Differential Invariants and Hidden Symmetry

Irina YEHORCHENKO

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: iyegorch@imath.kiev.ua

Abstract

We show an algorithm for description of classes of equations having specific conditional or hidden symmetry, and/or reducible with a specific ansatz. We consider reductions that are due to Lie, conditional and Type II hidden symmetry. We also discuss relations between the concepts of hidden and conditional symmetry. As examples, we describe general classes equations having hidden and conditional symmetry under rotations and boosts in the Lorentz and Euclid groups.

1 Background Concepts

One of the key problems within the field of symmetry analysis of differential equations is description of equations with particular pre-determined symmetry properties. Choosing equations with particular symmetry properties may mean that we have equations with solutions of some particular structure.

It was proved in [1] that reduction of PDE (direct ansatz approach) is equivalent to the non-classical (conditional) symmetry. Thus, the conditional invariance of a differential equation under an involutive family of first-order differential operators Q_a is equivalent to possibility of reduction of this equation by means of the ansatz corresponding to this family of operators.

So, description of all equations from some class having a particular conditional symmetry (we will deal, speaking more rigourously, with Q-conditional symmetry, as defined in [2]) will give all equations from this class that can be reduced by means of the specific ansatz corresponding to conditional symmetry operators being considered.

Definition 1. *The equation $F(x, u, \underset{1}{u}, \ldots, \underset{l}{u}) = 0$, where $\underset{k}{u}$ is the set of all kth-order partial derivatives of the function $u = (u^1, u^2, \ldots, u^m)$, is called Q-conditionally invariant [2] under the operator*

$$Q = \xi^i(x, u)\partial_{x_i} + \eta^r(x, u)\partial_{u^r}$$

if there is an additional condition

$$Qu = 0, \tag{1}$$

such that the system of two equations $F = 0$, $Qu = O$ is invariant under the operator Q. All differential consequences of the condition $Qu = 0$ shall be taken into account up to the order $l - 1$.

Note that we will need such definition of conditional symmetry if we specifically want to describe equations reducible with some form of ansatz; however, we can use a more general definition of conditional symmetry and describe classes of equations having such symmetry.

Definition 2. *The equation $F(x, u, \underset{1}{u}, \ldots, \underset{l}{u}) = 0$, where $\underset{k}{u}$ is the set of all kth-order partial derivatives of the function $u = (u^1, u^2, \ldots, u^m)$, is called conditionally invariant [2], if there is an additional condition $G(x, u, \underset{1}{u}, \ldots, \underset{l_1}{u}) = 0$ such that the system of two equations $F = 0$, $G = 0$ is invariant under some operator*

$$Q = \xi^i(x, u)\partial_{x_i} + \eta^r(x, u)\partial_{u^r}$$

that is not a Lie invariance operator of the equation $F = 0$. All differential consequences of the condition $G = 0$ shall be taken into account up to the order $l - l_1$.

In this paper we give an outlook of description of partial differential equations possessing certain conditional and hidden symmetries.

Group classification for classes of differential equations is aimed at identification of equations having wider symmetries than the equations of the class in general. For an overview of the group classification problems and the extensive list of related references see [3]. Usually two types of such problems are considered – finding the equations within a general class that are invariant under specific symmetry group, and description of all symmetries (up to appropriately chosen equivalence) of equations that belong

to the specific class. On the basis of the known algorithms for group classification of differential equations in the Lie's sense we develop approaches for a systematic description of classes of nonlinear PDEs that display conditional and hidden symmetry.

The definition of conditional differential invariants presented below refers to the Definition of the conditional symmetry.

Definition 3. *A class of equations can be regarded as general if any local transformations of dependent and independent variables transform the equations into an equation within the same class, e.g. the class of all kth-order PDEs $F = F(x, u, u_1, \ldots, u_k) = 0$ with x, u being respectively n- and m-dimensional independent and dependent variables, u_r being the set of all r-th-order partial derivatives of the function $u = \left(u^1, u^2, \ldots, u^m\right)$.*

Group classification even with respect to the Lie symmetry for the general classes is usually an overwhelming task, and, to our knowledge, such problem was completely solved only for single ordinary differential equations by S. Lie [4]. A restricted, but practically important problem for the general classes of equations would be description of all equations within the class invariant under some specified symmetry group that can be done by describing all differential invariants for such group. Similarly, description of equations having specified Lie symmetry and specified conditional symmetry may be done by means of conditional differential invariants, as shown in [5].

