Contact classification of linear ordinary differential equations. I.

V.A. YUMAGUZHIN

Abstract. It is known that a linear ordinary differential equation of order $n \geq 3$ can be transformed to the Laguerre–Forsyth form $y^{(n)} = \sum_{i=3}^{n} a_{n-i}(x)y^{(n-i)}$ by a point transformation of variables. The classification of equations of this form in a neighborhood of a regular point up to a contact transformation is given.

1. Introduction

This paper is devoted to the problem of local classification of n-th order linear ordinary differential equations (ODE) up to a contact transformation. For $n \leq 2$, it is well known (for example, see [2]) that any n-th order linear ODE can be transformed locally to the form $y^{(n)} = 0$ by a point transformation. For $n \geq 3$, this statement is incorrect: there is infinite number of different equivalence classes of linear ODEs.

First this problem was posed by classics of the XIX century E. Laguerre, G.-H. Halphen and others. They obtained results concerning classification of third and forth orders linear ODEs, see [7, 3]. Essentially, this problem was forgotten after that.

Here, we solve the problem for $n \geq 3$ in a neighborhood of a regular point. We considered the case $n = 3$ in [12, 13]. Our approach to the problem is as follows.

In their paper [8], F.M. Mahomed and P.G.L. Leach proved that dimension of the algebra of point symmetries of an $n-$th order linear ODE equals either $n + 4$ or $n + 2$, or $n + 1$. We prove (Theorem 3.2) that dimension of the algebra of point symmetries of a linear ODE is an invariant of contact transformations that take the set of linear ODEs to itself.

It is well known (see [11, 4, 9]) that any linear ODE can be transformed by a point transformation to the *Laguerre–Forsyth form*

$$(1) \qquad y^{(n)} = a_{n-3}(x)\, y^{(n-3)} + a_{n-4}(x) y^{(n-4)} + \cdots + a_0(x) y.$$

2000 *Mathematics Subject Classification.* 34A30, 34A26, 34C20,53A55, 58F35.

Key words and phrases. Linear ordinary differential equation, contact transformation, point transformation, projective transformation, equivalence problem, scalar differential invariants, classical symmetry, point symmetry, Laguerre–Forsyth form.

This work was supported by RFBR Grant 99-01-00353.

We prove (Theorem 3.3) that the Laguerre–Forsyth form of a linear ODE with $n+4$-dimensional algebra of point symmetries is $y^{(n)} = 0$.

For linear ODEs with $n+2$ and $n+1$-dimensional algebras of point symmetries, we prove (Theorem 3.6) that a contact transformation that takes one of these equation to the other one is a point transformation. Further, for any two equations $\mathcal{E}_1$ and $\mathcal{E}_2$ of the form (1), we prove (Theorem 3.7) that if there exists a point transformation that takes $\mathcal{E}_1$ to $\mathcal{E}_2$, then there exists a point transformation of the form

$$(2) \qquad f(x) = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \hat{f}(x, y) = |f'|^{(n-1)/2} \cdot y, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

that takes $\mathcal{E}_1$ to $\mathcal{E}_2$. Transformations (2) take the set of all ODEs of the form (1) to itself. Thus, the problem of local classification of linear ODEs with respect to contact transformation is reduced to classification of ODEs (1) with respect to transformations (2).

A transformation $(f, \hat{f})$ of the form (2) is generated by a projective transformation f of $\mathbb{R}^1$. The correspondence $f \mapsto (f, \hat{f})$ is an isomorphism from the group G of projective transformations of $\mathbb{R}^1$ to the group of point transformations of form (2).

Further, we identify ODE (1) with the section

$$S_{\mathcal{E}} \colon x \mapsto (a_{n-3}(x), a_{n-4}(x), \dots, a_0(x))$$

of the trivial bundle $\pi \colon E = \mathbb{R}^1 \times \mathbb{R}^{n-2} \to \mathbb{R}^1$. The transformation law of coefficients of ODEs (1) under transformations (2) defines the lifting of every transformation $f \in G$ of the base of π to a diffeomorphism $f^{(0)}$ of the total space E of π. So, it is possible to consider the transformation law of coefficients of ODEs (1) under (2) as the transformation law $S \mapsto f(S) \stackrel{\text{def}}{=} f^{(0)} \circ S \circ f^{-1}$ for sections of π under projective transformations of the base $\mathbb{R}^1$. Obviously, the transformation $(f, \hat{f})$ takes $\mathcal{E}_1$ of the form (1) to ODE $\mathcal{E}_2$ iff $f(S_{\mathcal{E}_1}) = S_{\mathcal{E}_2}$. Thus the problem of local classification of linear ODEs reduces to the one of classification of germs of sections of π w.r.t. the group G. Since this group is transitive, the last problem is reduces to classification of germs at $0 \in \mathbb{R}^1$ of sections of π w.r.t. the isotropy group $G_0 \subset G$ of $0 \in \mathbb{R}^1$. We obtain this classification for regular germs in Theorem 4.8 and Corollary 4.9.

Finally, we calculate (Theorem 5.2) scalar differential invariants of the action of the group $G_+ = \{ f \in G \mid f' > 0 \}$ on π. This gives a solution to the equivalence problem for regular linear ODEs (Theorem 5.4) resulting in canonical forms for all nonequivalent regular linear ODEs (Theorem 5.8 and its corollary).

All manifolds below and maps are assumed to be smooth; $\mathbb{R}^n$ denotes the n-dimensional arithmetical space.

2. Preliminaries

Let us recall necessary notation and results of the geometry of differential equations [5, 6] and some facts concerning linear ODEs [8, 11].

2.1. **Jet bundles.**

2.1.1. *Cartan distribution.* Let E and M be smooth manifolds of dimensions $n+m$ and n respectively and $\pi\colon E \to M$ be a smooth bundle. Denote by $[S]^k_x$ the k-jet of a section S at $x \in M$. Let $\pi_k\colon J^k\pi \to M$, $\pi_k\colon [S]^k_x \mapsto x$, $k = 0, 1, 2, \ldots, \infty$, be the bundle of k-jets of all sections of π. Moreover, the projection $\pi_{k,r}\colon J^k\pi \to J^r\pi$, $k > r$, is defined by $\pi_{k,r}([S]^k_x) = [S]^r_x$. Every section S of π generates the section j_kS of π_k by the formula $j_kS\colon x \mapsto [S]^k_x$. Denote by $L^{(k)}_S$ the image of the section j_kS.

Let $T_{x_k}(J^k\pi)$ denote the tangent space to $J^k\pi$ at $x_k \in J^k\pi$, $T_{x_k}(L^{(k)}_S)$ denote the tangent space to $L^{(k)}_S$ at $x_k \in L^{(k)}_S$. Consider all $L^{(k)}_S$ containing x_k. The subspace $\mathcal{C}_{x_k} \subset T_{x_k}(J^k\pi)$ spanning the union of $T_{x_k}(L^{(k)}_S)$ is called the *Cartan plane* at x_k. The distribution $\mathcal{C}\colon x_k \mapsto \mathcal{C}_{x_k}$ is called the *Cartan distribution* on $J^k\pi$.

2.1.2. *Lie transformations.* A (local) diffeomorphism of $J^k\pi$ that takes the Cartan distribution to itself is called a *Lie transformation.* A Lie transformation of $J^0\pi$ (that is an arbitrary diffeomorphism of $J^0\pi$) is called a *point transformation.* A Lie transformation of $J^1\pi$ is called a *contact transformation* if $m = 1$.

Every Lie transformation $f\colon U \to U'$ of $J^k\pi$ can be lifted canonically to the Lie transformation $f^{(r)}\colon \pi^{-1}_{k+r,k}(U) \to \pi^{-1}_{k+r,k}(U')$ of $J^{k+r}\pi$, $r = 1, 2, \ldots$, such that $\pi_{k+r,k+l} \circ f^{(r)} = f^{(l)} \circ \pi_{k+r,k+l}$ for $r \geq l$. Indeed, $f^{(r)}$ is defined in the following way. A point $x_{k+1} = [S]^{k+1}_x \in J^{k+1}\pi$ is identified by $K_{x_{k+1}} = T_{x_k}(L^{(k)}_S)$, where $x_k = \pi_{k+1,k}(x_{k+1})$. The differential f_* maps $K_{x_{k+1}}$ to the subspace $f_*(K_{x_{k+1}})$. If $f_*(K_{x_{k+1}})$ is projected on M nondegenerately, then there is $x'_{k+1} \in J^{k+1}\pi$ such that $K_{x'_{k+1}} = f_*(K_{x_{k+1}})$ and we set $f^{(1)}(x_{k+1}) = x'_{k+1}$. Obviously, $f^{(1)}$ is a Lie transformation of $J^{k+1}\pi$ defined almost everywhere in $\pi^{-1}_{k+1,k}(U)$. Setting $f^{(r+1)} = (f^{(r)})^{(1)}$, we define the Lie transformation $f^{(r)}$ for all $r = 1, 2, \ldots$ Clearly, $f^{(r)}$ is defined almost everywhere in $\pi^{-1}_{k+r,k}(U)$. We shall say for brevity that $f^{(r)}$ is defined in $\pi^{-1}_{k+r,k}(U)$.

It is well known [5, 6]) that any Lie transformation is the lifting of some point transformation if $m > 1$ and, if $m = 1$, any Lie transformation is the lifting of a contact transformation.

2.1.3. *Lie fields.* A vector field ξ on $J^k\pi$ is called a *Lie field* if its flow is generated by Lie transformations. A vector field in $J^0\pi$ is said to be a *point vector field.* A Lie field on $J^1\pi$ is called a *contact vector field* if $m = 1$. Let ξ be a Lie field in $J^k\pi$ and let f_t be its flow. Then the flow $f_t^{(r)}$ of Lie transformations defines the Lie field $\xi^{(r)}$ on $J^{k+r}\pi$, $r = 1, 2, \dots$ such that $(\pi_{k+r,k+l})_*\xi^{(r)} = \xi^{(l)}$, $r \geq l$.

