

# Introduction to Differential Invariants

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## 1. Introduction

These lectures constitute an introduction to the theory of differential invariants. Calculation of differential invariants is explained by means of examples from the theory of ordinary differential equations.

In contrast to the Cartan's method [11], we perform all calculations and constructions of differential invariants in natural bundles of considered objects. It seems to us that this approach is more natural.

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## CHAPTER 1

# Jet bundles

In our approach, differential invariants are defined on jet bundles. Therefore, following [9], in this chapter we introduce jets and jet bundles. Next we introduce the Cartan distribution, which is a necessary tool to investigate jet bundles.

A different approach to differential invariants, connected with  $G$ -structures, depends on the notion of  $k$ -frames. For later use (in Chapter 8) we also introduce differential groups, their Lie algebras, and bundles of  $k$ -frames on a smooth manifold. Here we follow [1].

Below, all manifolds and maps are supposed to be smooth. By  $\mathbb{R}$  we denote the field of real numbers and by  $\mathbb{R}^n$  we denote the  $n$ -dimensional arithmetic space.

We assume summation over repeated indices in all formulas. For example, we will write

$$u_{j_1 \dots j_k}^i \frac{\partial}{\partial u_{j_1 \dots j_k}^i} \quad \text{instead of} \quad \sum_{i=1}^n \sum_{j_1=1}^m \dots \sum_{j_k=1}^m u_{j_1 \dots j_k}^i \frac{\partial}{\partial u_{j_1 \dots j_k}^i}.$$

### 1. Jets

Let  $X$  be a smooth  $n$ -dimensional manifold and let  $p$  be some point of  $X$ , let  $Y$  be a smooth  $m$ -dimensional manifold and let  $q$  be some point of  $Y$ . By  $F$  we denote the set of all smooth maps from  $X$  to  $Y$  defined in neighborhoods of  $p$  and sending  $p$  to  $q$ . Fix some coordinate system  $x^1, \dots, x^n$  in a neighborhood of  $p$  and some coordinate system  $y^1, \dots, y^m$  in a neighborhood of  $q$ . Consider two maps  $f, g \in F$ . In terms of these coordinate systems, these maps are defined by collections of functions:

$$\begin{aligned} f &= (f^1(x^1, \dots, x^n), \dots, f^m(x^1, \dots, x^n)), \\ g &= (g^1(x^1, \dots, x^n), \dots, g^m(x^1, \dots, x^n)). \end{aligned}$$

We say that  $f$  and  $g$  are  $k$ -equivalent,  $k = 0, 1, 2, \dots$ , if

$$f(p) = g(p), \quad \frac{\partial f^i}{\partial x^j}(p) = \frac{\partial g^i}{\partial x^j}(p), \quad \dots, \quad \frac{\partial^k f^i}{\partial x^{j_1} \dots \partial x^{j_k}}(p) = \frac{\partial^k g^i}{\partial x^{j_1} \dots \partial x^{j_k}}(p)$$

$i = 1, \dots, m; \quad j, j_1, \dots, j_k = 1, \dots, n$

Clearly, the introduced relation is an equivalence relation on  $F$ . By  $[f]_p^k$  denote the equivalence class of  $f$  w.r.t. this relation. It is easy to prove that  $[f]_p^k$  is well defined, that is  $[f]_p^k$  is independent of the choice of the coordinate systems in neighborhoods of  $p$  and  $q$ . The equivalence class  $[f]_p^k$  is called the *k-jet of  $f$  at  $p$* . The point  $p$  and  $f(p)$  is said to be the *source* and the *target* of the *k-jet*  $[f]_p^k$ , respectively.

Consider two smooth maps of smooth manifolds  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . Let  $p \in X$ . Suppose that the point  $f(p)$  is contained in the domain of  $g$ . Then the formula

$$[f]_p^k \cdot [g]_{f(p)}^k = [g \circ f]_p^k$$

defines the multiplication of jets.

Some constructions based on jets:

1. Consider the set of all diffeomorphisms of  $\mathbb{R}^n$  to itself preserving the point  $0 \in \mathbb{R}^n$ . By  $D_k(n)$  we denote the set of all *k-jets* at 0 of these diffeomorphisms.

The jet multiplication

$$[d_1]_0^k \cdot [d_2]_0^k = [d_1 \circ d_2]_0^k$$

defines the group structure on  $D_k(n)$ . Obviously, in this group,

$$([d]_0^k)^{-1} = [d^{-1}]_0^k \quad \text{and} \quad [\text{id}]_0^k \text{ is the unity of the group } D_k(n).$$

The standard coordinates  $x^1, \dots, x^n$  on  $\mathbb{R}^n$  generate standard coordinates  $x_j^i, \dots, x_{j_1 \dots j_n}^i$  on  $D_k(n)$  so that for any  $[d]_0^k \in D_k(n)$

$$x_j^i([d]_0^k) = \frac{\partial d^i}{\partial x^j}(0), \dots, x_{j_1 \dots j_n}^i([d]_0^k) = \frac{\partial^k d^i}{\partial x^{j_1} \dots \partial x^{j_n}}(0).$$

Obviously, in this group, the group operation  $D_k(n) \times D_k(n) \rightarrow D_k(n)$  and the operation  $[d^{-1}]_0^k \mapsto ([d]_0^k)^{-1}$  are smooth maps. It follows that  $D_k(n)$  is a Lie group.

The Lie group  $D_k(n)$  is called the *differential group of order  $k$* . Obviously,  $D_1(n)$  is the complete linear group  $\text{GL}(n)$ .

By  $D_k^r(n)$ ,  $r = 0, 1, 2, \dots, k$  we denote the subgroup of  $D_k(n)$  defined by

$$D_k^r(n) = \{ [d]_0^k \in D_k(n) \mid [d]_0^r = [\text{id}]_0^r \}.$$

2. Consider the set of all vector fields  $\xi$  in  $\mathbb{R}^n$  such that  $\xi|_0 = 0$ . By  $L_k^0(n)$  we denote the set of all *k-jets* at 0 of these vector fields. There exist a natural structure of Lie algebra over  $\mathbb{R}$  on  $L_k^0(n)$ . This structure is defined by the



operations

$$\begin{aligned}\lambda[\xi]_0^k &\stackrel{df}{=} [\lambda\xi]_0^k, & [\xi_1]_0^k + [\xi_2]_0^k &\stackrel{df}{=} [\xi_1 + \xi_2]_0^k, \\ & & [[\xi_1]_0^k, [\xi_2]_0^k] &\stackrel{df}{=} [[\xi_1, \xi_2]]_0^k \\ & & \forall \lambda \in \mathbb{R}, \quad \forall [\xi_1]_0^k, [\xi_2]_0^k &\in L_k^0.\end{aligned}$$

By  $L_k^r$ ,  $r = 0, 1, 2, \dots, k$ , we denote the subalgebra in  $L_k^0$  defined by

$$L_k^r = \{ [X]_0^k \in L_k^0 \mid [X]_0^r = 0 \}.$$

The algebras  $L_k^0$  and  $L_k^r$  are identified with the Lie algebras of  $D_k(n)$  and  $D_k^r(n)$  respectively.

3. By  $J^k(X, Y)$  we denote the set of all  $k$ -jets of all smooth local maps from  $X$  to  $Y$ . Obviously,  $J^0(X, Y) = X \times Y$ . Introduce the map

$$\pi : J^k(X, Y) \rightarrow J^0(X, Y)$$

by the formula  $\pi : [f]_p^k \mapsto [f]_p^0 = (p, f(p))$ .

Let  $x^1, \dots, x^n$  be a coordinate system on some open set  $U \subset X$  and let  $u^1, \dots, u^m$  be a coordinate system on some open subset  $V \subset Y$ . These coordinate systems generate the coordinate system

$$\begin{aligned}x^j, u^i, u_{j_1 \dots j_k}^i, \dots, u_{j_1 \dots j_k}^i, \\ j, j_1 \dots j_k = 1, \dots, n; i = 1, \dots, m\end{aligned}$$

on the subset  $\pi^{-1}(U \times V) \subset J^k(X, Y)$  in the following way. Let  $[f]_p^k \in \pi^{-1}(U \times V)$ , then

$$\begin{aligned}x^j([f]_p^k) &= x^j(p), u^i([f]_p^k) = u^i(f(p)), \\ u_{j_1 \dots j_k}^i([f]_p^k) &= \frac{\partial f^i}{\partial x^{j_1 \dots j_k}}(p), \dots, u_{j_1 \dots j_k}^i([f]_p^k) = \frac{\partial^k f^i}{\partial x^{j_1} \dots \partial x^{j_k}}(p).\end{aligned}$$

The obtained coordinate systems generate the structure of a smooth manifold on  $J^k(X, Y)$ .

## 2. Bundles

Suppose  $B$  and  $F$  are smooth manifolds. The map

$$\text{pr}_1 : B \times F \rightarrow B, \quad \text{pr}_1 : (b, f) \mapsto b$$

is called a trivial bundle.

A map  $S : B \rightarrow B \times F$  is called a *section of the bundle*  $\text{pr}_1$  if  $\text{pr}_1 \circ S = \text{id}_B$ .

Let  $x^1, \dots, x^n$  be a coordinate system on some open set  $U \subset B$  and let  $u^1, \dots, u^m$  be a coordinate system on some open subset  $V \subset F$ . Then the coordinate system  $x^1, \dots, x^n, u^1, \dots, u^m$  is defined on the open subset  $U \times V$  of the manifold  $B \times F$ . Let  $S$  be a section of  $\text{pr}_1$  such that its image

is contained in  $U \times V$ . Then its domain contains in  $U$  and  $S$  is described in terms of the coordinates  $x^1, \dots, x^n, u^1, \dots, u^m$  on  $U \times V$  by the formula

$$S(x^1, \dots, x^n) = (x^1, \dots, x^n, S^1(x^1, \dots, x^n), \dots, S^m(x^1, \dots, x^n)).$$

As usual, we write

$$S(x^1, \dots, x^n) = (S^1(x^1, \dots, x^n), \dots, S^m(x^1, \dots, x^n)).$$

Suppose there are given three smooth manifolds  $E$ ,  $B$ , and  $F$  and a smooth map  $\pi : E \rightarrow B$ . The quadruple  $(E, \pi, B, F)$  is called a *locally trivial bundle* or simply a *bundle* if the following conditions hold:

- (1)  $\pi$  is surjective,
- (2) for any point  $p \in B$ , there exist a neighborhood  $U$  of  $p$  and a diffeomorphism  $\varphi : \pi^{-1}(U) \rightarrow U \times F$ , such that for any  $e \in \pi^{-1}(U)$

$$\pi(e) = \text{pr}_1 \circ \varphi(e), \quad (2.1)$$

where  $\text{pr}_1 : U \times F \rightarrow U$  is the projection onto the first component, that is  $\text{pr}_1 : (x, y) \mapsto x$ .

Instead of  $(E, \pi, B, F)$ , we can write  $\pi : E \rightarrow B$ , or simply  $E$ .

$E$  is called a total space,  $B$  a base,  $F$  a standard fiber, and diffeomorphisms  $\varphi : \pi^{-1}(U) \rightarrow U \times F$  a local trivializations. For any point  $b \in B$ , the set  $\pi^{-1}(b)$  is called the fiber over  $b$ .

From the second condition of this definition we get that the bundle  $\pi$  is organized as a trivial bundle locally, on every subset  $\pi^{-1}(U)$ .

A map  $S : B \rightarrow E$  is called a *section of the bundle*  $\pi$  if  $\pi \circ S = \text{id}_B$ .

Suppose  $\dim B = n$  and  $\dim E = n + m$ . Consider a local trivialisation  $\varphi : \pi^{-1}(U) \rightarrow U \times F$  of  $\pi$ . Let  $x^1, \dots, x^n$  be a coordinate system on  $U$  and let  $u^1, \dots, u^m$  be a coordinate system on some open subset  $V \subset F$ . Then  $x^1, \dots, x^n, u^1, \dots, u^m$  is a coordinate system on the open subset  $U \times V \subset U \times F$ . This coordinate system is transferred on the open subset  $\varphi^{-1}(U \times V) \subset E$  by the inverse diffeomorphism  $\varphi^{-1}$ . The coordinate system  $x^1, \dots, x^n, u^1, \dots, u^m$  obtained on  $\varphi^{-1}(U \times V)$  is called *special*.

From (2.1), we get that in terms of the special coordinate system, the map  $\pi$  is described by the formula

$$\pi(x^1, \dots, x^n, u^1, \dots, u^m) = (x^1, \dots, x^n)$$

and a section  $S$  of  $\pi$  such that its image contains in  $\varphi^{-1}(U \times V)$  is described by

$$S(x^1, \dots, x^n) = (S^1(x^1, \dots, x^n), \dots, S^m(x^1, \dots, x^n)). \quad (2.2)$$

Examples.

1. Let  $M$  be a smooth  $n$ -dimensional manifold,  $T_p(M)$  its tangent space at  $p \in M$ . Put  $T(M) = \bigcup_{p \in M} T_p M$ . Then the natural projection  $\pi : T(M) \rightarrow M$  sending a tangent vector  $v_p$  at the point  $p$  to this point  $p$

is a locally trivial bundle. The standard fiber of this bundle is  $\mathbb{R}^n$ . This bundle is called the *tangent bundle of  $M$* . Any section of a tangent bundle is a *vector field*.

2. The cotangent bundle  $T^*(M)$  of  $M$ . More generally, the bundle of tensors of type  $(p, q)$  over  $M$ .

3. Let  $M$  be an  $n$ -dimensional smooth manifold. Consider all diffeomorphisms of neighborhoods of  $0 \in \mathbb{R}^n$  to  $M$ . By  $P_k(M)$  we denote the set of  $k$ -jets at  $0$  of all these diffeomorphisms. The following natural projection holds:

$$\pi_k : P_k(M) \rightarrow M, \quad \pi_k : [s]_0^k \mapsto s(0).$$

A local chart  $(U, (x^1, \dots, x^n))$  in  $M$  generates the local chart in  $P_k(M)$   $(\pi_k^{-1}(U), (x^i, x_j^i, \dots, x_{j_1 \dots j_k}^i))$ . In this chart, the coordinates of a point  $[s]_0^k \in \pi_k^{-1}(U)$  are calculated by the formula

$$x_{j_1 \dots j_r}^i([s]_0^k) = \frac{\partial^r (x^i \circ s)}{\partial t^{j_1} \dots \partial t^{j_r}},$$

$$i, j_1, \dots, j_r = 1, \dots, n, \quad r = 0, 1, \dots, k,$$

where  $t^1, \dots, t^n$  are the standard coordinates on  $\mathbb{R}^n$ . Now we see that  $P_k(M)$  is a smooth manifold.

It is easy to prove that the quadruple  $(P_k(M), \pi_k, M, D_k(n))$  is a smooth locally trivial bundle. It is called the *bundle of  $k$ -frames of  $M$* .

The group  $D_k(n)$  acts freely and transitively on the fibers of this bundle:

$$[s]_0^k \cdot [d]_0^k = [s \circ d]_0^k \quad \forall [s]_0^k \in P_k(M), \quad \forall [d]_0^k \in D_k(n).$$

### 3. Jet bundles

By  $J^k \pi$  denote the set of all  $k$ -jets of all sections of  $\pi$ . Obviously,  $J^0 \pi = E$ . Consider the following natural maps:

$$\pi_{k,r} : J^k \pi \rightarrow J^r \pi, \quad \pi_{k,r} : [S]_p^k \mapsto [S]_p^r,$$

$$\pi_k : J^k \pi \rightarrow B, \quad \pi_k : [S]_p^k \mapsto p$$

The quadruple  $(J^k \pi, \pi_k, B, J^k(B, F))$  is a locally trivial bundle. Indeed, a special coordinate system  $x^j, u^i$ , on  $W = \varphi^{-1}(U \times V) \subset E$  defines the coordinate system  $x^j, u^i, u_j^i, \dots, u_{j_1 \dots j_k}^i$  on the subset  $\pi_{k,0}^{-1}(W)$  of  $J^k \pi$  in the following way. Let  $\theta_k \in \pi_{k,0}^{-1}(W)$  and  $\pi_k(\theta) = p$ . Then there exist a section  $S$  of  $\pi$  such that  $[S]_p^k = \theta_k$ . In the special coordinate system  $x^j, u^i$ ,  $S$  is described by (2.2). Then

$$x^j(\theta_k) = x^j(p), \quad u^i(\theta_k) = S^i(p),$$

$$u_j^i(\theta_k) = \frac{\partial S^i}{\partial x^j}(p), \quad \dots, \quad u_{j_1 \dots j_k}^i(\theta_k) = \frac{\partial^k S^i}{\partial x^{j_1} \dots \partial x^{j_k}}(p).$$

This coordinate system in  $J^k\pi$  is called *special*, too. These special coordinate systems define the structure of a smooth manifold on  $J^k\pi$ . It is easy now to check all the other conditions of a locally trivial bundle.

Any section  $S$  of  $\pi$  generate the section of  $J^k\pi$

$$j_k S : B \rightarrow J^k\pi, \quad j_k S : p \mapsto [S]_p^k.$$

In terms of special coordinates, this section is described by

$$S(x) = \left( S^i(x), \frac{\partial S^i}{\partial x^j}(x), \dots, \frac{\partial^k S^i}{\partial x^{j_1} \dots \partial x^{j_k}}(x) \right).$$

By  $L_S^k$  we denote the graph of the section  $j_k S$ .

#### 4. Cartan distributions

Let  $L_S^k$  be graph of some section  $j_k S$  of  $\pi_k$  and let  $\theta_k \in L_S^k$ . By  $T_{\theta_k} L_S^k$  denote the tangent space to  $L_S^k$  at  $\theta_k$ . Suppose  $\theta_k = (x^j, u^i, u_j^i, \dots, u_{j_1 \dots j_k}^i)$  in a special coordinate system, then  $T_{\theta_k} L_S^k$  is spanned by the vectors

$$\begin{aligned} \frac{\partial}{\partial x^j} + u_j^i \frac{\partial}{\partial u^i} + \dots + u_{j_1 \dots j_{k-1} j}^i \frac{\partial}{\partial u_{j_1 \dots j_{k-1}}^i} \\ + \frac{\partial^{k+1} S^i}{\partial x^{j_1} \dots \partial x^{j_k} \partial x^j}(p) \frac{\partial}{\partial u_{j_1 \dots j_k}^i}, \quad j = 1, 2, \dots, n. \end{aligned} \quad (4.1)$$

Let  $\theta_k \in J^k\pi$ . Consider all graphs  $L_S^k$  passing through  $\theta_k$ . By  $\mathcal{C}_{\theta_k}$  denote the subspace of  $T_{\theta_k} J^k\pi$  spanned by all subspaces  $T_{\theta_k} L_S^k$  of these graphs. The space  $\mathcal{C}_{\theta_k}$  is called the *Cartan plane* at the point  $\theta_k$ .

From the description of  $T_{\theta_k} L_S^k$  in a special coordinate system we get that  $\mathcal{C}_{\theta_k}$  is spanned by the vectors

$$\frac{\partial}{\partial x^j} + u_j^i \frac{\partial}{\partial u^i} + \dots + u_{j_1 \dots j_{k-1} j}^i \frac{\partial}{\partial u_{j_1 \dots j_{k-1}}^i}, \quad j = 1, 2, \dots, n$$

and all vectors of the form

$$\frac{\partial}{\partial u_{j_1 \dots j_k}^i}.$$

Obviously,  $\mathcal{C}_{\theta_0} = T_{\theta_0} J^0\pi$ .

The distribution  $\mathcal{C}_k : \theta_k \rightarrow \mathcal{C}_{\theta_k}$  is called the *Cartan distribution* on  $J^k\pi$ .

In special coordinate system, the distribution  $\mathcal{C}_k$  is defined by the following differential 1-forms, which are called the *Cartan forms*:

$$\begin{aligned} du^i - u_j^i dx^j, \quad du_{j_1}^i - u_{j_1 j}^i dx^j, \quad \dots, \quad du_{j_1 \dots j_{k-1}}^i - u_{j_1 \dots j_{k-1} j}^i dx^j, \\ i = 1, 2, \dots, m; \quad j, j_1, \dots, j_k = 1, 2, \dots, n. \end{aligned}$$

In other words,  $\xi \in T_{\theta_k}$  belongs to  $\mathcal{C}_{\theta_k}$  iff  $\xi$  is a solution of the system of linear homogeneous equations

$$\begin{cases} (du^i - u_j^i dx^j)(\xi) = 0, \\ \dots \\ (du_{j_1 \dots j_{k-1}}^i - u_{j_1 \dots j_{k-1} j}^i dx^j)(\xi) = 0 \\ i = 1, 2, \dots, m; j, j_1, \dots, j_k = 1, 2, \dots, n. \end{cases}$$

### 5. Exercises

- (1) Prove that the set  $J^k(X, Y)$  consisting of all  $k$ -jets of all smooth maps  $X \rightarrow Y$  is a smooth manifold.
- (2) Prove that special coordinate systems in  $J^k\pi$  define the structure of a smooth manifold on  $J^k\pi$ .
- (3) Prove that  $(J^k\pi, \pi_k, B, J^k(B, F))$  is a locally trivial bundle.
- (4) Prove that the bundle of  $k$ -frames of  $M$  is a locally trivial bundle.



## CHAPTER 2

### Lie transformations

Differential invariants are objects which are invariant w.r.t. Lie transformations of jet bundles. Therefore in this chapter, following [9], we introduce Lie transformations of jet bundles. In particular, we introduce point and contact transformations. We study lifts of these transformations both from the geometric point of view and from the coordinate point of view.

#### 1. Point transformations

An arbitrary diffeomorphism  $f : J^0\pi \rightarrow J^0\pi$  is called a *point transformation*.

For any  $k = 1, 2, \dots$  a point transformation  $f$  can be lifted in a unique way to the diffeomorphism  $f^{(k)} : J^k\pi \rightarrow J^k\pi$  so that for any  $k_1 > k_2 \geq 0$  the diagram

$$\begin{array}{ccc} J^{k_1}\pi & \xrightarrow{f^{(k_1)}} & J^{k_1}\pi \\ \pi_{k_1, k_2} \downarrow & & \downarrow \pi_{k_1, k_2} \\ J^{k_2}\pi & \xrightarrow{f^{(k_2)}} & J^{k_2}\pi \end{array}$$

is commutative (in the domain of  $f^{(k_1)}$ ), here we suppose  $f^{(0)} = f$ .

**1.1. The geometric description.** Let us describe  $f^{(k)}$  in geometric terms. Let  $\theta_{k+1} \in J^{k+1}\pi$ , let  $\theta_k = \pi_{k+1, k}(\theta_{k+1})$ , and let  $S$  be a section of  $\pi$  such that  $[S]_p^{k+1} = \theta_{k+1}$ .

LEMMA 2.1. *The  $k + 1$ -jet  $\theta_{k+1}$  is identified in the natural way with the tangent space  $T_{\theta_k} L_S^k$*

PROOF. The proof follows from (4.1). □

The tangent space  $T_{\theta_k} L_S^k$  identified with  $\theta_{k+1}$  is called the *R-plane of  $\theta_{k+1}$* .

Let us construct  $f^{(1)}$ . Consider an arbitrary point  $\theta_1 = [s]_p^1$ . The map  $f_*$  transforms the R-plane  $T_{\theta_0} L_S^0$  of  $\theta_1$  onto the tangent plane  $f_*(T_{\theta_0} L_S^0)$  at the point  $f(p)$ . If this transformed plane projected onto  $T_p B$  without degeneration (that is  $\pi_* : f_*(T_{\theta_0} L_S^0) \rightarrow T_p B$  is an isomorphism), then some

neighborhood of  $s(p)$  in  $L_s^0$  is transformed to  $L^0S$  for some section  $S$  and the transformed plane  $f_*(T_{\theta_0}L_s^0)$  is an R-plane of  $[S]_{f(p)}^1$ . By definition, put

$$f^{(1)}(\theta_1) = [S]_{f(p)}^1.$$

It is easy to check that  $f^{(1)}$  is defined on some open almost everywhere dense subset  $U \subset J^1\pi$ .

Obviously,  $\pi_{1,0} \circ f^{(1)} = f \circ \pi_{1,0}$ .

Let us prove the uniqueness of  $f^{(1)}$ . Suppose there exist diffeomorphism  $h : J^1\pi \rightarrow J^1\pi$  such that  $\pi_{1,0} \circ h = f \circ \pi_{1,0}$ . Then the diffeomorphism  $f^{(1)} \circ h^{-1}$  preserves all fibers of the bundle  $\pi_{1,0}$ . It is easy to check now that  $f^{(1)} \circ h^{-1} = \text{id}$

Let us construct  $f^{(2)}$ . Consider an arbitrary point  $\theta_2 = [s]_p^2 \in U$ . Suppose  $f^{(1)}(\theta_1) = [S]_{f(p)}^1$ , where  $\theta_1 = \pi_{2,1}(\theta_2)$ . In the first step, we proved that  $f^{(1)}$  transforms  $L_s^1$  onto  $L_S^1$ . It follows that  $f_*^{(1)}$  transforms the tangent space to  $L_s^1$  at  $\theta_1$  to the tangent space to  $L_S^1$  at  $f^{(1)}(\theta_1)$ . In other words,  $f_*^{(1)}$  transforms the R-plane of  $\theta_2 = [s]_p^2$  to the R-plane of  $[S]_{f(p)}^2$ . By definition, put

$$f^{(2)}(\theta_2) = [S]_{f(p)}^2.$$

Thus  $f^{(2)}$  is defined on the open set  $(\pi_{2,1})^{-1}(U)$ . Obviously,  $\pi_{2,1} \circ f^{(2)} = f^{(1)} \circ \pi_{2,1}$

The uniqueness is proved analogously. Continuing in the same way, we obtain  $f^{(2)}, f^{(3)}, \dots, f^{(k)}$  satisfying required condition.

