FINITE TYPE INTEGRABLE GEOMETRIC STRUCTURES

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Abstract. In this paper, we consider the finite type geometric structures of arbitrary order. The aim of this paper is to solve the integrability problem for these structures. This problem is equivalent to the integrability problem for the corresponding G-structures. The latter problem is solved by constructing the structure functions for G-structures of order ≥ 1. These functions coincide with the well-known ones, see [1], for the first order G-structures, although their constructions are different.

We prove that a finite type G-structure is integrable iff the structure functions of the corresponding number of its first prolongations are equal to zero.

Applications of this result to second and third-order ordinary differential equations are noted.

Introduction. This paper is devoted to the integrability problem of finite type geometric structures of arbitrary orders, that is, the problem of local equivalence between these structures and the flat ones is investigated.

Following the paper [4], we interpret an arbitrary geometric structure on a smooth manifold M as a map $\Omega : P_k(M) \to \mathbb{R}^N$, where $P_k(M)$ is the bundle of k-frames of M and $\mathbb{R}^N$ is the N-dimensional arithmetic space. We suppose that the differential group of order k acts on $\mathbb{R}^N$ and assume that this action is compatible with the natural action of the group on $P_k(M)$, see Sec. 1.4.

Let $q \in \operatorname{Im}\Omega$. The inverse image $B = \Omega^{-1}(q)$ is a G-structure of order k. Let $\Omega^{(r)}$ be the differential prolongation of order r for the structure Ω and let q_r be its value that projects naturally to q. Then $B^{(r)} = (\Omega^{(r)})^{-1}(q_r)$ is a $G^{(r)}$-structure of order $k+r$. This $B^{(r)}$ is called the *prolongation of order* r for the G-structure B.

Obviously, the integrability problem for geometric structures is equivalent to the integrability problem for their G-structures.

In order to solve the latter problem, we follow the well-known approach to the equivalence problem for the first-order G-structures (see,

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e.g., [1]). We construct the structure functions of G-structures of arbitrary orders. These structure functions are defined on G-structures and their values are the Spencer δ-cohomologies. For structures of order 1, these functions coincide with the well-known structure functions of G-structures of order 1, see [1], although their constructions are different.

In Theorem 3.2 we prove that a finite type G-structure is integrable iff the structure functions of the corresponding number of its first prolongations are equal to zero.

In the last section, we consider the applications of this theorem. We note that the known point transformations linearizability condition for second-order ordinary differential equations and a condition for a third-order ordinary differential equation to be reducible to the form $y''' = 0$ by contact transformations are the integrability conditions for the correspondence finite type G-structures.

Throughout this paper, all manifolds and maps are supposed to be smooth. By $[f]_p^k$ we denote the k-jet of a map f at a point p, by $\mathbb{R}$ we denote the field of real numbers, and by $\mathbb{R}^n$ we denote the n-dimensional arithmetic space.

1. Preliminaries

In this section, we recall all necessary preliminary facts from the papers [2]-[6].

1.1. **Formal vector fields.** By W_n we denote the set of ∞-jets at $0 \in \mathbb{R}^n$ of all vector fields defined in neighborhoods of 0 in the space $\mathbb{R}^n$. The operations

$$\lambda \cdot [X]_0^\infty = [\lambda \cdot X]_0^\infty,$$
$$[X]_0^\infty + [Y]_0^\infty = [X+Y]_0^\infty,$$
$$[\,[X]_0^\infty, [Y]_0^\infty\,] = [\,[X,Y]\,]_0^\infty$$

define the Lie algebra structure on W_n.

By L_k, $k = -1, 0, 1, 2, \ldots$, we denote the subalgebra in W_n defined by

$$L_k = \{\, [X]_0^\infty \in W_n \mid [X]_0^k = 0 \,\}, \ k \geq 0, \quad L_{-1} = W_n.$$

By definition, put

$$V = W_n / L_0 .$$

Obviously, $V \cong \mathbb{R}^n$. We have the filtration

$$W_n = L_{-1} \supset L_0 \supset L_1 \supset \ldots \supset L_k \supset L_{k+1} \supset \ldots .$$

The formula

$$[\, L_i \,,\ L_j \,] \;=\; L_{i+j} \,, \quad i \geq -1, \ j \geq 0$$

makes possible to define the bracket operations on the quotient spaces:

$$[\,\cdot\,,\,\cdot\,]: W_n/L_k \times W_n/L_k \to W_n/L_{k-1}\,, \tag{1}$$

$$[\,\cdot\,,\,\cdot\,]: V \times L_k/L_{k+1} \to L_{k-1}/L_k\,. \tag{2}$$

The last formula leads to the canonical isomorphism

$$L_k/L_{k+1} \cong V \otimes S^k(V^*)\,.$$

Let $g_k \subset L_{k-1}/L_k$. Then the subspace $g_k^{(i)} \subset L_{k-1+i}/L_{k+i}$ is defined by

$$g_k^{(i)} = \{\, X \in L_{k-1+i}/L_{k+i} \mid \\ \mid \forall\, v_1, \dots, v_i \in V\ [\, v_1, \dots\,,\ [\, v_i,\ X\,] \dots] \in g_k \,\},$$

and is called the *i-th prolongation of g_k*.