2.2. **Ordinary differential equations.**

2.2.1. *Contact classification.* Let $\pi\colon \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ and $x, y, p_1, \dots, p_k$ be standard coordinates on $J^k\pi$. Any k-th order ODE

$$F(x, y(x), \frac{dy}{dx}, \dots, \frac{d^k y}{dx^k}) = 0$$

is identified with the submanifold

$$\mathcal{E} = \{\ F(x, y, p_1, \dots, p_k) = 0\ \} \subset J^k\pi.$$

A "usual" solution $S(x)$ is identified with the submanifold $L_S^{(k)} \subset \mathcal{E}$ corresponding to a section $S\colon x \mapsto S(x)$ of π. Obviously, $L_S^{(k)}$ is a 1-dimensional integral manifold of the Cartan distribution on $J^k\pi$. A "multivalued" solution of $\mathcal{E}$ is a 1-dimensional integral manifold L of the Cartan distribution on $J^k\pi$ such that $L \subset \mathcal{E}$. Locally, almost everywhere, a "multivalued" solution has the form $L_S^{(k)}$.

It is natural to classify k-th order ODEs up to a diffeomorphism of $J^k\pi$ that takes the set of all solutions of ODEs to itself. Such diffeomorphisms are Lie transformations. Hence, they are liftings of contact transformation. So, we come to the problem of ODE classification up to a contact transformation.

Let $\mathcal{E}_1, \mathcal{E}_2 \subset J^k\pi$ be k-th order ODEs and f be a point (contact) transformation. We say that f (locally) takes $\mathcal{E}_1$ to $\mathcal{E}_2$ if $f^{(k)}$ ($f^{(k-1)}$) takes (locally) the submanifold $\mathcal{E}_1 \subset J^k\pi$ to the submanifold $\mathcal{E}_2 \subset J^k\pi$. We say that ODEs $\mathcal{E}_1$ and $\mathcal{E}_2$ are *equivalent* if there exists a point (contact) transformation that takes (locally) $\mathcal{E}_1$ to $\mathcal{E}_2$.

2.2.2. *Point and contact transformations.* Any point transformation f is defined in coordinates by the formulas

$$X = X(x, y), \quad Y = Y(x, y). \tag{3}$$

Obviously, the lifting $f^{(k)}$ is defined in standard coordinates by

$$X = X(x, y),\ Y = Y(x, y),\ P_1 = \frac{DY}{DX}, \dots, P_k = \frac{DP_{k-1}}{DX}, \tag{4}$$

where

$$D = \frac{\partial}{\partial x} + p_1\frac{\partial}{\partial y} + p_2\frac{\partial}{\partial p_1} + \dots + p_{k+1}\frac{\partial}{\partial p_k} + \cdots \tag{5}$$

is the operator of the total derivative over x.

It is easy to show that a contact transformation is defined in standard coordinates by the formulas

$$X = X(x, y, p_1), \quad Y = Y(x, y, p_1), \quad P_1 = \frac{Y_x + p_1 Y_y}{X_x + p_1 X_y}, \tag{6}$$

where the functions $X(x, y, p_1)$, $Y(x, y, p_1)$ are connected by the relation

$$Y_{p_1}(X_x + p_1 X_y) - X_{p_1}(Y_x + p_1 Y_y) = 0.$$

2.2.3. *Point and contact vector fields.* Let ξ be a contact vector field. Then ξ can be represented in standard coordinates as

$$\xi = \xi_\varphi = -\varphi_{p_1}\frac{\partial}{\partial x} + (\varphi - p_1\varphi_{p_1})\frac{\partial}{\partial y} + (\varphi_x + p_1\varphi_y)\frac{\partial}{\partial p_1}, \tag{7}$$

where the function $\varphi = \varphi(x, y, p_1)$ is the *generating function* of ξ.

Let ζ be an arbitrary point vector field. It has the form

$$\zeta = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y} \tag{8}$$

in standard coordinates. The lifting $\zeta^{(1)}$ is a contact vector field. The generating function of $\zeta^{(1)}$ is

$$b(x, y) - a(x, y) \cdot p_1. \tag{9}$$

Conversely, if the generating function of a contact vector field has the form (9), then this vector field is the lifting of some point vector field.

Let us transform a contact vector field by contact transformation (6). It is easy to verify that the generating functions Φ of the obtained vector field and φ are connected by the formula

$$\frac{X_x + p_1 X_y}{X_x Y_y - X_y Y_x}\Phi(X, Y, P_1) = \varphi(x, y, p_1). \tag{10}$$

2.2.4. *Classical symmetries.* A point vector field ζ in $J^0\pi$ is called a *point symmetry* of a differential equation $\mathcal{E} \subset J^n\pi$ if $\zeta^{(n)}$ is tangent to $\mathcal{E}$. By $\operatorname{Pnt}\mathcal{E}$ we denote the set of all point symmetries of $\mathcal{E}$. A contact vector field ξ in $J^1\pi$ is called a *contact symmetry* of $\mathcal{E} \subset J^n\pi$ if $\xi^{(n-1)}$ is tangent to the submanifold $\mathcal{E}$. Point and contact symmetries are called *classical symmetries.* By $\operatorname{Sym}\mathcal{E}$ we denote the set of all classical symmetries of $\mathcal{E}$.

The space of generating functions of classical symmetries for an ODE $p_n - F(x, y, p_1, \ldots, p_{n-1}) = 0$ coincide with the space of solutions $\varphi = \varphi(x, y, p_1)$ of the linear PDE [5, 6]

$$(\bar{D}^n - \frac{\partial F}{\partial p_{n-1}}\bar{D}^{n-1} - \cdots - \frac{\partial F}{\partial p_1}\bar{D}^1 - \frac{\partial F}{\partial y})(\varphi) = 0, \tag{11}$$

where

$$\bar{D} = \frac{\partial}{\partial x} + p_1\frac{\partial}{\partial y} + p_2\frac{\partial}{\partial p_1} + \cdots + p_{n-1}\frac{\partial}{\partial p_{n-2}} + F\frac{\partial}{\partial p_{n-1}}.$$

2.3. **Linear ordinary differential equations.**

2.3.1. *Point transformations of linear ODEs.* Any linear ODE can reduced to the form

$$P_n = A_{n-2}(X)P_{n-2} + A_{n-3}(X)P_{n-3} + \cdots + A_0(X)Y \tag{12}$$

by a point transformation. It is known [11] that an arbitrary equation (12) of order $n \geq 3$ is reduced to the *Laguerre–Forsyth form* (1)

$$p_n = a_{n-3}(x)p_{n-3} + a_{n-4}(x)p_{n-4} + \cdots + a_0(x)y$$

by the point transformation

$$X = f(x), \quad Y = |f'|^{(n-1)/2}y,$$

where f is a solution of the ODE

$$2f'f''' - 3(f'')^2 - 24\frac{(n-2)!}{(n+1)!}(f')^4 A_{n-2}(f) = 0.$$

This transformation is called the *Laguerre–Forsyth transformation* of (12). It follows from this result that the problem of local classification of linear ODEs up to a contact transformation reduces to classification of ODEs of the form (1). The following proposition holds [11]:

Proposition 2.1. *Let $\mathcal{E}$ be an ODE of the form* (1)*. Then a point transformation X takes $\mathcal{E}$ to an ODE of the same form iff*

$$X = \frac{\alpha \cdot x + \beta}{\gamma \cdot x + \delta}, \quad Y = C \cdot |X'|^{(n-1)/2} \cdot y, \quad \alpha, \beta, \gamma, \delta, C \in \mathbb{R}\,. \tag{13}$$

2.3.2. *Point symmetries of linear ODEs.* Let $\mathcal{E}$ be an arbitrary ODE (1)). In [8] it was proved that a point symmetry of $\mathcal{E}$ has the form

$$(\varphi(x)\frac{\partial}{\partial x} + \frac{n-1}{2}\varphi' y\frac{\partial}{\partial y}) + Cy\frac{\partial}{\partial y} + \gamma(x)\frac{\partial}{\partial y},$$

where $\gamma(x)$ is a solution of $\mathcal{E}$, $C \in \mathbb{R}$, and $\varphi(x)$ is a solution of

$$\begin{cases} \varphi''' = 0 \\ 3a_{n-3}\varphi' + a'_{n-3}\varphi = 0 \\ \dfrac{(k-1)(n-(k-1))}{2}a_{n-k+1}\varphi'' + ka_{n-k}\varphi' + a'_{n-k}\varphi = 0, \\ k = 4, 5, \ldots, n. \end{cases} \tag{14}$$

Dimension of the solution space of system (14) can be equal to either 3, 1, or 0. It follows that $\dim \operatorname{Pnt} \mathcal{E}$ can be equal to either $n+4$, $n+2$, or $n+1$. Obviously, dimension of the algebra of point symmetries is an invariant of point transformations. Below, we prove (Theorem 3.2) that this dimension is an invariant of contact transformations that take the set of ODEs of the form (1) to itself. Thus the set of all linear ODEs $\mathcal{E}$ of the form (1) is divided into three nonintersecting families according to $\dim \operatorname{Pnt} \mathcal{E}$. These families are invariant w.r.t. contact transformations.

3. Symmetries and transformations of linear ODEs

3.1. **Classical symmetries.** Let $\mathcal{E}$ be an n-th order linear ODE. In [13], we proved for $n = 3$ the following results: $\dim \operatorname{Sym} \mathcal{E}$ can be 10, 5, or 4; if $\dim \operatorname{Sym} \mathcal{E} = 10$, then $\operatorname{Sym} \mathcal{E}$ is generated by three contact and seven point symmetries; if $\dim \operatorname{Sym} \mathcal{E} = 5$ or 4, then $\operatorname{Sym} \mathcal{E} = \operatorname{Pnt} \mathcal{E}$.

Proposition 3.1. *If* $n > 3$, *then* $\operatorname{Sym} \mathcal{E} = \operatorname{Pnt} \mathcal{E}$.

Proof. We can assume without loss of generality that $\mathcal{E}$ has the form (1).

