Equivalence problems for Lagrangians on the line

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Abstract

We review Cartan's method for determining whether two G-structures are locally equivalent. This method is based on the reduction of the structure group to the identity, giving rise to a complete set of local invariants. There are several techniques such as group reduction, prolongation, involutivity test, absorbtion of torsion, and normalization needed in this procedure. Next we introduce the equivalence problem for first-order Lagrangians on the line. We apply this method to the solution of the equivalence problem for the first order Lagrangians under fiber-preserving transformations, point transformations, fiber-preserving transformations up to a divergence, and point transformations up to a divergence. All the local invariants are explicitly computed.

Résumé

Nous passons en revue la méthode de Cartan visant à déterminer si deux G-structures sont localement équivalentes. Cette méthode est basée sun la réduction du groupe structurel à l'identité, donnant naissance à un ensemble complet d'invariants locaux. Diverses techniques telles que la réduction de groupes, la continuation, le test d'involutivité, l'absorbtion de torsion et la normalisation sont nécessaires à cette procédure. Ensuite, nous introduisons le problème d'équivalence pour les Lagrangiens de premier order sur la droite. Nous employons cette méthode à la résolution du problème d'équivalence pour les Lagranigiens de premier ordre par des transformations qui présent les fibres, des transformations de points, des transformations qui préservent les fibres à une divergence près. Tous les invariants locaux sont explicitement Calculés.

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Contents

Introduction

Broadly stated, the equivalence problem of Elie Cartan is to determine if two geometric structures defined by reductions of two frame bundle of a manifold are locally isomorphic. Elie Cartan studied widely this problem and developed a method known as Cartan's equivalence method for solving it [2].

The modern formulation of the Cartan equivalence problem is given in terms of G-structure, Chern [3], Sternberg [12]. Gardner [5] has developed an algorithmic implementation of the method of equivalence through the formulation of the geometric of absorbtion and normalization of torsion which were implicit in Cartan's work. One of Gardner's aims was to apply the method of equivalence to problems of feedback linearization in control theory. Other applications to ordinary differential equations and variational problems can be formed in References [6], [8], and [10].

One of the main goals of Cartan's method is to construct a complete set local invariants of the G-structure. These invariants are then used to derive the necessary and sufficient conditions for local equivalence.

In Chapter 1, we introduce the Cartan equivalence problem. We formulate the structure equations satisfied by the tautological 1-form. The set of solutions of the equivalence problem remains unchanged under group reduction. In order to solve the Cartan equivalence problem, one has to carry out a sequence of reductions and prolongations of the structure group. The involutivity test of Cartan, which experiences the Cartan-Kahler theorem, plays a critical role in the analysis.

In Chapter 2, we introduce the equivalence problem for the Lagrangians on the line. We then apply the method of equivalence to find a complete set of local invariants for first order Lagrangians on the line, following the results of Reference [7]. We explicitly derive the invariants under fiber-preserving transformations, point transformations, and fiber-preserving transformations up to a divergence. We also give the main results on point transformations up to a divergence.

We also have to mention that all the results reviewed in this paper are local and apply to smooth Lagrangians and maps, except for the case where the involutivity test is being applied. This requires the Cartan-Kaehler theorem and is therefore a result about analytic Lagrangirans.

Chapter 1

The Cartan equivalence problem

1.1 Introduction

Elie Cartan developed a method in the early of 20th century which makes it possible to determine the invariants of many geometric structures. This method is called the Method of Equivalence.

1.2 Formulation and solution in the case of $\{e\}$-structures

1.2.1 The coframe bundle

Let M be a smooth n-manifold. A coframe at $x \in M$ is a linear isomorphism $u : T_xM \to \mathbb{R}^n$. The set of such coframes based at x will be denoted by $F_x^*(M)$. (or simply F_x^* when the manifold M is clear from the context.) The disjoint union of F_x^* as x varies on M will be denoted by $F^*(M)$ (or simply F^*), and is called the space of coframes of M. The base point map $\pi : F^* \to M$ is defined by $\pi(F^*x) = x$. The group $GL(n, \mathbb{R})$ acts on F^* on the right by $u.g = g^{-1}u$ for $u \in F^*$ and $g \in GL(n, \mathbb{R})$. Let $U \subset M$ be an open set. Suppose that the η_i's $i = 1, ..., n$ are (smooth) linearly independent 1-forms on U. $\eta = \{\eta_1, ..., \eta_n\}$ is called a (smooth) coframing of U. Now, let $\Phi : U \times GL(n, \mathbb{R}) \to F^*(U)$ be defined by the formula

$$\Phi(x, g) = g^{-1}\eta_x.$$

It is clear that

$$\Phi(x, gh) = h^{-1}\Phi(x, g) = \Phi(x, g)h.$$

The base point map $\pi : F^* \to M$ is then a smooth submersion since if (U, ϕ) is a chart at $u_x \in F^*$ and (V, ψ) is a chart at $\pi(u_x) = x \in M$ then $\psi\pi\phi^{-1}$ is a projection map; on the other hand, if u is a (local) section of F^* then u is a coframing on its domain. Let N be another smooth n-manifold and $f : M \to N$ be a local diffeomorphism. We define a smooth bundle map $f_1 : F^*(M) \to F^*(N)$ by

$$f_1(u) = u \circ (f'(\pi(u)))^{-1}. \tag{1.1}$$

For the diffeomorphism f_1 the following diagram is commutative:

$$\begin{array}{ccc} F^*(M) & \xrightarrow{f_1} & F*(N) \\ \pi\big\downarrow & & \big\downarrow\pi' \\ M & \xrightarrow{f} & N \end{array}$$

1.2.2 G-structures

Let M be an m-manifold, and G be a Lie subgroup of $GL(n, \mathbb{R})$. A (smooth) G-structure on M is a (smooth) submanifold $B \subset F^*(M)$ such that the restricted base point map $\pi : B \to M$ is a surjective submersion whose fibers $B_x = B \cap F_x$ are G-orbits.

Two G-structures $B \subset F^*(M)$ and $\tilde{B} \subset F^*(\tilde{M})$ are said to be equivalent if there exists a (local) diffeomorphism $f : M \to \tilde{M}$ so that $f_1(B) = \tilde{B}$ where f_1 is a (smooth) bundle map defined by (1.1). By the equivalence problem for G-structures we mean the method of determining whether or not two given G-structures are equivalent (and, if so, in how many ways). Before discussing this method, we will illustrate its connections with geometry by the use of several examples of geometric structures described in terms of G-structures.

Definition 1.2.1 ***[Sternberg,*** [12], ***p.*** 312*] A manifold is said to have a complex structure if the coordinates of the atlas* $(U_\alpha, \varphi_\alpha)$ *are complex and the transition functions* $\varphi_\alpha = (\varphi_\alpha^1, ..., \varphi_\alpha^n, \psi_\alpha^1, ..., \psi_\alpha^n)$, $\varphi_\alpha^i, \psi_\alpha^i$ *are complex.*

Definition 1.2.2 ***[Sternberg,***[12], ***p.*** 312*] Structure A is called almost complex if each tangent space is a vector space over* $\mathbb{C}$.

Note that a complex structure is an almost complex structure. However, the converse is not true in general.

Example 1.2.1 ***[Sternberg,***[12], ***p.*** 312*] Suppose now that* $n = 2m$ *and let*

$$J_m = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$$

and define $G \subset GL(2m, \mathbb{R})$ *to be the subgroup of matrices that commute with* J_m. *One can identify* $\mathbb{R}^{2m}$ *with* $\mathbb{C}^m$ *in such a way that* J_m *becomes multiplied by* i *and* G *is then shown to be isomorphic to* $GL(m, \mathbb{C})$.
Suppose now that J *is an almost complex structure on a manifold* M^{2m}*,i.e.,* $J : TM \to TM$ *is a bundle map satisfying* $J^2 = -id$. *Since* m*-dim complex vector spaces are unique up to isomorphism, then the set*

$$B_J = \{u \in F_x^*(M) \mid u(J_x v) = J_m u(\nu) \ for \ all \ \nu \in T_x M\}$$

has the property that each fiber $(B_J)_x$ *is a* $GL(m, \mathbb{C})$*-orbit in* F_x^*. *Moreover, it is not difficult to show that when* J *is smooth, then so is* B_J. *Conversely, given a* $GL(m, \mathbb{C})$*-structure* $B \subset F^*(M)$, *there is a unique almost complex structure* J *for which* $B = B_J$. *Thus, the two kinds of structures are equivalent.*

Definition 1.2.3 ***[Sternberg,*[12]*, p.* 312*]*** *(An almost complex) structure J on a $2m$-manifold M is called a complete first-order invariant if*

i) for another almost complex structure K on a $2m$-manifold N, there exists a diffeomorphism $f : U \longrightarrow N$ defined on an x-neighborhood U which satisfies $f(x) = y$ where $y \in N$, and

ii) $f^*(K_J)$ vanishes to second order at x if and only if there exists a linear isomorphism $L : T_xM \rightarrow T_yN$ which satisfies $L^*(K_y) = J_x$ and $L^*(N_y) = N_x$.

Example 1.2.2 ***[Sternberg,*[12]*, p.* 312*]*** *Again, suppose that $n = 2m$ and let J_m be defined as in the previous example. Now, however, consider the subgroup $Sp(m, \mathbb{R}) \in GL(2m, \mathbb{R})$ consisting of those matrices $A \in GL(2m, \mathbb{R})$ that satisfy $A^t J_m A = J_m$. This group is known as the symplectic group of rank m and is a matrix group of dimension $2m^2 + m$. Given a $Sp(m, \mathbb{R})$-structure B on a $2m$-manifold M, one can define a non-degenerate, 2-form Ω on M by the rule*

$$\Omega(\nu, \omega) = J_m(u(\nu)) \cdot u(\omega) \text{ for all } \nu, \omega \in T_xM, \ u \in B_x.$$

Conversely, the uniqueness up to isomorphism of symplectic vector spaces of a given dimension implies that any non-degenerate 2-form on M corresponds to a unique $Sp(m, \mathbb{R})$-structure via this construction.

The method of equivalence in this case will show that $d\Omega$ is a complete first-order invariant of 2-forms, i.e., if Ω and Υ are non-degenerate 2-forms on $2m$-manifolds M and N respectively, then for given points $x \in M$ and $y \in N$, there exists a local diffeomorphism $f : U \longrightarrow N$ where U is an x-neighborhood satisfying $f(x) = y$ and $f^*\Upsilon - \Omega$ vanishes to second order at x if and only if there exists a linear map $L : T_xM \rightarrow T_yN$ satisfying $L^*(\Upsilon_y) = \Omega_x$ and $L^*(d\Upsilon) = d\Omega_x$.

As the last example in this section, we introduce the G-equivalence problem for the Lagrangians.

Example 1.2.3 *Let G be a Lie subgroup of $GL(3,\mathbb{R})$. In $\mathbb{R}^3$, we take two coordinate systems (x,u,u'), and $(\overline{x},\overline{u},\overline{u}')$. Consider the following variational problems:*

$$I_L[u] = \int_\Omega L(x,u,u')dx, \tag{1.2}$$

$$\overline{I}_{\overline{L}}[\overline{u}] = \int_{\overline{\Omega}} \overline{L}(\overline{x},\overline{u},\overline{u}')d\overline{x}. \tag{1.3}$$

where the Lagrangians are defined on $J^1(\mathbb{R},\mathbb{R})$, and Ω, $\overline{\Omega} \subset \mathbb{R}$, and the dependent variables u and $\overline{u}$ are in $\mathbb{R}$. We define the coframes as

$$\{\omega_U^1 = du - u'dx,\ \omega_U^2 = L(x,u,u')dx,\ \omega_U^3 = du'\},$$

$$\{\overline{\omega}_{\overline{U}}^1 = d\overline{u} - \overline{u}'d\overline{x},\ \overline{\omega}_{\overline{U}}^2 = \overline{L}(\overline{x},\ \overline{u},\ \overline{u}')d\overline{x},\ \overline{\omega}_{\overline{U}}^3 = d\overline{u}'\}.$$

By the equivalence problem of a G-structure, we mean to find a map $f : U \to \overline{U}$ such that $f^*(\overline{I}_{\overline{L}}) = I_L$. In terms of coframes, this amounts to determining

$$f^*\tilde{\omega}_{\overline{U}}^i = \sum_j \gamma_j^i \omega_U^j, \tag{1.4}$$

where $\gamma_j^i \in G$.

We consider the product $U \times G$ and the vector valued 1-form defined by $\omega = g\theta$ on $U \times G$

$$\omega_{(x,g)} = g\theta_x, \quad \forall x \in U, \quad \forall g \in G, \tag{1.5}$$

where θ_x is a column vector $(\theta_x^1, ..., \theta_x^n)^t$. G acts on $U \times G$ on the left by the action defined by

$$h(x,g) = (x, hf), \quad \forall x \in U, \quad \forall g, h \in G.$$

Theorem 1.2.1 *[**Kamran,** [6], **p.** 31, **Prop.**3.1] A diffeomorphism $f : U \to \tilde{U}$ satisfies (1.4) if and only if there exists a diffeomorphism $F : U \times G \to \tilde{U} \times G$ satisfying*

i) $F^*\tilde{\omega} = \omega$.

ii) the following diagram commutes:

$$\begin{array}{ccc} U \times G & \xrightarrow{F} & \tilde{U} \times G \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & \tilde{U} \end{array}$$

iii) $F(x, gh) = gF(x, h)$, *for each* $x \in U$, *and* $g, h \in G$.

Proof Suppose that f satisfies $f^*\tilde{\theta} = g_0\theta$, where g_0 is a G-valued function on M. We define $F : U \times G \to \tilde{U} \times G$ by $F(x, g) = (f(x), gg_0^{-1})$.

Then F satisfies $ii)$ and $iii)$. Moreover,

$$F^*\tilde{\omega} = F^*(\tilde{\theta}\tilde{\omega}) = (gg_0^{-1})f^*\tilde{\theta} = (gg_0^{-1})g_0\theta = g\theta = \omega.$$

Conversely, suppose that $F : U \times G \to \tilde{U} \times G$ satisfies $i) - iii)$. We define $f : U \to \tilde{U}$ and $g_0 : U \to G$ by $F(x; e) = (f(x), g_0^{-1}(x))$, where e is the identity of G. Then $F(x, g) = gF(x, e) = (f(x), gg_0^{-1})$, and $i)$ implies that

$$g\theta = F^*(\tilde{\theta}\tilde{\omega}) = (gg_0^{-1})f^*\tilde{\theta}.$$

Therefore, $f^*\tilde{\theta} = g_0\theta$.

Applying d to (1.5), we obtain

$$d\omega = dg \wedge \theta + gd\theta.$$

By substituting $\theta = g^{-1} \cdot \omega$, we obtain

$$d\omega = dgg^{-1} \wedge \omega + gd\theta. \tag{1.6}$$

■

Definition 1.2.4 *[**Sternberg,** [12], **p.** 312] Let $G = \{e\}$ be the identity matrix in $GL(n, \mathbb{R})$. An $\{e\}$-structure on M is a submersion $B \subset F^*(M)$ that intersects each fiber at one point and projects onto M.*

Hence, B is simply the image of a smooth global section of $F^*(M)$, i.e., a coframing $\eta = \{\eta_i \ , \ i = 1, ..., n\}$ of M. Thus, an $\{e\}$-structure can be identified with a global coframing of M.

