GENERAL CRITERION OF INVARIANCE FOR INTEGRO-DIFFERENTIAL EQUATIONS

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General criterion of invariance of integro-differential equations under Lie symmetry group of point transformations is derived. It is a generalization of the previous form of the criterion to the case of a moving range of integration. This is the situation when a region of integration depends on external, with respect to integration, variables which leads to its explicit dependence on a group parameter.

Key words: symmetry, integro-differential equations, Lie groups.

1. Introduction

We present a general and direct method of determination of symmetry groups of point transformations for integro-differential equations. The method is a natural generalization of the Ovsiannikov method for differential equations [1]–[5]. We consider a system of integro-differential equations (IDE's) of the form

$$F(x^1,\ldots,x^n,y,\underset{1}{y},\ldots,\underset{m}{y}) + \int\limits_{X(x^{l+1},\ldots,x^n)} dx'^1\cdots dx'^l f(x'^1,\ldots,x'^l,x^1,\ldots,x^n,y,\underset{1}{y},\ldots,\underset{k}{y}) = 0, \quad (1)$$

where n,m,k,l are arbitrary natural numbers ($l \leq n$), $x=(x^1,\ldots,x^n)$, functions F and f are arbitrary but sufficiently regular to secure the existence of solutions to (1), limits of integrations (region X) are also arbitrary and can depend on external, with respect to integration, variables $x^{l+1},\ldots,x^n$. The symbol $\underset{m}{y}$ denotes the set of all partial derivatives of m-order:

$$\underset{m}{y} = \left\{ \frac{\partial^m y}{\partial x^{i_1}\cdots\partial x^{i_m}} \equiv \partial_{x^{i_1}}\cdots\partial_{x^{i_m}} y \equiv y_{i_1\cdots i_m} \right\}.$$

For $f=0$ the equation (1) reduces to differential equation, thus our method contains the Ovsiannikov method as a particular case.

The case of IDE's with a region of integration independent on external variables $x^{l+1},\ldots,x^n$ was considered in [6, 7]. The aim of this work is to generalize the previous results to the case when the boundary ∂X of the region of integration X can move under point transformations due to dependence on external variables $x^{l+1}, \ldots, x^n$:

$$\partial X \ni \underline{x} = (\underline{x}^1, \ldots, \underline{x}^l), \qquad \underline{x}^i = \phi^i(x^{l+1}, \ldots, x^n). \tag{2}$$

The functions ϕ^i defining limits of integrations are arbitrary, sufficiently smooth, functions.

In order to simplify the notation we restrict our considerations to the one scalar equation of the type (1). This reduces a number of indices in subsequent formulae. For a system of equations with p dependent variables $y = (y^1, \ldots, y^p)$ some minor changes are evident and the resulting criterion is to be applied to each equation of the system. Possible generalizations of the form of (1) are discussed in section 3.

We look for a Lie symmetry group of the *point transformations*

$$\begin{aligned} \widetilde{x}^i &= e^{\epsilon G} x^i = x^i + \epsilon \xi^i(x, y) + \mathcal{O}(\epsilon^2) \\ \widetilde{y} &= e^{\epsilon G} y = y + \epsilon \eta(x, y) + \mathcal{O}(\epsilon^2), \end{aligned} \tag{3}$$

with the *infinitesimal generator* (summation over repeated indices is assumed)

$$G = \xi^i(x, y)\partial_{x^i} + \eta(x, y)\partial_y, \tag{4}$$

admitted by the system of IDE's (1). For dummy variables $x^{'1}, \ldots, x^{'l}$ of integration we have the same law of transformation, the same functions ξ^i, as for variables $x^1, \ldots, x^l$

$$\widetilde{x}^{'i} = e^{\epsilon G} x^{'i} = x^{'i} + \epsilon \xi^i(x^{'}, y) + \mathcal{O}(\epsilon^2) \quad \text{for } i = 1, \ldots, l. \tag{5}$$

