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Contact classification of 3-order linear ODEs

by

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Abstract. A local classification of regular 3-order linear ordinary differential equations with respect to contact transformations is given.

0. Introduction

It is well known that any two 2–order linear ordinary differential equations (ODEs) are locally equivalent. The similar statement for 3-order linear ODEs is not correct. The point is that a necessary condition for one differential equation to be locally equivalent to another one is that dimensions of the classical symmetry algebras of these equations are the same; but dimension of the classical symmetry algebra of an arbitrary 3–order linear ODE can be equal to one of the numbers 10, 5, or 4 (see the corollary of Theorem 2.1).

The aim of this paper is to obtain a local classification of 3–order linear ODEs at a regular point with respect to contact transformations.

Our approach to this problem is the following.

It is easy to prove that any 3–order linear ODE can be transformed to the form

$$y''' = v(x) \cdot y' + w(x) \cdot y. \tag{0.1}$$

Therefore our problem is reduced to the classification of equations (0.1).

We identify every ODE (0.1) with the geometric structure γ on $\mathbf{R}^1$ with components $v(x)$ and $w(x)$ in the coordinate system x on $\mathbf{R}^1$. Now our problem is reduced to a local classification of those geometric structures with respect to diffeomorphisms of $\mathbf{R}^1$.

To this end we calculate differential invariants of these structures. The invariant differential form ω and the scalar differential invariant I obtained here make possible to solve the problem completely.

As a result we obtain the following local classification of 3–order linear ODEs at a regular point with respect to contact transformations. Let $\mathcal{E}$ be an ODE of form (0.1) and let $\operatorname{Sym}\mathcal{E}$ be its classical symmetriy algebra; then

(1) If $\dim \operatorname{Sym}\mathcal{E} = 10$, then $\mathcal{E}$ is locally equivalent to the equation

$$y''' = 0. \tag{0.2}$$

(2) If $\dim \operatorname{Sym}\mathcal{E} = 5$, then $\mathcal{E}$ is locally equivalent to one of the following equations

$$y''' = K \cdot y' + y, \quad K \in \mathbf{R}^1. \tag{0.3}$$

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(3) If $\dim \operatorname{Sym} \mathcal{E} = 4$, then $\mathcal{E}$ is locally equivalent to one of the following equations

$$y''' = \left[xu^2 - 2\left(\frac{u'}{u}\right)' + \left(\frac{u'}{u}\right)^2 \right] y' + \frac{1}{2}\left[\left(xu^2 - 2\left(\frac{u'}{u}\right)' + \left(\frac{u'}{u}\right)^2 \right)' - u^3 \right] y, \quad (0.4)$$

where functions u are nowhere vanishing smooth functions of x and any two of them don't have equal germs.

Note, that this complete list (0.1) – (0.3) of "simplest" locally nonequivalent 3–order linear ODEs with respect to contact transformations concides with the complete list of "simplest" locally nonequivalent 3–order linear ODEs with respect to point transformations, obtaned in [8].

Recall that the general method to investigate local equivalence problem based on the theory of differential invariants was originated by classics of the end of XIX century S.Lie, A.Tresse, G.–H.Halphen, E.Laguerre, R.Liouville and others. In particular, first the above-mentioned invariant differential form ω was used by E.Laguerre in [4] to solve the local equivalence problem of 3–order linear ODEs with respect to point transformations of the form

$$X = f(x), \qquad Y = g(x) \cdot y.$$

In [3], G.–H.Halphen used scalar differential invariants to obtain a criterion for a 3–order linear ODE to be locally equivalent to a linear ODE with constant coefficients with respect to those transformations.

This paper is organized as follows. In section 1 we recall preliminary definitions and facts. In section 2 we calculate the classical symmetry algebra for ODEs of the form (0.1), for ODEs admitting a 5 or 4 – dimensional classical symmetry algebra we reduce the problem of local classification with respect to contact transformations to the problem of local classification with respect to point transformations. In section 3 we solve the local equivalence problem for ODEs of the form (0.1). In section 4 we calculate differential invariants and give the local classification of 3–order linear ODEs with respect to contact transformations.

Below everything is supposed to be smooth.

By $\mathbf{R}^n$ denote the n – dimensional arithmetical space, by $[f]_x^k$ denote the k–jet of a map f at the point x, and by definition, put $A_{x_1 \ldots x_k} = \dfrac{\partial^k A}{\partial x_1 \ldots \partial x_k}$.

1. Preliminaries

In this section we recall necessary notations and results of the geometry of differential equations (see [6]) and the theory of geometric structures and differential invariants (see [1],[2]).

1.1. Lie transformations. Let $\lambda : E \to M$ be a smooth bundle, let $\dim M = n$, and let $\dim E = n + m$. By $J^k\lambda$ $k = 0, 1, 2, \dots$, denote the bundle of k–jets $[s]^k_x$ of all cross-sections s of λ. Let $\lambda_k : J^k\lambda \to M$ be the projection that takes each $[s]^k_m$ to m and let $\lambda_{k,r} : J^k\lambda \to J^r\lambda$, $k > r$, be the projection that takes each k-jet $[s]^k_m$ to the r-jet $[s]^r_m$.

Every cross-section $s : U \to E$, $U \subset M$, of λ generates the cross-section $j_k s : U \to J^k\lambda$ of the jet bundle λ_k by the formula $j_k s : x \mapsto [s]^k_m$. By $L^{(k)}_s$ denote the image of the cross-section $j_k s$.

By $T_{x_k}(J^k\lambda)$ denote the tangent space to $J^k\lambda$ at $x_k \in J^k\lambda$, by $T_{x_k}(L^{(k)}_s)$ denote the tangent space to $L^{(k)}_s$ at $x_k \in L^{(k)}_s$.

The *Cartan plane* $C_{x_k} \subset T_{x_k}(J^k\lambda)$ at $x_k \in J^k\lambda$ is the subspace spanned on the union of all $T_{x_k}(L^{(k)}_s)$ such that $x_k \in L^{(k)}_s$.

The *Cartan distribution* C on $J^k\lambda$ is the distribution defined by the formula

$$C : x_k \longmapsto C_{x_k}.$$

A (local) diffeomorphism of $J^k\lambda$ conserving the Cartan distribution is called a *Lie transformation.*

Every Lie transformation $f : U \to U'$ of $J^k\lambda$ can be lifted canonically up to the Lie transformation $f^{(r)} : \lambda^{-1}_{k+r,k}(U) \to \lambda^{-1}_{k+r,k}(U')$ of $J^{k+r}\lambda$, $r = 1, 2, \dots$, such that the diagram

$$\begin{array}{ccc} \lambda^{-1}_{k+r,k}(U) & \xrightarrow{f^{(r)}} & \lambda^{-1}_{k+r,k}(U') \\ \lambda_{k+r,k}\big\downarrow & & \big\downarrow\lambda_{k+r,k} \\ U & \xrightarrow[f]{} & U' \end{array}$$

is commutative.

Recall that $f^{(r)}$ is defined in the following way. A point $x_{k+1} = [s]^{k+1}_x \in J^{k+1}\lambda$ is identified with $K_{x_{k+1}} = T_{x_k}(L^{(k)}_s)$, where $x_k = \lambda_{k+1,k}(x_{k+1})$. The differential f_* maps $K_{x_{k+1}}$ onto the subspace $f_*(K_{x_{k+1}})$. If $f_*(K_{x_{k+1}})$ is projected on M without a degeneration, then there is $x'_{k+1} \in J^{k+1}\lambda$ such that $K_{x'_{k+1}} = f_*(K_{x_{k+1}})$ and we set $f^{(1)}(x_{k+1}) = x'_{k+1}$. It is obvious that $f^{(1)}$ is a Lie transformation of $J^{k+1}\lambda$ defined almost everywhere in $\lambda^{-1}_{k+1,k}(U)$. Setting $f^{(r+1)} = (f^{(r)})^{(1)}$, we define the Lie transformation $f^{(r)}$ for all $r = 1, 2, \dots$. Clearly, that $f^{(r)}$ is defined almost everywhere in $\lambda^{-1}_{k+r,k}(U)$. (We shall say for brevity that $f^{(r)}$ is defined in $\lambda^{-1}_{k+r,k}(U)$.)

A Lie transformation of $J^1\lambda$ is called a *contact transformation* if $m = 1$.

A Lie transformation of $J^0\lambda$ (that is an arbitrary diffeomorphism of $J^0\lambda$) is called a *point transformation*.

It is well known (see [6]) that if $m = 1$, then every Lie transformation of $J^k\lambda$, where $k > 1$, is the lifting of some contact transformation.

A vector field ξ in $J^k\lambda$ is called a *Lie field* if flow of ξ is generated by Lie transformations.

Obviously, the lifting of Lie transformations defines the lifting of every Lie field ξ in $J^k\lambda$ up to the Lie field $\xi^{(r)}$ in $J^{k+r}\lambda$ for all $r = 1, 2, \dots$.