Let $\varphi(x, y, y^{(1)})$ be the generating function of a classical symmetry. The generating function of a point symmetry is $\alpha(x, y)y^{(1)} + \beta(x, y)$. Hence, we must check that $\varphi_{y^{(1)}y^{(1)}} \equiv 0$. To this end, let us consider equation (11) for $\mathcal{E}$:

$$(\bar{D}^n - a_{n-3}\bar{D}^{n-3} - a_{n-4}\bar{D}^{n-4} - \cdots - a_1\bar{D}^1 - a_0)\varphi = 0. \tag{15}$$

Obviously,

$$\bar{D}^n(\varphi) = \bar{D}^{n-2}(\varphi_{y^{(1)}}y^{(3)} + \varphi_{y^{(1)}y^{(1)}}(y^{(2)})^2 + \text{low degree terms}) =$$

$$= \begin{cases} \bar{D}(3\varphi_{y^{(1)}y^{(1)}}y^{(2)}y^{(3)} + \text{l.d.t.}), & n = 4 \\ \bar{D}^{n-3}(\varphi_{y^{(1)}}y^{(4)} + 3\varphi_{y^{(1)}y^{(1)}}y^{(2)}y^{(3)} + \text{l.d.t.}), & n > 4 \end{cases}$$

$$= \begin{cases} 3\varphi_{y^{(1)}y^{(1)}}(y^{(3)})^2 + \text{l.d.t.}, & n = 4 \\ \left(\binom{n-3}{n-5} + 3\binom{n-3}{n-4} + 3\right)\varphi_{y^{(1)}y^{(1)}}y^{(3)}y^{(n-1)} + \text{l. d. t.}, & n > 4 \end{cases}$$

It now follows from (15) that $\varphi_{y^{(1)}y^{(1)}} \equiv 0$. □

Obviously, dimension of the algebra of classical symmetries is invariant under contact transformations. From the above mentioned results of [13] and this proposition, we get

Theorem 3.2. *Dimension of the algebra of point symmetries of a linear ODE is invariant under contact transformations that preserve the set of linear ODEs.*

3.2. **Linear ODEs with $n+4$-dimensional point symmetry algebra.** The following theorem gives local classification of all linear ODEs with $n + 4$-dimensional algebra of point symmetries.

Theorem 3.3. *The Laguerre–Forsyth form of a linear ODE with* $n+4$*-dimensional algebra of point symmetries is* $p_n = 0$.

Proof. Let $\mathcal{E} = \{\, p_n = a_{n-3}(x)p_{n-3} + a_{n-4}(x)p_{n-4} + \cdots + a_0(x)y \,\}$ be anODE of the form (1) with $\dim \operatorname{Pnt} \mathcal{E} = n + 4$. From the result of [8] cited in Subsection 2.3.2, we get that $\operatorname{Pnt} \mathcal{E}$ contains symmetries

$$\varphi_i(x)\frac{\partial}{\partial x} + \frac{n-1}{2}\varphi_i' y\frac{\partial}{\partial y},$$

where φ_i, $i = 1, 2, 3$, are linear independent solutions of system (14). From the second equation of the system, we have $a_{n-3} \equiv 0$; from the third equation, we have $a_{n-4} \equiv 0$, etc. □

Corollary 3.4. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be n-the order linear ODEs with $n+4$-dimensional algebras of point symmetries. Then there exists a point transformation that takes (locally) one of them to the other.*

Corollary 3.5.
1. *The equation $\mathcal{E} = \{ p_n = 0 \}$ is the only one in the set of all ODEs of the form (1) that has $n+4$-dimensional algebra of point symmetries.*
2. *The equation $\mathcal{E} = \{ p_n = 0 \}$ is invariant w.r.t. all contact transformations that preserve the equations (1).*

3.3. Contact transformations of linear ODEs.

Theorem 3.6. *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be linear ODEs with $n+2$ or $n+1$-dimensional algebras of point symmetries and f be a contact transformation that takes $\mathcal{E}_1$ to $\mathcal{E}_2$. Then f is the lifting of a point transformation.*

Proof. We can assume without loss of generality that $\mathcal{E}_1$ and $\mathcal{E}_2$ have the form (1).

Assume $\dim \operatorname{Pnt} \mathcal{E}_1 = n+2$. The transformation f is defined in standard coordinates by (6). Let $\Gamma_1(X)$, $\Gamma_2(X)$, $\Gamma_3(X)$ be linear independent solutions of $\mathcal{E}_1$. We can consider these solutions as generating functions of point symmetries of $\mathcal{E}_1$ (see Subsection 2.3.2). Eaech of these functions is connected with the corresponding generating function of a point symmetry of $\mathcal{E}_2$ by (10). Taking into account the form of generating functions of the $n+2$-dimensional algebra $\operatorname{Pnt} \mathcal{E}_2$ (see Subsection 2.3.2), we obtain:

$$\Delta\Gamma_1(X(x,y,y^{(1)})) = K_1(\varphi(x)y^{(1)} - \frac{n-1}{2}\varphi' y) + C_1 y + \gamma_1(x)$$
$$\Delta\Gamma_2(X(x,y,y^{(1)})) = K_2(\varphi(x)y^{(1)} - \frac{n-1}{2}\varphi' y) + C_2 y + \gamma_2(x)$$
$$\Delta\Gamma_3(X(x,y,y^{(1)})) = K_3(\varphi(x)y^{(1)} - \frac{n-1}{2}\varphi' y) + C_3 y + \gamma_3(x)$$

where $\Delta = \dfrac{X_x + y^{(1)} X_y}{X_x Y_y - X_y Y_x}$; $K_j, C_j \in \mathbb{R}$, $j = 1, 2, 3$. If one of the numbers K_1, K_2, K_3 does not vanish, say $K_1 \neq 0$, then

$$\Delta(\Gamma_2 - \frac{K_2}{K_1}\Gamma_1) = (C_2 - \frac{K_2}{K_1}C_1)y + \gamma_2 - \frac{K_2}{K_1}\gamma_1$$
$$\Delta(\Gamma_3 - \frac{K_3}{K_1}\Gamma_1) = (C_3 - \frac{K_3}{K_1}C_1)y + \gamma_3 - \frac{K_2}{K_1}\gamma_1$$

Then $\dfrac{K_1\Gamma_2 - K_2\Gamma_1}{K_1\Gamma_3 - K_3\Gamma_1}$ is independent of $y^{(1)}$. Therefore

$$\frac{\partial}{\partial y^{(1)}}\left(\frac{K_1\Gamma_2 - K_2\Gamma_1}{K_1\Gamma_3 - K_3\Gamma_1}\right) = \frac{d}{dX}\left(\frac{K_1\Gamma_2 - K_2\Gamma_1}{K_1\Gamma_3 - K_3\Gamma_1}\right) X_{y^{(1)}} = 0.$$

Suppose that $X_{y^{(1)}} \neq 0$. Then

$$K_1\Gamma_2 - K_2\Gamma_1 = K(K_1\Gamma_3 - K_3\Gamma_1),$$

where $K \in \mathbb{R}$. This means that the solutions Γ_1, Γ_2, Γ_3 are linear dependent. From this contradiction, we have $X_{y^{(1)}} \equiv 0$. Hence f is the lifting of some point transformation.

Obviously, the proofs for $K_1 = K_2 = K_3 = 0$ and $\dim \operatorname{Pnt} \mathcal{E} = n+1$ are similar. □

From this result, Theorem 3.3, and Corollary 3.5, we have that classification problem for linear ODEs w.r.t. contact transformations reduces to that for equations of the form (1) w.r.t. point transformations. Proposition 2.1 shows that the last problem reduced to classification of equation of the form (1) w.r.t. transformations (13).

3.4. **Reduction to the projective group.** From (4), we have that the lifting of transformation (13) to Lie transformation of $J^n\pi$ is defined by

$$X = \frac{\alpha \cdot x + \beta}{\gamma \cdot x + \delta}, \quad Y = C \cdot |X'|^{(n-1)/2} \cdot y,$$
$$P_k = C \cdot \nabla^k(|X'|^{(n-1)/2} \cdot y), \quad k = 1, 2, \ldots, n,$$

where $\nabla = \dfrac{1}{DX} \cdot D$ and D is operator (5). Consider two ODEs of the form (1):

$$\mathcal{E}_1 = \{\, P_n = A_{n-3}(X)P_{n-3} + A_{n-4}(X)P_{n-4} + \cdots + A_0(X)Y \,\}$$

and

$$\mathcal{E}_2 = \{\, p_n = a_{n-3}(x)p_{n-3} + a_{n-4}(x)p_{n-4} + \cdots + a_0(x)y \,\}.$$

Suppose, transformation (13) takes $\mathcal{E}_1$ to $\mathcal{E}_2$. This means that

$$C \cdot \nabla^n(|X'|^{(n-1)/2} \cdot y) = C \cdot \sum_{i=3}^{n} A_{n-i}(X(x)) \cdot \nabla^{n-i}(|X'|^{(n-1)/2} \cdot y).$$

Theorem 3.7. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be ODEs of the form* (1)*. Then if there exists a point transformation that takes $\mathcal{E}_1$ to $\mathcal{E}_2$, then there exist a point transformation of the form* (2)

$$f(x) = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \hat{f}(x, y) = |f'|^{(n-1)/2} \cdot y, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R},$$

that takes $\mathcal{E}_1$ to $\mathcal{E}_2$.

Thus the problem of local contact classification of linear ODEs reduces to that of local classification of ODEs of the form (1) w.r.t. point transformations (2). From (2), we have that the map $f \mapsto (f, \hat{f})$ is an isomorphism from the group of projective transformations of $\mathbb{R}^1$ to the group of point transformations of the form (2).

4. Classification of linear ODEs

Here we reduce the classification problem for linear ODEs to that for germs of sections of the bundle, related to an ODE, up to projective transformations.

4.1. **Bundles of linear ODEs.**

4.1.1. *The projective group.* Denote by G the Lie group of all projective transformations of $\mathbb{R}^1$, i.e.,

$$G = \left\{ f(x) = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} \;\middle|\; \alpha, \beta, \gamma, \delta \in \mathbb{R} \text{ and } \det \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \neq 0 \right\}.$$

It is easy to check that the set of nonconstant solutions of the equation

$$2f'''f' - 3(f'')^2 = 0 \tag{16}$$

coincides with G. Let

$$G_+ = \{\, f \in G \mid f' > 0 \,\}, \quad G_- = \{\, f \in G \mid f' < 0 \,\}.$$

Obviously, G_+ is the connected component of the unit in G, $G = G_+ \cup G_-$. Let $\mu \in G_-$ be defined by $\mu(x) = -x$, $x \in \mathbb{R}$. Then $G_- = \mu \circ G_+$.

Denote by $\mathfrak{g}$ the Lie algebra of G. It is easy to check that $\mathfrak{g}$ as a vector space over $\mathbb{R}$ is generated by the vector fields

$$\xi_0 = \frac{\partial}{\partial x}, \quad \xi_1 = x\frac{\partial}{\partial x}, \quad \xi_2 = x^2\frac{\partial}{\partial x}. \tag{17}$$

4.1.2. *Bundles of linear ODEs of the Laguerre–Forsyth form.* Let $\pi\colon E = \mathbb{R}^1 \times \mathbb{R}^{n-2} \to \mathbb{R}^1$ be the product bundle. Denote by x coordinates on the base $\mathbb{R}^1$ and by $a_{n-3}, a_{n-4}, \dots, a_0$ coordinates on the fiber $\mathbb{R}^{n-2}$.