Primed and unprimed variables are distinguished only by the value of a variable. All points of the l-dimensional subspace of independent variables obey the same transformation law. This for instance, concerns points of the region X: inner points (dummy variables) $x \in X$ transform like boundary points $\underline{x} \in \partial X$. Another possibility for the boundary points appears only when they depend on external variables $x^{l+1}, \ldots, x^n$. Then, additional mechanism of a change of boundary points $\underline{x}$ comes through functions ϕ^i (2). If we want to explicitly distinguish dummy variables of integrations in the law of transformation (3), by adding the formula (5), we have to add to the definition (4) of the infinitesimal generator the following term

$$\sum_{i=1}^{l} \xi^i(x^{'}, y)\partial_{x^{'i}}. \tag{6}$$

We do not use the summation convention when summation does not go over the whole range of an index, here $1, \ldots, l \leq n$ only.

The determination of this group reduces to finding its algebra spanned by independent generators of the form (4). As in the Ovsiannikov method we extend the group of point transformations (3) to a jet space of independent and dependent variables and derivatives of dependent variables in the usual way [1]–[6]

$$\begin{aligned}
&\widetilde{x}^i = e^{\epsilon G^{(m)}} x^i = x^i + \epsilon \xi^i(x,y) + \mathcal{O}(\epsilon^2)\\
&\widetilde{y} = e^{\epsilon G^{(m)}} y = y + \epsilon \eta(x,y) + \mathcal{O}(\epsilon^2)\\
&\widetilde{y}_i = e^{\epsilon G^{(m)}} y_i = y_i + \epsilon \eta_i(x,y,\underset{1}{y}) + \mathcal{O}(\epsilon^2)\\
&\vdots\\
&\widetilde{y}_{i_1\cdots i_m} = e^{\epsilon G^{(m)}} y_{i_1\cdots i_m} = y_{i_1\cdots i_m} + \epsilon \eta_{i_1\cdots i_m}(x,y,\underset{1}{y},\ldots,\underset{m}{y}) + \mathcal{O}(\epsilon^2),
\end{aligned} \tag{7}$$

where the *extended generator* is of the form

$$G^{(m)} = G + \eta_i \partial_{y_i} + \cdots + \eta_{i_1\cdots i_m} \partial_{y_{i_1\cdots i_m}}. \tag{8}$$

The coefficients $\eta_i, \ldots, \eta_{i_1\cdots i_m}$, defining the *extended group*, are given by the recursion relations:

$$\begin{aligned}
&\eta_i = D_i \eta - y_j D_i \xi^j\\
&\vdots\\
&\eta_{i_1\cdots i_m} = D_{i_m} \eta_{i_1\cdots i_{m-1}} - y_{i_1\cdots i_{m-1} j} D_{i_m} \xi^j
\end{aligned} \tag{9}$$

and the total derivative D_i is defined as follows

$$D_i = \partial_i + y_i \partial_y + y_{ij} \partial_{(y_j)} + \cdots + y_{i i_1\cdots i_n} \partial_{(y_{i_1\cdots i_n})} + \cdots .$$

The relations (9) follow from the requirement of preservation of the contact structure of a jet bundle. Simply, we demand that the extended variables $\partial_i y \equiv y_i$, $\partial_i \partial_j y \equiv y_{ij}$, ... now formally treated as independent variables, are transformed under the group action (7) as ordinary derivatives of the function y. The above procedure of lifting the group of point transformations (3) to a jet bundle is the essence of the Ovsiannikov method. In terms of a jet space a differential equation is equivalent to an algebraic equation, and thus is much easier tractable. In the case of the integro-differential equations (1) we use the same method to deal with derivatives.

2. Criterion

Invariance of an equation means invariance of the space of its solutions. Thus, point transformation (3) maps any solution $y(x)$ of the equation (1) into another solution $\widetilde{y}(\widetilde{x})$ of the equation. New solutions of (1) can be constructed in this way. In our geometric language solutions $y(x)$ are represented by their graphs in a jet space. In terms of a jet space the notion of the form of equation has the precise meaning [3], and the invariance of the space of solutions is equivalent to the invariance of the form of the equation under extended transformations (7). Thus, to obtain our criterion of invariance of IDE's of the type (1), we act on (1) by extended transformations (7) writing down explicitly only terms that are linear with respect to the group parameter ϵ. Next, by expanding functions F, f, and ϕ^i in their Taylor series and changing variables in the integral, we express the change of (1) in terms of the extended generator (8). This change must be equal to zero for all values of ϵ.

