METHOD OF THE INVERSE SCATTERING PROBLEM WITH SPECTRAL PARAMETER ON AN ALGEBRAIC CURVE

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In all versions of the method of the inverse scattering problem (MISP), nonlinear integrable equations arise as compatibility conditions of an overdetermined system of linear equations. If one is concerned with the integration of evolution equations on a line (in two-dimensional space–time), the compatibility of the overdetermined linear system must be identical with respect to the complex parameter λ, which, by analogy with the simplest version of the method (the Lax scheme) can be called the spectral parameter.

In the simplest formulation of MISP ([1], cf. also [2-4]) the overdetermined linear system has the form

$$\Psi_x = U(x, t, \lambda)\Psi, \quad \Psi_t = V(x, t, \lambda)\Psi, \tag{1}$$

where U and V are matrix rational functions with given disposition of a finite number of poles.

The compatibility condition for (1), the equation

$$U_t - V_x + [U, V] = 0 \tag{2}$$

for a suitable number and disposition of poles gives the majority of nonlinear systems which are integrable with the help of MISP.

It seems quite natural to generalize the scheme given in [1], by saying that the functions U and V are rational functions on an arbitrary algebraic curve. The parameter here is the uniformization parameter, and the poles must be given in a fundamental domain of the corresponding Fuchsian group. A conjecture about the possibility of constructing equations which are integrable in this way was made in [5]. For the simplest case when U and V are elliptic functions of λ (the genus of the curve is one), an interesting example was given in this same paper. Other examples of integrable equations with "elliptic" spectral parameter (which are also of physical interest) were constructed in [6, 7]. For the effective integration of these equations the development of the technique of the matrix Riemann problem on the torus is necessary. In application to the most interesting example [6] (the anisotropic Landau–Lifschitz equation), this was done recently in [8, 9]. An example of an integrable system with spectral parameter on an elliptic curve, constructively different from that mentioned above, was constructed in [10]. In addition, there exists another approach which allows one to construct integrable systems with spectral parameter on an algebraic curve, whose central point is the construction of infinite-dimensional Lie algebras of strictly identical growth over the ring of rational functions on the curve. Investigations in this direction were carried out by one of the authors, and the results will be published quickly.

The goal of the present paper is to present examples of integrable nonlinear equations, whose spectral parameter is situated on a curve of arbitrary genus (specifically on hyperelliptic curves). Along with this we shall show that the transfer of the spectral parameter to curves of genus $\rho \geqslant 1$ is associated with serious difficulties, the nature of which is clarified by the Riemann–Roch theorem. In the present paper we shall not use uniformization, but rather we represent algebraic curves by systems of quadrics.

1. Construction of Integrable Systems with the Help of Quadrics

In (1) let the functions U and V have simple poles λ_n ($n = 1, \ldots, N$) (which may coincide). We represent U and V as sums of partial fractions

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$$U = U_0 + \sum_{n=1}^{N} \frac{Z_n}{Z_0} U_n; \quad V = V_0 + \sum_{n=1}^{N} \frac{Z_n}{Z_0} V_n, \tag{3}$$

where

$$Z_0 = 1; \quad Z_n = \frac{1}{\lambda - \lambda_n}. \tag{4}$$

Among the Z_i there are $N(N-1)/2$ relations

$$Z_i Z_j = -\frac{Z_0}{\lambda_i - \lambda_j}(Z_j - Z_i), \quad i, j \neq 0, \quad i \neq j. \tag{5}$$

The relations (5) are linearly independent, any $N-1$ of them define in complex-projective space CP^N a rational curve, whose explicit uniformization is given by (4). The remaining $(N-1)(N-2)/2$ quadratic conditions (5) turn out to hold automatically on this curve. It is easy to see that it is impossible to add any additional equations which are quadratic in Z_i, and which are linearly independent from the equations of (5) but compatible with them.

To get equations which are integrable by means of (1), (3), it is necessary to substitute (3) into (2) and express all products $Z_i Z_j (i \neq j)$ by (5). Setting the coefficients of Z_i^2 and $Z_0 Z_i$ equal to zero gives $2N + 1$ equations in $2N + 2$ unknown functions U_i, V_i. The excess by one of the number of unknown functions over the number of equations explains the gauge freedom [2, 11] – the possibility of performing the transformation

$$\Psi' = g\Psi; \; U' = g_x g^{-1} + gUg^{-1}; \; V' = g_t g^{-1} + gVg^{-1}, \tag{6}$$

where $g = g(x, t)$ is an arbitrary matrix function.