**1.2. The coordinate description.** Let us describe  $f^{(k)}$  in terms of special coordinates.

At first consider the simplest example. Let  $\pi = \text{pr}_1 : \mathbb{R}^1 \times \mathbb{R}^1 \rightarrow \mathbb{R}^1$  and let a point transformation  $f$  be defined by

$$X = X(x, u), \quad U = U(x, u) \tag{1.1}$$

Consider an arbitrary section  $s : x \mapsto u = s(x)$  of  $\pi$ . Suppose  $f$  transforms the graph  $L_s^0$  of this section to the graph  $L_S^0$  of some section  $S : X \mapsto U = S(X)$ . Then corresponding points  $(x, s(x))$  and  $f(x, s(x)) = (X, S(X))$  of these graphs satisfy the equations

$$X = X(x, s(x)), \tag{1.2}$$

$$S(X(x, s(x))) = U(x, s(x)). \tag{1.3}$$

The first one means transformation of variable  $x$  of the section  $s$  to variable  $X$  of the section  $S$ . The second one means that the a value of the section  $S$  is expressed in the terms of the corresponding value of  $s$ . It means that derivatives of the function  $S$  are expressed in the terms of derivatives of  $s$ .



In other words, every jet  $[S]_X^k$  of  $S$  is expressed in terms of the jet  $[s]_x^k$  of  $s$ . Indeed, differentiating equation (1.3) w.r.t.  $x$ , we obtain

$$S'_X \cdot (X_x + s'_x \cdot X_u) = U_x + s'_x \cdot U_u.$$

It follows

$$S' = \frac{U_x + s'_x U_u}{X_x + s'_x X_u}.$$

Differentiating equation (1.3) two times w.r.t.  $x$ , we obtain

$$S'' = \frac{1}{(X_x + s'_x X_u)^2} \left[ (U_{xx} + s'' U_u + s'(U_{xu} + s' U_{uu}))(X_x + s'_x X_u) - (U_x + s'_x U_u)(X_{xx} + s'' X_u + s'(X_{xu} + s' X_{uu})) \right]$$

And so on.

Note that the transformed jet  $[S]_X^k$  is expressed in the terms of  $k$ -jet  $[s]_x^k$  and  $k$ -jet of  $f$  at the point  $(x, s(x))$ , which is the source and the target of  $[s]_x^k$ . This means that the equations

$$\begin{aligned} X &= X(x, u), \quad U = U(x, u), \quad U_1 = \frac{U_x + u_1 U_u}{X_x + u_1 X_u}, \\ U_2 &= \frac{1}{(X_x + u_1 X_u)^2} \left[ (U_{xx} + u_2 U_u + u_1 (U_{xu} + u_1 U_{uu}))(X_x + u_1 X_u) - (U_x + u_1 U_u)(X_{xx} + u_2 X_u + u_1 (X_{xu} + u_1 X_{uu})) \right], \\ &\dots \\ U_k &= \frac{1}{(X_x + u_1 X_u)^k} [\dots] \end{aligned}$$

define the diffeomorphism  $f^{(k)} : J^k \pi \rightarrow J^k \pi$ .

The operator

$$D = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \dots + u_{k+1} \frac{\partial}{\partial u_k} + \dots \quad (1.4)$$

is called the *operator of total derivative w.r.t.  $x$* . It makes possible to rewrite the equations defining  $f^{(k)}$  in the following way

$$X = X(x, u), \quad U = U(x, u), \quad U_1 = \nabla(U), \quad \dots, \quad U_k = \nabla^k(U), \quad (1.5)$$

where  $\nabla = (1/D(X))D$ .

Denominators of equations (1.5) contain  $D(X) = X_x + u_1 X_u$ . It follows that  $f^{(1)}$  is defined only on the open set  $W = J^1 \pi \setminus V$ , where  $V = \{(x, u, u_1) \in J^1 \pi \mid X_x + u_1 X_u = 0\}$ . Clearly, every transformation  $f^{(k)}$ ,  $k > 1$ , is defined on the open set  $\pi_{k,1}^{-1}(W)$ .

It follows from the definition that  $f^{(k)}$  preserves the Cartan distribution  $\mathcal{C}_k$ .

Consider the general case. Let  $f : J^0\pi \rightarrow J^0\pi$  be a point transformation, let  $x^j, u^i$  be a special coordinate system on an open set  $W$  belonging to domain of  $f$ , and let  $X^j, U^i$  be a special coordinate system on the open set  $f(W)$ . Then in terms of these coordinates,  $f$  is described by the formulas

$$\begin{aligned} X^{j'} &= X^{j'}(x^j, u^i), \\ U^{i'} &= U^{i'}(x^j, u^i), \quad i, i' = 1, \dots, n; \quad j, j' = 1, \dots, m \end{aligned} \quad (1.6)$$

By the same way as above, we get

$$S_{j'}^{i'}(X_{x^j}^{j'} + X_{u^i}^{j'} u_j^i) = U_{x^j}^{i'} + U_{u^i}^{i'} u_j^i$$

Finally, introducing *operator of total derivative w.r.t.  $x^j$*

$$D_j = \frac{\partial}{\partial x^j} + u_j^i \frac{\partial}{\partial u^i} + \dots + u_{j_1 \dots j_k}^i \frac{\partial}{\partial u_{j_1 \dots j_k}^i} + \dots, \quad (1.7)$$

we obtain the following formula describing  $f^{(1)}$ :

$$\begin{aligned} X^{j'} &= X^{j'}(x^j, u^i), \\ U^{i'} &= U^{i'}(x^j, u^i), \quad i, i' = 1, \dots, n; \quad j, j' = 1, \dots, m, \\ \begin{pmatrix} U_1^1 \dots U_1^m \\ \dots \\ U_n^1 \dots U_n^m \end{pmatrix} &= \begin{pmatrix} D_1(X^1) \dots D_1(X^1) \\ \dots \\ D_n(X^n) \dots D_n(X^n) \end{pmatrix}^{-1} \cdot \begin{pmatrix} D_1(U^1) \dots D_1(U^m) \\ \dots \\ D_n(U^1) \dots D_n(U^m) \end{pmatrix} \end{aligned}$$

It is clear that we can obtain formulas describing  $f^{(2)}, \dots, f^{(k)}$ . But they are very cumbersome.

Clearly,  $f^{(1)}$  is defined only on the open set  $W = J^1\pi \setminus V$ , where

$$V = \left\{ (x^j, u^i, u_j^i) \in \pi_{1,0}^{-1}(W) \mid \det \begin{pmatrix} D_1(X^1) \dots D_1(X^1) \\ \dots \\ D_n(X^n) \dots D_n(X^n) \end{pmatrix} = 0 \right\}$$

Similarly, every transformation  $f^{(k)}$ ,  $k > 1$ , is defined on the open set  $\pi_{k,1}^{-1}(W)$ .

Thus a lifted point transformation  $f^{(k)}$ ,  $k = 1, 2, \dots$ , is defined on some open everywhere dense subset of  $J^k\pi$

Obviously,  $f^{(k)}$  preserves the Cartan distribution  $\mathcal{C}_k$ .

## 2. Contact transformations

**2.1.** Consider diffeomorphisms from  $J^1\pi$  to itself. It is natural to consider diffeomorphisms transforming every section of  $\pi_1$  of the form  $j_1S$  to some section of the same form. Clearly, these diffeomorphisms preserve the Cartan distribution on  $J^1\pi$ . It can be proved that a diffeomorphism of  $J^1\pi$  preserving the Cartan distribution on  $J^1\pi$  transforms every section of the form  $j_1S$  to some section of the same form.

An arbitrary diffeomorphism  $f : J^1\pi \rightarrow J^1\pi$  preserving the Cartan distribution  $\mathcal{C}_1$  is called a *contact transformation*.

Examples:

- (1) Let  $f : J^0\pi \rightarrow J^0\pi$  be a point transformation, then  $f^{(1)}$  is a contact transformation.
- (2) Let  $\pi = \text{pr}_1 : \mathbb{R}^1 \times \mathbb{R}^1 \rightarrow \mathbb{R}^1$ . Consider a transformation  $f : J^1\pi \rightarrow J^1\pi$  defined by

$$X = -u_1, \quad U = u - xu_1 \quad U_1 = x$$

It is a contact transformation. Indeed,  $dU - U_1dX = -u_1dx + du - xdu_1 - x(-du_1) = du - u_1dx$ . This transformation is called the Legendre transformation. Obviously, there is no point transformation  $g$  such that  $f = g^{(1)}$ .

Let  $n$  be dimension of the base  $B$  of the bundle  $\pi$  and let  $m$  be dimension of the fiber of  $\pi$ .

**THEOREM 2.2.** *If  $m > 1$ , then for every contact transformation  $f : J^1\pi \rightarrow J^1\pi$ , there exist a point transformation  $g : J^0\pi \rightarrow J^0\pi$  such that  $f = g^{(1)}$ .*

**2.2.** Let us obtain the general form of a contact transformation  $f$  in terms of special coordinates.

Consider the case  $n = m = 1$ . In special coordinates,  $f$  is described by

$$X = X(x, u, u_1), \quad U = U(x, u, u_1), \quad U_1 = U_1(x, u, u_1)$$

From definition of a contact transformation, we get  $dU - U_1dX = \lambda(du - u_1dx)$ . It follows

$$X = X(x, u, u_1), \quad U = U(x, u, u_1), \quad U_1 = D(U)/D(X) \quad (2.1)$$

where  $D$  is the operator of total derivative w.r.t.  $x$ , see (1.4). In the same way, we obtain the expression of a contact transformation in the general case.

**2.3.** In the same way as a point transformation, any contact transformation  $f$  can be lifted to the unique diffeomorphism  $f^{(k)} : J^k\pi \rightarrow J^k\pi$ ,  $k = 2, 3, \dots$ , so that  $\pi_{k,1} \circ f^{(k)} = f \circ \pi_{k,1}$ .

### 3. Lie transformations

Consider diffeomorphisms from  $J^k\pi$  to itself. It is naturally consider diffeomorphisms transforming every section of  $\pi_k$  of the form  $j_k S$  to some section of the same form. Clearly, these diffeomorphisms preserve the Cartan distribution on  $J^k\pi$ . It can be proved that a diffeomorphism of  $J^k\pi$  preserving the Cartan distribution on  $J^k\pi$  transforms every section of  $\pi_k$  of the form  $j_k S$  to some section of the same form.

**THEOREM 2.3.** *If  $m = 1$ , then for every Lie transformation is a lifted contact transformation.*

*If  $m > 1$ , then for every Lie transformation is a lifted point transformation.*

#### 4. Exercises

- (1) Let a point transformation  $f$  be defined by

$$X = u, \quad U = x$$

Find  $f^{(1)}$ ,  $f^{(2)}$ .

- (2) Applying the computer-algebraic system MAPLE, prove that an arbitrary point transformation (1.1) transforms an arbitrary linear ODE  $y'' = a(x)y' + b(x)y + c(x)$  to ODE of the form  $y'' = a(x, y)(y')^3 + b(x, y)(y')^2 + c(x, y)y' + d(x, y)$ .
- (3) Consider a general point transformation (1.6). Prove that if

$$\det \begin{pmatrix} D_1(X^1) & \dots & D_1(X^1) \\ & \dots & \\ D_n(X^n) & \dots & D_n(X^n) \end{pmatrix} = 0$$

on an open set of  $J^1\pi$ , then  $f$  is not a diffeomorphism.

- (4) Let a contact transformation  $f$  be defined by

$$X^i = -u_i, \quad U = u - x^1u_1 - x^2u_2, \quad U_i = x^i, \quad i = 1, 2$$

Find  $f^{(1)}$ .

- (5) Prove formula (2.1)

## CHAPTER 3

### Lie vector fields

Scalar differential invariants are 1st integrals of the corresponding Lie vector fields. This provides a general approach to calculate scalar differential invariants. Therefore in this chapter, we derive the formula describing Lie vector fields in special coordinates of jet bundles. Here we follow [8].

#### 1. Liftings of vector fields

Let  $\pi : E \rightarrow B$  be an arbitrary bundle. Any vector field on  $J^0\pi$  is called a *point vector field* if its flow consists of point transformations. A vector field on  $J^1\pi$  is called a *contact vector field* if its flow consists of contact transformations.

Let

$$\xi = a^j \frac{\partial}{\partial x^j} + b \frac{\partial}{\partial u} + c_j \frac{\partial}{\partial u_j}$$

be an arbitrary contact vector field. Put  $\varphi = b - u_j a^j$ . It is easy to check that

$$\xi = -\varphi_{u_j} \frac{\partial}{\partial x^j} + (\varphi - u_j \varphi_{u_j}) \frac{\partial}{\partial u^i} + (\varphi_{x^j} + u_j \varphi_u) \frac{\partial}{\partial u_j} \quad (1.1)$$

The function  $\varphi$  is called the generating function of contact vector field  $\xi$ . Often we will write  $\xi_\varphi$  instead  $\xi$ .

Finally, a vector field on  $J^k\pi$  is called a *Lie vector field* if its flow consists of Lie transformations.

The lifting of point (contact) transformations to Lie transformations induces the lifting of point (contact) vector fields to Lie vector fields. Indeed, let  $\xi$  be a point (contact) vector field and let  $f_t$  be its flow. Then  $f_t^{(k)}$  is a flow on  $J^k\pi$  ( $J^{k+1}\pi$ ). By  $\xi^{(k)}$  we denote the vector field generated by  $f_t^{(k)}$ . This field is said to be a lifted vector field. Obviously, for  $k_1 > k_2$

$$(\pi_{k_1, k_2})_*(\xi^{(k_1)}) = \xi^{(k_2)}$$

Clearly, a Lie vector field is either a lifted point vector field or a lifted contact vector field.

## 2. The coordinate description of lifted vector fields

Let us obtain the formula describing a Lie vector field  $\xi^{(k)}$  in terms of special coordinates. Suppose for definiteness that  $\xi$  is a point vector field

$$\xi = a^j \frac{\partial}{\partial x^j} + b^i \frac{\partial}{\partial u^i}, \quad (2.1)$$

where  $a^j$  and  $b^i$  are smooth functions of  $x^1, \dots, x^n, u^1, \dots, u^m$ . Then

$$\xi^{(\infty)} = a^j \frac{\partial}{\partial x^j} + b^i \frac{\partial}{\partial u^i} + b_j^i \frac{\partial}{\partial u_j^i} + \dots + b_{j_1 \dots j_k}^i \frac{\partial}{\partial u_{j_1 \dots j_k}^i} + \dots$$

where  $b_j^i, \dots, b_{j_1 \dots j_k}^i, \dots$  are smooth functions in corresponding jet bundles  $J^k \pi$ . Consider the value  $\xi^{(\infty)}|_{\theta_\infty}$  of the vector field  $\xi^{(\infty)}$  at the point  $\theta_\infty \in J^\infty \pi$ . Let  $s$  be a section of  $\pi$  such that  $[s]_p^\infty = \theta_\infty$ . Then  $\xi^{(\infty)}|_{\theta_\infty}$  can be decomposed in the sum

$$\xi^{(\infty)}|_{\theta_\infty} = a^j D_j|_{\theta_\infty} + (\xi^{(\infty)}|_{\theta_\infty} - a^j D_j|_{\theta_\infty}),$$

where  $D_j$  is the operator of total derivative w.r.t.  $x^j$ , see (1.7). The vector  $a^j D_j|_{\theta_\infty}$  tangents to the graph  $L_s^\infty$  of section  $j_\infty s$  at the point  $\theta_\infty$ . Obviously,

$$\xi^{(\infty)}|_{\theta_\infty} - a^j D_j|_{\theta_\infty} = (b^i - u_j^i a^j) \frac{\partial}{\partial u^i} + \dots \quad (2.2)$$

Let  $f_t$  be the flow of vector field  $\xi$ . Then the flow  $f_t^{(\infty)}$  transforms the graph  $L_s^\infty$  to the graph  $f_t^{(\infty)}(L_s^\infty)$ . The vector  $a^j D_j|_{\theta_\infty}$  is tangent to  $L_s^\infty$ . This means that its contribution in the transformation velocity of  $L_s^\infty$  by  $f_t^{(\infty)}$  is zero. Therefore (2.2) is the transformation velocity  $L_s^\infty$  by  $f_t^{(\infty)}$ . Let us calculate this velocity. In the special coordinates  $x^j, u^i, \dots, u_{j_1 \dots j_k}^i, \dots$  the transformed graph  $f_t^{(\infty)}(L_s^\infty)$  is described by parametric equations

$$\begin{aligned} x \mapsto & \left( S^i(t, x), \frac{\partial S^i}{\partial x^j}(t, x), \dots, \frac{\partial^k S^i}{\partial x^{j_1} \dots \partial x^{j_k}}(t, x), \dots \right) \\ & = \left( S^i(t, x), D_j(S^i)(t, x), \dots, D_{j_1 \dots j_k}(S^i)(t, x), \dots \right), \end{aligned}$$

here  $x = x^1, \dots, x^n$  and  $D_{j_1 \dots j_k} = D_{j_1} \circ \dots \circ D_{j_k}$ . Therefore the transformation velocity of  $L_s^\infty$  by  $f_t^{(\infty)}$  is the following vector

$$\begin{aligned} & \left. \frac{d}{dt} S^i|_{t=0} \frac{\partial}{\partial u^i} + \dots + \frac{d}{dt} D_{j_1 \dots j_k}(S^i)|_{t=0} \frac{\partial}{\partial u_{j_1 \dots j_k}^i} + \dots \right) \\ & = \left( \frac{d}{dt} S^i|_{t=0} \frac{\partial}{\partial u^i} + \dots + D_{j_1 \dots j_k} \left( \frac{d}{dt} S^i|_{t=0} \right) \frac{\partial}{\partial u_{j_1 \dots j_k}^i} + \dots \right) \quad (2.3) \end{aligned}$$

Comparing (2.2) and (2.3) and putting  $\varphi^i = b^i - u_j^i a^j$ , we get that the transformation velocity of  $L_s^\infty$  by  $f_t^{(\infty)}$  is the vector

$$\mathfrak{D}_\varphi = \varphi^i \frac{\partial}{\partial u^i} + \dots + D_{j_1 \dots j_k}(\varphi^i) \frac{\partial}{\partial u_{j_1 \dots j_k}^i} + \dots \quad (2.4)$$

A vector field of the form (2.4) is called an operator of evolution derivative and  $\varphi = (\varphi^1, \dots, \varphi^m)$  is called the *generating function* of this operator.

Thus for the point vector field  $\xi = a^j \partial / \partial x^j + b^i \partial / \partial u^i$  we obtain the formula describing the lifted vector field  $\xi^{(\infty)}$ :

$$\xi^{(\infty)} = a^j D_j + \mathfrak{D}_{(b^i - u_j^i a^j)}. \quad (2.5)$$

Obviously,

$$\begin{aligned} \xi^{(k)} &= (\pi_{\infty, k})_*(a^j D_j + \mathfrak{D}_{(b^i - u_j^i a^j)}) \\ &= a^j \left( \frac{\partial}{\partial x^j} + u_j^i \frac{\partial}{\partial u^i} + \dots + u_{j_1 \dots j_k}^i \frac{\partial}{\partial u_{j_1 \dots j_k}^i} \right) \\ &\quad + (b^i - u_j^i a^j) \frac{\partial}{\partial u^i} + \dots + D_{j_1 \dots j_k} (b^i - u_j^i a^j) \frac{\partial}{\partial u_{j_1 \dots j_k}^i} \end{aligned} \quad (2.6)$$

Obviously, if  $\xi = a^j \partial / \partial x^j + b^i \partial / \partial u^i + c_j^i \partial / \partial u_j^i$  is a contact vector field, then taking into account (1.1), the vector field  $\xi^{(\infty)}$  is described by the formula

$$\xi^{(\infty)} = -\varphi_{u_j} D_j + \mathfrak{D}_\varphi, \quad (2.7)$$

where  $\varphi$  is the generating function of the contact vector field  $\xi$  and

$$\xi^{(k)} = (\pi_{\infty, k})_*(-\varphi_{u_j} D_j + \mathfrak{D}_\varphi). \quad (2.8)$$

### 3. The Lie algebra of lifted vector fields

It is easy to prove the following statements:

**PROPOSITION 3.1.** *For any  $i, j = 1, 2, \dots, n$  and any generating function  $\varphi = (\varphi^1, \dots, \varphi^m)$  the following equalities hold*

$$[D_i, D_j] = 0, \quad [D_i, \mathfrak{D}_\varphi] = 0$$

**PROPOSITION 3.2.** *The map*

$$\xi \mapsto \xi^{(k)}$$

*is a homomorphism of the Lie algebra of all point (contact) vector fields into the Lie algebra of all vector fields on  $J^k \pi$  ( $J^{k+1} \pi$ ).*

**4. Exercises**

- (1) Prove formula (1.1).
- (2) Prove proposition 3.1.
- (3) Prove proposition 3.2



## CHAPTER 4

# Lie pseudogroups

Sets of transformations considered during calculation of differential invariants are traditionally called groups of transformations. Remark that, as a rule, they are not groups in the present day sense of the word. For example, consider the “group”  $\Gamma$  of all contact transformations. Let  $f, g \in \Gamma$ . Then by definition the group operation in  $\Gamma$  we get  $f \cdot g = f \circ g$ . But the transformation  $f \circ g$  is defined iff image of  $g$  coincides with domain of  $f$ . From (2.1) we see that, in general, image of  $g$  does not coincide with domain of  $f$ . Hence the “group operation” is not everywhere defined on  $\Gamma$ . The considered “groups” of transformations are actually pseudogroups.

In this chapter, following [13] and [12] we introduce Lie pseudogroups, their Lie algebras, and consider examples of these notions.

### 1. Pseudogroups

DEFINITION 4.1. *Let  $M$  be a smooth manifold and let  $\Gamma$  be a collection of diffeomorphisms of open subsets of  $M$  into  $M$ .  $\Gamma$  is called a pseudogroup if the following hold:*

- (1)  $\Gamma$  is closed under restriction: if  $f \in \Gamma$  and  $U$  is domain of  $f$ , then  $f|_V \in \Gamma$  for any open  $V \subset U$ .
- (2) if  $f : U \rightarrow M$  is a diffeomorphism,  $U = \cup_{\alpha} U_{\alpha}$ , and  $f|_{U_{\alpha}} \in \Gamma$ , then  $f \in \Gamma$ .
- (3)  $\Gamma$  is closed under inverse: if  $f \in \Gamma$ , then  $f^{-1} \in \Gamma$ .
- (4)  $\Gamma$  is closed under composition:  $f : U \rightarrow M$  and  $g : f(U) \rightarrow M$  both belong to  $\Gamma$ , then  $g \circ f \in \Gamma$ .
- (5) The identity diffeomorphism  $M \rightarrow M$  belongs to  $\Gamma$ .

By  $J^k(M)$  we denote the manifold of all  $k$ -jets of all diffeomorphisms of open subsets of  $M$  into  $M$ . By  $J^k\Gamma$  we denote the set of all  $k$ -jets of all diffeomorphisms belonging to  $\Gamma$ .

DEFINITION 4.2. *A pseudogroup  $\Gamma$  is a Lie pseudogroup if there exists an integer  $k \geq 0$ , called the order of  $\Gamma$ , such that*

- (1) *The set  $J^k\Gamma$  is a smooth submanifold of  $J^k(M)$ .*
- (2) *A diffeomorphism  $f : U \rightarrow M$  belongs to  $\Gamma$  iff  $[f]_p^k \in J^k\Gamma$  for all  $p \in U$ .*

The submanifold  $J^k\Gamma$  of a Lie pseudogroup  $\Gamma$  is called a *system of PDEs defining*  $\Gamma$ .

Examples:

- (1) The set of all diffeomorphisms of  $M$  is a Lie pseudogroup.  $J^1(M)$  is the system of PDEs defining this pseudogroup.
- (2) Let  $\Gamma$  be the pseudogroup of all contact transformations in the case  $n = m = 1$ . In special coordinates, a contact transformation is described by

$$X = X(x, u, u_1), \quad U = U(x, u, u_1), \quad U_1 = U_1(x, u, u_1)$$

where vector-function  $(X(x, u, u_1), U(x, u, u_1), U_1(x, u, u_1))$  satisfies to the system of PDEs

$$\begin{cases} U_1(X_x + u_1 X_u) - (U_x + u_1 U_u) = 0, \\ U_1 X_{u_1} - U_{u_1} = 0 \end{cases}$$

Thus  $\Gamma$  is a Lie pseudogroup.

- (3) Let  $\Gamma$  be the pseudogroup of all point transformations of the form

$$X = x, \quad U = u\varphi(x).$$

This pseudogroup acts on the arithmetic space  $\mathbb{R}^2$  of variables  $x$  and  $u$ . Let  $f \in \Gamma$ , then it is defined by some function  $\varphi$ . By  $U$  we denote the open subset of  $\mathbb{R}$  such that  $\varphi(x) \neq 0$  for all  $x \in U$ . Obviously,  $f$  is defined on the open set  $U \times \mathbb{R} \subset \mathbb{R}^2$ . Clearly, the system of PDEs

$$X - x = 0, \quad \frac{\partial^2 U}{\partial u \partial u} = 0, \quad \frac{\partial U}{\partial u} u - U = 0$$

is defined  $\Gamma$ . Hence,  $\Gamma$  is a Lie pseudogroup.

A pseudogroup  $\Gamma$  is *transitive* if for any  $p_1, p_2 \in M$  there exists  $f \in \Gamma$  such that  $f(p_1) = p_2$ .

Obviously the pseudogroups of the first two examples are transitive. The pseudogroup of the last example is not transitive.

## 2. Lie algebras of pseudogroups

Let  $\Gamma$  be a Lie pseudogroup acting on manifold  $M$ , let  $\xi$  be a vector field in  $M$ , and let  $\varphi_t$  be the flow of  $\xi$ .

The vector field  $\xi$  is  $\Gamma$ -*vector field* if its flow consists of diffeomorphisms belonging to  $\Gamma$ , that is  $\xi \in \Gamma$  for all  $t$ .

**PROPOSITION 4.3.** *The set of all  $\Gamma$ -vector fields is a Lie subalgebra in the Lie algebra of all vector fields in  $M$ .*

PROOF. Let  $\xi$  be  $\Gamma$ -vector field and  $\varphi_t$  be its flow. Then for any  $\lambda \in \mathbb{R}$   $\varphi_{\lambda t}$  is the flow of  $\lambda \xi$  and  $\varphi_{\lambda t} \in \Gamma$ .