For a more specific class, it may be possible to perform a full group classification of the system that consists of the original equation together with the reduction conditions of the type $Q_a[u] = 0$ (with appropriate prolongations of the conditions).

Definition 4. [5] *A function $F(x, u, u_1, \ldots, u_k)$ is a conditional differential invariant of the operator Q, if under the conditions $G(x, u, u_1, \ldots, u_r) = 0$ the relations $Q[F] = 0$, $Q[G] = 0$ are satisfied. We take prolongations of the operators of the order $\max(k, r)$.*

A set of invariants of the order $r \leq k$ of the operator Q with the conditions $G = 0$ is called a *generating set* of the kth-order conditional differential invariants of the algebra Q if all other invariants can be represented

as functions of invariants in this set.

Invariants in such generating set may be both absolute invariants of Q and G-conditional of the form $G^{(l)} \times R_{(l)}$, where $G^{(l)}$ are derivatives of G of the order $l \leq k - r$ and $R_{(l)}$ are arbitrary functions determined on the manifold $G^{(k)} = 0$ for all values of k.

The number of functionally independent Q-absolute invariants in the generating set of conditional differential invariants can be calculated similarly to the number of invariants in a functional basis of absolute differential invariants, as $s - 1$, where s is the number of variables in the set $x, u, \underset{1}{u}, \ldots, \underset{k}{u}$. Number of independent purely G-conditional invariants is equal to the number of independent conditions of the type $G^{(l)} = 0$ and their differential consequences.

In some cases we would be able to construct a functional basis of conditional invariants, i.e. the maximal set of functionally independent conditional invariants. That is possible e.g. in the case when we put a requirement that our conditional invariants should be also absolute invariants of some Lie algebra L, and additional conditions $G = 0$ in the Definition 3 and their relevant differential consequences are not invariant under L.

2   Relation of Hidden Symmetry and Conditional Symmetry

Further we will treat Type II hidden symmetry as a partial case of conditional symmetry, and discuss a systematic approach to description of equations with some specific hidden symmetry, or equations that may have such symmetries, within the lines of such conditional symmetry approach.

A very close relation of the hidden symmetry (for some initial papers on the subject see [6]) to conditional symmetry (for definitions and examples see e.g. [7]) is well-known. The concept of "hidden symmetry" has quite a few different meanings in various contexts, and it is usually a symmetry not obtainable by some standard and straightforward procedure applicable to the models in this context. This term shares the usage of other related terms like "conditional symmetry", "approximate symmetry", and "symmetry" or "invariance" when the same words may denote rather different concepts.

4

Here we will consider hidden symmetry of partial differential equations similarly to Type II hidden symmetry of ordinary differential equations generally within the context of [6]. With respect to ODE such symmetry arises as symmetry of equations with reduced order that is not a symmetry of the original equations. In the same way, for a PDE it is symmetry of the reduced equation (with reduced number of independent variables) not present in the original equation. However, we would like to point out that we will consider all possible reductions to find hidden symmetries, not only symmetry reductions.

Definition 5. *A differential equation is said to have hidden symmetry under an operator X if after the process of reduction of the number of independent variables the resulting reduced equation is invariant under the operator X_1 (being the projection of the operator X in new variables) while the original equation is not invariant under the operator X.*

Such symmetry may be "classical" in the sense that full hidden symmetry of either ODE or PDE may be found by consecutive symmetry reduction of the original equation and investigation of Lie symmetries of the reduced equations, as provided by L. Ovsyannikov's *Submodels* programme [8]. "Symmetry" or "Lie symmetry" is determined in accordance to the procedures that may be found e.g. in [9, 10].

An interesting discussion of the origin and nature of the Type II hidden symmetry was presented in [11]. To see clearly the origin of hidden symmetries, it is necessary always to keep in mind the additional conditions (representing Lie or non-classical symmetries) that resulted in reduction to the new equations having new symmetries that are hidden symmetries for an original equation.

The additional symmetry under the operator X_1 of the reduced equation (hidden symmetry under the operator X for the original equation) turns out to be a conditional symmetry of the original equation under conditions $Q_a[u] = 0$ ($Q_a[u]$ designate characteristics of the vector field Q_a) with all appropriate differential consequences. Note that X_1 is a Lie symmetry of the reduced equation, and we do not add the condition $X[u] = 0$ to the set of conditions, so such operator will not present a proper Q-conditional symmetry in the sense of [2].