The change ΔF of the differential term of (1) is calculated by expanding the function F in a Taylor series and substituting the extended transformations (7)

$$\begin{aligned}\Delta F &= F(\widetilde{x},\widetilde{y},\underset{1}{\widetilde{y}},\dots,\underset{m}{\widetilde{y}}) - F(x,y,\underset{1}{y},\dots,\underset{m}{y})\\ &= F\big(x^1+\epsilon\xi^1+\mathcal{O}(\epsilon^2),\dots,x^n+\epsilon\xi^n+\mathcal{O}(\epsilon^2), y+\epsilon\eta+\mathcal{O}(\epsilon^2), y_1+\epsilon\eta_1+\mathcal{O}(\epsilon^2),\\ &\qquad \dots, y_n+\epsilon\eta_n+\mathcal{O}(\epsilon^2), y_{i_1\dots i_m}+\epsilon\eta_{i_1\cdots i_m}+\mathcal{O}(\epsilon^2)\big) - F(x,y,\underset{1}{y},\dots,\underset{m}{y})\\ &= \epsilon\big[\xi^i\partial_{x^i}F+\eta\partial_y F+\eta_i\partial_{y_i}F+\cdots\eta_{i_1\cdots i_m}\partial_{y_{i_1}}\cdots\partial_{y_{i_m}}F\big]+\mathcal{O}(\epsilon^2).\end{aligned}$$

Due to the definition of the extended generator (8) we can rewrite the above result in the form

$$\Delta F = \epsilon\, G^{(m)} F(x,y,\underset{1}{y},\dots,\underset{m}{y}) + \mathcal{O}(\epsilon^2). \tag{10}$$

Thus, the condition $\Delta F = 0$ leads to the Ovsiannikov infinitesimal criterion of invariance for differential equation $G^{(m)}F(x,y,\underset{1}{y},\dots,\underset{m}{y}) = 0$.

Let us consider the change of an integral term in the equation (1)

$$\Delta I = \int\limits_{\widetilde{X}(\widetilde{x}^{l+1},\dots,\widetilde{x}^n)} d\widetilde{x}'^1\cdots d\widetilde{x}'^l f(\widetilde{x}',\widetilde{x},\widetilde{y},\underset{1}{\widetilde{y}},\dots,\underset{k}{\widetilde{y}}) - \int\limits_{X(x^{l+1},\dots,x^n)} dx'^1\cdots dx'^l f(x',x,y,\underset{1}{y},\dots,\underset{k}{y})$$

under the extended transformations (7). To this end, we change variables in the first integral according to this transformations (see (5))

$$\{\widetilde{x}'^1,\dots,\widetilde{x}'^l\}\mapsto\{x'^1,\dots,x'^l\}.$$

By virtue of (5) the elements of Jacobi's matrix are equal

$$\frac{\partial\widetilde{x}'^i}{\partial x'^j} = \delta_{ij} + \epsilon\frac{\partial\xi^i}{\partial x'^j} + \mathcal{O}(\epsilon^2), \quad i,j=1,\dots,l.$$

Since the off-diagonal elements of the matrix are of the order $\mathcal{O}(\epsilon^2)$, the linear contribution to the Jacobian comes only from the product of the diagonal elements

$$\frac{\partial(\widetilde{x}'^1\cdots\widetilde{x}'^l)}{\partial(x'^1\cdots x'^l)} = \left(1+\epsilon\frac{\partial\xi^1}{\partial x'^1}\right)\cdots\left(1+\epsilon\frac{\partial\xi^l}{\partial x'^l}\right)+\mathcal{O}(\epsilon^2) = 1+\epsilon\sum_{i=1}^{l}\frac{\partial\xi^i}{\partial x'^i}+\mathcal{O}(\epsilon^2).$$