1.2.3 The tautological 1-form

For the coframe bundle F^* we can assign an $\mathbb{R}^n$-valued 1-form which has some important functional properties. This vector-valued 1-form is known as the tautological 1-form or the canonical 1-form. [**Sternberg,** [12], **p.** 294]

Definition 1.2.5 *Suppose $B \subset F^*(M)$ is a G-structure and u is a coframe. A tautological 1-form ω is defined by*

$$\omega(\nu) = u \circ \pi'((u)(v)) \quad \textit{for all } \nu \in T_u B. \tag{1.7}$$

On the other hand, a tautological 1-form ω is defined by the composition of the maps

$$\begin{array}{ccc} T_u B & & \\ {\scriptstyle \pi'(u)} \downarrow & & \\ T_{\pi(u)} M & \xrightarrow{\ u\ } & \mathbb{R}^n \end{array}$$

Let η be a local section of G-structure B. Suppose $\Phi : U \times G \to B$ is defined by

$$\Phi(x, g) = g^{-1}\eta_x, \qquad \text{where} \qquad U \subset M. \tag{1.8}$$

Therefore Φ^* is defined on TB by

$$\Phi^*(\omega) = g^{-1}\eta, \quad \text{where} \quad \omega = \{\omega_1, ..., \omega_n\}. \tag{1.9}$$

Note that the components ω_i's are linearly independent, and their kernel is the vector v that is tangent to the π-fibers of B. So $\eta^*(\omega) = \eta$ for any local section η of B. The following theorem shows that by having the tautological 1-form, one can determine how the method of equivalence works.

Theorem 1.2.2 ***[Sternberg,** [12], **pp.** 313 − 314]* *Let B_1 and B_2 be G-structures over manifolds M_1 and M_2, respectively. If $f : M_1 \longrightarrow M_2$ is a diffeomorphism satisfying $f_1^*(B_1) = B_2$, then $f_1^*(\omega_2) = \omega_1$.*

Proof The proof is achieved by applying the chain rule. The following diagram of maps is commutative.

$$\begin{array}{ccc} B_1 & \xrightarrow{f_1} & B_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

By starting with a vector $\nu \in T_u B$, from (1.7) we have

$$\begin{aligned} f_1^*(\omega_2)(v) &= \omega_2(f_1'(u))(v) \\ &= f_1(u)(\pi_2'(f_1(u))(f_1'(u)(v))) = f_1(u)((\pi_2 \circ f_1)(u)(v)) \\ &= f_1(u)((f \circ \pi_1)(u)(v)) = f_1(u)(f'(\pi_1(u))(\pi_1'(u)(v))) \\ &= u((f'(\pi_1(u)))^{-1} f'(\pi_1(u))(\pi_1'(u)(v))) = u((\pi_1'(u)(v))) \\ &= \omega_1(v). \end{aligned}$$

■

Now we are able to verify whether or not two G-structures B_1 and B_2 are locally equivalent by looking for integral manifolds of the ideal generated by the 1-form $\theta = \pi_1^*\omega_1 - \pi_2^*\omega_2$ on the product manifold $B_1 \times B_2$.

If we can find such an integral manifold $\Gamma \subset B_1 \times B_2$ that projects (diffeomorphically) onto each of the factors, then it will be the graph of a smooth map $g : B_1 \longrightarrow B_2$ that satisfies $g^*(\omega_2) = \omega_1$. Therefore, by the previous theorem, the diffeomorphism $f : M_1 \longrightarrow M_2$ induces an equivalence between the two G-structures.

1.3 Structure equations

The purpose of this section is to show how to find a coframing on a G-structure B. Let V be an n-vector space with the standard basis $\mathbf{v}_i$, $(i = 1, ..., n)$. We denote the basis elements of V^* by $\mathbf{v}^i$'s. The tautological 1-form ω on $F^*(M)$ can be represented as a vector-valued 1-form

$$\omega = \omega^i \mathbf{v}_i.$$

where the ω^i's are 1-forms on $F^*(M)$.

Let G be an s-dimensional Lie group and $g \subset gl(n, \mathbb{R}) = V \otimes V^*$ be the corresponding Lie algebra. Let u_α, $1 \leq \alpha \leq s$ denote the basis elements of the Lie algebra g. By canonical inclusion $g \hookrightarrow V \otimes V^*$, each u_α can be written as

$$u_\alpha = u^i_{\alpha_j} \mathbf{v}_i \otimes \mathbf{v}^j.$$

Let $\eta = \eta^i \mathrm{v}_i$ be a local section of B with domain $U \subset M$. There are unique functions $T^i_{jk} = -T^i_{kj}$ such that

$$d\eta^i = \frac{1}{2} T^i_{jk} \eta^j \wedge \eta^k. \tag{1.10}$$

By writing

$$T = \frac{1}{2} T^i_{jk} \mathrm{v}_i \otimes \mathrm{v}^j \wedge \mathrm{v}^k, \tag{1.11}$$

and by considering T as a function from U to $V \otimes \Lambda^2(V^*)$, this can be written as a vector equation in the form

$$d\eta = \frac{1}{2} T(\eta \wedge \eta). \tag{1.12}$$

Consider $\Phi : U \times G \to B$ defined by (1.8). Let θ be a connection on B, i.e. there exists a g-valued 1-form θ_0 on U such that $\Phi^*(\theta) = g^{-1} dg + g^{-1} \theta_0 g$. On the other hand, according to equation (1.9) we have $\Phi^*(\omega) = g^{-1} \eta$.

Hence

$$\begin{aligned}\Phi^*(d\omega) &= d(g^{-1}\eta) = -g^{-1}dg \wedge g^{-1}\eta + g^{-1}d\eta = -g^{-1}dg \wedge g^{-1}\eta + \frac{1}{2}g^{-1}T(\eta \wedge \eta) \\ &= (-g^{-1}dg - g^{-1}\theta_0 g) \wedge (g^{-1}\eta) + g^{-1}(\theta_0 \wedge \eta) + \frac{1}{2}g^{-1}T(\eta \wedge \eta) \\ &= \Phi^*(-\theta \wedge \omega) + g^{-1}(\theta_0 \wedge \eta + \frac{1}{2}T(\eta \wedge \eta)) \\ &= \Phi^*(-\theta \wedge \omega + \frac{1}{2}T(\omega \wedge \omega)).\end{aligned}$$

Therefore, the first structure equation of Elie Cartan holds

$$d\omega = -\theta \wedge \omega + \frac{1}{2}T(\omega \wedge \omega).$$

The function $T = \frac{1}{2}T^i_{jk}\mathrm{v}_i \otimes \mathrm{v}^j \wedge \mathrm{v}^k$ is called the torsion function of the connection θ. We can also represent the first structure equation by

$$d\omega^i = \sum_j \sum_k a^i_{jk}\theta^k \wedge \omega^j + \frac{1}{2}\sum_j \sum_l \gamma^i_{jl}\omega^j \wedge \omega^l. \tag{1.13}$$

Now consider the effect on T of changing the connection. For a connection θ we have $\theta(X_v) = v$ for any $v \in g$ (Here, X_v denotes the vector field on B). Now, let θ^* be another connection on B. The difference $\theta^* - \theta$ is a g-valued 1-form on B that vanishes on vectors tangent to the fibers of $\pi : B \to M$. Hence, there exists a unique function $p : B \to \mathrm{g} \otimes V^*$ so that $\theta^* = \theta + p(\omega)$.

On the other hand, for any G-equivalent $p : B \to \mathrm{g} \otimes V^*$ and any connection 1-form θ, the formula $\theta^* = \theta + p(\omega)$ defines a connection 1-form on B.

Therefore

$$d\omega = -\theta \wedge \omega + \frac{1}{2}T(\omega \wedge \omega) = -\theta^* \wedge \omega + \frac{1}{2}T^*(\omega \wedge \omega).$$

where T^* is the torsion function associated to θ^*. It follows that

$$\frac{1}{2}(T^* - T)(\omega \wedge \omega) = (\theta^* - \theta) \wedge \omega = p(\omega) \wedge \omega = \frac{1}{2}\delta(p)(\omega \wedge \omega),$$

where

$$\delta : \mathrm{g} \otimes V^* \to V \otimes \Lambda^2(V^*) \tag{1.14}$$

is the G-equivalence linear map defined as the composition

$$g \otimes V^* \to (V \otimes V^*) \otimes V^* \to V \otimes \Lambda^2(V^*), \tag{1.15}$$

where the first map is the tensor product with V^* of the inclusion $\mathrm{g} \hookrightarrow V \otimes V^*$ and the second map is skew symmetrization of the second two factors.
Therefore

$$T^* = T - \delta(p).$$

This formula necessitates studying the kernel and co-kernel of the map δ. These spaces have special notations and names

$$ker\delta = \mathrm{g}^{(1)} \quad and \quad coker\delta = \Pi_{\mathrm{g}}. \tag{1.16}$$

The space $\mathrm{g}^{(1)}$ is known as the first prolongation of g and the space Π_{g} is known as the intrinsic torsion space of g.

1.4 Reduction of the structure group

Now Consider the exact sequence constructed by (1.15) and (1.16)

$$0 \to \mathrm{g}^{(1)} \to \mathrm{g} \otimes V^* \to V \otimes \Lambda^2(V^*) \to \Pi_{\mathrm{g}} \to 0 \tag{1.17}$$

Definition 1.4.1 *[**Kobayashi,** [8]**, p.** 16] The kernel* $\mathrm{g}^{(1)}$ *is called the first prolongation of the Lie algebra* g.

Let T be a function from U to $V \otimes \Lambda^2(V^*)$ as defined in (1.11). We define *the structure tensor* τ_U as the composition of the project map $pr : U \times G \to U$ and T. Therefore

$$\tau_U = T \circ pr. \tag{1.18}$$

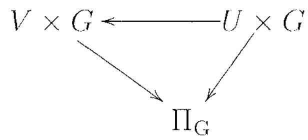
Therefore, $\tau(x, gh) = \tilde{\rho}(g)\tau(x, h)$, where $\tilde{\rho}$ is the G-action in Π_g. Now we define

$$G_{(1)} = \{g \in G \mid \tilde{\rho}(g)\tau_{(1)} = \tau_{(1)}\}.$$

as the stability group of $\tau_{(1)}$ for a fixed vector $\tau_{(1)} \in \tau_U(U \times G)$.

The following theorem gives us a necessary condition for solving the G-equivalence problem (1.4).

Theorem 1.4.1 ***[Sternberg,*** **[12],** ***p.*** *328] For the equivalence problem (1.4) the following diagram is commutative.*



In practice one computes τ_U and τ_V by choosing an equivariant splitting of the sequence (1.17). This procedure is known as absorbtion of torsion and is described in the following.

Absorbtion of torsion

Consider the coframe

$$\omega = \{\omega_1, ..., \omega_n\},$$

and the corresponding G-equivalence problem. To solve the equivalence problem, we introduced the lifted coframe

$$\theta_i = \sum_{j=1}^{m} g_j^i \omega_j, \quad i = 1, ..., n.$$

We have

$$d\theta_i = \sum_{j=1}^{m} (dg_j^i \wedge \omega_j + g_i^j d\omega_j).$$

By rewriting the above formula in terms of θ_i's we obtain

$$d\theta_i = \sum_{j=1}^{m} \gamma_j^i \wedge \theta_j + \sum_{j=1}^{m} \sum_{k=1}^{m} \tau_{jk}^i \theta_j \wedge \theta_k, \qquad i = 1, ..., m.$$

Note that the torsion coefficients, γ_j^i, are defined by the structure group G which is evaluated by

$$\gamma_j^i = \sum_{k=1}^{s} a_{jk}^i \alpha_k.$$

Rewriting the structure equations with respect to Maurer-Cartan forms,

$$d\theta_i = \sum_{j=1}^{m}\sum_{k=1}^{s} a^i_{jk}\alpha_k \wedge \theta_j + \sum_{j=1}^{m}\sum_{k=1}^{m} \tau^i_{jk}\theta_j \wedge \theta_k, \qquad i, j = 1, .., m.$$

We recall that we are ultimately looking for 1-forms that are invariant under the action of the diffeomorphism $\Phi : J^1 \longrightarrow J^1$ such that $\Phi^*(\overline{L}) = L$. Suppose the coframe corresponding to $\overline{L}$ is defined by $\overline{\omega}$ and the corresponded lifted coframe is $\overline{\theta} = \overline{g}.\overline{\omega}$, where $\overline{g} \in \overline{G}$

The structure equations corresponding to the equivalence problem are given by

$$d\overline{\theta}_i = \sum_{j=1}^{m} \overline{\gamma}^i_j \wedge \overline{\theta}_j + \sum_{j=1}^{m}\sum_{k=1}^{m} \overline{\tau}^i_{jk}\theta_j \wedge \theta_k, \qquad i = 1, ..., m.$$

Now, in order to solve the G-equivalence problem, we reduce the problem to the $\{e\}$-equivalence case. This is done by absorbtion and normalization.

By absorbtion of torsion, we eliminate as many of the torsion coefficients as possible by replacing the Maurer-Cartan coefficients α_k by

$$\alpha_k + \sum_{k=1}^{s} \sigma^k_j \theta_j.$$

However, there might still be some un-normalized non-invariant components left. We will repeat the absorbtion-normalization process for the new coframe rebuilt by the previous absorbtion-normalization process. Two cases might occur.

In the first case, we would be able to eliminate all torsion components, and all group parameters would then be determined; this means that we could reduce the problem to the $\{e\}$-equivalence case. Then we could easily find the invariants. In the second case, there would be some torsion coefficients remaining which are acted on trivially by the structure group. We would then need to prolong the group structure and at each step of prolongation, apply the absorbtion and normalization method.

We now return to the main problem. Suppose $F : U \to \mathbb{R}$. We define the *covariant derivatives* $F_{,i}$ of function F by

$$dF = \sum_i F_{,i}\omega^i_U. \tag{1.19}$$

By equation (1.10) we stated that

$$d\omega^i_U = \frac{1}{2}T^i_{jk}(x^1, ..., x^n)\omega^j_U \wedge \omega^k_U, \qquad (1.20)$$

$$d\overline{\omega}^i_{\overline{U}} = \frac{1}{2}\overline{T}^i_{jk}(\overline{x}^1, ..., \overline{x}^n)\overline{\omega}^j_{\overline{U}} \wedge \overline{\omega}^k_{\overline{U}}. \qquad (1.21)$$

Therefore

$$f^*(\overline{T}^i_{jk}\overline{\omega}^j_{\overline{U}} \wedge \overline{\omega}^k_{\overline{U}}) = f^*(\overline{T}^i_{jk})f^*(\overline{\omega}^j_{\overline{U}}) \wedge f^*(\overline{\omega}^k_{\overline{U}}) = f^*(\overline{T}^i_{jk})\omega^j_U \wedge \omega^k_U.$$

Moreover

$$f^*(d\overline{\omega}^i_{\overline{U}}) = df^*(\overline{\omega}^i_{\overline{U}}) = d\omega^i_U.$$

Therefore $f^*(\overline{T}^i_{jk}) = T^i_{jk}$. Now, we define $T^i_{jk,l} = X_{\omega^l}T^i_{jk}$, where $\omega^k(X^l_\omega) = \delta^k_l$. Therefore

$$dT^i_{jk} = T^i_{jk,l}\omega^l_U,$$
$$d\overline{T}^i_{jk,l} = \overline{T}^i_{jk,l}\overline{\omega}^l_{\overline{U}}.$$

The same argument will show that

$$f^*(\overline{T}^i_{jk,l}) = T^i_{jk,l},$$

and in general

$$f^*(\overline{T}^i_{jk,l}) = T^i_{jk,l_1,...,l_p} \quad \text{where } 1 \le i, j, k, l_1, ..., l_p \le n.$$

Now, suppose $\{I_1, ...I_{l_p}\}$ is a maximal functionally independent set of functions such that for any linear combination of $T^i_{jk}, T^i_{jk,l_1}, ..., T^i_{jk,l_p}$, say F, we have $F = F(I_1, ...I_{l_p})$. Therefore either $l_p = n$ or $l_p = l_{p+1}$. Similarly, we can get the same maximal independent set of $\{\overline{I}_1, ..., \overline{I}_{l_p}\}$ corresponding to the other coframe. By the implicit function theorem, $X_{\omega^k}(I_l) = I_{a,k} = F(I_1, ..., I_{l_p})$. Likewise, $X_{\overline{\omega}^k}(\overline{I}_l) = \overline{I}_{a,k} = \overline{F}(\overline{I}_1, ..., \overline{I}_{l_p})$.