Suppose  $\xi_1$  and  $\xi_2$  are  $\Gamma$ -vector fields and  $\varphi_{1t}, \varphi_{2t}$  are their flows respectively. Then  $\varphi_{1t} \circ \varphi_{2t} \in \Gamma$  and it is the flow of  $\xi_1 + \xi_2$ .

Let us prove if  $\xi_1$  and  $\xi_2$  are  $\Gamma$ -vector fields, then  $[\xi_1, \xi_2]$  is a  $\Gamma$ -vector field. Let  $\xi$  be a vector field in  $M$  and let  $\varphi_t$  be its flow. Then the flow  $\varphi_t^k$  is defined in  $J^k(M)$  by the formula

$$\varphi_t^k([f]_p^k) = [\varphi_t]_{f(p)}^k \cdot [f]_p^k = [\varphi_t \circ f]_p^k$$

This flow generate the vector field  $\xi^k$  in  $J^k(M)$ . The following lemmas are easily proved:

LEMMA 4.4. *Suppose  $\xi_1$  and  $\xi_2$  are vector fields in  $M$ . Then*

$$[\xi_1^k, \xi_2^k] = [\xi_1, \xi_2]^k$$

LEMMA 4.5. *A vector field  $\xi_1$  in  $M$  is a  $\Gamma$ -vector field iff the vector field  $\xi^k$  is tangent to the equation  $J^k\Gamma$ .*

Our statement follows now from these lemmas. □

The Lie algebra of all  $\Gamma$ -vector fields is called the *Lie algebra of  $\Gamma$* . We denote it by  $\mathcal{G}$

### 3. Exercises

- (1) Prove lemma 4.4.
- (2) Prove lemma 4.5.
- (3) Complete proposition 4.3.



## Differential invariants

In this chapter, we introduce differential invariants of the action of a Lie pseudogroup on jet bundles of a bundle  $\pi$  and differential invariants of sections of  $\pi$ . We formulate the equivalence problem. Following [16], we introduce scalar differential invariants as well as the general approach to calculate them. We also explain the method of reduction to a fiber. We point out connection between differential invariants of a section and its symmetries. Finally, we illustrate the general approach by a simple example.

### 1. Differential invariants on the jet bundles. The equivalence problem

Let  $\pi : E \rightarrow B$  be a locally-trivial bundle, let  $\Gamma$  be a Lie pseudogroup acting on  $E$ , and let  $\mathcal{G}$  be the Lie algebra of  $\Gamma$ .

The pseudogroup  $\Gamma$  acts on every  $J^k\pi$  by its lifted transformations.

A function or a vector field or a differential form or any other object defined in  $J^k\pi$  is a *differential invariant of the action of  $\Gamma$  on  $J^k\pi$*  if for any  $f \in \Gamma$  the lifted transformation  $f^{(k)}$  preserves this object. These differential invariants are called also *differential invariants (of order  $k$ )* or simply *differential invariants (of order  $k$ )*.

Let  $S : U \rightarrow E$  be a section of  $\pi$  and  $I$  a differential invariant of order  $k$ . Then the restriction  $I|_{L_S^{(k)}}$  is called a *differential invariant of order  $k$  of the section  $S$* . As  $j_k S : U \rightarrow L_S^{(k)} \subset J^k\pi$  is always a diffeomorphism,  $I|_{L_S^{(k)}}$  is essentially an object on the domain of  $S$ .

Suppose  $S_1$  and  $S_2$  are an arbitrary sections of  $\pi$ . Consider the following problem.

Find necessary and sufficient conditions to exist a transformation  $f \in \Gamma$  such that it transforms locally (in its domain)  $S_1$  to  $S_2$ . This problem is called *the equivalence problem of sections of  $\pi$  w.r.t. the pseudogroup  $\Gamma$* .

Suppose  $S_1$  and  $S_2$  are (locally) equivalent, that is there exist a transformation  $f \in \Gamma$  transforms (locally)  $S_1$  to  $S_2$ . Then obviously, the lifted transformation  $f^{(k)}$  transforms  $I|_{L_{S_1}^{(k)}}$  to  $I|_{L_{S_2}^{(k)}}$ , for any  $k$ th order differential invariant  $I$ ,  $k = 0, 1, \dots$

This means that differential invariants of sections give necessary conditions to solve the equivalence problem. In many of cases, differential invariants give sufficient conditions to solve this problem. Below, we consider that cases.

## 2. Scalar differential invariants

Functions that are differential invariants are also called *scalar differential invariants*.

As a result of the action of  $\Gamma$  on the jet bundles  $J^k\pi$ , every  $J^k\pi$  is divided into nonintersecting orbits of this action. Obviously, a scalar differential invariant of order  $k$  is constant on every orbit of the action  $\Gamma$  on  $J^k\pi$ .

By  $A_k$  we denote the set of all scalar differential invariants of order  $\leq k$ . It is clear that  $A_k$  is an  $\mathbb{R}$ -algebra. This means that  $A_k$  satisfies to the following two conditions:

- (1)  $A_k$  is a vector space over the field  $\mathbb{R}$ .
- (2) if  $I_1, I_2 \in A_k$ , then  $I_1 \cdot I_2 \in A_k$ .
- (3) In addition,  $A_k$  satisfies to the condition: let  $I_1, \dots, I_r \in A_k$  and let  $\varphi(\cdot, \dots, \cdot)$  be an arbitrary smooth function of  $r$  arguments, then  $\varphi(I_1, \dots, I_r) \in A_k$ .

We have a sequence of inclusions

$$A_0 \subset A_1 \subset \dots \subset A_k \subset A_{k+1} \subset \dots$$

The  $\mathbb{R}$ -algebra  $A = \bigcup_{k=0}^{\infty} A_k$  is called the *algebra of scalar differential invariants*.

The following obvious statement is very important.

**THEOREM 5.1.** *Let  $I$  be a  $k$ th order scalar differential invariant of the action of  $\Gamma$  on  $J^k\pi$ . Then for any  $\Gamma$ -vector field  $\xi$ , the Lie derivative of  $I$  w.r.t. the vector field  $\xi^{(k)}$  is equal to zero:*

$$L_{X^{(k)}}(I) = 0.$$

This theorem means, that  $k$ th order scalar differential invariants are 1st integrals of all  $\Gamma$ -vector fields lifted to  $J^k\pi$ . This gives the general method to calculate scalar differential invariants:

- (1) Lift all  $\Gamma$ -vector fields  $\xi$  to vector fields  $\xi^{(k)}$  in  $J^k\pi$ . At every point  $\theta_k \in J^k\pi$ , these lifted fields generate the subspace  $D_{\theta_k}^k$  spanned by all vectors  $\xi_{\theta_k}^{(k)}$ . Thus the distribution  $\mathcal{D}^k : \theta_k \mapsto D_{\theta_k}^k$  is generated in  $J^k\pi$ . This distribution has constant maximal dimension almost everywhere on  $J^k\pi$ .
- (2) Find vector fields  $\nu_1, \dots, \nu_N$  in  $J^k\pi$  so that they generate  $\mathcal{D}^k$  on some open subset and they are linear independent at every point of this subset.

- (3) Calculate all functionally independent common 1-st integrals of the vector fields  $\nu_1, \dots, \nu_N$ . The number of these integrals is given by

**PROPOSITION 5.2.** *The number of functionally independent common 1-st integrals of the vector fields  $\nu_1, \dots, \nu_N$  is equal to  $\dim J^k \pi - \dim \mathcal{D}^k$ .*

**PROOF.** By the same way as proposition 3.2 it can be proved that the map  $\xi \mapsto \xi^{(k)}$  is a Lie algebra homomorphism. It follows that the distribution  $\mathcal{D}^k$  is completely integrable. By the Frobenius theorem, in a neighborhood of almost every point of  $J^k \pi$ , there exist a coordinate system  $y^1, \dots, y^N, \dots, y^{\dim J^k \pi}$  such that  $\mathcal{D}^k$  is generated by the vector fields  $\partial/\partial y^1, \dots, \partial/\partial y^N$ . In the terms of this coordinate system, the proposition is obvious now.  $\square$

**REMARK 5.3.** Let  $\tilde{A}_k$  be the  $\mathbb{R}$ -algebra of all 1-st integrals calculated by this method. Obviously,

$$A_k \subset \tilde{A}_k.$$

We have the equality  $A_k = \tilde{A}_k$  if, for example,  $\Gamma$  is a connected Lie group, or the pseudogroup  $\Gamma$  satisfies to some regularity conditions, see [12].

**2.1. The reduction to a fiber.** Suppose  $\Gamma$  acts transitively on the base  $B$  of  $\pi$ . It follows that every orbit of the action of  $\Gamma$  on  $J^k \pi$  intersects the fiber  $J_p^k \pi = (\pi_k)^{-1}(p)$ . As a result  $J_p^k \pi$  is divided into nonintersect subsets. Clearly, these subsets are orbits w.r.t. the action of the group

$$G_p^k = \{ f^{(k)} \mid f \in \Gamma, f(p) = p \}.$$

Suppose we have an scalar differential invariant  $I$  on  $J_p^k \pi$  w.r.t. the action of  $G_p^k$ . Then, acting by all lifted transformations  $f^{(k)}$  on  $I$ , we obtain scalar differential invariant on  $J^k \pi$ .

Let  $U \subset B$  be a coordinate domain with coordinate  $x^1, \dots, x^n$  in neighborhood of  $p$  and let  $x^1(p) = 0, \dots, x^n(p) = 0$ . Suppose there exist a neighborhood  $V \subset U$  of  $p$  such that  $\Gamma$  contains all transformations  $f : V \rightarrow U$  of the form  $x \mapsto x + x_0$ , where  $x_0 \in V$ . Then if the scalar invariant  $I$  on  $J_p^k \pi$  has the form  $I = I(u, \dots, u_k)$ , then the function  $I(u, \dots, u_k)$  is scalar differential invariant on  $(\pi_k)^{-1}(V) \subset J^k \pi$  w.r.t. the action of  $\Gamma$ . Indeed, shifts  $f(x) = x - x_0$  transfer  $I$  from  $J_p^k \pi$  to  $(\pi_k)^{-1}(V)$  so that the expression of the obtained invariant is the same  $I = I(u, \dots, u_k)$ .

Thus in the case when  $\Gamma$  contains shifts, the problem of calculation of  $k$ -order scalar differential invariants w.r.t.  $\Gamma$  is reduced to the problem of calculation of  $k$ -order scalar differential invariants on  $J_p^k \pi$  w.r.t.  $G_p^k$ .

Further details of this method we explain by examples in following chapters.

### 3. Symmetries and invariants

Let  $\xi$  be a vector field in  $J^0\pi$  and  $S$  be a section of  $\pi$ .

We say that  $\xi$  is a *symmetry of  $S$*  if  $\xi$  tangent to the graph  $L_S^0$ .

Suppose  $\xi$  is a symmetry of  $S$ , and  $I|_{L_S^{(k)}}$  be a scalar differential invariant of order  $k$  of  $S$ . Taking into account that the lifted vector fields  $\xi^{(k)}$  tangent to the graph  $L_S^k$  of the section  $j_k S$  of  $\pi_k$ , we get that

$$\xi^{(k)}|_{L_S^{(k)}}(I|_{L_S^{(k)}}) = 0$$

This means that the function  $I|_{L_S^{(k)}}$  is constant along integral lines of the vector field  $\xi^{(k)}|_{L_S^{(k)}}$ . The following statement is obvious now.

**PROPOSITION 5.4.** *Suppose a section  $S$  of  $\pi$  has  $n$  symmetries linearly independent in every point of  $L_S^0$ . Then every scalar differential invariant  $I|_{L_S^{(k)}}$  is a constant.*

### 4. Example

Let us illustrate the general method of calculation of scalar differential invariants by the calculation of invariants of ODEs of the form

$$y' = a(x)y + b(x) \quad (4.1)$$

w.r.t. transformations of the form

$$x = X, \quad y = Yg(X) \quad (4.2)$$

According to our approach, we consider the natural bundle of these ODEs:

$$\pi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \pi : (x, u^1, u^2) \mapsto x$$

Any section  $S(x) = (S^1(x), S^2(x))$  of  $\pi$  is identified with the ODE  $y' = S^1(x)y + S^2(x)$ . Obviously this identification is a bijection between the set of all considering ODEs and the set of all sections of  $\pi$ . The transformation law of coefficients of these equations under considering transformations of variables generates the Lie pseudogroup  $\Gamma$  acting on  $J^0\pi$  by the formula

$$X = x, \quad U^1 = u^1 - g'(x)/g(x), \quad U^2 = u^2/g(x),$$

where  $g(x)$  is an arbitrary smooth function. Hence an arbitrary  $\Gamma$ -vector field has the form

$$\xi = -h'(x)\frac{\partial}{\partial u^1} - u^2h(x)\frac{\partial}{\partial u^2},$$

where  $h(x)$  is an arbitrary smooth function of  $x$  and  $h'(x)$  its derivative w.r.t.  $x$ . Below, it is convenient use notation  $\xi_h$  instead  $\xi$ . It follows that

$$\xi_h^{(\infty)} = \mathfrak{D}_\varphi = -D^j(\varphi^1)\frac{\partial}{\partial u_j^1} - D^j(\varphi^2)\frac{\partial}{\partial u_j^2},$$



where  $\varphi = (\varphi^1, \varphi^2) = (h'(x), u^2 h(x))$ . Thus

$$\begin{aligned} \xi_h^{(\infty)} = & -h' \frac{\partial}{\partial u^1} - u^2 h \frac{\partial}{\partial u^2} \\ & - h^{(2)} \frac{\partial}{\partial u_1^1} - (u_1^2 h + u^2 h') \frac{\partial}{\partial u_1^2} \\ & - \dots - h^{(j+1)} \frac{\partial}{\partial u_j^1} - D^j(u^2 h) \frac{\partial}{\partial u_j^2} - \dots \end{aligned}$$

Let us find zero order scalar differential invariants. It is easy that the collection of vector fields

$$\nu_1 = \xi_1 = -u^2 \frac{\partial}{\partial u^2}, \quad \nu_2 = \xi_x - x \xi_1 = -\frac{\partial}{\partial u^1}$$

generates the distribution  $\mathcal{D}^0$ . From these formulas, we get that the action  $\Gamma$  on  $J^0\pi$  divides  $J^0\pi$  onto two intersected orbits:

$$J^0\pi = \{u^2 = 0\} \cup \{u^2 \neq 0\}$$

On the 1-st orbit, dimension of  $\mathcal{D}$  is equal to 1, on the 2-nd one it is equal to 2.

Let us investigate the case  $u^2 \neq 0$ .

Obviously the common 1-st integrals of these vector fields are functions of  $x$ . Thus

$$A_0 = \{\text{All smooth functions of } x\}$$

These zero-order scalar differential invariants are trivial for our pseudogroup  $\Gamma$ .

Let us find 1-st order scalar differential invariants. It is clear that the collection of vector fields

$$\begin{aligned} \nu_1 = \xi_1^{(1)} &= -u^2 \frac{\partial}{\partial u^2} - u_1^2 \frac{\partial}{\partial u_1^2}, \\ \nu_2 = \xi_x^{(1)} - x \xi_1^{(1)} &= -\frac{\partial}{\partial u^1} - u^2 \frac{\partial}{\partial u_1^2}, \\ \nu_3 = \xi_{x^2}^{(1)} - 2x \nu_2 - x^2 \nu_1 &= -2 \frac{\partial}{\partial u_1^1} \end{aligned}$$

generates the distribution  $\mathcal{D}^1$ . Common 1-st integrals of the vector fields  $\nu_1$ ,  $\nu_2$ , and  $\nu_3$  are solutions of the following PDEs system w.r.t. an unknown function  $I$  of  $x$ ,  $u^1$ ,  $u^2$ ,  $u_1^1$ , and  $u_1^2$

$$\begin{cases} \nu_1(I) = 0 \\ \nu_2(I) = 0 \\ \nu_3(I) = 0 \end{cases}$$

From the 3-rd equation, we have  $I = I(x, u^1, u^2, u_1^2)$ . From the 1-st one, we have  $I = I(x, u^1, u_1^2/u^2)$ . Finally, from the 2-nd equation, we have  $I = I(x, u_1^2/u^2 - u^1)$ . Thus

$$A_1 = \{\text{All smooth functions of two arguments: } x \text{ and } u_1^2/u^2 - u^1\}$$

In particular,  $u_1^2/u^2 - u^1$  is 1-st order scalar differential invariant.

### 5. Exercises

- (1) Prove that all ODEs of the form  $y' = a(x)y$  are equivalent w.r.t. transformations (4.2).
- (2) Classify all ODEs (4.1) up to equivalence w.r.t. pseudogroup of all transformations (4.2).
- (3) Calculate the algebra of all scalar invariants of ODEs (4.1) w.r.t. pseudogroup of transformations (4.2).
- (4) Prove that the scalar differential invariant  $I = b'/b - a$  of an equation  $y' = a(x)y + b(x)$  is a constant iff this ODE has symmetry of the form  $h(x)\partial/\partial x$ .
- (5) Prove that dimension of the algebra of symmetries of the form  $h(x)\partial/\partial x$  for an equation  $y' = a(x)y$  is  $\geq 2$ .
- (6) Solve the equivalence problem of ODEs  $y'' = a(x)y' + b(x)y + c(x)$  w.r.t. of transformations (4.2).
- (7) Classify all ODEs  $y'' = a(x)y' + b(x)y + c(x)$  up to equivalence w.r.t. pseudogroup of all transformations (4.2).

## CHAPTER 6

### Classification of linear ODEs up to equivalence

From theorem 2.3 of chapter 2, we know that a most general transformation of variables for ODEs is a contact transformation. It is natural to have a classification of all linear ODEs up to a contact transformation.

In this chapter, we solve this problem for linear ODEs of order 3 for the simplicity. The solution of this problem for an arbitrary order can be found in [18]. Here on the example of these equations, we clarify in details our general approach to calculate scalar differential invariants on a natural bundles. We also investigate in details the equivalence problem for these ODEs. Finally, we get the complete classification of 3rd order linear ODEs up to equivalence.

#### 1. Reduction to the lesser pseudogroup

Solving the problem of classification of some objects up to a transformation of some pseudogroup, it is useful to investigate the possibility to reduce the problem to a lesser pseudogroup. As usually, the solution of the reduced problem is more easy.

The problem of classification of linear ODEs up to a contact transformations can be reduced to lesser pseudogroups. We use one of these reductions. The other one is represented in exercises in the end of this chapter.

Consider an arbitrary 3rd order linear ODEs

$$y^{(3)} = a_2(x)y^{(2)} + a_1(x)y' + a_0(x)y + b(x),$$

Obviously that the point transformation

$$x = X, \quad y = Y + y_0(X),$$

where  $y_0(x)$  is a solution of the initial equation, transforms this ODE to the homogeneous equation

$$Y^{(3)} = a_2(X)Y^{(2)} + a_1(X)Y' + a_0(X)Y$$

It is easy to check that the point transformation

$$X = x, \quad Y = ye^{\frac{1}{3} \int a_2(x) dx}$$

reduces the obtained ODE to an ODE of the form

$$y^{(3)} = a_1(x)y' + a_0(x)y. \tag{1.1}$$

The following important statement is proved by direct calculations using some facts about symmetries of linear ODEs, see [17] or [18].

**THEOREM 6.1.** *Let  $f$  be a contact transformation, transforming an ODE  $\mathcal{E}_1$  of the form (1.1) to an ODE  $\mathcal{E}_2$  of the same form. Then there exist a point transformation of the following form transforming  $\mathcal{E}_1$  to  $\mathcal{E}_2$*

$$X = f(x), \quad Y = y \cdot f'. \quad (1.2)$$

It is easy to check that an arbitrary point transformation (1.2) transforms an arbitrary ODE (1.1) to an ODE of the same form.

Thus, the problem of classification of 3rd order linear ODEs w.r.t. contact transformations is reduced to the classification of ODEs (1.1) w.r.t. the Lie pseudogroup of all transformations (1.2).

## 2. The natural bundle of linear ODEs

Consider the trivial bundle

$$\pi : E = \mathbb{R}^1 \times \mathbb{R}^2 \rightarrow \mathbb{R}^1, \quad \pi : (x, u^1, u^2) \mapsto x,$$

where  $x$  is the standard coordinate on the base  $\mathbb{R}^1$  and  $u^1, u^2$  are the standard coordinates on the fiber  $\mathbb{R}^2$ .

We identify any linear ODE of form (1.1)

$$\mathcal{E} = \{ y^{(3)} = a_1(x)y' + a_0(x)y. \}$$

with the section  $S_{\mathcal{E}}$  of  $\pi$  defined by the formula

$$S_{\mathcal{E}} : x \mapsto ( a_1(x), a_0(x) ).$$

Clearly, this identification  $\mathcal{E} \mapsto S_{\mathcal{E}}$  is a bijection. By  $\mathcal{E}_S$  we denote the equation corresponding to the section  $S$  under this identification and by  $S_{\mathcal{E}}$  the section corresponding to the ODE  $\mathcal{E}$  under this identification.

Let us obtain the transformation law of coefficients of equations (1.1) w.r.t. transformations (1.2). Suppose the ODE  $Y^{(3)} = A_1(X)Y' + A_0(X)Y$  is transformed to  $y^{(3)} = a_1(x)y' + a_0(x)y$ . Applying equations (1.5), we get

$$\begin{aligned} a_1(x) &= -2 \left( \frac{f''(x)}{f'(x)} \right)' + \left( \frac{f''(x)}{f'(x)} \right)^2 + (f'(x))^2 A_1(f(x)), \\ a_0(x) &= - \left( \frac{f''(x)}{f'(x)} \right)'' + \frac{f''(x)}{f'(x)} \left( \frac{f''(x)}{f'(x)} \right)^2 \\ &\quad + f'(x) f''(x) A_1(f(x)) + (f'(x))^3 A_0(f(x)) \end{aligned} \quad (2.1)$$

This transformation law can be considered as the transformation law of the sections of  $\pi$  generated by the transformation  $f$  of the base of  $\pi$ . More exactly, let  $\Gamma$  be the Lie pseudogroup of all transformations of the base  $\mathbb{R}$  of  $\pi$ . Then we can define the lifting of every  $f \in \Gamma$  to diffeomorphism

$$f^{(0)} : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R} \times \mathbb{R}^2$$

of the total space  $\mathbb{R}^1 \times \mathbb{R}^2$  of  $\pi$  by the formula

$$\begin{aligned} X &= f(x) \\ U^1 &= -2\left(\frac{g''}{g'}\right)' + \left(\frac{g''}{g'}\right)^2 + (g')^2 u^1, \\ U^2 &= -\left(\frac{g''}{g'}\right)'' + \frac{g''}{g'}\left(\frac{g''}{g'}\right)' + g'g''u^1 + (g')^3 u^2, \end{aligned} \quad (2.2)$$

where  $g = f^{-1}$ . We can now represent the transformation of sections of  $\pi$  corresponding the transformation of ODEs (2.1) by the formula

$$S = f^{(0)} \circ s \circ f^{-1} \quad (2.3)$$

Thus, the problem of classification of 3rd order linear ODEs w.r.t. contact transformations is reduced to the classification of sections of  $\pi$  w.r.t. the action of  $\Gamma$  on the total space of  $\pi$ .

### 3. Differential invariants of linear ODEs

**3.1. Lifts of  $\Gamma$ -vector fields.** The Lie algebra of all  $\Gamma$ -vector fields is the Lie algebra of all vector fields on the base  $\mathbb{R}$  of  $\pi$ .

Let

$$\xi_\varphi = \varphi(x) \frac{\partial}{\partial x}$$

be an arbitrary field on the base  $\mathbb{R}$  of  $\pi$ . Then from (2.2), we get

$$\begin{aligned} \xi_\varphi^{(0)} &= \varphi(x) \frac{\partial}{\partial x} + (2\varphi^{(3)} - 2\varphi' u^1 - \varphi u_1^1) \frac{\partial}{\partial u^1} \\ &\quad + (\varphi^{(4)} - \varphi'' u^1 - 3\varphi' u^2 - \varphi u_1^2) \frac{\partial}{\partial u^2} \end{aligned} \quad (3.1)$$

It follows

$$\begin{aligned} \xi_\varphi^{(k)} &= \varphi(x) D \Big|_{J^k \pi} + \mathfrak{D}_\psi \Big|_{J^k \pi} \\ &= \varphi(x) \left( \frac{\partial}{\partial x} + u_1^1 \frac{\partial}{\partial u^1} + u_1^2 \frac{\partial}{\partial u^2} + \dots + u_{k+1}^1 \frac{\partial}{\partial u_k^1} + u_{k+1}^2 \frac{\partial}{\partial u_k^2} \right) \\ &\quad + \left( \psi^1 \frac{\partial}{\partial u^1} + \psi^2 \frac{\partial}{\partial u^2} + \dots + D^k(\psi^1) \frac{\partial}{\partial u_k^1} + D^k(\psi^2) \frac{\partial}{\partial u_k^2} \right) \end{aligned} \quad (3.2)$$

where  $\psi = (\psi^1, \psi^2)$  and

$$\psi^1 = 2\varphi^{(3)} - 2\varphi' u^1 - \varphi u_1^1, \quad \psi^2 = \varphi^{(4)} - \varphi'' u^1 - 3\varphi' u^2 - \varphi u_1^2$$

**3.2. The reduction to a fiber.** Obviously,  $\Gamma$  acts transitively on the base  $\mathbb{R}$  of  $\pi$ . Indeed, for any point  $x_0 \in \mathbb{R}$  there exist  $f \in \Gamma$  such that  $f(0) = x_0$ , for example  $f(x) = x + x_0$ . It follows that every orbit of the action of  $\Gamma$  on  $J^k\pi$  intersects the fiber  $J_0^k\pi = (\pi_k)^{-1}(0)$ . As a result  $J_0^k\pi$  is divided into nonintersect subsets. Clearly, these subsets are orbits w.r.t. the action of the group

$$G^k = \{ f^{(k)} \mid f \in \Gamma, f(0) = 0 \}.$$

Suppose we have an invariant  $I = I(u, \dots, u_k)$  on  $J_0^k\pi$  w.r.t. the action of  $G^k$ . Then,  $I = I(u, \dots, u_k)$  is invariant on  $J^k\pi$  w.r.t. the action of  $\Gamma$ . Indeed, the transformations of the form  $f(x) = x - x_0$  transfer  $I$  from  $J_0^k\pi$  to  $J^k\pi$  so that the expression of the obtained invariant is the same  $I = I(u, \dots, u_k)$ .