We identify any linear ODE of the form (1)

$$\mathcal{E} = \{\, p_n = a_{n-3}(x)p_{n-3} + a_{n-4}(x)p_{n-4} + \cdots + a_0(x)y \,\}$$

with the section $S_{\mathcal{E}}$ of π defined by the formula

$$S_{\mathcal{E}}\colon x \mapsto (x, a_{n-3}(x), a_{n-4}(x), \dots, a_0(x)).$$

This identification $\mathcal{E} \mapsto S_{\mathcal{E}}$ is a bijection. We denote by $\mathcal{E}_S$ the equation corresponding to the section S under this identification.

Let

$$\mathcal{E}_2 = \{\, P_n = A_{n-3}(X)P_{n-3} + A_{n-4}(X)P_{n-4} + \cdots + A_0(X)Y \,\}$$

be an ODE of form the (1). Subjecting $\mathcal{E}_2$ to an arbitrary transformation $(f, \hat{f})$ of the form (2), we obtain linear ODE

$$\mathcal{E}_1 = \{ p_n = a_{n-3}(x)p_{n-3} + a_{n-4}(x)p_{n-4} + \cdots + a_0(x)y \}.$$

The coefficients of $\mathcal{E}_1$ are expressed in terms of the coefficients of $\mathcal{E}_2$ and the projective transformation f by equations of the following form

$$a_{n-j} = F_{n-j}(A_{n-3}, \ldots, A_{n-j}; \frac{df}{dx}, \ldots, \frac{d^{j+1}f}{dx^{j+1}}), \quad j = 3, 4, \ldots, n.$$

Obviously, the coefficients of $\mathcal{E}_2$ are expressed in terms of the coefficients of $\mathcal{E}_1$ and the projective transformation f^{-1} by the same equations

$$A_{n-j} = F_{n-j}(a_{n-3}, \ldots, a_{n-j}; \frac{df^{-1}}{dX}, \ldots, \frac{d^{j+1}f^{-1}}{dX^{j+1}}), \tag{18}$$

$j = 3, 4, \ldots, n$. Equations (18) define the lifting of any projective transformation f to a diffeomorphism $f^{(0)}$ of the bundle π such that $\pi \circ f^{(0)} = f \circ \pi$ (in the domain of $f^{(0)}$).

For any $f \in G$, we define the transformation of sections of π by the formula

$$S \mapsto f(S) = f^{(0)} \circ S \circ f^{-1}. \tag{19}$$

Now equations (18) can be represented as $S_{\mathcal{E}_2} = f(S_{\mathcal{E}_1})$. Obviously, the following statement holds.

Proposition 4.1. *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be equations of the form* (1)*. Then a transformation $(f, \hat{f})$ of the form* (2) *takes $\mathcal{E}_1$ to $\mathcal{E}_2$ iff $f(S_{\mathcal{E}_1}) = S_{\mathcal{E}_2}$.*

Thus, the problem of local classification of linear ODEs of the form (1) up to transformations of the form (2) reduces to classification of germs of sections of π up to a projective transformation of $\mathbb{R}^1$.

Since the action of G in $\mathbb{R}^1$ is transitive, the last problem reduces to classification of germs at $0 \in \mathbb{R}^1$ of sections of π w.r.t. the isotropy group $G_0 = \{ f \in G \mid f(0) = 0 \} \subset G$ of $0 \in \mathbb{R}^1$.

4.1.3. *Jet bundles.* Let $x; a_{n-3}, \ldots, a_0; a'_{n-3}, \ldots, a'_0; \ldots; a^{(k)}_{n-3}, \ldots, a^{(k)}_0$ be standard coordinates on the jet bundle $\pi_k \colon J^k\pi \to \mathbb{R}^1$, $k = 0, 1, 2, \ldots, \infty$. Any diffeomorphism $f^{(0)}$ $f \in G$ can be lifted to the Lie transformation $f^{(k)}$ of $J^k\pi$, $k = 1, 2, \ldots, \infty$ by the formula

$$f^{(k)}([S]^k_p) = [f^{(0)} \circ S \circ f^{-1}]^k_{f(p)}. \tag{20}$$

Obviously, for any $l > m$, one has $\pi_{l,m} \circ f^{(l)} = f^{(m)} \circ \pi_{l,m}$ (in the domains of $f^{(l)}$). In particular, $\mu^{(k)}$ is defined in standard coordinates by the formula

$$\mu^{(k)}((x, a^{(r)}_{n-j})) = (-x, (-1)^{j+r} a^{(r)}_{n-j}), \tag{21}$$

$j = 3, 4, \ldots, n$, $r = 0, 1, \ldots, k$.

Let $G^{(k)} = \{f^{(k)} \mid f \in G\}$, $k = 0, 1, 2, \dots, \infty$, and

$$G_+^{(k)} = \{\, f^{(k)} \mid f \in G_+ \,\}, \quad G_-^{(k)} = \{\, f^{(k)} \mid f \in G_- \,\}.$$

Obviously, $G_+^{(k)}$ is the connected component of the unit of $G^{(k)}$ and

$$G^{(k)} = G_+^{(k)} \cup G_-^{(k)}, \quad G_-^{(k)} = \mu^{(k)} \circ G_+^{(k)}.$$

The lifting of projective transformations of the base $\mathbb{R}^1$ to diffeomorphisms of $J^k\pi$ generates the lifting of any vector field $\xi \in \mathfrak{g}$ to the vector field $\xi^{(k)}$ on $J^k\pi$. By definition, $\xi^{(k)}$ is the vector field defined by the flow $f_t^{(k)}$, where f_t is the flow of ξ. Obviously $(\pi_{l,m})_*(\xi^{(l)}) = \xi^{(m)}$ for $l > m$.

Let $\xi = \varphi(x)\dfrac{\partial}{\partial x}$ be an arbitrary element of $\mathfrak{g}$. The vector field $\xi^{(\infty)}$ is defined by the formula (see [5])

$$\xi^{(\infty)} = \varphi D_x + \mathfrak{Э}_\psi, \tag{22}$$

where

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty}\sum_{j=3}^{n} a_{n-j}^{(k+1)} \frac{\partial}{\partial a_{n-j}^{(k)}}$$

is the operator of total derivative over x in $J^\infty\pi$,

$$\mathfrak{Э}_\psi = \sum_{k=0}^{\infty}\sum_{j=3}^{n} D_x^k(\psi_{n-j}) \frac{\partial}{\partial a_{n-j}^{(k)}}$$

is the evolutionary derivation with $\psi = (\psi_{n-3}, \dots, \psi_0)^t$ being its generating function. This function is defined in the following way. Let $x_1 = [S]_x^1 \in J^1\pi$, $x = \pi_1(x_1)$; then

$$\psi(x_1) = \begin{pmatrix} \psi_{n-3}(x_1) \\ \dots\dots\dots \\ \psi_0(x_1) \end{pmatrix} = \frac{d}{dt}(f_t^{(0)} \circ S \circ f_t^{-1})\Big|_{t=0}(x) \tag{23}$$

Let $S(x) = (x, a_{n-3}(x), \dots, a_0(x))$. Then, taking into account that $\dfrac{df_t}{dt}\Big|_{t=0} = \varphi$ and $\varphi''' = 0$, we obtain

$$\psi = \begin{pmatrix} -3a_{n-3}\varphi' - a_{n-3}^{(1)}\varphi \\ -\frac{3(n-3)}{2}a_{n-3}\varphi'' - 4a_{n-4}\varphi' - a_{n-4}^{(1)}\varphi \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -\frac{(k-1)(n-(k-1))}{2}a_{n-k+1}\varphi'' - ka_{n-k}\varphi' - a_{n-k}^{(1)}\varphi \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -\frac{n-1}{2}\cdot 1 \cdot a_1\varphi'' - na_0\varphi' - a_0^{(1)}\varphi \end{pmatrix} \tag{24}$$

Now it follows from (22) and (24) that for any $k = 0, 1, 2, \dots, \infty$,

$$\xi_0^{(k)} = \frac{\partial}{\partial x}, \tag{25}$$

$$\xi_1^{(k)} = x\frac{\partial}{\partial x} - \sum_{r=0}^{k}\sum_{j=3}^{n}(j+r)a_{n-j}^{(r)}\frac{\partial}{\partial a_{n-j}^{(r)}}, \tag{26}$$

$$\begin{aligned}\xi_2^{(k)} &= x^2\frac{\partial}{\partial x} - \sum_{r=0}^{k}\sum_{j=3}^{n}\Big[2x(j+r)a_{n-j}^{(r)}\frac{\partial}{\partial a_{n-j}^{(r)}} \\ &\quad + (j-1)(n-(j-1))a_{n-(j-1)}^{(r)}\frac{\partial}{\partial a_{n-j}^{(r)}} \\ &\quad + (2j+r-1)ra_{n-j}^{(r-1)}\frac{\partial}{\partial a_{n-j}^{(r)}}\Big],\end{aligned} \tag{27}$$

where $a_{n-2}^{(r)} = 0$.

4.2. **Projective symmetries.** Let S be a section of π and ξ be a vector field from $\mathfrak{g}$. By f_t we denote the flow of ξ. We say that ξ is a *projective symmetry* of S if one of the following equivalent conditions is fulfilled:

(1) the vector field $\xi^{(0)}$ is tangent to the image $L_S^{(0)}$ of S;
(2) $f_t(S) \stackrel{\text{def}}{=} f_t^{(0)} \circ S \circ f_t^{-1} = S$;
(3) $\frac{d}{dt}(f_t(S)\Big|_{t=0} = 0$.

Denote by $\operatorname{Prj} S$ the Lie algebra of all projective symmetries of S.

Proposition 4.2. *Consider the section* $S(x) = (x, a_{n-3}(x), \dots, a_0(x))$ *and let* $\xi = \varphi(x)\frac{\partial}{\partial x}$. *Then:*

1. ξ *is a projective symmetry of* S *iff* $\varphi(x)$ *is a solution of system* (14);
2. $\dim \operatorname{Prj} S$ *is equal to either* 3 *or* 1, *or* 0;
3. $\varphi(x)\frac{\partial}{\partial x}$ *is a projective symmetry of* S *iff* $\varphi(x)\frac{\partial}{\partial x} + \frac{n-1}{2}\varphi' y\frac{\partial}{\partial y}$ *is a point symmetry of the equation* $\mathcal{E}_S$;
4. $\dim \operatorname{Pnt} \mathcal{E}_S = \dim \operatorname{Prj} S_{\mathcal{E}} + n + 1$.