Theorem 1.4.2 *(Cartan's theorem)[**Sternberg,** [12]**, p.** 344] Let $(p, q) \in \Gamma(f)$, the graph of a diffeomorphism $f : U \to V$. The formula*

$$f^*\overline{\omega}^i_V = \omega^i_U \qquad (1.22)$$

is satisfied if and only if

i) Ranks and labels match, i.e. $l = \bar{l}$.

ii) $I_i(p) = \bar{I}_i(q)$ *for* $1 \leq i \leq l_p$.

iii) F *and* $\overline{F}$ *are the same functions of their respective arguments; i.e.* $F(I_1, ..., I_{l_p}) = \overline{F}(\bar{I}_1, ..., \bar{I}_{l_p})$.

We now state a key theorem which describes the above concept precisely.

Theorem 1.4.3 ***[Gardner,** [5], **p.** 38]* *The Cartan equivalence problems*

$$f^* \tilde{\omega}_V^i = \sum_j \gamma_j^i \omega_U^j,$$

where $(\gamma_j^i) \in G$, *and*

$$f^{*(1)} \tilde{\omega}_V^{i(1)} = \sum_j \gamma_j^{i(1)} \omega_U^{j(1)}.$$

where $(\gamma_j^{i(1)}) \in G_{(1)}$, *have the same solution.*

Proof Define the G-valued diffeomorphisms $\sigma_U^{(1)} : U \to G$ and $\sigma_V^{(1)} : V \to G$ such that

$$\sum_j \sigma_{U_j}^{i(1)} \omega_U^j = \omega_U^{i(1)}, \quad \text{and} \quad \sum_j \sigma_{V_j}^{i(1)} \tilde{\omega}_V^j = \tilde{\omega}_V^{i(1)},$$

respectively. Now,

$$\begin{aligned} f^{*(1)} \tilde{\omega}_V^{i(1)} &= f^{*(1)}(\sum_j \sigma_{V_j}^{i(1)} \tilde{\omega}_V^j) = f^{*(1)}(\sum_j \sigma_{V_j}^{i(1)} f^{*^{-1}}(\sum_k \gamma_k^j \omega_U^k)) \\ &= \sum_j ((\sigma_{V_j}^{i(1)} \circ f_j^i) \sum_k \gamma_k^j \omega_U^k)) = \sum_j ((\sigma_{V_j}^{i(1)} \circ f_j^i) \sum_k \gamma_k^j \sum_l \sigma_{U_l}^{k(1)^{-1}} \omega_U^{l(1)}) \\ &= \sum_j \sum_k \sum_l ((\sigma_{V_j}^{i(1)} \circ f_j^i) \gamma_k^j \sigma_{U_l}^{k(1)^{-1}}) \omega_U^{l(1)}. \end{aligned}$$

Hence in order for f to be a solution of the second equivalence problem, we need to show that

$$\sum_k \sum_l ((\sigma_{V_j}^{i(1)} \circ f_j^i) \gamma_k^j \sigma_{U_l}^{k(1)^{-1}})$$

is a $G^{(1)}$-valued function.

Note that for all $x \in U$, $\tau_U(x, \sigma_U^{(1)}(x))$ is in $\tau_U(U \times G)$. On the other hand, by (1.18),

$$\begin{aligned}
\tau_{(1)} &= \tau_U(x, \sigma_U^{(1)}(x)) = (\tau_V \circ \tilde{f})(x, \sigma_U^{(1)}(x)) = \tau_V(f(x), \sigma_U^{(1)}(x).\gamma(x)^{-1}) \\
&= \tau_V(f(x), \sigma_U^{(1)}(x).\gamma(x)^{-1}.(\sigma_V^{(1)} \circ f)^{-1}(x).(\sigma_V^{(1)} \circ f)(x)) \\
&= \tilde{\rho}(\sigma_U^{(1)}(x).\gamma(x)^{-1}.(\sigma_V^{(1)} \circ f)^{-1}(x)).\tau_V(f(x), (\sigma_V^{(1)} \circ f)(x)) \\
&= \tilde{\rho}(\sigma_U^{(1)}(x).\gamma(x)^{-1}.(\sigma_V^{(1)} \circ f)^{-1}(x))\tau_{(1)}.
\end{aligned}$$

■

We can now apply the above theorems to show that the G-equivalence problem given by (1.22) can be reduced to the $G_{(1)}$-equivalence problem such that both have the same solution.

The process of reducing an equivalence problem to a new one by reducing the structure group G to its stability group $G_{(1)}$ is called a *group reduction.*

Note that in the case of $\{e\}$-structures, the equivalence problem reduces to finding diffeomorphisms $f : U \to V$ such that

$$f^* \tilde{\omega}_V^i = \omega_U^i. \tag{1.23}$$

Theorem (1.4.3) in fact claims that we can reduce the initial G-equivalence problem to the $G_{(k)}$-equivalence problem by induction.

$$G_{(k)} \subset G_{(k-1)} \subset ...G_{(1)} \subset G$$

We keep reducing $G_{(k)}$ until either $G_{(k)} = \{e\}$ or $G_{(k)} \neq \{e\}$, though the $G_{(k)}$-action on $\Pi_{\mathfrak{g}_{(k)}}$ is trivial. In the first case, we can solve the $\{e\}$-equivalence problem by using theorem 1.4.2.

1.5 Prolongation and involutivity test

The other case is when $G_{(k)} \neq \{e\}$, but the action is trivial.

Now we define the first prolonged group.

$$G^{(1)}_{(k)} = \{ \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ v & \mathbf{1} \end{pmatrix} \mid v \in ker\delta\}$$

Recalling equation (1.16), $G^{(1)}_{(k)}$ is the first prolongation of the stability group $G_{(k)}$. Consider the tableau matrix [5] taken from the equation (1.13)

$$\Sigma = (\sum_k a^i_{jk}\theta_k).$$

We define the reduced Cartan characters [5] $\sigma_1, ..., \sigma_k$ by

$$\sigma_p = \dim \Sigma_p,$$

where

$$\Sigma_p = \Sigma \cap \text{span}(\{\theta_{p+1}, ..., \theta_n\} \otimes \Lambda^2(V^*)).$$

Theorem 1.5.1 *(Cartan's involutivity test)[**Gardner,** [5]**, p.** 74] The system is in involution if and only if*

$$dim\ \mathfrak{g}^{(1)}_{(k)} = \sum_{p=1}^{n} \sigma_p.$$

Theorem 1.5.2 *[**Kamran,** [6]**, p.** 47**, Prop.**3.8] The Cartan equivalence problems*

$$f^*\tilde{\omega}^i_V = \sum_j \gamma^i_j \omega^j_U,$$

where $(\gamma^i_j) \in G$, and

$$\widehat{f}^{*(k)} \tilde{\omega}^{i(k)}_{V\times G_{(k)}} = \widehat{\gamma}^{i(k)}_j \omega^{j(k)}_{U\times G_{(k)}},$$

where $(\widehat{\gamma}^{i(k)}_j) \in G^{(1)}_{(k)}$, have the same solution.

On the other hand the solution of the initial equivalence problem and the equivalence problem of the first prolonged group reduction are the same.

1.6 Conclusion

We saw that the solution of the initial equivalence problem is the same as the solution of the $G_{(k)}$-equivalence problem. We also showed that if $G_{(k)} = \{e\}$, then by theorem (1.4.2) we can easily find the solution. On the other hand, if $G_{(k)} \neq \{e\}$, the group action is trivial, and the involutivity condition is not satisfied, we must the prolong the system once to obtain the prolonged group $G^{(1)}_{(k)}$. By theorem (1.5.2) we know that the solutions of the initial problems are the same as those of the $G^{(1)}_{(k)}$-equivalence problems. At this point we reduce the first prolonged group to $G^{(1)}_{(k)(l)}$. Again, if $G^{(1)}_{(k)(l)} = \{e\}$, we can easily find the solutions. If $G^{(1)}_{(k)(l)} \neq \{e\}$, the group action is trivial, and the involutivity condition is not satisfied, we must then prolong the group for the second time and we repeat the above algorithm. According to the theorem Cartan-Kuranishi [1], we should obtain an involutive system after a finite number of prolongations and reductions.

Theorem 1.6.1 *[**Kamran,** [6], **p.** 46, **Prop.**3.7] If the inolutivity condition is satisfied and the coefficients a^i_{jp} and $T^i_j k$ are constant then the set of self-equivalence forms a Lie pseudogroup of infinite type.*

In this chapter, Cartan's method was introduced to find the solutions of the equivalence problem. In the next chapter we apply Cartan's equivalence method to find the solutions of a classical equivalence problem in Calculus of variations, first order Lagrangians on the line. Example (1.2.3) expressed the G-structure of the Lagrangians equivalence problem; in the beginning of chapter 2, the equivalence problem for first order Lagrangians on the line will be derived. Later on in the chapter we will show how Cartan's method gives rise to a solution to the Lagrangian equivalence problem.

Chapter 2

Equivalence of first order Lagrangian on the line

2.1 Introduction

Given functionals $I_L[u]$, and $\bar{I}_{\bar{L}}[\bar{u}]$ corresponding to the Lagrangians, the principal question in equivalence of first order Lagrangians on the line is to know if there exists a local diffeomorphism $\Phi : J^n \to J^n$ such that

$$\Phi^*(\bar{I}_{\bar{L}}[\bar{u}]) = I_L[u]. \tag{2.1}$$

Equivalently, given ω_U^j and $\bar{\omega}_U^i$ we would like to know if there exists any Φ such that

$$\Phi^*(\bar{\omega}_U^i) = \sum_j \gamma_j^i \omega_U^j, \tag{2.2}$$

where

$$\gamma_j^i : U \to GL(n, \mathbb{R}).$$

We consider the special case of equivalence of first-order Lagrangians on the line, i.e.

$$I_L[u] = \int_\Omega L(x, u^n) dx,$$

where $n = 1$, $u \in \mathbb{R}$, and $\Omega \subseteq \mathbb{R}$. Hence the problem reduces to

$$I_L[u] = \int_{\mathbb{R}} L(x, u, u')dx. \tag{2.3}$$

Definition 2.1.1 ***[Kamran, and Olver,*** **[7],** ***pp 33-34]*** *There exists at least six versions of the equivalence problem for the Lagrangians on the line.*

1.Standard fiber preserving map:

In this case, the two Lagrangians are related by

$$\bar{L} = \frac{L}{detJ},$$

where J is the Jacobian of the matrix $D_i\varphi^j$, and $\bar{x} = \varphi(x), \bar{u} = \psi(x.u)$.

2. Standard general point transformations:

In this case, the relation between the two Lagrangians is the same as in the previous case; but x and u are transformed by

$$\bar{x} = \varphi(x, u), \quad \bar{u} = \psi(x, u).$$

3.Standard contact transformations:

In this case,

$$\bar{L} = \frac{L}{detJ},$$

and $\bar{x} = \varphi(x, u, u')$, $\bar{u} = \psi(x, u, u')$, $\bar{u}' = \chi(x, u, u')$

4. Divergence equivalence problem for fiber-preserving map:

In this case,

$$\bar{L} = \frac{L + divF}{detJ}$$

and $\bar{x} = \varphi(x)$, $\bar{u} = \psi(x, u)$.

5. Divergence equivalence for point transformations:

In this case,

$$\bar{L} = \frac{L + divF}{detJ},$$

and $\bar{x} = \varphi(x, u)$, $\bar{u} = \psi(x, u)$.

6. Divergence equivalence for contact transformations:

In this case,

$$\bar{L} = \frac{L + divF}{detJ},$$

and $\bar{x} = \varphi(x, u, u'), \bar{u} = \psi(x, u, u'), \bar{u}' = \chi(x, u, u')$

Theorem 2.1.1 ***[Kamran, and Olver,*** [7]***, p.*** 43***, Prop.***3.5*]* L *and* $\bar{L}$ *have the same Euler-Lagrange equations if and only if*

$$\bar{L} = \frac{L + divF}{detJ}.$$

In the divergence equivalence problem, we are dealing with the Euler-Lagrange equations of the two Lagrangians instead of the Lagrangians, themselves.

Theorem 2.1.2 *For the first order Lagrangian equivalence problem on the line, any two Lagrangians* L *and* $\bar{L}$ *are divergence equivalent under contact transformations.*

Theorem 2.1.3 *For the standard equivalence problem, contact and point transformations are the same.*

Definition 2.1.2 ***[Kamran, and Olver,*** [7]***, p.*** 35***, Def.***1.1*]* *The Lagrangians* L *and* $\bar{L}$ *are* m*-equivalent if* L *is mapped to* $\bar{L}$ *by a*

$m = 1$: *standard fiber-preserving map.*

$m = 2$: *standard contact transformations.*

$m = 3$: *fiber-preserving map up to divergence.*

$m = 4$: *point transformations up to divergence.*

2.2 Formulation of various notions of equivalence

Consider the basic coframe $\{\omega_1,\ \omega_2,\ \omega_3,\ \omega_4\}$ defined by

$$\omega_1 = du - pdx,\quad \omega_2 = Ldx,\quad \omega_3 = dp,\quad \omega_4 = dw.$$

We define [**Kamran, and Olver,**[7], **p.** 41] four Lie subgroups G_m, $m = 1, 2, 3, 4$ of $GL(3, \mathbb{R})$ and $GL(4, \mathbb{R})$ by

$$G_1 = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \\ a_4 & a_5 & a_6 \end{pmatrix} : a_i \in \mathbb{R},\ a_1.a_6 \neq 0 \right\},$$

$$G_2 = \left\{ \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 1 & 0 \\ b_4 & b_5 & b_6 \end{pmatrix} : b_i \in \mathbb{R},\ b_1.b_6 \neq 0 \right\},$$

$$G_3 = \left\{ \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_3 & 0 & 0 \\ c_4 & c_5 & c_6 & 0 \\ c_7 & c_3 - 1 & 0 & 1 \end{pmatrix} : c_i \in \mathbb{R},\ c_1.c_3.c_6 \neq 0 \right\},$$

$$G_4 = \left\{ \begin{pmatrix} d_1 & 0 & 0 & 0 \\ d_2 & d_3 & 0 & 0 \\ d_4 & d_5 & d_6 & 0 \\ d_7 & d_3 - 1 & 0 & 1 \end{pmatrix} : d_i \in \mathbb{R},\ d_1.d_3.d_6 \neq 0 \right\}.$$

By the following theorem, we obtain the relationships between the equivalence of two Lagrangians and the Lie groups G_m.

Theorem 2.2.1 *[**Kamran, and Olver,** [7], **p.** 42, **Prop.**2.1] There exists a diffeomorphism*

$$\Phi : J^1 \times \mathbb{R} \to J^1 \times \mathbb{R}$$

such that it satisfies (2.2) if and only if L and $\overline{L}$ are equivalent.

2.3 Inductive method

Before starting to explain this method, we should mention that solving the Lagrangian equivalence problem in general is not very easy. The task becomes easier if we use the solutions of the simpler case to attack the more complicated one. Suppose we have two equivalence problems with the same coframe ω.

Let G, and H corresponding to the equivalence problems, be two different structure groups such that $G \subset H$. Suppose that we could solve the G-equivalence problem. The inductive method suggests that instead of solving the H-equivalence problem, which is more complicated, we use the solution of the G-equivalence problem directly. Therefore, instead of solving the G-equivalence problem, by determining a map $g_0 : U \to G$ such that

$$\theta = g_0.\omega, \tag{2.4}$$

we use the adapted coframe $\eta = h.g_0.\omega$ as the starting point.

