ON THE OBSTRUCTION TO LINEARIZABILITY OF 2-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the action of pseudogroup of all point transformations on the bundle of equations

$$y'' = u^0(x,y) + u^1(x,y)y' + u^2(x,y)(y')^2 + u^3(x,y)(y')^3 \, .$$

We calculate the 1-st nontrivial differential invariant of this action. It is a horizontal differential 2-form with values in some algebra, it is defined on the bundle of 2–jets of sections of the considered bundle. We prove that this form is a unique obstruction to linearizability of these equations by point transformations.

Contents

1. Introduction

It is well known that any point transformation takes a 2-order linear ordinary differential equation to an equation of the form

$$y'' = u^0(x,y) + u^1(x,y)y' + u^2(x,y)(y')^2 + u^3(x,y)(y')^3 \, . \tag{1}$$

By π we denote the bundle of equations (1).

It is well known that any point transformation takes an arbitrary equation (1) to the equation of the same form. This means that the pseudogroup Γ of

Date: 29 November 2002.

1991 *Mathematics Subject Classification.* 53A55, 53C10, 53C15, 34A30, 34A26, 34C20, 58F35.

Key words and phrases. 2-nd order ordinary differential equation, point transformation, equivalence problem, differential invariant, Spenser cohomology.

all point transformations acts on π. This action can be lifted in the natural way to the action on the bundle $J^k\pi$ of k–jets of sections of π , $k = 1, 2 \dots$

In this paper, we investigate these actions. Earlier in [2], we obtained the following:

1. $J^k\pi$ is an orbit of the action of Γ iff $k = 0, 1$,

2. $J^2\pi$ is divided into two orbits of the action $J^2\pi = \mathrm{Orb}_1 \cup \mathrm{Orb}_2$ with $\dim \mathrm{Orb}_1 = \dim J^k\pi$ and $\dim \mathrm{Orb}_2 = \dim J^k\pi - 2$,

3. Equation (1) can be reduced to the linear form by a point transformation iff the collection of its coefficients is a solution of the equations defining the submanifold Orb_2 in $J^2\pi$.

This means that the first nontrivial differential invariant of the actions of Γ "lives" on $J^2\pi$ and it is a unique obstruction to the linearizability of equations (1) by point transformations.

The aim of this paper is to construct this obstruction. We constructed it in subsection 4.2 of this paper. It is a horizontal differential 2-form on $J^2\pi$ with values in some algebra. This form is nontrivial at any point of Orb_1 and it is zero at any point of Orb_2.

Recall that in [1], Cartan proved that equation (1) is equivalent to some projective connection and the equation can be reduced to the linear form by a point transformation iff the curvature form of this connection is equal to zero. We do not use projective connections to construct the obstruction form. Our construction recall the well known construction of structure functions of prolongations of G–structures (see [3]).

Below, all manifolds and maps are supposed to be smooth. By $[f]^k_p$ denote the k–jet of the map f at the point p, by $\mathbb{R}$ denote the field of real numbers, and by $\mathbb{R}^n$ denote the n–dimensional arithmetic space.

2. Bundles of equations

2.1. **Liftings of point transformations.**

2.1.1. *The lifting to the bundle of equations.* Let

$$\pi : E = \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}^2$$

be a product bundle. By x^1, x^2 denote the standard coordinate on the base of π, by u^0, u^1, u^2, u^3 denote the standard coordinates on the fiber of π.

Let $\mathcal{E}$ be an arbitrary equation (1). We identify $\mathcal{E}$ with the section $S_\mathcal{E}$ of π defined by the formula

$$S_\mathcal{E} : (x^1,\, x^2) \mapsto \big(\, x^1,\, x^2,\, u^0(x^1, x^2),\, u^1(x^1, x^2),\, u^2(x^1, x^2),\, u^3(x^1, x^2)\,\big).$$

Clearly, this identification is a bijection between the set of all equations (1) and the set of all sections of π.

It is well known (see [4]) that an arbitrary point transformation

$$f : (x^1, x^2) \mapsto \big(\, \tilde{x}^1 = f^1(x^1, x^2),\ \tilde{x}^2 = f^2(x^1, x^2)\,\big)\,. \tag{2}$$

transforms an equation of form (1) to the equation of the same form. The coefficients of the obtained equation are expressed in terms of the coefficients of the initial one and the derivatives of order ≤ 2 of the inverse transformation to f:

$$\tilde{u}^{\alpha} = \Phi^{\alpha}\Big(\, u^{\beta}, \frac{\partial g^i}{\partial \tilde{x}^j}, \frac{\partial^2 g^i}{\partial \tilde{x}^{j_1} \partial \tilde{x}^{j_2}} \Big)\,, \tag{3}$$
$$\alpha, \beta = 0, 1, 2, 3\,,\ g = \big(g^1, g^2\big) = f^{-1}\,,\ i, j, j_1, j_2 = 1, 2\,.$$

Equations (2) and (3) defines the diffeomorphism $f^{(0)}$ of the bundle π which is called *the lifting of f to the bundle π*.

Obviously, the following diagram

$$\begin{array}{ccc} E & \xrightarrow{f^{(0)}} & E \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathbb{R} & \xrightarrow[f]{} & \mathbb{R} \end{array}$$

is commutative (in the domain of $f^{(0)}$).

For any point transformation f, we define the transformation of sections of π by the formula

$$S \mapsto f(\,S\,) = f^{(0)} \circ S \circ f^{-1}\,. \tag{4}$$

Equations (3) can be represented now as

$$S_{\tilde{\mathcal{E}}} = f(\,S_{\mathcal{E}}\,)\,.$$

Now the following statement is obvious.

Proposition 2.1. *Let $\mathcal{E}$, $\tilde{\mathcal{E}}$ be equations of form* (1)*. Then a point transformation f takes $\mathcal{E}$ to $\tilde{\mathcal{E}}$ iff $S_{\tilde{\mathcal{E}}} = f(\,S_{\mathcal{E}}\,)$.*

2.1.2. *The lifting to jet bundles.* By $[S]^k_p$ denote the k–jet of a section S of π at the point p, $k = 0, 1, 2, \ldots, \infty$. By

$$\pi_k : J^k\pi \to \mathbb{R}^2\,,\ \pi_k : [S]^k_p \mapsto p\,,$$

denote the bundle of all k–jets of sections of π. The projection $\pi_{k,r} : J^k\pi \to J^r\pi\,,\ k > r$, is defined by $\pi_{k,r}(\,[S]^k_p\,) = [S]^r_p$. By definition, put $J^k_p\pi = \pi_k^{-1}(p)$.

