defined by 1-forms $\omega_1, \ldots, \omega_{d-r}$ such that $\omega_1 \wedge \cdots \wedge \omega_{d-r} \neq 0$ at each point, where r is the rank of D;

$$D = \{ \omega_1 = \cdots = \omega_{d-r} = 0 \}.$$

For two differential systems (M, D) and $(\hat{M}, \hat{D})$, a diffeomorphism φ of M onto $\hat{M}$ is called an **isomorphism** of (M, D) onto $(\hat{M}, \hat{D})$ if the differential map φ_* of φ sends D onto $\hat{D}$.

By the **Frobenius Theorem**, we know that D is completely integrable if and only if

$$d\omega_i \equiv 0 \pmod{\omega_1, \ldots, \omega_s} \qquad \text{for } i = 1, \ldots, s,$$

or equivalently, if and only if

$$[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}.$$

where $s = d - r$ and $\mathcal{D} = \Gamma(D)$ denotes the space of sections of D.

Thus, for a non-integrable differential system D, we are led to consider the **Derived System** ∂D of D, which is defined, in terms of sections, by

$$\partial \mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

Furthermore the **Cauchy Characteristic System** $\mathrm{Ch}\,(D)$ of (M,D) is defined at each point $x \in M$ by

$$\mathrm{Ch}\,(D)(x) = \{X \in D(x) \mid X\rfloor d\omega_i \equiv 0 \pmod{\omega_1,\ldots,\omega_s} \quad \text{for } i = 1,\ldots,s\ \},$$

where $\rfloor$ denotes the interior multiplication, i.e., $X\rfloor d\omega(Y) = d\omega(X.Y)$. When $\mathrm{Ch}\,(D)$ is a differential system (i.e., has constant rank), it is always completely integrable (see §1.4.).

Moreover **Higher Derived Systems** $\partial^k D$ are usually defined successively (cf. $[BCG_3]$) by

$$\partial^k D = \partial(\partial^{k-1} D),$$

where we put $\partial^0 D = D$ for convention.

On the other hand we define the k-th **Weak Derived System** $\partial^{(k)} D$ of D inductively by

$$\partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}],$$

where $\partial^{(0)} D = D$ and $\partial^{(k)}\mathcal{D}$ denotes the space of sections of $\partial^{(k)} D$. This notion is one of the key point in the **Tanaka Theory** ([T1]).

1.4. **Proof of the Darboux Theorem.** First of all, we will show that, for a differntial system (M,D), the Cachy characteristic system $\mathrm{Ch}\,(D)$ is completely integrable if $\mathrm{Ch}\,(D)$ is of constant rank, i.e., if $\mathrm{Ch}\,(D)$ is a subbundle of $T(M)$, where we assume D is locally defined by

$$D = \{\,\omega_1 = \cdots = \omega_s = 0\ \}.$$

We will show that $[X,Y] \in \Gamma(\mathrm{Ch}\,(D)) = \mathrm{Ch}\,(\mathcal{D})$ for $X, Y \in \mathrm{Ch}\,(\mathcal{D})$. From

$$d\omega_\alpha(X,Y) = X(\omega_\alpha(Y)) - Y(\omega_\alpha(X)) - \omega_\alpha([X,Y]),$$

it follows that

$$\omega_\alpha([X,Y]) = -d\omega_\alpha(X,Y) = -(X\rfloor d\omega_\alpha)(Y) = 0$$

for $X, Y \in \mathrm{Ch}\,(\mathcal{D}))$. Hence $[X,Y] \in \mathcal{D}$. Moreover, from $[L_X, i_Y] = i_{[X,Y]}$, where i_Y denotes the interior multiplication by Y, we calculate

$$[X,Y]\rfloor d\omega_\alpha = [L_X, i_Y](d\omega_\alpha) = L_X i_Y d\omega_\alpha - i_Y L_X d\omega_\alpha = L_X(Y\rfloor d\omega_\alpha) - Y\rfloor d(L_X\omega_\alpha)$$

The first term of the last equality vanishes because

$$L_X\omega_\beta = d(i_X\omega_\beta) + i_X d\omega_\beta = X\rfloor d\omega_\beta \equiv 0 \pmod{\omega_1,\ldots,\omega_s}.$$

for $X \in \mathrm{Ch}\,(\mathcal{D})$. As for the second term, writing $L_X\omega_\alpha = \sum A^\alpha_\beta\omega_\beta$, we get

$$Y\rfloor d(L_X\omega_\alpha) = \sum dA^\alpha_\beta(Y)\ \omega_\beta + \sum A^\alpha_\beta\ (Y\rfloor d\omega_\beta) \equiv 0 \pmod{\omega_1,\ldots,\omega_s}$$

from $d(L_X\omega_\alpha) = \sum dA^\alpha_\beta \wedge \omega_\beta + \sum A^\alpha_\beta d\omega_\beta$. Thus we obtain $[X,Y]\rfloor d\omega_\alpha \equiv 0 \pmod{\omega_1,\ldots,\omega_s}$. This implies $[X,Y] \in \mathrm{Ch}\,(\mathcal{D})$.

Now let (J,C) be a contact manifold of dimension $2n+1$. Let us fix a point u_o of J. Then there exists a coframe $\{\,\varpi, \omega_1, \ldots, \omega_n, \pi_1, \ldots, \pi_n\}$ defined around $u_o \in J$ such that the following holds;

$$d\varpi \equiv \omega_1 \wedge \pi_1 + \cdots + \omega_n \wedge \pi_n \pmod{\varpi}$$

Then, from the definition of $\mathrm{Ch}\,(C)$, it follows

$$\mathrm{Ch}\,(C) = \{\,\varpi = \omega_1 = \cdots = \omega_n = \pi_1 = \cdots = \pi_n = 0\,\} = \{0\}.$$

In fact, (J,C) is a contact manifold if and only if $\mathrm{Ch}\,(C)$ is trivial.

Let us take a function x_1 defined aroud u_o such that $\varpi\wedge dx_1\wedge\omega_2\wedge\cdots\wedge\omega_n\wedge\pi_1\wedge\cdots\wedge\pi_n\neq 0$ around u_o and consider the differential system C^1 defined by

$$C^1=\{\,\varpi=dx_1=0\,\}.$$

We can write

$$\omega_1\equiv\sum_{i=2}^{n}a_i\omega_i+\sum_{i=1}^{n}b_i\pi_i\quad(\text{mod }\varpi,dx_1)$$

Then we calculate

$$\begin{aligned}
d\varpi &\equiv (\sum_{i=2}^{n}a_i\omega_i+\sum_{i=1}^{n}b_i\pi_i)\wedge\pi_1+\omega_2\wedge\pi_2+\cdots+\omega_n\wedge\pi_n \qquad (\text{mod }\varpi,dx_1)\\
&= (\sum_{i=2}^{n}b_i\pi_i)\wedge\pi_1+\omega_2\wedge(\pi_2+a_2\pi_1)+\cdots+\omega_n\wedge(\pi_n+a_n\pi_1)\\
&= (\omega_2-b_2\pi_1)\wedge(\pi_2+a_2\pi_1)+\cdots+(\omega_n-b_n\pi_1)\wedge(\pi_n+a_n\pi_1)
\end{aligned}$$

Thus, putting $\hat{\omega}_i=\omega_i-b_i\pi_1$, $\hat{\pi}_i=\pi_i+a_i\pi_1$ for $2\leqq i\leqq n$, we get

$$d\varpi\equiv\hat{\omega}_2\wedge\hat{\pi}_2+\cdots+\hat{\omega}_n\wedge\hat{\pi}_n\quad(\text{mod }\varpi,dx_1)$$

Hence we obtain

$$\begin{aligned}
\mathrm{Ch}\,(C^1) &= \{X\in C^1(u)\mid X\rfloor d\varpi\equiv 0\ (\text{mod }\varpi,dx_1)\}\\
&= \{\varpi=dx_1=\hat{\omega}_2=\cdots=\hat{\omega}_n=\hat{\pi}_2=\cdots=\hat{\pi}_n=0\}
\end{aligned}$$

Moreover we have

$$\{0\}=\mathrm{Ch}\,(C)\subset\mathrm{Ch}\,(C^1)\subset C^1\subset C.$$

Now let us take a first integral x_2 of $\mathrm{Ch}\,(C^1)$ such that $\varpi\wedge dx_1\wedge dx_2\wedge\hat{\omega}_3\wedge\cdots\wedge\hat{\omega}_n\wedge\hat{\pi}_2\wedge\cdots\wedge\hat{\pi}_n\neq 0$ around u_o and consider the differential system C^2 defined by

$$C^2=\{\,\varpi=dx_1=dx_2=0\,\}.$$

Thus

$$\mathrm{Ch}\,(C^1)=\{\varpi=dx_1=dx_2=\hat{\omega}_3=\cdots=\hat{\omega}_n=\hat{\pi}_2=\cdots=\hat{\pi}_n=0\},$$

so that we can write

$$\hat{\omega}_2\equiv\sum_{i=3}^{n}\hat{a}_i\hat{\omega}_i+\sum_{i=2}^{n}\hat{b}_i\hat{\pi}_i\quad(\text{mod }\varpi,dx_1,dx_2).$$

Then, as in the above calculation, we get

$$d\varpi\equiv\tilde{\omega}_3\wedge\tilde{\pi}_3+\cdots+\tilde{\omega}_n\wedge\tilde{\pi}_n\quad(\text{mod }\varpi,dx_1,dx_2)$$

where we put $\tilde{\omega}_i=\hat{\omega}_i-\hat{b}_i\hat{\pi}_2$, $\tilde{\pi}_i=\hat{\pi}_i+\hat{a}_i\hat{\pi}_2$ for $3\leqq i\leqq n$. Thus we obtain

$$\begin{aligned}
\mathrm{Ch}\,(C^2) &= \{X\in C^2(u)\mid X\rfloor d\varpi\equiv 0\ (\text{mod }\varpi,dx_1,dx_2)\}\\
&= \{\varpi=dx_1=dx_2=\tilde{\omega}_3=\cdots=\tilde{\omega}_n=\tilde{\pi}_3=\cdots=\tilde{\pi}_n=0\}
\end{aligned}$$

Moreover we have

$$\{0\}=\mathrm{Ch}\,(C)\subset\mathrm{Ch}\,(C^1)\subset\mathrm{Ch}\,(C^2)\subset C^2\subset C^1\subset C.$$

If we repeat this procedure n times, we obtain first integrals x_i of $\mathrm{Ch}\,(C^{i-1})$ defined around u_o for $i=2,\ldots,n$ such that $\varpi\wedge dx_1\wedge\cdots\wedge dx_n\neq 0$ around u_o, and that

$$C^i=\{\,\varpi=dx_1=\cdots=dx_i\}\qquad\text{for }\ i=1,\ldots,n.$$

Moreover we have

$$d\varpi\equiv 0\qquad(\text{mod }\varpi,dx_1,\ldots,dx_n),$$

i.e., $C^n = \mathrm{Ch}\,(C^n)$ is completely integrable.

Finally let us take a first integral z of C^n such that $dz \wedge dx_1 \wedge \cdots \wedge dx_n \neq 0$ around u_o. Then we have

$$C^n = \{\, dz = dx_1 = \cdots = dx_n \,\},$$

so that

$$\varpi = a(dz - \sum_{i=1}^{n} p_i dx_i),$$

for some functions $a, p_1, \ldots, p_n$ defined around u_o such that $a(u_o) \neq 0$. Hence we obtain

$$C = \{\, dz - \sum_{i=1}^{n} p_i dx_i = 0 \,\}.$$

Then, from $X \rfloor d\hat{\varpi} = \sum_{i=1}^{n}(dx_i(X)dp_i - dp_i(X)dx_i)$ for $\hat{\varpi} = dz - \sum_{i=1}^{n} p_i dx_i$, we get

$$\begin{aligned} &\{\hat{\varpi} = dx_1 = \cdots = dx_n = dp_1 = \cdots = dp_n = 0\} \\ &\qquad\qquad = \{\, dz = dx_1 = \cdots = dx_n = dp_1 = \cdots = dp_n = 0\} \subset \mathrm{Ch}\,(C) = \{0\} \end{aligned}$$

which implies that $dz \wedge dx_1 \wedge \cdots \wedge dx_n \wedge dp_1 \wedge \cdots \wedge dp_n \neq 0$ aroud $u_o \in J$. This completes the proof of the Darboux Theorem.