Thus the problem of calculation of  $k$ -order scalar differential invariants w.r.t.  $\Gamma$  is reduced to the problem of calculation of  $k$ -order scalar differential invariants on  $J_0^k\pi$  w.r.t.  $G^k$ .

The last problem, is solved in the following way:

- (1) Calculate the Lie algebra of the group  $G^k$ , that is calculate all lifted  $\Gamma$ -vector fields  $\xi^{(k)}$  such that their values at  $0 \in \mathbb{R}$  is a zero vector, that is  $\xi|_0 = 0$ . We call these fields as  $G^k$ -vector fields.
- (2) At every point  $\theta_k \in J_0^k\pi$ ,  $G^k$ -vector fields generate the subspace  $D_{\theta_k}^k$  spanned by all their values at  $\theta_k$ . Thus the distribution  $\mathcal{D}^k : \theta_k \mapsto D_{\theta_k}^k$  is generated in  $J_0^k\pi$ . This distribution has constant maximal dimension almost everywhere on  $J_0^k\pi$ .
- (3) Find vector fields  $\nu_1, \dots, \nu_N$  in  $J_0^k\pi$  so that they generate  $\mathcal{D}^k$  on some open subset and they are linear independent at every point of this subset.
- (4) Calculate the algebra of all common 1st integrals of  $G^k$ -vector fields. This problem is reduced to the calculation of all functionally independent common 1-st integrals of the vector fields  $\nu_1, \dots, \nu_N$ .

**3.3. The calculation of distributions  $\mathcal{D}^k$ .** Let  $\xi_\varphi^{(k)}|_{x=0}$  be a  $G^k$ -vector field. Decomposing the function  $\varphi$  in the Taylor series with remainder term at the point  $0 \in \mathbb{R}$ , we obtain

$$\varphi = \lambda_1 x + \lambda_2 x^2 + \dots + \lambda_{k+4} x^{k+4} + R(x) x^{k+5},$$

where  $\lambda_i = \frac{1}{i!} \frac{d^i \varphi}{(dx)^i}(0)$ . Obviously,

$$\xi_\varphi^{(k)}|_{x=0} = \lambda_1 \xi_x^{(k)}|_{x=0} + \lambda_2 \xi_{x^2}^{(k)}|_{x=0} + \dots + \lambda_{k+4} \xi_{x^{k+4}}^{(k)}|_{x=0}$$

It follows that the distribution  $\mathcal{D}^k$  is generated by the  $G^k$ -vector fields

$$\xi_x^{(k)}|_{x=0}, \quad \xi_{x^2}^{(k)}|_{x=0}, \quad \dots, \quad \xi_{x^{k+4}}^{(k)}|_{x=0}$$

Reducing this system of vector fields to the step form, we obtain vector fields  $\nu_1, \dots, \nu_N$ .

It is easy to get that the distribution  $\mathcal{D}^0$  on  $J_0^0\pi$  is spanned by the following vector fields linearly independent at every point of  $J_0^0\pi$ :

$$\nu_1 = \frac{\partial}{\partial u^1}, \nu_2 = \frac{\partial}{\partial u^2}. \quad (3.3)$$

This means,  $\dim J_0^0\pi = \dim \mathcal{D}^0$ . It follows,  $J_0^0\pi$  is an orbit of the action  $G^0$  on  $J_0^0\pi$ . Hence trivial (that is constant) scalar differential invariants only live on  $J^0\pi$ .

The distribution  $\mathcal{D}^1$  is spanned by the following vector fields on  $J_0^1\pi$ :

$$\nu_1 = \frac{\partial}{\partial u^1}, \nu_2 = \frac{\partial}{\partial u^2} + 2\frac{\partial}{\partial u_1^1}, \nu_3 = (u_1^1 - 2u^2)\frac{\partial}{\partial u_1^1}, \nu_4 = \frac{\partial}{\partial u_1^2}. \quad (3.4)$$

These fields are linearly independent at every point of the subset of  $J_0^1\pi$  describing by the inequality  $u_1^1 - 2u^2 \neq 0$ . Obviously, this subset is an orbit of the action of  $G$  on  $J_0^1\pi$ . We denote it by  $\text{Orb}_1$ . System (3.4) is degenerated to the system

$$\nu_1 = \frac{\partial}{\partial u^1}, \nu_2 = \frac{\partial}{\partial u^2} + 2\frac{\partial}{\partial u_1^1}, \nu_4 = \frac{\partial}{\partial u_1^2}.$$

on the subset of  $J_0^1\pi$  describing by the equation  $u_1^1 - 2u^2 = 0$ . Obviously, this subset is an orbit of the action of  $G$  on  $J_0^1\pi$ . We denote it by  $\text{Orb}_2$ . Thus  $J_0^1\pi$  is divided into two orbits of the action of  $G^1$ :  $J_0^1\pi = \text{Orb}_1 \cup \text{Orb}_2$  so that  $\dim \text{Orb}_1 = \dim J_0^1\pi$ ,  $\dim \text{Orb}_2 = \dim J_0^1\pi - 1$ , and  $u_1^1 - 2u^2 = 0$  is the equation describing  $\text{Orb}_2$  as a submanifold of  $J_0^1\pi$ .

The equality  $\dim \text{Orb}_1 = \dim J_0^1\pi$  means that  $\text{Orb}_1$  is open everywhere dense subset of  $J_0^1\pi$ . This means that trivial scalar differential invariants only live on  $J^1\pi$ .

Recall that an orbit of maximal dimension is called a *generic orbit*. An orbit of lesser dimension is called a *degenerate orbit*.

The distribution  $\mathcal{D}^2$  is spanned by the following vector fields linearly independent almost everywhere on  $J_0^2\pi$ :

$$\begin{aligned} \nu_1 &= 2\frac{\partial}{\partial u^1}, \nu_2 = \frac{\partial}{\partial u^2} + 2\frac{\partial}{\partial u_1^1}, \\ \nu_3 &= -3(u_1^1 - 2u^2)\frac{\partial}{\partial u_1^1} - 4(u_2^1 - 2u_1^2)\frac{\partial}{\partial u_2^1}, \\ \nu_4 &= \frac{\partial}{\partial u_1^2} + 2\frac{\partial}{\partial u_2^1}, \nu_5 = -3(u_1^1 - 2u^2)\frac{\partial}{\partial u_2^1}, \nu_6 = \frac{\partial}{\partial u_2^2}. \end{aligned} \quad (3.5)$$

Clearly,  $J_0^2\pi$  is divided into some orbits and dimension of the unique generic orbit is equal to  $\dim J_0^2\pi$ . Therefore trivial scalar differential invariants only live on  $J^2\pi$ .

The distribution  $\mathcal{D}^3$  is spanned by the following vector fields linearly independent almost everywhere on  $J_0^3\pi$ :

$$\begin{aligned} \nu_1 &= 2\frac{\partial}{\partial u^1} - 3h\frac{\partial}{\partial u^3}, & \nu_2 &= \frac{\partial}{\partial u^2} + 2\frac{\partial}{\partial u^1}, \\ \nu_3 &= -3h\frac{\partial}{\partial u^1} - 4h'\frac{\partial}{\partial u^2} - (5h'' + 3u^1h)\frac{\partial}{\partial u^3}, & \nu_4 &= \frac{\partial}{\partial u^2} + 2\frac{\partial}{\partial u^1}, \\ \nu_5 &= -3h\frac{\partial}{\partial u^1} - 7h'\frac{\partial}{\partial u^2}, & \nu_6 &= \frac{\partial}{\partial u^2} + 2\frac{\partial}{\partial u^3}, & \nu_7 &= \frac{\partial}{\partial u^3}, \end{aligned} \quad (3.6)$$

where  $h = u_1^1 - 2u^2$ . Clearly,  $J_0^3\pi$  is divided into some orbits and dimension of the unique generic orbit is equal to  $\dim J_0^3\pi - 1$ . Therefore nontrivial scalar differential invariants live on  $J^3\pi$ . More exactly, the algebra  $A_3$  of all scalar differential invariants of order 3 is generated by one functionally independent invariant.

It is not hard to get now the following table, where by  $N_k$  we denote the number of functionally independent scalar differential invariants on  $J^k\pi$ ,  $N_k = \dim J_0^k\pi - \dim \mathcal{D}^k$ :

$k$	$\dim J_0^k\pi$	$\dim \mathcal{D}^k$	$N^k$
0	2	2	0
1	4	4	0
2	6	6	0
3	8	7	1
4	10	8	2
5	12	9	3
...	...	...	...
$r$	$2r + 2$	$r + 4$	$r - 2$
...	...	...	...

(3.7)

**3.4. The calculation of the 1st nontrivial invariant.** It is not hard to calculate a common 1st integral of vector fields (3.6). As a result, we obtain the following scalar differential invariant on  $J^3\pi$

$$\mathcal{J} = \left[ \frac{2}{3}D\left(\frac{D(u_1^1 - 2u^2)}{u_1^1 - 2u^2}\right) - \frac{1}{9}\left(\frac{D(u_1^1 - 2u^2)}{u_1^1 - 2u^2}\right)^2 + u^1 \right] (u_1^1 - 2u^2)^{-2/3}, \quad (3.8)$$

where  $D$  is the total derivative operator.

**3.5. Nonscalar differential invariants.** Consider equations (2.2) defining the action of  $\Gamma$  on the total space  $J^0\pi$  of  $\pi$ . Subtracting the second equation differentiated by  $x$  from the doubled third equation, we get

$$(U_1^1 - 2U^2)^{1/3} = g'(u_1^1 - 2u^2)^{1/3}.$$



From the transformation law of coefficients of differential 1-form over a diffeomorphism, we get that the differential form on  $J^0\pi$

$$\omega = (u_1^1 - 2u^2)^{1/3} dx \quad (3.9)$$

is invariant w.r.t. the action of  $\Gamma$  on  $J^1\pi$ . Thus  $\omega$  is differential invariant of order 1 of the action of  $\Gamma$ .

From (3.4), we have

**THEOREM 6.2.** *The unique degenerate orbit  $\text{Orb}_2$  of  $J^1\pi$  is defined by the equation  $\omega = 0$ , that is*

$$\text{Orb}_2 = \{ \theta_1 \in J^1\pi \mid \omega|_{\theta_1} = 0 \}$$

Let  $s : x \mapsto (s_1(x), s_2(x))$  be a section of  $\pi$ . Then the restriction

$$\omega|_{L_s^1} = \left( \frac{ds_1}{dx}(x) - 2s_2(x) \right)^{1/3} dx \quad (3.10)$$

of  $\omega$  on the graph  $L_s^1$  of the section

$$j_1s : x \mapsto \left( s_1(x), s_2(x), \frac{ds_1}{dx}(x), \frac{ds_2}{dx}(x) \right)$$

is a differential 1-form on  $L_s^1$ . It is a differential invariant of order 1 of  $s$  w.r.t. the action of  $\Gamma$  on  $\pi$ . Tacking into account that  $\pi$  projects  $L_s^1$  onto the base of  $\pi$  diffeomorphically, we can think that  $\omega|_{L_s^1}$  is a differential 1-form on the base of  $\pi$ . It makes possible to think that this form is a (nonscalar) differential invariant of order 1 of the equation  $y^{(3)} = s_1(x)y' + s_2(x)y$ , corresponding the section  $s$ , w.r.t. transformations (1.2).

The following vector field on the graph  $L_s^1$

$$\left( \frac{ds_1}{dx}(x) - 2s_2(x) \right)^{-1/3} \frac{\partial}{\partial x}. \quad (3.11)$$

is dual to form (3.10). Therefore it is a differential invariant of order 1 of  $s$  w.r.t. the action of  $\Gamma$  on  $\pi$  (or invariant of the equation  $y^{(3)} = s_1(x)y' + s_2(x)y$  w.r.t. transformations (1.2)). We stress that this means the following: let  $f \in \Gamma$  then  $f^{(1)}$  transforms  $L_s^1$  to  $L_S^1$  for some section  $S$  of  $\pi$  and  $(f^{(1)})_*$  transforms the vector field  $(s_1'(x) - 2s_2(x))^{-1/3} \partial/\partial x$  on  $L_s^1$  to the vector field  $(S_1'(X) - 2S_2(X))^{-1/3} \partial/\partial X$  on  $L_S^1$ .

Note that the vector field on  $J^1\pi$

$$(u_1^1 - 2u^2)^{-1/3} \frac{\partial}{\partial x}$$

is not invariant w.r.t. action of  $\Gamma$  on  $J^1\pi$ .

**3.6. Algebras of scalar differential invariants.** Consider the operator

$$\zeta = (u_1^1 - 2u^2)^{-1/3} D,$$

where  $D$  is the total derivative operator. It is easy to prove the following

**THEOREM 6.3.** *Let  $I$  be a  $k$ -order scalar differential invariant of the action of  $\Gamma$  on  $J^k\pi$ . Then  $\zeta(I)$  is a  $k+1$ -order scalar differential invariant of the action of  $\Gamma$  on  $J^{k+1}\pi$ .*

From table (3.7) and this theorem, we get

**THEOREM 6.4.** *The algebra  $A_{3+k}$ ,  $k = 0, 1, \dots$ , of  $3+k$ - order scalar differential invariants of the action of  $\Gamma$  on  $J^{3+k}\pi$  is generated by the invariants*

$$\mathcal{J}, \quad \zeta(\mathcal{J}), \quad \dots, \quad \zeta^k(\mathcal{J}).$$

#### 4. The equivalence problem

Suppose  $s : x \mapsto (s_1(x), s_2(x))$  and  $S : X \mapsto (S_1(X), S_2(X))$  are an arbitrary sections of  $\pi$ . The sections  $s$  and  $S$  are locally equivalent if there exist transformation (2.2) transforming locally  $s$  to  $S$ .

Clearly,  $s$  and  $S$  are locally equivalent iff there exist a transformation  $g(X) = x$  in  $\Gamma$  satisfying to the system of ODEs

$$\begin{aligned} S_1 &= -2\left(\frac{g''}{g'}\right)' + \left(\frac{g''}{g'}\right)^2 + (g')^2 s_1(g), \\ S_2 &= -\left(\frac{g''}{f'}\right)'' + \frac{g''}{g'}\left(\frac{g''}{g'}\right)' + g'g''s_1(g) + (g')^3 s_2(g). \end{aligned}$$

Subtracting the first equation differentiated by  $x$  from the doubled second one, we get the equivalent system of ODEs

$$S_1 = -2\left(\frac{g''}{g'}\right)' + \left(\frac{g''}{g'}\right)^2 + (g')^2 s_1(g), \quad (4.1)$$

$$S_1' - 2S_2 = (g')^3 (s_1'(g) - 2s_2(g)). \quad (4.2)$$

**THEOREM 6.5.** (1) *If  $\omega|_{L_s^1} = 0$  and  $\omega|_{L_S^1} = 0$ , then  $s$  and  $S$  are locally equivalent.*

(2) *If one of the forms  $\omega|_{L_s^1}$  and  $\omega|_{L_S^1}$  is equivalent and the other one is not equivalent to zero, then  $s$  and  $S$  are not locally equivalent.*

(3) *If  $\omega|_{L_s^1} \neq 0$  and  $\omega|_{L_S^1} \neq 0$ , then  $s$  and  $S$  are locally equivalent iff a solution  $g(X)$  of ODE (4.2) satisfies to the equality*

$$\mathcal{J}|_{L_S^3}(X) = \mathcal{J}|_{L_s^3}(g(X)), \quad (4.3)$$

where  $\mathcal{J}$  is scalar differential invariant (3.8)

PROOF. The first statement follows from the existence of solution of ODE (4.1).

The second statement holds because one of the sections  $j_1s$  and  $j_1S$  belongs to the orbit  $\text{Orb}_1$  and the other one belongs to  $\text{Orb}_2$ .

Substituting

$$g' = \left( \frac{S'_1 - 2S_2}{s'_1 - 2s_2} \right)^{1/3}$$

in ODE (4.1), we obtain (4.3). It follows the third statement.  $\square$

## 5. The classification of linear ODEs

Let  $\mathcal{E}$  be an arbitrary ODE (1.1) and let  $S$  be the section of  $\pi$  identified with  $\mathcal{E}$ . We will write  $\omega|_{\mathcal{E}}$  and  $\mathcal{J}|_{\mathcal{E}}$  instead of  $\omega|_{L^1_S} = 0$  and  $\mathcal{J}|_{L^3_S}$  respectively.

### 5.1. The case $\omega|_{\mathcal{E}} = 0$ .

**THEOREM 6.6.** *Suppose  $\mathcal{E}$  satisfies to the condition  $\omega|_{\mathcal{E}} \equiv 0$ . Then  $\mathcal{E}$  is locally equivalent to the ODE*

$$y^{(3)} = 0.$$

PROOF. Obviously, the zero-section  $Z : x \mapsto (0, 0)$  of  $\pi$  satisfies to condition  $\omega|_{L^1_Z} = 0$ . The theorem follows now from theorem 6.5.  $\square$

### 5.2. The case $\omega|_{\mathcal{E}} \neq 0$ and $(\mathcal{J}|_{\mathcal{E}})' = 0$ .

**THEOREM 6.7.** (1) *Suppose  $\mathcal{E}$  satisfies to the conditions  $\omega|_{\mathcal{E}} \neq 0$  and  $\mathcal{J}|_{\mathcal{E}} \equiv C$ , where  $C$  is a constant. Then  $\mathcal{E}$  is locally equivalent to the ODE*

$$y^{(3)} = 2^{2/3}Cy' + y.$$

(2) *If the constants  $C_1$  and  $C_2$  are not equal, then the ODEs  $y^{(3)} = 2^{2/3}C_1y' + y$  and  $y^{(3)} = 2^{2/3}C_2y' + y$  are not locally equivalent.*

PROOF. It is easy to check that  $\omega|_{L^1_S} = C$  for the constant section  $S : x \mapsto (2^{2/3}C, 1)$ . The theorem follows now from theorem 6.5.  $\square$

**5.3. The case  $\omega|_{\mathcal{E}} \neq 0$  and  $(\mathcal{J}|_{\mathcal{E}})' \neq 0$ .** Suppose  $\mathcal{E}$  satisfies to the conditions  $\omega|_{\mathcal{E}} \neq 0$  and  $(\mathcal{J}|_{\mathcal{E}})' \neq 0$ . Then  $\mathcal{J}|_{\mathcal{E}}$  can be considered as the new independent variables. The transformation  $I = \mathcal{J}|_{\mathcal{E}}(x)$  generates the point transformation of the form (1.2)

$$I = \mathcal{J}|_{\mathcal{E}}(x), \quad Y = y \cdot (\mathcal{J}|_{\mathcal{E}})'(x).$$

The inverse transformation transforms  $\mathcal{E}$  to ODE, which we call the *canonical form* of  $\mathcal{E}$ .

PROPOSITION 6.8. *Suppose  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are ODEs (1.1) satisfying to conditions  $\omega|_{\mathcal{E}_i} \neq 0$  and  $(\mathcal{J}|_{\mathcal{E}_i})' \neq 0$ ,  $i = 1, 2$ . Then  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are locally equivalent iff their canonical forms are the same.*

PROOF. Let us denote by  $S_i$  the section corresponding to  $\mathcal{E}_i$  and denote by  $\mathcal{J}_i$  the restriction  $\mathcal{J}|_{\mathcal{E}_i}$ ,  $i = 1, 2$ . Then

$$(\mathcal{J}_i)^{(0)} \circ S_i \circ (\mathcal{J}_i)^{-1}$$

is the *canonical* form of section  $S_i$ .

Suppose  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are locally equivalent, that is  $S_1$  and  $S_2$  are locally equivalent. This means that there exist  $f \in \Gamma$  such that  $S_2 = f^{(0)} \circ S_1 \circ f^{-1}$ . Substituting this expression of  $S_2$  in the canonical form of  $S_2$ , we get

$$\begin{aligned} (\mathcal{J}_2)^{(0)} \circ S_2 \circ (\mathcal{J}_2)^{-1} &= (\mathcal{J}_2)^{(0)} \circ f^{(0)} \circ S_1 \circ f^{-1} \circ (\mathcal{J}_2)^{-1} \\ &= (\mathcal{J}_2 \circ f)^{(0)} \circ S_1 \circ (\mathcal{J}_2 \circ f)^{-1} = (\mathcal{J}_1)^{(0)} \circ S_1 \circ (\mathcal{J}_1)^{-1}, \end{aligned}$$

where the last equality follows from (4.3) of theorem 6.5. This means that the canonical forms of  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are the same. The necessity is proved. The sufficiency is obvious.  $\square$

Let  $\mathcal{E}$  be the equation  $y^{(3)} = a_1(x)y' + a_0(x)y$ . It is obvious that  $\mathcal{E}$  is an ODE of canonical form iff the following equality holds

$$\mathcal{J}_{\mathcal{E}}(x) \equiv x. \quad (5.1)$$

Suppose the equation  $\mathcal{E}$  has the canonical form. We set  $w = (a_1' - 2a_0)^{1/3}$ . Then from (3.8) it follows, that the coefficients  $a_1$  and  $a_0$  of this equation can be expressed in the terms of  $w$

$$a_1 = xw^2 - 2\left(\frac{w'}{w}\right)' + \left(\frac{w'}{w}\right)^2, \quad a_0 = \frac{1}{2}\left[\left(xw^2 - 2\left(\frac{w'}{w}\right)' + \left(\frac{w'}{w}\right)^2\right)' - w^3\right].$$

The following statement is proved by a direct verification of identity (5.1)

PROPOSITION 6.9. *Any ODE*

$$\begin{aligned} y^{(3)} &= \left[xw^2 - 2\left(\frac{w'}{w}\right)' + \left(\frac{w'}{w}\right)^2\right]y' \\ &\quad + \frac{1}{2}\left[\left(xw^2 - 2\left(\frac{w'}{w}\right)' + \left(\frac{w'}{w}\right)^2\right)' - w^3\right]y, \end{aligned} \quad (5.2)$$

where  $w$  is an arbitrary nowhere vanishing function of  $x$ , is an ODE of canonical form.

From this proposition, we get

THEOREM 6.10. (1) *Let  $\mathcal{E}$  be an arbitrary ODE of the form (1.1) satisfying to the conditions  $\omega|_{\mathcal{E}} \neq 0$  and  $(\mathcal{J}|_{\mathcal{E}})' = 0$ . Then  $\mathcal{E}$  is locally equivalent to some ODE of the form (5.2).*

- (2) Suppose functions  $w_1$  and  $w_2$  are nowhere vanishing and are not equal in any neighborhood, then the corresponding equations of form (5.2) are not locally equivalent.

## 6. Exercises

- (1) Check that transformation (1.2) preserves the form of equations (1.1).
- (2) Using computer-algebraic system MAPLE, calculate vector fields (3.3)-(3.6).
- (3) Find common 1st integrals of vector fields (3.6).
- (4) Let  $\xi = a(x)\partial/\partial x$  be an arbitrary vector field on the base  $\mathbb{R}$  of  $\pi$  and  $S$  be an arbitrary section of  $\pi$ . Then  $\xi$  is a *symmetry* of  $S$  if  $\xi^{(0)}$  tangent to the graph  $L_S^0$ .
- (a) Find all symmetries of  $S$ .
- (b) Prove that dimension of the algebra  $\text{Sym}_S$  of all symmetries of  $S$  is equal to one of the numbers 0, 1, 3.
- (c) Prove that a section  $S$  of  $\pi$  is locally equal to the zero-section iff  $\dim \text{Sym}_S = 3$ .
- (d) Prove that  $\mathcal{J}|_{L_S^3}$  is a constant iff  $\dim \text{Sym}_S = 1$ .
- (e) Prove that  $(\mathcal{J}|_{L_S^3})' \neq 0$  iff  $\dim \text{Sym}_S = 0$ .
- (5) \* Prove theorem 6.1
- (6) The problem of classification of linear ODEs up to a contact transformations can be reduced to the pseudogroup of projective transformations. This reduction is represented here by the following problems:
- (a) Prove that a point transformation of the form

$$x = f(X), \quad y = Y \cdot f',$$

where  $f'$  is a solution of the ODE

$$2f'f''' - 3(f'')^2 - (f')^4 a^1(f) = 0, \quad (6.1)$$

transforms an arbitrary ODE (1.1) to an ODE of the *Laguerre-Forsyth form*

$$y^{(3)} = a^0(x)y. \quad (6.2)$$

- (b) Prove that *projective transformation*

$$f = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}, \quad \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0. \quad (6.3)$$

is the general solution of equation (6.3).

- (c) Prove that point transformations of the form

$$X = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad Y = y \cdot X', \quad (6.4)$$

transform ODEs of the Laguerre-Forsyth form to ODEs of the same form. Thus, the problem of classification of 3rd order linear ODEs up to a contact transformation is reduced to the classification of ODEs (6.2) up to transformation (6.4).

- (d) Calculate the algebra of all scalar differential invariants of ODEs (6.2) w.r.t. the pseudogroup of all transformations (6.4).

## CHAPTER 7

### Differential invariants in natural bundles

In this chapter, we explain the general approach to construct nonscalar differential invariants of natural bundles by the example of the natural bundle of nonlinear equations

$$y'' = u^0(x, y) + u^1(x, y)y' + u^2(x, y)(y')^2 + u^3(x, y)(y')^3.$$

We construct in details the first nontrivial differential invariant of this bundle. It is a differential 2-form on the bundle of 2-jets of considering bundle. Its values belong to some algebra. We prove that this invariant is a unique obstruction to linearizability of the considering ODEs by a point transformations.

As a preliminary, in this chapter, we introduce geometric structures following [15] and [14]. We clarify in details the lifting of diffeomorphisms and vector fields in a natural bundle of geometric structures. Following [4], we explain necessary facts concerning formal vector fields and Spencer  $\delta$ -cohomologies.