Let $\lambda : \mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}^1$ be the product bundle and let $x,\ y,\ p_1,\ \ldots ,p_k$ be the standard coordinates on $J^k\lambda$. Then it is easily shown that a contact transformation is defined in the standard coordinates by the formulas

$$\begin{aligned} X &= X(x,y,p_1), \\ Y &= Y(x,y,p_1), \\ P_1 &= \frac{Y_x + p_1 Y_y}{X_x + p_1 X_y}, \end{aligned} \tag{1.1}$$

where the functions $X(x,y,p_1)$ and $Y(x,y,p_1)$ are connected by the relation

$$Y_{p_1}(X_x + p_1 X_y) - X_{p_1}(Y_x + p_1 Y_y) = 0.$$

Obviously, a point transformation is defined in the standard coordinates $x,\ y$ of $J^0\lambda$ by the formulas

$$\begin{aligned} X &= X(x,y), \\ Y &= Y(x,y). \end{aligned} \tag{1.2}$$

1.2. Geometric structures. Let $G^n_k,\ \ k = 1,2,\ldots,$ be the group of k-jets at the point $0 \in \mathbf{R}^n$ of all diffeomorphisms $\mathbf{R}^n \to \mathbf{R}^n$ conserving the point 0.

Let M be a smooth n – dimensional manifold and let $S_k(M)$ be the manifold of k-jets at $0 \in \mathbf{R}^n$ of all diffeomorphisms $s : \mathbf{R}^n \to M$. The projection $\rho_k : S_k(M) \to M$ is defined by $\rho_k(\,[s]^k_0\,) = s(0)$. The right action $S_k(M) \times G^n_k \to S_k(M)$ of the group G^n_k on $S_k(M)$ is defined by

$$(\,[s]^k_0,\ [g]^k_0\,) \longmapsto [s \circ g]^k_0 \qquad \forall\,[s]^k_0 \in S_k(M),\ \forall\,[g]^k_0 \in G^n_k.$$

It is easy to verify that $(\ S_k(M),\ \rho_k,\ M,\ G^n_k\ )$ is a principal bundle with structural group G^n_k.

Let Q be a smooth N – dimensional manifold and let $\mu : G^n_k \times Q \to Q$ be a left action of the group G^n_k on Q. We often will write $g_k \cdot q$ instead $\mu(\,g_k, q\,)$ $g_k \in G^n_k,\ q \in Q$.

Let $\rho_\mu : E \to M$ be the Q – associated bundle of the principal bundle $S_k(M)$ generated by μ.

We recall the definition of this bundle. The actions G^n_k on $S_k(M)$ and on Q generate the equivalence relation on $S_k(M) \times Q$ by the formula

$$(\,s_k,\ g_k \cdot q\,) \sim (\,s_k \cdot g_k,\ q\,) \qquad \forall\, g_k \in G^n_k,\ \forall\,(\,s_k,\ q\,) \in S_k(M) \times Q.$$

By $[s_k,\ q]$ denote the equivalence class of the element $(s_k,\ q)$, by E denote the manifold of all those equivalence classes, and the projection $\rho_\mu : E \to M$ is defined by the formula $\rho_\mu(\,[s_k,\ q]\,) = \rho_k(s_k)$.

A cross-section γ of ρ_μ is called a *geometric structure* of type μ on the manifold M. The bundle ρ_μ is called a *bundle of geometric structures of type* μ too.

Every coordinate system (local chart) $(\,U, h = (x^1,\ldots,x^n)\,)$ of M generates the local trivialisation $h_E : \rho_\mu^{-1}(U) \to U \times Q$ of the bundle ρ_μ in the following way. The coordinate system $(\,U, h = (x^1,\ldots,x^n)\,)$ generates the cross-section $h_k : U \to S_k(M)$ by the formula

$$h_k(m) = [h^{-1} \circ t_{h(m)}]^k_0,$$

where $t_{h(m)}$ is the translation of $\mathbf{R}^n$ defined by $t_{h(m)}(x) = x + h(m)$. Let $e \in \rho_\mu^{-1}(U)$ and $m = \rho_\mu(e)$; then there is $g_k \in G_k^n$ such that

$$e = [s_k,\, q] = [h_k(m) \cdot g_k,\, q] = [h_k(m),\, g_k^{-1} \cdot q].$$

By definition, put

$$h_E : e \mapsto (m,\, g_k^{-1} \cdot q).$$

Let γ be a geometric structure of type μ on M, let $pr_2 : U \times Q \to Q$ be the projection on the second factor; then for every coordinate system $(\,U, h = (x^1, \dots, x^n)\,)$ of M the map $\gamma_h = pr_2 \circ h_E \circ (\gamma \mid U) : U \to Q$ is called the *expression of* γ *in the coordinate system* $(\,U, h = (x^1, \dots, x^n)\,)$.

Let $(\,U, h = (x^1, \dots, x^n)\,)$ and $(\,U', h' = (y^1, \dots, y^n)\,)$ be coordinate systems of M such that $U \cap U' \neq \emptyset$ and let $(y^1(x^1, \dots, x^n), \dots, y^n(x^1, \dots, x^n))$ be the transformation of these coordinate systems. Then it follows easily that the expressions of γ in these coordinate systems are connected by the following formula.

$$\gamma_{h'}(\,y(m)\,) = \mu\Big(\frac{\partial y^i}{\partial x^j}(x(m)), \dots, \frac{\partial^k y^i}{\partial x^{j_1} \dots \partial x^{j_k}}(x(m)),\ \gamma_h(\,x(m)\,)\,\Big), \tag{1.3}$$

where $m \in U \cap U'$ and $\Big(\dfrac{\partial y^i}{\partial x^j}(x(m)), \dots, \dfrac{\partial^k y^i}{\partial x^{j_1} \dots \partial x^{j_k}}(x(m)\,)\Big) \in G_k^n$. Relation (1.3) is called the *transformation law of coordinate expressions.*

Obviously, every geometric structure is defined completely by the collection of all its coordinate expressions and the transformation law of these expressions.

Example. A function, an arbitrary tensor field, and a linear connection are examples of geometric structures of different types.

1.3. Differential invariants. Every diffeomorphism f of the manifold M generates the Lie transformation f_E of E by the formula

$$f_E : [\,[s]_0^k,\, q\,] \longmapsto [\,[f \circ s]_{s(0)}^k,\, q\,]. \tag{1.4}$$

Obviously, the following diagram is commutative

$$\begin{array}{ccc} E & \xrightarrow{\ f_E\ } & E \\ {\scriptstyle \rho_\mu}\big\downarrow & & \big\downarrow{\scriptstyle \rho_\mu} \\ M & \xrightarrow[\ f\ ]{} & M \end{array}$$

Therefore every diffeomorphism f of M generates the transformation of geometric structures by the formula

$$\gamma \longmapsto f_E \circ \gamma \circ f^{-1}. \tag{1.5}$$

Let $(y^1 = f^1(x^1, \dots, x^n), \dots, y^n = f^n(x^1, \dots, x^n)$ be the expression of diffeomorphism f in coordinate systems $(\,U, h = (x^1, \dots, x^n)\,)$ and $(\,U', h' = (y^1, \dots, y^n)\,)$ of M and let $\gamma' = f_E \circ \gamma \circ f^{-1}$; then it follows easily that

$$\gamma'_{h'}(m) = \mu\Big(\frac{\partial f^i}{\partial x^j}(\,f^{-1}(m)\,), \dots, \frac{\partial^k f^i}{\partial x^{j_1} \dots \partial x^{j_k}}(\,f^{-1}(m)\,),\ \gamma_h(\,f^{-1}(m)\,)\,\Big).$$

Let $(\rho_\mu)_r : J^r\rho_\mu \to M$ be the r–jet bundle of ρ_μ, $r = 0, 1, 2, \ldots$. By $f_E^{(r)}$ denote the lifting of f_E up to the Lie transformation of $J^r\rho_\mu$.

Obviously, this lifting defines the lifting of every vector field ξ on M up to the Lie vector field $\xi_E^{(r)}$ on $J^r\rho_\mu$, $r = 0, 1, 2, \ldots$.

A diffeomorphism f of M is called a *symmetry of the geometric structure* γ if $\gamma = f_E \circ \gamma \circ f^{-1}$.

A vector field ξ on M is called an *infinitesimal symmetry of the geometric structure* γ if the flow of ξ consists of symmetries for γ.

It is obvious that a vector field ξ is an infinitesimal symmetry for γ iff the vector field ξ_E tangents to $L_\gamma^{(0)}$.

A geometric structure on the manifold J^rE is called a *differential invariant of order* r if it is invariant with respect to all diffeomorphisms of the form $f_E^{(r)}$. A differential invariant is called a *scalar differential invariant* if it is a function.