2. Tanaka Theory of Linear Differential Sytems.

2.1. **Symbol algebras of** (M, D)**.** A differential system (M, D) is called **regular**, if $D^{-(k+1)} = \partial^{(k)} D$ are subbundles of $T(M)$ for every integer $k \geqq 1$. For a regular differential system (M, D), we have ([T2], Proposition 1.1)

$(S1)$ *There exists a unique integer* $\mu > 0$ *such that, for all* $k \geqq \mu$,

$$D^{-k} = \cdots = D^{-\mu} \supsetneqq D^{-\mu+1} \supsetneqq \cdots \supsetneqq D^{-2} \supsetneqq D^{-1} = D,$$

$(S2)$ $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$ *for all* $p, q < 0$.

where $\mathcal{D}^p$ denotes the space of sections of D^p. $(S2)$ can be checked easily by induction on q. Thus $D^{-\mu}$ is the smallest completely integrable differential system, which contains $D = D^{-1}$.

Let (M, D) be a regular differential system such that $T(M) = D^{-\mu}$. As a first invariant for non-integrable differential systems, we now define the **symbol algebra** $\mathfrak{m}(x)$ *associated with a differential system* (M, D) at $x \in M$, which was introduced by N. Tanaka [T2].

We put $\mathfrak{g}_{-1}(x) = D^{-1}(x)$, $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$ $(p < -1)$ and

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

Let ϖ_p be the projection of $D^p(x)$ onto $\mathfrak{g}_p(x)$. Then, for $X \in \mathfrak{g}_p(x)$ and $Y \in \mathfrak{g}_q(x)$, the bracket product $[X, Y] \in \mathfrak{g}_{p+q}(x)$ is defined by

$$[X, Y] = \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x),$$

where $\tilde{X}$ and $\tilde{Y}$ are any element of $\mathcal{D}^p$ and $\mathcal{D}^q$ respectively such that $\varpi_p(\tilde{X}_x) = X$ and $\varpi_q(\tilde{Y}_x) = Y$.

Endowed with this bracket operation, by $(S2)$ above, $\mathfrak{m}(x)$ becomes a nilpotent graded Lie algebra such that $\dim \mathfrak{m}(x) = \dim M$ and satisfies

$$\mathfrak{g}_p(x) = [\mathfrak{g}_{p+1}(x), \mathfrak{g}_{-1}(x)] \qquad \text{for } p < -1.$$

We call $\mathfrak{m}(x)$ the **symbol algebra of** (M, D) at $x \in M$ for short.

Furthermore, let $\mathfrak{m}$ be a FGLA (fundamental graded Lie algebra) of μ-th kind, that is,

$$\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$$

is a nilpotent graded Lie algebra such that

$$\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \qquad \text{for } p < -1.$$

Then (M, D) is called of type $\mathfrak{m}$ if the symbol algebra $\mathfrak{m}(x)$ is isomorphic with $\mathfrak{m}$ at each $x \in M$.

2.2. **Standard Differential System** $(M(\mathfrak{m}), D_{\mathfrak{m}})$ **of Type** $\mathfrak{m}$**.** Conversely, given a FGLA $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$, we can construct a model differential system of type $\mathfrak{m}$ as follows: Let $M(\mathfrak{m})$ be the simply connected Lie group with Lie algebra $\mathfrak{m}$. Identifying $\mathfrak{m}$ with the Lie algebra of left invariant vector fields on $M(\mathfrak{m})$, $\mathfrak{g}_{-1}$ defines a left invariant subbundle $D_{\mathfrak{m}}$ of $T(M(\mathfrak{m}))$. By definition of symbol algebras, it is easy to see that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is a regular differential system of type $\mathfrak{m}$. $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is called the standard differential system of type $\mathfrak{m}$. The Lie algebra $\mathfrak{g}(\mathfrak{m})$ of all infinitesimal automorphisms of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ can be calculated algebraically as the prolongation of $\mathfrak{m}$ ([T1], cf. [Y5]). We will discuss in §4 the question of *when* $\mathfrak{g}(\mathfrak{m})$ *becomes finite dimensional and simple* ?

As an example to calculate symbol algebras, let us show that $(L(J), E)$ is a regular differential system of type $\mathfrak{c}^2(n)$:

$$\mathfrak{c}^2(n) = \mathfrak{c}_{-3} \oplus \mathfrak{c}_{-2} \oplus \mathfrak{c}_{-1},$$

where $\mathfrak{c}_{-3} = \mathbb{R}$, $\mathfrak{c}_{-2} = V^*$ and $\mathfrak{c}_{-1} = V \oplus S^2(V^*)$. Here V is a vector space of dimension n and the bracket product of $\mathfrak{c}^2(n)$ is defined accordingly through the pairing between V and V^* such that V and $S^2(V^*)$ are both abelian subspaces of $\mathfrak{c}_{-1}$. This fact can be checked as follows: Let us take a canonical coordinate system $U; (x_i, z, p_i, p_{ij})$ $(1 \leqq i \leqq j \leqq n)$ of $(L(J), E)$. Then we have a coframe $\{\varpi, \varpi_i, dx_i, dp_{ij}\}$ $(1 \leqq i \leqq j \leqq n)$ at each point in U, where $\varpi = dz - \sum_{i=1}^n p_i\, dx_i$, $\varpi_i = dp_i - \sum_{j=1}^n p_{ij}\, dx_j$ $(i = 1, \cdots, n)$. Now take the dual frame $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial p_i}, \frac{d}{dx_i}, \frac{\partial}{\partial p_{ij}}\}$, of this coframe, where

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial z} + \sum_{j=1}^n p_{ij} \frac{\partial}{\partial p_j}$$

is the classical notation. Notice that $\{\frac{d}{dx_i}, \frac{\partial}{\partial p_{ij}}\}$ $(i = 1, \cdots, n)$ forms a free basis of $\Gamma(E)$. Then an easy calculation shows the above fact. Moreover we see that the derived system ∂E of E satisfies the following :

$$\partial E = \{\varpi = 0\} = \pi_*^{-1} C, \qquad \mathrm{Ch}\,(\partial E) = \mathrm{Ker}\,\pi_*.$$

These facts provide the proof of Theorem 1.3 (cf. Theorem 3.2 [Y1]).

Similarly we see that $(J(M, n), C)$ is a regular differential system of type $\mathfrak{c}^1(n, m)$:

$$\mathfrak{c}^1(n, m) = \mathfrak{c}_{-2} \oplus \mathfrak{c}_{-1},$$

where $\mathfrak{c}_{-2} = W$ and $\mathfrak{c}_{-1} = V \oplus W \otimes V^*$ for vector spaces V and W of dimension n and m respectively, and the bracket product of $\mathfrak{c}^1(n,m)$ is defined accordingly through the pairing between V and V^* such that V and $W \otimes V^*$ are both abelian subspaces of $\mathfrak{c}_{-1}$.

2.3. **Prolongation $\mathfrak{g}(\mathfrak{m})$ of Symbol Algebras $\mathfrak{m}$.** Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a fundamental graded Lie algebra of μ-th kind defined over a field K. Here K denotes the field of real numbers $\mathbb{R}$ or that of complex numbers $\mathbb{C}$. We put

$$\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p(\mathfrak{m}),$$

where $\mathfrak{g}_p(\mathfrak{m}) = \mathfrak{g}_p$ for $p < 0$, $\mathfrak{g}_0(\mathfrak{m})$ is the Lie algebra of all (gradation preserving) derivations of graded Lie algebra $\mathfrak{m}$ and $\mathfrak{g}_k(\mathfrak{m})$ is defined inductively by the following for $k \geqq 1$;

$$\mathfrak{g}_k(\mathfrak{m}) = \{\, u \in \bigoplus_{p<0} \mathfrak{g}_{p+k} \otimes \mathfrak{g}_p^* \mid u([Y,Z]) = [u(Y),Z] - [u(Z),Y] \,\}.$$

Thus, as a vector space over K, $\mathfrak{g}_k(\mathfrak{m})$ is a linear subspace of $\mathrm{End}\,(\mathfrak{m},\mathfrak{m}^k) = \mathfrak{m}^k \otimes \mathfrak{m}^*$, where $\mathfrak{m}^k = \mathfrak{m} \oplus \mathfrak{g}_0(\mathfrak{m}) \oplus \cdots \oplus \mathfrak{g}_{k-1}(\mathfrak{m})$. The bracket operation of $\mathfrak{g}(\mathfrak{m})$ is given as follows: First, since $\mathfrak{g}_0(\mathfrak{m})$ is the (gradation preserving) derivation algebra of graded Lie algebra $\mathfrak{m}$, we see that $\bigoplus_{p \leqq 0} \mathfrak{g}_p(\mathfrak{m})$ becomes a graded Lie algebra by putting

$$[u,X] = -[X,u] = u(X) \qquad \text{for } u \in \mathfrak{g}_0(\mathfrak{m}) \text{ and } X \in \mathfrak{m}.$$

Similarly, for $u \in \mathfrak{g}_k(\mathfrak{m}) \subset \mathfrak{m}^k \otimes \mathfrak{m}^*$ $(k > 0)$ and $X \in \mathfrak{m}$, we put $[u,X] = -[X,u] = u(X)$. Now, for $u \in \mathfrak{g}_k(\mathfrak{m})$ and $v \in \mathfrak{g}_\ell(\mathfrak{m})$ $(k,\ \ell \geqq 0)$, by induction on the integer $k + \ell \geqq 0$, we define $[u,v] \in \mathfrak{m}^{k+\ell} \otimes \mathfrak{m}^*$ by

$$[u,v](X) = [[u,X],v] + [u,[v,X]] \qquad \text{for } X \in \mathfrak{m}.$$

Here we note that, as the first case $k = \ell = 0$, this definition begins with that of the bracket product in $\mathfrak{g}_0(\mathfrak{m})$. It follows easily that $[u,v] \in \mathfrak{g}_{k+\ell}(\mathfrak{m})$. With this bracket product, $\mathfrak{g}(\mathfrak{m})$ becomes a graded Lie algebra. In fact the Jacobi identity

$$[[u,v],w] + [[v,w],u] + [[w,u],v] = 0,$$

for $u \in \mathfrak{g}_p(\mathfrak{m})$, $v \in \mathfrak{g}_q(\mathfrak{m})$ and $w \in \mathfrak{g}_r(\mathfrak{m})$, follows by definition when one of p, q or r is negative, and can be shown by induction on the integer $p + q + r \geqq 0$, when all of p, q and r are non-negative. The structure of the Lie algebra $\mathcal{A}(M(\mathfrak{m}), D_\mathfrak{m})$ of all infinitesimal automorphisms of $(M(\mathfrak{m}), D_\mathfrak{m})$ can be described by $\mathfrak{g}(\mathfrak{m})$. Especially $\mathcal{A}(M(\mathfrak{m}), D_\mathfrak{m})$ is isomorphic with $\mathfrak{g}(\mathfrak{m})$, when $\mathfrak{g}(\mathfrak{m})$ is finite dimensional ([T1], cf. [Y5]).

Let $\mathfrak{g}_0$ be a subalgebra of $\mathfrak{g}_0(\mathfrak{m})$. We define a subspace $\mathfrak{g}_k$ of $\mathfrak{g}_k(\mathfrak{m})$ for $k \geqq 1$ inductively by

$$\mathfrak{g}_k = \{\, u \in \mathfrak{g}_k(\mathfrak{m}) \mid [u,\mathfrak{g}_{-1}] \subset \mathfrak{g}_{k-1} \,\}.$$

Then, putting

$$\mathfrak{g}(\mathfrak{m},\mathfrak{g}_0) = \mathfrak{m} \oplus \bigoplus_{k \geqq 0} \mathfrak{g}_k,$$

we see, with the generating condition of $\mathfrak{m}$, that $\mathfrak{g}(\mathfrak{m},\mathfrak{g}_0)$ is a graded subalgebra of $\mathfrak{g}(\mathfrak{m})$. $\mathfrak{g}(\mathfrak{m},\mathfrak{g}_0)$ is called the prolongation of $(\mathfrak{m},\mathfrak{g}_0)$.

Remark 2.1 The notion of the prolongation of $\mathfrak{m}$ or $(\mathfrak{m},\mathfrak{g}_0)$ plays quite an important role in the equivalence problems for the geometric structures subordinate to regular differential systems of type $\mathfrak{m}$, e.g., CR-structures, pseudo-product structures or Lie contact structures. We could not touch upon the more important geometric aspect of the prolongation theory of these structures. On these subjects, we refer the reader to foundational papers [T2], [T3], [T4] of N.Tanaka.