Suppose that the sequence of subspaces

$$g_1\,,\ g_2\,,\ \dots\,,\ g_i\,,\ \dots\,,$$

where $g_i \subset L_{i-1}/L_i$, for $i = 1, 2, 3, \dots$, satisfies the condition

$$g_{i+1} \subset g_i^{(1)}\,.$$

Then for every g_i there exists the complex

$$0 \to g_i \xrightarrow{\partial_{i,0}} g_{i-1} \otimes V^* \xrightarrow{\partial_{i-1,1}} g_{i-2} \otimes \wedge^2 V^* \xrightarrow{\partial_{i-2,2}} \dots \\ \dots \xrightarrow{\partial_{2,i-2}} g_1 \otimes \wedge^{i-1} V^* \xrightarrow{\partial_{1,i-1}} V \otimes \wedge^i V^*\,, \tag{3}$$

where the operator $\partial_{k,l} : g_k \otimes \wedge^l V^* \to g_{k-1} \otimes \wedge^{l+1} V^*$ is defined in the following way: an element $\xi \in g_k \otimes \wedge^l V^*$ can be considered as the external form on V with values in g_k, then

$$(\,\partial_{k,l}(\xi)\,)(v_1, \dots, v_{l+1}) = \sum_{i=1}^{l+1} (-1)^{i+1} [\, v_i\,,\ \xi(v_1, \dots, \hat{v}_i, \dots, v_{l+1})\,]\,.$$

The cohomologies of this complex in the term $g_k \otimes \wedge^l V^*$ are denoted by $H^{k,l}$. These are the Spencer δ-cohomologies.

1.2. **Differential groups.** Let $\mathcal{D}$ be the set of all diffeomorphisms d defined in neighborhoods of $0 \in \mathbb{R}^n$ and satisfying the condition $D(0) = 0$. By definition, put

$$D_k = \{\ [d]_0^k \mid d \in \mathcal{D}\ \}\,.$$

The operation $[d_1]_0^k \cdot [d_2]_0^k = [d_1 \circ d_2]_0^k$ defines the Lie group structure on D_k. Obviously,

$$([d]_0^k)^{-1} = [d^{-1}]_0^k \quad \text{and} \quad [\mathrm{id}]_0^k \ \text{is the unity of the group}\ D_k\,.$$

The Lie group D_k is called the *differential group of order k*.

Obviously, the Lie algebra of D_k is identified with the Lie algebra L_0/L_k.

By D_k^{k-1} we denote the subgroup of D_k defined by

$$D_k^{k-1} = \{\, [d]_0^k \in D_k \mid [d]_0^{k-1} = [\mathrm{id}]_0^{k-1} \,\}\,.$$

Its Lie algebra is identified with L_{k-1}/L_k.

1.3. **Frame bundles.** Let M be an n-dimensional smooth manifold. Consider all diffeomorphisms of neighborhoods of $0 \in \mathbb{R}^n$ to M. By $P_k(M)$ we denote the set of k-jets at 0 of all these diffeomorphisms. The following natural projection holds:

$$\pi_k : P_k(M) \to M\,, \quad \pi_k : [s]_0^k \mapsto s(0)\,.$$

A local chart $(\,U,\, (x^1, \ldots, x^n)\,)$ in M generates the local chart in $P_k(M)$ $(\,\pi_k^{-1}(U),\, (x^i, x^i_j, \ldots, x^i_{j_1 \ldots j_k})\,)$. In this chart, the coordinates of a point $[s]_0^k \in \pi_k^{-1}(U)$ are calculated by the formula

$$x^i_{j_1 \ldots j_r}([s]_0^k) = \frac{\partial^r (x^i \circ s)}{\partial t^{j_1} \ldots \partial t^{j_r}},$$
$$i, j_1, \ldots, j_r = 1, \ldots, n\,, \quad r = 0, 1, \ldots, k\,,$$

where $t^1, \ldots, t^n$ are the standard coordinates on $\mathbb{R}^n$. Now we see that $P_k(M)$ is a smooth manifold.

It is easy to prove that $\pi_k : P_k(M) \to M$ is a smooth locally trivial bundle. The group D_k acts freely and transitively on the fibers of this bundle:

$$[s]_0^k \cdot [d]_0^k = [s \circ d]_0^k \quad \forall\, [s]_0^k \in P_k(M)\,,\ \forall\, [d]_0^k \in D_k\,.$$

Thus, $P_k(M)$ is a principal bundle over M with the structure group D_k. $P_k(M)$ is called the *bundle of k-frames of M*.

The natural projection $\pi_{l,m} : P_l(M) \to P_m(M)\,,\ l \geq m$, is defined by $\pi_{l,m}([s]_0^l) = [s]_0^m$.

Let $\theta_k \in P_k(M)$, let $p = \pi_k(\theta_k)$, and let $T_{\theta_k} P_k(M)$ be the tangent space to $P_k(M)$ at the point θ_k.

Proposition 1.1. *Let $\theta_{k+1} \in \pi_{k+1,k}^{-1}(\theta_k)$. Then:*

(1) *θ_{k+1} defines the isomorphism of vector spaces*

$$T_{\theta_k} P_k(M) \longrightarrow W_n/L_k\,.$$

We will denote this isomorphism by θ_{k+1} too.