Q-conditional symmetry can also be hidden – that is being a new Q-conditional symmetry of the reduced equation. For examples of such symmetries see e.g. [13].

When we look for equations with fully defined hidden symmetries, we can describe such equations by means of conditional differential invariants - both reduction conditions and hidden symmetry operators should be used as conditions for such invariants.

An algorithm for group classification with respect to hidden symmetry in the situation when hidden symmetries are not known from the start:

Step 1. Obtain reduced equations for the class of initial PDE, using possible Lie and conditional symmetries - this step requires a standard group classification with respect to Lie and conditional symmetries.

Step 2. Perform group classification of the reduced equations. This is the step when new symmetries of the reduced equations may be found (and may be not - in this case the class of equations will have no hidden symmetries). This step involves finding of equivalence group of the class and of subclasses.

Step 3. Multiply inequivalent invariant reduced equations by means of transformations from the equivalence group of this class (or subclasses).

Step 4. Go back to the original class of equations: find equations from the initial class of PDE corresponding to the multiplied reduced equations.

Step 5. Find all inequivalent equations with respect to transformations from the equivalence group of the initial class of PDE.

A nontrivial hidden symmetry for partial differential equations stems from the reduced equations having wider equivalence group than the original equations (see [14]). So group classification of the classes of equations with respect to hidden symmetry involves study of equivalence groups of such classes, in the similar way as it is done for classification with respect to the Lie symmetry.

"Simple" hidden symmetries (Lie symmetries of reduced equations) for a particular class of equations can be found by means of consecutive Lie reductions and consecutive finding Lie symmetries of the reduced equations. Group classification of such class with respect to hidden symmetries of reduced equations will involve description of all possible reductions and

group classification of the respective classes of reduced equations.

In [12] we considered a rather simple example of the nonlinear wave equation in two spatial dimensions

$$\Box u = f(t, x, y, u) \tag{2}$$

Here we use the usual notations for partial derivatives and the d'Alembert operator.

Group classification of equation (2) with respect to hidden symmetries for reduction by means of the operator ∂_y was presented. Such reduction leads to the two-dimensional wave equation $u_{tt} - u_{xx} = f(t, x, u)$. The next step is the usual group classification of the reduced equation. It was given in [9] ($f_{uu} = 0$) and in [15] ($f_{uu} \neq 0$). Such group classification was performed up to transformations from the equivalence group of the equation.

Let us note that conditional symmetry of equation (2) was studied in [16]. "Extension" of dimensions of the found equations with nontrivial conditional symmetries in this class will produce new multidimensional equations with hidden symmetries.

3   Example of a General Class: Hidden Symmetry with Respect to Translations

For simplicity we use an example of a general class of all second-order PDE for a scalar function u and three independent variables t, x, y. This class includes many physically interesting evolution and wave equations. The ideas presented can be easily extended to equations with larger number of dimensions.

Such class includes all equations of the form

$$F = F(t, x, y, u, \underset{1}{u}, \underset{2}{u}) = 0. \tag{3}$$

We will start with a straightforward example - description of all such equations having Lie symmetry with respect to the operator ∂_x and hidden symmetry with respect to the operator ∂_y after reduction by means of the operator ∂_x. The condition of such Lie and hidden symmetry, according

to Definition 1, is invariance of the equation (3) under the operator ∂_y on condition that $u_x = 0$:

$$\partial_x F\big|_{F=0} = 0, \qquad \partial_y F\big|_{F=0,\ u_x=0} = 0. \tag{4}$$

The general solution of the condition (4) will be a function of all invariants of the operators ∂_x and ∂_y, that is t, u, u_t, u_x, u_y(being an absolute invariant of ∂_x), and of the conditional invariants

$$q^1 = u_x R^1(t, y, u, \underset{1}{u}, \underset{2}{u}), \tag{5}$$
$$q^2 = u_{xt} R^2(t, y, u, \underset{1}{u}, \underset{2}{u}),$$
$$q^3 = u_{xx} R^3(t, y, u, \underset{1}{u}, \underset{2}{u}),$$
$$q^4 = u_{xy} R^4(t, y, u, \underset{1}{u}, \underset{2}{u}), \tag{6}$$

where R^k are arbitrary functions that is reasonably determined on the relevant manifolds $u_x = 0$, $u_{xt} = 0$, $u_{xx} = 0$, $u_{xy} = 0$:

$$F(q^k, t, u, \underset{1}{u}, \underset{2}{u}) = 0, \tag{7}$$

F has to be a function of the invariants of the hidden symmetry operator on the manifold determined by the reduction condition, and have arbitrary form elsewhere. Note that the functions q^k in (7) are not entirely arbitrary: we cannot take e.g.

$$q^1 = R^1(y, t, u, \underset{1}{u}, \underset{2}{u}) = u_x \frac{\widetilde{R}^1}{u_x}$$

as such $R^1 = \frac{\widetilde{R}^1}{u_x}$ will be in the general case undetermined on the manifold $u_x = 0$.