2.4. **Proof of the Bäcklund Theorem (Theorem 1.4 [Y3]).** Let $J(M,n)$ be the space of n-dimensional contact elements to M and C be the canonical system on $J(M,n)$. Recall that $(J(M,n),C)$ is a regular differential system of type $\mathfrak{c}^1(n,m)=\mathfrak{c}^1(V,W)$:

$$\mathfrak{c}^1(V,W)=\mathfrak{c}_{-2}\oplus\mathfrak{c}_{-1},$$

where $\mathfrak{c}_{-2}=W$ and $\mathfrak{c}_{-1}=V\oplus W\otimes V^*$ for vector spaces V and W of dimension n and m respectively. Put $\mathfrak{f}=W\otimes V^*$. First we will characterize the abelian subspace $\mathfrak{f}$ of $\mathfrak{c}_{-1}$. Namely we first claim : *If* $\dim W\geqq 2$, *then*

$$\mathfrak{f}=\langle\{\,X\in\mathfrak{c}_{-1}\mid \operatorname{rank}\operatorname{ad}(X)\leqq 1\,\}\rangle,$$

i.e., $\mathfrak{f}$ is the span of elements $X\in\mathfrak{c}_{-1}$ such that $\operatorname{rank}\operatorname{ad}(X)=1$. In fact, let $X=v_X+f_X$ be any element of $\mathfrak{c}_{-1}(V,W)$, where $v_X\in V$ and $f_X\in\mathfrak{f}=W\otimes V^*$. Then we have

$$\begin{aligned} \operatorname{ad}(X)(v)&=[X,v]=f_X(v) &&\text{for}\quad v\in V,\\ \operatorname{ad}(X)(f)&=[X,f]=-f(v_X) &&\text{for}\quad f\in W\otimes V^*. \end{aligned}$$

Thus we see that $\operatorname{rank}\operatorname{ad}(X)=\dim W$ if $v_X\neq 0$ and $\operatorname{rank}\operatorname{ad}(X)=\operatorname{rank}\, f_X$ if $v_X=0$. On the other hand it is clear that $\mathfrak{f}=W\otimes V^*$ is spanned by elements of rank 1. Put $E=\langle\{\,X\in\mathfrak{c}_{-1}\mid \operatorname{rank}\operatorname{ad}(X)\leqq 1\,\}\rangle$. Then it follows that $E=\mathfrak{c}_{-1}(V,W)$ if $\dim W=1$ and $E=\mathfrak{f}$ otherwise.

To prove Theorem 1.1, assume that $m=\dim W\geqq 2$ and let u be any point of $J(M,n)$. Let $\mathfrak{c}(u)$(resp. $\hat{\mathfrak{c}}(\Phi(u))$) be the symbol algebra of $(J(M,n),C))$ (resp. $(J(\hat{M},n),\hat{C})$) at u (resp. $\Phi(u)$). Then there exist graded Lie algebra isomorphisms $\nu:\mathfrak{c}^1(V,W)\to\mathfrak{c}(u)$ and $\hat{\nu}:\mathfrak{c}^1(V,W)\to\hat{\mathfrak{c}}(\Phi(u))$ such that $\nu(\mathfrak{f})=\operatorname{Ker}\pi_*$ and $\hat{\nu}(\mathfrak{f})=\operatorname{Ker}\hat{\pi}_*$. Then, by the above claim, we get $\Phi_*(\operatorname{Ker}\pi_*)=\operatorname{Ker}\hat{\pi}_*$. Since each fibre of $J(M,n)$ and $J(\hat{M},n)$ is connected, we see that Φ is fibre-preserving. Hence Φ induces a unique diffeomorphism φ of M onto $\hat{M}$ such that $\hat{\pi}\cdot\Phi=\varphi\cdot\pi$. Finally $\Phi=\varphi_*$ easily follows from $\Phi_*(C)=\hat{C}$ and the definiton of the canonical system on $J(M,n)$.

3. PD-manifolds of Second Order.

We will here formulate the submanifold theory for $(L(J),E)$ as the geometry of PD-manifolds ([Y1]).

3.1. **Submanifolds in $L(J)$.** Let R be a submanifold of $L(J)$ satisfying the following condition:

$$(R.0)\qquad p:R\to J\ ;\ \text{submersion},$$

where $p=\pi\mid_R$ and $\pi:L(J)\to J$ is the projection. There are two differential systems $C^1=\partial E$ and $C^2=E$ on $L(J)$. We denote by D^1 and D^2 those differential systems on R obtained by restricting these differential systems to R. Moreover we denote by the same symbols those 1-forms obtained by restricting the defining 1-forms $\{\varpi,\varpi_1,\cdots,\varpi_n\}$ of the canonical system E to R. Then it follows from $(R.0)$ that these 1-forms are independent at each point on R and that

$$D^1=\{\varpi=0\},\qquad D^2=\{\varpi=\varpi_1=\cdots=\varpi_n=0\}.$$

In fact $(R;D^1,D^2)$ further satisfies the following conditions:

$(R.1)$ D^1 *and* D^2 *are differential systems of codimension* 1 *and* $n+1$ *respectively.*

$(R.2)$ $\partial D^2 \subset D^1$.

$(R.3)$ $\mathrm{Ch}\,(D^1)$ *is a subbundle of* D^2 *of codimension* n.

$(R.4)$ $\mathrm{Ch}\,(D^1)(v) \cap \mathrm{Ch}\,(D^2)(v) = \{0\}$ *at each* $v \in R$.

The last condition follows easily from the Realization Lemma below.

3.2. **Realization Lemma.** Conversely these four conditions characterize submanifolds in $L(J)$ satisfying $(R.0)$. To see this , we first recall the following **Realization Lemma**, which characterize a submanifold of $(J(M,n), D)$.

Realization Lemma. *Let* R *and* M *be manifolds. Assume that the quadruple* (R, D, p, M) *satisfies the following conditions :*

(1) p *is a map of* R *into* M *of constant rank.*

(2) D *is a differential system on* R *such that* $F = Ker\ p_*$ *is a subbundle of* D *of codimension* n.

Then there exists a unique map ψ *of* R *into* $J(M,n)$ *satisfying* $p = \pi \cdot \psi$ *and* $D = \psi_*^{-1}(C)$, *where* C *is the canonical differential system on* $J(M,n)$ *and* $\pi : J(M,n) \to M$ *is the projection. Furthermore, let* v *be any point of* R. *Then* ψ *is in fact defined by*

$$\psi(v) = p_*(D(v)) \qquad \textit{as a point of } Gr\ (T_{p(v)}(M)),$$

and satisfies

$$Ker\ (\psi_*)_v = F(v) \cap Ch\,(D)(v).$$

where $Ch\,(D)$ *is the Cauchy Characteristic System of* D.

For the proof, see Lemma 1.5 [Y1].

In view of this Lemma, we call the triplet $(R; D^1, D^2)$ of a manifold and two differential systems on it a **PD-manifold** if these satisfy the above four conditions $(R.1)$ to $(R.4)$. We have the (local) Realization Theorem for PD-manifolds as follows: From conditions $(R.1)$ and $(R.3)$, it follows that the codimension of the foliation defined by the completely integrable system $\mathrm{Ch}\,(D^1)$ is $2n+1$. Assume that R is regular with respect to $\mathrm{Ch}\,(D^1)$, i.e., the space $J = R/\mathrm{Ch}\,(D^1)$ of leaves of this foliation is a manifold of dimension $2n+1$. Then D^1 drops down to J. Namely there exists a differential system C on J of codimension 1 such that $D^1 = p_*^{-1}(C)$, where $p : R \to J = R/\mathrm{Ch}\,(D^1)$ is the projection. Obviously (J, C) becomes a contact manifold of dimension $2n+1$. Conditions $(R.1)$ and $(R.2)$ guarantees that the image of the following map ι is a legendrian subspace of (J, C):

$$\iota(v) = p_*(D^2(v)) \subset C(u), \qquad u = p(v).$$

Finally the condition $(R.4)$ shows that $\iota : R \to L(J)$ is an immersion by Realization Lemma for (R, D^2, p, J). Furthermore we have (Corollary 5.4 [Y1])

Theorem 3.1. *Let* $(R; D^1, D^2)$ *and* $(\hat{R}; \hat{D}^1, \hat{D}^2)$ *be* PD*-manifolds. Assume that* R *and* $\hat{R}$ *are regular with respect to* $Ch\,(D^1)$ *and* $Ch\,(\hat{D}^1)$ *respectively. Let* (J, C) *and* $(\hat{J}, \hat{C})$ *be the associated contact manifolds. Then an isomorphism* $\Phi : (R; D^1, D^2) \to (\hat{R}; \hat{D}^1, \hat{D}^2)$ *induces a contact diffeomorphism* $\varphi : (J, C) \to (\hat{J}, \hat{C})$ *such that the following commutes;*

$$\begin{array}{ccc} R & \xrightarrow{\iota} & L(J) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\varphi_*} \\ \hat{R} & \xrightarrow{\hat{\iota}} & L(\hat{J}). \end{array}$$

By this theorem, the submanifold theory for $(L(J), E)$ is reformulated as the geometry of PD-manifolds.

3.3. **Reduction Theorem.** When $D^1 = \partial D^2$ holds for a PD-manifold $(R; D^1, D^2)$, the geometry of $(R; D^1, D^2)$ reduces to that of (R, D^2) and the Tanaka theory is directly applicable to this case (cf. [YY2]). Concerning about this situation, the following theorem is known under the compatibility condition (C) below:

$$(C) \qquad p^{(1)} : R^{(1)} \to R \textit{ is onto.}$$

where $R^{(1)}$ is the first prolongation of $(R; D^1, D^2)$,i.e.,

$$R^{(1)} = \{n\text{-dim. } \textit{integral elements} \text{ of } (R, D^2), \text{ transversal to } F = \operatorname{Ker} p_*\} \subset J(R, n),$$

(cf. Proposition 5.11 [Y1]).

Theorem 3.2. *Let $(R; D^1, D^2)$ be a PD-manifold satisfying the condition (C) above. Then the following equality holds at each point v of R:*

$$\dim D^1(v) - \dim \partial D^2(v) = \dim Ch\,(D^2)(v).$$

In particular $D^1 = \partial D^2$ holds if and only if $Ch\,(D^2) = \{0\}$.

When PD-manifold $(R; D^1, D^2)$ admits a non-trivial Cauchy characteristics, i.e.,when rank $\mathrm{Ch}\,(D^2) > 0$, the geometry of $(R; D^1, D^2)$ is further reducible to the geometry of a single differential system. Here we will be concerned with the local equivalence of $(R; D^1, D^2)$, hence we may assume that R is regular with respect to $\mathrm{Ch}\,(D^2)$, i.e., the leaf space $X = R/\mathrm{Ch}\,(D^2)$ is a manifold such that the projection $\rho : R \to X$ is a submersion and there exists a differential system D on X satisfying $D^2 = \rho_*^{-1}(D)$. Then the local equivalence of $(R; D^1, D^2)$ is further reducible to that of (X, D) as in the following

Theorem 3.3. *Let (R, D^1, D^2) and $(\hat{R}; \hat{D}^1, \hat{D}^2)$ be PD-manifolds satisfying the condition (C) such that $Ch\,(D^2)$ and $Ch\,(\hat{D}^2)$ are subbundles of rank r $(0 < r < n)$. Assume that R and $\hat{R}$ are regular with respect to $Ch\,(D^2)$ and $Ch\,(\hat{D}^2)$ respectively. Let (X, D) and $(\hat{X}, \hat{D})$ be the leaf spaces, where $X = R/Ch\,(D^2)$ and $\hat{X} = \hat{R}/Ch\,(\hat{D}^2)$. Let us fix points $v_o \in R$ and $\hat{v}_o \in \hat{R}$ and put $x_o = \rho(v_o)$ and $\hat{x}_o = \hat{\rho}(\hat{v}_o)$. Then a local isomorphism $\psi : (R; D^1, D^2) \to (\hat{R}; \hat{D}^1, \hat{D}^2)$ such that $\psi(v_o) = \hat{v}_o$ induces a local isomorphism $\varphi : (X, D) \to (\hat{X}, \hat{D})$ such that $\varphi(x_o) = \hat{x}_o$ and $\varphi_*(\kappa(x_o)) = \hat{\kappa}(\hat{x}_o)$, and vice versa.*

3.4. **Higher Order Jet Spaces.** The essential part of the Bäcklund's Theorem is to show that $F = \operatorname{Ker} \pi_*$ is the covariant system of $(J(M, n), C)$ for $m \geq 2$. Namely an isomorphism Φ sends F onto $\hat{F} = \operatorname{Ker} \hat{\pi}_*$ for $m \geq 2$.