2.4 The fiber preserving case

We consider the coordinates (x, u, u') on $\mathbb{R}^3$ (or $J^1(\mathbb{R}, \mathbb{R})$). The basic coframe on $\Omega \times \mathbb{R}$ is given by

$$\omega_U^1 = du - pdx, \quad \omega_U^2 = Ldx, \quad \omega_U^3 = dp. \tag{2.5}$$

For our case, the standard fiber-preserving and point transformation equivalence problems have the same coframe ω given in (2.5). The appropriate $g_0 \in G$ for solving the G_1-equivalence problem is given by G_1 from the previous page. Now, we can easily use this solution to derive the formulas for the invariants of the point transformation problem. To do this, it is enough to assume that the lifted coframe of the G_2-equivalence problem has the form

$$\eta = h.g_0.\omega_0.$$

By absorbtion and normalization of torsion we can then explicitly find the parameters of the lifted coframe η. For the cases G_3, and G_4, the dimension of the coframes is 4. So, we can add a component to the basic coframe ω of the standard fiber-preserving equivalence problem so as to obtain a similar coframe for the G_1, G_3, and G_4-equivalence problems since $G_1 \subset G_3$ and $G_3 \subset G_4$. We can then apply the method of induction to the case to find the adapted coframe which leads us to find the invariants of the structure equations.

Standard Fiber Preserving Equivalence Problem:

This is the most elementary case. Let ω_1, ω_2, ω_3 denote the base frame and be given by

$\omega_1 = du - pdx, \omega_2 = Ldx, \omega_3 = dp$.

Now, we define new variables

$$\begin{aligned} \xi_1 &= a_1\omega_1 = a_1 du - a_1 p dx, \\ \xi_2 &= \omega_2 = Ldx, \\ \xi_3 &= a_4\omega_1 + a_5\omega_2 + a_6\omega_3 = a_4 du - a_4 p dx - a_5 L dx + a_6 dp. \end{aligned} \tag{2.6}$$

By rewriting the above equations with respect to ω_i's, we obtain

$$\begin{aligned} \omega_1 &= \tfrac{1}{a_1}\xi_1, \\ \omega_2 &= \xi_2, \\ \omega_3 &= \tfrac{1}{a_6}\xi_3 - \tfrac{a_4}{a_1 a_6}\xi_1 - \tfrac{a_5}{a_6}\xi_2. \end{aligned} \tag{2.7}$$

We now compute the exterior derivative of ω_i's.

$$\begin{aligned} d\omega_1 &= -dp \wedge dx = \tfrac{1}{L}\omega_2 \wedge \omega_3 \\ d\omega_2 &= dL \wedge dx = (L_x dx + L_u du + L_p dp) \wedge dx = \tfrac{L_u}{L}\omega_1 \wedge \omega_2 + \tfrac{L_p}{L}\omega_3 \wedge \omega_2 \\ d\omega_3 &= 0 \end{aligned}$$

Determining the differentials $d\xi_i$ and rewriting the equations in terms of the ξ_i's, we obtain

$$
\begin{aligned}
d\xi_1 &= da_1 \wedge (\tfrac{1}{a})\xi_1 + \tfrac{a_1}{L}\xi_2 \wedge (\tfrac{1}{a_6}\xi_3 - \tfrac{a_4}{a_1 a_6}\xi_1 - \tfrac{a_5}{a_6}\xi_2) = \alpha_1 \wedge \xi_1 + \tau_{123}\xi_2 \wedge \xi_3 + \tau_{112}\xi_1 \wedge \xi_2, \\
d\xi_2 &= \tfrac{L_u}{L}(\tfrac{1}{a_1})\xi_1 \wedge \xi_2 + \tfrac{L_p}{L}(\tfrac{1}{a_6}\xi_3 - \tfrac{a_4}{a_1 a_6}\xi_1 - \tfrac{a_5}{a_6}\xi_2) \wedge \xi_2 = \tau_{212}\xi_1 \wedge \xi_2 + \tau_{223}\xi_2 \wedge \xi_3, \\
d\xi_3 &= da_4 \wedge (\tfrac{1}{a_1})\xi_1 + \tfrac{a_4}{L}\xi_2 \wedge (\tfrac{1}{a_6}\xi_3 - \tfrac{a_4}{a_1 a_6}\xi_1 - \tfrac{a_5}{a_6}\xi_2) + da_5 \wedge \xi_2 + \tfrac{a_5 L_u}{a_1 L}\xi_1 \wedge \xi_2 \\
&\quad + \tfrac{a_5 L_p}{L}(\tfrac{1}{a_6}\xi_3 - \tfrac{a_4}{a_1 a_6}\xi_1 - \tfrac{a_5}{a_6}\xi_2) \wedge \xi_2 + da_6 \wedge (\tfrac{1}{a_6}\xi_3 - \tfrac{a_4}{a_1 a_6}\xi_1 - \tfrac{a_5}{a_6}\xi_2) = \\
&\quad \alpha_4 \wedge \xi_1 + \alpha_5 \wedge \xi_2 + \alpha_6 \wedge \xi_3 + \tau_{312}\xi_1 \wedge \xi_2 + \tau_{323}\xi_2 \wedge \xi_3.
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_1 &= \frac{1}{a_1}, \quad \alpha_4 = \frac{a_6 da_4 - a_4 da_6}{a_1 a_6}, \quad \alpha_5 = \frac{a_6 da_5 - a_5 da_6}{a_6}, \quad \alpha_6 = \frac{da_6}{a_6} \\
\tau_{123} &= \frac{a_1}{L a_6}, \quad \tau_{112} = \frac{a_4}{a_6 L}, \quad \tau_{212} = \frac{a_6 L_u - a_4 L_p}{a_1 a_6 L}, \quad \tau_{223} = \frac{-L_p}{a_6 L} \\
\tau_{312} &= \frac{a_4^2 + a_5 a_6 L_u - a_4 a_5 L_p}{a_1 a_6 L}, \quad \tau_{323} = \frac{a_4 - a_5 L_p}{a_6 L}.
\end{aligned}
$$

As mentioned earlier, in this method we try to eliminate as many τ_{ijk}'s as possible. For example, by taking

$$\alpha_1 = \overline{\alpha}_1 + \tau_{112}\xi_2, \tag{2.8}$$

we have

$$
\begin{aligned}
d\xi_1 &= (\overline{\alpha}_1 + \tau_{112}\xi_2) \wedge \xi_1 + \tau_{123}\xi_2 \wedge \xi_3 + \tau_{112}\xi_1 \wedge \xi_2 \\
&= \overline{\alpha}_1 \wedge \xi_1 - \tau_{112}\xi_1 \wedge \xi_2 + \tau_{123}\xi_2 \wedge \xi_3 + \tau_{112}\xi_1 \wedge \xi_2 \\
&= \overline{\alpha}_1 \wedge \xi_1 + \tau_{123}\xi_2 \wedge \xi_3, \\
d\xi_2 &= \tau_{212}\xi_1 \wedge \xi_2 + \tau_{223}\xi_2 \wedge \xi_3, \\
d\xi_3 &= \alpha_4 \wedge \xi_1 + \alpha_5 \wedge \xi_2 + \alpha_6 \wedge \xi_3 + \tau_{312}\xi_1 \wedge \xi_2 + \tau_{323}\xi_2 \wedge \xi_3,
\end{aligned} \tag{2.9}
$$

where $\alpha_1, \alpha_4, \alpha_5, \alpha_6$ form a basis for the right-invariant 1-forms on Lie group G. The only remaining τ_{ijk}'s are

$$
\begin{aligned}
\tau_{123} &= \frac{a_1}{L a_6}, \\
\tau_{212} &= \frac{a_6 L_u - a_4 L_p}{a_1 a_6 L}, \\
\tau_{223} &= \frac{-L_p}{a_6 L}.
\end{aligned}
$$

Normalization:

We suppose that $L_p \neq 0$. We normalize τ_{123}, τ_{212}, τ_{223} to 1, 0, -1 by letting

$$\begin{aligned} a_1 &= L_p, \\ a_4 &= \tfrac{L_u}{L}, \\ a_6 &= \tfrac{L_p}{L}. \end{aligned} \tag{2.10}$$

By substituting the expressions (2.5) into the formulas of the lifted coframe, we obtain new expressions for the structure equations.

$$\begin{aligned} \xi_1 &= a_1\omega_1 \\ \xi_2 &= \omega_3 \\ \xi_3 &= a_4\omega_1 + a_5\omega_2 + a_6\omega_3 \end{aligned} \tag{2.11}$$

By substituting the new values for a_i's, we obtain

$$\begin{aligned} \xi_1 &= L_p\omega_1 \\ \xi_2 &= \omega_2 \\ \xi_3 &= \tfrac{L_u}{L}\omega_1 + a_5\omega_2 + \tfrac{L_p}{L}\omega_3. \end{aligned}$$

Repeating the previous process, we find new expressions for the $d\xi_i$'s.

$$\begin{aligned} d\xi_1 &= dL_p \wedge \omega_1 + L_p d\omega_1 \\ d\xi_2 &= d\omega_2 \\ d\xi_3 &= d(\tfrac{L_u}{L}) \wedge \omega_1 + \tfrac{L_u}{L}d\omega_1 + da_5 \wedge \omega_2 + a_5 d\omega_2 + d(\tfrac{L_p}{L}) \wedge \omega_3 + (\tfrac{L_p}{L})d\omega_3. \end{aligned}$$

where

$$dL = \frac{L_x - a_5L^2 + pL_u}{L}\xi_2 + L\xi_3.$$

Therefore

$$\begin{aligned} d\xi_2 &= d(Ldx) = dL \wedge dx = ((\tfrac{L_x - a_5L^2 + pL_u}{L})\xi_2 + L\xi_3) \wedge (\tfrac{1}{L})\xi_2 = -\xi_2 \wedge \xi_3 \\ d\xi_3 &= d(\tfrac{L_u}{L}) \wedge (\tfrac{1}{L_p})\xi_1 + \tfrac{L_u}{L}.\tfrac{1}{L}\xi_2 \wedge (\tfrac{1}{a_6}\xi_3 - \tfrac{a_4}{a_1a_6}\xi_1 - \tfrac{a_5}{a_6}\xi_2) + da_5 \wedge \xi_2 - a_5\xi_2 \wedge \xi_3 \\ &\quad + d(\tfrac{L_p}{L}) \wedge (\tfrac{1}{a_6}\xi_3 - \tfrac{a_4}{a_1a_6}\xi_1 - \tfrac{a_5}{a_6}\xi_2) \\ &\quad = \alpha_4 \wedge \xi_1 + \alpha_5 \wedge \xi_2 + \alpha_6 \wedge \xi_3 + \tau_{312}\xi_1 \wedge \xi_2 + \tau_{323}\xi_2 \wedge \xi_3. \end{aligned}$$

where

$$\tau_{112} = \frac{L_p(L_u - pL_{up} - L_{px})}{L.L_p^2}, \tau_{113} = -\frac{L.L_{pp}}{L_p^2}, \alpha_5 = da_5 + \frac{a_5(-LdL_p + L_pdL)}{a_6L^2}.$$

Now, we normalize τ_{112} to 0 by choosing an appropriate a_5.

$$\tau_{112} = 0 = \frac{L_p(L_u pL_{pu} - L_{px}) + a_5L^2L_{pp}}{L^2L_{pp}^2}$$

Therefore

$$a_5 = -\frac{L_p(L_u - pL_{pu} - L_{px})}{L^2L_{pp}^2}. \quad (2.12)$$

We now define the operator $\check{E}$ by

$$\check{E} = \frac{\partial}{\partial u} - \check{D}_x\frac{\partial}{\partial p}.$$

where

$$\check{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u}.$$

By substitution, we obtain

$$\check{E}(L) = L_u - L_{px} - pL_{pu}. \quad (2.13)$$

Now, we substitute

$$a_5 = -\frac{-L_pQ}{L^2},$$

where $\quad Q = \frac{\check{E}(L)}{L_{pp}}$.

Therefore,

$$\tau_{113} = -\frac{L_{pp}L}{L_p^2}, \quad (2.14)$$

which is the first invariant.

The invariant coframe is now determined by

$$\xi_1 = a_1\omega_1$$
$$\xi_2 = \omega_2$$
$$\xi_3 = a_4\omega_1 + a_5\omega_2 + a_6\omega_3.$$

where

$$a_1 = L_p, \quad a_4 = \frac{L_u}{L}, \quad a_5 = -\frac{L_pQ}{L^2}, \quad a_6 = \frac{L_p}{L}. \tag{2.15}$$

Hence

$$\begin{aligned}
\xi_1 &= L_p du - pL_p dx \\
\xi_2 &= L dx \\
\xi_3 &= \tfrac{L_u}{L}(du - pdx) - \tfrac{L_pQ}{L}dx + \tfrac{L_p}{L}dp = d(\log L) - D_x(\log L)dx,
\end{aligned}$$

where $D_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + Q\frac{\partial}{\partial p}$.

We have

$$\begin{aligned}
d\xi_1 &= -I_1\xi_1 \wedge \xi_3 + \xi_2 \wedge \xi_3 \\
d\xi_2 &= -\xi_2 \wedge \xi_3 \\
d\xi_3 &= I_2\xi_1 \wedge \xi_2 + I_3\xi_2 \wedge \xi_3,
\end{aligned} \tag{2.16}$$

where the explicit forms of the invariants I_1, I_2, I_3 will be given below.

The covariant derivatives of F with respect to the coframe $\{\xi_1, \xi_2, \xi_3\}$ will be denoted by $F_{,\xi_1}, F_{,\xi_2}, F_{,\xi_3}$, respectively. The explicit expression of the covariant derivatives are easily computed. We have

$$df = f_x dx + f_u du + f_p dp.$$

However,

$$dx = \frac{1}{L}\xi_2, \ du = \frac{1}{L_p\xi_1} + \frac{p}{L}\xi_2, \ dp = \frac{L}{L_p}\xi_3 - \frac{L_u}{L_p^2}\xi_1 + \frac{Q}{L}\xi_2.$$

Therefore

$$dF = \frac{1}{L}f_x\xi_2 + \frac{1}{L_p}f_u\xi_1 + \frac{p}{L}f_u\xi_2 + \frac{L}{L_p}f_p\xi_3 - \frac{L_u}{L_p}f_p\xi_1 + \frac{Q}{L}f_p\xi_2 = F_{,\xi_1}\xi_1 + F_{,\xi_2}\xi_2 + F_{,\xi_3}\xi_3. \tag{2.17}$$

So, we can easily obtain the explicit formulas for these derivatives.

$$\begin{aligned}
F_{,\xi_1} &= \tfrac{1}{L_p}f_u - \tfrac{L_u}{L_u^2}f_p = \tfrac{1}{L_p^2}\tfrac{\partial(L,f)}{\partial(p,u)} \\
F_{,\xi_2} &= \tfrac{f_x}{L} + \tfrac{p}{L}f_u + \tfrac{Qf_p}{L} = \tfrac{1}{L}D_x F \\
F_{,\xi_3} &= \tfrac{L}{L_p}f_p = \tfrac{L}{L_p}F_p
\end{aligned} \tag{2.18}$$

For $F = L$ we obtain

$$\begin{cases} F_{,\xi_1} = L_{,\xi_1} = \frac{L_u}{L_p} - \frac{L_u L_p}{L_p^2} = 0 \\ F_{,\xi 2} = L_{,\xi_2} = \frac{pL_u + QL_p + L_x}{L} = \frac{D_x L}{L} \\ F_{,\xi_3} = L_{,\xi_3} = \frac{L}{L_p}.L_p = L. \end{cases}$$

Now we can easily compute the invariants.