Let Γ be the Lie pseudogroup of all diffeomorphisms from M to itself. The Lie pseudogroup $\Gamma_E^{(r)} = \{ f_E^{(r)} \mid f \in \Gamma \}$ acts on the J^rE. As a result J^rE is divided into orbits. It is clear that a function I on J^rE is a scalar differential invariant iff I is constant on each orbit.

The coordinate definition of a scalar differential invariant is the following. Let $q^1, \ldots, q^N$ be a coordinate system on the manifold Q; then for every geometric structure γ we have $\gamma_h = (\gamma_h^1, \ldots, \gamma_h^N)$. A smooth function I of variables $q^\alpha; q^\alpha_{j_1}; \ldots; q^\alpha_{j_1\ldots j_r}$, $\alpha = 1, 2, \ldots, N$, $j_l = 1, 2, \ldots, n$, is said to be a scalar differential invariant of order r if for every geometrical structure γ of type μ and for every transformation of coordinate systems $y = y(x)$, the following condition holds

$$I(\ \gamma^\alpha(x), \frac{\partial \gamma^\alpha}{\partial x^j}(x), \ldots, \frac{\partial^r \gamma^\alpha}{\partial x^{j_1} \ldots \partial x^{j_r}}(x)\ ) \\ = I(\ \tilde{\gamma}^\alpha(y(x)), \frac{\partial \tilde{\gamma}^\alpha}{\partial y^j}(y(x)), \ldots, \frac{\partial^r \tilde{\gamma}^\alpha}{\partial y^{j_1} \ldots \partial y^{j_l}}(y(x))\ ),$$

where γ^α are the components of the expression of γ in the coordinates $x^1, \ldots x^n$ and $\tilde{\gamma}^\alpha$ are the components of the expression of γ in the transformed coordinates $y^1, \ldots y^n$.

Let I be a differential invariant of order r and let γ be a geometric structure of type μ on M; then by $I(\gamma)$ denote the restriction $I \mid L_\gamma^r$ of I on image of $j_r\gamma$. One can assume that $I(\gamma)$ is defined on M because the restriction $(\rho_\mu)_r \mid L_\gamma^r : L_\gamma^r \to M$ is a diffeomorphism. $I(\gamma)$ is said to be the *value of I on the geometric structure* γ.

Let ξ be infinitesimal symmetry of a geometric structure γ and let I be a scalar differential invariant. Then it is easy to prove that $I(\gamma)$ is an integral of the vector field ξ.

1.4. Equivalence problem for ODEs. Let $\pi : \mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}^1$ be the product bundle.

We identify every ODE of order k

$$F(x,\ y(x),\ \frac{dy}{dx}, \ldots, \frac{d^ky}{dx^k}) = 0$$

with the submanifold $\mathcal{E} \subset J^k\pi$ defined by the equation

$$F(x,\, y,\, p_1, \dots, p_k) = 0.$$

Let $\mathcal{E}$ and $\mathcal{E}'$ be ODEs of order k. We say that $\mathcal{E}$ and $\mathcal{E}'$ are *locally equivalent* if there exist a local contact transformation $f : U \to U',\ \ U, U' \subset J^1\pi$, transforming $\mathcal{E}$ to $\mathcal{E}'$ that is $f^{(k-1)}(\,\mathcal{E} \cap \pi_{k,1}^{-1}(U)\,) = \mathcal{E}' \cap \pi_{k,1}^{-1}(U')$; in this case for every point $x_k \in \mathcal{E} \cap \pi_{k,1}^{-1}(U)$ we say that the equation $\mathcal{E}$ *is locally equivalent to the equation* $\mathcal{E}'$ *at* x_k.

The *local equivalence problem* for ODEs consists in obtaining of a criterion for local equivalence of two equations.

1.5. Classical symmetries of ODEs. A Lie field on $J^1\pi$ is called a *contact vector field*. It is easily shown that a contact vector field ξ can be represented in the standard coordinates $x,\ y,\ p_1$ by the formula

$$\xi = \xi_\varphi = -\varphi_{p_1}\frac{\partial}{\partial x} + (\varphi - p_1\varphi_{p_1})\frac{\partial}{\partial y} + (\varphi_x + p_1\varphi_y)\frac{\partial}{\partial p_1},$$

where the function $\varphi = \varphi(x, y, p_1)$ is called the *generating function* of ξ. Obviously, the generating function defines the contact vector field completely.

Subjecting a contact vector field to an arbitrary contact transformation (1.4), we get a contact vector field. It is easy to verify that the generating functions Φ of the obtained vector field and the generating function φ of the initial one are connected by the formula

$$\Phi(\ X(x,y,p_1),\ Y(x,y,p_1),\ P_1(x,y,p_1)\ ) = \frac{X_xY_y - X_yY_x}{X_x + p_1X_y}\varphi(x,y,p_1)\,. \tag{1.6}$$

An arbitrary vector field ζ in $J^0\pi$ is called a *point vector field.* In the standard coordinates ζ is of the form

$$\zeta = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}. \tag{1.7}$$

The lifting $\zeta^{(1)}$ of ζ is a contact vector field. It is easy to verify that the generating function of $\zeta^{(1)}$ has the form

$$-a(x,y)\cdot p_1 + b(x,y). \tag{1.8}$$

Conversely, if the generating function of a contact vector field has form (1.8), then this contact vector field is the lifting of some point vector field.

We shall say that function (1.8) is the *generating function of point vector field* (1.7).

A contact transformation $f : U \to U'$ in $J^1\pi$ is called a *symmetry* of a differential equation $\mathcal{E} \subset J^k\pi$ if $\mathcal{E} \cap \pi_{k,1}^{-1}(U') = f^{(k-1)}(\,\mathcal{E} \cap \pi_{k,1}^{-1}(U)\,)$.

A contact vector field ξ_φ in $J^1\pi$ is called a *classical symmetry* of a differential equation $\mathcal{E} \subset J^k\pi$ if the lifting $\xi_\varphi^{(k-1)}$ is tangent to the submanifold $\mathcal{E}$.

Let $\operatorname{Sym}\mathcal{E}$ be the set of all classical symmetries of a differential equation $\mathcal{E}$. It is easy to verify that $\operatorname{Sym}\mathcal{E}$ is a Lie algebra over $\mathbf{R}^1$ with respect to the Lie bracket.

It follows from the general theory of symmetries of differential equations (see [6]) that the space of generating functions of classical symmetries for the equation

$$p_k - F(x,\, y,\, p_1, \dots, p_{k-1}) = 0.$$

coincides with the space of solutions of the form $\varphi = \varphi(x, y, p_1)$ for the following linear PDE

$$(D^k - \frac{\partial F}{\partial p_{k-1}} D^{k-1} - \dots - \frac{\partial F}{\partial p_1} D - \frac{\partial F}{\partial y})(\varphi) = 0, \tag{1.9}$$

where $D = \frac{\partial}{\partial x} + p_1 \frac{\partial}{\partial y} + p_2 \frac{\partial}{\partial p_1} + \dots + p_{k-1} \frac{\partial}{\partial p_{k-2}} + F \frac{\partial}{\partial p_{k-1}}$.

2. Classical symmetries of 3–order linear ODEs

2.1. Let $\mathcal{E}$ be an ODE of the form $p_3 = v(x) \cdot p_1 + w(x) \cdot y$.

Theorem 2.1. *Suppose coefficients v and u of $\mathcal{E}$ are defined in a domain U and $\xi_\varphi \in \operatorname{Sym} \mathcal{E}$; then the following statements hold:*

(1) *The generating function φ is of the form*

$$\varphi = a(x) p_1^2 + (b(x) - 2a_x y) p_1 + (2a_{xx} - v a(x)) y^2 + (K - b_x) y + c(x)$$

with $K \in \mathbf{R}^1$ and $a(x)$, $b(x)$, $c(x)$ are solutions of $\mathcal{E}$ iff $v_x - 2w \equiv 0$ in U.

(2) *The generating function φ is of the form*

$$\varphi = K^1 (v_x - 2w)^{-1/3} p_1 + [K^2 - K^1((v_x - 2w)^{-1/3})_x] y + c(x)$$

with $K^1, K^2 \in \mathbf{R}^1$ and $c(x)$ is a solution of $\mathcal{E}$ iff $(v_x - 2w)(x) \neq 0$ almost everywhere in U and $(v_x - 2w)^{-1/3}$ is a solution of the equation $p_3 = v \cdot p_1 + (1/2) \cdot v_x \cdot y$.

(3) *The generating function φ is of the form*

$$\varphi = K y + c(x)$$

with $K \in \mathbf{R}^1$ and $c(x)$ is a solution of $\mathcal{E}$ iff $(v_x - 2w)(x) \neq 0$ almost everywhere in U and $(v_x - 2w)^{-1/3}$ is not a solution of equation $p_3 = v \cdot p_1 + (1/2) \cdot v_x \cdot y$.