Consequently, the change ΔI of the integral term is equal

$$\int\limits_{X(\widetilde{x}^{l+1},\dots,\widetilde{x}^n)} dx'^1 \cdots dx'^l \Bigg[\Bigg(1+\epsilon\sum_{i=1}^{l}\frac{\partial\xi^i}{\partial x'^i}\Bigg) f\Big(x'^1+\epsilon\xi^1+\mathcal{O}(\epsilon^2),\dots,x'^l+\epsilon\xi^l+\mathcal{O}(\epsilon^2),$$
$$x^1+\epsilon\xi^1+\mathcal{O}(\epsilon^2),\dots,x^n+\epsilon\xi^n+\mathcal{O}(\epsilon^2),$$
$$y+\epsilon\eta+\mathcal{O}(\epsilon^2), y_1+\epsilon\eta_1+\mathcal{O}(\epsilon^2),\dots,y_n+\epsilon\eta_n+\mathcal{O}(\epsilon^2),$$
$$\dots,y_{i_1\cdots i_k}+\epsilon\eta_{i_1\cdots i_k}+\mathcal{O}(\epsilon^2)\Big)\Bigg] - \int\limits_{X(x^{l+1},\dots,x^n)} dx'^1\cdots dx'^l f(x', x, y, \underset{1}{y},\dots,\underset{k}{y}) + \mathcal{O}(\epsilon^2).$$

Due to dependence of X on external variables we have $X(\widetilde{x}^{l+1},\dots,\widetilde{x}^n) \neq X(x^{l+1},\dots,x^n)$, contrary to the previous case [6]. We represent the region $X(\widetilde{x}^{l+1},\dots,\widetilde{x}^n)$ as an union of the set $X(x^{l+1},\dots,x^n)$ and an oriented set $\Delta X(x^{l+1},\dots,x^n)$ in the following way

$$X(\widetilde{x}^{l+1},\dots,\widetilde{x}^n) = X(x^{l+1},\dots,x^n) \cup \Delta X(x^{l+1},\dots,x^n),$$

where the oriented set ΔX comes from the oriented boundary ∂X under the group transformation (3). We assume standard orientation of the boundary ∂X: the direction of the positive surface normal $\mathbf{n}$ is outward from the bounded volume X. Parts of the region ΔX which are build from ∂X in direction of the positive normal to ∂X under group action (3), that is the scalar product of the normal $\mathbf{n}$ and the vector $\boldsymbol{\xi} = (\xi^1,\dots,\xi^l)$ is greater than zero, are positive. They correspond to expansion of the region X and do not intersect with it. The other parts of ΔX have negative sign and correspond to shrinking of X. They are contained in X. The integral over oriented ΔX inherits the signs of parts of ΔX. Thus, the integral over the set $X(\widetilde{x}^{l+1},\dots,\widetilde{x}^n)$ is equal to the sum of integrals over $X(x^{l+1},\dots,x^n)$ and $\Delta X(x^{l+1},\dots,x^n)$. For the integral over $X(x^{l+1},\dots,x^n)$ we can proceed in the same way as in [6]. It leads to the change $\Delta I'$ of the integral term

$$\Delta I' = \int\limits_{X(x^{l+1},\dots,x^n)} dx'^1 \cdots dx'^l \Bigg[\Bigg(1+\epsilon\sum_{i=1}^{l}\frac{\partial\xi^i}{\partial x'^i}\Bigg) f\Big(x'^1+\epsilon\xi^1+\mathcal{O}(\epsilon^2),\dots,x'^l+\epsilon\xi^l+\mathcal{O}(\epsilon^2),$$
$$x^1+\epsilon\xi^1+\mathcal{O}(\epsilon^2),\dots,x^n+\epsilon\xi^n+\mathcal{O}(\epsilon^2),$$
$$y+\epsilon\eta+\mathcal{O}(\epsilon^2), y_1+\epsilon\eta_1+\mathcal{O}(\epsilon^2),\dots,y_n+\epsilon\eta_n+\mathcal{O}(\epsilon^2),$$
$$\dots,y_{i_1\cdots i_k}+\epsilon\eta_{i_1\cdots i_k}+\mathcal{O}(\epsilon^2)\Big) - f(x', x, y, \underset{1}{y},\dots,\underset{k}{y})\Bigg] + \mathcal{O}(\epsilon^2).$$