Equation (7) is reduced by means of the operator ∂_x to the equation

$$F_1(t, u, u_t, u_y, u_{tt}, u_{ty}, u_{yy}) = 0$$

that is invariant with respect to ∂_y. If $R_y^k \neq 0$ in at least one expression for q^k in (7), then this equation is not invariant with respect to the operator ∂_y, and the relevant hidden symmetry is a proper hidden symmetry.

This example is easily generalised for larger order or equations or number of reduction operators.

More specific examples of equations with hidden translational symmetry are listed below:

$$u_t + u_x K_1(t, y, u) + u_y K_2(t, u) + u_{xx} + u_{yy} = 0;$$
$$u_{tt} - (K_1(t, y, u)u_x)_x - (K_2(t, u)u_y)_y = 0.$$

These equations can be reduced by means of the operator ∂_x to new equations invariant under ∂_y - so they have hidden symmetry with respect to the operator ∂_y.

4 Equations Reducible using Radial Variables

As it was shown e.g. in [13] and then in [17], reduction using radial variables often results in the reduced equation with new symmetries as compared to the initial equation, thus contributing to description of its hidden symmetries.

In this section we will describe all equations of the type (3) that can be reduced by means of radial variables

$$r = x^2 + y^2 \tag{8}$$
$$\rho = t^2 - x^2 - y^2. \tag{9}$$

Reduction of equation (3) by means of the new variable (8) is equivalent to its conditional invariance under the rotation operator

$$J = x\partial_y - y\partial_x. \tag{10}$$

Conditional differential invariants with the condition

$$xu_y - yu_x = 0. \tag{11}$$

may be chosen as follows:

$$t, u, u_t, u_{tt}, r = x^2 + y^2, xu_x + yu_y, u_x^2 + u_y^2, u_{xx} + u_{yy}, \tag{12}$$
$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}, xu_x u_{xx} + (xu_y + yu_x)u_{xy} + yu_y u_{yy},$$
$$u_{xt}^2 + u_{yt}^2, xu_{xt} + yu_{yt},$$
$$\frac{u_k}{x_k}, \frac{u_{kt}}{x_k}, \frac{u_{kl}}{x_k x_l} - \epsilon_{kl}\frac{u_k}{x_k^3} \tag{13}$$

We used notations $x_1 = x, x_2 = y, u_1 = u_x, u_2 = u_y$ etc.; $\epsilon_{kl} = 1$ if $k = l$ or $\epsilon_{kl} = 0$ if $k \neq l$.

Invariants (12) represent a functional basis of absolute differential invariants for the operator (10) (see e.g. [18]), and invariants (13) are proper conditional differential invariants under condition (11). It is easy to check directly that they are really differential invariants under such condition. The listed proper conditional differential invariants do not actually represent a functional basis: e.g. from (11) $\frac{u_x}{x} = \frac{u_y}{y}$, but we adduced all such invariants just to show their general structure.

The general form of the equation (3) reducible with the ansatz

$$u = \phi(t, r), \tag{14}$$

will be

$$F(I_A, \frac{u_k}{x_k}, \frac{u_{kt}}{x_k}, \frac{u_{kl}}{x_k x_l} - \epsilon_{kl} \frac{u_k}{x_k^3}) = 0, \tag{15}$$

where I_A is the functional basis of absolute differential invariants (12), and the remaining variables are represented by proper conditional differential invariants.

It is easy to check that equation (15) can be reduced by means of the ansatz (14) to the form

$$f(t, r, \phi, \phi_t, \phi_r, \phi_{tt}, \phi_{tr}, \phi_{rr}) = 0, \tag{16}$$

and the class (16) may be studied to find equations having new symmetries. Equations with hidden symmetries then will be described by conditional differential invariants under (11) and these new symmetries.