(2) *The reduction of the inverse isomorphism $(\theta_{k+1})^{-1}$ to L_0/L_k is the canonical isomorphism of the Lie algebra of the structure group D_k to the space $T_{\theta_k}(\,\pi_k^{-1}(p)\,)$ tangent to the fiber of π_k over the point p.*

Proof. Let $[s]_0^{k+1} = \theta_{k+1}$ and $s(0) = p$. By $T_p^k(M)$ we denote the space of k-jets at p of all vector fields in M passing through p. Obviously, the map

$$\alpha : T_p^k(M) \to T_{\theta_k} P_k(M)\,, \quad \alpha : [X]_p^k \mapsto \frac{d}{dt}(\,[\varphi_t \circ s]_0^k\,)\Big|_{t=0},$$

where φ_t is the flow of X, is an isomorphism of vector spaces. Also, the map

$$\beta : T^k_p(M) \to T^k_0\mathbb{R}^n\,, \quad \beta : [X]^k_p \mapsto \frac{d}{dt}(\,[s^{-1}\circ\varphi_t\circ s]^k_0\,)\Big|_{t=0}$$

is an isomorphism of vector spaces. The isomorphism θ_{k+1} is defined now by the formula

$$\theta_{k+1} = \beta\circ\alpha^{-1}\,.$$

The canonical isomorphism $L_0/L_k \to T_{\theta_k}(\,\pi_k^{-1}(p)\,)$ is defined by the formula

$$d/dt(\,[d_t]^k_0\,)\big|_{t=0} \mapsto d/dt(\,[s\circ d_t]^k_0\,)\big|_{t=0}.$$

This formula can be rewritten in the following way:

$$d/dt(\,s^{-1}\circ(s\circ d_t\circ s^{-1})\circ s\,)\big|_{t=0} \mapsto \qquad\qquad d/dt(\,[(s\circ d_t\circ s^{-1})\circ s]^k_0\,)\big|_{t=0}.$$

This completes the proof. □

The diffeomorphism s^{-1} is a local chart in M. It generates the local chart $(x^i, x^i_j, \ldots, x^i_{j_1\ldots j_k})$ in $P_k(M)$ as stated above. Obviously, within this chart, the isomorphism θ_{k+1} is defined by

$$\theta_{k+1} : X^i\frac{\partial}{\partial x^i} + \ldots + X^i_{j_1\ldots j_k}\frac{\partial}{\partial x^i_{j_1\ldots j_k}} \longmapsto (X^i, \ldots, X^i_{j_1\ldots j_k})\,. \tag{4}$$

Let $\theta_{k+1}, \tilde\theta_{k+1} \in (\pi_{k+1,k})^{-1}(\theta_k)$. Then there exists a unique element $[d]^{k+1}_0 = (\delta^i_j, 0, \ldots, 0, d^i_{j_1\ldots j_{k+1}}) \in D^k_{k+1}$ such that $\tilde\theta_{k+1} = \theta_{k+1}\cdot[d]^{k+1}_0$. It is easy to prove the following statement.

Proposition 1.2. *Let $\xi \in T_{\theta_k}P_k(M)$ and*

$$\theta_{k+1}(\xi) = (X^i, \ldots, X^i_{j_1\ldots j_{k-1}}, X^i_{j_1\ldots j_k}).$$

Then

$$\tilde\theta_{k+1}(\xi) = (X^i, \ldots, X^i_{j_1\ldots j_{k-1}}, X^i_{j_1\ldots j_k} + d^i_{j_1\ldots j_k r}X^r).$$

Let f be an arbitrary diffeomorphism of M to itself. Then the diffeomorphism $f^{(k)} : P_k(M) \to P_k(M)$ is defined by

$$f^{(k)}(\,[s]^k_0\,) = [\,f\circ s\,]^k_0\,.$$

The diffeomorphism $f^{(k)}$ is called the *lift of f to the bundle $P_k(M)$*.

1.4. **Geometric structures.** Recall that a *geometric structure* on M is defined by the following three conditions:

(1) a collection of functions $q(x) = (\,q^1(x), \ldots, q^N(x)\,)$ is defined for every local coordinate system $x = (\,x^1, \ldots, x^n\,)$ in M. These functions are called the *components* of the geometric structure in the coordinate system x;
(2) some action $F\colon D_k\times\mathbb{R}^N \to \mathbb{R}^N$ of the group D_k is defined on $\mathbb{R}^N$;

(3) suppose $q(x)$ and $\tilde{q}(y)$ are the collections of components of the structure in a coordinate systems x and y respectively, suppose $y = y(x)$ is the transformation of these coordinates; then the collections $q(x)$ and $\tilde{q}(y)$ are related in the following way:

$$\tilde{q}(y) = F\left(\frac{\partial y^i}{\partial x^j}, \dots, \frac{\partial^k y^i}{\partial x^{j_1} \dots \partial x^{j_k}},\ q(x) \right). \tag{5}$$

The number k is called the *order* of this structure and F is called the *transformation law of the components of the structure.*

The following equivalent definition of a geometric structure for the first time was given by V. V. Vagner in his paper [4]. This definition is more convenient for us. Let $F : D_k \times \mathbb{R}^N \to \mathbb{R}^N$ be an action of D_k on $\mathbb{R}^N$. Then a map

$$\Omega : P_k(M) \to \mathbb{R}^N ,$$

is called a *geometric structure of order k on M* if

$$\Omega(\theta_k \cdot d_k) = F(\, d_k^{-1},\ \Omega(\theta_k)\,) \quad \forall\, \theta_k \in P_k(M)\,,\ \forall\, d_k \in D_k\,.$$

Any local coordinate system $(\,U,\, h = (x^1, \dots, x^n)\,)$ in M generates the section of $P_k(M)$ by the formula

$$U \to \pi_k^{-1}(U)\,,\ \ p \mapsto \left[\,(\, h - h(p)\,)^{-1}\right]_0^k . \tag{6}$$

The reduction of Ω to the image of this section is the collection of the components $q^1(x), \dots, q^N(x)$ of Ω in the coordinates $x^1, \dots, x^n$.

A geometric structure Ω is called *homogeneous* if $\operatorname{Im}\Omega$ is an orbit of the action F of the group D_k.