Every section S of π generates the section j_kS of the bundle π_k by the formula $j_kS : p \mapsto [S]^k_p$.

By $x^1, x^2, u^i_\sigma\,,\ i = 0, \ldots, 3\,,\ 0 \leq |\sigma| \leq k$, denote the standard coordinates in $J^k\pi$, here σ is the multi-index $\{j_1 \ldots j_r\}\,,\ |\sigma| = r\,,\ j_1, \ldots, j_r = 1, 2$. By definition, put $\sigma j = \{j_1 \ldots j_r j\}$

Any point transformation f can be lifted to the diffeomorphism $f^{(k)}$ of $J^k\pi$ by the formula

$$f^{(k)}(\,[S]^k_p\,) = \big[\,f^{(0)} \circ S \circ f^{-1}\,\big]^{\,k}_{f(p)}\,. \tag{5}$$

The diffeomorphism $f^{(k)}$ is called *the lifting of f to the jet bundle $J^k\pi$*.

Obviously, for any $l > m$, the diagram

$$\begin{array}{ccc} J^l\pi & \xrightarrow{f^{(l)}} & J^l\pi \\ {\scriptstyle \pi_{l,m}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{l,m}} \\ J^m\pi & \xrightarrow[f^{(m)}]{} & J^m\pi \end{array}$$

is commutative (in the domains of $f^{(l)}$).

By Γ we denote the pseudogroup of all point transformation of the base of π, by $\Gamma^{(k)}$ we denote the transformation pseudogroup in $J^k\pi$ generated by all diffeomorphisms $f^{(k)}$, $f \in \Gamma$.

2.2. **Liftings of vector fields.** Let X be a vector field in the base of π and let f_t be its flow. Then the flow $f_t^{(k)}$ in $J^k\pi$ defines the vector field $X^{(k)}$ in $J^k\pi$ which is called *the lifting of X to $J^k\pi$*. Obviously

$$(\,\pi_{l,m}\,)_*\big(\,X^{(l)}\,\big) = X^{(m)}\,, \quad \infty \geq l > m \geq -1\,, \tag{6}$$

where $X^{(-1)} = X$.

Let

$$X = X^1(x^1,x^2)\frac{\partial}{\partial x^1} + X^2(x^1,x^2)\frac{\partial}{\partial x^2}\,,$$

then we have the following formula (see [5])

$$X^{(\infty)} = X^1 D_1 + X^2 D_2 + \ni_{\psi(X)}\,, \tag{7}$$

where

$$D_j = \frac{\partial}{\partial x^j} + \sum_{|\sigma|\geq 0}\sum_{i=0}^{3} u^i_{\sigma j}\frac{\partial}{\partial u^i_\sigma}\,,$$

is the operator of total derivation w.r.t. x^j,

$$\ni_{\psi(X)} = \sum_{|\sigma|\geq 0}\sum_{i=0}^{3} D_\sigma\big(\,\psi^i(X)\,\big)\frac{\partial}{\partial u^i_\sigma} \tag{8}$$

is the operator of evolution differentiation corresponding to the generating function $\psi(X) = (\psi^0(X),\ldots,\psi^3(X))^t$, $\sigma = \{j_1 \ldots j_r\}$, $D_\sigma = D_{j_1}\circ\ldots\circ D_{j_r}$. The function $\psi(X)$ is defined in the following way. Let S be a section of π defined in the domain of X, let $\theta_1 = [S]^1_p$, and let $p = \pi_1(\theta_1)$; then

$$\psi(X)(\theta_1) = \begin{pmatrix} \psi^0(X)(\theta_1) \\ \cdots \\ \psi^3(X)(\theta_1)\,. \end{pmatrix} = \frac{d}{dt}\big(\,f_t^{(0)} \circ S \circ f_t^{-1}\,\big)\Big|_{t=0}(p) \tag{9}$$

Obviously, $\psi(X)(\theta_1)$ is the deformation velocity of the section S at the point p under the action of the flow f_t.

Let $\theta_1 = (\,x^1, x^2, u^i, u^i_j\,)$, $i = 0,1,2,3$, $j = 1,2$; then it can be calculated that

$$\psi(X)(\theta_1) = \begin{pmatrix} -u^0_1X^1 - u^0_2X^2 \\ -2u^0X^1_1 + u^0X^2_2 - u^1X^2_1 + X^2_{11} \\ -u^1_1X^1 - u^1_2X^2 \\ -3u^0X^1_2 - u^1X^1_1 - 2u^2X^2_1 - X^1_{11} + 2X^2_{12} \\ -u^2_1X^1 - u^2_2X^2 \\ -2u^1X^1_2 - u^2X^2_2 - 3u^3X^2_1 - 2X^1_{12} + X^2_{22} \\ -u^3_1X^1 - u^3_2X^2 \\ -u^2X^1_2 + u^3X^1_1 - 2u^3X^2_2 - X^1_{22} \end{pmatrix}, \qquad (10)$$

where $X^i_j = \dfrac{\partial X^i}{\partial x^j}(p)$ and $X^i_{j_1j_2} = \dfrac{\partial^2 X^i}{\partial x^{j_1}\partial x^{j_2}}(p)\,.$

Let $\operatorname{Vect}\mathbb{R}^2$ and $\operatorname{Vect}J^k\pi$ be the Lie algebras of all vector fields in $\mathbb{R}^2$ and $J^k\pi$ respectively.

Proposition 2.2. *The map*

$$\operatorname{Vect}\mathbb{R}^2 \to \operatorname{Vect}J^k\pi\,, \quad X \mapsto X^{(k)}\,,$$

is a Lie algebra homomorphism.