Reduction of equation (3) by means of the new variable (9) is equivalent to its conditional invariance under the operators of Lorentz algebra

$$J_{01} = t\partial_x + x\partial_t, \qquad J_{02} = t\partial_y + y\partial_t, \qquad J = x\partial_y - y\partial_x. \tag{17}$$

Conditional differential invariants with the conditions

$$tu_x + xu_t = 0, \qquad tu_y + yu_t = 0, \qquad xu_y - yu_x = 0. \tag{18}$$

may be chosen as follows:

$$u, x_\mu x_\mu, x_\mu u_\mu, u_\mu u_\mu, \Box u, u_\mu u_{\mu\nu} u_\nu, u_\mu u_{\mu\nu} u_{\nu\alpha} u_\alpha, \qquad (19)$$

$$u_{\mu\nu} u_{\nu\alpha} u_{\mu\alpha}, x_\mu u_{\mu\nu} u_\nu, x_\mu u_{\mu\nu} u_{\nu\alpha} u_\alpha,$$

$$\frac{u_\mu}{x_\mu}, \frac{u_{\mu\nu}}{x_\mu x_\nu} - g_{\mu\nu} \frac{u_\mu}{x_\mu^3}. \qquad (20)$$

Here μ, ν, α take values from 0 to 2, and we used notations $x_0 = t, x_1 = x, x_2 = y, u_0 = u_t, u_1 = u_x, u_2 = u_y$ etc.; $g_{\mu\nu} = (1, -1, -1)$.

Invariants (19) represent a functional basis of absolute differential invariants for the operator (17), and invariants (20) are proper conditional differential invariants under condition (18). It is easy to check directly that they are really differential invariants under such condition. The listed proper conditional differential invariants do not actually represent a functional basis: e.g. from (18) $\frac{u_x}{x} = \frac{u_y}{y}$, but we adduced such invariants just to show their general structure.

The general form of the equation (3) reducible with the ansatz

$$u = \phi(\rho), \qquad (21)$$

will be

$$F(I_A, \frac{u_\mu}{x_\mu}, \frac{u_{\mu\nu}}{x_\mu x_\nu} - g_{\mu\nu} \frac{u_\mu}{x_\mu^3}) = 0, \qquad (22)$$

where I_A is the functional basis of absolute differential invariants (19), and the remaining variables are represented by proper conditional differential invariants.

It is easy to check that equation (22) can be reduced by means of the ansatz (21) to the form

$$f(\rho, \phi, \phi', \phi'') = 0, \qquad (23)$$

and the class (23) may be studied to find equations having new symmetries. Equations with hidden symmetries then will be described by conditional differential invariants under (18) and these new symmetries.

The presented results can be naturally extended to arbitrary number of space dimensions.

An example of a nonlinear wave equation having conditional symmetry with respect to the Lorentz group of the type (17) with n space dimensions was given in [19]:

$$\Box u = \frac{\lambda_0 u_0^2}{x_0^2} + \frac{\lambda_1 u_1^2}{x_1^2} + \cdots + \frac{\lambda_n u_n^2}{x_n^2}.$$

It is easy to see that this equation is actually constructed with first-order conditional differential invariants of the type $\frac{u_\mu}{x_\mu}$.

5   Conclusions

We presented examples related to description of equations with some specified conditional and hidden symmetry, as well as algorithms for group classification of classes of equations with respect to conditional and hidden symmetry and possible reductions.

Further research in this direction shall involve classification with respect to Lie and conditional symmetry of the remarkable classes of reduced equations.

References

[1] Zhdanov R.Z., Tsyfra I.M. and Popovych R.O., A precise definition of reduction of partial differential equations, *J. Math. Anal. Appl.*, 1999, V.238, N 1, 101–123.

[2] Fushchych W.I., Shtelen W.M. and Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kyiv, Naukova Dumka, 1989 (in Russian); Kluwer Publishers, 1993 (in English).

[3] Popovych R.O. and Ivanova N.M., New results on group classification of nonlinear diffusion-convection equations, *J. Phys. A*, 2004, V.37, 7547; math-ph/0306035.

[4] Lie S., Über die Integration durch bestimmte Integrale von einer Klasse linear partieller Differentialgleichung, *Arch. for Math.*, 1881, V.6, N 3, 328–368 (Translation by N.H. Ibragimov: Lie S., On integration of a class of linear partial differential equations by means of definite integrals, in CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 2, 473–508).