Proof. Equation (1.9) corresponding to an equation $p_3 = v(x) \cdot p_1 + w(x) \cdot y$ is the following one

$$(D^3 - v(x) \cdot D - w(x))(\varphi) = 0, \tag{2.1}$$

where $D = \frac{\partial}{\partial x} + p_1 \frac{\partial}{\partial y} + p_2 \frac{\partial}{\partial p_1} + (v(x) \cdot p_1 + w(x) \cdot y) \frac{\partial}{\partial p_2}$.

Taking into account that φ depends on x, y, p_1, we see that the left hand side of (2.1) is a polynomial of degree 3 on p_2. Therefore (2.1) holds iff coefficients of this polynomial are equal to zero. Therefore equation (2.1) is equivalent to the following system of four PDEs

$$\varphi_{p_1 p_1 p_1} = 0, \tag{2.2}$$

$$\varphi_{x p_1 p_1} + p_1 \varphi_{y p_1 p_1} + \varphi_{y p_1} = 0, \tag{2.3}$$

$$(v \cdot p_1 + w \cdot y) \varphi_{p_1 p_1} + \varphi_{xy} + p_1 \varphi_{yy} + \varphi_{xx p_1} + 2 p_1 \varphi_{x y p_1} + p_1^2 \varphi_{y y p_1} = 0, \tag{2.4}$$

$$(v_x \cdot p_1 + w_x \cdot y) \varphi_{p_1} - w(\varphi - p_1 \varphi_{p_1}) - v(\varphi_x + p_1 \varphi_y) + (v \cdot p_1 + w \cdot y)(3\varphi_{x p_1} + 3 p_1 \varphi_{y p_1} + \varphi_y) + \varphi_{xxx} + 3 p_1 \varphi_{xxy} + 3 p_1^2 \varphi_{xyy} + p_1^3 \varphi_{yyy} = 0. \tag{2.5}$$

From equation (2.2) we get $\varphi(x, y, p_1) = a(x, y)p_1^2 + b^1(x, y)p_1 + c^1(x, y)$.

Substituting this expression for φ in equation (2.3), we get the equation $4a_y p_1 + 2a_x + b^1_y = 0$. Taking into account that the functions a, b depend on x, y, we see that this equation is equivalent to the following system of PDEs

$$a_y = 0, \qquad 2a_x + b^1_y = 0$$

. From this system we get $\varphi(x, y, p_1) = a(x)p_1^2 + (b(x) - 2a_x y)p_1 + c^1(x, y)$.

In the same way, substituting the obtained expression for φ in equation (2.4), we get the following system

$$c^1_{yy} + 2va - 4a_{xx} = 0, \quad c^1_{xy} + 2way - 2a_{xxx}y + b_{xx} = 0.$$

From the first equation we get $c^1(x, y) = (2a_{xx} - va)y^2 + c^{11}(x)y + c(x)$. Substituting this expression for c in the second equation, we get the following system

$$b_{xx} + c^{11}_x = 0 \tag{2.6}$$

$$a_{xxx} - va_x - wa - (v_x - 2w)a = 0 \tag{2.7}$$

From equation (2.6) we obtain $c^{11} = K - b_x$, where $K \in \mathbf{R}^1$. Therefore we have

$$\varphi(x, y, p_1) = a(x)p_1^2 + (b(x) - 2a_x y)p_1 + (2a_{xx} - va)y^2 + (K - b_x)y + c(x). \tag{2.8}$$

Substituting the obtained expression for φ in equation (2.5), we see that left hand side of the obtained equation is a polynomial of degree 2 on p_1. Therefore this equation is equivalent to a system of three equations. The first one corresponding to the coefficient of p_1^2 is the following equation

$$7(\, a_{xxx} - va_x - wa\,) - 4(v_x - 2w)a = 0. \tag{2.9}$$

Comparing equations (2.7) and (2.9), we get

$$(v_x - 2w)a = 0. \tag{2.10}$$

Let $v_x - 2w \equiv 0$ in U.

Then it follows from (2.7) or (2.9) that a is a solution of $\mathcal{E}$. Now substituting expression (2.8) for φ in equation (2.5), we get

$$\begin{aligned}
[-5(\, a_{xxx} - va_x - wa\,)_x y + 2(\, b_{xxx} - vb_x - wb\,)\,]p_1 \\
- (\, a_{xxx} - va_x - wa\,)_{xx} y^2 + (\, b_{xxx} - vb_x - wb\,)_x y \\
- (\, c_{xxx} - vc_x - wc\,) = 0.
\end{aligned}$$

From this equation we see that b and c are solutions of $\mathcal{E}$. This completes the proof of the first statement of Theorem 2.1.

Now let $(v_x - 2w)(x) \neq 0$ almost everywhere in U.

Then it follows from (2.10) that $a \equiv 0$ in U. Now from (2.8) we see that

$$\varphi(x, y, p_1) = b(x)p_1 + (K - b_x)y + c(x).$$

Substituting this expression for φ in equation (2.5), we get the following system

$$b_{xxx} - vb_x - \frac{1}{2}v_x b = 0 \tag{2.11}$$

$$2(b_{xxx} - vb_x - \frac{1}{2}v_x b)_x - 3(v_x - 2w)b_x - (v_x - 2w)_x b = 0 \tag{2.12}$$

$$c_{xxx} - vc_x - wc = 0. \tag{2.13}$$

Equation (2.13) means that c is a solution of $\mathcal{E}$. From equations (2.11) and (2.12) we obtain $3(v_x - 2w)b_x + (v_x - 2w)_x b = 0$. From this equation we see that $b = K^1(v_x - 2w)^{-1/3}$, where $K^1 \in \mathbf{R}^1$.

Now we see that if $b = (v_x - 2w)^{-1/3}$ is a solution of equation (2.11), then from (2.8) we have $\varphi = K^1(v_x - 2w)^{-1/3}p_1 + [K^2 - K^1((v_x - 2w)^{-1/3})_x]y + c(x)$ else $b \equiv 0$ and from (2.8) we have $\varphi = Ky + c(x)$. ■

Corollary. $\dim \operatorname{Sym} \mathcal{E}$ *is equal to one of the numbers 10, 5, 4, moreover*

(1) $\dim \operatorname{Sym} \mathcal{E} = 10$ *iff* $v_x - 2w \equiv 0$ *in* U.
(2) $\dim \operatorname{Sym} \mathcal{E} = 5$ *iff* $(v_x - 2w)(x) \neq 0$ *almost everywhere in* U *and* $(v_x - 2w)^{-1/3}$ *is a solution of the equation* $p_3 = v \cdot p_1 + \frac{1}{2}v_x \cdot y$.
(3) $\dim \operatorname{Sym} \mathcal{E} = 4$ *iff* $(v_x - 2w)(x) \neq 0$ *almost everywhere in* U *and* $(v_x - 2w)^{-1/3}$ *is not a solution of the equation* $p_3 = v \cdot p_1 + \frac{1}{2}v_x \cdot y$.

Proof. Let $v_x - 2w \equiv 0$ in U and let $\{\ a^1(x),\ a^2(x),\ a^3(x)\ \}$, $\{\ b^1(x),\ b^2(x),\ b^3(x)\ \}$, $\{\ c^1(x),\ c^2(x),\ c^3(x)\ \}$ be three collections of linear independent (over $\mathbf{R}^1$) solutions of $\mathcal{E}$. Then it follows from Theorem 2.1 that the collection of the following functions

$$\begin{aligned}
\mathcal{A}^i &= a^i(x)p_1^2 - 2a^i_x y p_1 + (2a^i_{xx} - va^i(x))y^2, \quad i = 1,2,3;\\
\mathcal{B}^j &= b^j(x)p_1 - b^j_x y, \quad j = 1,2,3;\\
\mathcal{C}^k &= c^k(x), \quad k = 1,2,3;\\
\mathcal{D} &= y
\end{aligned}$$

is a basis for the space (over $\mathbf{R}^1$) of all generating functions of classical symmetries for $\mathcal{E}$. Therefore the collection of corresponding classical symmetries $\xi_{\mathcal{A}^1}$, $\xi_{\mathcal{A}^2}$, $\xi_{\mathcal{A}^3}$, $\xi_{\mathcal{B}^1}$, $\xi_{\mathcal{B}^2}$, $\xi_{\mathcal{B}^3}$, $\xi_{\mathcal{C}^1}$, $\xi_{\mathcal{C}^2}$, $\xi_{\mathcal{C}^3}$, $\xi_{\mathcal{D}}$ is a basis for $\operatorname{Sym}\mathcal{E}$. It follows that $\dim \operatorname{Sym} \mathcal{E} = 10$.

A proof of statements (2) and (3) are analogous. ■

2.2.