Suppose Ω_1 and Ω_2 are geometric structures with the same transformation law of their components. We say that these structures are *equivalent* if there exists a diffeomorphism f of M such that

$$\Omega_1 = \Omega_2 \circ f^{(k)} .$$

1.5. **Prolongations of structures.** Suppose Ω is a geometric structure and the transformation law of its components is defined by (5). Then its *first prolongation*

$$\Omega^{(1)} : P_{k+1}(M) \to \mathbb{R}^{N(1+n)}$$

is defined in the following way. Suppose $q^1(x), \dots, q^N(x)$ are the components of Ω in the coordinates $x^1, \dots, x^n$. Then

$$q^\alpha(x)\,,\ \ \frac{\partial}{\partial x^j}(\, q^\alpha(x)\,)\,,\ \ \alpha = 1, \dots, N\,,\ j = 1, \dots, n\,,$$

are the components of $\Omega^{(1)}$ in the coordinates $x^1, \dots, x^n$. Obviously, the transformation law of components of $\Omega^{(1)}$ is defined by

$$\begin{aligned} \tilde{q}^\alpha &= F^\alpha(\, d^i_{j_1}, \dots, d^i_{j_1 \dots j_k},\ q^1, \dots, q^N\,)\,, \\ \partial_i \tilde{q}^\alpha \cdot d^i_j &= \frac{\partial F^\alpha}{\partial d^i_{j_1}} d^i_{j_1 j} + \dots + \frac{\partial F^\alpha}{\partial d^i_{j_1 \dots j_k}} d^i_{j_1 \dots j_k j} + \frac{\partial F^\alpha}{\partial q^\beta} \partial_j q^\beta\,. \end{aligned} \tag{7}$$

The i-th *prolongation* of Ω is defined by induction on i:

$$\Omega^{(i+1)} = (\,\Omega^{(i)}\,)^{(1)}\,,\ i = 1, 2, \dots .$$

1.6. G**-structures.** Let $G \subset D_k$ be a closed Lie subgroup and let $B \subset P_k(M)$ be a reduction of $P_k(M)$ to G. Then B is called a *G-structure of order k over M.*

Let $\Omega : P_k(M) \to \mathbb{R}^N$ be an arbitrary homogeneous geometric structure, $q_0 \in \operatorname{Im}\Omega$, and $G \subset D_k$ be the isotropy group of q_0. Then the inverse image $B = \Omega^{-1}(q_0) \subset P_k(M)$ is a G-structure of order k over M.

Suppose B_1 and B_2 are G-structures over M. They are *equivalent* if there exists a diffeomorphism f of M such that

$$f^{(k)}(B_1) = B_2\,.$$

It is easy to prove the following statement.

Theorem 1.3. *Suppose Ω_1 and Ω_2 are homogeneous geometric structures with the same transformation law of the components, suppose that $\operatorname{Im}\Omega_1 = \operatorname{Im}\Omega_2$, and suppose $q \in \operatorname{Im}\Omega_1$. Then Ω_1 and Ω_2 are equivalent iff the G-structures $\Omega_1^{-1}(q)$ and $\Omega_2^{-1}(q)$ are equivalent.*

Let B be a G-structure of order k over M and let $\mathfrak{g} \subset L_0/L_k$ be the Lie algebra of G. By definition, put

$$g_k = \mathfrak{g} \cap (L_{k-1}/L_k)\,.$$

By $g_k^{(i)}$, $i = 0, 1, \dots$ denote the i-th prolongation of g_k, where $g_k^{(0)} = g_k$.

By definition, B is a *finite type* G-structure if there exists a nonnegative integer r such that $g_k^{(r)} = \{0\}$. Obviously, $g_k^{(i)} = \{0\}$ if $i > r$. By $r(B)$ we denote the least nonnegative integer r such that $g_k^{(r)} = \{0\}$.

1.7. **Flat structures.** Let $F : D_k \times \mathbb{R}^N \to \mathbb{R}^N$ be an arbitrary action of D_k on $\mathbb{R}^N$, let $q \in \mathbb{R}^N$, and let $G \subset D_k$ be the isotropy group of q.

The standard coordinate system on $\mathbb{R}^n$ generates the section $P_k(\mathbb{R}^n)$ by formula (6). Subjecting image of this section to the action of G, we obtain the G-structure B over $\mathbb{R}^n$. It is called *flat*. Obviously, the G-structure B, q, and the transformation law F define the geometric structure $\Omega : P_k(\mathbb{R}^n) \to \mathbb{R}^N$ uniquely. This geometric structure is called a *flat structure* too.

A geometric structure (G-structure) on M is called a *locally-flat* or *integrable* if it is locally equivalent to a flat structure (G-structure).

Obviously, a G-structure B on M is integrable iff there exists a local chart of M such that the section of $P_k(M)$ generated by this chart is a section of B. In other words, a geometric structure on M is integrable iff there exists a local chart in M such that the components of this structure are constants in this chart.

In the sequel, we use the following

Theorem 1.4. *Let Ω be an arbitrary geometric structure and let q be some value of Ω. Then Ω is integrable iff the G-structure $B = \Omega^{-1}(q)$ is integrable.*

2. Structure functions

2.1. Consider a homogeneous geometric structure $\Omega\colon P_k(M) \to \mathbb{R}^N$. Transformation law (5) of its components can be interpreted as the system of partial differential equations w.r.t. unknown functions $y^i(x^1, \ldots, x^n)$, $i = 1, 2, \ldots, n$. We treat this PDE system as the submanifold $\mathcal{E}$ in the bundle of k-jets $J^k\tau$ of sections of the trivial bundle

$$\tau : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n .$$

In this paper, we suppose that $\mathcal{E}$ satisfies the condition

$$\tau_{k,k-1}(\ \mathcal{E}\ ) = J^{k-1}\tau\,, \tag{8}$$

where $\tau_{l,m} : J^l\tau \to J^m\tau$, $l \geq m$, is the natural projection that takes a l-jet to its m-jet.