#### 1. Differential groups and geometric structures

Let  $M$  be an  $n$ -dimensional smooth manifold and  $D_k(n)$  be the differential group of order  $k$ . Now we shall give the classic definition of a *geometric structure*. One says that a geometric structure is defined on  $M$  if the following conditions hold:

- (1) a collection of functions  $q(x) = (q^1(x), \dots, q^N(x))$  is defined for every local coordinate system  $x = (x^1, \dots, x^n)$  in  $M$ . These functions are called the *components* of the geometric structure in the coordinate system  $x$ ;
- (2) an action  $F: D_k(n) \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  of the group  $D_k(n)$  is defined on  $\mathbb{R}^N$ ;
- (3) suppose  $q(x)$  and  $\tilde{q}(y)$  are the collections of components of the structure in a coordinate systems  $x$  and  $y$  respectively, suppose  $y = y(x)$  is the transformation of these coordinates; then the collections  $q(x)$  and  $\tilde{q}(y)$  are related in the following way:

$$\tilde{q}(y) = F\left(\frac{\partial y^i}{\partial x^j}, \dots, \frac{\partial^k y^i}{\partial x^{j_1} \dots \partial x^{j_k}}, q(x)\right). \quad (1.1)$$

The number  $k$  is called the *order* of this structure and  $F$  is called the *transformation law of the components of the structure*.

Examples.

1. Any smooth function  $f$  on  $M$  is a  $k$ -order geometric structure on  $M$ . The expression of  $f$  in terms of a coordinate system  $x = (x^1, \dots, x^n)$  in  $M$  is the component of the structure  $f$  in this coordinate system. The transformation law  $F: D_k(n) \times \mathbb{R}^1 \rightarrow \mathbb{R}^1$  of components of the structure  $f$  is trivial, that is  $F(d_k, q) = q$ .

2. Any vector field  $\xi$  on  $M$  is an 1-order geometric structure on  $M$ . For any coordinate system  $x = (x^1, \dots, x^n)$  in  $M$ , we have

$$\xi = \xi^1(x^1, \dots, x^n) \frac{\partial}{\partial x^1} + \dots + \xi^n(x^1, \dots, x^n) \frac{\partial}{\partial x^n}$$

It means, that the functions  $\xi^1(x), \dots, \xi^n(x)$  are the components of the structure  $\xi$  in the coordinate system  $x = (x^1, \dots, x^n)$ . The transformation law of components of the structure  $\xi$  is defined by the formula

$$\tilde{\xi}^i(\tilde{x}(x)) = \frac{\partial \tilde{x}^i}{\partial x^j}(x) \xi^j(x), \quad i, j = 1, \dots, n,$$

where the functions  $\tilde{\xi}^1(\tilde{x}), \dots, \tilde{\xi}^n(\tilde{x})$  are the components of the structure  $\xi$  in the coordinate system  $\tilde{x} = (\tilde{x}^1, \dots, \tilde{x}^n)$ .

3. By the same way it can be shown that any smooth differential 1-form on  $M$  or more generally any smooth tensor field of type  $(p, q)$  on  $M$  is a 1st order geometric structure.

4. A classical example of 2nd order geometric structure on  $M$  is a linear connection on  $M$ . It is defined in every coordinate system  $x = (x^1, \dots, x^n)$  by  $n^2(n+1)/2$  components  $\Gamma_{11}^1(x), \dots, \Gamma_{nn}^n(x)$ , satisfying the conditions  $\Gamma_{jk}^i(x) = \Gamma_{kj}^i(x)$ ,  $i, j, k = 1, \dots, n$ . The transformation law of components of this structure is defined by the formula

$$\tilde{\Gamma}_{jk}^i(\tilde{x}) = \frac{\partial^2 x^r}{\partial \tilde{x}^j \partial \tilde{x}^k} \frac{\partial \tilde{x}^i}{\partial x^r} + \Gamma_{lm}^r(x) \frac{\partial x^l}{\partial \tilde{x}^j} \frac{\partial x^m}{\partial \tilde{x}^k} \frac{\partial \tilde{x}^i}{\partial x^r}$$

5. Differential equations can be considered as a geometric structures. For example, from previous lectures we have that an arbitrary 3rd order linear ODE  $\mathcal{E}$  of the form  $y^{(3)} = a_1(x)y' + a_0(x)y$  can be considered as a 4th order geometric structure on  $\mathbb{R}^1$ . Its transformation law of components is defined by formula (2.1).

In this chapter, we will investigate the 2nd order geometric structure on  $\mathbb{R}^2$  generated by coefficients of the ODE

$$y'' = u^0(x, y) + u^1(x, y)y' + u^2(x, y)(y')^2 + u^3(x, y)(y')^3$$



The following equivalent definition of a geometric structure for the first time was given by V. V. Vagner in his paper [15]. Recall that by  $P_k(M)$  we denote the bundle of  $k$ -frames of  $M$ , see chapter 1, section 2.

A map

$$\Omega : P_k(M) \rightarrow \mathbb{R}^N,$$

is called a *geometric structure of type  $F$  on  $M$*  if

$$\Omega(\theta_k \cdot d_k) = F((d_k)^{-1}, \Omega(\theta_k)) \quad \forall \theta_k \in P_k(M), \forall d_k \in D_k(n).$$

Consider the manifold  $P_k(M) \times \mathbb{R}^N$ . Let us define the action of the group  $D_k(n)$  on this manifold by the rule

$$d_k : (\theta_k, q) \mapsto (\theta_k \cdot d_k, F(d_k, q))$$

On the manifold  $P_k(M) \times \mathbb{R}^N$ , we introduce the equivalence relation

$$(\theta_k, q) \sim (\theta'_k, q')$$

if there is  $d_k \in D_k$  such that  $(\theta'_k, q') = (\theta_k \cdot d_k, F(d_k, q))$ .

By  $F(M)$  we denote the quotient space of  $P_k(M) \times \mathbb{R}^N$  by this equivalence relation. By  $\pi$  we denote the natural projection of  $F(M)$  onto  $M$ . It is easy to verify that the quadruple  $(F(M), \pi, M, \mathbb{R}^N)$  is a locally-trivial bundle. This bundle is called the *bundle of geometric structures of type  $F$*

Now we can give a third equivalent definition of a geometric structure. Any section of the bundle  $\pi$  is a geometric structure of type  $F$  on  $M$ .

## 2. Natural bundles

### 2.1. The definition and examples. Let

$$\pi : E \rightarrow B$$

be a bundle. The bundle  $\pi$  is called a *natural bundle* if the following conditions hold:

- (1) Any diffeomorphism  $f : B \rightarrow B$  can be lifted to a diffeomorphism  $f^{(0)} : E \rightarrow E$  such that the following diagram is commutative

$$\begin{array}{ccc} E & \xrightarrow{f^{(0)}} & E \\ \pi \downarrow & & \downarrow \pi \\ B & \xrightarrow{f} & B \end{array}$$

- (2) The lifted identity diffeomorphism is the identity diffeomorphism,

$$(\text{id}_B)^{(0)} = \text{id}_E.$$

- (3) For any two diffeomorphisms  $f, g$  of  $B$ , it is held

$$(f \circ g)^{(0)} = f^{(0)} \circ g^{(0)}.$$

Obviously, these conditions define  $f^{(0)}$  uniquely.

Let  $S$  be an arbitrary section of  $\pi$ . Then any diffeomorphism  $f$  of  $B$  generates the transformation of this section to the section  $f(S)$  defined by

$$f(S) = f^{(0)} \circ S \circ f^{-1}. \quad (2.1)$$

Examples.

1. Let  $M$  be a smooth manifold and  $T(M)$  its tangent bundle. Then the natural projection

$$\pi : T(M) \rightarrow M, \quad \pi : \xi_p \mapsto p,$$

is a natural bundle. The lifted diffeomorphism  $f^{(0)}$  is denoted here as  $f_*$ .

2. By the same way, a cotangent bundle  $T^*(M)$  over  $M$  is a natural bundle. More generally, a bundle of tensors of type  $(p, q)$  over  $M$  is a natural bundle.

3. Consider the bundle of  $k$ -frames of  $M$ , see chapter 1, section 2,

$$\pi : P_k(m) \rightarrow M, \quad \pi : [s]_0^k \mapsto s(0).$$

An arbitrary diffeomorphism  $f$  of  $M$  is lifted to diffeomorphism

$$f^{(0)} : P_k(M) \rightarrow P_k(M)$$

by the formula  $f^{(0)}([s]_0^k) = [f]_{s(0)}^k \cdot [s]_0^k = [f \circ s]_0^k$ . It is easy to check, that  $P_k(m)$  is a natural bundle.

4. The following example gives the main way to construct natural bundles. Let  $F : D_k(n) \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  be an action of the differential group  $D_k(n)$  on  $\mathbb{R}^N$ . Then the bundle of geometric structures  $F(M)$  is a natural bundle. The action  $F$  defines the lifting of diffeomorphisms of  $M$  to diffeomorphisms of  $F(M)$ .

**2.2. The lifting of diffeomorphisms in jet bundles.** Any lifted diffeomorphism  $f^{(0)}$  can be lifted to the Lie transformation  $f^{(k)}$  of  $J^k\pi$ ,  $k = 1, 2, \dots$ , by the formula

$$f^{(k)}([S]_p^k) = [f^{(0)} \circ S \circ f^{-1}]_{f(p)}^k. \quad (2.2)$$

Obviously, for any  $l > m$ , the diagram

$$\begin{array}{ccc} J^l\pi & \xrightarrow{f^{(l)}} & J^l\pi \\ \pi_{l,m} \downarrow & & \downarrow \pi_{l,m} \\ J^m\pi & \xrightarrow{f^{(m)}} & J^m\pi \end{array}$$

is commutative (in the domains of  $f^{(l)}$ ).

By  $\Gamma$  we denote the Lie pseudogroup of all diffeomorphisms of the base of  $\pi$ .  $\Gamma$  acts on every  $J^k\pi$  by its lifted transformations.

**2.3. The lifting of vector fields.** Let  $\xi$  be a vector field in the base of  $\pi$  and let  $f_t$  be its flow. Then the flow  $f_t^{(k)}$  in  $J^k\pi$  defines the vector field  $\xi^{(k)}$  in  $J^k\pi$  which is called *the lifting of  $\xi$  to  $J^k\pi$* . Obviously

$$(\pi_{l,m})_*(\xi^{(l)}) = \xi^{(m)}, \quad \infty \geq l > m \geq -1, \quad (2.3)$$

where  $\xi^{(-1)} = \xi$ .

By  $x^1, \dots, x^n, u_\sigma^i$ ,  $i = 1, \dots, m$ ,  $0 \leq |\sigma| \leq k$ , we denote the standard coordinates in the jet bundle  $J^k\pi$ , here  $\sigma$  is the multi-index  $\{j_1 \dots j_r\}$ ,  $|\sigma| = r$ ,  $j_1, \dots, j_r = 1, \dots, n$ . By definition, put  $\sigma j = \{j_1 \dots j_r j\}$

Let

$$\xi = a^1(x^1, \dots, x^n) \frac{\partial}{\partial x^1} + \dots + a^n(x^1, \dots, x^n) \frac{\partial}{\partial x^n},$$

then the infinite lifted vector field  $\xi^{(\infty)}$  is defined by (2.5)

$$\xi^{(\infty)} = a^1 D_1 + \dots + a^n D_n + \mathfrak{D}_{\psi(\xi)},$$

where  $D_j$  is the operator of total derivation w.r.t.  $x^j$ ,  $\mathfrak{D}_{\psi(\xi)}$  is the operator of evolution differentiation corresponding to the generating function  $\psi(\xi) = (\psi^0(\xi), \dots, \psi^m(\xi))^t$ , see (2.4). The function  $\psi(\xi)$  is defined in the following way. Let  $S$  be a section of  $\pi$  defined in the domain of  $\xi$ , let  $\theta_1 = [S]_p^1$ ; then

$$\psi(\xi)(\theta_1) = \begin{pmatrix} \psi^0(\xi)(\theta_1) \\ \dots \\ \psi^m(\xi)(\theta_1) \end{pmatrix} = \left. \frac{d}{dt} (f_t^{(0)} \circ S \circ f_t^{-1}) \right|_{t=0} (p) \quad (2.4)$$

Obviously,  $\psi(\xi)(\theta_1)$  is the transformation velocity of the section  $S$  at the point  $p$  under the action of the flow  $f_t$ .

It is clear that

$$\begin{aligned} \xi^{(k)} = X^1 & \left( \frac{\partial}{\partial x^1} + \sum_{|\sigma|=0}^k \sum_{i=1}^m u_{\sigma 1}^i \frac{\partial}{\partial u_{\sigma}^i} \right) + \dots \\ & + X^n \left( \frac{\partial}{\partial x^n} + \sum_{|\sigma|=0}^k \sum_{i=1}^m u_{\sigma n}^i \frac{\partial}{\partial u_{\sigma}^i} \right) \\ & + \sum_{|\sigma|=0}^k \sum_{i=1}^m D_{\sigma} (\psi^i(\xi)) \frac{\partial}{\partial u_{\sigma}^i}. \end{aligned} \quad (2.5)$$

Let  $\text{Vect } B$  and  $\text{Vect } J^k\pi$  be the Lie algebras of all vector fields in the base  $B$  of  $\pi$  and in  $J^k\pi$  respectively.

PROPOSITION 7.1. *The map*

$$\text{Vect } B \rightarrow \text{Vect } J^k \pi, \quad \xi \mapsto \xi^{(k)},$$

*is a Lie algebra homomorphism.*

PROOF. The map  $\Gamma \rightarrow \Gamma^{(k)}$ ,  $f \mapsto f^{(k)}$ , is a homomorphism of Lie pseudogroups. It has as a consequence the statement of the proposition. Indeed, let  $X, Y$  be vector fields on  $B$  and let  $f_t, g_s$  be their flows respectively. Then

$$\begin{aligned} [X^{(k)}, Y^{(k)}] &= \lim_{t \rightarrow 0} \frac{1}{t} \left( Y^{(k)} - (f_t^{(k)})_*(Y^{(k)} \circ f_{-t}^{(k)}) \right) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left( \frac{d}{ds} \Big|_{s=0} g_s^{(k)} - (f_t^{(k)})_* \left( \frac{d}{ds} \Big|_{s=0} g_s^{(k)} \circ f_{-t}^{(k)} \right) \right) = \lim_{t \rightarrow 0} \frac{1}{t} \left( \frac{d}{ds} \Big|_{s=0} g_s^{(k)} \right. \\ &\quad \left. - \frac{d}{ds} \Big|_{s=0} f_t^{(k)} \circ g_s^{(k)} \circ f_{-t}^{(k)} \right) = \lim_{t \rightarrow 0} \frac{1}{t} \frac{d}{ds} \Big|_{s=0} \left( g_s^{(k)} \circ f_t^{(k)} \circ g_s^{(k)} \circ f_{-t}^{(k)} \right) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \frac{d}{ds} \Big|_{s=0} \left( g_s \circ f_t \circ g_s \circ f_{-t} \right)^{(k)} = \lim_{t \rightarrow 0} \frac{1}{t} \left( \frac{d}{ds} \Big|_{s=0} g_s \right. \\ &\quad \left. - \frac{d}{ds} \Big|_{s=0} f_t \circ g_s \circ f_{-t} \right)^{(k)} = \lim_{t \rightarrow 0} \frac{1}{t} \left( Y - (f_t)_*(Y \circ f_{-t}) \right)^{(k)} = [X, Y]^{(k)}. \end{aligned}$$

The  $\mathbb{R}$ -linearity of the map  $X \mapsto X^{(k)}$  is obvious.  $\square$

### 3. Formal vector fields. Spencer cohomologies

Let  $M$  be an arbitrary  $n$ -dimensional smooth manifold. By  $W_p$  we denote the set of  $\infty$ -jets at  $p \in M$  of all vector fields defined in a neighborhoods of  $p$ . There exist a natural structure of Lie algebra over  $\mathbb{R}$  on  $W_p$ . This structure is defined by the operations

$$\begin{aligned} \lambda[X]_p^\infty &\stackrel{df}{=} [\lambda X]_p^\infty, \quad [X]_p^\infty + [Y]_p^\infty \stackrel{df}{=} [X + Y]_p^\infty, \\ [[X]_p^\infty, [Y]_p^\infty] &\stackrel{df}{=} [[X, Y]]_p^\infty \\ \forall \lambda \in \mathbb{R}, \quad \forall [X]_p^\infty, [Y]_p^\infty &\in W_p. \end{aligned}$$

By  $L_p^k$ ,  $k = -1, 0, 1, 2, \dots$ , we denote the subalgebra in  $W_p$  defined by

$$L_p^k = \{ [X]_p^\infty \in W_n \mid [X]_p^k = 0 \}, \quad k \geq 0, \quad L_p^{-1} = W_p.$$

Let  $M = \mathbb{R}^n$ . In this case, we write  $W$  and  $L^k$  instead of  $W_0$  and  $L_0^k$  respectively. Recall, see chapter 1, section 1, that the Lie algebra of differential group  $D_k(n)$  is identified with the Lie algebra  $L^0/L^k$  and the Lie algebra of subgroup  $D_k^{k-1}(n) \subset D_k(n)$  is identified with the Lie algebra  $L^{k-1}/L^k$ .

By definition, put

$$V_p = W_p/L_p^0.$$

Obviously,  $V_p \cong T_p M$ . We have the filtration

$$W_p = L_p^{-1} \supset L_p^0 \supset L_p^1 \supset \dots \supset L_p^k \supset L_p^{k+1} \supset \dots$$

For any  $i > j \geq 0$ , we denote by  $\rho_{i,j}$  the natural projection

$$\rho_{i,j} : W_p/L_p^i \rightarrow W_p/L_p^j, \quad \rho_{i,j} : [X]_p^i \mapsto [X]_p^j$$

and by definition, put

$$\rho_i = \rho_{i,0}.$$

Taking into account that

$$[L_p^i, L_p^j] = L_p^{i+j}, \quad i, j = -1, 0, 1, 2, \dots,$$

we see that the bracket operation  $[\cdot, \cdot]$  on  $W_p$  generates the following maps

$$[\cdot, \cdot] : W_p/L_p^k \times W_p/L_p^k \rightarrow W_p/L_p^{k-1}, \quad (3.1)$$

$$[\cdot, \cdot] : V_p \times L_p^k/L_p^{k+1} \rightarrow L_p^{k-1}/L_p^k. \quad (3.2)$$

Let  $g_k$  be a subspace of  $L_p^{k-1}/L_p^k$ . The subspace  $g_k^{(1)} \subset L_p^k/L_p^{k+1}$  defined by

$$g_k^{(1)} = \{ X \in L_p^k/L_p^{k+1} \mid [v, X] \in g_k \quad \forall v \in V_p \}$$

is called *the 1-st prolongation of  $g_k$* .

Suppose the sequence of subspaces

$$g_1, g_2, \dots, g_i, \dots$$

satisfies to the property

$$[V, g_{i+1}] \subset g_i.$$

Then for every  $g_i$ , we have the complex

$$0 \rightarrow g_i \xrightarrow{\partial_{i,0}} g_{i-1} \otimes V_p^* \xrightarrow{\partial_{i-1,1}} g_{i-2} \otimes \wedge^2 V_p^* \xrightarrow{\partial_{i-2,2}} 0, \quad (3.3)$$

where the operators  $\partial_{k,l} : g_k \otimes \wedge^l V_p^* \rightarrow g_{k-1} \otimes \wedge^{l+1} V_p^*$  are defined in the following way: any element  $\xi \in g_k \otimes \wedge^l V_p^*$  can be considered as an exterior form on  $V_p$  with values in  $g_k$ , then

$$(\partial_{k,l}(\xi))(v_1, \dots, v_{l+1}) = \sum_{i=1}^{l+1} (-1)^{i+1} [v_i, \xi(v_1, \dots, \hat{v}_i, \dots, v_{l+1})].$$

We denote by  $H_p^{k,l}$  the cohomology group of this complex in the term  $g_k \otimes \wedge^l V_p^*$ . It is called a *Spencer  $\delta$ -cohomology group*.

#### 4. The construction of differential invariants of natural bundles

In this section, we construct in details the first nontrivial differential invariant of the natural bundle of nonlinear equations

$$y'' = u^0(x, y) + u^1(x, y)y' + u^2(x, y)(y')^2 + u^3(x, y)(y')^3. \quad (4.1)$$

This invariant is differential 2-form with values in some algebra.

The set of all ODEs (4.1) is invariant w.r.t. the pseudogroup of all point transformations, that is an arbitrary point transformation transforms an arbitrary ODE (4.1) to an ODE of the same form. It can be easily checked by direct calculations. This set contains the set of all 2nd order linear ODEs. This implies the following problem. For ODE (4.1) find necessary and sufficient conditions to exist a point transformation transforming this ODE to linear form.

The above mentioned invariant gives the solution of this problem. We prove that this invariant is a unique obstruction to linearizability of ODE (4.1) by a point transformation, that is ODE (4.1) is linearizable by a point transformation iff the invariant is zero for this ODE.

Note that the equivalence problem of 2nd order ODEs w.r.t. contact transformations is trivial because any two 2nd order ODEs are equivalent w.r.t. contact transformations, see, for example, [3].

**4.1. Natural bundles of ODEs. The lifting of diffeomorphisms.** Consider the trivial bundle

$$\pi : E = \mathbb{R}^2 \times \mathbb{R}^4 \rightarrow \mathbb{R}^2, \quad \pi : (x^1, x^2, u^0, u^1, u^2, u^3) \mapsto (x^1, x^2),$$

where  $x^1, x^2$  are the standard coordinate on the base of  $\pi$  and  $u^0, u^1, u^2, u^3$  are the standard coordinates on the fiber of  $\pi$ .

Let  $\mathcal{E}$  be an arbitrary equation (4.1). We identify  $\mathcal{E}$  with the section  $S_{\mathcal{E}}$  of  $\pi$  defined by the formula

$$S_{\mathcal{E}} : (x^1, x^2) \mapsto (x^1, x^2, u^0(x^1, x^2), u^1(x^1, x^2), u^2(x^1, x^2), u^3(x^1, x^2)).$$

Clearly, this identification is a bijection between the set of all equations (4.1) and the set of all sections of  $\pi$ .

An arbitrary point transformation

$$f : (x^1, x^2) \mapsto (\tilde{x}^1 = f^1(x^1, x^2), \tilde{x}^2 = f^2(x^1, x^2)). \quad (4.2)$$

transforms an arbitrary equation  $\mathcal{E}$  of form (4.1) to the equation  $\tilde{\mathcal{E}}$  of the same form. The coefficients of the obtained equation are expressed in terms of the coefficients of the initial one and the derivatives of order  $\leq 2$  of the inverse transformation to  $f$ :

$$\tilde{u}^{\alpha} = \Phi^{\alpha} \left( u^{\beta}, \frac{\partial g^i}{\partial \tilde{x}^j}, \frac{\partial^2 g^i}{\partial \tilde{x}^{j_1} \partial \tilde{x}^{j_2}} \right), \quad (4.3)$$

$$\alpha, \beta = 0, 1, 2, 3, \quad g = (g^1, g^2) = f^{-1}, \quad i, j, j_1, j_2 = 1, 2.$$

Equations (4.2) and (4.3) define the lift of  $f$  to the diffeomorphism  $f^{(0)}$  of the total space of  $\pi$ .

For any point transformation  $f$ , the transformation of sections of  $\pi$  is defined by formula (2.1)

$$S \mapsto f(S) = f^{(0)} \circ S \circ f^{-1}.$$

Equations (4.3) can be represented now as

$$S_{\tilde{\mathcal{E}}} = f(S_{\mathcal{E}}).$$

Now the following statement is obvious.

**PROPOSITION 7.2.** *Let  $\mathcal{E}, \tilde{\mathcal{E}}$  be equations of form (4.1). Then a point transformation  $f$  takes  $\mathcal{E}$  to  $\tilde{\mathcal{E}}$  iff  $S_{\tilde{\mathcal{E}}} = f(S_{\mathcal{E}})$ .*

By  $x^1, x^2, u_{\sigma}^i$ ,  $i = 0, \dots, 3$ ,  $0 \leq |\sigma| \leq k$ , we denote the standard coordinates in the jet bundle  $J^k\pi$ , here  $\sigma$  is the multi-index  $\{j_1 \dots j_r\}$ ,  $|\sigma| = r$ ,  $j_1, \dots, j_r = 1, 2$ . By definition, put  $\sigma j = \{j_1 \dots j_r j\}$

Any point transformation  $f$  can be lifted to the diffeomorphism  $f^{(k)}$  of  $J^k\pi$  by formula (2.2)

$$f^{(k)}([S]_p^k) = [f^{(0)} \circ S \circ f^{-1}]_{f(p)}^k.$$

By  $\Gamma$  we denote the Lie pseudogroup of all point transformation of the base of  $\pi$ .  $\Gamma$  acts on every  $J^k\pi$  by its lifted transformations.

#### 4.2. The lifting of vector fields. Let

$$X = X^1(x^1, x^2) \frac{\partial}{\partial x^1} + X^2(x^1, x^2) \frac{\partial}{\partial x^2},$$

then we have the following formula

$$X^{(\infty)} = X^1 D_1 + X^2 D_2 + \mathfrak{D}_{\psi(X)}, \quad (4.4)$$

where where  $D_j$  is the operator of total derivation w.r.t.  $x^j$ ,  $\mathfrak{D}_{\psi(\xi)}$  is the operator of evolution differentiation, see (2.4), corresponding to the generating function  $\psi(\xi) = (\psi^0(\xi), \dots, \psi^m(\xi))^t$ . The function  $\psi(\xi)$  is defined by formula (2.4).