Proof. The map $\Gamma \to \Gamma^{(k)}$, $f \mapsto f^{(k)}$, is a homomorphism of Lie pseudogroups. It has as a consequence the statement of the proposition. Indeed, let $X\,,Y$ be vector fields on $\mathbb{R}^2$ and let $f_t\,, g_s$ be their flows respectively. Then

$$\begin{aligned}
&[\,X^{(k)}, Y^{(k)}\,] = \lim_{t\to0}\frac{1}{t}\Big(Y^{(k)} - (f^{(k)}_t)_*(Y^{(k)}\circ f^{(k)}_{-t})\Big) \\
&= \lim_{t\to0}\frac{1}{t}\Big(\frac{d}{ds}\Big|_{s=0}g^{(k)}_s - (f^{(k)}_t)_*\big(\frac{d}{ds}\Big|_{s=0}g^{(k)}_s\circ f^{(k)}_{-t}\big)\Big) = \lim_{t\to0}\frac{1}{t}\Big(\frac{d}{ds}\Big|_{s=0}g^{(k)}_s \\
&\quad - \frac{d}{ds}\Big|_{s=0}f^{(k)}_t\circ g^{(k)}_s\circ f^{(k)}_{-t}\Big) = \lim_{t\to0}\frac{1}{t}\frac{d}{ds}\Big|_{s=0}\Big(g^{(k)}_s\circ f^{(k)}_t\circ g^{(k)}_s\circ f^{(k)}_{-t}\Big) \\
&\qquad = \lim_{t\to0}\frac{1}{t}\frac{d}{ds}\Big|_{s=0}\Big(g_s\circ f_t\circ g_s\circ f_{-t}\Big)^{(k)} = \lim_{t\to0}\frac{1}{t}\Big(\frac{d}{ds}\Big|_{s=0}g_s \\
&\quad - \frac{d}{ds}\Big|_{s=0}f_t\circ g_s\circ f_{-t}\Big)^{(k)} = \lim_{t\to0}\frac{1}{t}\big(Y - (f_t)_*(Y\circ f_{-t})\big)^{(k)} = [\,X, Y\,]^{(k)}\,.
\end{aligned}$$

The $\mathbb{R}$ – linearity of the map $X \mapsto X^{(k)}$ is obvious. ☐

3. Isotropy algebras and spaces

3.1. **Preliminaries.** In this subsection, we recall some necessary notions concerning formal vector fields, prolongations of subspaces, Spenser cohomologies and the decomposition of tangent spaces to $J^k\pi$ (see [6], [7], and [5]).

3.1.1. *Formal vector fields.* By W_p we denote the Lie algebra of ∞–jets at $p \in \mathbb{R}^2$ of all vector fields defined in a neighborhoods of p. Recall that the structure of Lie algebra on W_p is defined by the operations

$$\lambda[X]_p^\infty \stackrel{df}{=} [\lambda X]_p^\infty\,, \quad [X]_p^\infty + [Y]_p^\infty \stackrel{df}{=} [X+Y]_p^\infty\,,$$
$$\big[\,[X]_p^\infty, [Y]_p^\infty\,\big] \stackrel{df}{=} \big[\,[X,Y]\,\big]_p^\infty$$
$$\forall\, \lambda \in \mathbb{R}\,, \quad \forall\ [X]_p^\infty\,, [Y]_p^\infty \in W_p\,.$$

By $L_p^k\,,\ k=-1,0,1,2,\dots$, we denote the subalgebra in W_p defined by

$$L_p^k = \big\{\,[X]_p^\infty \in W_n \,\big|\, [X]_p^k = 0\,\big\}\,,\ k \geq 0\,, \quad L_p^{-1} = W_p\,.$$

By definition, put

$$V_p = W_p/L_p^0\,.$$

Obviously, $V_p \cong T_p\,\mathbb{R}^2$. We have the filtration

$$W_p = L_p^{-1} \supset L_p^0 \supset L_p^1 \supset \ldots \supset L_p^k \supset L_p^{k+1} \supset \ldots\,.$$

For any $i > j \geq 0$, we denote by $\rho_{i,j}$ the natural projection

$$\rho_{i,j} : W_p/L_p^i \to W_p/L_p^j\,, \quad \rho_{i,j} : [X]_p^i \mapsto [X]_p^j$$

and by definition, put

$$\rho_i = \rho_{i,0}\,.$$

Taking into account that

$$[\,L_p^i\,,\ L_p^j\,]\ ,=\ L_p^{i+j}\,, \quad i,j = -1,0,1,2,\dots\,,$$

we see that the bracket operation $[\,\cdot\,,\cdot\,]$ on W_p generates the following maps

$$[\,\cdot\,,\,\cdot\,] : W_p/L_p^k \times W_p/L_p^k \to W_p/L_p^{k-1}\,, \tag{11}$$
$$[\,\cdot\,,\,\cdot\,] : V_p \times L_p^k/L_p^{k+1} \to L_p^{k-1}/L_p^k\,. \tag{12}$$

The last map generates the isomorphism

$$L_p^k/L_p^{k+1} \cong V_p \otimes S^k(V_p^*)\,.$$

Let g_k be a subspace of L_p^{k-1}/L_p^k. The subspace $g_k^{(1)} \subset L_p^k/L_p^{k+1}$ defined by

$$g_k^{(1)} = \big\{\,X \in L_p^k/L_p^{k+1} \,\big|\, [\,v\,,\,X\,] \in g_k \ \ \forall\, v \in V_p\,\big\}$$

is called *the 1-st prolongation of* g_k.

Suppose the sequence of subspaces

$$g_1\,,\ g_2\,,\ \dots\,,\ g_i\,,\ \dots$$

satisfies to the property

$$[\,V\,,\ g_{i+1}\,] \subset g_i\,.$$

Then for every g_i, we have the complex

$$0 \to g_i \xrightarrow{\partial_{i,0}} g_{i-1} \otimes V_p^* \xrightarrow{\partial_{i-1,1}} g_{i-2} \otimes \wedge^2 V_p^* \xrightarrow{\partial_{i-2,2}} 0\,, \tag{13}$$

where the operators $\partial_{k,l} : g_k \otimes \wedge^l V_p^* \to g_{k-1} \otimes \wedge^{l+1} V_p^*$ are defined in the following way: any element $\xi \in g_k \otimes \wedge^l V_p^*$ can be considered as an exterior form on V_p with values in g_k, then

$$(\,\partial_{k,l}(\xi)\,)(v_1,\ldots,v_{l+1}) = \sum_{i=1}^{l+1}(-1)^{i+1}[\,v_i\,,\;\xi(v_1,\ldots,\hat{v}_i,\ldots,v_{l+1})\,]\,.$$