[5] Yehorchenko I.A., Differential invariants and construction of conditionally invariant equations, Symmetry in nonlinear mathematical physics, in Proceedings of Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (9–15 July, 2001, Kyiv), Editors A.G. Nikitin, V.M. Boyko and R.O. Popovych, Kyiv, Insitute of Mathematics, 2002, V.43, Part 1, 256–262; math-ph/0304029.

[6] Abraham-Shrauner B., Hidden symmetries and nonlocal group generators for ordinary differential equations, *IMA J.Appl.Math.*, 1996, V.56, 235–252; Abraham-Shrauner B., Hidden symmetries, first integrals and reduction of order of nonlinear ordinary differential equations, *J. Nonlin. Math. Phys.*, 2002, V.9, Suppl. 2, 1–9.

[7] Bluman G.W. and Cole J.D., The general similarity solution of the heat equation, *J. Math. Mech.*, 1969, V.18, 1025–1042; Olver P.J. and Rosenau P., The construction of special solutions to partial differential equations, *Phys. Lett. A*, 1986, V.114, 107–112; Fushchych W.I. and Zhdanov R.Z., Symmetry and exact solutions of nonlinear spinor

equations, *Phys. Reports*, 1989, V.172, 123–174; Clarkson P. and Kruskal M.D., New similarity solutions of the Boussinesq equation, *J. Math. Phys.*, 1989, V.30, 2201–2213; Levi D. and Winternitz P., Non-classical symmetry reduction: example of the Boussinesq equation, *J. Phys. A*, 1989, V.22, 2915–2924.

[8] Ovsyannikov L.V., Program SUBMODELS. Gas dynamics, *Appl. Math. Mech.*, 1994, V.58, N 4, 30–55.

[9] Ovsyannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982.

[10] Olver P., Application of Lie groups to differential equations, New York, Springer Verlag, 1987; Bluman G.W. and Kumei S., Symmetries and differential equations, New York, Springer Verlag, 1989.

[11] Abraham-Shrauner B. and Govinder K.S. On the origins of symmetries of partial differential equations: the example of the Kortewegde Vries equation, *J. Nonlin. Math. Phys.*, 2008, V.15, Suppl.1, 60-68.

[12] Yehorchenko I. A., Group classification with respect to hidden symmetry, in Proceedings of Fifth International Conference "Symmetry in Nonlinear Mathematical Physics" (June 23-29, 2003, Kyiv), Editors A.G. Nikitin, V.M. Boyko, R.O. Popovych and I.A. Yehorchenko, *Proceedings of Institute of Mathematics, Kyiv*, 2004, **50**, Part 1, 290–297.

[13] Yehorchenko I. A., Vorobyova A. I., Sets of conditional symmetry operators and exact solutions for wave and generalised heat equations, in Proceedings of Fifth International Conference "Symmetry in Nonlinear Mathematical Physics" (June 23-29, 2003, Kyiv), Editors A.G. Nikitin, V.M. Boyko, R.O. Popovych and I.A. Yehorchenko, *Proceedings of Institute of Mathematics, Kyiv*, 2004, **50**, Part 1, 298–303.

[14] Lisle I.G., Equivalence transformations for classes of differential equations, Thesis, University of British Columbia, 1992; http://www.ise.canberra.edu.au/mathstat/StaffPages/LisleDissertation.pdf;
Lisle I.G. and Reid G.J., Symmetry classification using noncommutative invariant differential operators, *Foundations of Computational Mathematics*, 2006, V.6, 353–386.

[15] Lahno V., Zhdanov R. and Magda O., Group classification and exact solutions of nonlinear wave equations, *Acta Appl. Math.*, 2006, V.251, 253-313.

[16] Yehorchenko I.A. Conditional Symmetry and reductions for the two-dimensional nonlinear wave equation. I. general case. arXiv:1010.4913

[17] Abraham-Shrauner B., Govinder K.S. and Arrigo D.J. Type-II hidden symmetries of the linear 2D and 3D wave equations, *J. Phys. A*, 2006, V.39, 5739-5747; Type II hidden symmetries through weak symmetries for some wave equations, *Comm.Nonlin.Sci.and Num.Simulation* 2010, V. 15, 291–299.

[18] Fushchych W.I. and Yegorchenko I.A., Second-order differential invariants of the rotation group $O(n)$ and of its extension $E(n)$, $P(l, n)$, *Acta Appl. Math.*, 1992, V.28, 69–92.

[19] Fushchych W.I. and Tsyfra I.M., On a reduction and solutions of the nonlinear wave equations with broken symmetry, *J. Phys. A*, 1987, V.20, L45–L48.