Theorem 2.2. *Let* $\mathcal{E}$ *and* $\mathcal{E}'$ *be ODEs of the form* $p_3 = v(x)\cdot p_1 + w(x)\cdot y$ *and let both* $\dim \operatorname{Sym} \mathcal{E}$ *and* $\dim \operatorname{Sym} \mathcal{E}'$ *be equal to 4 or 5; then if* f *is a contact transformation transforming* $\mathcal{E}$ *to* $\mathcal{E}'$*, then* f *is the lifting of some point transformation.*

Proof. Suppose $\mathcal{E}$ is the equation $P_3 = V(X) \cdot P_1 + W(X) \cdot Y$, $\mathcal{E}'$ is the equation $p_3 = v(x) \cdot p_1 + w(x) \cdot y$, f is defined by (1.1), and f transforms $\mathcal{E}$ to $\mathcal{E}'$. Suppose $\dim \operatorname{Sym} \mathcal{E} = 5$, then obviously, $\dim \operatorname{Sym} \mathcal{E}' = 5$. Let the collection $\Phi_1(X)$, $\Phi_2(X)$, $\Phi_3(X)$ be a fundamental system of solutions for $\mathcal{E}$. It follows from Theorem 2.1 that they are generating functions for some classical symmetries of $\mathcal{E}$. Therefore transformation (1.1) connects each of these functions with a generating function of some classical symmetry of $\mathcal{E}'$ by formula (1.6). Taking into account the form of last generating functions (see Theorem 2.1), we obtain from formula (1.6):

$$\Phi_1(X(x,y,p_1)) = \mathcal{F}\cdot\{\ K_1^1(v_x-2w)^{-1/3}p_1 + [K_1^2 - K_1^1((v_x-2w)^{-1/3})']y + c_1\ \},$$
$$\Phi_2(X(x,y,p_1)) = \mathcal{F}\cdot\{\ K_2^1(v_x-2w)^{-1/3}p_1 + [K_2^2 - K_2^1((v_x-2w)^{-1/3})']y + c_2\ \},$$
$$\Phi_3(X(x,y,p_1)) = \mathcal{F}\cdot\{\ K_3^1(v_x-2w)^{-1/3}p_1 + [K_3^2 - K_3^1((v_x-2w)^{-1/3})']y + c_3\ \},$$

where $\mathcal{F} = \dfrac{X_xY_y - X_yY_x}{X_x + p_1X_y}$ and $c_1,\ c_2,\ c_3$ are some solutions of $\mathcal{E}'$ and $K_j^i \in \mathbf{R}^1,\ i = 1,\ 2,\ j = 1,\ 2,\ 3$. If one of the numbers $K_1^1,\ K_2^1,\ K_3^1$ is not equal to zero, say $K_1^1 \neq 0$, then the following relations hold

$$\Phi_2 - \frac{K_2^1}{K_1^1}\Phi_1 = \mathcal{F}\cdot\{\ [K_2^2 - \frac{K_2^1}{K_1^1}K_1^2]y + c_2(x) - \frac{K_2^1}{K_1^1}c_1(x)\ \},$$
$$\Phi_3 - \frac{K_3^1}{K_1^1}\Phi_1 = \mathcal{F}\cdot\{\ [K_3^2 - \frac{K_3^1}{K_1^1}K_1^2]y + c_3(x) - \frac{K_3^1}{K_1^1}c_1(x)\ \}.$$

It follows that $\dfrac{K_1^1\Phi_2 - K_2^1\Phi_1}{K_1^1\Phi_3 - K_3^1\Phi_1}$ does not depend on p_1. Therefore

$$\frac{\partial}{\partial p_1}\left(\frac{K_1^1\Phi_2 - K_2^1\Phi_1}{K_1^1\Phi_3 - K_3^1\Phi_1}\right) = \frac{d}{dX}\left(\frac{K_1^1\Phi_2 - K_2^1\Phi_1}{K_1^1\Phi_3 - K_3^1\Phi_1}\right)\cdot X_{p_1} = 0\,.$$

If $X_{p_1} \neq 0$ in some neighborhood, then

$$K_1^1\Phi_2 - K_2^1\Phi_1 = K(\ K_1^1\Phi_3 - K_3^1\Phi_1\ ),$$

where $K \in \mathbf{R}^1$, in this neighborhood. But it is impossible, because the collection $\Phi_1,\ \Phi_2,\ \Phi_3$ is the fundamental system of solutions. Hence $X_{p_1} \equiv 0$. This means that f is the lifting of some point transformation.

The above-mentioned arguments are valid if all numbers $K_1^1,\ K_2^1,\ K_3^1$ are equal to zero.

Obviously, a proof for the case $\dim \mathrm{Sym}\,\mathcal{E} = 4$ is analogous. ∎

Below we prove that for any 3–order linear ODE $\mathcal{E}$ with $\dim \mathrm{Sym}\,\mathcal{E} = 10$ there is a point transformation transforming $\mathcal{E}$ to the equation $y''' = 0$. Taking into account this fact and the last theorem, we obtain that the problem of local classification of 3-order linear ODEs with respect to contact transformations is reduced to the problem of local classification of these equations with respect to point transformations.

3. The equivalence problem for 3–order linear ODEs

3.1. Let $\mathcal{E}$ be the equation $P_3 = V(X)\cdot P_1 + W(X)\cdot Y$, let $\mathcal{E}'$ be the equation $p_3 = v(x)\cdot p_1 + w(x)\cdot y$, and let f be an arbitrary point transformation defined by (1.2). It is not hard to prove that the transformation f transforms $\mathcal{E}$ to the following equation

$$p_3 = u^1\cdot(p_2)^2 + u^2\cdot p_2 + u^3,$$

where the coefficients $u^1,\ u^2,\ u^3$ are defined by the formulas

$$u^1 = \frac{X_x + p_1 X_y}{X_x Y_y - Y_x X_y}[\, X_y Z_{p_1} - (X_x + p_1 X_y) Z_{p_1 p_1}\,],$$

$$\begin{aligned} u^2 = \frac{X_x + p_1 X_y}{X_x Y_y - Y_x X_y}[\, & Z_{p_1}(X_{xx} + 2p_1 X_{xy} + (p_1)^2 X_{yy}) \\ & - 2(Z_{xp_1} + p_1 Z_{yp_1})(X_x + p_1 X_y) \\ & - Z_y(X_x + p_1 X_y) + (Z_x + p_1 Z_y) X_y\,], \end{aligned}$$

$$\begin{aligned} u^3 = \frac{X_x + p_1 X_y}{X_x Y_y - Y_x X_y}[\, & (X_{xx} + 2p_1 X_{xy} + (p_1)^2 X_{yy})(Z_x + p_1 Z_y) \\ & - (Z_{xx} + 2p_1 Z_{xy} + (p_1)^2 Z_{yy})(X_x + p_1 X_y) \\ & + (VZ + WY)(X_x + p_1 X_y)^3\,], \end{aligned}$$

where $Z = \dfrac{Y_x + p_1 Y_y}{X_x + p_1 X_y}$.

It is clear that f transforms $\mathcal{E}$ to $\mathcal{E}'$ iff the collection of functions $X(x,y),\ Y(x,y)$ defining f is a solution of the following system of PDEs

$$X_y Z_{p_1} - (X_x + p_1 X_y) Z_{p_1 p_1} = 0, \tag{3.1}$$

$$\begin{aligned} & Z_{p_1}(X_{xx} + 2p_1 X_{xy} + (p_1)^2 X_{yy}) \\ & \quad - 2(Z_{xp_1} + p_1 Z_{yp_1})(X_x + p_1 X_y) \\ & \quad\quad - Z_y(X_x + p_1 X_y) + (Z_x + p_1 Z_y) X_y = 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} & \frac{X_x + p_1 X_y}{X_x Y_y - Y_x X_y}[(X_{xx} + 2p_1 X_{xy} + (p_1)^2 X_{yy})(Z_x + p_1 Z_y) \\ & \quad - (Z_{xx} + 2p_1 Z_{xy} + (p_1)^2 Z_{yy})(X_x + p_1 X_y) \\ & \quad\quad + (VZ + WY)(X_x + p_1 X_y)^3] = vp_1 + wy. \end{aligned} \tag{3.3}$$

Therefore the local equivalence problem for $\mathcal{E}$ and $\mathcal{E}'$ with respect to point transformations consists in obtaining of an existence criterion of a point transformation $(\, X(x,y),\ Y(x,y)\,)$ satisfying to system (3.1) – (3.3).