Proof. The first statement follows from (23), (24), and (14). The second one follows from (17) and (14). From Proposition 4.1, we obtain the third statement. The last statement follows from the results of [8]. □

4.3. **Invariant subbundles.** Let E^i, $i = n-3, n-4, \dots, 0, -1$, be the subspaces of the total space E of π defined by

$$E^i = \{\, (x, a_{n-3}, a_{n-4}, \dots, a_0) \in E \mid a_j = 0 \text{ if } j > i \,\}.$$

Consider the subbundle $\pi|_{E^i} \colon E^i \to \mathbb{R}$ of the bundle π.

Proposition 4.3. *Every subbundle* E^i *is* $G^{(0)}$*-invariant.*

Proof. From (25)–(27), we have that the restrictions of the vector fields $\xi_0^{(0)}, \xi_1^{(0)}, \xi_2^{(0)}$ to E^i are defined by

$$\xi_0^{(0)}\Big|_{E^i} = \frac{\partial}{\partial x}, \tag{28}$$

$$\xi_1^{(0)}\Big|_{E^i} = x\frac{\partial}{\partial x} - ((n-i)a_i\frac{\partial}{\partial a_i} + \cdots + na_0\frac{\partial}{\partial a_0}), \tag{29}$$

$$\begin{aligned}\xi_2^{(0)}\Big|_{E^i} &= x^2\frac{\partial}{\partial x} - 2x((n-i)a_i\frac{\partial}{\partial a_i} + \cdots + na_0\frac{\partial}{\partial a_0}) \\ &\quad - (i(n-i)a_i\frac{\partial}{\partial a_{i-1}} + \cdots + (n-1)a_1\frac{\partial}{\partial a_0}).\end{aligned} \tag{30}$$

Clearly, $\xi_0^{(0)}\Big|_{E^i}$, $\xi_1^{(0)}\Big|_{E^i}$, $\xi_2^{(0)}\Big|_{E^i}$ are tangent to E^i. Therefore every subbundle E^i is $G_+^{(0)}$-invariant. From (21), we have $\mu^{(0)}(E^i) = E^i$. ☐

Thus, we have the following sequence of the $G^{(0)}$-invariant subbundles: $E = E^{n-3} \supset E^{n-4} \supset \cdots \supset E^0 \supset E^{-1}$. Let E_i, $i = n-3, n-4, \ldots, 0, -1$, be the subsets of the total space E of π defined by

$$E_i = E^i \setminus E^{i-1} \text{ if } i \geq 0 \text{ and } E_{-1} = E^{-1}.$$

Consider the subbundle $\pi^i = \pi|_{E_i} \colon E_i \to \mathbb{R}$ of the bundle π.

Corollary 4.4. *Every subbundle E_i is $G^{(0)}$-invariant.*

Thus, E is the union

$$E = E_{n-3} \cup E_{n-4} \cup \cdots \cup E_0 \cup E_{-1} \tag{31}$$

of nonintersecting $G^{(0)}$-invariant subbundles.

The following proposition is needed for the sequel.

Proposition 4.5. *The symmetric differential $n-i$-form $\omega_i = a_i\, dx^{n-i}$ on E_i is $G^{(0)}$-invariant.*

Proof. Let us calculate the Lie derivatives of ω_i w.r.t. vector fields $\xi_0^{(0)}|_{E_i}$, $\xi_1^{(0)}|_{E_i}$, $\xi_2^{(0)}|_{E_i}$. From (28)–(30), we have

$$\xi_0^{(0)}\Big|_{E_i}(\omega_i) = 0,$$

$$\xi_1^{(0)}\Big|_{E_i}(\omega_i) = a_i(n-i)\, dx^{n-i} - (n-i)a_i\, dx^{n-i} = 0,$$

$$\xi_2^{(0)}\Big|_{E_i}(\omega_i) = a_i(n-i)2x\, dx^{n-i} - 2x(n-i)a_i\, dx^{n-i} = 0.$$

Hence ω_i is $G_+^{(0)}$-invariant. It follows from (21) that $(\mu^{(0)})^*(\omega_i) = \omega_i$. Thus ω_i is $G^{(0)}$-invariant. ☐

This result gives us the transformation law for the first nonzero component:

Corollary 4.6. *Let $\theta_0 = (x, 0, \ldots, 0, a_i, \ldots, a_0) \in E_i$, let $f \in G$, and $f^{(0)}(\theta_0) = (f(x), 0, \ldots, 0, A_i, \ldots, A_0) \in E_i$. Then $a_i = (f'(x))^{n-i}A_i$.*

4.4. **Classification of regular germs.**

4.4.1. *Regular germs.* Let S be a section of π, p be a point in a domain of S. Denote by $\{S\}_p$ the germ of S at p. Let $\{S_1\}_{p_1}$ and $\{S_2\}_{p_2}$ be germs of sections S_1 and S_2 respectively. We say that $\{S_1\}_{p_1}$ and $\{S_2\}_{p_2}$ are $G_+(G)$-*equivalent* if there exists $f \in G_+(G)$ with $\{\, f(S_1)\,\}_{f(p_1)} = \{S_2\}_{p_2}$. A germ $\{S\}_p$ is *regular of class* i if there exist a neighborhood V of p and subbundle E_i with $\operatorname{Im} S|_V \subset E_i$.

If $\{S\}_p$ is a regular germ of class $i \geq 0$, then one has $S(x) = (x, 0, \dots, 0, a_i(x), \dots, a_0(x))$ in a neighborhood of p. For this reason, we will often denote $\{S\}_p$ by $\{\, a_i, \dots, a_0\,\}_p$. If $\{S\}_p$ is a regular germ, then p is a *regular point of* S (*regular point of* $\mathcal{E}_S$).

Let $\mathcal{F}$ be the set of all regular germs at $0 \in \mathbb{R}^1$ of sections of π. Obviously, dimension of the algebra of projective symmetries of a section of π^i is an invariant of transformations (19). Then $\mathcal{F} = \mathcal{F}_3 \cup \mathcal{F}_1 \cup \mathcal{F}_0$, where the subsets

$$\mathcal{F}_r = \{\, \{S\}_0 \mid \dim \operatorname{Prj} S = r\,\}, \quad r = 3, 1, 0,$$

are nonintersecting G_0-invariant subsets. Obviously, $\mathcal{F}_3$ consists of the germ of the zero section only.

4.4.2. *Classification.* Here we classify regular germs from $\mathcal{F}_r$, $r = 0, 1$, w.r.t. G_0. Recall that

$$G_0 = \{\, f \in G \mid f(0) = 0\,\} = \left\{ \frac{\beta x}{\gamma x + 1},\ \beta, \gamma \in \mathbb{R},\ \beta \neq 0 \right\}.$$

By definition, put $G_{0+} = G_0 \cap G_+$, $G_{0-} = G_0 \cap G_-$. Then $G_0 = G_{0+} \cup G_{0-}$ and $G_{0-} = \mu \circ G_{0+}$.

Let $\mathcal{F}_{r,i} \subset \mathcal{F}_r$ be the subset of all regular germs of class i. It follows from Corollary 4.4 that $\mathcal{F}_{r,i}$ is G_0-invariant. Thus $\mathcal{F}_r$ is the union $\mathcal{F}_r = \cup_{i=0}^{n-3} \mathcal{F}_{r,i}$ of nonintersecting invariant subsets.

Let $\mathbb{R}_+ = \{\, a \in \mathbb{R} \mid a > 0\,\}$ and let $\mathbb{R}_- = \{\, a \in \mathbb{R} \mid a < 0\,\}$. Define the map $\ell_{r,i} \colon \mathcal{F}_{r,i} \to (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$ by the formula

$$\{\, a_i, \dots, a_0\,\}_0 \mapsto (a_i(0), a_i'(0)).$$

Consider the action $G_{0+} \times \mathcal{F}_{r,i} \to \mathcal{F}_{r,i}$, $(f, \{\, S\}_0) \mapsto \{f(S)\}_0$, of the group G_{0+} on $\mathcal{F}_{r,i}$. This action divides $\mathcal{F}_{r,i}$ into nonintersecting orbits. Let Θ be one of these orbits.

Proposition 4.7. *The map $\ell_{r,i}|_\Theta$ is a bijection from the orbit Θ either to $(\mathbb{R}_+) \times \mathbb{R}$ or to $(\mathbb{R}_-) \times \mathbb{R}$.*

Proof. Let $\{S\}_0 = \{\, a_i, \dots, a_0\,\}_0 \in \Theta$, let $f = \frac{\beta x}{\gamma x + 1}$ be an element of G_{0+}, and let $f(\{S\}_0) = \{\, A_i, \dots, A_0\,\}_0$. Then from Corollary 4.6 we have $a_i(x) = (f'(x))^{n-i} A_i(f(x))$. Since $\beta > 0$, the points $A_i(0)$ and $a_i(0)$ belong either to $\mathbb{R}_+$ or to $\mathbb{R}_-$. This means that $\ell_{r,i}(\Theta)$ belongs either to $(\mathbb{R}_+) \times \mathbb{R}$ or to $(\mathbb{R}_-) \times \mathbb{R}$.

Assume $\ell_{r,i}(\Theta) \subset (\mathbb{R}_+) \times \mathbb{R}$ and prove that the map $\ell_{r,i}|_\Theta \colon \Theta \to (\mathbb{R}_+) \times \mathbb{R}$ is an injection. Let $\{S_1\}_0 = \{\, a_i, \dots, a_0 \,\}_0 \in \Theta$ and $\{S_2\}_0 = \{\, A_i, \dots, A_0 \,\}_0 \in \Theta$. Then there exist a transformation $f(x) = d\frac{\beta x}{\gamma x+1} \in G_{0+}$ that takes $\{S_1\}_0$ to $\{S_2\}_0$. This means that in some neighborhood of 0, we have

$$a_i(x) = (f'(x))^{n-i} A_i(f(x)).$$

Differentiating both sides of this equation w.r.t. x, we obtain

$$a_i'(x) = -2(n-i)(f'(x))^{n-i-1} f''(x) A_i(f(x)) + (f'(x))^{n-i} A_i'(f(x)) f'(x).$$

Suppose $a_i(0) = A_i(0)$ and $a_i'(0) = A_i'(0)$. Then from the last two equations we have

$$\begin{cases} A_i(0) = & \beta^{n-i} A_i(0), \\ A_i'(0) = & -2(n-i)\beta^{n-i}\gamma A_i(0) + \beta^{n-i+1} A_i'(0). \end{cases}$$

From this system, we obtain $\beta = 1$ and $\gamma = 0$. This means that f is the identical transformation. It follows that $\{S_1\}_0 = \{S_2\}_0$. Therefore $\ell_{r,i}|_\Theta$ is an injection.