It is easy to see that the general rational case does not differ from the one considered. If the degree of the divisor of poles of the functions U and V is N, then among the N partial fractions characterizing it there are $N(N-1)/2$ linearly independent quadratic relations, which can actually be used to calculate the equations which are integrable by the scheme of [1].

We try to generalize the scheme given by saying that U and V are represented in the form (3) as before, while the Z_i $(i = 0, 1, \ldots, N)$ are linearly independent and subject to the system of quadratic, linearly independent relations (quadrics)

$$\sum_{i,j=0}^{N} a_{ij}^l Z_i Z_j = 0, \quad l = 1, \ldots, M, \tag{7}$$

which intersect in a variety of dimensions one of which is an algebraic curve G. Then the Z_i are rational functions on G; we require that they be linearly independent (if not one can pass to a smaller collection of the Z_i). Now the compatibility condition (2) leads to a system of $L = (N+1)(N+2)/2 - M$ nonlinear equations in $2N + 2$ coefficient functions U_i, V_i. Taking account of the gauge invariance, this system will be completely determined if $L = 2N + 1$, so that $M = N(N-1)/2$. One has the following theorem.

THEOREM 1. Let the system of linearly independent quadrics of the form (7), where $M = N(N-1)/2$, intersect in an irreducible algebraic curve G, where the Z_i, as functions on this curve, are linearly independent. Then the curve G is rational.

As V. V. Shokurov pointed out to us, Theorem 1 was already known to mathematicians of the last century, including Bertini. At the present time it is clear that this theorem has important applications to the method of the inverse scattering problem, including some which have no relation to the topic of this paper. Hence we consider it appropriate to give two proofs of Theorem 1 which we have found.

The first proof is based on the Riemann–Roch theorem and requires additional a priori assumptions about the genus of the curve G. Let the curve G be realized by quadrics in N-dimensional space, and let its genus ρ satisfy the condition $2\rho + 1 < N$. We consider the quantities $y_i = Z_i/Z_0$. Due to the linear independence of the Z_i these quantities are linearly independent on the curve G and form a basis for the space $\mathcal{L}_y$ of nonconstant functions which are multiples of their common divisor (y). The pairwise products $y_i y_j$ are rational functions lying in the space $\mathcal{L}_{2y}$ of functions which are multiples of the divisor (2y). As shown in [12], for $N > 2\rho + 1$ any element of $\mathcal{L}_{2y}$ can be represented in the form $\sum_{i,j} a_{ij} y_i y_j + \sum_i b_i y_i + c$.

It follows from the Riemann–Roch theorem that the degree deg(y) of the divisor (y) is equal to $\deg(y) = N + \rho$. Hence the dimension of the space $\mathcal{L}_{2y}$ is equal to $N_1 = 2N + \rho$. The quadrics are linear relations among the $N(N + 1)/2$ elements of the form $y_i y_j$ and N elements of the form y_i. The total number of these relations is

$$M = \frac{N(N+1)}{2} + N - N_1 = \frac{N(N-1)}{2} - \rho. \tag{8}$$

Thus, the value $M = N(N - 1)/2$ is achieved only for $\rho = 0$. Actually, we have proved a more general assertion, getting for $N > 2\rho + 1$ the sharp estimate (8) for the maximal number of linearly independent quadrics which intersect in a curve of genus ρ.

The second proof is more elementary, and it does not allow us to get (8), but makes it possible to construct an explicit uniformization of the rational curve G.

Since the affine coordinates $y_i = Z_i/Z_0$ ($i = 1, \ldots, N$) on the curve are linearly independent, the curve cannot lie in a hyperplane of lower dimension. Let P be a point in general position on the curve. We carry the origin to P and we perform a linear orthogonal transformation to the moving frame of the curve so that the new coordinate X_1 is directed along the tangent, X_2 along the normal, etc. By dilation of axes one can arrange that

$$X_k = X_1^k + O(X_1^{k+1}), \quad k = 2, \ldots, N.$$

Passing to projective coordinates $X_k = W_k/W_0$, we note that in the new equations of the quadrics the monomials W_0^2 and W_0W_1 do not appear, so that the total number of monomials in the quadrics is equal to $(N + 1)(N + 2)/2 - 2$. We choose among them the $2N - 1$ basis monomials $W_1^2, W_1W_2, W_2^2, \ldots, W_{N-1}W_N, W_N^2$ and we expand the other monomials in terms of the basis. For the monomial W_iW_j we define the height to be $h = i + j$ and we denote the basis monomials by Φ_h, $2 \leqslant h \leqslant 2N$.