First we compute the exterior derivative of L, which is

$$dL = L_{,\xi_2}\xi_2 + L\xi_3.$$

Hence

$$0 = d^2 L = dL_{,\xi_2}\xi_2 \wedge \xi_2 + L_{,\xi_2} d\xi_3 + dL \wedge \xi_3 + L d\xi_3 = (L_{,\xi_2,\xi_1} + LI_2)\xi_1 \wedge \xi_2 + (-L_{,\xi_2,\xi_3} + I_3 L)\xi_2 \wedge \xi_3.$$

We can obtain the invariants I_2 and I_3 from the above equation.

$$I_2 = -\frac{L_{,\xi_2,\xi_1}}{L}, \qquad I_3 = \frac{L_{,\xi_2,\xi_3}}{L} \tag{2.19}$$

To find I_1, we compute $d\xi_1$:

$$\begin{aligned} d\xi_1 &= dL_p \wedge du - d(pL_p) \wedge dx \\ &= (L_{p,\xi_1}\xi_1 + L_{p,\xi_2}\xi_2 + L_{p,\xi_3} d\xi_3) \wedge (\frac{1}{L_p}\xi_1 + \frac{p}{L}\xi_2) - L_p dp \wedge dx - p dp \wedge dx \\ &= (\frac{pL_{p,\xi_1}}{L})\xi_1 \wedge \xi_2 - \frac{L_{p,\xi_2}}{L_p}\xi_1 \wedge \xi_2 - \frac{L_{p,\xi_3}}{L_p}\xi_1 \wedge \xi_3 - \frac{pL_{p,\xi_3}}{L}\xi_2 \wedge \xi_3 \\ &\quad - L_p(\frac{L}{L_p}\xi_3 \frac{L_u}{L_p^2}\xi_1 + \frac{Q}{L}\xi_2) \wedge (\frac{1}{L})\xi_2 - p(L_{p,\xi_1}\xi_1 + L_{p,\xi_2}\xi_2 + L_{p,\xi_3}\xi_3) \wedge (\frac{1}{L})\xi_2 \\ &= -I_1 \xi_1 \wedge \xi_3 + \xi_2 \wedge \xi_3, \end{aligned}$$

where

$$I_1 = \frac{L_{p,\xi_3}}{L_p} \qquad and \qquad L_u = L_{p,\xi_2} L.$$

On the other hand, we know that

$$F_{,\xi_2} = \frac{1}{L} D_x F.$$

By substituting L_p for F, we obtain

$$L_{p,\xi_2} = \frac{1}{L} D_x L_p \quad \Rightarrow \frac{L_u}{L} = \frac{1}{L} D_x L_p \quad \Rightarrow L_u = D_x L_p.$$

which is the Euler-Lagrange equation and which forms another invariant.

The Bianchi identities are obtained as follows.

We have

$$d\xi_2 = -\xi_2 \wedge \xi_3 \quad \Rightarrow d^2\xi_2 = d\xi_2 \wedge \xi_3 + \xi_2 \wedge d\xi_3.$$

This equation gives a trivial case since

$$d\xi_2 = -\xi_2 \wedge \xi_3.$$

Computing the differential of ξ_1

$$d\xi_1 = -I_1 \xi_1 \wedge \xi_3 + \xi_2 \wedge \xi_3 \quad \Rightarrow$$

$$d^2\xi_1 = -dI_1 \wedge \xi_1 \wedge \xi_3 - I_1 d\xi_1 \wedge \xi_3 + I_1 \xi_1 \wedge d\xi_3 + d\xi_2 \wedge \xi_3 - \xi_2 \wedge d\xi_3 = 0.$$

Hence

$$-dI_1 \wedge \xi_1 \wedge \xi_3 - I_1 I_3 \xi_2 \wedge \xi_1 \wedge \xi_3 = 0. \tag{2.20}$$

On the other hand

$$dI_1 = I_{1,\xi_1}\xi_1 + I_{1,\xi_2}\xi_2 + I_{1,\xi_3}\xi_3.$$

Therefore

$$(-I_{1,\xi_1}\xi_1 - I_{1,\xi_2}\xi_2 - I_{1,\xi_3}\xi_3) \wedge \xi_1 \wedge \xi_3 - I_3 I_1 \xi_2 \wedge \xi_1 \wedge \xi_3 = 0.$$

So

$$I_3 = -\frac{I_{1,\xi_2}}{I_1}. \tag{2.21}$$

Another invariant can be obtained by differentiating ξ_3

$$d\xi_3 = I_2 \xi_1 \wedge \xi_2 + I_3 \xi_2 \wedge \xi_3 \quad \Rightarrow$$

$$d^2\xi_3 = dI_2 \wedge \xi_1 \wedge \xi_2 + I_2 d\xi_1 \wedge \xi_2 - I_2 \xi_1 \wedge d\xi_2 + dI_3 \xi_2 \wedge \xi_3 + I_3 d\xi_2 \wedge \xi_3 - I_3 \xi_2 \wedge d\xi_3 = 0.$$

Hence

$$I_{2.\xi_3} + I_1 I_2 + I_2 + I_{3.\xi_1} = 0. \tag{2.22}$$

Therefore

$$I_{2.\xi_3} + I_{3.\xi_1} + (I_1 + 1) I_2 = 0.$$

This is the full set of Bianchi identities for the adapted coframe.

2.5 Main results in the remaining cases

(a) Standard point transformations equivalence (case $m = 2$):

In this section, we try to obtain the invariants of the standard point transformations equivalence problem.

As explained in Section 2.1, the general formula for the point transformations case is

$$\widetilde{x} = \varphi(x, u), \quad \widetilde{u} = \psi(x, u).$$

In this section, we discuss the standard equivalence problem which has the form

$$\overline{L} = \frac{L}{\det J}.$$

The coframe corresponding to the given G_1-equivalence problem is given by

$$\theta = g_0.\omega,$$

where ω is the initial coframe.

We recall that the coframe expressions have the form

$$\begin{aligned}
\xi_1 &= L_p(du - pdx) = L_p\omega_1 = a_1\omega_1 \\
\xi_2 &= Ldx = \omega_2 \\
\xi_3 &= d(\log L) - \widetilde{D}_x(\log L)dx = a_4\omega_1 + a_5\omega_2 + a_6\omega_3.
\end{aligned}$$

As mentioned earlier, the group element of G_1 has the matrix form

$$g_0 = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \\ a_4 & a_5 & a_6 \end{pmatrix}.$$

and the lifted coframe has the form

$$h = \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 1 & 0 \\ b_4 & b_5 & b_6 \end{pmatrix}.$$

Therefore, for the G_2-equivalence problem, the lifted coframe can be represented by

$$\begin{aligned} \eta_1 &= b_1\xi_1 \\ \eta_2 &= b_2\xi_1 + \xi_2 \\ \eta_3 &= b_4\xi_1 + b_5\xi_2 + b_6\xi_3. \end{aligned} \tag{2.23}$$

The differentials are

$$\begin{aligned} d\eta_1 &= db_1 \wedge \xi_1 + b_1 d\xi_1 \\ d\eta_2 &= db_2 \wedge \xi_1 + b_2 d\xi_1 + d\xi_2 \\ d\eta_3 &= db_4 \wedge \xi_1 + db_5 \wedge \xi_2 + db_6 \wedge \xi_3 + b_4 d\xi_1 + b_5 d\xi_2. \end{aligned}$$

By (2.21), we can obtain ξ_i's in terms of η_i's

$$\xi_1 = \tfrac{1}{b_1}\eta_1$$

$$\xi_2 = -\tfrac{b_2}{b_1}\eta_1 + \eta_2$$

$$\xi_3 = (\tfrac{b_2 b_5 - b_4}{b_1 b_6})\eta_1 + (-\tfrac{b_5}{b_6})\eta_2 + \tfrac{1}{b_6}\eta_3.$$

By computing the differentials, we have

$$d\xi_1 = (\tfrac{I_1 b_5 + b_4}{b_1 b_6})\eta_1 \wedge \eta_2 + (\tfrac{-I_1 - b_2}{b_1 b_6})\eta_1 \wedge \eta_3 + (\tfrac{1}{b_6})\eta_2 \wedge \eta_3$$

$$d\xi_2 = (-\tfrac{b_4}{b_1 b_6})\eta_1 \wedge \eta_2 + (\tfrac{b_2}{b_1 b_6})\eta_1 \wedge \eta_3 + (-\tfrac{1}{b_6})\eta_2 \wedge \eta_3$$

$$d\xi_3 = (\tfrac{I_2 b_6 + I_3 b_4}{b_1 b_6})\eta_1 \wedge \eta_2 + (\tfrac{I_3}{b_6})\eta_2 \wedge \eta_3 + (-\tfrac{b_2 I_3}{b_1 b_6})\eta_1 \wedge \eta_3.$$

Following Cartan's method, we can compute the differentials of η_i's.

$$d\eta_1 = \beta_1 \wedge \eta_1 + \tau_{113}\eta_1 \wedge \eta_3 + \tau_{123}\eta_2 \wedge \eta_3 + \tau_{112}\eta_1 \wedge \eta_2,$$

where

$$\beta 1 = \frac{db_1}{b_1}, \quad \tau_{113} = \frac{-I_1 - b_2}{b_6}, \quad \tau_{123} = \frac{b_1}{b_6}, \quad \tau_{112} = \frac{I_1 b_5 + b_4}{b_6}.$$

and

$$d\eta_2 = \beta_2 \wedge \eta_1 + \tau_{213}\eta_1 \wedge \eta_3 + \tau_{212}\eta_1 \wedge \eta_2 + \tau_{223}\eta_2 \wedge \eta_3,$$

where

$$\beta_2 = \frac{db_2}{b_1}, \quad \tau_{213} = \frac{-I_1 b_2 + b_2 - b_2^2}{b_1 b_6}, \quad \tau_{212} = \frac{I_1 b_2 b_5 + b_2 b_4 - b_4}{b_1 b_6}, \quad \tau_{223} = \frac{b_2 - 1}{b_6}.$$

and

$$d\eta_3 = \beta_4 \wedge \eta_1 + \beta_5 \wedge \eta_2 + \beta_6 \wedge \eta_3 + \tau_{312}\eta_1 \wedge \eta_2 + \tau_{323}\eta_2 \wedge \eta_3 + \tau_{313}\eta_1 \wedge \eta_3,$$

where

$$\beta_4 = \frac{b_6 db_4 - b_2 b_6 db_5 + b_2 b_5 db_6 - b_4 db_6}{b_1 b_6}, \quad \beta_5 = \frac{b_6 db_5 - b_5 db_6}{b_6}, \quad \beta_6 = \frac{db_6}{b_6},$$

$$\tau_{312} = \frac{I_1 b_5 b_4 + b_4^2 - b_4 b_5 + I_2 b_6^2 + I_3 b_4 b_6}{b_1 b_6}, \quad \tau_{323} = \frac{b_4 - b_5 + I_3 b_6}{b_6},$$

$$\tau_{313} = \frac{-b_4 I_1 - b_2 b_4 + b_2 b_5 - b_2 b_6 I_3}{b_1 b_6}.$$

Note that L is not an affine function of p, i.e. $L \neq ap+b$. Therefore $I_1 \neq 0$ ($L_{pp} \neq 0$). We assume that I_1 does not vanish on Ω.

Following a similar procedure as the one we used in the fiber-preserving case, we can eliminate two of the torsion coefficients τ_{ijk} by absorbtion of torsion. By using the structure equations

$$d\eta_i = \sum_{j,k} T^i_{jk}\eta_j \wedge \eta_k,$$

we eliminate as many η_{ijk}'s as possible.

$$\begin{aligned}
d\eta_1 &= \overline{\beta}_1 \wedge \eta_1 + \tau_{123}\eta_2 \wedge \eta_3 \\
d\eta_2 &= \overline{\beta}_2 \wedge \eta_1 + \tau_{223}\eta_2 \wedge \eta_3 \\
d\eta_3 &= \overline{\beta}_4 \wedge \eta_1 + \overline{\beta}_5 \wedge \eta_2 + \overline{\beta}_6 \wedge \eta_3
\end{aligned}$$

The next step is to normalize the lifted coframe by the simplest normalization.

Normalization (phase1)

$i)$ $\tau_{123} = 1 \quad \Rightarrow \frac{b_1}{b_6} = 1 \quad \Rightarrow b_1 = b_6.$

$ii)$ $\tau_{223} = 0 \quad \Rightarrow \frac{b_2 - 1}{b_6} = 0 \quad \Rightarrow b_2 = 1.$

Therefore

$$\begin{aligned}
d\eta_1 &= \tfrac{db_1}{b_1} \wedge \eta_1 + \tfrac{-I_1-1}{b_1}\eta_1 \wedge \eta_3 + \eta_2 \wedge \eta_3 + \tfrac{I_1 b_5 + b_4}{b_1}\eta_1 \wedge \eta_2 \\
d\eta_2 &= \tfrac{-I_1}{b_1^2}\eta_1 \wedge \eta_3 + \tfrac{I_1 b_5}{b_1^2}\eta_1 \wedge \eta_2 \\
d\eta_3 &= \left(\tfrac{b_1 db_4 - b_1 db_5 + b_5 db_1 - b_4 db_1}{b_1^2}\right) \wedge \eta_1 + \left(\tfrac{b_1 db_5 - b_5 db_1}{b_1}\right) \wedge \eta_2 + \left(\tfrac{db_1}{b_1}\right) \wedge \eta_3 + \tau_{312}\eta_1 \wedge \eta_2 \\
&\quad + \tau_{323}\eta_2 \wedge \eta_3 + \tau_{313}\eta_1 \wedge \eta_3.
\end{aligned}$$

where

$$\begin{aligned}
\eta_{312} &= \frac{I_1 b_5 b_4 + b_4^2 - b_4 b_5 + I_2 b_1^2 + I_3 b_4 b_1}{b_1^2} \\
\eta_{313} &= \frac{-b_4 I_1 - b_4 + b_5 - b_1 I_3}{b_1^2} \\
\eta_{323} &= \frac{b_4 - b_5 + I_3 b_1}{b_1}.
\end{aligned}$$

At this stage, $\eta_2 = \xi_1 + \xi_2$ is an invariant.

So

$$\eta_2 = \xi_1 + \xi_2 = (L - pL_p)dx + L_p du.$$

This is Hilbert's invariant integral. **[Kamran, and Olver, [7], p. 55]**

Absorbtion

We can now apply the second step of absorbtion.

$$d\eta_1 = \overline{\overline{\beta}}_1 \wedge \eta_1 + \eta_2 \wedge \eta_3$$

$$d\eta_2 = -\tfrac{I_1}{b_1^2}\eta_1 \wedge gh_3 + \tfrac{I_1 b_5}{b_1^2}\eta_1 \wedge \eta_2$$

$$d\eta_3 = \overline{\overline{\beta}}_4 \wedge \eta_1 + \overline{\overline{\beta}}_5 \wedge \eta_2 + \overline{\overline{\beta}}_6 \wedge \eta_3$$

The second step of normalization follows directly.