Let us prove that $\ell_{r,i}|_\Theta$ is a surjection. Let $(a, a') \in (\mathbb{R}_+) \times \mathbb{R}$ and let $\{S_1\}_0 = \{A_i, \dots, A_0\}_0 \in \Theta$. Obviously the equations

$$\begin{cases} a = & \beta^{n-i} A_i(0), \\ a' = & -2(n-i)\beta^{n-i}\gamma A_i(0) + \beta^{n-i+1} A_i'(0). \end{cases}$$

define the transformation $f(x) = \dfrac{\beta x}{\gamma x + 1} \in G_{0+}$ uniquely. Clearly, $\{S_2\}_0 = \{(f^{-1})^{(0)} \circ S_1 \circ f\}_0 \in \Theta$ and $\ell_{r,i}(\{S_2\}_0) = (a, a')$.

Obviously, the proof for the case $\ell_{r,i}(\Theta) \subset (\mathbb{R}_-) \times \mathbb{R}$ is the same. □

Let $\mathcal{L}^+_{r,i} = \ell^{-1}_{r,i}((1,0))$ and let $\mathcal{L}^-_{r,i} = \ell^{-1}_{r,i}((-1,0))$. Denote by $\mathcal{M}_{r,i}$ the subset of $\mathcal{L}^+_{r,i} \cup \mathcal{L}^-_{r,i}$ defined in the following way:

(1) if $i = 0$, then $\mathcal{M}_{r,0} = \mathcal{L}^+_{r,0} \cup \mathcal{L}^-_{r,0}$,
(2) if $i > 0$, then $\mathcal{M}_{r,i}$ consists of all germs $\{\, a_i, a_{i-1}, \dots, a_0 \,\}_0$ from $\mathcal{L}^+_{r,i} \cup \mathcal{L}^-_{r,i}$ satisfying one of the following conditions:
  (a) $a_{i-j}(0) = 0$ for all odd numbers j with $1 \le j \le i$,
  (b) there exists an odd number r with $1 \le r \le i$ such that $a_{i-r}(0) > 0$ and if $r > 1$, then $a_{i-j}(0) = 0$ for all odd numbers j with $1 \le j < r$.

Classification of regular germs from the family $\mathcal{F}_{r,i}$ is given by

Theorem 4.8. 1. *The set $\mathcal{L}^+_{r,i} \cup \mathcal{L}^-_{r,i}$ is a family of all germs from $\mathcal{F}_{r,i}$ nonequivalent w.r.t. G_{0+}.*

2. *If $n - i$ is odd, then $\mathcal{L}^+_{r,i}$ is a family of all germs from $\mathcal{F}_{r,i}$ nonequivalent w.r.t. G_0.*

3. *If $n-i$ is even, then $\mathcal{M}_{r,i}$ is a family of all germs from $\mathcal{F}_{r,i}$ nonequivalent w.r.t. G_0.*

Proof. From Proposition 4.7, we have the first statement. From (21), we have $\mu(\mathcal{L}^-_{r,i}) = \mathcal{L}^+_{r,i}$ from where the second statement follows. The third statement also follows immediately from (21). □

Corollary 4.9. *Classification of regular germs of sections is as follows*:

1. *The family of germs of the form*
$$\{\,\pm 1 + b(x)x^2, a_{i-1}(x), \dots, a_0(x)\,\}_0$$
is a family of all regular germs of class i nonequivalent w.r.t. G_{0+}.
2. *If $n-i$ is odd, then the family of germs of the form*
$$\{\,1 + b(x)x^2, a_{i-1}(x), \dots, a_0(x)\,\}_0$$
is a family of all regular germs of class i nonequivalent w.r.t. G_0.
3. *If $n-i$ is even, then the family of germs of the form*
$$\{\,\pm 1 + b(x)x^2, a_{i-1}(x), \dots, a_0(x)\,\}_0,$$
satisfying to one of the following conditions:
   a) *$a_{i-j}(0) = 0$ for all odd numbers j with $1 \le j \le i$,*
   b) *there exist an odd number r with $1 \le r \le i$ such that $a_{i-r}(0) > 0$ and if $r > 1$, then $a_{i-j}(0) = 0$ for all odd numbers j with $1 \le j < r$*

   is the family of all regular germs of class i nonequivalent w.r.t. G_0.

5. The equivalence problem

5.1. **Scalar differential invariants of linear ODEs.** Here we calculate scalar differential invariants of linear ODEs. For a general theory of scalar differential invariants refer to [1, 10].

It was proved in Subsection 4.3 that the bundle $\pi^i = \pi|_{E_i} \colon E_i \to \mathbb{R}$ is $G^{(0)}$-invariant. It follows that the jet bundles $J^k\pi^i$ are $G^{(k)}$-invariant, $k = 1, 2 \dots, \infty$. Hence $J^k\pi^i$ are invariant w.r.t. the subgroup $G^{(k)}_+ \subset G^{(k)}$, $k = 0, 1, 2 \dots, \infty$.

A function $I \in C^\infty(J^k\pi^i)$ is called a *scalar differential invariant* of G (G_+) if
$$(f^{(k)})^* I = I \quad \forall f \in G\ (G_+).$$
Let I be a scalar differential invariant of G (G_+) and S be a section of π^i. By definition, put $I(S) = (j_k S)^* I$. For any $f \in G$, we have

$$I(f(S)) \circ f = I(S). \tag{32}$$

Indeed,

$$
\begin{aligned}
I(f(S)) &= (j_k f(S))^* I = (j_k(f^{(0)} \circ S \circ f^{-1}))^* I \\
&= (f^{(k)} \circ j_k S \circ f^{-1})^* I = (f^{-1})^* \circ (j_k S)^* \circ (f^{(k)})^* I \\
&= (f^{-1})^* \circ (j_k S)^* I = (f^{-1})^* I(S) = I(S) \circ f^{-1}.
\end{aligned}
$$

Let S be a section of π admitting a 1-dimensional algebra of projective symmetries. Then $I(S)$ is a constant for any scalar differential invariant I. Indeed, let ξ be a projective symmetry of S and let f_t be its flow. Then

$$I(S) = I(f_t(S)) = I(f_t(S)) \circ f_t = I(S) \circ f_t.$$

It is not hard to prove that $I \in C^\infty(J^k\pi^i)$ is a scalar differential invariant of G_+ iff I is a solution of the system of linear PDEs

$$
\begin{cases}
\bar{\xi}_0^{(k)}(I) &= 0 \\
\bar{\xi}_1^{(k)}(I) &= 0 \\
\bar{\xi}_2^{(k)}(I) &= 0,
\end{cases}
\tag{33}
$$

where $\bar{\xi}_0^{(k)}$, $\bar{\xi}_1^{(k)}$ $\bar{\xi}_2^{(k)}$ are the restrictions of $\xi_0^{(k)}$, $\xi_1^{(k)}$, $\xi_2^{(k)}$ to $J^k\pi^i$. From (25)–(27), we have

(34)

$$\bar{\xi}_0^{(k)} = \xi_0^{(k)}\Big|_{J^k\pi^i} = \frac{\partial}{\partial x},$$

(35)

$$\bar{\xi}_1^{(k)} = \xi_1^{(k)}\Big|_{J^k\pi^i} = x\frac{\partial}{\partial x} - \sum_{r=0}^{k}\sum_{j=i}^{0}(n-j+r)a_j^{(r)}\frac{\partial}{\partial a_j^{(r)}},$$

$$\bar{\xi}_2^{(k)} = \xi_2^{(k)}\Big|_{J^k\pi^i} = x^2\frac{\partial}{\partial x} - \sum_{r=0}^{k}\sum_{j=0}^{i}\Big[2x(n-j+r)a_j^{(r)}\frac{\partial}{\partial a_j^{(r)}}$$

(36)

$$+ (n-j-1)(j+1)a_{j+1}^{(r)}\frac{\partial}{\partial a_j^{(r)}} + (2(n-j)+r-1)ra_j^{(r-1)}\frac{\partial}{\partial a_j^{(r)}}\Big].$$

Denote by $\mathcal{A}_i^k$ the algebra of scalar differential invariants of G_+ on $J^k\pi^i$. We identify A_i^k with its image $(\pi_{l,k}^i)^*(\mathcal{A}_i^k)$, $l > k$. As a result, we have the following filtration

$$A_i = A_i^\infty \supset \cdots \supset A_i^k \supset \cdots \supset A_i^1 \supset A_i^0.$$

Let $\mathcal{D}_i^k$ be the distribution on $J^k\pi^i$ generated by vector fields $\bar{\xi}_0^{(k)}$, $\bar{\xi}_1^{(k)}$, $\bar{\xi}_2^{(k)}$. From (34)–(36), we have that $\dim \mathcal{D}_i^k = 2$ if $i = 0$ and $k = 0$ otherwise $\dim \mathcal{D}_i^k = 3$.

Denote by N_i^k the number of functionally independent scalar differential invariant in A_i^k. Clearly,

$$N_i^k = \dim J^k\pi^i - \dim \mathcal{D}_i^k.$$

It is easy to prove that

(37) $N_0^0 = 0, \quad N_0^1 = 0, \quad N_0^k = k-1, \qquad$ if $k \geq 2$,

(38) $N_1^0 = 0, \quad N_1^k = 2k, \qquad$ if $k \geq 1$,

(39) $N_i^0 = i-1, \quad N_i^k = (k+1)(i-1)+2k, \qquad$ if $k \geq 1$.

Consider the vector field on $J^\infty\pi^i$

$$\zeta_i = |a_i|^{-1/(n-i)}\bar{D}_x, \tag{40}$$

where $\bar{D}_x = D_x\big|_{J^\infty\pi^i} = \partial/\partial x + \sum_{r=0}^{\infty}\sum_{j=i}^{0} a_j^{(r+1)}\partial/\partial a_j^{(r)}$ is the operator of total derivative over x restricted to $J^\infty\pi^i$.