In case $m = 1$, it is a well known fact that the group of isomorphisms of $(J(M, n), C)$, i.e., the group of contact transformations, is larger than the group of diffeomorphisms of M. Therefore, when we consider the geometric 2-jet spaces, the situation differs according to whether the number m of dependent variables is 1 or greater.

(1) **Case** $m = 1$**.** We should start from a contact manifold (J, C) of dimension $2n + 1$, which is locally a space of 1-jet for one dependent variable by Darboux's theorem. Then we

can construct the geometric second order jet space $(L(J), E)$ as the Lagrange- Grassmann bundle $L(J)$ over J consisting of all n-dimensional integral elements of (J, C), while E is the restriction to $L(J)$ of the canonical system on $J(L(J), n)$.

Now we put

$$(J^2(M, n), C^2) = (L(J(M, n)), E),$$

where M is a manifold of dimension $n + 1$.

(2) **Case** $m \geq 2$**.** Since $F = \operatorname{Ker} \pi_*$ is a covariant system of $(J(M, n), C)$, we define $J^2(M, n) \subset J(J(M, n), n)$ by

$$J^2(M, n) = \{n\text{-dim. } \textit{integral elements} \text{ of } (J(M, n), C), \text{ transversal to } F\},$$

C^2 is defined as the restriction to $J^2(M, n)$ of the canonical system on $J(J(M, n), n)$.

Now the higher order (geometric) jet spaces $(J^{k+1}(M, n), C^{k+1})$ for $k \geq 2$ are defined (simultaneously for all m) by induction on k. Namely, for $k \geq 2$, we define $J^{k+1}(M, n) \subset J(J^k(M, n), n)$ and C^{k+1} inductively as follows:

$$J^{k+1}(M, n) = \{n\text{-dim. } \textit{integral elements} \text{ of } (J^k(M, n), C^k), \text{ transversal to Ker } (\pi^k_{k-1})_* \},$$

where $\pi^k_{k-1} : J^k(M, n) \to J^{k-1}(M, n)$ is the projection. Here we have

$$\text{Ker } (\pi^k_{k-1})_* = \operatorname{Ch}(\partial C^k),$$

and C^{k+1} is defined as the restriction to $J^{k+1}(M, n)$ of the canonical system on $J(J^k(M, n), n)$. Then we have ([Y1],[Y3])

$$\begin{array}{ccccccc}
 & C^k & \subset \cdots \subset & \partial^{k-2} C^k & \subset \partial^{k-1} C^k \subset & \partial^k C^k = T(J^k(M, n)) \\
 & \cup & & \cup & \cup & \\
\{0\} = \operatorname{Ch}(C^k) \subset & \operatorname{Ch}(\partial C^k) & \subset \cdots \subset & \operatorname{Ch}(\partial^{k-1} C^k) \subset & F &
\end{array}$$

where $\operatorname{Ch}(\partial^{i+1} C^k)$ is a subbundle of $\partial^i C^k$ of codimension n for $i = 0, \ldots, k-2$ and, when $m \geq 2$, F is a subbundle of $\partial^{k-1} C^k$ of codimension n. The transversality conditions are expressed as

$$C^k \cap F = \operatorname{Ch}(\partial C^k) \quad \text{for } m \geqq 2, \qquad C^k \cap \operatorname{Ch}(\partial^{k-1} C^k) = \operatorname{Ch}(\partial C^k) \quad \text{for } m = 1$$

By the above diagram together with the rank condition, Jet spaces $(J^k(M, n), C^k)$ can be characterized as higher order contact manifolds as in [Y1]and [Y3].

Here we observe that, if we drop the transversality condition in our definition of $J^k(M, n)$ and collect all n-dimensional integral elements, we may have some singularities in $J^k(M, n)$ in general. However, since every 2-form vanishes on 1-dimensional subspaces, in case $n = 1$, the integrability condition for $v \in J(J^{k-1}(M, 1), 1)$ reduces to $v \subset C^{k-1}(u)$ for $u = \pi^k_{k-1}(v)$. Hence, in this case, we can safely drop the transversality condition in the above construction as in the following "Rank 1 Prolongation" , which constitutes the key construction for the Drapeau theorem for m-flags (see [SY]).

We say that (R, D) is an m-flag of length k, if $\partial^i D$ is a subbundle of $T(R)$ for any i and has a derived length k, i.e., $\partial^k D = T(R)$;

$$D \subset \partial D \subset \cdots \subset \partial^{k-2} D \subset \partial^{k-1} D \subset \partial^k D = T(R),$$

such that rank $D = m + 1$ and rank $\partial^i D =$ rank $\partial^{i-1} D + m$ for $i = 1, \ldots, k$. In particular $\dim R = (k + 1)m + 1$.

Moreover, for a differential system (R, D), the **Rank** 1 **Prolongation** $(P(R), \widehat{D})$ is defined as follows;

$$P(R) = \bigcup_{x \in R} P_x \subset J(R, 1),$$

where

$$P_x = \{\text{1-dim. integral elements of } (R, D)\} = \{u \subset D(x) \mid \text{1-dim. subspaces}\} \cong \mathbb{P}^m.$$

We define the canonical system $\widehat{D}$ on $P(R)$ as the restriction to $P(R)$ of the canonical system on $J(R, 1)$. It can be shown that the Rank 1 Prolongation of an m-flag of length k becomes a m-flag of length $k + 1$.

Especially (R, D) is called a *Goursat flag* (un drapeau de Goursat) of length k when $m = 1$. Historically, by Engel, Goursat and Cartan, it is known that a Goursat flag (R, D) of length k is locally isomorphic, at a generic point, to the canonical system $(J^k(M, 1), C^k)$ on the k-jet spaces of 1 independent and 1 dependent variable. The characterization of the canonical (contact) systems on jet spaces was given by R. Bryant in [B] for the first order systems and in [Y1] and [Y3] for higher order systems for n independent and m dependent variables. However, it was first explicitly exhibited by A.Giaro, A. Kumpera and C. Ruiz in [GKR] that a Goursat flag of length 3 has singuralities and the research of singularities of Goursat flags of length k ($k \geq 3$) began as in [M]. To this situation, R. Montgomery and M. Zhitomirskii constructed the "Monster Goursat manifold" by successive applications of the "Cartan prolongation of rank 2 distributions [BH]" to a surface and showed that every germ of a Goursat flag (R, D) of length k appears in this "Monster Goursat manifold" in [MZ] , by first exhibitting the following Sandwich Lemma for (R, D);

$$\begin{array}{ccccccc} & D & \subset & \partial D & \subset \cdots \subset & \partial^{k-2} D & \subset \partial^{k-1} D \subset \partial^k D = T(R) \\ & \cup & & \cup & & \cup & \\ \mathrm{Ch}\,(D) \subset & \mathrm{Ch}\,(\partial D) & \subset & \mathrm{Ch}\,(\partial^2 D) & \subset \cdots \subset & \mathrm{Ch}\,(\partial^{k-1} D) & \end{array}$$

where $\mathrm{Ch}\,(\partial^i D)$ is the Cauchy characteristic system of $\partial^i D$ and $\mathrm{Ch}\,(\partial^i D)$ is a subbundle of $\partial^{i-1} D$ of corank 1 for $i = 1, \ldots, k-1$. Moreover, after [MZ], P.Mormul defined the notion of a *special* m*- flag* of length k for $m \geq 2$ to characterize those m-flags which are obtained by successive applications of the Rank 1 Prolongations to the space of 1-jets of 1 independent and m dependent variables.

To be precise, starting from a manifold M of dimension $m + 1$, we put, for $k \geqq 2$,

$$(P^k(M), C^k) = (P(P^{k-1}(M)), \widehat{C}^{k-1})$$

where $(P^1(M), C^1) = (J(M, 1), C)$. When $m = 1$, $(P^k(M), C^k)$ are called **"Monster Goursat Manifolds"** in [MZ].

Then we have (Corollary 5.8. [SY])

Theorem 3.4. *An m-flag (R, D) of length k for $m \geq 3$ is locally isomorphic to $(P^k(M), C^k)$ if and only if $\partial^{k-1} D$ is of Cartan rank* 1*, and, moreover for $m \geq 4$, if and only if $\partial^{k-1} D$ is of Engel rank* 1.

Here, the *Cartan rank* of (R, C) is the smallest integer ρ such that there exist 1-forms $\{\pi^1, \ldots, \pi^\rho\}$, which are independent modulo $\{\omega_1, \ldots, \omega_s\}$ and satisfy

$$d\alpha \wedge \pi^1 \wedge \cdots \wedge \pi^\rho \equiv 0 \pmod{\omega_1, \ldots, \omega_s} \qquad \text{for } \forall \alpha \in \mathcal{C}^\perp = \Gamma(C^\perp),$$

where $C = \{\omega_1 = \cdots = \omega_s = 0\ \}$. Furthermore the *Engel (half) rank* of (R, C) is the smallest integer ρ such that

$$(d\alpha)^{\rho+1} \equiv 0 \pmod{\omega_1, \ldots, \omega_s} \qquad \text{for } \forall \alpha \in \mathcal{C}^\perp,$$

Moreover we have for an m-flag of length k for $m \geq 2$ (Corollary 6.3. [SY]) ,

Theorem 3.5. *An m-flag (R, D) of length k is locally isomorphic to $(P^k(M), C^k)$ if and only if there exists a completely integrable subbundle F of $\partial^{k-1}D$ of corank 1.*

4. Differential Sytems associated with Simple Graded Lie Algebras.

4.1. **Gradation of $\mathfrak{g}$ in terms of Root Space Decomposition.** Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$. Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and choose a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of the root system Φ of $\mathfrak{g}$ relative to $\mathfrak{h}$. Then every $\alpha \in \Phi$ is an (all non-negative or all non-positive) integer coefficient linear combination of elements of Δ and we have the root space decomposition of $\mathfrak{g}$;

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha},$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [h, X] = \alpha(h)X \quad \text{for } h \in \mathfrak{h}\}$ is (1-dimensional) root space (corresponding to $\alpha \in \Phi$) and Φ^+ denotes the set of positive roots.

Now let us take a nonempty subset Δ_1 of Δ. Then Δ_1 defines the partition of Φ^+ as in the following and induces the gradation of $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ as follows:

$$\Phi^+ = \cup_{p \geqq 0} \Phi_p^+, \qquad \Phi_p^+ = \{\alpha = \sum_{i=1}^{\ell} n_i \alpha_i \mid \sum_{\alpha_i \in \Delta_1} n_i = p\},$$

$$\mathfrak{g}_p = \bigoplus_{\alpha \in \Phi_p^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_0 = \bigoplus_{\alpha \in \Phi_0^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{g}_{-p} = \bigoplus_{\alpha \in \Phi_p^+} \mathfrak{g}_{-\alpha},$$

$$[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} \qquad \text{for} \quad p, q \in \mathbb{Z}.$$

Moreover the negative part $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ satisfies the following generating condition :

$$\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \quad \text{for} \quad p < -1$$

We denote the SGLA (simple graded Lie algebra) $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ obtained from Δ_1 in this manner by (X_ℓ, Δ_1), when $\mathfrak{g}$ is a simple Lie algebra of type X_ℓ. Here X_ℓ stands for the Dynkin diagram of $\mathfrak{g}$ representing Δ and Δ_1 is a subset of vertices of X_ℓ. Moreover we have

$$\mu = \sum_{\alpha_i \in \Delta_1} n_i(\theta),$$

where $\theta = \sum_{i=1}^{\ell} n_i(\theta)\, \alpha_i$ is the highest root of Φ^+.