Normalization (phase 2)

$\tau_{212} = 0 \ \Rightarrow \ b_5 = 0.$

$\tau_{213} = -\varepsilon \ \Rightarrow \ -\frac{I_1}{b_1^2} = -\varepsilon \ \Rightarrow \ b_1^2 = \frac{I_1}{\varepsilon} \ \Rightarrow \ b_1^2 = \varepsilon I_1 \ \Rightarrow \ b_1 = \pm\sqrt{|I_1|} = \pm k\sqrt{\varepsilon I_1}.$

$b_1 = b_6 = k\sqrt{|I_1|}.$

Therefore, the differential structure equations of the lifted coframe is

$$d\eta_1 = \frac{kd(\sqrt{|I_1|})}{k(\sqrt{|I_1|})} \wedge \eta_1 + \frac{-I_1 - 1}{k(\sqrt{|I_1|})}\eta_1 \wedge \eta_3 + \eta_2 \wedge \eta_3 + \frac{b_4}{k\sqrt{|I_1|}}\eta_1 \wedge \eta_2$$

$$d\eta_2 = \tfrac{-I_1}{|I_1|}\eta_1 \wedge \eta_3$$

$$d\eta_3 = \frac{k\sqrt{|I_1|}db_4 - b_4 d(k\sqrt{|I_1|})}{|I_1|} \wedge \eta_1 + \frac{kd(\sqrt{|I_1|})}{k\sqrt{|I_1|}} \wedge \eta_3 + \tau_{312}\eta_1 \wedge \eta_2 + \tau_{323}\eta_2 \wedge \eta_3 + \tau_{313}\eta_1 \wedge \eta_3.$$

where

$$\tau_{312} = \frac{b_4^2+I_2|I_1|+I_3b_4\sqrt{|I_1|}}{|I_1|}$$

$$\tau_{323} = \frac{b_4+I_3k\sqrt{|I_1|}}{k\sqrt{|I_1|}}$$

$$\tau_{313} = \frac{b_4I_1-b_4-k\sqrt{|I_1|}}{k\sqrt{|I_1|}}$$

$$\tau_{313} = \frac{-b_4I_1-b_4-k\sqrt{|I_1|}I_3}{|I_1|}.$$

On the other hand,

$$kd(\sqrt{|I_1|}) = \frac{1}{\sqrt{|I_1|}}.\frac{\epsilon k}{2}.(\frac{k\sqrt{|I_1|}(I_{1,\xi_1}-I_{1,\xi_2})-b_4I_{1,\xi_3}}{|I_1|}\eta_1+I_{1,\xi_2}\eta_2+k\frac{I_{1,\xi_3}}{\sqrt{|I_1|}}\eta_3).$$

Therefore

$$d\eta_1 = \frac{-\varepsilon I_{1,\xi_2}+2kb_4\sqrt{|I_1|}}{2|I_1|}\eta_1\wedge\eta_2+\eta_2\wedge\eta_3+(\frac{-k\varepsilon I_{1,\xi_3}}{2|I_1|^{3/2}}+\frac{-I_1-1}{k\sqrt{|I_1|}})\eta_1\wedge\eta_3$$

$$d\eta_2 = (-\varepsilon)\eta_1\wedge\eta_3$$

$$d\eta_3 = \frac{kdb_4}{\sqrt{|I_1|}}\wedge\eta_1+\tau_{312}\eta_1\wedge\eta_2+\tau_{313}\eta_1\wedge\eta_3+\tau_{323}\eta_2\wedge\eta_3.$$

where

$$\tau_{312} = \frac{k\varepsilon b_4I_{1,\xi_2}}{2|I_1|^{3/2}}+\frac{b_4^2+I_2|I_1|+I_3b_4k\sqrt{|I_1|}}{|I_1|}$$

$$\tau_{323} = \frac{\varepsilon I_{1,\xi_2}}{2|I_1|}+\frac{b_4^2+I_3k\sqrt{|I_1|}}{k\sqrt{|I_1|}}$$

$$\tau_{313} = \frac{k^2\varepsilon b_4I_{1,\xi_3}}{2|I_1|^2}+\frac{k\varepsilon\sqrt{|I_1|}(I_{1,\xi_1}-I_{1,\xi_2})-\varepsilon b_4I_{1,\xi_3}}{2|I_1|^2}+\frac{-b_4I_1-b_4-k\sqrt{|I_1|}I_3}{|I_1|}.$$

Absorbtion of torsion

For the third time, we apply the method of absorbtion to eliminate more torsion coefficients.

$$d\eta_3 = \overline{\overline{\overline{\beta}}}_4 \wedge \eta_1 + \tau'_{323}\eta_2 \wedge \eta_3$$

Now, we normalize the remaining coefficients.

Normalization (Phase 3)

$\tau_{323} = 0 \;\Rightarrow\; b_4 = -\frac{1}{2}kI_3\sqrt{|I_1|}$.

$\tau_{112} = 0 \;\Rightarrow\; b_4 = \frac{\varepsilon k I_{1.\xi_2}}{2\sqrt{|I_1|}}$.

By rewriting the differential structure equations of τ_i's, we obtain

$$d\eta_1 = -\varepsilon k J_1 \eta_1 \wedge \eta_3 + \eta_2 \wedge \eta_3,$$

where

$$J_1 = \frac{1/2 I_{1.\xi_3} + I_1^2 + I_1}{|I_1|^{3/2}}, \tag{2.24}$$

and

$$d\eta_2 = -\varepsilon \eta_1 \wedge \eta_3.$$

In order to compute $d\eta_3$, we first need to determine db_4.

$$\begin{aligned} db_4 &= -1/2kI_3.\frac{\varepsilon dI_1}{2\sqrt{|I_1|}} - 1/2k\sqrt{|I_1|}(I_{3.\xi_1}\xi_1 + I_{3.\xi_2}\xi_2 + I_{3.\xi_3}\xi_3) \\ &= \frac{\varepsilon}{4|I_1|}(I_3 I_{1.\xi_2} + 2I_1 I_{3.\xi_2})\eta_1 \wedge \eta_2 + \frac{\varepsilon k}{4\sqrt{|I_1|}}(\frac{I_3 I_{1.\xi_3} + 2I_1 I_{3.\xi_3}}{|I_1|})\eta_1 \wedge \eta_3 \end{aligned}$$

Therefore

$$d\eta_3 = J_2 \eta_1 \wedge \eta_2 + \varepsilon k J_3 \eta_1 \wedge \eta_3,$$

where

$$J_2 = 1/2 I_{3.\xi_2} - 1/4 I_3^2 + I_2, \tag{2.25}$$

and

$$J_3 = \frac{I_{1.\xi_1} + I_1 I_{3.\xi_3} + I_1^2 I_3 + 1/2 I_3 I_{1.\xi_3}}{2|I_1|^{3/2}}. \tag{2.26}$$

Now we compute the Bianchi identity.

$$0 = d^2\eta_2 = -\varepsilon d(\eta_1 \wedge \eta_3)$$

This is a trivial case.

$$0 = d^2\eta_1 = -\varepsilon k(dJ_1)\wedge(\eta_1\wedge\eta_3) - \varepsilon kJ_1 d(\eta_1\wedge\eta_3) + d(\eta_2\wedge\eta_3) = -\varepsilon k(J_{1,\eta_2} + J_3)\eta_2\wedge\eta_1\wedge\eta_3$$

So

$$J_{1,\eta_2} = -J_3. \tag{2.27}$$

Next, we evaluate the second derivative of η_3.

$$\begin{aligned} 0 &= d^2\eta_3 = (dJ_2)\wedge\eta_1\wedge\eta_2 + J_2(d\eta_1)\wedge\eta_2 - J_2\eta_1\wedge d\eta_2 \\ &\quad +\varepsilon k(dJ_3)\wedge\eta_1\wedge\eta_3 + \varepsilon kJ_3(d\eta_1)\wedge\eta_3 - \varepsilon kJ_3\eta_1\wedge(d\eta_3) \\ &= (-J_{2,\eta_3} + \varepsilon kJ_1J_2 - \varepsilon kJ_{3,\eta_2})\eta_1\wedge\eta_2\wedge\eta_3 \end{aligned}$$

Hence

$$J_1J_2 + J_{2,\eta_3} - J_{3,\eta_2} = 0. \tag{2.28}$$

By computing the covariant derivative of J with respect to the lifted coframe (2.21)

$$dJ = J_{,\eta_1}\eta_1 + J_{,\eta_2}\eta_2 + J_{,\eta_3}\eta_3 = J_{,\xi_1}\xi_1 + J_{,\xi_2}\xi_2 + J_{,\xi_3}\xi_3.$$

we obtain three more invariants.

$$\begin{aligned} J_{,\eta_1} &= k.\frac{J_{,\xi_1} - J_{,\xi_2} + 1/2I_3J_{,\xi_3}}{\sqrt{|I_1|}} \\ J_{,\eta_2} &= J_{,\xi_2} \\ J_{,\eta_3} &= k.\frac{J_{,\xi_3}}{\sqrt{|I_1|}} \end{aligned} \tag{2.29}$$

(b) Fiber-preserving divergence equivalence (case $m = 3$):

We start with the coframe $\{\xi_1,\ \xi_2,\ \xi_3\}$ discussed in the fiber-preserving equivalence case and we then add the coframe component ξ_4 to the previous coframe. In the end, we apply the involutive method for $G_1 \subset G_3$.

We evaluate the lifted coframe ξ_i for the G_3-equivalence problem.

$$\begin{aligned}
\zeta_1 &= c_1\xi_1 \\
\zeta_2 &= c_3\zeta_2 \\
\zeta_3 &= c_4\xi_1 + c_5\xi_2 + c_6\xi_3 \\
\zeta_4 &= c_7\xi_1 + (c_3 - 1)\xi_2 + \xi_4.
\end{aligned} \tag{2.30}$$

where the new coframe can be obtained by

$$\begin{aligned}
\xi_1 &= a_1\omega_1 = L_p\omega_1 = L_p\omega_1 = L_p(du - pdx) \\
\xi_2 &= \omega_2 = Ldx \\
\xi_3 &= a_4\omega_1 + a_5\omega_2 + a_6\omega_3 = d(\log L) - \widehat{D}_x(\log L)dx \\
\xi_4 &= dw.
\end{aligned}$$

By rewriting the equations and acting the differential operator d on them we obtain

$$\begin{aligned}
d\zeta_1 &= dc_1 \wedge \xi_1 + c_1 d\xi_1 \\
d\zeta_2 &= dc_3 \wedge \xi_2 + c_3 d\xi_2 \\
d\zeta_3 &= dc_4 \wedge \xi_1 + c_4 d\xi_1 + c_5 d\xi_2 + dc_5 \wedge \xi_2 + dc_6 \wedge \xi_3 + c_6 d\xi_3 \\
d\zeta_4 &= dc_7 \wedge \xi_1 + c_7 d\xi_1 + dc_3 \wedge \xi_2 + (c_3 - 1)d\xi_2 + d\xi_4.
\end{aligned}$$

Therefore

$$\begin{aligned}
d\zeta_1 &= \gamma_1 \wedge \zeta_1 + \tau_{112}\zeta_1 \wedge \zeta_2 + \tau_{123}\zeta_2 \wedge \zeta_3 + \tau_{113}\zeta_1 \wedge \zeta_3 \\
d\zeta_2 &= \gamma_3 \wedge \zeta_2 + \tau_{212}\zeta_1 \wedge \zeta_2 + \tau_{223}\zeta_2 \wedge \zeta_3 \\
d\zeta_3 &= \gamma_4 \wedge \zeta_1 + \gamma_5 \wedge \zeta_2 + \gamma_6 \wedge \zeta_3 + \tau_{312}\zeta_1 \wedge \zeta_2 + \tau_{323}\zeta_2 \wedge \zeta_3 + \tau_{313}\zeta_1 \wedge \zeta_3 \\
d\zeta_4 &= \gamma_7 \wedge \zeta_1 + \gamma_3 \wedge \zeta_2 + \tau_{413}\zeta_1 \wedge \zeta_3 + \tau_{423}\zeta_2 \wedge \zeta_3 + \tau_{412}\zeta_1 \wedge \zeta_2,
\end{aligned} \tag{2.31}$$

where

$$\begin{aligned}
\gamma_1 &= \frac{dc_1}{c_1},\ \gamma_3 = \frac{dc_3}{c_3},\ \gamma_4 = \frac{c_6dc_4 - c_4dc_6}{c_1c_6},\ \gamma_5 = \frac{c_6dc_5 - c_5dc_6}{c_3c_6},\ \gamma_6 = \frac{dc_6}{c_6},\ \gamma_7 = \frac{dc_7}{c_1} \\
\tau_{112} &= \frac{c_5I_1 + c_4}{c_3c_6},\ \tau_{123} = \frac{c_1}{c_3c_6},\ \tau_{113} = \frac{-I_1}{c_6} \\
\tau_{212} &= \frac{-c_4}{c_1c_6},\ \tau_{223} = \frac{-1}{c_6} \\
\tau_{312} &= \frac{I_1c_4c_5 + c_4^2 - c_4c_5 + c_4c_6I_3 + I_2c_6^2}{c_1c_3c_6},\ \tau_{323} = \frac{c_4 - c_5 + I_3c_6}{c_3c_6},\ \tau_{313} = \frac{-I_1c_4}{c_1c_6} \\
\tau_{413} &= \frac{-I_1c_7}{c_1c_6},\ \tau_{423} = \frac{1 - c_3 + c_7}{c_3c_6},\ \tau_{412} = \frac{I_1c_5c_7 + c_4 - c_3c_4 + c_4c_7}{c_1c_3c_6}.
\end{aligned}$$

Absorbtion of torsion

Similar to the previous case, we use the structure equations to absorb the torsion coefficients.

$$\begin{aligned}
d\zeta_1 &= \overline{\gamma}_1 \wedge \zeta_1 + \tau_{123}\zeta_2 \wedge \zeta_3 \\
d\zeta_2 &= \overline{\gamma}_3 \wedge \zeta_2 \\
d\zeta_3 &= \overline{\gamma}_4 \wedge \zeta_1 + \overline{\gamma}_5 \wedge \zeta_2 + \overline{\gamma}_6 \wedge \zeta_3 \\
d\zeta_4 &= \gamma_7 \wedge \zeta_1 + (\overline{\gamma}_3 - \tau_{212}\zeta_1 + \tau_{223}\zeta_3) \wedge \zeta_2 + \tau_{413}\zeta_1 \wedge \zeta_3 + \tau_{423}\zeta_2 \wedge \zeta_3 + \tau_{412}\zeta_1 \wedge \zeta_2.
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1 &= \overline{\gamma}_1 + \tau_{112}\zeta_2 + \tau_{113}\zeta_3, \\
\gamma_3 &= \overline{\gamma}_3 - \tau_{212}\zeta_1 + \tau_{223}\zeta_3.
\end{aligned}$$

On the other hand,

$$d\zeta_4 = \overline{\gamma}_7 \wedge \zeta_1 + \overline{\gamma}_3 \wedge \zeta_2 + (-\tau_{223} + \tau_{423})\xi_2 \wedge \zeta_3,$$

where

$$\gamma_7 = \overline{\gamma}_7 + (\tau_{212} - \tau_{412})\zeta_2 - \tau_{413}\zeta_3.$$

Normalization (phase1)

We now apply the normalization method. The only remained terms are

$\tau_{123} = \frac{c_1}{c_3c_6} = 1 \quad \Rightarrow c_1 = c_3c_6.$

$-\tau_{223} + \tau_{423} = \frac{1}{c_6} + \frac{1-c_3+c_7}{c_3c_6} = 0 \quad \Rightarrow c_7 = -1.$

Hence

$$c_1 = c_3c_6, c_7 = -1.$$

By evaluating the new version of the equations (2.31), we can obtain the new coefficients.

$$\begin{aligned}
\gamma_1 &= \frac{dc_1}{c_1},\ \tau_{112} = \frac{c_5I_1 + c_4}{c_1},\ \tau_{123} = 1,\ \tau_{113} = \frac{-I_1}{c_6} \\
\gamma_3 &= \frac{d(\frac{c_1}{c_6})}{\frac{c_1}{c_6}},\ \tau_{212} = \frac{-c_4}{c_1c_6},\ \tau_{223} = -\frac{1}{c_6} \\
\gamma_4 &= d(\frac{c_4}{c_6}).\frac{c_6}{c_1},\ \gamma_5 = \frac{c_6dc_5 - c_5dc_6}{c_1},\ \gamma_6 = \frac{dc_6}{c_6} \\
\tau_{312} &= \frac{I_1c_4c_5 + c_4^2 - c_4c_5 + c_4c_6I_3 + I_2c_6^2}{c_1^2} \\
\tau_{323} &= \frac{c_4 - c_5 + I_3c_6}{c_1},\ \tau_{313} = \frac{-I_1c_4}{c_1c_6} \\
\gamma_7 &= 0,\ \tau_{413} = \frac{I_1}{c_1c_6},\ \tau_{423} = -\frac{1}{c_6},\ \tau_{412} = \frac{-I_1c_5 - \frac{c_1c_4}{c_6}}{c_1^2}
\end{aligned}$$

Absorbtion of torsion

We now indicate the second step of the absorbtion.

$$\begin{aligned}
d\zeta_1 &= \overline{\gamma}_1 \wedge \zeta_1 + \zeta_1 \wedge \zeta_3 \\
d\zeta_2 &= \overline{\gamma}_3 \wedge \zeta_2 \\
d\zeta_3 &= \overline{\gamma}_4 \wedge \zeta_1 + \overline{\gamma}_5 \wedge \zeta_2 + \overline{\gamma}_6 \wedge \zeta_3 \\
d\zeta_4 &= \gamma_3 \wedge \zeta_2 + \tau_{413}\zeta_1 \wedge \zeta_3 + \tau_{423}\zeta_2 \wedge \zeta_3 + \tau_{412}\zeta_1 \wedge \zeta_2.
\end{aligned}$$

where

$$\gamma_3 = \overline{\gamma}_3 + \tau_{423}\zeta_3 - \tau_{412}\zeta_1.$$

The remaining term is

$$-\tau_{212}\zeta_1 + \tau_{223}\zeta_3 = -\tau_{412}\zeta_1 + \tau_{423}\zeta_3 \quad \Rightarrow \quad \tau_{212} = \tau_{412}.$$

Note that we have already shown that

$$\tau_{223} = \tau_{423}.$$

Normalization (phase2)

$\tau_{413} = \frac{I_1}{c_1 c_6} = 1 \Rightarrow c_3 = \frac{I_1}{c_6^2}$ or $c_6 = \frac{I_1}{c_1}$.