Proposition 5.1. *The vector field ζ_i is invariant w.r.t. $G_+^{(\infty)}$.*

Proof. By $\bar{\xi}_r^{(\infty)}$, $r = 0, 1, 2$, we denote the restriction of $\xi_r^{(\infty)}$ to $J^\infty\pi^i$. Let us check that $[\zeta_i, \bar{\xi}_r^{(\infty)}] = 0$ for all r. By (25), we have

$$[\zeta_i, \bar{\xi}_0^{(\infty)}] = \left[|a_i|^{-1/(n-i)}\bar{D}_x, \frac{\partial}{\partial x}\right] = 0.$$

Using (24), consider the vector fields $\bar{\xi}_1^{(\infty)}$ and $\bar{\xi}_2^{(\infty)}$ in the form (22):

$$\bar{\xi}_1^{(\infty)} = x\bar{D}_x + \bar{\ni}_{((n-i)a_i + xa_i^{(1)})},$$
$$\bar{\xi}_2^{(\infty)} = x^2\bar{D}_x + \bar{\ni}_{((i+1)(n-i+1)a_{i+1} + 2(n-i)xa_i + x^2a_i^{(1)})},$$

where $\bar{\ni}_\psi$ is the restriction of $\ni_\psi$ on $J^\infty\pi^i$. Now taking into account that $[\bar{D}_x, \bar{\ni}_\psi] = 0$ for any ψ, we easily obtain that $[\zeta_i, \bar{\xi}_1^{(\infty)}] = 0$ and $[\zeta_i, \bar{\xi}_2^{(\infty)}] = 0$. □

Obviously, for any $I \in \mathcal{A}_i$, its Lie derivative $\zeta_i(I) \in \mathcal{A}_i$. Thus, ζ_i and I generate the sequence I, $\zeta_i(I), \ldots, \zeta_i^k(I), \ldots$ of scalar differential invariants from $\mathcal{A}_i$.

Theorem 5.2. *The algebra $\mathcal{A}_i$ is generated by the following free generators*

$$\zeta_i^k(I_{i-m}), \qquad m = 0, 1, \ldots, i, \quad k = 0, 1, 2, \ldots,$$

where

$$I_i = \left[2a_ia_i^{(2)} - \frac{2(n-i)+1}{n-i}(a_i^{(1)})^2\right] \cdot (a_i)^{-2(n-i+1)/(n-i)}; \tag{41}$$

$$I_{i-1} = \left[a_{i-1} - \frac{i}{2}a_i^{(1)}\right] \cdot |a_i|^{-(n-i+1)/(n-i)}; \tag{42}$$

for $2 \leq m \leq i$,

$$I_{i-m} = \Bigg[a_{i-m} + \frac{(-1)^m}{m!}\prod_{r=n-i+1}^{n-i+m-1}\frac{(n-r)r}{(n-i)i}(a_i)^{1-m}(a_{i-1})^m$$

$$+\sum_{l=n-i+1}^{n-i+m-1}\frac{(-1)^{n-i+m-l}}{(n-i+m-l)!}\prod_{r=l}^{n-i+m-1}\frac{(n-r)r}{(n-i)i}(a_i)^{i-n+l-m}\cdot$$

$$\cdot(a_{i-1})^{n-i+m-l}a_{n-l}\Big]\cdot|a_i|^{-(n-i+m)/(n-i)}. \tag{43}$$

Proof. It is not hard to check that $I_i,\dots,I_0$ are solutions of (33).

Let $i=0$, then $I_0\in\mathcal{A}_0^2$. For any $k=0,1,2,\dots$, the invariants I_0, $\zeta_0(I_0),\zeta_0^2(I_0),\dots,\zeta_0^k(I_0)$ belong to $\mathcal{A}_0^{k+2}$ and are functionally independent. The number of them equals $(k+2)-1$. Now from (37) we obtain that $(k+2)-1=N_0^{k+2}$. This concludes the proof for $i=0$.

Suppose $i\geq 1$. We have

$$\zeta_i(I_{i-1})=[-\frac{i}{2}|a_i|a_i^{(2)}+\dots]\cdot(a_i)^{-2(n-i+1)/(n-i)}$$

The manifold $J^\infty\pi^i$ has two connected components defined by the inequalities $a_i>0$ and $a_i<0$. Comparing $\zeta_i(I_{i-1})$ with I_i, we can define the scalar differential invariant $J\in\mathcal{A}_i^1$ by the formula

$$J=\begin{cases}I_i+\frac{4}{i}\zeta_i(I_{i-1}), & \text{if } a_i>0\\ I_i-\frac{4}{i}\zeta_i(I_{i-1}), & \text{if } a_i<0.\end{cases}$$

It is easy to see that

$$J=\left[\frac{4}{i}a_ia_{i-1}^{(1)}-\frac{4(n-i+1)}{i(n-i)}a_i^{(1)}a_{i-1}\right.$$
$$\left.+\frac{1}{n-i}(a_i^{(1)})^2\right](a_i)^{-2(n-i+1)/(n-i)}.$$

Let $i=1$. Then $I_{i-1},J\in\mathcal{A}_i^1$ and they are functionally independent. The invariants

$$I_{i-1},\ J,\ \zeta_i(I_{i-1}),\ \zeta_i(J),\ \dots,\ \zeta_i^k(I_{i-1}),\ \zeta_i^k(J)$$

belong to $\mathcal{A}_i^{k+1}$, $k=0,1,2,\dots$, they are functionally independent, and the number of them equals $2(k+1)$. Now from (38), we obtain $2(k+1)=N_1^{k+1}$. This concludes the proof for $i=1$.

Let $i>1$. Then the invariants $I_{i-2},\dots,I_0$ are functionally independent and they belong to $\mathcal{A}_i^0$. The invariants $I_{i-2},\dots,I_0,I_{i-1},J$ are functionally independent and lie in $\mathcal{A}_i^1$. Finally, the invariants

$$I_{i-2},\ \dots,\ I_0,\ I_{i-1},\ J,\ \dots,\ \zeta_i^k(I_{i-2}),\ \dots,\ \zeta_i^k(I_0),\ \zeta_i^k(I_{i-1}),\ \zeta_i^k(J)$$

are functionally independent, they belong to $\mathcal{A}_i^k$, $k=1,2,\dots$, and the number of them is equal to $(k+1)(i-1)+2k$. Now from (39), we obtain $(k+1)(i-1)+2k=N_i^k$. □

Remark 5.3. From (21), we obtain that

$$I_i=\left[2a_ia_i^{(2)}-\frac{2(n-i)+1}{n-i}(a_i^{(1)})^2\right]\cdot(a_i)^{-2(n-i+1)/(n-i)}$$

is an invariant of the group G.

5.2. **The equivalence problem of linear ODEs.** Let

$$S_1 \colon x \mapsto (x, 0, \ldots, 0, a_i(x), \ldots, a_0(x))$$

and

$$S_2 \colon X \mapsto (X, 0, \ldots, 0, A_i(X), \ldots, A_0(X))$$

be sections of π^i in neighborhoods of points $p \in \mathbb{R}$ and $P \in \mathbb{R}$ respectively. The sections S_1 and S_2 are *locally G_+-equivalent at (p, P)* if there exist $f \in G_+$ and neighborhoods V of p and U of P such that $f(p) = P$ and $f(S_1|_V) \stackrel{\text{def}}{=} f^{(0)} \circ S_1|_V \circ f^{-1} = S_2|_U$. G-equivalence is defined in the same way.

Theorem 5.4. *Sections S_1 and S_2 of π^i are locally G_+-equivalent at (p, P) iff the following conditions hold*:

1. $a_i(p) \cdot A_i(P) > 0$,
2. *the solution f of the Cauchy problem*

$$\begin{cases} f' = |a_i(x)|^{1/(n-i)} \cdot |A_i(f(x))|^{-1/(n-i)}, \\ f(p) = P \end{cases} \tag{44}$$

satisfies to the equations

$$I_m(S_2) \circ f = I_m(S_1), \quad m = i, i-1, \ldots, 0 \tag{45}$$

in some neighborhood of p.

Proof. Suppose S_1 and S_2 are locally G_+-equivalent at (p, P). Then there exist $f \in G_+$ and neighborhoods V of p and U of P such that $f(p) = P$ and $f(S_1|_V) = S_2|_U$. Consider the symmetric differential $n-i$-form ω_i on E_i (Proposition 4.5). We have

$$f^*(S_2^*(\omega_i)) = S_1^*(\omega_i). \tag{46}$$

Indeed,

$$f^*(f(S_1)^*(\omega_i)) = f^*((f^{(0)} \circ S_1 \circ f^{-1})^*(\omega_i))$$
$$= S_1^*((f^{(0)})^*(\omega_i)) = S_1^*(\omega_i).$$

Equality (46) means that $a_i(x) = (f')^{n-i} A_i(f(x))$. It also follows that either $a_i(p), A_i(P) > 0$ or $a_i(p), A_i(P) < 0$ and that f is a solution of Cauchy problem (44). Further, from (32), we have that equations (45) hold.

Conversely, let $a_i(p) \cdot A_i(P) > 0$, f be a solution of Cauchy problem (44), and f be a solution of equations (45). Let us show that $f \in G_+$. From (44), we can obtain f'' and f''' in terms of a_i, A_i and their 1-st and 2-nd derivatives:

$$f'' = \frac{1}{r}\left[|a_i|^{\frac{1-r}{r}} |A_i|^{\frac{-1}{r}} \operatorname{sgn}(a_i) a_i' - |a_i|^{\frac{2}{r}} |A_i|^{\frac{-r-2}{r}} \operatorname{sgn}(A_i) A_i' \right], \tag{47}$$

$$f''' = \frac{1}{r}\Bigg[\frac{1-r}{r}|a_i|^{\frac{1-2r}{r}}|A_i|^{\frac{-1}{r}}(a_i')^2 + |a_i|^{\frac{1-r}{r}}|A_i|^{\frac{-1}{r}}\operatorname{sgn}(a_i)a_i''$$

$$-\frac{3}{r}|a_i|^{\frac{2-r}{r}}|A_i|^{\frac{-r-2}{r}}a_i'A_i' + \frac{2+r}{r}|a_i|^{\frac{3}{r}}|A_i|^{\frac{-2r-3}{r}}(A_i')^2$$

$$-|a_i|^{\frac{3}{r}}|A_i|^{\frac{-r-3}{r}}\operatorname{sgn}(A_i)A_i''\Bigg], \tag{48}$$

where $r = n - i$. Substituting expressions (44), (47), and (48) for f', f'', and f''' in the left-hand side of equation (16), we obtain

$$2f'''f' - 3(f'')^2 = \frac{1}{n-i}|a_i|^{4/(n-i)}|A_i|^{-2/(n-i)}(I_i(S_1) - I_i(S_2)\circ f) = 0.$$

Thus, $f \in G_+$.