Expanding the function f into a Taylor series we obtain

$$\Delta I' = \epsilon \int\limits_{X(x^{l+1},\dots,x^n)} dx'^1 \cdots dx'^l \Bigg[\sum_{i=1}^{l}\xi^i\partial_{x'^i}f + \xi^i\partial_{x^i}f + \eta\partial_y f + \eta_i\partial_{y_i}f + \cdots$$
$$+\eta_{i_1\cdots i_k}\partial_{y_{i_1}}\cdots\partial_{y_{i_k}}f + f\sum_{i=1}^{l}\frac{\partial\xi^i}{\partial x'^i}\Bigg] + \mathcal{O}(\epsilon^2).$$

In view of the definition of the extended generator (8) with (4) and (6) we can rewrite the above result as follows

$$\Delta I' = \epsilon \int\limits_{X(x^{l+1},\dots,x^n)} dx'^1 \cdots dx'^l \Bigg[G^{(k)} f(x', x, y, \underset{1}{y}, \dots, \underset{k}{y}) + f(x', x, y, \underset{1}{y}, \dots, \underset{k}{y}) \sum_{i=1}^{l} \frac{\partial \xi^i}{\partial x'^i} \Bigg] + \mathcal{O}(\epsilon^2). \quad (11)$$

Integration over $\Delta X(x^{l+1}, \dots, x^n)$ leads to the following change $\Delta I''$ of the integral term

$$\begin{aligned}\Delta I'' = \int\limits_{\Delta X(x^{l+1},\dots,x^n)} dx'^1 \cdots dx'^l \Bigg[\Bigg(1 + \epsilon \sum_{i=1}^{l} \frac{\partial \xi^i}{\partial x'^i}\Bigg) f\Big(& x'^1 + \epsilon\xi^1 + \mathcal{O}(\epsilon^2), \dots, x'^l + \epsilon\xi^l + \mathcal{O}(\epsilon^2), \\ & x^1 + \epsilon\xi^1 + \mathcal{O}(\epsilon^2), \dots, x^n + \epsilon\xi^n + \mathcal{O}(\epsilon^2), \\ & y + \epsilon\eta + \mathcal{O}(\epsilon^2), y_1 + \epsilon\eta_1 + \mathcal{O}(\epsilon^2), \dots, y_n + \epsilon\eta_n + \mathcal{O}(\epsilon^2), \\ & \dots, y_{i_1\cdots i_k} + \epsilon\eta_{i_1\cdots i_k} + \mathcal{O}(\epsilon^2)\Big) \Bigg] + \mathcal{O}(\epsilon^2).\end{aligned}$$

Integration over $\Delta X(x^{l+1}, \dots, x^n)$ can be split into integration over the boundary ∂X and inner one-dimensional integration in perpendicular direction to ∂X along the normal $\mathbf{n}$. The integration range in this direction is infinitesimal and is equal to the projection of the change of a boundary point

$$\Delta \underline{x}^i = \underline{\widetilde{x}}^i - \underline{x}^i = \sum_{j=l+1}^{n} \frac{\partial \phi^i}{\partial x^j} \Delta x^j = \epsilon \sum_{j=l+1}^{n} \frac{\partial \phi^i}{\partial x^j} \xi^j(\underline{x}, y) + \mathcal{O}(\epsilon^2).$$

on the direction $\mathbf{n}$. Applying the mean value theorem to this one-dimensional integral over the infinitesimal range

$$\sum_{i=1}^{l} n^i \Delta \underline{x}^i$$

we express $\Delta I''$ as a surface integral over ∂X

$$\Delta I'' = \epsilon \int\limits_{\partial X} d\sigma \sum_{i=1}^{l} n^i \sum_{j=l+1}^{n} \frac{\partial \phi^i}{\partial x^j} \xi^j(\underline{x}, y) f(\underline{x}, x, y, \underset{1}{y}, \dots, \underset{k}{y}) + \mathcal{O}(\epsilon^2),$$