Let  $\theta_1 = (x^1, x^2, u^i, u_j^i)$ ,  $i = 0, 1, 2, 3$ ,  $j = 1, 2$ ; then it can be calculated that

$$\psi(X)(\theta_1) = \begin{pmatrix} -u_1^0 X^1 - u_2^0 X^2 \\ -2u^0 X_1^1 + u^0 X_2^2 - u^1 X_1^2 + X_{11}^2 \\ -u_1^1 X^1 - u_2^1 X^2 \\ -3u^0 X_2^1 - u^1 X_1^1 - 2u^2 X_1^2 - X_{11}^1 + 2X_{12}^2 \\ -u_1^2 X^1 - u_2^2 X^2 \\ -2u^1 X_2^1 - u^2 X_2^2 - 3u^3 X_1^2 - 2X_{12}^1 + X_{22}^2 \\ -u_1^3 X^1 - u_2^3 X^2 \\ -u^2 X_2^1 + u^3 X_1^1 - 2u^3 X_2^2 - X_{22}^1 \end{pmatrix}, \quad (4.5)$$

where  $X_j^i = \frac{\partial X^i}{\partial x^j}(p)$  and  $X_{j_1 j_2}^i = \frac{\partial^2 X^i}{\partial x^{j_1} \partial x^{j_2}}(p)$ .

**4.3. Orbits of the action of  $\Gamma$ .** Obviously,  $\Gamma$  acts transitively on the base  $\mathbb{R}^2$  of  $\pi$ . Indeed, for any point  $x_0 \in \mathbb{R}^2$  there exist  $f \in \Gamma$  such that  $f(0) = x_0$ , for example  $f(x) = x + x_0$ . It follows that every orbit of the action of  $\Gamma$  on  $J^k\pi$  intersects the fiber  $J_0^k\pi = (\pi_k)^{-1}(0)$ . As a result  $J_0^k\pi$  is divided into nonintersect subsets. Clearly, these subsets are orbits w.r.t. the action of the group

$$G^k = \{ f^{(k)} \mid f \in \Gamma, f(0) = 0 \}.$$

Suppose, some orbit  $\text{Orb}_{\theta_k} \cap J_0^k\pi$  of the action of  $G^k$  on  $J_0^k\pi$  is described as a submanifold of  $J_0^k\pi$  by the equations  $I^i(u, \dots, u_k) = 0$ . Then, the these equations  $I^i(u, \dots, u_k) = 0$  is described the orbit  $\text{Orb}_{\theta_k}$  of the action of  $\Gamma$  on  $J^k\pi$ . Indeed, for any  $x_0 \in \mathbb{R}^2$  the transformations  $f(x) = x_0 - x$  transfer  $\text{Orb}_{\theta_k} \cap J_0^k\pi$  onto  $\text{Orb}_{\theta_k} \cap J_{x_0}^k\pi$  so that the orbit of the fiber  $J_{x_0}^k\pi$  is described by the same equations  $I^i(u, \dots, u_k) = 0$ .

Thus the problem to describe orbits of the action of  $\Gamma$  on  $J^k\pi$  is reduced to the problem describe orbits of the action of  $G^k$  on  $J_0^k\pi$ .

The last problem, can be investigated in the following way: Consider the Lie algebra of the group  $G^k$ . It consists of all lifted  $\Gamma$ -vector fields  $X^{(k)}$  such that  $X(0) = 0$ . We call these fields as  $G^k$ -vector fields. At every point  $\theta_k \in J_0^k\pi$ ,  $G^k$ -vector fields generate the subspace  $D_{\theta_k}^k$  spanned by all their values at  $\theta_k$ . Thus the distribution  $\mathcal{D}^k : \theta_k \mapsto \mathcal{D}_{\theta_k}^k$  is generated in  $J_0^k\pi$ . Let  $X^{(k)}|_{x=0}$  be a  $G^k$ -vector field. Decomposing its components  $X^1$  and  $X^2$  in the Taylor series with remainder term at the point  $0 \in \mathbb{R}$ , we obtain

$$X^i = \lambda_j^i x^j + \dots + \lambda_{j_1 \dots j_k}^i x^{j_1} \dots x^{j_{k+2}} + R(x) x^{j_1} \dots x^{j_{k+3}}, \quad i = 1, 2$$

Setting

$$X_i^{j_1 \dots j_k} = x^{j_1} \dots x^{j_k} \frac{\partial}{\partial x^i}, \quad i = 1, 2, \quad k = 1, 2, \dots,$$

we get

$$X^{(k)}|_{x=0} = \lambda_j^i (X_i^j)^{(k)}|_{x=0} + \dots + \lambda_{j_1 \dots j_k}^i (X_i^{j_1 \dots j_{k+2}})^{(k)}|_{x=0}$$

From (4.5) and (2.5), it follows that the distribution  $\mathcal{D}^k$  is generated by the all  $G^k$ -vector fields

$$(X_i^j)^{(k)}|_{x=0}, \dots, (X_i^{j_1 \dots j_{k+2}})^{(k)}|_{x=0}, \quad i, j, j_1, \dots, j_{k+2} = 1, 2.$$

Reducing this system of vector fields to the step form, we can obtain description of orbits of the action of  $\Gamma$  on  $J^k\pi$ . The following theorem is obtained in this way.

- THEOREM 7.3.** (1)  $J^k\pi$  is an orbit of the action of  $\Gamma$  iff  $k = 0, 1$ ,  
 (2)  $J^2\pi$  is divided into two orbits  $J^2\pi = \text{Orb}_1 \cup \text{Orb}_2$  such that  $\dim \text{Orb}_1 = \dim J^k\pi$  and  $\dim \text{Orb}_2 = \dim J^k\pi - 2$ ,



**4.4. Isotropy algebras.** Let  $\theta_k \in J^k\pi$  and  $p = \pi(\theta_k)$ . By  $G_{\theta_k}$  we denote the *isotropy group* of  $\theta_k$ , that is

$$G_{\theta_k} = \{ [f]_p^{2+k} \mid f \in \Gamma, f^{(k)}(\theta_k) = \theta_k \}$$

By  $\mathfrak{g}_{\theta_k}$  we denote the Lie algebra of  $G_{\theta_k}$ . It can be considered as a Lie subalgebra in  $L_p^0/L_p^{2+k}$ :

$$\mathfrak{g}_{\theta_k} = \{ [X]_p^{2+k} \in L_p^0/L_p^{2+k} \mid X \in \text{Vect } \mathbb{R}^2, X_{\theta_k}^{(k)} = 0 \}$$

The subalgebra  $\mathfrak{g}_{\theta_k} \subset L_p^0/L_p^{2+k}$  is called the *isotropy algebra* of  $\theta_k$ .

From this definition and (4.12), (4.13), and (4.14), we get

**PROPOSITION 7.4.**  $[X]_p^{2+k} \in \mathfrak{g}_{\theta_k}$  iff it is a solution of the system of linear algebraic equations

$$(D_\sigma(\psi_X^i))(\theta_k) = 0, \quad 0 \leq |\sigma| \leq k. \quad (4.6)$$

(We write  $D_\sigma(\psi_X^i)(\theta_k)$  in (4.6) instead  $D_\sigma(\psi_X^i)(\theta_{k+1})$  because from  $X_p = 0$  we have that system (4.6) depends on  $\theta_k$  and it is independent of  $\theta_{k+1}$ .)

Let  $\theta_0 \in J^0\pi$  and  $p = \pi(\theta_0)$ . From (4.6), we get that the isotropy algebra  $\mathfrak{g}_{\theta_0}$  of the point  $\theta_0$  is defined by the equations

$$\begin{cases} -2u^0 X_1^1 + u^0 X_2^2 - u^1 X_1^2 + X_{11}^2 = 0 \\ -3u^0 X_2^1 - u^1 X_1^1 - 2u^2 X_1^2 - X_{11}^1 + 2X_{12}^2 = 0 \\ -2u^1 X_2^1 - u^2 X_2^2 - 3u^3 X_1^2 - 2X_{12}^1 + X_{22}^2 = 0 \\ -u^2 X_2^1 + u^3 X_1^1 - 2u^3 X_2^2 - X_{22}^1 = 0 \end{cases} \quad (4.7)$$

It follows from (4.7) that

$$\rho_{2,1}(\mathfrak{g}_{\theta_0}) = L_p^0/L_p^1.$$

Let

$$g_{\theta_0} = \mathfrak{g}_{\theta_0} \cap (L_p^1/L_p^2).$$

Obviously, it is a commutative subalgebra in  $\mathfrak{g}_{\theta_0}$ . From (4.7), we get that  $g_{\theta_0}$  is defined by the equations

$$\begin{cases} X_{11}^2 = 0 \\ X_{11}^1 - 2X_{12}^2 = 0 \\ 2X_{12}^1 - X_{22}^2 = 0 \\ X_{22}^1 = 0 \end{cases} \quad (4.8)$$

It is clear that  $g_{\theta_0}$  and  $g_{\tilde{\theta}_0}$  are isomorphic for any  $\theta_0, \tilde{\theta}_0 \in J^0\pi$ . Below, we shall write  $g$  instead  $g_{\theta_0}$ .

It follows from (4.8) that

$$\dim g = 2 \quad (4.9)$$

and we can choose

$$\begin{aligned} e_1 &= 2 \frac{\partial}{\partial x^1} \otimes (dx^1 \odot dx^1) + \frac{\partial}{\partial x^2} \otimes (dx^1 \odot dx^2), \\ e_2 &= 2 \frac{\partial}{\partial x^2} \otimes (dx^2 \odot dx^2) + \frac{\partial}{\partial x^1} \otimes (dx^1 \odot dx^2) \end{aligned} \quad (4.10)$$

as independent generators of  $g$ .

It is easy to check that the 1-prolongation  $g^{(1)}$  of  $g$  is trivial, that is

$$g^{(1)} = \{0\}. \quad (4.11)$$

**4.5. Isotropy spaces.** Let  $\theta_{k+1} \in J^{k+1}\pi$ , let  $\theta_k = \pi_{k+1,k}(\theta_{k+1})$ , and let  $[S]_p^{k+1} = \theta_{k+1}$ . Then the tangent space to the image of the section  $j_k S$  at the point  $\theta_k$  is defined by  $\mathcal{H}_{\theta_{k+1}}$ . We denote this tangent space by  $\mathcal{H}_{\theta_{k+1}}$ . We have the following direct sum decomposition of the tangent space to  $J^k\pi$  at the point  $\theta_k$

$$T_{\theta_k} J^k\pi = \mathcal{H}_{\theta_{k+1}} \oplus T_{\theta_k}(\pi^{-1}(p)).$$

Let  $X$  be a vector field in the base of  $\pi$  defined in a neighborhood of  $p$ . Then the value  $X_{\theta_k}^{(k)}$  of  $X^{(k)}$  at the point  $\theta_k$  has a unique decomposition

$$X_{\theta_k}^{(k)} = \mathcal{H}_{\theta_{k+1}} X^{(k)} + V_{\theta_{k+1}} X^{(k)}, \quad (4.12)$$

where  $\mathcal{H}_{\theta_{k+1}} X^{(k)} \in \mathcal{H}_{\theta_{k+1}}$  and  $V_{\theta_{k+1}} X^{(k)} \in T_{\theta_k}(\pi^{-1}(p))$ . It follows from (4.4) and (2.3) that if  $X = X^1 \partial / \partial x^1 + X^2 \partial / \partial x^2$ , then

$$\mathcal{H}_{\theta_{k+1}} X^{(k)} = X^1 D_1^{\theta_{k+1}} + X^2 D_2^{\theta_{k+1}}, \quad V_{\theta_{k+1}} X^{(k)} = \mathfrak{D}_{\psi(X)}^{\theta_{k+1}}, \quad (4.13)$$

where

$$\begin{aligned} D_j^{\theta_{k+1}} &= \frac{\partial}{\partial x^j} + \sum_{0 \leq |\sigma| \leq k} \sum_{i=0}^3 u_{\sigma j}^i(\theta_{k+1}) \frac{\partial}{\partial u_{\sigma}^i}, \\ \mathfrak{D}_{\psi(X)}^{\theta_{k+1}} &= \sum_{0 \leq |\sigma| \leq k} \sum_{i=0}^3 \left( D_{\sigma}(\psi^i(X)) \right) (\theta_{k+1}) \frac{\partial}{\partial u_{\sigma}^i}. \end{aligned} \quad (4.14)$$

It follows from (4.5) that the value  $X_{\theta_k}^{(k)}$  of the vector field  $X^{(k)}$  at the point  $\theta_k$  is depended on the jet  $[X]_p^{k+2}$ .

By definition, put

$$\begin{aligned} \mathcal{A}_{\theta_{k+1}} &= \{ [X]_p^{2+k} \in W_p / L_p^{2+k} \mid X_{\theta_k}^{(k)} \in \mathcal{H}_{\theta_{k+1}} \}, \\ &k = 0, 1, \dots, \infty. \end{aligned} \quad (4.15)$$

From (4.12), (4.13), and (4.14), we get

PROPOSITION 7.5.  $[X]_p^{2+k} \in \mathcal{A}_{\theta_{k+1}}$  iff  $[X]_p^{2+k}$  is a solution of the system of linear equations

$$(D_\sigma(\psi_X^i))(\theta_{k+1}) = 0, \quad 0 \leq |\sigma| \leq k. \quad (4.16)$$

We say that  $\mathcal{A}_{\theta_{k+1}}$  is the *isotropy spase* of  $\theta_{k+1}$ .

THEOREM 7.6. (1)  $\rho_{k+2,k+1}(\mathcal{A}_{\theta_{k+1}}) \subset \mathcal{A}_{\theta_k}$ .

(2)  $[\cdot, \cdot] : \mathcal{A}_{\theta_{k+1}} \times \mathcal{A}_{\theta_{k+1}} \rightarrow \mathcal{A}_{\theta_k}$ .

PROOF. The first statement is obvious.

Prove the second one. Let  $[X]_p^{2+k}, [Y]_p^{2+k} \in \mathcal{A}_{\theta_{k+1}}$ , let  $\theta_\infty \in \pi_\infty^{-1}(p)$ ,  $\theta_k = \pi_{\infty,k}(\theta_\infty)$ ,  $\theta_{k-1} = \pi_{k,k-1}(\theta_k)$ , and let

$$X = X^1 \frac{\partial}{\partial x^1} + X^2 \frac{\partial}{\partial x^2}, \quad Y = Y^1 \frac{\partial}{\partial x^1} + Y^2 \frac{\partial}{\partial x^2}.$$

Then

$$[[X]_p^{2+k}, [Y]_p^{2+k}] = [[X, Y]]_p^{2+k-1}$$

and

$$[[X, Y]]_p^{2+k-1} \in \mathcal{A}_{\theta_k} \quad \text{iff} \quad [X, Y]_{\theta_{k-1}}^{(k-1)} \in \mathcal{H}_{\theta_k}.$$

We have

$$\begin{aligned} [X, Y]_{\theta_{k-1}}^{(k-1)} &= (\pi_{\infty,k-1})_* [X, Y]_{\theta_\infty}^{(\infty)} = (\pi_{\infty,k-1})_* [X^{(\infty)}, Y^{(\infty)}]_{\theta_\infty} \\ &= (\pi_{\infty,k-1})_* [X^1 D_1 + X^2 D_2 + \mathfrak{D}_{\psi(X)}, Y^1 D_1 + Y^2 D_2 + \mathfrak{D}_{\psi(Y)}]_{\theta_\infty}. \end{aligned}$$

Taking into account the well known relations, see Proposition 3.1 of Chapter 3,

$$[D_1, D_2] = [D_1, \mathfrak{D}_\psi] = [D_2, \mathfrak{D}_\psi] = 0 \quad \text{and} \quad [\mathfrak{D}_\phi, \mathfrak{D}_\psi] = \mathfrak{D}_{\{\phi, \psi\}},$$

where  $\{\phi, \psi\} = \mathfrak{D}_\phi(\psi) - \mathfrak{D}_\psi(\phi)$ , we get

$$\begin{aligned} [X, Y]_{\theta_{k-1}}^{(k-1)} &= (\pi_{\infty,k-1})_* \left( (X^1 Y_1^1 + X^2 Y_2^1 - Y^1 X_1^1 - Y^2 X_2^1) D_1 + \right. \\ &\quad \left. + (X^1 Y_1^2 + X^2 Y_2^2 - Y^1 X_1^2 - Y^2 X_2^2) D_2 + [\mathfrak{D}_{\psi(X)}, \mathfrak{D}_{\psi(Y)}] \right)_{\theta_\infty} = \\ &= \mathcal{H}_{\theta_k} [X, Y]^{(k-1)} + \mathfrak{D}_{\{\psi(X), \psi(Y)\}}^{\theta_k}. \end{aligned}$$

From (4.5), we obtain

$$\begin{aligned} \{\psi(X), \psi(Y)\}^i &= \psi^{i'}(X) \frac{\partial \psi^i(Y)}{\partial u^{i'}} + D_j(\psi^{i'}(X)) \frac{\partial \psi^i(Y)}{\partial u_j^{i'}} \\ &\quad - \psi^{i'}(Y) \frac{\partial \psi^i(X)}{\partial u^{i'}} - D_j(\psi^{i'}(Y)) \frac{\partial \psi^i(X)}{\partial u_j^{i'}}. \end{aligned}$$

From (4.14), we get now that  $\mathfrak{D}_{\{\psi(X), \psi(Y)\}}^{\theta_k} = 0$ . □

**4.6. Horizontal subspaces.** We shall say that a 2-dimensional subspace  $H \subset W_p/L_p^k$  is *horizontal* if

$$\rho_k(H) = V_p.$$

Let  $\theta_k \in J^k\pi$  and  $\theta_{k+1} \in \pi_{k+1,k}^{-1}(\theta_k)$ ; then it is clear that

$$\mathfrak{g}_{\theta_k} \subset \mathcal{A}_{\theta_{k+1}} \quad \forall \theta_{k+1} \in \pi_{k+1,k}^{-1}(\theta_k). \quad (4.17)$$

It is obvious that a 2-dimensional subspace  $H \subset \mathcal{A}_{\theta_{k+1}}$  is horizontal iff

$$\mathcal{A}_{\theta_{k+1}} = H \oplus \mathfrak{g}_{\theta_k}.$$

Any two horizontal subspaces  $H, \tilde{H} \subset \mathcal{A}_{\theta_{k+1}}$  define the linear function

$$f_{H,\tilde{H}} : V_p \rightarrow \mathfrak{g}_{\theta_k}, \quad f_{H,\tilde{H}} : X \mapsto (\rho_{k+2}|_H)^{-1}(X) - (\rho_{k+2}|\tilde{H})^{-1}(X).$$

It is clear that for any horizontal subspace  $H \subset \mathcal{A}_{\theta_{k+1}}$  and for any linear function  $f : V \rightarrow \mathfrak{g}_{\theta_k}$ , there exist a unique horizontal subspace  $\tilde{H} \subset \mathcal{A}_{\theta_{k+1}}$  with  $f = f_{H,\tilde{H}}$ .

Further in this subsection, we shall investigate horizontal subspaces of  $\mathcal{A}_{\theta_1}$ .

By  $H_p$  we denote the horizontal subspace in  $W_p/L_p^1$  generated by constant vector fields.

By  $H_{\theta_1}$  we denote a horizontal subspace in  $\mathcal{A}_{\theta_1}$  with

$$\rho_{2,1}(H_{\theta_1}) = H_p. \quad (4.18)$$

From

$$\rho_{2,1}(\mathcal{A}_{\theta_1}) = W_p/L_p^1,$$

we have that horizontal subspaces  $H_{\theta_1}$  exist. Obviously,  $H_{\theta_1}$  is defined by

$$H_{\theta_1} = \{ [X]_p^2 = (X^i, 0, X_\sigma^i), i = 1, 2, |\sigma| = 2 \} \quad (4.19)$$

in the standard coordinates.

It is clear now that for any two horizontal subspaces  $H_{\theta_1}, \tilde{H}_{\theta_1}$  satisfying to (4.18), we get

$$f_{H_{\theta_1}, \tilde{H}_{\theta_1}} : V_p \rightarrow \mathfrak{g}.$$

Taking into account that  $\mathfrak{g} \neq \{0\}$ , we obtain that there exist a lot of horizontal subspaces satisfying to (4.18). We choose one of them in the following way.

A horizontal subspace  $H_{\theta_1}$  defines the form  $\omega_{H_{\theta_1}} \in L_p^0/L_p^1 \otimes \wedge^2 V_p^*$  by the formula

$$\omega_{H_{\theta_1}}(X, Y) = [(\rho|_{H_{\theta_1}})^{-1}(X), (\rho|_{H_{\theta_1}})^{-1}(Y)] \quad \forall X, Y \in V_p.$$

From the Spenser complex

$$0 \rightarrow g^{(1)} \xrightarrow{\partial_{3,0}} g \otimes V_p^* \xrightarrow{\partial_{2,1}} L_p^0/L_p^1 \otimes \wedge^2 V_p^* \xrightarrow{\partial_{1,2}} 0, \quad (4.20)$$

we get that  $\omega_{H_{\theta_1}}$  defines the Spenser cohomology class  $\{\omega_{H_{\theta_1}}\} \in H_p^{1,2}$ .

PROPOSITION 7.7. *The cohomology class  $\{\omega_{H_{\theta_1}}\}$  is trivial.*

PROOF. From (4.11) we get that  $\partial_{2,1}$  is an injection in (4.20). From (4.9), we obtain  $\dim g \otimes V_p^* = 4$ . Obviously,  $\dim L_p^0/L_p^1 \otimes \wedge^2 V_p^* = 4$ . As a result, we obtain  $\text{Im } \partial_{2,1} = \ker \partial_{1,2}$  in (4.20).  $\square$

COROLLARY 7.8. *There exists a unique horizontal subspace  $H_{\theta_1} \subset \mathcal{A}_{\theta_1}$  with  $\omega_{H_{\theta_1}} = 0$ .*

PROOF. Prove the uniqueness. Suppose  $H_{\theta_1}, \tilde{H}_{\theta_1}$  are horizontal subspaces of  $\mathcal{A}_{\theta_1}$  with  $\omega_{H_{\theta_1}} = \omega_{\tilde{H}_{\theta_1}} = 0$ . We have  $\omega_{H_{\theta_1}} = \omega_{\tilde{H}_{\theta_1}} + \partial_{2,1}(f_{H_{\theta_1}, \tilde{H}_{\theta_1}})$ . Therefore,  $\partial_{2,1}(f_{H_{\theta_1}, \tilde{H}_{\theta_1}}) = 0$ . Taking into account that  $\partial_{2,1}$  is an injection, we get that  $f_{H_{\theta_1}, \tilde{H}_{\theta_1}} = 0$ . This means that  $H_{\theta_1} = \tilde{H}_{\theta_1}$ .

Prove the existence. We have  $\{\omega_{H_{\theta_1}}\} = \{0\}$ . Therefore there exist  $h \in g \otimes V_p^*$  with  $\omega_{H_{\theta_1}} = \partial_{2,1}(h)$ . It follows that the horizontal subspace

$$\tilde{H}_{\theta_1} = \{ (\rho_2|_{H_{\theta_1}})^{-1}(X) - h(X), \quad X \in V_p \}$$

satisfies to the property  $\omega_{\tilde{H}_{\theta_1}} = 0$ .  $\square$

Now, we express the horizontal space  $H_{\theta_1}$  with  $\omega_{H_{\theta_1}} = 0$  in terms of standard coordinates  $x^1, x^2, u^i(\theta_1), u_j^i(\theta_1)$ . Let

$$(\rho_2|_{H_{\theta_1}})^{-1}(X) = (X^i, 0, f_{jk,r}^i X^r), \quad \forall X \in V_p.$$

Then the property  $\omega_{H_{\theta_1}} = 0$  means that

$$f_{jk,r}^i = f_{jr,k}^i. \quad (4.21)$$

From proposition 7.5 we obtain that elements  $(X^i, 0, f_{jk,r}^i X^r) \in H_{\theta_1}$  is a solutions of the system

$$\begin{cases} -u_1^0 X^1 - u_2^0 X^2 + f_{11,r}^2 X^r = 0 \\ -u_1^1 X^1 - u_2^1 X^2 - f_{11,r}^1 X^r + 2f_{12,r}^2 X^r = 0 \\ -u_1^2 X^1 - u_2^2 X^2 - 2f_{12,r}^1 X^r + f_{22,r}^2 X^r = 0 \\ -u_1^3 X^1 - u_2^3 X^2 - f_{22,r}^1 X^r = 0 \end{cases}$$

From this system and (4.21), we obtain

$$\left\{ \begin{array}{l} f_{11,1}^2 = u_1^0, \quad f_{11,2}^2 = f_{12,1}^2 = u_2^0, \\ f_{12,2}^2 = f_{22,1}^2 = \frac{1}{3}(2u_2^1 - u_1^0), \\ f_{22,2}^2 = -2u_1^3 + u_2^2, \\ f_{22,2}^1 = -u_2^3, \quad f_{22,1}^1 = f_{12,2}^1 = -u_1^3, \\ f_{12,1}^1 = f_{11,2}^1 = \frac{1}{3}(u_2^1 - 2u_1^2), \\ f_{11,1}^1 = 2u_2^0 - u_1^1. \end{array} \right. \quad (4.22)$$

**4.7. The obstruction to linearizability of ODEs.** Let  $\theta_2 \in J^2\pi$  and  $\theta_1 = \pi_{2,1}(\theta_2)$ . It is not difficult to prove that

$$\rho_{3,2}(\mathcal{A}_{\theta_2}) = \mathcal{A}_{\theta_1}. \quad (4.23)$$

Let  $H_{\theta_1}$  be the horizontal subspace of  $\mathcal{A}_{\theta_1}$  with  $\omega_{H_{\theta_1}} = 0$ . From (4.23) and  $\mathfrak{g}_{\theta_1} \cap (L_p^2/L_p^3) = \{0\}$ , we get that there exist a unique horizontal subspace  $H_{\theta_2} \subset \mathcal{A}_{\theta_2}$  with

$$\rho_{3,2}(H_{\theta_2}) = H_{\theta_1}. \quad (4.24)$$

It follows from item (2) of theorem 7.6 that  $H_{\theta_2}$  defines the 2-form  $\omega_{\theta_2} \in \mathcal{A}_{\theta_1} \otimes \wedge^2 V_p^*$  by the formula

$$\omega_{\theta_2}(X, Y) = [(\rho_3|_{H_{\theta_2}})^{-1}(X), (\rho_3|_{H_{\theta_2}})^{-1}(Y)] \quad \forall X, Y \in V_p.$$

From  $\omega_{H_{\theta_1}} = 0$  we obtain

$$\omega_{\theta_2} \in \mathfrak{g} \otimes (V_p^* \wedge V_p^*)$$

Now we can define the horizontal differential 2-form  $\omega^{(2)}$  on  $J^2\pi$  with values in  $\mathfrak{g}$  by the following formula

$$\omega^{(2)} : \theta_2 \longmapsto \pi_2^*(\omega_{\theta_2}). \quad (4.25)$$

Obviously,  $H_{\theta_2}$  is defined by

$$H_{\theta_2} = \{ [X]_p^2 = (X^i, 0, f_{j_1 j_2, r}^i X^r, f_{j_1 j_2 j_3, r}^i X^r) \}$$

in the standard coordinates. Hence,

$$\omega_{\theta_2} = 2f_{j_1 j_2 [k, r]}^i \left( \frac{\partial}{\partial x^i} \otimes (dx^{j_1} \odot dx^{j_2}) \right) \otimes (dx^k \wedge dx^r).$$

Taking into account (4.9) and (4.10), we get

$$\omega^{(2)} = (F^1 \cdot e_1 + F^2 \cdot e_2) \otimes (dx^1 \wedge dx^2), \quad (4.26)$$

where  $F^1 = f_{11[1,2]}^1$  and  $F^2 = f_{22[1,2]}^2$ .