Let $S_3 = f(S_1)$. Then

$$I_m(S_3)\circ f = I_m(f(S_1))\circ f = I_m(S_1) = I_m(S_2)\circ f,$$

$m = i, i-1, \dots, 0$. Hence

$$I_m(S_3) = I_m(S_2) \quad m = i, i-1, \dots, 0.$$

Let $S_3 \colon X \mapsto (X, 0, \dots, 0, B_i, \dots, B_0)$. Then one obviously has $B_i = (f')^{-(n-i)}a_i = A_i$ in some neighborhood of P. It now follows from $I_{i-1}(S_3) = I_{i-1}(S_2)$ that $B_{i-1} = A_{i-1}$ in this neighborhood. From $I_{i-2}(S_3) = I_{i-2}(S_2)$, we have $B_{i-2} = A_{i-2}$ in this neighborhood and so on. Thus $S_3 = S_2$ in some neighborhood of P. □

Corollary 5.5. *Sections S_1, S_2 of π^i are locally G-equivalent at (p, P) iff S_1 locally G_+-equivalent either to S_2 at (p, P) or to $\mu(S_2)$ at $(p, -P)$.*

Corollary 5.6. *Let the invariants $I_m(S_1), I_m(S_2)$, $m = i, i-1, \dots, 0$ be constants. Then S_1, S_2 are G_+-locally equivalent at (p, P) iff the following conditions hold*:

1. *either $a_i(p)\cdot A_i(P) > 0$,*
2. *$I_m(S_1) = I_m(S_2)$, $m = i, i-1, \dots, 0$.*

Proposition 5.7. *Let S be a section of π^i. Then* $\dim \operatorname{Prj} S = 1$ *iff $I_i(S), I_{i-1}(S), \dots, I_0(S)$ are constants.*

Proof. The necessity was proved in the beginning of this subsection. Prove the sufficiency. Let $S(x) = (x, 0, \dots, 0, a_i(x), \dots, a_0(x))$ and invariants $I_i(S), I_{i-1}(S), \dots, I_0(S)$ be constants. From Proposition 4.2, we have that a vector field $\varphi(x)\partial/\partial x \in \mathfrak{g}$ is a projective symmetry of section S iff $\varphi(x)$ is a solution of system (14):

$$\begin{cases} \varphi''' = 0 \\ (n-i)a_i\varphi' + a_i'\varphi = 0 \\ \dfrac{(n-j-1)(j+1)}{2}a_{j+1}\varphi'' + (n-j)a_j\varphi' + a_j'\varphi = 0, \end{cases} \tag{49}$$

$j = i-1, i-2, \dots, 0$. From the 2-nd equation of the system, we have that $\varphi = C|a_i|^{-1/(n-i)}$, $C \in \mathbb{R}$.

From the identities $dI_m(S)/dx \equiv 0$, $m = i, i-1, \dots, 0$, we can obtain by direct calculations that $|a_i|^{-1/(n-i)}$ is a solution of system (49). Thus, the vector field $|a_i|^{-1/(n-i)}\partial/\partial x$ is a symmetry of S. □

5.3. **The canonical forms of linear ODEs.** In this subsection, we use the invariants $I_i, I_{i-1}, \dots, I_0$ to obtain canonical form of all nonequivalent regular germs at 0 of sections of π.

Denote by $\mathcal{Y}_i^{\pm}$ the system of equations on unknown section $S\colon x \mapsto (x, 0, \dots, 0, a_i(x), \dots, a_0(x))$ of π^i in a neighborhood of $0 \in \mathbb{R}^1$:

$$\begin{cases} I_i(S)(x) = K_i(x), \quad a_i(0) = \pm 1\,,\ a_i'(0) = 0 \\ I_{i-1}(S)(x) = K_{i-1}(x) \\ \dots\dots\dots\dots\dots\dots\dots \\ I_0(S)(x) = K_0(x), \end{cases}$$

where $K_i, K_{i-1}, \dots, K_0$ are smooth functions defined in neighborhoods of $0 \in \mathbb{R}$.

There exists a unique solution of the system $\mathcal{Y}_i^{\pm}$. Indeed, it follows from (41) that the first equation of the system is a Cauchy problem for 2-nd order ODE on unknown function $a_i(x)$. There exists a unique solution, say $a_i(x, K_i)$, to this problem. From (42), (43), we have that all other unknown functions $a_{i-1}, \dots, a_0$ are defined recursively, uniquely, and in an explicit form from the next equations of the system. Denote them by $a_{i-1}(x, K_i, K_{i-1}), \dots, a_0(x, K_i, K_{i-1}, \dots, K_0)$. Thus, the section

$$S(K_i, \dots, K_0)\colon x \mapsto (x, 0, \dots, 0, a_i(x, K_i), \dots, a_0(x, K_i, \dots, K_0))$$

is a unique solution of system $\mathcal{Y}_i^{+}$ ($\mathcal{Y}_i^{-}$).

Below we denote by $S^+(K_i, \dots, K_0)$ and $S^-(K_i, \dots, K_0)$ the solutions of $\mathcal{Y}_i^{+}$ and $\mathcal{Y}_i^{-}$ respectively. Now from Theorem 4.8, we obtain canonical forms of nonequivalent linear ODEs:

Theorem 5.8. 1. *The set* $\{\, S^{\pm}(K_i, \dots, K_0) \,\}_0$ *is the family of all regular germs of class* i *nonequivalent w.r.t.* G_{0+}.

2. *If* $n-i$ *is odd, then* $\{\, S^{+}(K_i, \dots, K_0) \,\}_0$ *is the family of all regular germs of class* i *nonequivalent w.r.t.* G_0.
3. *If* $n-i$ *is even, then* $\{\, S^{\pm}(K_i, \dots, K_0) \,\}_0$ *satisfying one of the conditions*:
   a) $a_{i-j}(0, K_i(0), \dots, K_{i-j}(0)) = 0$ *for all odd* j, $1 \le j \le i$,
   b) *there exist an odd number* r *with* $1 \le r \le i$ *such that*

$$a_{i-r}(0, K_i(0), \dots, K_{i-r}(0)) > 0$$

and if $r > 1$, *then for all odd numbers* j *with* $1 \le j < r$, $a_{i-j}(0, K_i(0), \dots, K_{i-j}(0)) = 0$

is a family of all regular germs of class i nonequivalent w.r.t. G_0*.*

Let $\{K_i, \dots, K_0\}_0$ be germ of the vector function $(K_i, \dots, K_0)$ at $0 \in \mathbb{R}^1$, $\mathcal{M}$ be the set of all these germs, and $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of germs of constant vector functions. Then canonical forms of nonequivalent linear ODEs with 1 and 0-dimensional algebras of projective symmetries are given by

Corollary 5.9. 1. *Suppose all vector functions* $(K_i, \dots, K_0)$ *are constant in Theorem* 5.8*; then the theorem gives the family of all regular germs of class i from* $\mathcal{F}_1$ *nonequivalent w.r.t.* G_0^+ *and* G_0*.*

2. *Suppose all vector functions* $(K_i, \dots, K_0)$ *in Theorem* 5.8 *satisfy the condition* $(K_i, \dots, K_0)_0 \in \mathcal{M} \setminus \mathcal{M}_1$*. Then the theorem gives the family of all regular germs of class i from* $\mathcal{F}_0$ *nonequivalent w.r.t.* G_{0+} *and* G_0*.*

References

[1] D.V. Alekseevskiy, V.V. Lychagin, and A.M. Vinogradov, *Fundamental ideas and conceptions of differential geometry, Sovremennye problemy matematiki. Fundamental'nye napravleniy, Vol. 28* (Itogi nauki i techniki, VINITI, AN SSSR, Moscow, 1988 (Russian)) [English transl.: Encyclopedia of Math. Sciences, Vol.28 (Springer, Berlin, 1991)]

[2] E. Cartan, Sur les varietes a connexion projective, *Bull. Soc. Math. France* **52** (1924) 205–241.

[3] G.-H. Halphen, Memoires sur la reduction des equations differentielles lineaires aux formes integrables, *Memoires presentes par duvers savants a l'Acad. des sci. de l'inst. math. de France*, **28** (1884) No. 1, 1–301.

[4] N.H. Ibragimov (ed), *New trends in theoretical developments and computational methods*, CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 3, CRC Press, Boca Raton, Fl., 1996.

[5] I.S. Krasil'shchik, V.V. Lychagin, and A.M. Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations*, Gordon and Breach, New York, 1986.

[6] I.S. Krasil'shchik and A.M. Vinogradov (eds.), *Symmetries and conservation laws for differential equations of mathematical physics*, American Mathematical Society, Providence, RI, USA, 1999.

[7] E.Laguerre, Sur les equations differentielles lineaires du troisieme ordre, *Comptes Rendus. Acad. Sci. Paris*, **88** (1879) 116–119.

[8] F.M. Mahomed and P.G.L. Leach, Symmetry Lie algebras of nth order ordinary differential equations, *J. Math. Analysis and Appl.*, **151** (1990) 80-107.

[9] F. Neuman, *Global properties of linear ordinary differential equations*, Kluwer Academic Publishers, Dordrecht, 1991.

[10] A.M. Vinogradov, Scalar differential invariants, diffieties and characteristic classes, in: M. Francaviglia (ed.) *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, North-Holland, 1991, 379–414.

[11] E.J. Wilczynski, *Projective differential geometry of curves and ruled surfaces*, B.G. Teubner, Leipzig, 1906.

[12] V.A. Yumaguzhin, Point transformations and classification of 3-order linear ODEs, *Russian Journal of Mathematical Physics*, **4** (1996) No. 3, 403–410.

[13] V.A. Yumaguzhin, Classification of 3-rd order linear ODEs up to equivalence, *Journal of Differential Geometry and its Applications*, **6** (1996) No. 4, 343–350.

E-mail address: `yuma@diffiety.botik.ru`

Program System Institute, m. Botik, Pereslavl-Zalessky, 152020, Russia.