$\tau_{412} = \tau_{212} \quad \Rightarrow \quad I_1 c_5 = 0 \ \Rightarrow \ c_5 = 0.$

Similar to the previous case, we can assume that $I_1 \neq 0$.

Again, by evaluating the equations (2.31), we can compute the coefficients.

$$\begin{aligned}
\gamma_1 &= \frac{d(\frac{I_1}{c_6})}{\frac{I_1}{c_6}}, \ \tau_{112} = \frac{c_4 c_6}{I_1}, \ \tau_{123} = 1, \ \tau_{113} = -\frac{I_1}{c_6} \\
\gamma_3 &= \frac{d(\frac{I_1}{c_6^2})}{\frac{I_1}{c_6^2}}, \ \tau_{212} = -\frac{c_4}{I_1}, \ \tau_{223} = -\frac{1}{c_6} \\
\gamma_4 &= d(\frac{c_4}{c_6}).\frac{c_6^2}{I_1}, \ \gamma_5 = 0, \ \gamma_6 = \frac{dc_6}{c_6}, \ \tau_{323} = \frac{(c_4 + I_3 c_6)c_6}{I_1} \\
\tau_{313} &= -c_4, \ \tau_{312} = \frac{(c_4^2 + c_4 c_6 I_3 + I_2 c_6^2)c_6^2}{I_1^2}, \ \gamma_7 = 0 \\
\tau_{413} &= 1, \ \tau_{423} = -\frac{1}{c_6} = \tau_{223}, \ \tau_{412} = \frac{-c_4}{I_1} = \tau_{212}
\end{aligned}$$

Hence

$$\begin{aligned}
d\zeta_1 &= \gamma_1 \wedge \zeta_1 + \tau_{112}\zeta_1 \wedge \zeta_2 + \tau_{113}\zeta_1 \wedge \zeta_3 + \zeta_2 \wedge \zeta_3 \\
d\zeta_2 &= \gamma_3 \wedge \zeta_2 + \tau_{212}\zeta_1 \wedge \zeta_2 + \tau_{213}\zeta_2 \wedge \zeta_3 \\
d\zeta_3 &= \gamma_4 \wedge \zeta_1 + \gamma_6 \wedge \zeta_3 + \tau_{323}\zeta_2 \wedge \zeta_3 + \tau_{313}\zeta_1 \wedge \zeta_3 + \tau_{312}\zeta_1 \wedge \zeta_2 \\
d\zeta_4 &= \gamma_3 \wedge \zeta_2 + \tau_{412}\zeta_1 \wedge \zeta_2 + \tau_{423}\zeta_2 \wedge \zeta_3 + \zeta_1 \wedge \zeta_3.
\end{aligned}$$

where

$$
\begin{aligned}
\gamma_1 &= \frac{-1}{c_6}dc_6,\ \tau_{112} = \frac{c_4c_6I_1 - c_6^2I_{1.\xi_2}}{I_1^2},\ \tau_{113} = \frac{-I_1^2 - I_{1.\xi_3}}{I_1c_6}.\\
\gamma_3 &= \frac{-2}{c_6}dc_6 = 2\gamma_1,\ \tau_{212} = \frac{-c_4I_1 + I_{1,\xi_1}c_6 - I_{1,\xi_3}c_4}{I_1^2}, \tau_{223} = \frac{-I_{1.\xi_3} - I_1}{I_1c_6}.\\
\gamma_4 &= \frac{c_6^2}{I_1}d(\frac{c_4}{c_6}),\ \gamma_6 = \frac{dc_6}{c_6} = -\gamma_1,\ \tau_{313} = -c_4\tau_{323} = \frac{c_4c_6 + I_3c_6^2}{I_1}.\\
\tau_{312} &= \frac{c_6^2(c_4^2 + c_4c_6 + I_3c_6^2I_2)}{I_1^2}, \gamma_3 = 2\gamma_1 = -\frac{2}{c_6}dc_6.\\
\tau_{412} &= \frac{I_{1,\xi_1}c_6 - I_{1,\xi_3}c_4 - I_1c_4}{I_1^2},\ \tau_{413} = 1,\ \tau_{423} = \frac{-I_{1,\xi_3} - I_1}{I_1c_6}.
\end{aligned}
$$

Absorbtion of torsion (phase3)

$$
\begin{aligned}
d\zeta_1 &= \overline{\gamma}_1 \wedge \zeta_1 + \zeta_2 \wedge \zeta_3\\
d\zeta_2 &= \overline{\gamma}_3 \wedge \zeta_2\\
d\zeta_3 &= \overline{\gamma}_4 \wedge \zeta_1 + \overline{\gamma}_6 \wedge \zeta_3\\
d\zeta_4 &= \overline{\gamma}_3 \wedge \zeta_2 + \zeta_1 \wedge \zeta_3.
\end{aligned}
$$

where

$$\gamma_1 = \overline{\gamma}_1 + \tau_{112}\zeta_2 + \tau_{113}\zeta_3.$$

$$\gamma_3 = \overline{\gamma}_3 - \tau_{212}\zeta_1 + \tau_{223}\zeta_3 \quad \Rightarrow \quad \tau_{223} = 2\tau_{113} + 1.$$

$$\overline{\gamma}_4 = \gamma_4 + \tau_{313}\zeta_3 + \tau_{312}\zeta_2.$$

$$\gamma_6 = \overline{\gamma}_6 - \tau_{323}\zeta_2 = -\gamma_1 = -\overline{\gamma}_1 - \tau_{112}\zeta_2 - \tau_{113}\zeta_3 \quad \Rightarrow \quad \tau_{323} = \tau_{112}.$$

Normalization (phase3)

Let $\tau_{323} = \tau_{112}$.

This gives us the Bianchi identity which was obtained earlier.

$$I_{1,\xi_2} = -I_1I_3.$$

Therefore this does not give us any new identity.

Let

$2\tau_{113} + 1 = \tau_{223} \qquad \Rightarrow \qquad c_6 = \frac{I_{1.\xi_3}}{I_1} + 2I_1 - 1.$

Therefore

$$c_6 = \frac{L.L_{ppp}}{L_p.L_{pp}}. \tag{2.32}$$

$\tau_{412} = \tau_{212}$.

$\tau_{423} = \tau_{223} = 2\tau_{113} + 1$.

Note that the latest equation gives us nothing new.

At this moment, we assume L_{ppp} does not vanish. Equivalently, c_6 is not zero. Note that if c_6 does vanish we cannot evaluate any further group reductions at this step and we must prolong the system. This is because a_{33} is not 1 in the following group element.

$$\begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_3 & 0 & 0 \\ c_4 & c_5 & 0 & 0 \\ c_7 & c_3 - 1 & 0 & 1 \end{pmatrix}$$

So, now we assume that $c_6 \neq 0$. We rewrite equations (2.31).

$$d\zeta_1 = \gamma_1 \wedge \zeta_1 + \tau_{112}\zeta_1 \wedge \zeta_2 + \tau_{113}\zeta_1 \wedge \zeta_3 + \zeta_2 \wedge \zeta_3.$$

where

$$\zeta_1 = -\frac{dI_4}{I_4}, \qquad \tau_{112} = \frac{c_4 I_4 I_1 - I_4^2 I_{1.\xi_2}}{I_1^2}, \qquad \tau_{113} = \frac{-I_1^2 - I_{1.\xi_3}}{I_1 I_4}.$$

and

$$d\zeta_2 = \gamma_3 \wedge \zeta_2 + \tau_{212}\zeta_1 \wedge \zeta_2 + \tau_{223}\zeta_2 \wedge \zeta_3.$$

where

$$\gamma_3 = -\frac{2}{I_4} dI_4 = 2\gamma_1, \qquad \tau_{212} = \frac{-c_4 I_1 + I_{1.\xi_1} 2I_4 - c_4 I_{1.\xi_3}}{I_1^2}, \qquad \tau_{223} = \frac{-I_{1.\xi_3} - I_1}{I_1 I_4}.$$

and

$$d\zeta_3 = \gamma_4 \wedge \zeta_1 + \gamma_6 \wedge \zeta_3 + \tau_{323}\zeta_2 \wedge \zeta_3 + \tau_{313}\zeta_1 \wedge \zeta_3 + \tau_{312}\zeta_1 \wedge \zeta_2.$$

where

$$\gamma_4 = \frac{I_4^2}{I_1} d(\frac{c_4}{I_4}), \quad \gamma_6 = \frac{dI_4}{I_4} = -\gamma_1, \quad \tau_{313} = -c_4,$$
$$\tau_{323} = \frac{c_4 I_4 + I_3 I_4^2}{I_1}, \quad \tau_{312} = \frac{I_4^2(c_4^2 + c_4 I_4 I_3 + I_4^2 I_2)}{I_1^2}.$$

and finally

$$d\zeta_4 = \gamma_3 \wedge \zeta_2 + \tau_{412}\zeta_1 \wedge \zeta_2 + \tau_{423}\zeta_2 \wedge \zeta_3 + \zeta_1 \wedge \zeta_3,$$

where

$$\gamma_3 = \frac{-2dI_4}{I_4}, \qquad \tau_{413} = 1; \tau_{423} = \frac{-I_{1,\xi_3} - I_1}{I_1 I_4}, \qquad \tau_{412} = \frac{I_{1,\xi_1} I_4 - c_4 I_{1,\xi_3} - c_4 I_1}{I_1^2}.$$

The differentials of the lifted coframe are given by

$$d\zeta_1 = (\tau_{112} + \tfrac{I_4 I_{4,\xi_2}}{I_1})\zeta_1 \wedge \zeta_2 + \zeta_2 \wedge \zeta_3 + (\tau_{113} + \tfrac{I_{1,\xi_3}}{I_4^2})\zeta_1 \wedge \zeta_3$$

$$d\zeta_2 = (\tau_{212} + \tfrac{2I_{4,\xi_3}}{I_1 I_4} c_4 - \tfrac{2I_{4,\xi_1}}{I_1})\zeta_1 \wedge \zeta_2 + (\tau_{223} + \tfrac{2I_{4,\xi_3}}{I_4^2})\zeta_2 \wedge \zeta_3$$

$$d\zeta_3 = (\tfrac{I_4}{I_1} dc_4) \wedge \zeta_1 + (\tau_{312} + \tfrac{I_4^2 c_4 I_{4,\xi_2}}{I_1^2})\zeta_1 \wedge \zeta_2 + (\tau_{323} + \tfrac{I_4 I_{4,\xi_2}}{I_1})\zeta_2 \wedge \zeta_3 + (\tau_{313} + \tfrac{I_{1,\xi_1}}{I_1})\xi_1 \wedge \xi_3$$

$$d\xi_4 = (\tau_{412} + \tfrac{-2I_{1,\xi_1}}{I_1} + \tfrac{2c_4 I_{4,\xi_3}}{I_1 I_4})\zeta_1 \wedge \zeta_2 + \zeta_1 \wedge \zeta_3 + (\tau_{423} + \tfrac{2I_{1,\xi_3}}{I_4^2})\zeta_2 \wedge \zeta_3.$$

Absorbtion of torsion (phase4)

So far, everything has been absorbed except $d\zeta_3$.

$$d\zeta_3 = \overline{\gamma}_4 \wedge \zeta_1 + \tau_{323}\zeta_2 \wedge \zeta_3,$$

where

$$\gamma_4 = \overline{\gamma}_4 + \tau_{312}\xi_2 + \tau_{313}\xi_3.$$

Normalization (phase4)

The only remaining term is

$$\tau_{323}^{new} = \tau_{323} + \frac{I_4 I_{4.\xi_2}}{I_1} = \frac{c_4 I_4 + I_3 I_4^2 + I_4 I_4.\xi_2}{I_1} = 0.$$

Therefore

$$c_4 = -I_3 I_4 - I_{4.\xi_2}. \tag{2.33}$$

We now use the Bianchi identities and normalize the group parameter of transformations (2.29) to obtain the structure equations explicitly.

Let

$$d\zeta_1 = -K_1 \zeta_1 \wedge \zeta_3 + \zeta_2 \wedge \zeta_3,$$

where

$$K_1 = \frac{I_1}{I_4} + \frac{1}{I_1}.(\frac{I_1}{I_4})_{.\xi_3}.$$

Note that

$$\tau_{113} + \frac{I_{4.\xi_3}}{I_4^2} = -K_1.$$

Also suppose that

$$d\zeta_2 = (\tau'_{212})\zeta_1 \wedge \zeta_2 + (\tau'_{223})\zeta_2 \wedge \zeta_3,$$

where

$$\tau'_{223} = \tau_{223} + \frac{2I_{4.\xi_3}}{I_4^2} = 1 - 2K_1.$$

From the structure equations, we know that

$$\tau'_{212} = \tau_{212} + \frac{2I_5 I_{4.\xi_3}}{I_1 I_4} - \frac{2I_{4.\xi_1}}{I_1}.$$

$$\tau'_{412} = \tau_{412} + \frac{2I_5 I_{4.\xi_3}}{I_1 I_4} - \frac{2I_{4.\xi_1}}{I_1},$$

where

$$\tau_{412} = \tau_{212}.$$

will be obtained by absorbtion of torsion and normalization. Next, we deal with the differential of ξ_3.

$$\begin{aligned} d\xi_3 &= \frac{I_4}{I_1} dI_5 \wedge \zeta_1 + (\tau_{312} + \frac{I_4^2 I_5 I_{4.\xi_2}}{I_1^2})\zeta_1 \wedge \zeta_2 + (\tau_{323} + \frac{I_4 I_{4.\xi_2}}{I_1})\zeta_2 \wedge \zeta_3 \\ &+ (\tau_{313} + \frac{I_{4.\xi_1}}{I_1})\zeta_1 \wedge \zeta_3 = K_3 \zeta_1 \wedge \zeta_2 + K_4 \zeta_1 \wedge \zeta_3, \end{aligned}$$

where

$$K_3 = \frac{I_4^4 I_2 + I_4^3(I_5 I_3 - I_{5.\xi_2}) + I_4^2(I_5^2 + I_5 I_{4.\xi_2})}{I_1^2}, \tag{2.34}$$

$$K_4 = \frac{-I_{5.\xi_3} + I_{4.\xi_1}}{I_1} - I_5. \tag{2.35}$$

The last structure equation is

$$d\zeta_4 = (\tau'_{412})\zeta_1 \wedge \zeta_2 + \zeta_1 \wedge \zeta_3 + (\tau'_{423})\zeta_2 \wedge \zeta_3,$$

where

$$\tau'_{423} = \tau_{423} + \frac{2I_{4.\xi_3}}{I_4^2} = 1 - 2K_1.$$

Therefore, the resulting structure equations are

$$\begin{aligned} d\zeta_1 &= -K_1\zeta_1 \wedge \zeta_3 + \zeta_2 \wedge \zeta_3 \\ d\zeta_2 &= K_2\zeta_1 \wedge \zeta_2 + (1 - 2K_1)\zeta_2 \wedge \zeta_3 \\ d\zeta_3 &= K_3\zeta_1 \wedge \zeta_2 + K_4\zeta_1 \wedge \zeta_3 \\ d\zeta_4 &= K_2\zeta_1 \wedge \zeta_2 + \zeta_1 \wedge \zeta_3 + (1 - 2K_1)\zeta_2 \wedge \zeta_3. \end{aligned} \tag{2.36}$$

Note that

$$F_{.\xi_3} = \frac{L}{L_p} F_p.$$

We now compute I_4. Take $F = I_1$.

$$I_4 = \frac{I_{1.\xi_3}}{I_1} + 2I_1 - 1 = \frac{\frac{L}{L_p}.(I_1)_p}{I_1} + 2I_1 - 1 = \frac{L.L_{ppp}}{L_p.L_{pp}}.$$

Hence

$$I_4 = \frac{L.L_{ppp}}{L_p.L_{pp}}. \tag{2.37}$$

At this stage, we compute the formulas for the derived invariants.