Calculate the functions  $F^1$ ,  $F^2$ . From proposition 7.5 we obtain that elements

$$(X^i, 0, f_{j_1 j_2, r}^i X^r, f_{j_1 j_2 j_3, r}^i X^r) \in H_{\theta_2}$$

is a solutions of system (4.16) for  $k = 1$ . From this system and (4.22), we get

$$\begin{aligned} F^1 = & 3u_{22}^0 - 2u_{12}^1 + u_{11}^2 \\ & + 3u^3 u_1^0 - 3u^2 u_2^0 + 2u^1 u_2^1 - u^1 u_1^2 - 3u^0 u_2^2 + 6u^0 u_1^3, \end{aligned} \quad (4.27)$$

$$\begin{aligned} F^2 = & u_{22}^1 - 2u_{12}^2 + 3u_{11}^3 \\ & - 3u^0 u_2^3 + 3u^1 u_1^3 - 2u^2 u_1^2 + u^2 u_2^1 + 3u^3 u_1^1 - 6u^3 u_2^0. \end{aligned} \quad (4.28)$$

Note that first the coefficients  $F^1$  and  $F^1$  were obtained by Cartan in [2] as unique nonzero coefficients of the curvature form of the projective connection corresponding to equation (4.1).

Thus we obtain the following expression of  $\omega^{(2)}$  in the standard coordinates

$$\begin{aligned} \omega^{(2)} = & \left( F^1 \left( 2 \frac{\partial}{\partial x^1} \otimes (dx^1 \odot dx^1) + \frac{\partial}{\partial x^2} \otimes (dx^1 \odot dx^2) \right) \right. \\ & \left. + F^2 \left( 2 \frac{\partial}{\partial x^2} \otimes (dx^2 \odot dx^2) + \frac{\partial}{\partial x^1} \otimes (dx^1 \odot dx^2) \right) \right) \\ & \otimes (dx^1 \wedge dx^2), \end{aligned} \quad (4.29)$$

where  $F^1$  and  $F^2$  are defined by (4.27) and (4.28) respectively.

We recall, that a differential form defined on  $J^k \pi$  is a *differential invariant of the action of  $\Gamma$  on  $\pi$*  if it is invariant w.r.t. the pseudogroup  $\Gamma^{(k)}$ .

**THEOREM 7.9.** *The form  $\omega^{(2)}$  is a differential invariant of the action of  $\Gamma$  on  $\pi$ .*

**PROOF.** Let  $f \in \Gamma$ , let  $p$  be a point from the domain of  $f$ , and let  $\theta_2 \in J_p^2 \pi$ . We should check that

$$(f^{(2)})^* \left( \omega^{(2)}|_{f^{(2)}(\theta_2)} \right) = \omega^{(2)}|_{\theta_2}. \quad (4.30)$$

We shall check it in the standard coordinates. It is clear that the left side of (4.30) is depend of  $[f]_p^4$ . This jet can be represented in the following way

$$[f]_p^4 = [f_1]_p^4 \cdot [f_2]_p^4,$$

where  $[f_1]_p^4$  is jet of the affine transformation and  $[f_2]_p^1 = [\text{id}]_p^1$ . It can easily be checked that  $\omega^{(2)}$  is invariant w.r.t. affine transformations. Therefore it remains to check that equation (4.30) holds for an arbitrary point transformation  $f$  with  $[f]_p^1 = [\text{id}]_p^1$ . Taking into account that  $\omega^{(2)}$  is horizontal, we

get that equation (4.30) holds for a point transformation  $f$  with  $[f]_p^1 = [\text{id}]_p^1$  iff

$$F^1(f^{(2)}(\theta_2)) = F^1(\theta_2), \quad F^2(f^{(2)}(\theta_2)) = F^2(\theta_2)$$

It is clear that the last equations hold iff the restrictions of  $F^1$ ,  $F^2$  to  $J_p^2\pi$  are 1-st integrals for any vector field  $\xi^{(2)}|_{J_p^2\pi}$  with  $[\xi]_p^\infty \in L_p^1$ . The last statement about  $F^1$  and  $F^2$  can be easily checked by direct calculations in standard coordinates.  $\square$

In his paper [2], Cartan proved that equation (4.1) can be reduced to the linear form by a point transformation iff the collection of its coefficients is a solution of the system of PDEs

$$F^1 = 0, \quad F^2 = 0. \quad (4.31)$$

This means that  $\omega^{(2)}$  is a unique obstruction to the linearizability of equations (4.1) by point transformations.

Below, we give the independent proof of this fact.

Let

$$M = \left\{ \theta_2 \in J^2\pi \mid \omega^{(2)}|_{\theta_2} = 0 \right\}.$$

From (4.26), it follows that  $M$  is defined by system of algebraic equations (4.31).

By  $\mathbf{0}$  we denote the zero section of  $\pi$ , by  $0_k$  we denote  $[\mathbf{0}]_0^k$ ,  $k = 0, 1, 2, \dots$

LEMMA 7.10.  $M = \text{Orb}_{0_2}$ .

PROOF. It is clear that  $\dim \text{Orb}_{0_2} = \dim W_0/L_0^4 - \dim \mathfrak{g}_{0_2}$ . We have  $\dim W_0/L_0^4 = 30$ . It is easy to calculate that  $\dim \mathfrak{g}_{0_2} = 6$ . Hence  $\dim \text{Orb}_{0_2} = 24$ . From (4.31), we have that  $\dim M = \dim J^2\pi - 2 = 24$  too.

Obviously,  $\omega^{(2)}|_{0_2} = 0$ . Now from theorem 7.9, we get that  $\text{Orb}_{0_2} \subset M$ . At last, the sets  $M$  and  $\text{Orb}_{0_2}$  are connected subsets in  $J^2\pi$ . This concludes the proof.  $\square$

LEMMA 7.11. *Let  $\theta_2 \in \text{Orb}_{0_2}$  and let  $\theta_1 = \pi_{2,1}(\theta_2)$ ; then the natural projection of the isotropy groups of these points*

$$G_{\theta_2} \rightarrow G_{\theta_1}, \quad [f]_p^4 \mapsto [f]_p^3,$$

*is a bijection.*

PROOF. It is easy to prove that the natural projection  $G_{0_2} \rightarrow G_{0_1}$  is an injection and that  $\dim G_{0_1} = \dim G_{0_2} = 6$ . Therefore the natural projection  $G_{0_2} \rightarrow G_{0_1}$  is a bijection. The projection  $\pi_{2,1} : \text{Orb}_{0_2} \rightarrow J^1\pi$  is a surjection. This implies the proof.  $\square$

For any section  $S$  of  $\pi$ , by  $\omega_S^{(2)}$  we denote the form  $(j_2S)^*(\omega^{(2)})$ .



**THEOREM 7.12.** *The section  $S$  can be transformed (locally) to  $\mathbf{0}$  by a point transformation iff  $\omega_S^{(2)} \equiv 0$ .*

**PROOF.** The necessity is obvious.

Prove the sufficiency. To this end, we should prove that the system of PDEs w.r.t. an unknown point transformation  $f$

$$\mathbf{0} = f^{(0)} \circ S \circ f^{-1}$$

has a solution. By  $\mathcal{E}(\mathbf{0}, S)$  we denote this system. It is easy to prove that the symbol of this PDE system at any point is the same as the subalgebra  $g$  defined above by (4.8). From (4.11), we obtain that the first prolongation  $\mathcal{E}^{(1)}(\mathbf{0}, S)$  of  $\mathcal{E}(\mathbf{0}, S)$  has the zero symbol at every point. Therefore  $\mathcal{E}^{(1)}(\mathbf{0}, S)$  has a solution if the natural projection  $\mathcal{E}^{(2)}(\mathbf{0}, S) \rightarrow \mathcal{E}^{(1)}(\mathbf{0}, S)$ ,  $[f]_p^4 \mapsto [f]_p^3$ , is a surjection (see [10]).

Let us check that this projection is a surjection. Let  $[f]_p^3 \in \mathcal{E}^{(1)}(\mathbf{0}, S)$ . It takes  $[S]_p^1$  to  $[\mathbf{0}]_{f(p)}^1$ . By assumption,  $\omega^{(2)}([S]_p^2) = 0$ . It follows from lemma 7.10 that  $[S]_p^2 \in \text{Orb}_{0_2}$ . Obviously,  $[\mathbf{0}]_{f(p)}^2 \in \text{Orb}_{0_2}$  too. Hence there exist a point transformation  $f'$  such that its jet  $[f']_p^4$  takes  $[S]_p^2$  to  $[\mathbf{0}]_{f(p)}^2$ . This means that  $[f']_p^4 \in \mathcal{E}^{(2)}(\mathbf{0}, S)$ ,  $[f']_p^3 \in \mathcal{E}^{(1)}(\mathbf{0}, S)$ , and  $[f']_p^3$  takes  $[S]_p^1$  to  $[\mathbf{0}]_{f(p)}^1$ . From the last, we obtain that there exist  $g \in G_{[S]_p^1}$  with  $[f']_p^3 \cdot g = [f]_p^3$ . From lemma 7.11, we get that there exist  $g' \in G_{[S]_p^2}$  with  $\rho_{4,3}(g') = g$ . Obviously,  $[f']_p^4 \cdot g' \in \mathcal{E}^{(2)}(\mathbf{0}, S)$  and it is clear that  $[f]_p^3$  is the image of  $[f']_p^4 \cdot g'$  under the natural projection  $\mathcal{E}^{(2)}(\mathbf{0}, S) \rightarrow \mathcal{E}^{(1)}(\mathbf{0}, S)$ . Thus, this natural projection is a surjection.  $\square$

**COROLLARY 7.13.** *The form  $\omega^{(2)}$  is a unique obstruction to the linearizability of ODEs (4.1) by point transformations.*

**PROOF.** It is well known that any two 2–order linear ODEs are (locally) equivalent w.r.t. point transformations. This implies the proof.  $\square$

## 5. Exercises

- (1) Prove that a bundle of geometric structures is a locally trivial bundle.
- (2) Prove equivalence of all definitions of a geometric structure.
- (3) Prove that  $\mathfrak{o}(n)^{(1)} = \{0\}$ , where  $\mathfrak{o}(n) \subset L^0/L^1$  denotes the orthogonal algebra.
- (4) Denoting by  $\mathfrak{sl}(n) \subset L^0/L^1$  the special linear algebra, prove that  $\mathfrak{sl}(n)^{(k)} \neq \{0\}$  for all  $k = 1, 2, \dots$
- (5) Let  $g \subset L^0/L^1$  be a Lie subalgebra. It is called an *algebra of infinite type* if  $g^{(k)} \neq \{0\}$  for all  $k$ . Prove that if a Lie algebra  $g \subset L^0/L^1$  contains a matrix of range 1 then  $g$  is an algebra of infinite type.

- (6) Applying the computer-algebraic system MAPLE, write the program to calculate prolonged spaces  $g^{(r)}$ ,  $r = 1, 2, \dots$ , for an arbitrary space  $g \subset L^k/L^{k+1}$ .
- (7) Prove that the sequence (3.3) is a complex.
- (8) Applying the computer-algebraic system MAPLE, solve the following problems:
- Prove that an arbitrary point transformation (4.2) transform an arbitrary ODE (4.1) to an equation of the same form.
  - Calculate the transformation velocity of section of the natural bundle of ODEs (4.1) w.r.t. a vector field on the base, (that is calculate formula (4.5)).
  - Prove theorem 7.3.
  - Find the equations describing the degenerate orbit  $\text{Orb}_2$  in theorem 7.3 as a submanifold of  $J^2\pi$ .
- (9) \* Consider the bundle of all symmetric linear connections on  $M$ . Locally, this bundle can be represented as the trivial bundle

$$\pi : \mathbb{R}^n \times \mathbb{R}^{n^2(n+1)/2} \longrightarrow \mathbb{R}^n.$$

- Following to the construction of the invariant 2-form  $\omega^{(2)}$ , see section 4.7, to construct invariant 2-form on the 1-jet bundle  $J^1\pi$  of the trivial bundle of all symmetric linear connections on  $M$ .
- Prove that the 2-form obtained in item (9a) is the curvature form considering as a 2-form on  $J^1\pi$ .

## CHAPTER 8

### $G$ -structures

This chapter is devoted to the theory of  $G$ -structures, which is an alternative approach to investigate differential invariants and the equivalence problem.

In this chapter, we state necessary facts concerning bundles of  $k$ -frames. We introduce  $G$ -structures of higher orders and investigate connection between the equivalence problem of geometric structures and the equivalence problem of the corresponding  $G$ -structures. Further, we construct in details the structure function of a  $G$ -structure of higher order. This function is a map from the  $G$ -structure to some Spencer  $\delta$ -cohomology group and is a differential invariant. Further, we apply this invariant to solve the integrability problem of  $G$ -structures. Finally, we illustrate the obtained solution of the integrability problem by examples from the theory of ordinary differential equations.

In this part, by  $W$  we denote the Lie algebra of  $\infty$ -jets at  $0 \in \mathbb{R}^n$  of all vector fields defined in a neighborhoods of 0. By  $L^k$ ,  $k = -1, 0, 1, 2, \dots$ , we denote the subalgebra in  $W$  defined by

$$L^k = \{ [X]_0^\infty \in W \mid [X]_0^k = 0 \}, \quad k \geq 0, \quad L^{-1} = W.$$

Put

$$V = W/L^0.$$

#### 1. Frame bundles

Let  $M$  be a smooth  $n$ -dimensional manifold. Consider the bundle

$$\pi_k : P_k(M) \rightarrow M$$

of  $k$ -frames of  $M$ , see chapter 1, section 2. By  $\pi_{l,m}$ ,  $l \geq m$ , we denote the natural projection

$$\pi_{l,m} : P_l(M) \rightarrow P_m(M), \quad \pi_{l,m}([s]_0^l) = [s]_0^m.$$

Let  $\theta_k \in P_k(M)$ , let  $p = \pi_k(\theta_k)$ , and let  $T_{\theta_k}P_k(M)$  be the tangent space to  $P_k(M)$  at the point  $\theta_k$ .

**PROPOSITION 8.1.** *Let  $\theta_{k+1} \in \pi_{k+1,k}^{-1}(\theta_k)$ . Then:*

- (1)  $\theta_{k+1}$  defines the isomorphism of vector spaces

$$T_{\theta_k} P_k(M) \longrightarrow W/L^k.$$

We will denote this isomorphism by  $\theta_{k+1}$  too.

- (2) The reduction of the inverse isomorphism  $(\theta_{k+1})^{-1}$  to  $L^0/L^k$  is the canonical isomorphism of the Lie algebra of the structure group  $D_k(n)$  to the space  $T_{\theta_k}(\pi_k^{-1}(p))$  tangent to the fiber of  $\pi_k$  over the point  $p$ .

PROOF. Let  $[s]_0^{k+1} = \theta_{k+1}$  and  $s(0) = p$ . By  $T_p^k(M)$  we denote the space of  $k$ -jets at  $p$  of all vector fields in  $M$  passing through  $p$ . Obviously, the map

$$\alpha : T_p^k(M) \rightarrow T_{\theta_k} P_k(M), \quad \alpha : [X]_p^k \mapsto \left. \frac{d}{dt}([\varphi_t \circ s]_0^k) \right|_{t=0},$$

where  $\varphi_t$  is the flow of  $X$ , is an isomorphism of vector spaces. Also, the map

$$\beta : T_p^k(M) \rightarrow T_0^k \mathbb{R}^n, \quad \beta : [X]_p^k \mapsto \left. \frac{d}{dt}([s^{-1} \circ \varphi_t \circ s]_0^k) \right|_{t=0}$$

is an isomorphism of vector spaces. The isomorphism  $\theta_{k+1}$  is defined now by the formula

$$\theta_{k+1} = \beta \circ \alpha^{-1}.$$

The canonical isomorphism  $L^0/L^k \rightarrow T_{\theta_k}(\pi_k^{-1}(p))$  is defined by the formula

$$d/dt([d_t]_0^k) \Big|_{t=0} \mapsto d/dt([s \circ d_t]_0^k) \Big|_{t=0}.$$

This formula can be rewritten in the following way:

$$d/dt(s^{-1} \circ (s \circ d_t \circ s^{-1}) \circ s) \Big|_{t=0} \mapsto d/dt([(s \circ d_t \circ s^{-1}) \circ s]_0^k) \Big|_{t=0}.$$

This completes the proof.  $\square$

The diffeomorphism  $s^{-1}$  is a local chart in  $M$ . It generates the local chart  $(x^i, x_j^i, \dots, x_{j_1 \dots j_k}^i)$  in  $P_k(M)$  as stated above. Obviously, within this chart, the isomorphism  $\theta_{k+1}$  is defined by

$$\theta_{k+1} : X^i \frac{\partial}{\partial x^i} + \dots + X_{j_1 \dots j_k}^i \frac{\partial}{\partial x_{j_1 \dots j_k}^i} \mapsto (X^i, \dots, X_{j_1 \dots j_k}^i). \quad (1.1)$$

Suppose  $\theta_{k+1}, \tilde{\theta}_{k+1} \in (\pi_{k+1,k})^{-1}(\theta_k)$ . Then there exists a unique element  $[d]_0^{k+1} = (\delta_j^i, 0, \dots, 0, d_{j_1 \dots j_{k+1}}^i) \in D_{k+1}^k$  such that  $\tilde{\theta}_{k+1} = \theta_{k+1} \cdot [d]_0^{k+1}$ . It is easy to prove the following statement.

PROPOSITION 8.2. *Let  $\xi \in T_{\theta_k} P_k(M)$  and*

$$\theta_{k+1}(\xi) = (X^i, \dots, X_{j_1 \dots j_{k-1}}^i, X_{j_1 \dots j_k}^i).$$

Then

$$\tilde{\theta}_{k+1}(\xi) = (X^i, \dots, X_{j_1 \dots j_{k-1}}^i, X_{j_1 \dots j_k}^i + d_{j_1 \dots j_k}^i X^r).$$

Let  $f$  be an arbitrary diffeomorphism of  $M$  to itself. Then the diffeomorphism  $f^{(k)} : P_k(M) \rightarrow P_k(M)$  is defined by

$$f^{(k)}([s]_0^k) = [f \circ s]_0^k.$$

The diffeomorphism  $f^{(k)}$  is called the *lift of  $f$  to the bundle  $P_k(M)$* .

## 2. Geometric structures and their prolongations

Let  $M$  be an arbitrary smooth  $n$ -dimensional manifold. Recall the convenient here definition of a geometric structure on  $M$ . Let

$$F : D_k(n) \times \mathbb{R}^N \rightarrow \mathbb{R}^N \quad (2.1)$$

be an action of  $D_k(n)$  on  $\mathbb{R}^N$ . Then a map

$$\Omega : P_k(M) \rightarrow \mathbb{R}^N,$$

is called a geometric structure of order  $k$  on  $M$  if

$$\Omega(\theta_k \cdot d_k) = F((d_k)^{-1}, \Omega(\theta_k)) \quad \forall \theta_k \in P_k(M), \forall d_k \in D_k(n).$$

The number  $k$  is called the *order* of this structure and  $F$  is called the transformation law of the components of the structure.

Any local coordinate system  $(U, h = (x^1, \dots, x^n))$  in  $M$  generates the section of  $P_k(M)$  by the formula

$$U \rightarrow \pi_k^{-1}(U), \quad p \mapsto [(h - h(p))^{-1}]_0^k. \quad (2.2)$$

The reduction of  $\Omega$  to the image of this section is the collection of the components  $q^1(x), \dots, q^N(x)$  of  $\Omega$  in the coordinates  $x^1, \dots, x^n$ .

A geometric structure  $\Omega$  is called *homogeneous* if  $\text{Im } \Omega$  is an orbit of the action  $F$  of the group  $D_k(n)$ .

Suppose  $\Omega_1$  and  $\Omega_2$  are geometric structures with the same transformation law of their components. We say that these structures are *equivalent* if there exists a diffeomorphism  $f$  of  $M$  such that

$$\Omega_1 = \Omega_2 \circ f^{(k)}.$$

Suppose  $\Omega$  is a geometric structure and the transformation law of its components is defined by (2.1). Then its *first prolongation*

$$\Omega^{(1)} : P_{k+1}(M) \rightarrow \mathbb{R}^{N(1+n)}$$

is defined in the following way. Suppose  $q^1(x), \dots, q^N(x)$  are the components of  $\Omega$  in the coordinates  $x^1, \dots, x^n$ . Then

$$q^\alpha(x), \quad \frac{\partial}{\partial x^j}(q^\alpha(x)), \quad \alpha = 1, \dots, N, \quad j = 1, \dots, n,$$

are the components of  $\Omega^{(1)}$  in the coordinates  $x^1, \dots, x^n$ . Obviously, the transformation law of components of  $\Omega^{(1)}$  is defined by

$$\begin{aligned} \tilde{q}^\alpha &= F^\alpha(d_{j_1}^i, \dots, d_{j_1 \dots j_k}^i, q^1, \dots, q^N), \\ \partial_i \tilde{q}^\alpha \cdot d_j^i &= \frac{\partial F^\alpha}{\partial d_{j_1}^i} d_{j_1 j}^i + \dots + \frac{\partial F^\alpha}{\partial d_{j_1 \dots j_k}^i} d_{j_1 \dots j_k j}^i + \frac{\partial F^\alpha}{\partial q^\beta} \partial_j q^\beta. \end{aligned} \quad (2.3)$$

The  $i$ -th prolongation of  $\Omega$  is defined by induction on  $i$ :

$$\Omega^{(i+1)} = (\Omega^{(i)})^{(1)}, \quad i = 1, 2, \dots$$

### 3. G-structures and geometric structures. The equivalence problem

Let  $G \subset D_k(n)$  be a closed Lie subgroup and let  $B \subset P_k(M)$  be a reduction of  $P_k(M)$  to  $G$ . Then  $B$  is called a  $G$ -structure of order  $k$  over  $M$ .

Let  $\Omega : P_k(M) \rightarrow \mathbb{R}^N$  be an arbitrary homogeneous geometric structure,  $q_0 \in \text{Im } \Omega$ , and  $G \subset D_k(n)$  be the isotropy group of  $q_0$ . Then the inverse image  $B = \Omega^{-1}(q_0) \subset P_k(M)$  is a  $G$ -structure of order  $k$  over  $M$ .

Examples

1. If  $G = D_k(n)$ , then  $G$ -structure  $B$  is  $P_k(M)$ .
2. Let  $\Omega : P_1(M) \rightarrow \mathbb{R}^{n(n+1)/2}$  be an arbitrary Riemannian metric on  $M$ . Then  $\Omega^{-1}(E)$ , where  $E$  is the unite  $n \times n$ -matrix, is the a principal  $O(n)$ -bundle of orthonormal frames on  $M$ .
3. Let  $\Omega : P_1(M) \rightarrow \mathbb{R}^{n^2(n+1)/2}$  be an arbitrary linear symmetric connection on  $M$ . Then the group  $G$  of the  $G$ -structure  $B = \Omega^{-1}(0)$  is defined by

$$G = \{ [d]_0^2 = (d_j^i, 0) \in D_2(n) \}.$$

Let  $[s]_0^2 \in B$ , then  $(\pi_{2,1})^{-1}([s]_0^1) \cap B = \{[s]_0^2\}$ . It follows that  $G$ -structure  $B$  is a section of the bundle  $\pi_{2,1} : P_2(M) \rightarrow P_1(M)$ .

Suppose  $B_1$  and  $B_2$  are  $G$ -structures over  $M$ . They are *equivalent* if there exists a diffeomorphism  $f$  of  $M$  such that

$$f^{(k)}(B_1) = B_2.$$

It is easy to prove the following statement.

**THEOREM 8.3.** *Suppose  $\Omega_1$  and  $\Omega_2$  are homogeneous geometric structures with the same transformation law of the components, suppose that  $\text{Im } \Omega_1 = \text{Im } \Omega_2$ , and suppose  $q \in \text{Im } \Omega_1$ . Then  $\Omega_1$  and  $\Omega_2$  are equivalent iff the  $G$ -structures  $\Omega_1^{-1}(q)$  and  $\Omega_2^{-1}(q)$  are equivalent.*

Let  $B$  be a  $G$ -structure of order  $k$  over  $M$  and let  $\mathfrak{g} \subset L^0/L^k$  be the Lie algebra of  $G$ . By definition, put

$$g_k = \mathfrak{g} \cap (L^{k-1}/L^k).$$

By  $g_k^{(i)}$ ,  $i = 0, 1, \dots$  denote the  $i$ -th prolongation of  $g_k$ , where  $g_k^{(0)} = g_k$ .