Let $q_0 \in \mathbb{R}^N$ be some value of Ω. Consider the G-structure $B = \Omega^{-1}(q_0)$. Then condition (8) means that

$$\pi_{k,k-1}(\ B\ ) = P_{k-1}(M). \tag{9}$$

For the group G, condition (8) means that

$$\rho_{k,k-1}(\ G\ ) = D_{k-1}\,. \tag{10}$$

For the Lie algebra $\mathfrak{g}$ of the group G, the last condition means that

$$\rho_{k,k-1}(\ \mathfrak{g}\ ) = L_0/L_{k-1}\,. \tag{11}$$

2.2. Let $\theta_k \in B$ and let $\theta_{k-1} = \pi_{k,k-1}(\theta_k)$. Then θ_k defines the linear isomorphism $\theta_k : T_{\theta_{k-1}}P_{k-1}(M) \to W_n/L_{k-1}$ as it was shown above. By H_{k-1} we denote the subspace in W_n/L_{k-1} which is generated by the vectors of the form $(\,X^i, 0, \ldots, 0\,)$. Obviously, the quotient space W_n/L_{k-1} is decomposable to the direct sum

$$W_n/L_{k-1} = H_{k-1} \oplus L_0/L_{k-1}\,.$$

Consider the subspace $H_{\theta_{k-1}} \subset T_{\theta_{k-1}}P_{k-1}(M)$ which is defined by

$$H_{\theta_{k-1}} = (\,\theta_k\,)^{-1}(\,H_{k-1}\,)\,. \tag{12}$$

We say that $H \subset T_{\theta_i}P_i(M)\,,\ i = 1, 2, \ldots, \infty$ is *horizontal* if it is n-dimensional and is naturally projected onto the space tangent to M without degeneration.

Clearly, subspace (12) is horizontal.

Let $\theta_{k+1} \in P_{k+1}(M)$ and let $\pi_{k+1,k}(\theta_{k+1}) = \theta_k \in B$. Then the isomorphism $\theta_{k+1} : T_{\theta_k}P_k(M) \to W_n/L_k$ defines the injective linear map

$$\theta_{k+1}\big|_{T_{\theta_k}B} : T_{\theta_k}B \to W_n/L_k$$

such that the following diagram is commutative:

$$\begin{array}{ccc} T_{\theta_k}B & \xrightarrow{\theta_{k+1}\big|_{T_{\theta_k}B}} & W_n/L_k \\ {\scriptstyle (\pi_{k,k-l})_*}\big\downarrow & & \big\downarrow{\scriptstyle \rho_{k,k-l}} \\ T_{\theta_{k-1}}P_{k-1}(M) & \xrightarrow[\theta_k]{} & W_n/L_{k-1}\,. \end{array}$$

Let us choose a horizontal subspace $H_{\theta_k} \subset T_{\theta_k}B$ such that

$$(\pi_{k,k-1})_*(H_{\theta_k}) = H_{\theta_{k-1}}\,. \tag{13}$$

Then

$$\forall X \in H_{\theta_{k+1}}, \quad \theta_k(X) = (\,X^i, 0, \ldots, 0, X^i_{j_1\ldots j_k}\,)\ .$$

The pair $(\,H_{\theta_k},\ \theta_{k+1}\,)$ defines the linear map

$$f_{(\,H_{\theta_k},\,\theta_{k+1}\,)} : V \to L_{k-1}/L_k$$

by the formula

$$f_{(\,H_{\theta_k},\,\theta_{k+1}\,)} : X^i \;\mapsto (\,X^i_{j_1\ldots j_k}\,) \;= (\,f^i_{j_1\ldots j_k,r}X^r\,)\,.$$

Suppose H_{θ_k} , $\tilde{H}_{\theta_k} \subset T_{\theta_k}B$ are horizontal subspaces satisfying Eq. (13). Then, obviously,

$$(\,f_{(\,H_{\theta_k},\,\theta_{k+1}\,)} - f_{(\,\tilde{H}_{\theta_k},\,\theta_{k+1}\,)}\,) : V \to g_k\,, \tag{14}$$

where $g_k = \mathfrak{g} \cap (L_{k-1}/L_k)$.

Let $\theta_k \in B$ and $\theta_{k+1}, \tilde{\theta}_{k+1} \in (\pi_{k+1,k})^{-1}(\theta_k)$. Then there exists a unique element $[d]_0^{k+1} = (\delta^i_j, 0, \ldots, 0, d^i_{j_1\ldots j_{k+1}}) \in D^k_{k+1}$ such that $\tilde{\theta}_{k+1} = \theta_{k+1}\cdot [d]_0^{k+1}$.