Expressing the remaining monomials in terms of the basis, we note that monomials of height h_0 can be expressed in terms of basis monomials of height $h \geqslant h_0$ only. We consider the collection of $N - 1$ monomials W_iW_N, $0 \leqslant i \leqslant N - 2$. The expansions of these monomials in terms of the basis completely determine the curve G. The inverse triangular transformations of the form $W_N' = W_N$; $W_{N-1}' = W_{N-1}$; $W_i' = W_i + \sum_{k=i+1}^{N} c_i^k W_k$, $0 \leqslant i \leqslant N - 2$ allow us to reduce the system of expansions mentioned to diagonal form $W_i'W_N' = \Phi_{i+N}'$. In this basis the entire collection of quadrics looks like this:

$$W_\alpha' W_\beta' = W_\gamma' W_\delta'; \quad \alpha + \beta = \gamma + \delta, \tag{9}$$

so that $W_k' = (W_1'/W_0')^k W_0'$, and the curve G is rational. The original projective coordinates can be uniformized as polynomials of degree N in the parameter W_1'/W_0'.

Since (9) means that all monomials of given height are equal to one another, it is impossible to add to the system (9) another quadric which is linearly independent from the preceding quadrics and the number $N(N - 1)/2$ is maximal.

2. Case of a Hyperelliptic Curve

We shall make the rest of the analysis on the concrete example when G is a hyperelliptic curve of genus n, defined by an equation

$$y^2 = X^{2n} + a_{2n+1}X^{2n-1} + \ldots + a_3X + a_2. \tag{10}$$

Introducing projective coordinates $y = Z_0/Z_1$, $X^k = Z_{k+1}/Z_k$, $1 \leqslant k \leqslant n$, we represent the curve (10) as the intersection of quadrics

$$\begin{gathered} Z_0^2 = Z_{n+1}^2 + a_{2n+1}Z_{n+1}Z_n + \ldots + a_2Z_1^2; \\ Z_\alpha Z_\beta = Z_\gamma Z_\delta, \ \alpha + \beta = \gamma + \delta, \ \ 1 \leqslant \alpha, \ \beta, \ \gamma, \ \delta \leqslant n + 1. \end{gathered} \tag{11}$$

The genus of the curve (10) is equal to $n - 1$, the number of quadrics in (11) is the maximum possible for a curve with this genus.

The compatibility condition (2) leads to the following equations ($N = n + 1$):

$$\frac{\partial U_i}{\partial t} - \frac{\partial V_i}{\partial x} = [U_i, V_0] + [U_0, V_i]; \tag{12}$$

$$\frac{\partial U_0}{\partial t} - \frac{\partial V_0}{\partial x} = [U_0, V_0] + [U_{n+1}, V_{n+1}]; \tag{13}$$

$$\sum_{s+l=k} [U_s, V_l] = a_k [U_{n+1}, V_{n+1}]; \tag{14}$$
$$s,\ l \geqslant 1, \quad k \leqslant 2n+1.$$

The systems (12), (13) give n + 2 differentials, and the system (13) gives 2n algebraic equations in 2n + 4 coefficient functions U_i, V_i ($i = 0, 1, \ldots, n + 1$). In all there are $n - 2$ equations more than functions. Considering the gauge freedom this difference coincides with the genus, equal to $n - 1$, and is in accord with (8). System (12)-(14) can be considered on any matrix algebra. Below we restrict ourselves to the case of the algebra $sl(2)$. In order to clarify the question of the consistency of this system, we note that it has the particular solution

$$U_i = \alpha_i \sigma_3, \quad V_i = \beta_i \sigma_3, \text{where} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{15}$$

and α_i, β_i are arbitrary complex constants. Further, considering $U_i = \alpha_i\sigma_3 + \delta U_i$, $V_i = \beta_i\sigma_3 + \delta V_i$, we linearize the system (12)-(14), taking δU_i, δV_i small. The direct analysis which we made for the simplest nontrivial case $n = 2$, shows that for any choice of constants α_i, β_i, the linearized system is consistent only in the trivial case when δU_i, δV_i commute with the matrix σ_3. This result can be explained as follows. In trying to effect the "dressing up" of the solution (15) in the spirit of [1], we are led to a matrix Riemann problem on an algebraic curve (for $n = 2$ on the torus), whose solvability requires additional conditions (cf., e.g., [13]), which can turn out to be unsatisfied identically in the variables x and t.