Let us find this criterion. To this end we consider equation (3.1). Solving this equation with respect to Z_{p_1} and taking into account that $(\, X(x,y),\ Y(x,y)\,)$ define a point transformation, we get

$$Z_{p_1} = A(x,y)\cdot(X_x + p_1 X_y) \quad \text{and} \quad A(x,y) \neq 0 \quad \text{everywhere.}$$

It follows that,

$$Z = \frac{1}{2} A X_y p_1^2 + A X_x p_1 + B(x,y).$$

Taking into account that $Z = \dfrac{Y_x + p_1 Y_y}{X_x + p_1 X_y}$, we get the following equation

$$\frac{1}{2} A (X_y)^2 p_1^3 + \frac{1}{2} A X_x X_y p_1^2 + (A X_x^2 + B X_y - Y_y) p_1 + B X_x - Y_x = 0.$$

The functions X, Y, A, B do not depend on p_1. Therefore the last equation is equivalent to the following system of PDEs

$$A(X_y)^2 = 0, \quad AX_xX_y = 0, \quad AX_x^2 + BX_y - Y_y = 0, \quad BX_x - Y_x = 0.$$

From this system we easily get the following results:

$$X = X(x), \quad Z = A(x,y)X_xp_1 + B(x,y), \quad \forall x, y\ A(x,y) \neq 0, \tag{3.4}$$

$$Y_x = BX_x, \quad Y_y = AX_x^2, \tag{3.5}$$

$$B_y = 2AX_{xx} + A_xX_x. \tag{3.6}$$

Now substituting obtained expressions for X and Z in equation (3.2), we get

$$3A_yX_x^2p_1 + (\,AX_{xx} + 2A_xX_x + B_y\,)X_x = 0.$$

Obviously, this equation is equivalent to the following system of PDEs

$$A_y = 0, \quad AX_{xx} + 2A_xX_x + b_y = 0.$$

It follows from the first equation of this system that $A = A(x)$. Comparing the second equation of this system with equation (3.6), we get $AX_{xx} + A_xX_x = 0$. Whence

$$A = \frac{K}{X_x}, \quad K \in \mathbf{R}^1 \setminus \{0\}.$$

Substituting this expression for A in the second equation of (3.5), we get $Y_y = KX_x$. Whence $Y = KX_xy + C(x)$. Thus we have

$$X = X(x), \quad Y = KX_xy + C(x), \quad K \in \mathbf{R}^1 \setminus \{0\}. \tag{3.7}$$

Now substituting the obtained expressions for X and Y in equation (3.3), we get

$$\left[-2\left(\frac{X_{xx}}{X_x}\right)_x + \left(\frac{X_{xx}}{X_x}\right)^2 + X_x^2V - v\right]p_1 + \left[-\left(\frac{X_{xx}}{X_x}\right)_{xx} + \frac{X_{xx}}{X_x}\left(\frac{X_{xx}}{X_x}\right)_x + X_{xx}X_xV + X_x^3W - w\right]y + \frac{1}{K}\left[-\left(\frac{C_x}{X_x}\right)_{xx} + \left(\frac{C_x}{X_x}\right)_x\frac{X_{xx}}{X_x} + X_xVC_x + X_x^2WC\right] = 0$$

Obviously, this equation is equivalent to the following system of PDEs

$$-2\left(\frac{X_{xx}}{X_x}\right)_x + \left(\frac{X_{xx}}{X_x}\right)^2 + X_x^2V - v = 0, \tag{3.8}$$

$$-\left(\frac{X_{xx}}{X_x}\right)_{xx} + \frac{X_{xx}}{X_x}\left(\frac{X_{xx}}{X_x}\right)_x + X_{xx}X_xV + X_x^3W - w = 0, \tag{3.9}$$

$$-\left(\frac{C_x}{X_x}\right)_{xx} + \left(\frac{C_x}{X_x}\right)_x\frac{X_{xx}}{X_x} + X_xVC_x + X_x^2WC = 0. \tag{3.10}$$

Now it is clear that if there exist a point transformation satisfying system (3.1) – (3.3), then it is defined by formulas (3.7), where $X(x)$ satisfies to system (3.8) – (3.9) and $C(x)$ is a solution of the ODE obtained by substitution the solution $X(x)$ in (3.10). Therefore our local equivalence problem for $\mathcal{E}$ and $\mathcal{E}'$ is reduced to obtaining of an existence criterion of a local diffeomorphism $X(x)$ satisfying to system (3.8) – (3.9).

Let us obtain this criterion. To this end we differentiate equation (3.8) with respect to x and subtract the obtained equation from doubled equation (3.9). As a result we get the following equation

$$(V_X - 2W)X_x^3 - (v_x - 2w) = 0. \tag{3.11}$$

It is clear that system of ODEs (3.8) – (3.9) is equivalent to system of ODEs (3.8), (3.11).

Suppose both $\dim \operatorname{Sym} \mathcal{E}$ and $\dim \operatorname{Sym} \mathcal{E}'$ are equal to 10. Then it follows from Theorem 2.1 that $V_X - 2W \equiv 0$ and $v_x - 2w \equiv 0$. Therefore system (3.8), (3.11) is reduced to ODE (3.8). Obviously, there is local diffeomorphisms satisfying to this equation.

Thus in particular we obtain the following statements:

Theorem 3.1. *Let $\mathcal{E}$ and $\mathcal{E}'$ be 3–order linear ODEs admitting a 10–dimensional algebras of classical symmetries; then there exist a point transformation transforming $\mathcal{E}$ to $\mathcal{E}'$.*

Corollary. *(The local contact classification of 3–order linear ODEs admitting a 10–dimensional classical symmetry algebra.)*

Let $\mathcal{E}$ be a 3–order linear ODE with $\dim \operatorname{Sym} \mathcal{E} = 10$; then the equation $\mathcal{E}$ is locally equivalent to the equation $p_3 = 0$ at every point $x_3 \in \mathcal{E}$.

Now let $\dim \operatorname{Sym} \mathcal{E} \neq 10$ and $\dim \operatorname{Sym} \mathcal{E}' \neq 10$. Then it follows from Theorem 2.1 that $(V_X - 2W)(X) \neq 0$ and $(v_x - 2w)(x) \neq 0$ almost everywhere. Therefore every solution $X(x)$ of (3.11) is a local diffeomorphism.

It follows from (3.11) that $X_x = \left(\dfrac{v_x - 2w}{V_X - 2W}\right)^{1/3}$. Substituting this expression for X_x in (3.8), we obtain the following relation

$$\left[\frac{2}{3}\left(\frac{(V_X - 2W)_X}{V_X - 2W}\right)_X - \frac{1}{9}\left(\frac{(V_X - 2W)_X}{V_X - 2W}\right)^2 + V\right](V_X - 2W)^{-2/3}(\,X(x)\,) = \tag{3.12}$$
$$= \left[\frac{2}{3}\left(\frac{(v_x - 2w)_x}{v_x - 2w}\right)_x - \frac{1}{9}\left(\frac{(v_x - 2w)_x}{v_x - 2w}\right)^2 + v\right](v_x - 2w)^{-2/3}(x).$$

Thus we obtain the following statement

Theorem 3.2. *Let $\mathcal{E}$ be the equation $P_3 = V(X) \cdot P_1 + W(X) \cdot Y$, let $\mathcal{E}'$ be the equation $p_3 = v(x) \cdot p_1 + w(x) \cdot y$, and let both $\dim \operatorname{Sym} \mathcal{E}$ and $\dim \operatorname{Sym} \mathcal{E}'$ be equal to 5 or 4. Then $\mathcal{E}$ and $\mathcal{E}'$ are locally equivalent iff there is a solution of (3.11) satisfying to equation (3.12).*

The following statement follows from above-mentioned arguments.

Theorem 3.3. *If the equations* $P_3 = V(X)\cdot P_1 + W(X)\cdot Y$ *and* $p_3 = v(x)\cdot p_1 + w(x)\cdot y$ *are locally equivalent, then a transformation transforming the first equation to the second one can be chosen in the form*

$$\begin{aligned} X &= X(x), \\ Y &= X_x \cdot y. \end{aligned} \tag{3.13}$$

Thus we have reduced the problem of local classification of 3–order linear ODEs with respect to contact transformation to the problem of local classification of equations $p_3 = v(x) \cdot p_1 + w(x) \cdot y$ with respect to transformations of the form (3.13).

4. Differential invariants of 3-order linear ODEs

4.1. Geometric structures of equations $\mathbf{p_3 = v(x) \cdot p_1 + w(x) \cdot y}$. Let $\mathcal{E}$ be an arbitrary equation $P_3 = V(X) \cdot P_1 + W(X) \cdot Y$. It is easy to verify that every transformation (3.13) transforms this equation to the equation $p_3 = v(x)\cdot p_1 + w(x)\cdot y$ of the same form. The coefficients v and w of the last equation are expressed through the coefficients V and W of the initial equation by the formulas

$$v = -2\left(\frac{X_{xx}}{X_x}\right)_x + \left(\frac{X_{xx}}{X_x}\right)^2 + X_x^2 V, \tag{4.1}$$

$$w = -\left(\frac{X_{xx}}{X_x}\right)_{xx} + \frac{X_{xx}}{X_x}\left(\frac{X_{xx}}{X_x}\right)_x + X_{xx}X_x V + X_x^3 W. \tag{4.2}$$

Note that these formulas are the same as (3.8), (3.9).

Obviously, the family of all transformation of the form (3.13) is a Lie pseudogroup. It follows that formulas (4.1), (4.2) define the action $\mu : G_4^1 \times \mathbf{R}^2 \to \mathbf{R}^2, \quad \mu : (X_x, X_{xx}, X_{xxx}, X_{xxxx}) \times (V, W) \longmapsto (v, w)$.