We denote by $H_p^{k,l}$ the cohomology group of this complex in the term $g_k \otimes \wedge^l V_p^*$. It is called a *Spenser cohomology group.*

3.1.2. *The decomposition of tangent spaces.* Let $\theta_{k+1} \in J^{k+1}\pi$, let $\theta_k = \pi_{k+1,k}(\theta_{k+1})$, and let $[S]_p^{k+1} = \theta_{k+1}$. Then the tangent space to the image of the section $j_k S$ at the point θ_k is defined by θ_{k+1}. We denote this tangent space by $\mathcal{H}_{\theta_{k+1}}$. We have the following direct sum decomposition of the tangent space to $J^k\pi$ at the point θ_k

$$T_{\theta_k} J^k\pi = \mathcal{H}_{\theta_{k+1}} \oplus T_{\theta_k}\big(\pi^{-1}(p)\big)\,.$$

Let X be a vector field in the base of π defined in a neighborhood of p. Then the value $X^{(k)}_{\theta_k}$ of $X^{(k)}$ at the point θ_k has a unique decomposition

$$X^{(k)}_{\theta_k} = \mathcal{H}_{\theta_{k+1}} X^{(k)} + V_{\theta_{k+1}} X^{(k)}\,, \tag{14}$$

where $\mathcal{H}_{\theta_{k+1}} X^{(k)} \in \mathcal{H}_{\theta_{k+1}}$ and $V_{\theta_{k+1}} X^{(k)} \in T_{\theta_k}\big(\pi^{-1}(p)\big)$. It follows from (7) and (6) that if $X = X^1\partial/\partial x^1 + X^2\partial/\partial x^2$, then

$$\mathcal{H}_{\theta_{k+1}} X^{(k)} = X^1 D_1^{\theta_{k+1}} + X^2 D_2^{\theta_{k+1}}\,, \quad V_{\theta_{k+1}} X^{(k)} = \mathfrak{D}^{\theta_{k+1}}_{\psi(X)}\,, \tag{15}$$

where

$$D_j^{\theta_{k+1}} = \frac{\partial}{\partial x^j} + \sum_{0\le|\sigma|\le k}\sum_{i=0}^{3} u^i_{\sigma j}(\theta_{k+1})\frac{\partial}{\partial u^i_\sigma}\,,$$

$$\mathfrak{D}^{\theta_{k+1}}_{\psi(X)} = \sum_{0\le|\sigma|\le k}\sum_{i=0}^{3}\Big(\,D_\sigma\big(\,\psi^i(X)\,\big)\Big)(\theta_{k+1})\frac{\partial}{\partial u^i_\sigma}\,. \tag{16}$$

It follows from (10) that the value $X^{(k)}_{\theta_k}$ of the vector field $X^{(k)}$ at the point θ_k is depended on the jet $[X]_p^{k+2}$.

3.2. **Isotropy algebras.** Let $\theta_k \in J^k\pi$ and $p = \pi(\theta_k)$. By G_{θ_k} we denote the *isotropy group* of θ_k, that is

$$G_{\theta_k} = \{\ [f]_p^{2+k} \mid f \in \Gamma\,,\ f^{(k)}(\theta_k) = \theta_k\ \}$$

By $\mathfrak{g}_{\theta_k}$ we denote the Lie algebra of G_{θ_k}. It can be considered as a Lie subalgebra in L_p^0/L_p^{2+k}:

$$\mathfrak{g}_{\theta_k} = \{\ [X]_p^{2+k} \in L_p^0/L_p^{2+k} \mid X \in \operatorname{Vect}\mathbb{R}^2\,,\ X^{(k)}_{\theta_k} = 0\ \}$$

The subalgebra $\mathfrak{g}_{\theta_k} \subset L_p^0/L_p^{2+k}$ is called the *isotropy algebra* of θ_k.

From this definition and (14), (15), and (16), we get

Proposition 3.1. $[X]_p^{2+k} \in \mathfrak{g}_{\theta_k}$ *iff it is a solution of the system of linear algebraic equations*

$$\big(D_\sigma(\psi_X^i)\big)(\theta_k) = 0\,, \quad 0 \le |\sigma| \le k\,. \tag{17}$$

(We write $D_\sigma(\psi_X^i))(\theta_k)$ in (17) instead $D_\sigma(\psi_X^i))(\theta_{k+1})$ because from $X_p = 0$ we have that system (17) depends on θ_k and it is independent of θ_{k+1}.)

Let $\theta_0 \in J^0\pi$ and $p = \pi(\theta_0)$. From (17), we get that the isotropy algebra $\mathfrak{g}_{\theta_0}$ of the point θ_0 is defined by the equations

$$\begin{cases} -2u^0X_1^1 + u^0X_2^2 - u^1X_1^2 + X_{11}^2 = 0 \\ -3u^0X_2^1 - u^1X_1^1 - 2u^2X_1^2 - X_{11}^1 + 2X_{12}^2 = 0 \\ -2u^1X_2^1 - u^2X_2^2 - 3u^3X_1^2 - 2X_{12}^1 + X_{22}^2 = 0 \\ -u^2X_2^1 + u^3X_1^1 - 2u^3X_2^2 - X_{22}^1 = 0 \end{cases} \tag{18}$$

It follows from (18) that

$$\rho_{2,1}(\mathfrak{g}_{\theta_0}) = L_p^0/L_p^1\,.$$

Let

$$g_{\theta_0} = \mathfrak{g}_{\theta_0} \cap (L_p^1/L_p^2)\,.$$

Obviously, it is a commutative subalgebra in $\mathfrak{g}_{\theta_0}$. From (18), we get that g_{θ_0} is defined by the equations

$$\begin{cases} X_{11}^2 = 0 \\ X_{11}^1 - 2X_{12}^2 = 0 \\ 2X_{12}^1 - X_{22}^2 = 0 \\ X_{22}^1 = 0 \end{cases} \tag{19}$$

It is clear that g_{θ_0} and $g_{\tilde\theta_0}$ are canonically isomorphic for any $\theta_0, \tilde\theta_0 \in J^0\pi$. Therefore we shall write g instead g_{θ_0}.