Take the covariant derivative of K.

$$dK = K_{.\zeta}\zeta_1 + K_{.\zeta_2}\zeta_2 + K_{.\zeta_3}\zeta_3 + K_{.\zeta_4}\zeta_4 = K_{.\xi_1}\xi_1 + K_{.\xi_2}\xi_2 + K_{.\xi_3}\xi_3 + K_{.\zeta_4}\zeta_4$$

We evaluate the derived invariants explicitly.

$$\begin{aligned} K_{,\zeta_1} &= \tfrac{I_4}{I_1} K_{,\xi_1} - \tfrac{I_5}{I_1} K_{,\xi_3} + \tfrac{I_4}{I_1} K_{,\xi_1} \\ K_{,\zeta_2} &= \tfrac{I_4^2}{I_1} K_{,\xi_2} + (\tfrac{I_4^2}{I_1} - 1) K_{,\xi_1} \\ K_{,\zeta_3} &= \tfrac{K_{,\xi_3}}{I_4} \\ K_{,\zeta_4} &= K_{,\xi_4} \end{aligned} \tag{2.38}$$

Bianchi's identities:

$$\begin{aligned} 0 &= d^2\xi_1 = -dK_1 \wedge \xi_1 \wedge \xi_3 + K_2\xi_1 \wedge \xi_2 \wedge \xi_3 - K_4\xi_2 \wedge \xi_1 \wedge \xi_3 \\ &= (K_{1,\zeta_2} + K_2 + K_4)\zeta_1 \wedge \zeta_2 \wedge \zeta_3. \end{aligned}$$

So

$$K_{1,\zeta_2} + K_2 + K_4 = 0. \tag{2.39}$$

The next equation is

$$\begin{aligned} 0 &= d^2\zeta_2 = dK_2 \wedge \zeta_1 \wedge \zeta_2 + K_2(-K_1\zeta_1 \wedge \zeta_3 + \zeta_2 \wedge \zeta_3) \wedge \zeta_2 \\ &\quad -K_2\zeta_1 \wedge (K_2\zeta_1 \wedge \zeta_2 + (1 - 2K_1)\zeta_2 \wedge \zeta_3) - 2dK_1 \wedge \zeta_2 \wedge \zeta_3 \\ &\quad +(1 - 2K_1)(K_2\zeta_1 \wedge \zeta_2 + (1 - 2K_1)\zeta_2 \wedge \zeta_3) \wedge \zeta_3 \\ &\quad -(1 - 2K_1)\zeta_2 \wedge (K_3\zeta_1 \wedge \zeta_2 + K_4\zeta_1 \wedge \zeta_3). \end{aligned}$$

Therefore

$$K_{2,\zeta_3} + K_1K_2 - 2K_{1,\zeta_1} + K_4 - 2K_1K_4 = 0.$$

The last two equations are

$$0 = d^2\zeta_3 \Rightarrow K_{3,\zeta_3} - K_{4,\zeta_1} - K_1K_3 + (4K_1K_3 - K_3) = 0.$$

$$0 = d^2\zeta_4 \Rightarrow K_{2,\zeta_3} + K_1K_2 - 2K_{1,\zeta_1} + K_4 - 2K_1K_4 = 0.$$

Therefore

$$K_{2,\zeta_3} + 2K_{1,\zeta_1} + K_1K_2 + 2K_1K_4 + (5K_4 - 4K_1K_4 + 4K_2) = 0.$$

The other case to be considered is when $L_{ppp} = 0$ i.e. $I_4 = 0$.
Now after phase 3 of the absorbtion of torsion process, the parameter c_6 is zero. We should reduce the structure group G_4 to $\widetilde{G}_3 \subset G_3$.

$$\begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_7 & c_3 - 1 & 0 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_3 & 0 \\ c_7 & c_3 - 1 & 1 \end{pmatrix}$$

Modifying (2.29) we obtain

$$\begin{aligned} d\zeta_1 &= -\beta \wedge \zeta_1 + \zeta_2 \wedge \zeta_3 \\ d\zeta_2 &= -2\beta \wedge \zeta_2 \\ d\zeta_3 &= \alpha \wedge \zeta_1 + \beta \wedge \zeta_3 \\ d\zeta_4 &= -2\beta \wedge \zeta_2 + \zeta_1 \wedge \zeta_3, \end{aligned} \tag{2.40}$$

where α and β are equivalent, modulo the base coframe to (the right invariant coframe) γ_4 and γ_6 on $\widetilde{G}_3$. The action of the reduced group $\widetilde{G}_3$ on the torsion is trivial, since there is no non-constant torsion left in (2.38).
Therefore, there is no further possible group reduction.

$$\beta = -\gamma_1 + \tau_{112}\zeta_3 = \gamma_6 + \tau_{112}\zeta_2 + \tau_{113}\zeta_3$$

Hence

$$\alpha = \gamma_4 - \tau_{313}\zeta_3 - \tau_{312}\zeta_2 - m\zeta_1 - \frac{1}{2}\tau_{212}\zeta_3, \tag{2.41}$$

$$\beta = \gamma_6 + \tau_{112}\zeta_2 + \tau_{113}\zeta_3 - \frac{1}{2}\tau_{212}\zeta_1. \tag{2.42}$$

Now we compute the differentials and then absorb these two new structure equations.

$$\begin{aligned} d\alpha &= B\zeta_1 \wedge \zeta_2 + 2\alpha \wedge \beta, \\ d\beta &= -\alpha \wedge \zeta_2. \end{aligned}$$

By adding up the above equations to the previous structure equations obtained from (2.40), we reduce the prolonged system to an $\{e\}$-structure.
If $B = 0$ we obtain

$$I_1^2 I_{6,\zeta_2} = I_{2,\zeta_1}.$$

Otherwise, we can normalize B to 1, and obtain

$$c_6 = I_9.$$

(c) point transformation divergence equivalence (case $m = 4$):
The lifted coframe is given by

$$\begin{aligned}
\theta_1 &= d_1\eta_1\\
\theta_2 &= d_2\eta_1 + d_3\eta_2\\
\theta_3 &= d_4\eta_1 + d_5\eta_2 + d_6\eta_3\\
\theta_4 &= d_7\eta_1 + (d_3 - 1)\eta_2 + \eta_4,
\end{aligned}$$

where

$$\begin{aligned}
\eta_1 &= b_1\xi_1\\
\eta_2 &= \xi_1 + \xi_2\\
\eta_3 &= b_4\xi_1 + b_6\xi_3\\
\eta_4 &= dw,
\end{aligned}$$

and

$$b_1 = b_6 = k\sqrt{|I_1|},\ b_4 = -\frac{1}{2}kI_3\sqrt{|I_1|},$$

So

$$\begin{aligned}
d\theta_1 &= \delta_1 \wedge \theta_1 + \tau_{113}\theta_1 \wedge \theta_3 + \tau_{112}\theta_1 \wedge \theta_2 + \tau_{123}\theta_2 \wedge \theta_3\\
d\theta_2 &= \delta_2 \wedge \theta_1 + \delta_3 \wedge \theta_2 + \tau_{212}\theta_1 \wedge \theta_2 + \tau_{213}\theta_1 \wedge \theta_3 + \tau_{223}\theta_2 \wedge \theta_3\\
d\theta_3 &= \delta_4 \wedge \theta_1 + \delta_5 \wedge \theta_3 + \tau_{312}\theta_1 \wedge \theta_2 + \tau_{313}\theta_1 \wedge \theta_3 + \tau_{323}\theta_2 \wedge \theta_3.\\
d\theta_4 &= \delta_7 \wedge \theta_1 + \delta_3 \wedge \theta_2 + \tau_{412}\theta_1 \wedge \theta_2 + \tau_{423}\theta_2 \wedge \theta_3 + \tau_{413}\theta_1 \wedge \theta_3.
\end{aligned}$$

where

$$
\begin{aligned}
\delta_1 &= \frac{d(d_1)}{d_1}, \ \tau_{113} = \frac{-\varepsilon k J_1 d_3 - d_2}{d_3 d_6}, \ \tau_{112} = \frac{\varepsilon k d_5 d_3 J_1 + d_3 d_4}{d_3^2 d_6}, \ \tau_{123} = \frac{d_1}{d_3 d_6}. \\
\delta_2 &= \frac{d_3 d(d_2) - d_2 d(d_3)}{d_1 d_3}, \ \delta_3 = \frac{d(d_3)}{d_3}, \ \tau_{223} = \frac{d_1 d_2}{d_1 d_3 d_6}. \\
\tau_{213} &= \frac{-\varepsilon k d_2 J_1 d_3 - d_2^2 - \varepsilon d_3^2}{d_1 d_3 d_6}, \ \tau_{212} = \frac{\varepsilon k d_2 d_5 J_1 + d_2 d_4 + \varepsilon d_3 d_5}{d_1 d_3 d_6}, \\
\delta_7 &= \frac{d_3 d(d_7) - d_2 . d(d_3)}{d_1 . d_3}, \ \delta_3 = \frac{d(d_3)}{d_3}, \ \tau_{413} = \frac{d_3 - d_2 d_7}{d_1 d_3 d_6}. \\
\tau_{412} &= \frac{\epsilon k d_5 d_7 J_1 + d_3 d_5 - d_5 + d_4 d_7}{d_1 d_3 d_6}.
\end{aligned}
$$

The first three phases of absorbtion and normalization of torsion are as follows

Phase 1.

$$
\begin{aligned}
\tau_{123} &= 1 \ \Rightarrow \ d_1 = d_3 d_6. \\
\tau_{223} &= \tau_{423} \ \Rightarrow \ d_2 = d_7.
\end{aligned}
$$

Phase 2.

$$
\begin{aligned}
\tau_{212} &= \tau_{412} \ \Rightarrow \ d_5 = 0, \\
\tau_{213} &= \tau_{413} - 1 \ \Rightarrow \ d_3 = \frac{\epsilon}{d_6^2}.
\end{aligned}
$$

Phase 3.

$$
\begin{aligned}
\tau_{112} &= \tau_{323} \ \text{(This equation is always satisfied.)} \\
\tau_{223} &= 2\tau_{113} \ \Rightarrow \ d_7 = -\frac{2\epsilon k J_1}{3 d_6^2}.
\end{aligned}
$$

The fourth case is

$$\tau_{413} = 1 + \mathrm{sign}.\tilde{J}_4 \ \Rightarrow \ d_6 = \tilde{\kappa}\sqrt{|\tilde{J}_4|}.$$

where $\tilde{\kappa}$ denotes the ambiguity in the sign, and

$$\tilde{J}_4 = \epsilon - \frac{2}{3}\epsilon k J_{1,\eta_3} - \frac{2}{9} J_1^2$$

is not zero, otherwise we must prolong the system. The last phase is obtained by

Phase 5.

$$\tau_{323} = 0 \ \Rightarrow \ d_4 = -\tilde{\kappa} J_{4,\eta_2}.$$

After absorbtion of torsion, we can obtain the structure equations

$$
\begin{aligned}
d\theta_1 &= -\tilde{\kappa} M_1 \theta_1 \wedge \theta_3 + \theta_2 \wedge \theta_3, \\
d\theta_2 &= \tilde{\kappa} M_2 \theta_1 \wedge \theta_2 + \tilde{c}\theta_1 \wedge \theta_3 - 2\tilde{\kappa} M_1 \theta_2 \wedge \theta_3, \\
d\theta_3 &= M_3 \theta_1 \wedge \theta_2 + \tilde{\kappa} M_4 \theta_1 \wedge \theta_3, \\
d\theta_4 &= \tilde{\kappa} M_2 \theta_1 \wedge \theta_2 + (1+\tilde{c})\theta_1 \wedge \theta_3 - 2\tilde{\kappa} M_1 \theta_2 \wedge \theta_3,
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 &= \frac{\frac{1}{3}J_1 J_4 - J_{4,\eta_3}}{J_4^2}, \\
M_2 &= -2J_{4,\eta_1} + \frac{2}{3}(J_4 J_{1,\eta_2} - J_1 J_{4,\eta_2}) - 2\frac{J_{4,\eta_2} J_{4,\eta_3}}{J_4}, \\
M_3 &= J_4^4 J_2 + J_4^3 J_{4,\eta_2,\eta_3}, \\
M_4 &= J_{4,\eta_1} + J_1 J_{4,\eta_2} + J_{4,\eta_2,\eta_3} + J_3 J_4.
\end{aligned}
$$

We can now compute the derived invariants.

$$
\begin{aligned}
M_{,\theta_1} &= \tilde{\kappa}\{J_4 M_{,\eta_1} + \frac{2}{3}J_1 J_4 M_{,\eta_2} + J_{4,\eta_2} M_{,\eta_3} - \frac{2}{3}J_1 J_4 M_{,\eta_4}\}, \\
M_{,\theta_1} &= J_4^2 M_{,\eta_2} + (J_4^2 - 1) M_{,\eta_4}, \\
M_{,\theta_1} &= \tilde{\kappa}\frac{M_{,\eta_3}}{J_4}, \\
M_{,\theta_1} &= M_{,\eta_4}.
\end{aligned}
$$

Therefore, the Bianchi identities are obtained.

$$
\begin{aligned}
M_{1,\theta_2} + M_2 + M_4 &= 0, \\
M_{2,\theta_3} - 2M_{1,\theta_1} - M_1(M_2 + 2M_4) &= 0, \\
M_{3,\theta_3} - M_{4,\theta_2} - (M_1 + 1)M_3 &= 0, \\
M_{,\eta_4} &= 0 \text{ (which is always satisfied).}
\end{aligned}
$$

In the phase 4, if $\tilde{J}_4 = 0$ we cannot obtain any more group reductions and we need to prolong the system and then absorb and normalize the torsion coefficients of the prolonged system [7].

Conclusions

In this thesis, we reviewed the main steps in the solution of the equivalence problem of Elie Cartan, and carried out an explicit implementation of Cartan's method in the case of first order Lagrangians on the line. The solution of the equivalence problem is based on the construction of a complete set of local invariants, that were first obtained in [7]. An obvious question that arises from this work is whether these invariants have practical applications in other areas of mathematics. The answer to this question is a clear "yes". Indeed, the solution of the Lagrangian equivalence problem reviewed in this thesis leads to a highly original approach to the classification problem of binary forms in classical invariant theory, [10]. The local invariants of the equivalence problem for Lagrangians give rise to a complete set of local invariants for binary forms which can be used to derive necessary and sufficient conditions for equivalence which would not have been readily attainable in the content of classical methods in invariant theory.

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