Let $f_{(\,H_{\theta_k},\,\theta_{k+1}\,)} = (\,f^i_{j_1\ldots j_k,r}\,)$ and $f_{(\,H_{\theta_k},\,\tilde{\theta}_{k+1}\,)} = (\,\tilde{f}^i_{j_1\ldots j_k,r}\,)$. Then from Proposition 1.2 it follows that

$$(\,\tilde{f}^i_{j_1\ldots j_k,r}\,) \,= (\,f^i_{j_1\ldots j_k,r}\,+\,d^i_{j_1\ldots j_k r})\,. \tag{15}$$

Suppose $X, Y \in H_{\theta_k}$. Consider the bracket $[\,\theta_{k+1}(X),\,\theta_{k+1}(Y)\,]$, see Eq. (1). We have

$$\begin{aligned}[\,\theta_{k+1}(X),\,\theta_{k+1}(Y)\,] &= (\,X^rY^i_{j_1\ldots j_{k-1}r} - Y^rX^i_{j_1\ldots j_{k-1}r}\,) \\ &= (\,X^rY^s(\,f^i_{j_1\ldots j_{k-1}r,s} - f^i_{j_1\ldots j_{k-1}s,r}\,)\,)\ . \end{aligned} \tag{16}$$

By definition, put

$$c(\,H_{\theta_k},\ \theta_{k+1}\,) = (\,f^i_{j_1\ldots j_{k-1}r,s} - f^i_{j_1\ldots j_{k-1}s,r}\,)\ .$$

From (15) it follows that $c(\,H_{\theta_k},\ \theta_{k+1}\,)$ is independent of the choice of the point θ_{k+1} over $\theta_k \in B$. Therefore we will write $c(\,H_{\theta_k}\,)$ instead of $c(\,H_{\theta_k},\ \theta_{k+1}\,)$.

Consider the Spencer complex

$$0 \to g_k^{(1)} \xrightarrow{\partial_{k+1,0}} g_k \otimes V^* \xrightarrow{\partial_{k,1}} L_{k-2}/L_{k-1} \otimes \wedge^2 V^* \xrightarrow{\partial_{k-1,2}} \ \cdots\ . \tag{17}$$

Obviously,

$$c(\,H_{\theta_k}\,) \in L_{k-2}/L_{k-1} \otimes \wedge^2 V^*\,.$$

From (14) it follows that if H_{θ_k} and $\tilde{H}_{\theta_k}$ are horizontal subspaces in $T_{\theta_k}B$ and satisfy (13), then

$$c(\,H_{\theta_k}\,) - c(\,\tilde{H}_{\theta_k}\,) \in \operatorname{Im}\partial_{k,1}$$

This means that the class $c(\,H_{\theta_k}\,) \mod (\operatorname{Im}\partial_{k,1})$ is independent of the choice of the horizontal subspace H_{θ_k} over $H_{\theta_{k-1}}$. We denote this class by $c(\theta_k)$. It is easy to check that

$$c(\,H_{\theta_k}\,) \in \ker\partial_{k-1,2}\,.$$

Consequently, $c(\theta_k)$ is a Spencer δ-cohomology class, that is,

$$c(\theta_k) \in H^{k-1,2}.$$

We say that the map

$$c : B \to H^{k-1,2}\,, \quad c : \theta_k \mapsto c(\theta_k)$$

is the *structure function* of the G-structure B.

Proposition 2.1. *Structure functions of flat G-structures are trivial.*

Proof. Let B be a flat G-structure of order k on $\mathbb{R}^n$ and let $(\,h = (x^1, \ldots, x^n)\,)$ be the standard chart in $\mathbb{R}^n$. An arbitrary element $g \in G$ defines the diffeomorphism $\hat{g}$ of $\mathbb{R}^n$ to itself by the formula

$$\hat{g}(x^1, \ldots, x^n) = \frac{1}{1!}g^i_j x^j + \ldots + \frac{1}{k!}g^i_{j_1 \ldots j_k} x^{j_1} \ldots x^{j_k}\,,$$

where $(\,g^i_j, \ldots, g^i_{j_1 \ldots j_k}\,) = g^{-1}$. By s^g_r, $r = 0, 1, \ldots$, we denote the section of $P_r(\mathbb{R}^n)$ that is generated by the chart $(\,\hat{g} \circ h = (y^1, \ldots, y^n)\,)$ on $\mathbb{R}^n$. Then s^g_k is a section of B. Indeed, let e be the unit of G, then s^e_k is a section of $P_k(\mathbb{R}^n)$ generated by the standard chart in $\mathbb{R}^n$. This section is a section of B. It is clear that

$$s^g_k(p) = s^e_k(p) \cdot g \quad \forall\, p \in \mathbb{R}^n\,.$$

Let $\theta_k = s^g_k(p)$ and let $\theta_{k+1} = s^g_{k+1}(p)$. Then it is obvious that $H_{\theta_k} = (s^g_k)_*(T_p\mathbb{R}^n)$ is a horizontal subspace in $T_{\theta_k}B$ and

$$\theta_{k+1} : X \mapsto (\,X^i, 0, \ldots, 0\,) \quad \forall\, X \in H_{\theta_k}\,.$$

It is clear now that the structure function of the G-structure B is equal to zero for any point of $\operatorname{Im} s^g_k$. Taking into account that images of sections $\operatorname{Im} s^g_k$, $g \in G$, cover B completely, we conclude that the structure function is equal to zero at each point of B. □

In general, the structure functions give only necessary conditions to solve the local equivalence problem for G-structures.