By definition,  $B$  is a *finite type*  $G$ -structure if there exists a nonnegative integer  $r$  such that  $g_k^{(r)} = \{0\}$ , otherwise  $B$  is an *infinite type*  $G$ -structure. Obviously,  $g_k^{(i)} = \{0\}$  if  $i > r$ . By  $r(B)$  we denote the least nonnegative integer  $r$  such that  $g_k^{(r)} = \{0\}$ .

Let  $\Omega$  be an arbitrary geometric structure and let  $B = \Omega^{-1}(q)$  be one of its  $G$ -structures. We say that  $\Omega$  has *finite or infinite type* if  $B$  has respectively finite or infinite type. Clearly, the type of a geometric structure is well defined.

Examples.

1.  $O(n)$ -structure is a finite type structure such that  $r(B) = 1$ .
2.  $SL(n)$ -structure, almost complex structure are 1st order structures of infinite type.
3. The geometric structure on  $\mathbb{R}^1$  generated by coefficients of a linear ODE of order  $k$  is a  $k$ th order finite type structure such that  $r(B) = 0$ .
4. The geometric structure on  $\mathbb{R}^2$  generated by coefficients of an equation

$$y'' = u^0(x, y) + u^1(x, y)y' + u^2(x, y)(y')^2 + u^3(x, y)(y')^3$$

is a 2nd order finite type structure such that  $r(B) = 1$ .

5. Geometric structure on  $\mathbb{R}^3$  generated by coefficients of an arbitrary equation of the form

$$y''' = u^0(x, y, y') + u^1(x, y, y')y'' + u^2(x, y, y')(y'')^2 + u^3(x, y, y')(y'')^3$$

is a 3rd order infinite type structure.

#### 4. The integrability problem

Let  $F : D_k(n) \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  be an arbitrary action of  $D_k(n)$  on  $\mathbb{R}^N$ , let  $q \in \mathbb{R}^N$ , and let  $G \subset D_k(n)$  be the isotropy group of  $q$ .

The standard coordinate system on  $\mathbb{R}^n$  generates the section  $P_k(\mathbb{R}^n)$  by formula (2.2). Subjecting image of this section to the action of  $G$ , we obtain the  $G$ -structure  $B$  over  $\mathbb{R}^n$ . It is called *flat*. Obviously, the  $G$ -structure  $B$ ,  $q$ , and the transformation law  $F$  define the geometric structure  $\Omega : P_k(\mathbb{R}^n) \rightarrow \mathbb{R}^N$  uniquely. This geometric structure is called a *flat structure* too.

A geometric structure ( $G$ -structure) on  $M$  is called a *locally-flat* or *integrable* if it is locally equivalent to a flat structure ( $G$ -structure).

Obviously, a  $G$ -structure  $B$  on  $M$  is integrable iff there exists a local chart of  $M$  such that the section of  $P_k(M)$  generated by this chart is a section of  $B$ . In other words, a geometric structure on  $M$  is integrable iff there exists a local chart in  $M$  such that the components of this structure are constants in this chart.

In the sequel, we use the following

**THEOREM 8.4.** *Let  $\Omega$  be an arbitrary geometric structure and let  $q$  be some value of  $\Omega$ . Then  $\Omega$  is integrable iff the  $G$ -structure  $B = \Omega^{-1}(q)$  is integrable.*

### 5. Structure functions of $G$ -structures and their prolongations

Consider a homogeneous geometric structure  $\Omega: P_k(M) \rightarrow \mathbb{R}^N$ . Transformation law (2.1) of its components can be interpreted as the system of partial differential equations w.r.t. unknown functions  $y^i(x^1, \dots, x^n)$ ,  $i = 1, 2, \dots, n$ . We treat this PDE system as the submanifold  $\mathcal{E}$  in the bundle of  $k$ -jets  $J^k\tau$  of sections of the trivial bundle

$$\tau: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

We suppose that  $\mathcal{E}$  satisfies the condition

$$\tau_{k,k-1}(\mathcal{E}) = J^{k-1}\tau, \quad (5.1)$$

where  $\tau_{l,m}: J^l\tau \rightarrow J^m\tau$ ,  $l \geq m$ , is the natural projection that takes a  $l$ -jet to its  $m$ -jet.

Let  $q_0 \in \mathbb{R}^N$  be some value of  $\Omega$ . Consider the  $G$ -structure  $B = \Omega^{-1}(q_0)$ . Then condition (5.1) means that

$$\pi_{k,k-1}(B) = P_{k-1}(M). \quad (5.2)$$

For the group  $G$ , condition (5.1) means that

$$\rho_{k,k-1}(G) = D_{k-1}. \quad (5.3)$$

For the Lie algebra  $\mathfrak{g}$  of the group  $G$ , the last condition means that

$$\rho_{k,k-1}(\mathfrak{g}) = L^0/L^{k-1}. \quad (5.4)$$

**5.1. Structure functions of  $G$ -structures.** Let  $\theta_k \in B$ . Then  $\theta_k$  defines the linear isomorphism  $\theta_k: T_{\theta_{k-1}}P_{k-1}(M) \rightarrow W/L^{k-1}$  as it was shown above. By  $H_{k-1}$  we denote the subspace in  $W/L^{k-1}$  which is generated by the vectors of the form  $(X^i, 0, \dots, 0)$ . Obviously, the quotient space  $W/L^{k-1}$  is decomposable to the direct sum

$$W/L^{k-1} = H_{k-1} \oplus L^0/L^{k-1}.$$

Consider the subspace  $H_{\theta_{k-1}} \subset T_{\theta_{k-1}}P_{k-1}(M)$  which is defined by

$$H_{\theta_{k-1}} = (\theta_k)^{-1}(H_{k-1}). \quad (5.5)$$

We say that  $H \subset T_{\theta_i}P_i(M)$ ,  $i = 1, 2, \dots, \infty$  is *horizontal* if it is  $n$ -dimensional and is naturally projected onto the space tangent to  $M$  without degeneration.

Clearly, subspace (5.5) is horizontal.



Let  $\theta_{k+1} \in P_{k+1}(M)$  and let  $\pi_{k+1,k}(\theta_{k+1}) = \theta_k \in B$ . Then the isomorphism  $\theta_{k+1} : T_{\theta_k} P_k(M) \rightarrow W/L^k$  defines the injective linear map

$$\theta_{k+1}|_{T_{\theta_k} B} : T_{\theta_k} B \rightarrow W/L^k$$

such that the following diagram is commutative:

$$\begin{array}{ccc} T_{\theta_k} B & \xrightarrow{\theta_{k+1}|_{T_{\theta_k} B}} & W/L^k \\ (\pi_{k,k-1})_* \downarrow & & \downarrow \rho_{k,k-1} \\ T_{\theta_{k-1}} P_{k-1}(M) & \xrightarrow{\theta_k} & W/L^{k-1}. \end{array}$$

Let us choose a horizontal subspace  $H_{\theta_k} \subset T_{\theta_k} B$  such that

$$(\pi_{k,k-1})_*(H_{\theta_k}) = H_{\theta_{k-1}}. \quad (5.6)$$

Then

$$\forall X \in H_{\theta_{k+1}}, \quad \theta_k(X) = (X^i, 0, \dots, 0, X_{j_1 \dots j_k}^i).$$

The pair  $(H_{\theta_k}, \theta_{k+1})$  defines the linear map

$$f_{(H_{\theta_k}, \theta_{k+1})} : V \rightarrow L^{k-1}/L^k$$

by the formula

$$f_{(H_{\theta_k}, \theta_{k+1})} : X^i \mapsto (X_{j_1 \dots j_k}^i) = (f_{j_1 \dots j_k, r}^i X^r).$$

Suppose  $H_{\theta_k}, \tilde{H}_{\theta_k} \subset T_{\theta_k} B$  are horizontal subspaces satisfying Eq. (5.6). Then, obviously,

$$(f_{(H_{\theta_k}, \theta_{k+1})} - f_{(\tilde{H}_{\theta_k}, \theta_{k+1})}) : V \rightarrow g_k, \quad (5.7)$$

where  $g_k = \mathfrak{g} \cap (L^{k-1}/L^k)$ .

Let  $\theta_k \in B$  and  $\theta_{k+1}, \tilde{\theta}_{k+1} \in (\pi_{k+1,k})^{-1}(\theta_k)$ . Then there exists a unique element  $[d]_0^{k+1} = (\delta_j^i, 0, \dots, 0, d_{j_1 \dots j_{k+1}}^i) \in D_{k+1}^k$  such that  $\tilde{\theta}_{k+1} = \theta_{k+1} \cdot [d]_0^{k+1}$ .

Let  $f_{(H_{\theta_k}, \theta_{k+1})} = (f_{j_1 \dots j_k, r}^i)$  and  $f_{(H_{\theta_k}, \tilde{\theta}_{k+1})} = (\tilde{f}_{j_1 \dots j_k, r}^i)$ . Then from Proposition 8.2 it follows that

$$(\tilde{f}_{j_1 \dots j_k, r}^i) = (f_{j_1 \dots j_k, r}^i + d_{j_1 \dots j_k, r}^i). \quad (5.8)$$

Suppose  $X, Y \in H_{\theta_k}$ . Consider the bracket  $[\theta_{k+1}(X), \theta_{k+1}(Y)]$ , see Eq. (3.1). We have

$$\begin{aligned} [\theta_{k+1}(X), \theta_{k+1}(Y)] &= (X^r Y_{j_1 \dots j_{k-1} r}^i - Y^r X_{j_1 \dots j_{k-1} r}^i) \\ &= (X^r Y^s (f_{j_1 \dots j_{k-1} r, s}^i - \tilde{f}_{j_1 \dots j_{k-1} r, s}^i)). \end{aligned} \quad (5.9)$$

By definition, put

$$c(H_{\theta_k}, \theta_{k+1}) = (f_{j_1 \dots j_{k-1} r, s}^i - f_{j_1 \dots j_{k-1} s, r}^i).$$

From (5.8) it follows that  $c(H_{\theta_k}, \theta_{k+1})$  is independent of the choice of the point  $\theta_{k+1}$  over  $\theta_k \in B$ . Therefore we will write  $c(H_{\theta_k})$  instead of  $c(H_{\theta_k}, \theta_{k+1})$ .

Consider the Spencer complex

$$0 \rightarrow g_k^{(1)} \xrightarrow{\partial_{k+1,0}} g_k \otimes V^* \xrightarrow{\partial_{k,1}} L^{k-2}/L^{k-1} \otimes \wedge^2 V^* \xrightarrow{\partial_{k-1,2}} \dots \quad (5.10)$$

Obviously,

$$c(H_{\theta_k}) \in L^{k-2}/L^{k-1} \otimes \wedge^2 V^*.$$

From (5.7) it follows that if  $H_{\theta_k}$  and  $\tilde{H}_{\theta_k}$  are horizontal subspaces in  $T_{\theta_k} B$  and satisfy (5.6), then

$$c(H_{\theta_k}) - c(\tilde{H}_{\theta_k}) \in \text{Im } \partial_{k,1}$$

This means that the class  $c(H_{\theta_k}) \bmod (\text{Im } \partial_{k,1})$  is independent of the choice of the horizontal subspace  $H_{\theta_k}$  over  $H_{\theta_{k-1}}$ . We denote this class by  $c(\theta_k)$ . It is easy to check that

$$c(H_{\theta_k}) \in \ker \partial_{k-1,2}.$$

Consequently,  $c(\theta_k)$  is a Spencer  $\delta$ -cohomology class, that is,

$$c(\theta_k) \in H^{k-1,2}.$$

We say that the map

$$c : B \rightarrow H^{k-1,2}, \quad c : \theta_k \mapsto c(\theta_k)$$

is the *structure function* of the  $G$ -structure  $B$ .

**PROPOSITION 8.5.** *Structure functions of flat  $G$ -structures are trivial.*

**PROOF.** Let  $B$  be a flat  $G$ -structure of order  $k$  on  $\mathbb{R}^n$  and let  $(h = (x^1, \dots, x^n))$  be the standard chart in  $\mathbb{R}^n$ . An arbitrary element  $g \in G$  defines the diffeomorphism  $\hat{g}$  of  $\mathbb{R}^n$  to itself by the formula

$$\hat{g}(x^1, \dots, x^n) = \frac{1}{1!} g_j^i x^j + \dots + \frac{1}{k!} g_{j_1 \dots j_k}^i x^{j_1} \dots x^{j_k},$$

where  $(g_j^i, \dots, g_{j_1 \dots j_k}^i) = g^{-1}$ . By  $s_r^g$ ,  $r = 0, 1, \dots$ , we denote the section of  $P_r(\mathbb{R}^n)$  that is generated by the chart  $(\hat{g} \circ h = (y^1, \dots, y^n))$  on  $\mathbb{R}^n$ . Then  $s_k^g$  is a section of  $B$ . Indeed, let  $e$  be the unit of  $G$ , then  $s_k^e$  is a section of  $P_k(\mathbb{R}^n)$  generated by the standard chart in  $\mathbb{R}^n$ . This section is a section of  $B$ . It is clear that

$$s_k^g(p) = s_k^e(p) \cdot g \quad \forall p \in \mathbb{R}^n.$$

Let  $\theta_k = s_k^g(p)$  and let  $\theta_{k+1} = s_{k+1}^g(p)$ . Then it is obvious that  $H_{\theta_k} = (s_k^g)_*(T_p \mathbb{R}^n)$  is a horizontal subspace in  $T_{\theta_k} B$  and

$$\theta_{k+1} : X \mapsto (X^i, 0, \dots, 0) \quad \forall X \in H_{\theta_k}.$$

It is clear now that the structure function of the  $G$ -structure  $B$  is equal to zero for any point of  $\text{Im } s_k^g$ . Taking into account that images of sections  $\text{Im } s_k^g$ ,  $g \in G$ , cover  $B$  completely, we conclude that the structure function is equal to zero at each point of  $B$ .  $\square$

In general, the structure functions give only necessary conditions to solve the local equivalence problem for  $G$ -structures.

**THEOREM 8.6.** *Suppose  $B$  and  $\tilde{B}$  are  $G$ -structures on  $M$ ,  $c$  and  $\tilde{c}$  are their structure functions, respectively, and let  $f$  be a diffeomorphism of  $M$  to itself such that  $f^{(k)}(B) = \tilde{B}$ . Then  $(f^{(k)})^*(\tilde{c}) = c$ .*

**PROOF.** Let  $[s]_0^k = \theta_k \in B$  and let  $X \in T_{\theta_k}B$ . Then for any point  $\theta_{k+1} \in \pi_{k+1,k}^{-1}(\theta_k)$  we have

$$\theta_{k+1}(X) = f^{(k+1)}(\theta_{k+1})((f^{(k)})_*(X)).$$

Indeed, from the construction of the isomorphism  $\theta_{k+1}$ , see the proof of proposition 8.1, it follows that there exists a vector field  $\xi$  with the flow  $\varphi_t$  in  $M$  such that  $X = d/dt([\varphi_t \circ s]_0^k)|_{t=0}$  and

$$\theta_{k+1}(X) = d/dt([s^{-1} \circ \varphi_t \circ s]_0^k)|_{t=0}.$$

It follows that

$$\begin{aligned} f^{(k+1)}(\theta_{k+1})((f^{(k)})_*(X)) &= \frac{d}{dt}([(f \circ s)^{-1} \circ (f \circ \varphi_t \circ f^{-1}) \circ (f \circ s)]_0^k)|_{t=0} \\ &= \frac{d}{dt}([s^{-1} \circ \varphi_t \circ s]_0^k)|_{t=0} = \theta_{k+1}(X). \end{aligned}$$

It is obvious now that the cohomology classes  $c(\theta_k)$  and  $c(f^{(k)}(\theta_k))$  coincide.  $\square$

**5.2. Structure functions of prolongations.** Let  $\Omega$  be an arbitrary geometric structure on  $M$ ,  $F$  be its components' transformation law, and  $q_0 \in \mathbb{R}^N$  be some value of  $\Omega$ . Consider a  $G$ -structure  $B = \Omega^{-1}(q_0)$ . Let  $\mathfrak{g} \subset L^0/L^k$  be the Lie algebra of  $G$  and let  $g_k = \mathfrak{g} \cap L^{k-1}/L^k$ . Suppose that the structure function of  $B$  is equal to zero. Let  $\theta_k \in B$  and let  $\theta_{k+1} \in (\pi_{k+1,k})^{-1}(\theta_k)$ . Consider an arbitrary horizontal subspace  $H_{\theta_k} \subset T_{\theta_k}B$  satisfying (5.6). Let  $f(H_{\theta_k}, \theta_{k+1}) = (f_{j_1 \dots j_k, s}^i)$ . From the Spencer complex in Eq. (5.10) and the equation  $c(H_{\theta_k}, \theta_{k+1}) = 0 \pmod{(\text{Im } \partial_{k,1})}$ , it follows that there exists  $(g_{j_1 \dots j_k, s}^i) \in g_k \otimes V^*$  such that

$$(f_{j_1 \dots j_{k-1} r, s}^i - f_{j_1 \dots j_{k-1} s, r}^i) = \partial_{k,1}((g_{j_1 \dots j_k, s}^i)).$$

Therefore,

$$f_{j_1 \dots j_k, s}^i = g_{j_1 \dots j_k, s}^i + d_{j_1 \dots j_k, s}^i,$$

where  $(d_{j_1 \dots j_{k_s}}^i) \in g_k^{(1)}$ . By  $\tilde{H}_{\theta_k}$  we denote a horizontal subspace in  $T_{\theta_k}B$  such that  $f(\tilde{H}_{\theta_k}, \theta_{k+1}) = (d_{j_1 \dots j_{k_s}}^i)$ . Let  $\tilde{\theta}_{k+1} = \theta_{k+1} \cdot d$ , where  $d = (-d_{j_1 \dots j_{k_s}}^i) \in G \cap D_k(n)^{k+1}$ . Then it is clear that

$$\forall X \in \tilde{H}_{\theta_k} \quad \tilde{\theta}_{k+1}(X) = (X^i, 0, \dots, 0). \quad (5.11)$$

By  $B^{(1)}$  we denote the set of all  $\tilde{\theta}_{k+1}$ , which are obtained in this way. Obviously,

$$\pi_{k+1,k}(B^{(1)}) = B.$$

PROPOSITION 8.7. *We have*

$$B^{(1)} = (\Omega^{(1)})^{-1}((q_0, 0)),$$

i.e.,  $B^{(1)}$  is a  $G^{(1)}$ -structure. Here  $G^{(1)}$  is the isotropy group of the point  $(q_0, 0) \in \mathbb{R}^{N(1+n)}$ .

PROOF. Let  $[s]_0^{k+1} = \theta_{k+1} \in B^{(1)}$ . The local chart  $s^{-1} = (y^1, \dots, y^n)$  generates the local chart in  $P_k(M)$ . From (2.1) it follows that the  $G$ -structure  $B$  is defined within this chart by the equations

$$\tilde{q}^\alpha(y) = F^\alpha(y_j^i, \dots, y_{j_1 \dots j_k}^i, q_0). \quad (5.12)$$

Let  $H_{\theta_k} \subset T_{\theta_k}B$  be a horizontal subspace that satisfies (5.6) and (5.11). Then a vector  $X \in H_{\theta_k}$  is

$$X = X^i \frac{\partial}{\partial y^i} + 0 \cdot \frac{\partial}{\partial y_j^i} + \dots + 0 \cdot \frac{\partial}{\partial y_{j_1 \dots j_k}^i}$$

within this chart. From (5.12) we deduce that  $X$  satisfies the equation

$$\partial_j q^\alpha(0) \cdot X^j = 0.$$

This means that

$$\partial_j q^\alpha(0) = 0 \quad \forall \alpha = 1, 2, \dots, N, \quad j = 1, 2, \dots, n.$$

whence,

$$\Omega^{(1)}(\theta_{k+1}) = (q_0, 0).$$

Thus we obtain

$$B^{(1)} \subset (\Omega^{(1)})^{-1}(q_0, 0).$$

From (2.3) it follows that  $G^{(1)}$ -structure  $(\Omega^{(1)})^{-1}(q_0, 0)$  is defined by the equations

$$\begin{aligned} \tilde{q}^\alpha(y) &= F^\alpha(d_{j_1}^i, \dots, d_{j_1 \dots j_k}^i, q_0), \\ \partial_i \tilde{q}^\alpha(y) \cdot d_j^i &= \frac{\partial F^\alpha}{\partial y_{j_1}^i} y_{j_1 j}^i + \dots + \frac{\partial F^\alpha}{\partial y_{j_1 \dots j_k}^i} y_{j_1 \dots j_k j}^i. \end{aligned}$$

Therefore,

$$B^{(1)} \cap \pi_{k+1,k}^{-1}(\theta_k) = (\Omega^{(1)})^{-1}(q_0, 0) \cap \pi_{k+1,k}^{-1}(\theta_k) \quad \forall \theta_k \in B.$$

Now it is clear that

$$B^{(1)} = (\Omega^{(1)})^{-1}(q_0, 0).$$

□

In the same way as above, we construct the structure function

$$c^{(1)} : B^{(1)} \rightarrow H^{k,2}$$

of the  $G^{(1)}$ -structure  $B^{(1)}$ . If  $c^{(1)} = 0$ , then, in the same way as above, we can construct  $G^{(2)}$ -structure  $B^{(2)} = (\Omega^{(2)})^{-1}((q_0, 0, 0))$  and its structure function  $c^{(2)}$ , and so on.

### 5.3. Integrability of the finite type structures.

**THEOREM 8.8.** *Let  $B$  be a finite type  $G$ -structure and let  $c$  be its structure function. Then  $B$  is integrable iff  $c = 0$ ,  $c^{(1)} = 0, \dots, c^{(r(B))} = 0$ .*

**PROOF.** First, we consider the case  $r(B) = 0$ . Let  $\Omega$  be a geometric structure of order  $k$  such that  $B = \Omega^{-1}(q_0)$  and let  $y = (y^1, \dots, y^n)$  be a local chart of  $M$ . This chart generates the local chart of  $P_k(M)$ . In terms of this chart, the submanifold  $B$  is defined by the equations

$$\tilde{q}(y) = F(y_j^i, \dots, y_{j_1 \dots j_k}^i, q_0). \quad (5.13)$$

We interpret these equations as a system of partial differential equations  $\mathcal{E}$  w.r.t. the unknown functions  $y^1(x^1, \dots, x^n), \dots, y^n(x^1, \dots, x^n)$  that define the coordinate transformation  $x \rightarrow y$ . If there exists a solution of this PDE system, then  $x = (x^1, \dots, x^n)$  is a local chart of  $M$  and the components of  $\Omega$  in this chart are constant, *i.e.*,  $\Omega$  is integrable.

The condition  $g_k = \{0\}$  means that the symbol of the PDEs system is equal to zero. Thence, the natural projection  $\pi_{k+1,k} : B^{(1)} \rightarrow B$  is surjective. This means that the natural projection  $\mathcal{E}^{(1)} \rightarrow \mathcal{E}$ , where  $\mathcal{E}^{(1)}$  is the first prolongation of  $\mathcal{E}$ , is surjective too. Thus the system  $\mathcal{E}$  of partial differential equations is completely integrable, see [10], therefore it has a solution. This completes the proof for the case  $r(B) = 0$ .

The proof for the case  $r(B) > 0$  is obvious now. □

## 6. Applications of $G$ -structures to ordinary equations

**6.1. Second order equations.** Consider an ODEs of the form (4.1)

$$y'' = u^0(x, y) + u^1(x, y)y' + u^2(x, y)(y')^2 + u^3(x, y)(y')^3.$$

It is well-known that an arbitrary point transformation takes equation (4.1) to the equation of the same form. This means that equation (4.1) defines the second-order geometric structure on  $\mathbb{R}^2$  such that the coefficients of the equation are the components of this structure in the standard coordinates in  $\mathbb{R}^2$ . We denote this structure by  $\Omega$ . Thus,

$$\Omega : P_2(\mathbb{R}^2) \rightarrow \mathbb{R}^4.$$

This structure has finite type and  $r(B) = 1$ .

Consider the  $G$ -structure  $B = \Omega^{-1}(0)$ . Its structure function  $c$  is equal to zero. It can be proved that equation (4.1) can be reduced to linear form by a point transformation iff the structure function  $c^{(1)}$  of its first prolongation  $B^{(1)}$  is equal to zero (see [5], [20]).

**6.2. Third order equations.** Consider ordinary differential equations of the form

$$y''' = u^0(x, y, y') + u^1(x, y, y')y'' + u^2(x, y, y')(y'')^2 + u^3(x, y, y')(y'')^3 \quad (6.1)$$

It is easy to prove that an arbitrary contact transformation takes equation (6.1) to the equation of the same form. This means that equation (6.1) defines the geometric structure of third order on the space  $\mathbb{R}^3$  such that the coefficients of the equation are the components of this structure in the standard coordinates in space  $\mathbb{R}^3$ . We denote this structure by  $\Omega$ . Thus,

$$\Omega : P_3(\mathbb{R}^3) \rightarrow \mathbb{R}^4.$$

This structure has infinite type.

Let  $\Omega^{(\infty)}$  be the infinite prolongation of the structure  $\Omega$  and let  $B = \pi_{\infty,3}((\Omega^{(\infty)})^{-1}(0))$ . Then it can be proved that  $B$  is a finite type  $G$ -structure such that  $r(B) = 1$ . Its structure function  $c$  is equal to zero. It can be proved that equation (6.1) can be reduced to the form  $y''' = 0$  by a contact transformation iff the structure function  $c^{(1)}$  of its first prolongation  $B^{(1)}$  is equal to zero (see [6]).

## 7. Exercises

- (1) Prove theorem 8.3
- (2) Prove that the type of geometric structure (finite or infinite) is well defined.
- (3) Prove that geometric structure on  $\mathbb{R}^2$  generated by coefficients of an equation

$$y'' = u^0(x, y) + u^1(x, y)y' + u^2(x, y)(y')^2 + u^3(x, y)(y')^3$$

is a finite type structure.

- (4) \* Prove that geometric structure on  $\mathbb{R}^3$  generated by coefficients of an arbitrary equation of the form

$$y''' = u^0(x, y, y') + u^1(x, y, y')y'' + u^2(x, y, y')(y'')^2 + u^3(x, y, y')(y'')^3$$

is an infinite type structure.

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