It follows from (19) that

$$\dim g = 2 \tag{20}$$

and we can choose

$$\begin{aligned} e_1 &= 2\frac{\partial}{\partial x^1} \otimes (dx^1 \odot dx^1) + \frac{\partial}{\partial x^2} \otimes (dx^1 \odot dx^2)\,, \\ e_2 &= 2\frac{\partial}{\partial x^2} \otimes (dx^2 \odot dx^2) + \frac{\partial}{\partial x^1} \otimes (dx^1 \odot dx^2) \end{aligned} \tag{21}$$

as independent generators of g.

It is easy to check that the 1–prolongation $g^{(1)}$ of g is trivial, that is

$$g^{(1)} = \{0\}\,. \tag{22}$$

3.3. **Isotropy spaces.** By definition, put

$$\mathcal{A}_{\theta_{k+1}} = \{ \, [X]_p^{2+k} \in W_p/L_p^{2+k} \mid X_{\theta_k}^{(k)} \in \mathcal{H}_{\theta_{k+1}} \, \} , \quad k = 0, 1, \ldots, \infty . \tag{23}$$

From (14), (15), and (16), we get

Proposition 3.2. *$[X]_p^{2+k} \in \mathcal{A}_{\theta_{k+1}}$ iff $[X]_p^{2+k}$ is a solution of the system of linear equations*

$$\big(D_\sigma(\psi_X^i) \big)(\theta_{k+1}) = 0 , \quad 0 \le |\sigma| \le k . \tag{24}$$

We say that $\mathcal{A}_{\theta_{k+1}}$ is the *isotropy spase* of θ_{k+1}.

Theorem 3.3. (1) $\rho_{k+2,k+1}(\mathcal{A}_{\theta_{k+1}}) \subset \mathcal{A}_{\theta_k}$.
(2) $[\,\cdot\,,\,\cdot\,] : \mathcal{A}_{\theta_{k+1}} \times \mathcal{A}_{\theta_{k+1}} \to \mathcal{A}_{\theta_k}$.

Proof. The first statement is obvious.

Prove the second one. Let $[X]_p^{2+k}, [Y]_p^{2+k} \in \mathcal{A}_{\theta_{k+1}}$, let $\theta_\infty \in \pi_\infty^{-1}(p)$, $\theta_k = \pi_{\infty,k}(\theta_\infty)$, $\theta_{k-1} = \pi_{k,k-1}(\theta_k)$, and let

$$X = X^1 \frac{\partial}{\partial x^1} + X^2 \frac{\partial}{\partial x^2} , \; Y = Y^1 \frac{\partial}{\partial x^1} + Y^2 \frac{\partial}{\partial x^2} .$$

Then

$$\big[\, [X]_p^{2+k}, [Y]_p^{2+k} \, \big] = \big[\, [\, X \, , Y \,] \, \big]_p^{2+k-1}$$

and

$$\big[\, [\, X \, , Y \,] \, \big]_p^{2+k-1} \in \mathcal{A}_{\theta_k} \quad \text{iff} \quad [\, X , \, Y \,]_{\theta_{k-1}}^{(k-1)} \in \mathcal{H}_{\theta_k} .$$

We have

$$[\, X , \, Y \,]_{\theta_{k-1}}^{(k-1)} = (\pi_{\infty,k-1})_* [\, X , \, Y \,]_{\theta_\infty}^{(\infty)} = (\pi_{\infty,k-1})_* [\, X^{(\infty)} , \, Y^{(\infty)} \,]_{\theta_\infty}$$
$$= (\pi_{\infty,k-1})_* [\, X^1 D_1 + X^2 D_2 + \ni_{\psi(X)} , \, Y^1 D_1 + Y^2 D_2 + \ni_{\psi(Y)} \,]_{\theta_\infty} .$$

Taking into account the well known relations (see [5])

$$[\, D_1 , \, D_2 \,] = [\, D_1 , \, \ni_\psi \,] = [\, D_2 , \, \ni_\psi \,] = 0 \quad \text{and} \quad [\, \ni_\phi , \, \ni_\psi \,] = \ni_{\{\phi,\psi\}} ,$$

where $\{\phi, \psi\} = \ni_\phi(\psi) - \ni_\psi(\phi)$, we get

$$[\, X , \, Y \,]_{\theta_{k-1}}^{(k-1)} = (\pi_{\infty,k-1})_* \Big(\, (X^1 Y_1^1 + X^2 Y_2^1 - Y^1 X_1^1 - Y^2 X_2^1) D_1 +$$
$$+ (X^1 Y_1^2 + X^2 Y_2^2 - Y^1 X_1^2 - Y^2 X_2^2) D_2 + [\, \ni_{\psi(X)} \, , \ni_{\psi(Y)} \,] \Big)_{\theta_\infty} =$$
$$= \mathcal{H}_{\theta_k} [\, X , \, Y \,]^{(k-1)} + \ni^{\theta_k}_{\{\psi(X),\psi(Y)\}} .$$

From (10), we obtain

$$\{\psi(X), \psi(Y)\}^i = \psi^{i'}(X) \frac{\partial \psi^i(Y)}{\partial u^{i'}} + D_j(\psi^{i'}(X)) \frac{\partial \psi^i(Y)}{\partial u_j^{i'}}$$
$$- \psi^{i'}(Y) \frac{\partial \psi^i(X)}{\partial u^{i'}} - D_j(\psi^{i'}(Y)) \frac{\partial \psi^i(X)}{\partial u_j^{i'}} .$$

From (16), we get now that $\ni^{\theta_k}_{\{\psi(X),\psi(Y)\}} = 0$. □

4. Differential invariants

4.1. **Horizontal subspaces.** We shall say that a 2-dimensional subspace $H \subset W_p/L_p^k$ is *horisontal* if

$$\rho_k(H) = V_p\,.$$

Let $\theta_k \in J^k\pi$ and $\theta_{k+1} \in \pi_{k+1,k}^{-1}(\theta_k)$; then it is clear that

$$\mathfrak{g}_{\theta_k} \subset \mathcal{A}_{\theta_{k+1}} \quad \forall\, \theta_{k+1} \in \pi_{k+1,k}^{-1}(\theta_k)\,. \tag{25}$$

It is obvious that a 2-dimensional subspace $H \subset \mathcal{A}_{\theta_{k+1}}$ is horizontal iff

$$\mathcal{A}_{\theta_{k+1}} = H \oplus \mathfrak{g}_{\theta_k}\,.$$

Any two horizontal subspaces $H, \tilde{H} \subset \mathcal{A}_{\theta_{k+1}}$ define the linear function

$$f_{H,\tilde{H}} : V_p \to \mathfrak{g}_{\theta_k}\,, \quad f_{H,\tilde{H}} : X \mapsto (\rho_{k+2}|_H)^{-1}(X) - (\rho_{k+2}|_{\tilde{H}})^{-1}(X)\,.$$

It is clear that for any horizontal subspace $H \subset \mathcal{A}_{\theta_{k+1}}$ and for any linear function $f : V \to \mathfrak{g}_{\theta_k}$, there exist a unique horizontal subspace $\tilde{H} \subset \mathcal{A}_{\theta_{k+1}}$ with $f = f_{H,\tilde{H}}$.

