Differential Invariants for Infinite-Dimensional Algebras[1]

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Abstract

We present an approach for construction of functional bases of differential invariants for some infinite-dimensional algebras with coefficients of generating operators depending on arbitrary functions. An example for the infinite-dimensional Poincare-type algebra is given.

1 Introduction

Our studies in differential invariants (see e.g. [3]) started from the problem of description of equations invariant under certain algebras.If we speak about single equations, we can say that all equations invariant under certain algebras can be presented as functions of absolute differential invariants.

The theory and methods for searching differential invariants of finite-dimensional Lie algebras are well-developed. See for the relevant definitions e.g. the classical books [7, 8].

All absolute invariants can be presented as functions of invariants from a functional basis. The number of invariants (of a certain particular order r) is determined as difference between the number of all derivatives up to the r-th order and both dependent and independent variables, and of the rank of r-th Lie prolongation of the basis operators of the algebra under consideration.

The case of infinite-dimensional algebras is more complicated, as their bases contain infinite (countable) number of operators (e.g. Virasoro and Kac-Moody algebras), or contain infinitesimal operators having arbitrary functions as coefficients.

However, it appears that despite the name "infinite-dimensional" ranks of r-th Lie prolongations of basis operators are finite for each fixed r. Unlike finite-dimensional algebras these ranks do not stabilise, or do not reach any fixed value. Finiteness of such rank is discussed in [6]. Such finiteness is obvious as the rank of the r-th prolongation of the basis operators cannot exceed the number of all derivatives up to the r-th order and both dependent and independent variables.

Calculation of differential invariants for infinite-dimensional algebras is specifically interesting in application to equivalence algebras of classes of differential equations, as knowledge of such invariants gives criteria for equivalence of different equations from the same class with respect to local transformations of variables. In the case of ODE knowledge of such invariants gives both necessary and sufficient conditions of equivalence [1].

In a number of papers by N.H. Ibragimov and his coauthors (see e.g. [4, 5]) invariants for equivalence algebras of classes of differential equations are sought for directly.

Here we suggest a systematic procedure that allows considerable simplification of these calculations. Instead of arbitrary functions usually found in basis operators of equivalence algebras, we use expansions of these functions into Taylor series. We have to remember that we deal with

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arbitrary functions. Though they have to be infinitely differentiable (due to commutation condition in the definition of the Lie algebra), that does not mean that they are analytical. However, for the purpose of calculation of differential invariants of infinite-dimensional algebras we can reasonably limit our consideration by analytical functions in coefficients of basis operators, and with finite number of such arbitrary functions in coefficients of basis operators.

Using expansion of coefficients into series allows replacement of operators with arbitrary functions with infinite series of infinitesimal operators without such arbitrary functions. This approach allows much more straightforward calculation of the prolongations' rank (in some cases the rank is equal to the number of variables and derivatives, and then it is easy to see without any further calculations that there are no absolute invariants of the respective order).

Statement. For a fixed order r there is a functional basis of any Lie algebra, including infinite-dimensional algebras with finite number of such arbitrary functions in coefficients of basis operators or with countable infinite sequences of basis operators with no arbitrary functions.

2 Differential Invariants for Infinite-Dimensional Poincaré-Type Algebra

We will illustrate our approach to searching absolute differential invariants of infinite-dimensional algebras by the example of infinite-dimensional Poincaré-type algebra that is an invariance algebra of the eikonal equation.

It is well-known that the simplest first-order relativistic equation — the eikonal or Hamilton equation, for n independent space variables x_n and time variable x_0, and scalar dependent variable u,

$$u_\alpha u_\alpha \equiv u_0^2 - u_1^2 - \cdots - u_n^2 = 0 \tag{1}$$

is invariant under the infinite-dimensional algebra generated by the operators [2]

$$X = (b^{\mu\nu} x_\nu + a^\mu)\partial_\mu + \eta(u)\partial_u, \tag{2}$$

$-b^{\mu\nu} = b^{\nu\mu}$, a^μ, η being arbitrary differentiable functions on u, $\partial_\mu = \partial/\partial x_\mu$. Usual summation is implied over the repeated Greek indices: $u_\mu u_\mu = u_0^2 - u_1^2 - \cdots - u_n^2$. Equation (1) is widely used e.g. in geometrical optics.

Here we construct differential invariants of orders 1 and 2. Instead of the operators (2), after expansion of the functions $-b^{\mu\nu} = b^{\nu\mu}$, a^μ, η into Taylor series, we can consider the following sequences of operators:

$$J^k_{\mu\nu} = u^k(x_\mu\partial_\nu - x_\nu\partial_\mu), \quad P^k_\mu = u^k\partial_\mu, \quad P^k_u = u^k\partial_u. \tag{3}$$

For the order 1 we have $n+2$ variables and $n+1$ first derivatives, and the rank of the first prolongation of (2) is equal to $2n+3$. There is no absolute invariants, and one obvious relative invariant $u_\mu u_\mu$. It is interesting to note that for the first order invariance under (2) implies invariance under the dilation operator $D = x_\mu\partial_\mu$.

The generating set of operators for the first prolongation will be

$$J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu,\ D,\ P^0_u,\ P^0_\mu.$$

Rank of the first prolongation of this set is $2n+3$, and invariance under prolongations of these operators is equivalent to invariance under the first prolongation of the algebra (3).

Finding such simple generating set (with the rank and number of operators equal to the rank of the first prolongation) makes finding invariants much simpler, and would allow using computer software to do so).

The tensor of the rank 2 [3]

$$\theta_{\mu\nu} = u_\mu u_{\lambda\nu} u_\lambda + u_\nu u_{\lambda\mu} u_\lambda - u_\mu u_\nu u_{\lambda\lambda} - u_\lambda u_\lambda u_{\mu\nu} \tag{4}$$

is covariant under the algebra (3) (for simplicity of the definition, we say that a tensor is covariant under a certain algebra, if all its convolutions are relative or absolute invariants of this algebra).

Covariance can be checked directly by application of the Lie algorithm.

Calculation of the rank of the second prolongation of (3) gives that there will be n second-order invariants:

$$S_k/(u_\mu u_\mu)^{(3/2)k}, \tag{5}$$

where $S_k = \theta_{\mu_1\mu_2}\theta_{\mu_2\mu_3}\cdots\theta_{\mu_{k-1}\mu_1}$, $k = 1, ..., n$.

Statement. The set (5) is a functional basis of second-order absolute differential invariants of the algebra (3).

Here we can make a comment on sufficiency of consideration of analytical functions in (2) and correctness of the transition to algebra (3).

We have found a set of functionally independent invariants using the algebra (3) and its rank; the rank of the second prolongation of (2) cannot be larger, as otherwise there would be only a smaller set of functionally independent invariants.

3 Conclusion

Here we present the steps for calculation of functional bases of absolute differential invariants for infinite-dimensional algebras with arbitrary functions in basis operators:

1. Expand functions into Taylor series.
2. Transform the set of operators with arbitrary functions into discrete infinite set without arbitrary functions.
3. Find needed prolongations of the algebra.
4. Calculate rank of the prolongation of the algebra.
5. Find a minimal "generating set" of operators with the rank of their prolongation equal to that of the prolongation of the algebra.
6. Find a functional basis using the "generating set".

Further research in this direction, beside calculation of invariants for other algebras, includes studying dependence of the ranks and orders of prolongations, and structures of the generation sets for the prolongations that may be of interest for study of the algebraic properties of invariant equations.

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