Therefore the collections $V(X), W(X)$ and $v(x), w(x)$ are the expressions of some geometric structure of type μ on $\mathbf{R}^1$ in the coordinate systems X and x respectively. We denote this structure by $\gamma_{\mathcal{E}}$.

Obviously, $\mathcal{E}$ is identified with $\gamma_{\mathcal{E}}$.

We see that the bundle of geometric structures of type μ in our case is the product bundle $\rho_\mu : \mathbf{R}^2 \times \mathbf{R}^1 \to \mathbf{R}^1$. We denote by $x;\ u^1, u^2;\ u^1_1, u^2_1; \ldots ; u^1_k, u^2_k$ the standard coordinates in $J^k\rho_\mu$.

Let $\operatorname{Sym}\gamma_{\mathcal{E}}$ be algebra of all infinitesimal symmetries for $\gamma_{\mathcal{E}}$.

Proposition 4.1.

(1) *if* $\dim\operatorname{Sym}\mathcal{E} = 10$, *then* $\dim\operatorname{Sym}\gamma_{\mathcal{E}} = 3$*;*
(2) *if* $\dim\operatorname{Sym}\mathcal{E} = 5$, *then* $\dim\operatorname{Sym}\gamma_{\mathcal{E}} = 1$*;*
(3) *if* $\dim\operatorname{Sym}\mathcal{E} = 4$, *then* $\dim\operatorname{Sym}\gamma_{\mathcal{E}} = 0$*;*

Proof. It is obvious that a transformation (3.13) is a symmetry for $\mathcal{E}$ iff the diffeomorphism $x \mapsto X(x)$ is a symmetry for $\gamma_{\mathcal{E}}$.

Let ζ be a vector field on $J^0\pi$ and let the flow of ζ be generated by transformations (3.13). Then ζ is expressed in the standard coordinates x, y on $J^0\pi$ by the following formula

$$\zeta = a(x)\frac{\partial}{\partial x} + a_x \cdot y\frac{\partial}{\partial y}.$$

It is obvious that ζ is a classical symmetry for $\mathcal{E}$ iff the vector field $a(x)\dfrac{\partial}{\partial x}$ is an infinitesimal symmetry for $\gamma_{\mathcal{E}}$.

Generating function of ζ is

$$-a(x) \cdot p_1 + a_x \cdot y \tag{4.3}$$

.

Let φ be the generating function of an arbitrary classical symmetry for $\mathcal{E}$.

It follows from Theorem 2.1 that if $\dim \operatorname{Sym} \mathcal{E} = 10$, then φ is of the form (4.3) iff $\varphi = b(x) \cdot p_1 - b_x \cdot y$, where b is a solution of $\mathcal{E}$. Taking into account that there are three linear independent solutions for $\mathcal{E}$, we see that $\dim \operatorname{Sym} \gamma_{\mathcal{E}} = 3$.

In just the same way, if $\dim \operatorname{Sym} \mathcal{E} = 5$, then φ is of the form (4.3) iff $\varphi = K \cdot (\ (v_x - 2w)^{-1/3} \cdot p_1 + (\ (v_x - 2w)^{-1/3}\ )' \cdot y\ )$, where $K \in \mathbf{R}^1$. This means that $\dim \operatorname{Sym} \gamma_{\mathcal{E}} = 1$.

If $\dim \operatorname{Sym} \mathcal{E} = 4$, then there is no φ of the form (4.3). Therefore $\dim \operatorname{Sym} \gamma_{\mathcal{E}} = 0$.
■

4.2. Differential invariants of equations $\mathbf{p_3 = v(x) \cdot p_1 + w(x) \cdot y}$. Let geometric structure $\gamma_{\mathcal{E}}$ of type μ on $\mathbf{R}^1$ be defined by components $u^1 = V(X)$, $u^2 = W(X)$ in the coordinate system X on $\mathbf{R}^1$. We subject $\gamma_{\mathcal{E}}$ to an arbitrary diffeomorphism $x \mapsto X(x)$. Suppose that obtained geometric structure $\gamma_{\mathcal{E}'}$ is defined by components $u^1 = v(x)$, $u^2 = w(x)$; then components of these structures are connected by formulas (4.1), (4.2).

We differentiate equation (4.1) with respect to x and subtract the obtained equation from doubled equation (4.2). As a result we get the relation $(V_X - 2W)(X_x)^3 - (v_x - 2w) = 0$. Therefore,

$$X_x \cdot (V_X - 2W)^{1/3} = (v_x - 2w)^{1/3}. \tag{4.4}$$

This means that the differential form

$$\omega = (\ u^1_1 - 2 \cdot u^2\ )^{1/3} dx$$

on $J^1\rho_\mu$ is a differential invariant for geometric structures of type μ.

We say that ω is a differential invariant for ODEs of the form $p_3 = v(x) \cdot p_1 + w(x) \cdot y$.

We denote by $\omega(\mathcal{E})$ the value $\omega(\ \gamma_{\mathcal{E}}\ )$ of differential invariant ω on the geometric structure $\gamma_{\mathcal{E}}$ (see subsection 1.3). Thus if $\mathcal{E}$ is the equation $p_3 = v(x) \cdot p_1 + w(x) \cdot y$; then

$$\omega(\mathcal{E}) = (\ v_x - 2 \cdot w\ )^{1/3} dx.$$

Obviously, the following proposition is valid.

Proposition 4.2. *Let $\mathcal{E}$ be the equation $p_3 = v(x) \cdot p_1 + w(x) \cdot y$; then*

(1) $\dim \operatorname{Sym} \mathcal{E} = 10$ *iff* $\omega(\ \mathcal{E}\ ) = 0$;
(2) *if* $\omega(\mathcal{E}) = 0$, *then the equation* $\mathcal{E}$ *is locally equivalent to the equation* $p_3 = 0$ *at every point* $x_3 \in \mathcal{E}$.

Now let $\omega(\mathcal{E}) \neq 0$.

Taking into account (4.4), we have $X_x = \dfrac{(v_x - 2w)^{1/3}}{(V_X - 2W)^{1/3}}$. Substituting this expression for X_x in (4.1), we obtain relation (3.12). This relation means that the function I defined on $J^3\rho_\mu$ by formula (4.5) is a scalar differential invariant for geometric structures of type μ.

$$I = \left[\frac{2}{3}D\left(\frac{D(u^1_1 - 2u^2)}{u^1_1 - 2u^2}\right) - \frac{1}{9}\left(\frac{D(u^1_1 - 2u^2)}{u^1_1 - 2u^2}\right)^2 + u^1\right](u^1_1 - 2u^2)^{-2/3}, \qquad (4.5)$$

here $D = \sum_{i=1}^{2} \dfrac{\partial}{\partial x} + u^i_1\dfrac{\partial}{\partial u^i} + u^i_2\dfrac{\partial}{\partial u^i_1} + \ldots + u^i_k\dfrac{\partial}{\partial u^i_{k-1}} + \ldots$ is full derivative operator with respect to x.

We say that I is a scalar differential invariant for ODEs of the form $p_3 = v(x)\cdot p_1 + w(x)\cdot y$.

By definition, put $I(\,\mathcal{E}\,) = I(\,\gamma_{\mathcal{E}}\,)$. Thus if $\mathcal{E}$ is the equation $p_3 = v(x)\cdot p_1 + w(x)\cdot y$; then

$$I(\mathcal{E})(x) = \left[\frac{2}{3}\left(\frac{(v_x - 2w)_x}{v_x - 2w}\right)_x - \frac{1}{9}\left(\frac{(v_x - 2w)_x}{v_x - 2w}\right)^2 + v\right](v_x - 2w)^{-2/3}(x). \quad (4.6)$$

Now Theorem 3.2 can be rewritten in the following way.

Proposition 4.3. *Let $\mathcal{E}$ be the equation $P_3 = V(X)\cdot P_1 + W(X)\cdot Y$, let $\mathcal{E}'$ be the equation $p_3 = v(x)\cdot p_1 + w(x)\cdot y$, and let $\omega(\mathcal{E}) \neq 0$ and $\omega(\mathcal{E}') \neq 0$. Then $\mathcal{E}$ and $\mathcal{E}'$ are locally equivalent iff there is a solution of* (3.11) *satisfying to the equation*

$$I(\mathcal{E})(X(x)) = I(\mathcal{E}')(x). \qquad (4.7)$$

4.3. Equations with 5–dimensional algebra of classical symmetries.

Proposition 4.4. *Let $\mathcal{E}$ be the equation $p_3 = v(x)\cdot p_1 + w(x)\cdot y$; then* $\dim \operatorname{Sym} \mathcal{E} = 5$ *iff $I(\mathcal{E})$ is a constant.*

Proof. Suppose $\dim \operatorname{Sym} \mathcal{E} = 5$. It follows from Proposition 4.1 that $\gamma_{\mathcal{E}}$ admits a 1–dimensional algebra of infinitesimal symmetries. Hence $I(\mathcal{E})$ is an integral for any $\xi \in \operatorname{Sym} \gamma_{\mathcal{E}}$. It follows that $I(\mathcal{E})$ is a constant.