where $d\sigma$ denotes the measure on ∂X. By the use of the Gauss's integral theorem we obtain

$$\Delta I'' = \epsilon \int\limits_{X(x^{l+1},\dots,x^n)} dx'^1 \cdots dx'^l \sum_{i=1}^{l} \partial_{x'^i} \sum_{j=l+1}^{n} \frac{\partial \phi^i}{\partial x^j} \xi^j(x', y) f(x', x, y, \underset{1}{y}, \dots, \underset{k}{y}) + \mathcal{O}(\epsilon^2). \quad (12)$$

Combining the results (10)–(12) for ΔF and $\Delta I = \Delta I' + \Delta I''$ we finally arrive at:

$$\Delta = \Delta F + \Delta I = \epsilon\, G^{(m)} F(x, y, \underset{1}{y}, \ldots, \underset{m}{y})$$
$$+ \epsilon \int\limits_{X(x^{l+1}, \ldots, x^n)} dx'^1 \cdots dx'^l \left[G^{(k)} f(x', x, y, \underset{1}{y}, \ldots, \underset{k}{y}) + f(x', x, y, \underset{1}{y}, \ldots, \underset{k}{y}) \sum_{i=1}^{l} \frac{\partial \xi^i(x', y)}{\partial x'^i} \right.$$
$$\left. + \sum_{i=1}^{l} \partial_{x'^i} \sum_{j=l+1}^{n} \frac{\partial \phi^i}{\partial x^j}\, \xi^j(x', y) f(x', x, y, \underset{1}{y}, \ldots, \underset{k}{y}) \right] + \mathcal{O}(\epsilon^2)$$

By setting the total change Δ of the equation (1) to zero we derive the following *infinitesimal criterion of invariance* of integro-differential equations of the type of (1) under the point transformations (3):

THEOREM 1. *Necessary condition for invariance of the integro-differential equation of the type* (1) *under the point transformations* (3) *has the following form*

$$G^{(m)} F(x, y, \underset{1}{y}, \ldots, \underset{m}{y})$$
$$+ \int\limits_{X(x^{l+1}, \ldots, x^n)} dx'^1 \cdots dx'^l \left[G^{(k)} f(x', x, y, \underset{1}{y}, \ldots, \underset{k}{y}) + f(x', x, y, \underset{1}{y}, \ldots, \underset{k}{y}) \sum_{i=1}^{l} \frac{\partial \xi^i(x', y)}{\partial x'^i} \right.$$
$$\left. + \sum_{i=1}^{l} \partial_{x'^i} \sum_{j=l+1}^{n} \frac{\partial \phi^i}{\partial x^j}\, \xi^j(x', y) f(x', x, y, \underset{1}{y}, \ldots, \underset{k}{y}) \right] = 0 \quad \textit{on solutions of } (1). \tag{13}$$

According to the criterion (13) we have to take into account the equation (1), which is now a constraint on extended variables. Using this equation we can eliminate some of them. Remaining variables are independent, thus the equation (13) must be satisfied identically with respect to them. It means that the coefficients in front of independent expressions, involving these variables, must be equal to zero. This leads to the system of the so called *determining equations*. They are homogeneous and *linear* integro-differential equations for coefficients ξ^i, η determining the generator (4), and hence the point transformations (3). In applications, we have additional information in each particular case. Often, this information enables us to go to the integrands in the integral determining equations by using the Lagrange lemma of variational calculus [8]. This leads to differential determining equations.

The criterion (13) is a *necessary* condition for finding a symmetry group of the equation (1), so it allows us to find all *possible* symmetry transformations of (1). The difficult task is to find a sufficient condition for symmetry of an equation. To this end one needs a theorem on *global* existence and uniqueness of the solutions of the equation (1). The latter problem is far from being solved, see [9]. From a practical point of view the necessary condition is more important and useful than the sufficient one as the main task is to find symmetry transformations. A possible symmetry transformation of the equation (1) can be easily verified by inspection and this should be done anyway.