3. Reduction and General Covariance

We shall show that on the system of equations (12)-(14) one can impose additional conditions which guarantee its consistency. Let

$$\begin{gathered} U_0 = u_0\sigma_3; \quad V_0 = v_0\sigma_3; \quad \boldsymbol{\sigma} = (\sigma_1, \sigma_2); \\ U_i = \mathbf{u}_i\boldsymbol{\sigma}; \quad V_i = \mathbf{v}_i\boldsymbol{\sigma}; \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \end{gathered} \tag{16}$$

Here the two-dimensional vector $\boldsymbol{\sigma}$ has components σ_1, σ_2. The restrictions (16) are consistent with the system (12)-(14) and have the meaning of reductions in it (cf. [1] and [4]). Now each of the algebraic equations (13) imposes only one condition on the two-dimensional vectors $\mathbf{u}_i$, $\mathbf{v}_i$. Hence system (12)-(14) now defines 4n + 3 scalar equations in 4n + 6 real functions. We note that under reduction of (16) gauge freedom is restricted by multiplication of a simultaneous solution of (1), Ψ, by a matrix of the form $g(x, t)\,\sigma_3$ and allows us to decrease the number of unknown functions by one. Considering this circumstance independently from the genus of the curve the number of unknown functions is two more than the number of equations. This circumstance is explained by the general covariance of the system. In fact, in our case the linear system has the form (1)

$$\left.\begin{aligned} Z_0\left(\frac{\partial}{\partial x} - U_0\right)\Psi &= \sum_{i=1}^{n+1} Z_i U_i \Psi; \\ Z_0\left(\frac{\partial}{\partial t} - V_0\right)\Psi &= \sum_{i=1}^{n+1} Z_i V_i \Psi. \end{aligned}\right\} \tag{17}$$

Under a change of coordinates $x = x(t', x')$, $t = t(t', x')$, depending on two arbitrary functions, (17) preserves its form, while the substitution $U_i \to U_i'$, $V_i \to V_i'$ occurs, where

$$\left.\begin{aligned} U_i' &= \frac{\partial x}{\partial x'} U_i + \frac{\partial t}{\partial x'} V_i; \\ V_i' &= \frac{\partial x}{\partial t'} U_i + \frac{\partial t}{\partial t'} V_i. \end{aligned}\right\} \tag{18}$$

The general covariance of the consistency conditions is a characteristic feature of all nonlinear equations, which are consistency conditions of systems of type (1), if the divisors of poles of the functions U and V coincide. In particular, this is true on a rational curve also.

Thus, after eliminating gauge and covariant freedoms, we have gotten a system in which the number of equations is equal to the number of unknown functions. Linearization of it near the particular solution $u_i = v_i = 0$, $i \geqslant u_0 = \alpha$, $v_0 = \beta$ with α, β arbitrary constants, obviously gives a consistent linear system. We note further that for a special choice of the coefficients a_i the hyperelliptic curve (3) degenerates into a rational curve. In this case the system (12)-(14) under the condition (16) is a new type of reduction in problems with rational spectral parameter. In order to extract equations capable of having concrete physical applications from the system (12)-(14), it is necessary to solve the algebraic relations (14) explicitly, to make a choice of curvilinear system of coordinates in the space x, t, and also to fix the gauge freedom (the latter operation is performed with a great degree of freedom). In the simplest rational case n = 1 there arise here in particular the sine-Gordon equation and the familiar Redzhe—Lund system [2]. We have seen that in case n = 2 (ρ = 1) the algebraic conditions (13) can be solvable explicitly but we are still unable to get equations having specific physical applications. Hence we do not give the explicit (and rather involved) expressions for the final equations which arise under several tested methods of eliminating the nonuniqueness.

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