Conversely we have (Theorem 3.12 [Y5])

Theorem 4.1. *Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ satisfying the generating condition. Let X_ℓ be the Dynkin diagram of $\mathfrak{g}$. Then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is isomorphic with a graded Lie algebra (X_ℓ, Δ_1) for some $\Delta_1 \subset \Delta$. Moreover (X_ℓ, Δ_1) and (X_ℓ, Δ_1') are isomorphic if and only if there exists a diagram automorphism ϕ of X_ℓ such that $\phi(\Delta_1) = \Delta_1'$.*

In the real case, we can utilize the Satake diagram of $\mathfrak{g}$ to describe gradations of $\mathfrak{g}$ (Theorem 3.12 [Y5]).

4.2. **Gradation of $\mathfrak{g}$ in terms of Matrix Representations.** Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of the classical type. We shall describe gradations of $\mathfrak{g}$ in terms of matrices. Here we reproduce the matrices description of the root space decomposition of $\mathfrak{g}$ from §7 of [Tk] (cf. [K-A], [V, Chapter 4.4]), which gives us explicit pictures of $M_{\mathfrak{g}}$.

(1) A_ℓ type ($\ell \geqq 1$). $\mathfrak{g} = \mathfrak{sl}(\ell+1,\mathbb{C})$. We take a Cartan subalgebra $\mathfrak{h}$ consisting of all diagonal elements of $\mathfrak{sl}(\ell+1,\mathbb{C})$, whose member we denote by $\operatorname{diag}(a_1,\dots,a_{\ell+1})$. Let $\lambda_1, \dots, \lambda_{\ell+1}$ be the linear form on $\mathfrak{h}$ defined by $\lambda_i \operatorname{diag}(a_1,\dots,a_{\ell+1}) \mapsto a_i$. We write E_{ij} ($1 \leqq i,j \leqq \ell+1$) for the matrix whose (i,j)-component is 1 and all of whose other components are 0. Then we have

$$[H, E_{ij}] = (\lambda_i - \lambda_j)(H)\, E_{ij} \qquad \text{for } H \in \mathfrak{h}.$$

Hence $\Phi = \{\lambda_i - \lambda_j \in \mathfrak{h}^* \ (1 \leqq i,j \leqq \ell+1,\ i \neq j)\}$ and E_{ij} spans the root subspace for $\lambda_i - \lambda_j \in \Phi$. Let us choose a simple root system $\Delta = \{\alpha_1,\dots,\alpha_\ell\}$ by putting

$$\alpha_i = \lambda_i - \lambda_{i+1}.$$

We have $\lambda_i - \lambda_j = \alpha_i + \cdots + \alpha_{j-1}$ when $i < j$. Hence $\theta = \alpha_1 + \cdots + \alpha_\ell$. Then we see that the gradation of $(A_\ell, \{\alpha_i\})$ is given by $\mathfrak{sl}(\ell+1,\mathbb{C}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$;

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \,\middle|\, C \in M(j,i) \right\},\ \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \,\middle|\, D \in M(i,j) \right\},$$
$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \,\middle|\, A \in M(i,i),\ B \in M(j,j) \text{ and } \mathrm{tr}A + \mathrm{tr}B = 0 \right\},$$

where $j = \ell - i + 1$ and $M(p,q)$ denotes the set of $p \times q$ matrices. This decomposition can be described schematically by the following diagram;

$$\begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{i} & \multicolumn{1}{c}{j} \\ \cline{2-3} i & 0 & 1 \\ \cline{2-3} j & -1 & 0 \\ \cline{2-3} \end{array}$$

where the vertical (resp. horizontal) line stands for the i-th vertical (resp. horizontal) intermediate line of a matrix in $\mathfrak{sl}(\ell+1,\mathbb{C})$. Then, for example, the diagram of $(A_\ell, \{\alpha_i, \alpha_j\})$ $(i < j)$ is obtained by superposing the diagrams of $(A_\ell, \{\alpha_i\})$ and $(A_\ell, \{\alpha_j\})$;

$$\begin{array}{|c|c|} \hline 0 & 1 \\ \hline -1 & 0 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline -1 & 0 \\ \hline \end{array} \qquad \Rightarrow \qquad \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline -1 & 0 & 1 \\ \hline -2 & -1 & 0 \\ \hline \end{array}$$

In general the diagram of $(A_\ell, \{\alpha_{i_1},\dots,\alpha_{i_k}\})$ is obtained by superposing the k diagrams of $(A_\ell, \{\alpha_{i_1}\}), \dots, (A_\ell, \{\alpha_{i_k}\})$. Namely the gradation of $(A_\ell, \{\alpha_{i_1},\dots,\alpha_{i_k}\})$ is obtained by subdividing matrices by both vertical and horizontal k lines. Here i-th intermediate line corresponds to the simple root α_i.

By this description of gradations, we see that the model space $M_{\mathfrak{g}}$ of $(A_\ell, \{\alpha_i\})$ is the complex Grassmann manifold $Gr(i,V)$ consisting of all i-dimensional subspaces of $V = \mathbb{C}^{\ell+1}$. Furthermore the model space $M_{\mathfrak{g}}$ of $(A_\ell, \{\alpha_{i_1},\dots,\alpha_{i_k}\})$ $(1 \leqq i_1 < \cdots < i_k \leqq \ell)$

is the flag manifold $F(i_1, \ldots, i_k; V)$ consisting of all flags $\{V_1 \subset \cdots \subset V_k\}$ in V such that $\dim V_j = i_j$ for $j = 1, \ldots, k$ (cf. [Tt]).

(2) C_ℓ type ($\ell \geqq 2$). Let $(V, \langle\ ,\ \rangle)$ be a symplectic vector space over $\mathbb{C}$ of dimension 2ℓ, that is, $\langle\ ,\ \rangle$ is a non-degenerate skew symmetric bilinear form on V. Then $\mathfrak{g} = \mathfrak{sp}(V)$. Let us take a symlectic basis $\{e_1, \ldots, e_\ell, f_1, \ldots, f_\ell\}$ of V such that $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$ and $\langle f_i, e_{\ell+1-j}\rangle = \delta_{ij}$ for $i, j = 1, \ldots, \ell$. Thus we have a matrix representation

$$\mathfrak{g} = \{\, X \in \mathfrak{gl}(2\ell, \mathbb{C}) \mid {}^tXJ + JX = 0\,\}, \quad \text{where } J = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix},$$

and K is the $\ell \times \ell$ matrix whose (i,j)-component is $\delta_{i,\ell+1-j}$. We put $A' = KAK$ for $A \in \mathfrak{gl}(\ell, \mathbb{C})$. Namely A' is the "transposed" matrix of A with respect to the anti-diagonal line. Each $X \in \mathfrak{g}$ is expressed as a matrix of the following form;

$$X = \begin{pmatrix} A & B \\ C & -A' \end{pmatrix},$$

where A, B, C are $\ell \times \ell$ matrices such that B and C satisfy $B = B'$ and $C = C'$. Namely both B and C are symmetric with respect to the anti-diagonal line. Thus we see that X is determined by its upper anti-diagonal part. In the following we write $X = (A, B, C)$ in short.

We take a Cartan subalgebra $\mathfrak{h}$ consisting of all diagonal elements of the form $H = (\operatorname{diag}(a_1, \ldots, a_\ell), 0, 0)$. Let $\lambda_1, \ldots, \lambda_\ell$ be the linear form on $\mathfrak{h}$ defined by $\lambda_i\, H \mapsto a_i$. We put $F_{ij} = E_{ij} + E'_{ij}$, where $E'_{ij} = E_{\ell+1-j,\ell+1-i}$. Then we have

$$\begin{aligned}
[H, (E_{ij}, 0, 0)] &= (\lambda_i - \lambda_j)(H)(E_{ij}, 0, 0), \\
[H, (0, F_{ij}, 0)] &= (\lambda_i + \lambda_{\ell+1-j})(H)(0, F_{ij}, 0), \\
[H, (0, 0, F_{ij})] &= -(\lambda_{\ell+1-i} + \lambda_j)(H)(0, 0, F_{ij}).
\end{aligned}$$

Hence $\Phi = \{\lambda_i - \lambda_j\ (i \neq j),\ \pm(\lambda_i + \lambda_j)\ (1 \leqq i \leqq j \leqq \ell)\}$ and $(E_{ij}, 0, 0)$, $(0, F_{i,\ell+1-j}, 0)$, $(0, 0, F_{\ell+1-i,j})$ are root vectors for $\lambda_i - \lambda_j$, $\lambda_i + \lambda_j$, $-(\lambda_i + \lambda_j) \in \Phi$ respectively. Let us choose a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ by putting

$$\begin{cases} \alpha_i = \lambda_i - \lambda_{i+1} & \text{for } i = 1, \ldots, \ell - 1, \\ \alpha_\ell = 2\,\lambda_\ell. \end{cases}$$

We have

$$\begin{cases} \lambda_i - \lambda_j = \alpha_i + \cdots + \alpha_{j-1} & (1 \leqq i < j \leqq \ell), \\ \lambda_i + \lambda_j = (\alpha_i + \cdots + \alpha_{\ell-1}) + (\alpha_j + \cdots + \alpha_\ell) & (1 \leqq i \leqq j \leqq \ell). \end{cases}$$

Hence $\theta = 2\,\alpha_1 + \cdots + 2\,\alpha_{\ell-1} + \alpha_\ell$. Then we see that the gradation of $(C_\ell, \{\alpha_i\})$ is given by the following diagram;

$$\begin{array}{c|c|c|c|} & i & & i \\ \hline i & 0 & 1 & 2 \\ \hline & -1 & 0 & 1 \\ \hline i & -2 & -1 & 0 \\ \hline \end{array} \quad (1 \leqq i < \ell) \qquad\qquad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline -1 & 0 \\ \hline \end{array} \quad (i = \ell)$$

Then the diagram of $(C_\ell, \{\alpha_{i_1}, \dots, \alpha_{i_k}\})$ is obtained by superposing the k diagrams of $(C_\ell, \{\alpha_{i_1}\})$, …, $(C_\ell, \{\alpha_{i_k}\})$. Here two intermediate lines (i-th and $(2\ell - i)$-th lines) correspond to the simple root $\{\alpha_i\}$ for $i = 1, \dots, \ell - 1$ and the center line corresponds to $\{\alpha_\ell\}$.

By this description of gradation, we see that the model space $M_\mathfrak{g}$ of $(C_\ell, \{\alpha_i\})$ is the Grassmann manifold $Sp\text{-}Gr(i, V)$ consisting of all i-dimensional isotropic subspaces of $(V, \langle\ ,\ \rangle)$. Furthermore the model space $M_\mathfrak{g}$ of $(C_\ell, \{\alpha_{i_1}, \dots, \alpha_{i_k}\})$ $(1 \leqq i_1 < \dots < i_k \leqq \ell)$ is the flag manifold $Sp\text{-}F(i_1, \dots, i_k; V)$ consisting of all flags $\{V_1 \subset \dots \subset V_k\}$ in V such that V_j is an i_j dimensional isotropic subspace of $(V, \langle\ ,\ \rangle)$ (cf. [Tt]).

(3) B_ℓ $(\ell \geqq 3)$, D_ℓ $(\ell \geqq 4)$ type. Let $(V, (\ |\ ))$ be an inner product space over $\mathbb{C}$ of dimension 2ℓ or $2\ell + 1$, that is, $(\ |\ )$ is a non-degenerate symmetric bilinear form on V. Then $\mathfrak{g} = \mathfrak{o}(V)$. Let us take a basis $\{e_1, \dots, e_\ell, e_{\ell+1}, f_1, \dots, f_\ell\}$ of V such that $(e_i|e_j) = (e_{\ell+1}|e_i) = (e_{\ell+1}|f_i) = (f_i|f_j) = 0$, $(e_{\ell+1}|e_{\ell+1}) = 1$ and $(e_i|f_{\ell+1-j}) = \delta_{ij}$ for $i, j = 1, \dots, \ell$. Here we neglect $e_{\ell+1}$, when $\dim V = 2\ell$. Then we have a matrices representation

$$\mathfrak{g} = \{\, X \in \mathfrak{gl}(n, \mathbb{C}) \mid {}^tXS + SX = 0 \,\}, \quad \text{where } S = \begin{pmatrix} 0 & 0 & K \\ 0 & 1 & 0 \\ K & 0 & 0 \end{pmatrix}$$

and $n = 2\ell$ or $2\ell + 1$. Each $X \in \mathfrak{g}$ is expressed as a matrix of the form

$$X = \begin{pmatrix} A & a & B \\ \xi & 0 & -a' \\ C & -\xi' & -A' \end{pmatrix}$$

where A, B, C are $\ell \times \ell$ matrices such that $B = -B'$, $C = -C'$ and a, ξ are column and row ℓ-vector respectively such that a' and ξ' are given by $a' = (a_\ell, \dots, a_1)$, $\xi' = {}^t(\xi_\ell, \dots, \xi_1)$ for $a = {}^t(a_1, \dots, a_\ell)$, $\xi = (\xi_1, \dots, \xi_\ell)$ respectively. Here the center column and the center row of X should be deleted when $\dim V = 2\ell$. Both B and C are skew symmetric with respect to the anti-diagonal line. In particular all the anti-diagonal components $x_{i,n+1-i}$ of X are 0. Thus X is determined by its upper anti-diagonal part. We write $X = (A, B, C, a, \xi)$, in short.