Theorem 2.2. *Suppose B and $\tilde{B}$ are G-structures on M, c and $\tilde{c}$ are their structure functions, respectively, and let f be a diffeomorphism of M to itself such that $f^{(k)}(B) = \tilde{B}$. Then $(f^{(k)})^*(\tilde{c}) = c$.*

Proof. Let $[s]_0^k = \theta_k \in B$ and let $X \in T_{\theta_k}B$. Then for any point $\theta_{k+1} \in \pi_{k+1,k}^{-1}(\theta_k)$ we have

$$\theta_{k+1}(X) = f^{(k+1)}(\theta_{k+1})(\,(f^{(k)})_*(X)\,)\,.$$

Indeed, from the construction of the isomorphism θ_{k+1}, see the proof of proposition 1.1, it follows that there exists a vector field ξ with the flow φ_t in M such that $X = d/dt(\,[\varphi_t \circ s]_0^k\,)\big|_{t=0}$ and

$$\theta_{k+1}(X) = d/dt(\,[s^{-1} \circ \varphi_t \circ s]_0^k\,)\big|_{t=0}.$$

It follows that

$$\begin{aligned} &f^{(k+1)}(\theta_{k+1})(\,(f^{(k)})_*(X)\,) \\ &\quad = \frac{d}{dt}(\,[(f \circ s)^{-1} \circ (f \circ \varphi_t \circ f^{-1}) \circ (f \circ s)]_0^k\,)\Big|_{t=0} \\ &\qquad\qquad = \frac{d}{dt}(\,[s^{-1} \circ \varphi_t \circ s]_0^k\,)\Big|_{t=0} = \theta_{k+1}(X)\,. \end{aligned}$$

It is obvious now that the cohomology classes $c(\theta_k)$ and $c(\,f^{(k)}(\theta_k)\,)$ coincide. ☐

3. Integrability of the finite type structures

Let Ω be an arbitrary geometric structure on M, F be its components' transformation law, and $q_0 \in \mathbb{R}^N$ be some value of Ω. Consider a G-structure $B = \Omega^{-1}(q_0)$. Let $\mathfrak{g} \subset L_0/L_k$ be the Lie algebra of G and let $g_k = \mathfrak{g} \cap L_{k-1}/L_k$. Suppose that the structure function of B is equal to zero. Let $\theta_k \in B$ and let $\theta_{k+1} \in (\pi_{k+1,k})^{-1}(\theta_k)$. Consider an arbitrary horizontal subspace $H_{\theta_k} \subset T_{\theta_k}B$ satisfying (13). Let $f(\,H_{\theta_k},\ \theta_{k+1}\,) = (\,f^i_{j_1\dots j_k,s}\,)$. From the Spencer complex in Eq. (17) and the equation $c(\,H_{\theta_k},\ \theta_{k+1}\,) = 0 \mod (\operatorname{Im}\partial_{k,1})$, it follows that there exists $(\,g^i_{j_1\dots j_k,s}\,) \in g_k \otimes V^*$ such that

$$(\,f^i_{j_1\dots j_{k-1}r,s} - f^i_{j_1\dots j_{k-1}s,r}\,) = \partial_{k,1}(\,(\,g^i_{j_1\dots j_k,s}\,)\,)\,.$$

Therefore,

$$f^i_{j_1\dots j_k,s} = g^i_{j_1\dots j_k,s} + d^i_{j_1\dots j_k s}\,,$$

where $(\,d^i_{j_1\dots j_k s}\,) \in g_k^{(1)}$. By $\tilde{H}_{\theta_k}$ we denote a horizontal subspace in $T_{\theta_k}B$ such that $f(\,\tilde{H}_{\theta_k},\ \theta_{k+1}\,) = (\,d^i_{j_1\dots j_k s}\,)$. Let $\tilde{\theta}_{k+1} = \theta_{k+1} \cdot d$, where $d = (-d^i_{j_1\dots j_k s}) \in G \cap D_k^{k+1}$. Then it is clear that

$$\forall X \in \tilde{H}_{\theta_k} \quad \tilde{\theta}_{k+1}(X) = (\,X^i, 0, \dots, 0\,)\,. \tag{18}$$

By $B^{(1)}$ we denote the set of all $\tilde{\theta}_{k+1}$, which are obtained in this way. Obviously,

$$\pi_{k+1,k}(B^{(1)}) = B\,.$$

Proposition 3.1. *We have*

$$B^{(1)} = (\Omega^{(1)})^{-1}(\,(\,q_0, 0\,)\,)\,,$$

i.e., $B^{(1)}$ is a $G^{(1)}$-structure. Here $G^{(1)}$ is the isotropy group of the point $(\,q_0, 0\,) \in \mathbb{R}^{N(1+n)}$.

Proof. Let $[s]_0^{k+1} = \theta_{k+1} \in B^{(1)}$. The local chart $s^{-1} = (y^1, \ldots, y^n)$ generates the local chart in $P_k(M)$. From (5) it follows that the G-structure B is defined within this chart by the equations

$$\tilde{q}^\alpha(y) = F^\alpha(\,y^i_j, \ldots, y^i_{j_1\ldots j_k}\,,\, q_0\,)\,. \tag{19}$$

Let $H_{\theta_k} \subset T_{\theta_k}B$ be a horizontal subspace that satisfies (13) and (18). Then a vector $X \in H_{\theta_k}$ is

$$X = X^i\frac{\partial}{\partial y^i} + 0\cdot\frac{\partial}{\partial y^i_j} + \ldots + 0\cdot\frac{\partial}{\partial y^i_{j_1\ldots j_k}}$$

within this chart. From (19) we deduce that X satisfies the equation

$$\partial_j q^\alpha(0)\cdot X^j = 0\,.$$

This means that

$$\partial_j q^\alpha(0) = 0 \quad \forall\, \alpha = 1, 2, \ldots, N\,,\; j = 1, 2, \ldots, n\,.$$

whence,

$$\Omega^{(1)}(\theta_{k+1}) = (\,q_0, 0\,)\,.$$

Thus we obtain

$$B^{(1)} \subset (\Omega^{(1)})^{-1}(\,q_0, 0\,)\,.$$

From (7) it follows that $G^{(1)}$-structure $(\Omega^{(1)})^{-1}(\,q_0, 0\,)$ is defined by the equations

$$\tilde{q}^\alpha(y) = F^\alpha(\,d^i_{j_1}, \ldots, d^i_{j_1\ldots j_k}\,,\, q_0\,)\,,$$
$$\partial_i\tilde{q}^\alpha(y)\cdot d^i_j = \frac{\partial F^\alpha}{\partial y^i_{j_1}}y^i_{j_1 j} + \ldots + \frac{\partial F^\alpha}{\partial y^i_{j_1\ldots j_k}}y^i_{j_1\ldots j_k j}\,.$$