Further in this subsection, we shall investigate horizontal subspaces of $\mathcal{A}_{\theta_1}$.

By H_p we denote the horizontal subspace in W_p/L_p^1 generated by constant vector fields.

By H_{θ_1} we denote a horizontal subspace in $\mathcal{A}_{\theta_1}$ with

$$\rho_{2,1}(H_{\theta_1}) = H_p\,. \tag{26}$$

From

$$\rho_{2,1}(\mathcal{A}_{\theta_1}) = W_p/L_p^1\,,$$

we have that horizontal subspaces H_{θ_1} exist. Obviously, H_{θ_1} is defined by

$$H_{\theta_1} = \{\ [X]_p^2 = (\,X^i, 0, X_\sigma^i\,)\,,\ i = 1,2\,,\ |\sigma| = 2\ \} \tag{27}$$

in the standard coordinates.

It is clear now that for any two horizontal subspaces $H_{\theta_1}, \tilde{H}_{\theta_1}$ satisfying to (26), we get

$$f_{H_{\theta_1},\tilde{H}_{\theta_1}} : V_p \to g\,.$$

Taking into account that $g \neq \{0\}$, we obtain that there exist a lot of horizontal subspaces satisfying to (26). We choose one of them in the following way.

A horizontal subspace H_{θ_1} defines the form $\omega_{H_{\theta_1}} \in L_p^0/L_p^1 \otimes \wedge^2 V_p^*$ by the formula

$$\omega_{H_{\theta_1}}(X,Y) = \big[\,(\rho|_{H_{\theta_1}})^{-1}(X), (\rho|_{H_{\theta_1}})^{-1}(Y)\,\big] \quad \forall\, X, Y \in V_p\,.$$

From the Spenser complex

$$0 \to g^{(1)} \xrightarrow{\partial_{3,0}} g \otimes V_p^* \xrightarrow{\partial_{2,1}} L_p^0/L_p^1 \otimes \wedge^2 V_p^* \xrightarrow{\partial_{1,2}} 0\,, \tag{28}$$

we get that $\omega_{H_{\theta_1}}$ defines the Spenser cohomology class $\{\omega_{H_{\theta_1}}\} \in H_p^{1,2}$.

Proposition 4.1. *The cohomology class $\{\omega_{H_{\theta_1}}\}$ is trivial.*

Proof. From (22) we get that $\partial_{2,1}$ is an injection in (28). From (20), we obtain $\dim g\otimes V_p^* = 4$. Obviously, $\dim L_p^0/L_p^1\otimes \wedge^2 V_p^* = 4$. As a result, we obtain $\operatorname{Im}\partial_{2,1} = \ker \partial_{1,2}$ in (28). □

Corollary 4.2. *There exists a unique horizontal subspace $H_{\theta_1}\subset \mathcal{A}_{\theta_1}$ with $\omega_{H_{\theta_1}} = 0$.*

Proof. Prove the uniqueness. Suppose H_{θ_1}, $\tilde H_{\theta_1}$ are horizontal subspaces of $\mathcal{A}_{\theta_1}$ with $\omega_{H_{\theta_1}} = \omega_{\tilde H_{\theta_1}} = 0$. We have $\omega_{H_{\theta_1}} = \omega_{\tilde H_{\theta_1}} + \partial_{2,1}(f_{H_{\theta_1},\tilde H_{\theta_1}})$. Therefore, $\partial_{2,1}(f_{H_{\theta_1},\tilde H_{\theta_1}}) = 0$. Taking into account that $\partial_{2,1}$ is an injection, we get that $f_{H_{\theta_1},\tilde H_{\theta_1}} = 0$. This means that $H_{\theta_1} = \tilde H_{\theta_1}$

Prove the existence. We have $\{\omega_{H_{\theta_1}}\} = \{0\}$. Therefore there exist $h\in g\otimes V_p^*$ with $\omega_{H_{\theta_1}} = \partial_{2,1}(h)$. It follows that the horizontal subspace

$$\tilde H_{\theta_1} = \left\{\, (\rho_2|_{H_{\theta_1}})^{-1}(X) - h(X)\,,\ \ X\in V_p \,\right\}$$

satisfies to the property $\omega_{\tilde H_{\theta_1}} = 0$. □

Now, we express the horizontal space H_{θ_1} with $\omega_{H_{\theta_1}} = 0$ in terms of standard coordinate $x^1, x^2, u^i(\theta_1), u^i_j(\theta_1)$. Let

$$\left(\rho_2|_{H_{\theta_1}}\right)^{-1}(X) = (\,X^i,\ 0\,,\ f^i_{jk,r}X^r\,)\,,\ \ \forall X\in V_p\,.$$

Then the property $\omega_{H_{\theta_1}} = 0$ means that

$$f^i_{jk,r} = f^i_{jr,k}\,. \tag{29}$$

From proposition 3.2 we obtain that elements $(\,X^i,\ 0\,,\ f^i_{jk,r}X^r\,)\in H_{\theta_1}$ is a solutions of the system

$$\begin{cases} -u^0_1X^1 - u^0_2X^2 + f^2_{11,r}X^r = 0\\ -u^1_1X^1 - u^1_2X^2 - f^1_{11,r}X^r + 2f^2_{12,r}X^r = 0\\ -u^2_1X^1 - u^2_2X^2 - 2f^1_{12,r}X^r + f^2_{22,r}X^r = 0\\ -u^3_1X^1 - u^3_2X^2 - f^1_{22,r}X^r = 0\end{cases}$$