We take a Cartan subalgebra $\mathfrak{h}$ consisting of all diagonal elements of the form $H = (\operatorname{diag}(a_1, \dots, a_\ell), 0, 0, 0, 0)$. Let $\lambda_1, \dots, \lambda_\ell$ be the linear form on $\mathfrak{h}$ defined by $\lambda_i\ H \mapsto a_i$. We put $G_{ij} = E_{ij} - E'_{ij}$ and $E_i = (\delta_{1i}, \dots, \delta_{\ell i}) \in \mathbb{C}^\ell$. Then we have

$$\begin{aligned}
[H, (E_{ij}, 0, 0, 0, 0)] &= (\lambda_i - \lambda_j)(H)(E_{ij}, 0, 0, 0, 0), \\
[H, (0, G_{ij}, 0, 0, 0)] &= (\lambda_i + \lambda_{\ell+1-j})(H)(0, G_{ij}, 0, 0, 0), \\
[H, (0, 0, G_{ij}, 0, 0)] &= -(\lambda_{\ell+1-i} + \lambda_j)(H)(0, 0, G_{ij}, 0, 0), \\
[H, (0, 0, 0, E_i, 0)] &= \lambda_i(H)(0, 0, 0, E_i, 0), \\
[H, (0, 0, 0, 0, E_i)] &= -\lambda_i(H)(0, 0, 0, 0, E_i).
\end{aligned}$$

Hence we have

$$\Phi = \begin{cases} \{\lambda_i - \lambda_j\ (i \neq j),\ \pm(\lambda_i + \lambda_j)\ (1 \leqq i < j \leqq \ell)\} & \text{if } n = 2\ell, \\ \{\pm\lambda_i\ (1 \leqq i \leqq \ell),\ \lambda_i - \lambda_j\ (i \neq j), \\ \qquad\qquad \pm(\lambda_i + \lambda_j)\ (1 \leqq i < j \leqq \ell)\} & \text{if } n = 2\ell + 1. \end{cases}$$

$(E_{ij},0,0,0,0)$, $(0,G_{i,\ell+1-j},0,0,0)$, $(0,0,G_{\ell+1-i,j},0,0)$, $(0,0,0,E_i,0)$ and $(0,0,0,0,E_i)$ are root vectors for $\lambda_i-\lambda_j$, $\lambda_i+\lambda_j$, $-(\lambda_i+\lambda_j)$, λ_i and $-\lambda_i\in\Phi$ respectively. Let us choose a simple root system $\Delta=\{\alpha_1,\dots,\alpha_\ell\}$ by putting

$$\text{(i) } B_\ell \text{ type} \qquad \begin{cases} \alpha_i=\lambda_i-\lambda_{i+1} & \text{for } i=1,\,\dots,\,\ell-1, \\ \alpha_\ell=\lambda_\ell. \end{cases}$$

$$\text{(ii) } D_\ell \text{ type} \qquad \begin{cases} \alpha_i=\lambda_i-\lambda_{i+1} & \text{for } i=1,\,\dots,\,\ell-1, \\ \alpha_\ell=\lambda_{\ell-1}+\lambda_\ell. \end{cases}$$

Then we have

(i) B_ℓ type

$$\begin{cases} \lambda_i-\lambda_j=\alpha_i+\cdots+\alpha_{j-1} & (1\leqq i<j\leqq\ell), \\ \lambda_i=\alpha_i+\cdots+\alpha_\ell & (1\leqq i\leqq\ell), \\ \lambda_i+\lambda_j=\alpha_i+\cdots+\alpha_{j-1}+2\,\alpha_j+\cdots+2\,\alpha_\ell & (1\leqq i<j\leqq\ell). \end{cases}$$

Hence $\theta=\alpha_1+2\,\alpha_2+\cdots+2\,\alpha_\ell$.

(ii) D_ℓ type

$$\begin{cases} \lambda_i-\lambda_j=\alpha_i+\cdots+\alpha_{j-1} & (1\leqq i<j\leqq\ell), \\ \lambda_i+\lambda_\ell=\alpha_i+\cdots+\alpha_{\ell-2}+\alpha_\ell & (1\leqq i\leqq\ell-2), \\ \lambda_{\ell-1}+\lambda_\ell=\alpha_\ell & \\ \lambda_i+\lambda_{\ell-1}=\alpha_i+\cdots+\alpha_{\ell-1}+\alpha_\ell & (1\leqq i\leqq\ell-2), \\ \lambda_i+\lambda_j=\alpha_i+\cdots+\alpha_{j-1}+2\,\alpha_j+\cdots+2\,\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell & \\ & (1\leqq i<j\leqq\ell-2). \end{cases}$$

Hence $\theta=\alpha_1+2\,\alpha_2+\cdots+2\,\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell$.

Then we see that the gradation of $(B_\ell,\{\alpha_i\})$ is given by the following diagram;

$$\begin{array}{ccc} 1 & n-2 & 1 \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{*} \\ \hline \multicolumn{1}{|c|}{-1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} \\ \hline \multicolumn{1}{|c|}{*} & \multicolumn{1}{c|}{-1} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \quad (i=1) \qquad \begin{array}{ccc} i & n-2i & i \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{-1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} \\ \hline \multicolumn{1}{|c|}{-2} & \multicolumn{1}{c|}{-1} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \quad (1<i\leqq\ell)$$

The gradation of $(D_\ell,\{\alpha_i\})$ is given by the same diagram as above for $i=1,\,\dots,\,\ell-2$ and the above diagram with $i=\ell-1$ is that of $(D_\ell,\{\alpha_{\ell-1},\alpha_\ell\})$. Moreover the diagrams of $(D_\ell,\{\alpha_{\ell-1}\})$ and $(D_\ell,\{\alpha_\ell\})$ are given as follows

$$\begin{array}{c|c|c|c|c|} \cline{2-5} \ell-1 & 0 & 1 & 0 & 1 \\ \cline{2-5} 1 & -1 & 0 & * & 0 \\ \cline{2-5} 1 & 0 & * & 0 & 1 \\ \cline{2-5} \ell-1 & -1 & 0 & -1 & 0 \\ \cline{2-5} \end{array} \quad (i=\ell-1) \qquad \begin{array}{c|c|c|} \cline{2-3} \ell & 0 & 1 \\ \cline{2-3} \ell & -1 & 0 \\ \cline{2-3} \end{array} \quad (i=\ell)$$

Clearly, by interchanging e_ℓ and f_1, matrices representations of $(D_\ell, \{\alpha_{\ell-1}\})$ and $(D_\ell, \{\alpha_\ell\})$ transforms each other, i.e., $(D_\ell, \{\alpha_{\ell-1}\})$ and $(D_\ell, \{\alpha_\ell\})$ are conjugate. The other gradations of B_ℓ or D_ℓ type can be obtained by the principle of superposition as in the previous cases. Here two intermediate lines (i-th and $(n-i)$-th lines) correspond to the simple root $\{\alpha_i\}$ for $i = 1, \ldots, \ell$ in case of type B_ℓ and for $i = 1, \ldots, \ell-2$ in case of type D_ℓ. Moreover in case of type D_ℓ, $(\ell-1)$-th and $(\ell+1)$-th intermediate lines correspond to the pair $\{\alpha_{\ell-1}, \alpha_\ell\}$ and the center line corresponds to $\{\alpha_\ell\}$.

By this description of gradations, we see that the Grassmann manifold $O\text{-}Gr(i, V)$ consisting of all i-dimensional isotropic subspaces of $(V, (\,|\,))$ is the model space $M_{\mathfrak{g}}$ of $(B_\ell, \{\alpha_i\})$ or $(D_\ell, \{\alpha_i\})$ according as $\dim V = 2\ell + 1$ or 2ℓ, except for the case when $i = \ell - 1$ and $\dim V = 2\ell$. In the latter case $O\text{-}Gr(\ell-1, V)$ is the model space $M_{\mathfrak{g}}$ of $(D_\ell, \{\alpha_{\ell-1}, \alpha_\ell\})$, where $\dim V = 2\ell$. Thus, for D_ℓ type, we make a following convention for a subset Δ_1 of Δ: If $\alpha_{\ell-1} \in \Delta_1$ and $\alpha_\ell \notin \Delta_1$, we replace $\alpha_{\ell-1}$ by α_ℓ (the conjugacy class of (D_ℓ, Δ_1) does not change by this replacement), and if both $\alpha_{\ell-1}$ and $\alpha_\ell \in \Delta_1$, we write $\alpha^*_{\ell-1} = \{\alpha_{\ell-1}, \alpha_\ell\}$. Under this convention, we see that the model space $M_{\mathfrak{g}}$ of $(B_\ell, \{\alpha_{i_1}, \ldots, \alpha_{i_k}\})$ or $(D_\ell, \{\alpha_{i_1}, \ldots, \alpha_{i_k}\})$ $(1 \leqq i_1 < \cdots < i_k \leqq \ell)$ is the flag manifold $O\text{-}F(i_1, \ldots, i_k; V)$ consisting of all flags $\{V_1 \subset \cdots \subset V_k\}$ in V such that V_j is an i_j-dimensional isotropic subspace of $(V, (\,|\,))$, according as $\dim V = 2\ell + 1$ or 2ℓ (cf. [Tt]).

4.3. **Theorem on Prolongations.** By Theorem 4.1, the classification of gradations $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of simple Lie algebras $\mathfrak{g}$ satisfying the generating condition coincides with that of parabolic subalgebras $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ of $\mathfrak{g}$. Accordingly, to each SGLA (X_ℓ, Δ_1), there corresponds a unique R-space $M_{\mathfrak{g}} = G/G'$ (compact simply connected homogeneous complex manifold). Furthermore, when $\mu \geqq 2$, there exists the G-invariant differential system $D_{\mathfrak{g}}$ on $M_{\mathfrak{g}}$, which is induced from $\mathfrak{g}_{-1}$, and $(M(\mathfrak{m}), D_{\mathfrak{m}})$ (Standard differential system of type $\mathfrak{m}$) becomes an open submanifold of $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$. For the Lie algebras of all infinitesimal automorphisms of $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$, hence of $(M(\mathfrak{m}), D_{\mathfrak{m}})$, we have the following theorem (Theorem 5.2 [Y5]).