Conversely, suppose $I(\mathcal{E})$ is a constant. By definition, put $u = (v_x - 2w)^{-1/3}$, where v, w are components of expression of $\gamma_{\mathcal{E}}$ in the coordinate system x on $\mathbf{R}^1$. Then $I(\mathcal{E})$ can be rewritten in the form $I(\mathcal{E}) = -2u_{xx}u + (u_x)^2 + vu^2$. Therefore $\dfrac{dI(\mathcal{E})}{dx} = -2u(u_{xxx} - vu_x - \frac{1}{2}v_x u)$. Taking into account that the set $\{x \mid u(x) \neq 0\}$ is everywhere dense, we obtain that the condition $\dfrac{dI(\mathcal{E})}{dx} = 0$ is equivalent to the condition of corollary of Theorem 2.1 guaranteeing that $\dim \operatorname{Sym} \mathcal{E} = 5$. ∎

Proposition 4.5. *Suppose $\mathcal{E}$ is the equation $P_3 = V(X)\cdot P_1 + W(X)\cdot Y$, $\mathcal{E}'$ is the equation $p_3 = v(x)\cdot p_1 + w(x)\cdot y$, and $I(\mathcal{E})$ and $I(\mathcal{E}')$ are constants; then $\mathcal{E}$ and $\mathcal{E}'$ are locally equivalent iff $I(\mathcal{E}) = I(\mathcal{E}')$.*

Proof. This statement is an obvious consequence of Proposition 4.3. ∎

It follows from (4.6) that for every equation $\mathcal{E}$ of the form $p_3 = K\cdot p_1 + y$, $K \in \mathbf{R}^1$, we have $I(\mathcal{E}) = 2^{-2/3}\cdot K$.

Now the following theorem is obvious.

Theorem 4.6. *(The local contact classification of 3–order linear ODEs admitting a 5–dimensional classical symmetry algebra.)*

(1) *Let $\mathcal{E}$ be a 3-order linear ODE with* $\dim \operatorname{Sym} \mathcal{E} = 5$*; then $\mathcal{E}$ is locally equivalent to the equation $p_3 = 2^{2/3} \cdot I(\mathcal{E}) \cdot p_1 + y$ at almost every point $x_3 \in \mathcal{E}$.*

(2) *If numbers K and K' are not equal, then the equations $p_3 = K \cdot p_1 + y$ and $p_3 = K' \cdot p_1 + y$ are not local equivalent.*

(3) *The complete list of "simplest" locally nonequivalent regular 3–order linear ODEs with 5–dimensional algebra of classical symmetries consists of equations $p_3 = K \cdot p_1 + y, \quad K \in \mathbf{R}^1$.*

4.4. Equations with 4–dimensional algebra of classical symmetries. Let $\mathcal{E}$ be the equation $p_3 = v(x) \cdot p_1 + w(x) \cdot y$ with $\dim \operatorname{Sym} \mathcal{E} = 4$. Then $I(\mathcal{E})$ is not a constant function in domain of definition of the coefficients v and w. Therefore $I(\mathcal{E})$ can be considered as new independent variable in some neighborhood of point x such that $\dfrac{d\, I(\mathcal{E})}{dx}(x) \neq 0$.

Subjecting $\mathcal{E}$ to the point transformation

$$X = I(\mathcal{E})(x),$$
$$Y = \frac{d\, I(\mathcal{E})}{dx} \cdot y,$$

we obtain the equation $P_3 = V(X) \cdot P_1 + W(X) \cdot Y$, where the coefficients V and W considering as a functions of I are scalar differential invariants. We say that this equation is the *invariant form* of $\mathcal{E}$.

Let $\mathcal{E}$ be the equation $p_3 = v(x) \cdot p_1 + w(x) \cdot y$; then this equation is the invariant form of $\mathcal{E}$ iff the following identity holds

$$I(\mathcal{E})(x) \equiv x. \tag{4.8}$$

Proposition 4.7. *Suppose $\mathcal{E}$ and $\mathcal{E}'$ are equations of the form $p_3 = v(x) \cdot p_1 + w(x) \cdot y$ and* $\dim \operatorname{Sym} \mathcal{E} = \dim \operatorname{Sym} \mathcal{E}' = 4$*; then $\mathcal{E}$ and $\mathcal{E}'$ are locally equivalent iff the invariant forms for $\mathcal{E}$ and $\mathcal{E}'$ are the same.*

Proof. Let $P_3 = V(X) \cdot P_1 + W(X) \cdot Y$ be the invariant form of $\mathcal{E}$ and let $p_3 = v(x) \cdot p_1 + w(x) \cdot y$ be the invariant form of $\mathcal{E}'$.

Suppose $\mathcal{E}$ and $\mathcal{E}'$ are locally equivalent. Then equality (4.7) for these equations is $X = x$. Hence it follows from formulas (4.1) and (4.2) that $V = v$ and $W = w$.

The converse statement is obvious. ∎

Let the equation $p_3 = v(x) \cdot p_1 + w(x) \cdot y$ be the invariant form of some equation $\mathcal{E}$. By definition, put $u = (v_x - 2w)^{1/3}$. Then it follows from (4.6) that the coefficients v and w are represented as the functions of u in the following way:

$$v = xu^2 - 2\left(\frac{u_x}{u}\right)_x + \left(\frac{u_x}{u}\right)^2$$
$$w = \frac{1}{2}\left[\left(xu^2 - 2\left(\frac{u_x}{u}\right)_x + \left(\frac{u_x}{u}\right)^2\right)_x - u^3\right].$$

Consider an arbitrary ODE of the form

$$p_3 = \left[xu^2 - 2\left(\frac{u_x}{u}\right)_x + \left(\frac{u_x}{u}\right)^2 \right] \cdot p_1 + \frac{1}{2}\left[\left(xu^2 - 2\left(\frac{u_x}{u}\right)_x + \left(\frac{u_x}{u}\right)^2 \right)_x - u^3 \right] \cdot y, \quad (4.9)$$

where u is an arbitrary nowhere vanishing smooth function of x.

Proposition 4.8. *Equation* (4.9)*, is an invariant form.*

Proof. The proof is by direct verification of identity (4.8). ■

Theorem 4.9. *(The local contact classification of 3–order linear ODEs admitting a 4–dimensional classical symmetry algebra.)*

(1) *Let* $\mathcal{E}$ *be a 3-order linear ODE with* $\dim \operatorname{Sym} \mathcal{E} = 4$*; then* $\mathcal{E}$ *is locally equivalent to some equation* (4.9) *at almost every point* $x_3 \in \mathcal{E}$*.*

(2) *If functions* u *and* u' *are not equal one to another in some neighborhood and they are nowhere vanishing, then the corresponding equations of the form* (4.9) *are not locally equivalent.*

(3) *The complete list of "simplest" locally nonequivalent regular 3–order linear ODEs with 4–dimensional algebra of classical symmetries consists of equations* (4.9)*, where functions* u *are not equal one to another in some neighborhood and they are nowhere vanishing.*

Proof. (2) Assume the converse. Then the invariant forms of these equations are the same. Therefore $u = u'$.

Statements (1) and (3) are obvious . ■

References

1. D.V.Alekseevskiy, A.M.Vinogradov, V.V.Lychagin, *Fundamental ideas and conceptions of differential geometry,* Sovremennye problemy matematiki. Fundamental'nye napravleniy, Vol. 28, (Itogi nauki i techniki, VINITI, AN SSSR, Moscow, 1988 (Russian)) [English transl.: Encyclopedia of Math. Sciences, Vol.28 (Springer, Berlin, 1991)].
2. A.M.Vinogradov, *Scalar differential invariants, diffieties and characteristic classes, in: Mechanics, Analysis and Geometry: 200 Years after Lagrange, ed. M.Francaviglia (North-Holland)* (1991), 379–414.
3. Halphen G.–H., *La reduction des equations differentielles lineaires aux formes integrables*, Memoires presentes par duvers savants a l'Acad. des sci. de l'inst. math. de France **28** (1884), no. 1, 1–301.
4. Laguerre E., *Sur les equations differentielles lineaires du troisieme ordre*, Comptes Rendus. Acad. sci. Paris **88** (1879), 116–119.
5. Liouville R., *Sur les invariantes de certaines equationes differentielles*, Jour. de l'Ecole Politechnique **59** (1889), 7–88.
6. I.S.Krasil'shchik, V.V.Lychagin, A.M.Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations*, Gordon and Breach, New York, 1986.
7. Tresse A., *Sur les invariants differentiels des groupes countinues des transformations*, Acta mathematica **18** (1894), 1–88.
8. V.A.Yumaguzhin, *Point transformations and classification of 3-order linear ODEs*, Russian journal of mathematical physics **4** (1996), no. 3, 403-410.

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