From this system and (29), we obtain

$$\begin{cases} f^2_{11,1} = u^0_1\,,\ \ f^2_{11,2} = f^2_{12,1} = u^0_2\,,\\ f^2_{12,2} = f^2_{22,1} = \frac{1}{3}(\,2u^1_2 - u^0_1\,)\,,\\ f^2_{22,2} = -2u^3_1 + u^2_2\,,\\ f^1_{22,2} = -u^3_2\,,\ \ f^1_{22,1} = f^1_{12,2} = -u^3_1\,,\\ f^1_{12,1} = f^1_{11,2} = \frac{1}{3}(\,u^1_2 - 2u^2_1\,)\,,\\ f^1_{11,1} = 2u^0_2 - u^1_1\,.\end{cases} \tag{30}$$

4.2. **The obstruction form.** Let $\theta_2 \in J^2\pi$ and $\theta_1 = \pi_{2,1}(\theta_2)$. It is not difficult to prove that

$$\rho_{3,2}(\mathcal{A}_{\theta_2}) = \mathcal{A}_{\theta_1}. \tag{31}$$

Let H_{θ_1} be the horizontal subspace of $\mathcal{A}_{\theta_1}$ with $\omega_{H_{\theta_1}} = 0$. From (31) and (22), we get that there exist a unique horizontal subspace $H_{\theta_2} \subset \mathcal{A}_{\theta_2}$ with

$$\rho_{3,2}(H_{\theta_2}) = H_{\theta_1}\,. \tag{32}$$

It follows from item (2) of theorem 3.3 that H_{θ_2} defines the 2–form $\omega_{\theta_2} \in \mathcal{A}_{\theta_1} \otimes \wedge^2 V_p^*$ by the formula

$$\omega_{\theta_2}(X,Y) = \big[\,(\rho_3|_{H_{\theta_2}})^{-1}(X),\,(\rho_3|_{H_{\theta_2}})^{-1}(Y)\,\big] \quad \forall\, X,Y \in V_p\,.$$

From $\omega_{H_{\theta_1}} = 0$ we obtain

$$\omega_{\theta_2} \in g \otimes (V_p^* \wedge V_p^*)$$

Now we can define the horizontal differential 2-form $\omega^{(2)}$ on $J^2\pi$ with values in g by the following formula

$$\omega^{(2)} : \theta_2 \longmapsto \pi_2^*(\omega_{\theta_2})\,. \tag{33}$$

Obviously, H_{θ_2} is defined by

$$H_{\theta_2} = \big\{\ [X]_p^2 = (\,X^i,\ 0,\ f^i_{j_1j_2,r}X^r,\ f^i_{j_1j_2j_3,r}X^r\,)\ \big\}$$

in the standard coordinates. Hence,

$$\omega_{\theta_2} = 2f^i_{j_1j_2[k,r]}\Big(\frac{\partial}{\partial x^i} \otimes (dx^{j_1} \odot dx^{j_2})\Big) \otimes\ (dx^k \wedge\ dx^r)\,.$$

Taking into account (20) and (21), we get

$$\omega^{(2)} = (\ F^1 \cdot e_1 + F^2 \cdot e_2\ ) \otimes\ (dx^1 \wedge\ dx^2)\,, \tag{34}$$

where $F^1 = f^1_{11[1,2]}$ and $F^2 = f^2_{22[1,2]}$.

Calculate the functions F^1, F^2. From proposition 3.2 we obtain that elements

$$(\,X^i,\ 0,\ f^i_{j_1j_2,r}X^r,\ f^i_{j_1j_2j_3,r}X^r\,) \in H_{\theta_2}$$

is a solutions of system (24) for $k = 1$. From this system and (30), we get

$$\begin{aligned} F^1 = {} & 3u^0_{22} - 2u^1_{12} + u^2_{11} \\ & + 3u^3u^0_1 - 3u^2u^0_2 + 2u^1u^1_2 - u^1u^2_1 - 3u^0u^2_2 + 6u^0u^3_1\,, \end{aligned} \tag{35}$$

$$\begin{aligned} F^2 = {} & u^1_{22} - 2u^2_{12} + 3u^3_{11} \\ & - 3u^0u^3_2 + 3u^1u^3_1 - 2u^2u^2_1 + u^2u^1_2 + 3u^3u^1_1 - 6u^3u^0_2\,. \end{aligned} \tag{36}$$

Note that first the coefficients F^1 and F^1 were obtained by Cartan in [1] as unique nonzero coefficients of the curvature form of the projective connection corresponding to equation (1).

Thus we obtain the following expession of $\omega^{(2)}$ in the standard coordinates

$$\omega^{(2)} = \Big(F^1\big(2\frac{\partial}{\partial x^1}\otimes(dx^1\odot dx^1)+\frac{\partial}{\partial x^2}\otimes(dx^1\odot dx^2)\big) + F^2\big(2\frac{\partial}{\partial x^2}\otimes(dx^2\odot dx^2)+\frac{\partial}{\partial x^1}\otimes(dx^1\odot dx^2)\big)\Big) \otimes (dx^1\wedge dx^2)\,, \quad (37)$$

where F^1 and F^2 are defined by (35) and (36) respectively.

We recall, that a differential form defined on $J^k\pi$ is *a differential invariant of the action of* Γ *on* π if it is invariant w.r.t. the pseudogroup $\Gamma^{(k)}$.

Theorem 4.3. *The form* $\omega^{(2)}$ *is a differential invariant of the action of* Γ *on* π.