Therefore,

$$B^{(1)} \cap \pi^{-1}_{k+1,k}(\theta_k) = (\Omega^{(1)})^{-1}(\,q_0, 0\,) \cap \pi^{-1}_{k+1,k}(\theta_k) \;\; \forall\, \theta_k \in B\,.$$

Now it is clear that

$$B^{(1)} = (\Omega^{(1)})^{-1}(\,q_0, 0\,)\,.$$

□

In the same way as above, we construct the structure function

$$c^{(1)} : B^{(1)} \to H^{k,2}$$

of the $G^{(1)}$-structure $B^{(1)}$. If $c^{(1)} = 0$, then, in the same way as above, we can construct $G^{(2)}$-structure $B^{(2)} = (\Omega^{(2)})^{-1}(\,(\,q_0, 0, 0\,)\,)$ and its structure function $c^{(2)}$, and so on.

Theorem 3.2. *Let B be a finite type G-structure and let c be its structure function. Then B is integrable iff $c = 0\,,\ c^{(1)} = 0\,, \ldots, c^{(r(B))} = 0$.*

Proof. First, we consider the case $r(B) = 0$. Let Ω be a geometric structure of order k such that $B = \Omega^{-1}(q_0)$ and let $y = (\, y^1, \ldots, y^n\,)$ be a local chart of M. This chart generates the local chart of $P_k(M)$. In terms of this chart, the submanifold B is defined by the equations

$$\tilde{q}(y) = F(\, y^i_j, \ldots, y^i_{j_1 \ldots j_k}\,,\ q_0\,)\,. \tag{20}$$

We interpret these equations as a system of partial differential equations $\mathcal{E}$ w.r.t. the unknown functions $y^1(x^1, \ldots, x^n), \ldots, y^n(x^1, \ldots, x^n)$ that define the coordinate transformation $x \to y$. If there exists a solution of this PDE system, then $x = (\, x^1, \ldots, x^n\,)$ is a local chart of M and the components of Ω in this chart are constant, *i.e.*, Ω is integrable.

The condition $g_k = \{0\}$ means that the symbol of the PDEs system is equal to zero. Thence, the natural projection $\pi_{k+1,k} : B^{(1)} \to B$ is surjective. This means that the natural projection $\mathcal{E}^{(1)} \to \mathcal{E}$, where $\mathcal{E}^{(1)}$ is the first prolongation of $\mathcal{E}$, is surjective too. Thus the system $\mathcal{E}$ of partial differential equations is completely integrable, see [7], therefore it has a solution. This completes the proof for the case $r(B) = 0$.

The proof for the case $r(B) > 0$ is obvious now. □

4. Applications to ordinary equations

4.1. **Second order equations.** Consider an ordinary differential equations of the form

$$y'' = a_3(x,y)\,(y')^3 + a_2(x,y)\,(y')^2 + a_1(x,y)\,y' + a_0(x,y)\,. \tag{21}$$

It is well-known that an arbitrary point transformation takes equation (21) to the equation of the same form. This means that equation (21) defines the second-order geometric structure on $\mathbb{R}^2$ such that the coefficients of the equation are the components of this structure in the standard coordinates in $\mathbb{R}^2$. We denote this structure by Ω. Thus,

$$\Omega : P_2(\mathbb{R}^2) \to \mathbb{R}^4\,.$$

This structure is a finite type structure such that $r(B) = 1$.

Consider the G-structure $B = \Omega^{-1}(0)$. Its structure function c is equal to zero. It can be proved that equation (21) can be reduced to linear form by a point transformation iff the structure function $c^{(1)}$ of its first prolongation $B^{(1)}$ is equal to zero (see [8], [9]).

4.2. **Third order equations.** Consider ordinary differential equations of the form

$$y''' = a_3\,(y'')^3 + a_2\,(y'')^2 + a_1\,y'' + a_0\,, \tag{22}$$

where a_3, a_2, a_1, a_0 are functions of x, y, y'. It is easy to prove that an arbitrary contact transformation takes equation (22) to the equation of the same form. This means that equation (22) defines the geometric structure of third order on the space $\mathbb{R}^3$ such that the coefficients of the equation are the components of this structure in the standard coordinates in space $\mathbb{R}^3$. We denote this structure by Ω. Thus,

$$\Omega : P_3(\mathbb{R}^3) \to \mathbb{R}^4 .$$

This structure is an infinite type structure.

Let $\Omega^{(\infty)}$ be the infinite prolongation of the structure Ω and let $B = \pi_{\infty,3}(\,(\Omega^{(\infty)})^{-1}(0)\,)$. Then it can be proved that B is a finite type G-structure such that $r(B) = 1$. Its structure function c is equal to zero. It can be proved that equation (22) can be reduced to the form $y''' = 0$ by a contact transformation iff the structure function $c^{(1)}$ of its first prolongation $B^{(1)}$ is equal to zero (see [10]).

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