Theorem 4.2. *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ satisfying the generating condition. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ except for the following three cases.*

(1) $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *is of depth* 1 *(i.e., $\mu = 1$).*

(2) $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ *is a (complex) contact gradation.*

(3) $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is isomorphic with $(A_\ell, \{\alpha_1, \alpha_i\})$ $(1 < i < \ell)$ or $(C_\ell, \{\alpha_1, \alpha_\ell\})$.*

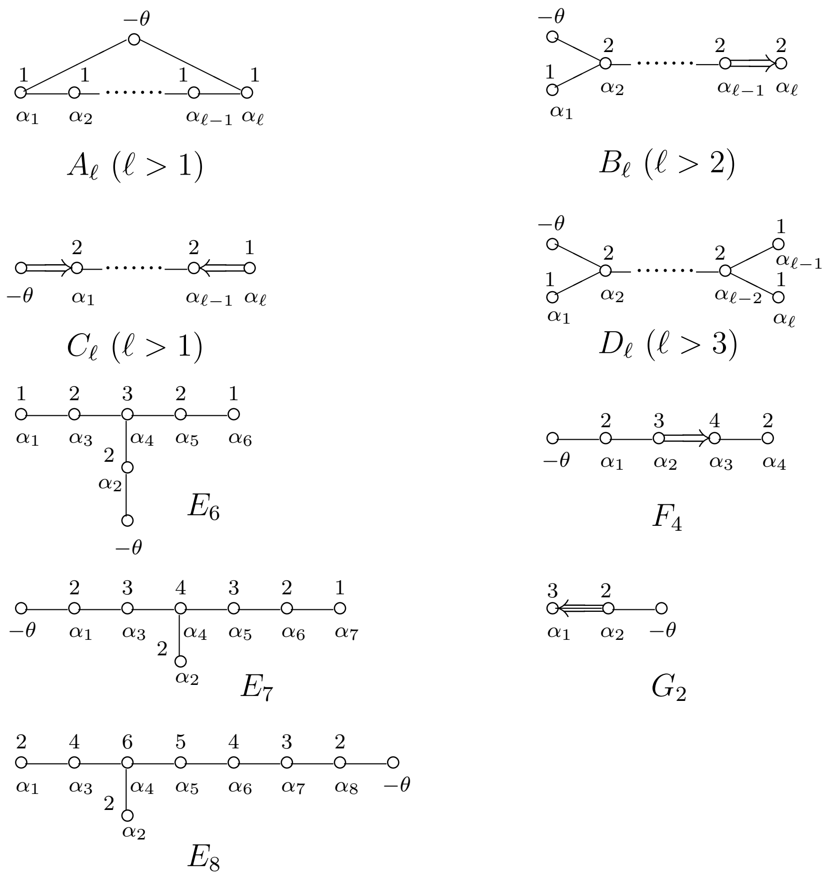
Here R-spaces corresponding to the above exceptions (1), (2) and (3) are as follows: (1) correspond to compact irreducible hermitian symmetric spaces. (2) correspond to contact manifolds of Boothby type (Standard contact manifolds), which exist uniquely for each simple Lie algebra other than $\mathfrak{sl}(2, \mathbb{C})$(see §5.1 below). In case of (3), $(J(\mathbb{P}^\ell, i), C)$ corresponds to $(A_\ell, \{\alpha_1, \alpha_i\})$ and $(L(\mathbb{P}^{2\ell-1}), E)$ corresponds to $(C_\ell, \{\alpha_1, \alpha_\ell\})$ $(1 < i < \ell)$, where $\mathbb{P}^\ell$ denotes the ℓ-dimensional complex projective space and $\mathbb{P}^{2\ell-1}$ is the Standard contact manifold of type C_ℓ. Here we note that R-spaces corresponding to (2) and (3) are all Jet spaces of the first or second order.

For the real version of this theorem, we refer the reader to Theorem 5.3 [Y5].

4.4. **Standard Contact Manifolds.** Each simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has the highest root θ. Let Δ_θ denote the subset of Δ consisting of all vertices which are connected to $-\theta$ in the Extended Dynkin diagram of X_ℓ ($\ell \geqq 2$). This subset Δ_θ of Δ, by the construction in §4, defines a gradation (or a partition of Φ^+), which distinguishes the highest root θ. Then, this gradation (X_ℓ, Δ_θ) turns out to be a contact gradation, which is unique up to conjugacy.

Moreover we have the adjoint (or equivalently coadjoint) representation, which has θ as the highest weight. The R-space $J_\mathfrak{g}$ corresponding to (X_ℓ, Δ_θ) can be obtained as the projectiviation of the (co-)adjoint orbit of G passing through the root vector of θ. By this construction, $J_\mathfrak{g}$ has the natural contact structure $C_\mathfrak{g}$ induced from the symplectic structure as the coadjoint orbit, which corresponds to the contact gradation (X_ℓ, Δ_θ) (cf. [Y5, §4]). Standard contact manifolds $(J_\mathfrak{g}, C_\mathfrak{g})$ were first found by Boothby ([Bo]) as compact simply connected homogeneous complex contact manifolds.

Extended Dynkin Diagrams with the coefficient of Highest Root (cf. [Bu])



5. G_2-Geometry of Overdetermined Systems.

This topic has its origin in the following paper of E. Cartan.

[C1] *Les systèmes de Pfaff à cinq variables et les équations aux derivèes partielles du second ordre*, Ann. Ec. Normale, 27 (1910), 109-192

In this paper, following the tradition of geometric theory of partial differential equations of 19th century, E.Cartan dealt with the equivalence problem of two classes of second order partial differential equations in two independent variables under "contact transformations". One class consists of overdetermined systems, which are involutive, and the other class consists of single equations of Goursat type, i.e., single equations of parabolic type whose Monge characteristic systems are completely integrable. Especially in the course of the investigation, he found out the following facts: the symmetry algebras (i.e., the Lie algebra of infinitesimal contact transformations) of the following overdetermined system (involutive system) (A) and the single Goursat type equation (B) are both isomorphic with the 14-dimensional exceptional simple Lie algebra G_2.

$$\text{(A)} \qquad \frac{\partial^2 z}{\partial x^2} = \frac{1}{3}\left(\frac{\partial^2 z}{\partial y^2}\right)^3, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^2 z}{\partial y^2}\right)^2.$$

$$\text{(B)} \qquad 9r^2 + 12t^2(rt - s^2) + 32s^3 - 36rst = 0,$$

where

$$r = \frac{\partial^2 z}{\partial x^2}, \qquad s = \frac{\partial^2 z}{\partial x \partial y}, \qquad t = \frac{\partial^2 z}{\partial y^2}$$

are the classical terminology.

5.1. **Gradation of G_2.** The Dynkin diagram of G_2 is given by

$$\underset{\alpha_1}{\circledcirc} \Lleftarrow \underset{\alpha_2}{\circledcirc}$$

and the set Φ^+ of positive roots consists of six elements (cf. [Bu]):

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

Here $\theta = 3\alpha_1 + 2\alpha_2$ and we have three choices for $\Delta_1 \subset \Delta = \{\alpha_1, \alpha_2\}$. Namely $\Delta_1 = \{\alpha_1\}$, $\{\alpha_2\}$ or $\{\alpha_1, \alpha_2\}$. Then the structure of each (G_2, Δ_1) is described as follows.

(1) $(G_2, \{\alpha_1\})$. We have $\mu = 3$ and Φ^+ decomposes as follows;

$$\Phi_3^+ = \{3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \qquad \Phi_2^+ = \{2\alpha_1 + \alpha_2\},$$
$$\Phi_1^+ = \{\alpha_1, \alpha_1 + \alpha_2\}, \qquad \Phi_0^+ = \{\alpha_2\}.$$

Thus $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_{-1} = 2$, $\dim \mathfrak{g}_{-2} = 1$ and $\dim \mathfrak{g}_0 = 4$. In the following section §5.2, we will see how the regular differential system of this type showed up historically.

(2) $(G_2, \{\alpha_2\})$. We have $\mu = 2$ and Φ^+ decomposes as follows;

$$\Phi_2^+ = \{3\alpha_1 + 2\alpha_2\}, \qquad \Phi_0^+ = \{\alpha_1\},$$
$$\Phi_1^+ = \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\}.$$

Thus $\dim \mathfrak{g}_{-2} = 1$ and $\dim \mathfrak{g}_{-1} = \dim \mathfrak{g}_0 = 4$. Hence this is a contact gradation (cf. §4.4).

(3) $(G_2, \{\alpha_1, \alpha_2\})$. We have $\mu = 5$ and Φ^+ decomposes as follows;

$$\Phi_5^+ = \{3\alpha_1 + 2\alpha_2\}, \qquad \Phi_4^+ = \{3\alpha_1 + \alpha_2\}, \qquad \Phi_3^+ = \{2\alpha_1 + \alpha_2\},$$
$$\Phi_2^+ = \{\alpha_1 + \alpha_2\}, \qquad \Phi_1^+ = \{\alpha_1, \alpha_2\}, \qquad \Phi_0^+ = \emptyset.$$

Namely $(G_2, \{\alpha_1, \alpha_2\})$ is a gradation according to the height of roots and $\mathfrak{g}' = \bigoplus_{p \geqq 0} \mathfrak{g}_p$ is a Borel subalgebra. This case shows up in connection with the Hilbert-Cartan equation([Y5, §1.3]).

5.2. **Classification of Symbol Algebras $\mathfrak{m}$ of Lower Dimension.** In this paragraph, following a short passage from Cartan's paper [C1], let us classify FGLAs $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$ such that $\dim \mathfrak{m} \leqq 5$, which gives us the first invariants towards the classification of regular differential system (M, D) such that $\dim M \leqq 5$.

In the case $\dim \mathfrak{m} = 1$ or 2, $\mathfrak{m} = \mathfrak{g}_{-1}$ should be abelian. To discuss the case $\dim \mathfrak{m} \geqq 3$, we further assume that $\mathfrak{g}_{-1}$ is nondegenerate, i.e., $[X, \mathfrak{g}_{-1}] = 0$ implies $X = 0$ for $X \in \mathfrak{g}_{-1}$. This condition is equivalent to say $\mathrm{Ch}\,(D) = \{0\}$ for regular differential system (M, D) of type $\mathfrak{m}$. When $\mathfrak{g}_{-1}$ is degenerate, $\mathrm{Ch}\,(D)$ is non-trivial, hence at least locally, (M, D) induces a regular differential system (X, D^*) on the lower dimensional space X, where $X = M/\mathrm{Ch}\,(D)$ is the leaf space of the foliation on M defined by $\mathrm{Ch}\,(D)$ and D^* is the differential system on X such that $D = p_*^{-1}(D^*)$. Here $p : M \to X = M/\mathrm{Ch}\,(D)$ is the projection. Moreover, for the following discussion, we first observe that the dimension of $\mathfrak{g}_{-2}$ does not exceed $\binom{m}{2}$, where $m = \dim \mathfrak{g}_{-1}$.

In the case $\dim \mathfrak{m} = 3$, we have $\mu \leqq 2$. When $\mu = 2$, $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the contact gradation, i.e., $\dim \mathfrak{g}_{-2} = 1$ and $\mathfrak{g}_{-1}$ is nondegenerate. In the case $\dim \mathfrak{m} = 4$, we see that $\mathfrak{g}_{-1}$ is degenerate when $\mu \leqq 2$. When $\mu = 3$, we have $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_{-2} = 1$ and $\dim \mathfrak{g}_{-1} = 2$. Moreover it follows that $\mathfrak{m}$ is isomorphic with $\mathfrak{c}^2(1)$ in this case. In the case $\dim \mathfrak{m} = 5$, we have $\dim \mathfrak{g}_{-1} = 4$, 3 or 2. When $\dim \mathfrak{g}_{-1} = 4$, $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the contact gradation. When $\dim \mathfrak{g}_{-1} = 3$, $\mathfrak{g}_{-1}$ is degenerate if $\dim \mathfrak{g}_{-2} = 1$, which implies that $\mu = 2$ and $\dim \mathfrak{g}_{-2} = 2$ in this case. Moreover, when $\mu = 2$, it follows that $\mathfrak{m}$ is isomorphic with $\mathfrak{c}^1(1, 2)$. When $\dim \mathfrak{g}_{-1} = 2$, we have $\dim \mathfrak{g}_{-2} = 1$ and $\mu = 3$ or 4. Moreover, when $\mu = 4$, it follows that $\mathfrak{m}$ is isomorphic with $\mathfrak{c}^3(1)$, where $\mathfrak{c}^3(1)$ is the symbol algebra of the canonical system on the third order jet spaces for 1 unknown function (cf. §3 [Y1]).

Summarizing the above discussion, we obtain the following classification of the FGLAs $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$ such that $\dim \mathfrak{m} \leqq 5$ and $\mathfrak{g}_{-1}$ is nondegenerate.

(1) $\dim \mathfrak{m} = 3 \Longrightarrow \mu = 2$

$$\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathfrak{c}^1(1) : \text{ contact gradation}$$

(2) $\dim \mathfrak{m} = 4 \Longrightarrow \mu = 3$

$$\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathfrak{c}^2(1)$$

(3) $\dim \mathfrak{m} = 5$, then $\mu \leqq 4$

(*a*) $\mu = 4$ $\qquad \mathfrak{m} = \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathfrak{c}^3(1)$

(*b*) $\mu = 3$ $\qquad \mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$
such that $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_{-1} = 2$ and $\dim \mathfrak{g}_{-2} = 1$

(*c*) $\mu = 2$ $\qquad \mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathfrak{c}^1(1, 2)$

(*d*) $\mu = 2$ $\qquad \mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \cong \mathfrak{c}^1(2)$: contact gradation

A notable and rather misleading fact is that, once the dimensions of $\mathfrak{g}_p$ are fixed, the Lie algebra structure of $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$ is unique in the above classification list. Moreover, except for the cases (*b*) and (*c*), every regular differential system (M, D) of type $\mathfrak{m}$ in the above list is isomorphic with the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$ by Darboux's theorem (cf. Corollary 6.6 [Y1]). The first non-trivial situation that cannot be analyzed on the basis of Darboux's theorem occurs in the cases (*b*) and (*c*) (see [C1], [St]). Regular differential systems of type (*b*) and (*c*) are mutually closely related to each other (cf. [Y6, §6.3] and [C1]). We encountered with the type (*b*) fundamental graded Lie algebra as the case (1) of §5.1. in connection with the root space decomposition of the exceptional simple Lie algebra G_2.