Proof. Let $f\in\Gamma$, let p be a point from the domain of f, and let $\theta_2\in J^2_p\pi$. We should check that

$$(f^{(2)})^*\Big(\omega^{(2)}\big|_{f^{(2)}(\theta_2)}\Big) = \omega^{(2)}\big|_{\theta_2}\,. \quad (38)$$

We shall check it in the standard coordinates. It is clear that the left side of (38) is depend of $[f]^4_p$. This jet can be represented in the following way

$$[f]^4_p = [f_1]^4_p\cdot[f_2]^4_p\,,$$

where $[f_1]^4_p$ is jet of the affine transformation and $[f_2]^1_p=[\mathrm{id}]^1_p$. It can easily be checked that $\omega^{(2)}$ is invariant w.r.t. affine transformations. Therefore it remains to check that equation (38) holds for an arbitrary point transformation f with $[f]^1_p=[\mathrm{id}]^1_p$. Taking into account that $\omega^{(2)}$ is horizontal, we get that equation (38) holds for a point transformation f with $[f]^1_p=[\mathrm{id}]^1_p$ iff

$$F^1\big(f^{(2)}(\theta_2)\big)=F^1(\theta_2)\,,\quad F^2\big(f^{(2)}(\theta_2)\big)=F^2(\theta_2)$$

It is clear that the last equations hold iff the restrictions of F^1, F^2 to $J^2_p\pi$ are 1-st integrals for any vector field $\xi^{(2)}\big|_{J^2_p\pi}$ with $[\xi]^\infty_p\in L^1_p$. The last statement about F^1 and F^2 can be easy checked by direct calculations in standard coordinates. □

In his paper [1], Cartan proved that equation (1) can be reduced to the linear form by a point transformation iff the collection of its coefficients is a solution of the system of PDEs

$$F^1=0\,,\quad F^2=0\,. \quad (39)$$

This means that $\omega^{(2)}$ is a unique obstruction to the linearizability of equations (1) by point transformations.

Below, we give the independent proof of this fact.

Let

$$M=\Big\{\,\theta_2\in J^2\pi\;\Big|\;\omega^{(2)}\big|_{\theta_2}=0\,\Big\}\,.$$

From (34), it follows that M is defined by system of algebraic equations (39).

By $\mathbf{0}$ we denote the zero section of π, by 0_k we denote $[\mathbf{0}]^k_0$, $k=0,1,2,\ldots$

Lemma 4.4. $M = \mathrm{Orb}_{0_2}$.

Proof. It is clear that $\dim \mathrm{Orb}_{0_2} = \dim W_0/L_0^4 - \dim \mathfrak{g}_{0_2}$. We have $\dim W_0/L_0^4 = 30$. It is easy to calculate that $\dim \mathfrak{g}_{0_2} = 6$. Therefore $\dim \mathrm{Orb}_{0_2} = 24$. From (39), we have that $\dim M = \dim J^2\pi - 2 = 24$ too.

Obviously, $\omega^{(2)}\big|_{0_2} = 0$. Now from theorem 4.3, we get that $\mathrm{Orb}_{0_2} \subset M$. At last, the sets M and Orb_{0_2} are connected subsets in $J^2\pi$. This concludes the proof. □

Lemma 4.5. *Let $\theta_2 \in \mathrm{Orb}_{0_2}$ and let $\theta_1 = \pi_{2,1}(\theta_2)$; then the natural projection of the isotropy groups of these points*

$$G_{\theta_2} \to G_{\theta_1}\,, \quad [f]_p^4 \mapsto [f]_p^3\,,$$

is a bijection.

Proof. It is easy to prove that the natural projection $G_{0_2} \to G_{0_1}$ is an injection and that $\dim G_{0_1} = \dim G_{0_2} = 6$. Therefore the natural projection $G_{0_2} \to G_{0_1}$ is a bijection. The projection $\pi_{2,1} : \mathrm{Orb}_{0_2} \to J^1\pi$ is a surjective. This implies the proof. □

For any section S of π, by $\omega_S^{(2)}$ we denote the form $\big(j_2S\big)^*(\omega^{(2)})$.

Theorem 4.6. *The section S can be transformed (locally) to $\mathbf{0}$ by a point transformation iff $\omega_S^{(2)} \equiv 0$.*

Proof. The necessity is obvious.

Prove the sufficiency. To this end, we should prove that the system of PDEs w.r.t. an unknown point transformation f

$$\mathbf{0} = f^{(0)} \circ S \circ f^{-1}$$

has a solution. By $\mathcal{E}(\mathbf{0}, S)$ we denote this system. It easy to prove that the symbol of this PDE system at any point is the same as the subalgebra g defined above by (19). From (22), we obtain that the first prolongation $\mathcal{E}^{(1)}(\mathbf{0}, S)$ of $\mathcal{E}(\mathbf{0}, S)$ has the zero symbol at every point. Therefore $\mathcal{E}^{(1)}(\mathbf{0}, S)$ has a solution if the natural projection $\mathcal{E}^{(2)}(\mathbf{0}, S) \to \mathcal{E}^{(1)}(\mathbf{0}, S)\,,\ [f]_p^4 \mapsto [f]_p^3$, is a surjection (see [8]).

Let us check that this projection is a surjection. Let $[f]_p^3 \in \mathcal{E}^{(1)}(\mathbf{0}, S)$. It takes $[S]_p^1$ to $[\mathbf{0}]_{f(p)}^1$. By assumption, $\omega^{(2)}([S]_p^2) = 0$. It follows from lemma 4.4 that $[S]_p^2 \in \mathrm{Orb}_{0_2}$. Obviously, $[\mathbf{0}]_{f(p)}^2 \in \mathrm{Orb}_{0_2}$ too. Hence there exist a point transformation f' such that its jet $[f']_p^4$ takes $[S]_p^2$ to $[\mathbf{0}]_{f(p)}^2$. This means that $[f']_p^4 \in \mathcal{E}^{(2)}(\mathbf{0}, S)\,,\ \ [f']_p^3 \in \mathcal{E}^{(1)}(\mathbf{0}, S)$, and $[f']_p^3$ takes $[S]_p^1$ to $[\mathbf{0}]_{f(p)}^1$. From the last, we obtain that there exist $g \in G_{[S]_p^1}$ with $[f']_p^3 \cdot g = [f]_p^3$. From lemma 4.5, we get that there exist $g' \in G_{[S]_p^2}$ with $\rho_{4,3}(g') = g$. Obviously, $[f']_p^4 \cdot g' \in \mathcal{E}^{(2)}(\mathbf{0}, S)$ and it is clear that $[f]_p^3$ is the image of $[f']_p^4 \cdot g'$ under the natural projection $\mathcal{E}^{(2)}(\mathbf{0}, S) \to \mathcal{E}^{(1)}(\mathbf{0}, S)$. Thus, this natural projection is a surjection. □

Corollary 4.7. *The form $\omega^{(2)}$ is a unique obstruction to the linearizability of ODEs (1) by point transformations.*

Proof. It is well known that any two 2–order linear ODEs are (locally) equivalent w.r.t. point transformations. This implies the proof. □

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