As for the diferential system of type (b) above, the following differential system (X, E) on $X = \mathbb{R}^5$ was constructed by E. Cartan [C1];

$$E = \{\, \omega_1 = \omega_2 = \omega_3 = 0 \,\},$$

where

$$\begin{cases} \omega_1 = dx_1 + (x_3 + \frac{1}{2}x_4x_5)\, dx_4, \\ \omega_2 = dx_2 + (x_3 - \frac{1}{2}x_4x_5)\, dx_5, \\ \omega_3 = dx_3 + \frac{1}{2}(x_4\, dx_5 - x_5\, dx_4), \end{cases}$$

and $(x_1, x_2, x_3, x_4, x_5)$ is a coordinate system of $X = \mathbb{R}^5$. We have

$$\begin{cases} d\omega_1 = \omega_3 \wedge \omega_4, \\ d\omega_2 = \omega_3 \wedge \omega_5, \\ d\omega_3 = \omega_4 \wedge \omega_5, \end{cases} \tag{5.1}$$

where $\omega_4 = dx_4$ and $\omega_5 = dx_5$. In this case we may calculate symbol algebras of (X, E) as follows. We take a dual basis $\{X_1, \ldots, X_5\}$ of vector fields on X to a basis of 1-forms $\{\omega_1, \ldots, \omega_5\}$ given above;

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \frac{\partial}{\partial x_3}, \\ X_4 &= \frac{\partial}{\partial x_4} + \frac{1}{2}x_5\frac{\partial}{\partial x_3} - (x_3 + \frac{1}{2}x_4x_5)\frac{\partial}{\partial x_1}, \\ X_5 &= \frac{\partial}{\partial x_5} - \frac{1}{2}x_4\frac{\partial}{\partial x_3} - (x_3 - \frac{1}{2}x_4x_5)\frac{\partial}{\partial x_2}. \end{aligned}$$

Then we calculate, or from (5.1),

$$[X_5, X_4] = X_3, \quad [X_5, X_3] = X_2, \quad [X_4, X_3] = X_1,$$

and $[X_i, X_j] = 0$ otherwise. This implies that $E^{-2} = \{\, \omega_1 = \omega_2 = 0 \,\}$, $E^{-3} = T(X)$ and that (X, E) is isomorphic with the standard differential system of type $\mathfrak{m}_5$, where

$$\mathfrak{m}_5 = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

is the fundamental graded algebra of third kind, whose Maurer-Cartan equation is given by (5.1). Here we note that the Lie algebra structure of $\mathfrak{m}_5$ is uniquely determined by the requirement that $\mathfrak{m}$ is fundamental, $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_{-1} = 2$ and $\dim \mathfrak{g}_{-2} = 1$ (cf. [C1], [T2]). In fact $\mathfrak{m}_5$ is the universal fundamental graded algebra of third kind with $\dim \mathfrak{g}_{-1} = 2$ (see [T2, §3]).

5.3. **G_2-Geometry.** Let $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ be the Standard contact manifold of type G_2, i.e., R-space corresponding to $(G_2, \{\alpha_2\})$. If we lift the action of the exceptional group G_2 to $L(J_{\mathfrak{g}})$, then we have the following orbit decomposition:

$$L(J_{\mathfrak{g}}) = O \cup R_1 \cup R_2,$$

where O is the open orbit and R_i is the orbit of codimension i. Here R_1 and R_2 can be considered as the global model of (B) and (A) respectively. Moreover R_2 is compact and is a R-space corresponding to $(G_2, \{\alpha_1, \alpha_2\})$. From this fact, it becomes possible to describe the PD-manifold $(R; D^1, D^2)$ corresponding to (A) in terms of the R-space corresponding to $(G_2, \{\alpha_1, \alpha_2\})$. In fact R_2 has double fibrations onto $J_{\mathfrak{g}}$ (corresponding to $(G_2, \{\alpha_2\})$) and onto $\tilde{X}$ (corresponding to $(G_2, \{\alpha_1\})$).

Now, utilizing the Reduction Theorem (Theorem 3.3), we will construct the model equation (A) from the standard differential system (X, E) in §5.2, which is the local model corresponding to $(G_2, \{\alpha_1\})$. In fact $(R; D^1, D^2)$ is constructed as follows; $R = R(X)$ is the collection of hyperplanes v in each tangent space $T_x(X)$ at $x \in X$ which contains the fibre $\partial E(x)$ of the derived system ∂E of E.

$$R(X) = \bigcup_{x \in X} R_x \subset J(X, 4),$$
$$R_x = \{v \in \mathrm{Gr}(T_x(X), 4) \mid v \supset \partial E(x)\} \cong \mathbb{P}^1,$$

Moreover D^1 is the canonical system obtained by the Grassmaniann construction and D^2 is the lift of E. Precisely, D^1 and D^2 are given by

$$D^1(v) = \nu_*^{-1}(v) \supset D^2(v) = \nu_*^{-1}(E(x)),$$

for each $v \in R(X)$ and $x = \nu(v)$, where $\nu : R(X) \to X$ is the projection.

We introduce a fibre coordinate λ by $\varpi = \omega_1 + \lambda\omega_2$, where

$$D^1 = \{\, \varpi = 0 \,\} \qquad \text{and} \qquad \partial E = \{\omega_1 = \omega_2 = 0\}.$$

Here $(x_1, \ldots, x_5, \lambda)$ constitutes a coordinate system on $R(X)$. Then we have

$$d\varpi = \omega_3 \wedge (\omega_4 + \lambda\omega_5) + d\lambda \wedge \omega_2,$$
$$\mathrm{Ch}\,(D^1) = \{\ \varpi = \omega_2 = \omega_3 = \omega_4 + \lambda\,\omega_5 = d\lambda = 0\ \},$$
$$D^2 = \{\ \varpi = \omega_2 = \omega_3 = 0\ \} \quad \text{and} \quad \partial D^2 = \{\ \varpi = \omega_2 = 0\ \}.$$

Hence $(R(X); D^1, D^2)$ is a PD-manifold of second order. Now we calculate

$$\begin{aligned}
\varpi &= \omega_1 + \lambda\omega_2 \\
&= dx_1 + \lambda\, dx_2 + (x_3 + \frac{1}{2}x_4x_5)dx_4 + \lambda\,(x_3 - \frac{1}{2}x_4x_5)dx_5 \\
&= d(x_1 + \lambda\, x_2) - x_2 d\lambda + (x_3 + \frac{1}{2}x_4x_5)(dx_4 + \lambda\, dx_5) - \lambda\, x_4x_5dx_5 \\
&= d(x_1 + \lambda\, x_2) - \{x_2 + x_5(x_3 + \frac{1}{2}x_4x_5)\}d\lambda + (x_3 + \frac{1}{2}x_4x_5)d(x_4 + \lambda\, x_5) - \lambda\, x_4x_5dx_5.
\end{aligned}$$

Moreover we have

$$\begin{aligned}
\lambda\, x_4x_5dx_5 &= \frac{1}{2}\lambda\, x_4dx_5^2 = \frac{1}{2}\{d(\lambda\, x_4x_5^2) - x_4x_5^2d\lambda - \lambda\, x_5^2dx_4\,\} \\
&= \frac{1}{2}\{d(\lambda\, x_4x_5^2) - x_4x_5^2d\lambda - \lambda\, x_5^2d(x_4 + \lambda\, x_5) + \lambda\, x_5^2d(\lambda\, x_5)\,\} \\
&= \frac{1}{2}\{d(\lambda\, x_4x_5^2) + (\lambda\, x_5^3 - x_4x_5^2)d\lambda - \lambda\, x_5^2d(x_4 + \lambda\, x_5) + \lambda^2\, x_5^2dx_5\,\} \\
&= \frac{1}{2}\{d(\lambda\, x_4x_5^2 + \frac{1}{3}\lambda^2x_5^3) - (\frac{2}{3}\lambda\, x_5^3 - \lambda\, x_5^3 + x_4x_5^2)d\lambda - \lambda\, x_5^2d(x_4 + \lambda\, x_5)\,\}.
\end{aligned}$$

Thus we obtain

$$\varpi = d(x_1+\lambda\, x_2-\frac{1}{2}\lambda\, x_4x_5^2-\frac{1}{6}\lambda^2\, x_5^3)-(x_2+x_3x_5+\frac{1}{6}\lambda\, x_5^3)d\lambda+(x_3+\frac{1}{2}x_4x_5+\frac{1}{2}\lambda\, x_5^2)d(x_4+\lambda\, x_5).$$

We put

$$\begin{cases} z = x_1 + \lambda\, x_2 - \dfrac{1}{2}\lambda\, x_4 x_5^2 - \dfrac{1}{6}\lambda^2\, x_5^3, \\ x = \lambda, \\ y = x_4 + \lambda\, x_5, \\ p = x_2 + x_3 x_5 + \dfrac{1}{6}\lambda\, x_5^3, \\ q = -(x_3 + \dfrac{1}{2} x_4 x_5 + \dfrac{1}{2}\lambda\, x_5^2). \end{cases}$$

Then

$$D^1 = \{\, dz - p\, dx - q\, dy = 0 \,\},$$

and (x, y, z, p, q) constitutes a canonical coordinate system on $J = R(X)/\mathrm{Ch}\,(D^1)$. Putting $x_5 = a$, we solve

$$\begin{cases} x_4 = y - xa, \\ x_3 = -q - \dfrac{1}{2}(y - xa)a - \dfrac{1}{2} xa^2 = -q - \dfrac{1}{2} ya, \\ x_2 = p + qa + \dfrac{1}{2} ya^2 - \dfrac{1}{6} xa^3, \\ x_1 = z - x(p + qa + \dfrac{1}{2} ya^2 - \dfrac{1}{6} xa^3) + \dfrac{1}{2} x(y - xa)a^2 + \dfrac{1}{6} x^2 a^3, \\ \qquad = z - xp - xqa - \dfrac{1}{6} x^2 a^3. \end{cases}$$

Then, from

$$\begin{cases} x_4 x_5 = ya - xa^2, \\ x_3 - \dfrac{1}{2} x_4 x_5 = -q - \dfrac{1}{2} ya - \dfrac{1}{2}(y - xa)a = -q - ya + \dfrac{1}{2} xa^2, \end{cases}$$

we calculate

$$\begin{aligned} \omega_3 &= -d(q + ya - \frac{1}{2} xa^2) + (y - xa)da = -dq + \frac{1}{2} a^2 dx - ady, \\ \omega_2 &= d(p + qa + \frac{1}{2} ya^2 - \frac{1}{6} xa^3) - (q + ya - \frac{1}{2} xa^2)da = dp + adq + \frac{1}{2} a^2 dy - \frac{1}{6} a^3 dx \\ &= a(dq - \frac{1}{2} a^2 dx + ady) + dp + \frac{1}{3} a^3 dx - \frac{1}{2} a^2 dy \\ &= dp + \frac{1}{3} a^3 dx - \frac{1}{2} a^2 dy - a\omega_3. \end{aligned}$$

Putting $a = -t$, we obtain

$$D^2 = \{\varpi = \hat{\omega}_2 = \hat{\omega}_3 = 0\},$$

where

$$\varpi = dz - pdx - qdy, \quad \hat{\omega}_2 = dp - \frac{1}{3} t^3 dx - \frac{1}{2} t^2 dy, \quad \hat{\omega}_3 = dq - \frac{1}{2} t^2 dx - tdy.$$

This implies

$$R(X) = \{\ r = \frac{1}{3} t^3, \quad s = \frac{1}{2} t^2\ \} \subset L(J),$$

in terms of the canonical coordinate (x, y, z, p, q, r, s, t) of $L(J)$.

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