3. Discussion

The presented method can be easily generalized [7] to IDE's of the form

$$W(F, I) = 0,$$

where F and I denote differential and integral parts of (1) respectively and W is an arbitrary smooth function. Then, the criterion (13) takes the form

$$\frac{\partial W}{\partial F} G^{(m)} F + \frac{\partial W}{\partial I} \int\limits_{X(x^{l+1},\ldots,x^n)} dx'^1 \cdots dx'^l \left[G^{(k)} f + f \sum_{i=1}^{l} \frac{\partial \xi^i}{\partial x'^i} + \sum_{i=1}^{l} \partial_{x'^i} \sum_{j=l+1}^{n} \frac{\partial \phi^i}{\partial x^j} \xi^j f \right] = 0$$

on solutions of $W(F, I) = 0$. Generalization to the case of more than one integral term is trivial [7]. The equation (1) corresponds to $W = F + I$.

As is shown in [10] dependent variable y can be complex and can contains functional, for example delayed, arguments.

Other methods of investigations of symmetries of IDE's can be found in [11], in CRC Handbook [12], and in references therein. *Indirect* methods are based on a transformation of a given set of IDE's to an equivalent set of auxiliary equations for which symmetries are known or can be found by known methods. Then symmetries of the initial system of IDE's can be reconstructed. Usually, this auxiliary set of equations consists of PDE's as, for example, in Taranov's method [13] for Vlasov–Maxwell equations. Another indirect approach is based on an extension of the Harrison and Estabrook method [14] to the case of IDE's. A given set of equations is transformed to an equivalent set of differential forms.

General *direct* method is presented in [5] and in Vol. 3 of CRC handbook [12]. The method makes use of the assumption that the derivative with respect to the group parameter of a transformed IDE vanishes at zero value of this parameter. This corresponds to vanishing of a Lie derivative of the equation. When this condition is properly evaluated, i.e. when the dependence of limits of an integral on the group parameter is taken into account, it leads to our criterion of symmetry of IDE's (1). However, this evaluation is not easy even for constant limits and must be done each time when this condition is used. This may be suitable for a computer (see [15]) but not for a man. The problem disappears for Bäcklund symmetries in the canonical form of vertical transformations [12], but only in the case of independence of integration limits on external variables. There is no transformation of independent variables in that case. However, the equivalence criterion (Vol. 3 of [12]) should be checked which is sometimes overlooked. The method was used in [16, 17] for finding symmetries of the Boltzmann equation of a special kind.

The sophisticated method of Vinogradov and Krasilshchik [18, 19] has arisen from a simple idea of elimination of integrals from IDE's by virtue of the fundamental theorem of calculus by prolongation to *nonlocal* variables: the primitive functions of dependent variables. This is natural for some types the of IDE's, for example Volterra type . However, the most important IDE's in physics contain integrals with constant limits for which this construction is somewhat artificial and complicated. The method becomes indirect since it leads to the so called *boundary-differential* equations [19]. The method requires advanced and sophisticated mathematics, for example the theory of coverings of a system of differential equations and the prolongation procedure for boundary–differential equations.

Another direct method of finding symmetries of Vlasov–Maxwell equations can be found in papers [20, 21]. It consist in treating some integrals defining moments of the distribution function as new variables. This is possible due to special form of the equations containing such quantities with the physical meaning of charge and current. By the proper and consequent use of this idea mathematical tricks with Dirac's δ made by the authors can be avoided. Treating some terms as new variables even in final result change an interpretation of a symmetry group. It may be interesting for parameters leading to the so called renormalization group but it requires reconstruction of a symmetry group corresponding to the standard variables.

The method presented in this paper is the most direct one and is a natural generalization of the Ovsiannikov theory to the case of IDE's. All local quantities in the equations, such as derivatives or delayed arguments, are treated with the use of extension (prolongation) procedure in the spirit of classical Lie theory. For nonlocal quantities (integrals) such a procedure is ineffective so we leave them unchanged and look for a new form of the criterion of invariance. In simple, one-dimensional special case with constant limits of integration the criterion (13) appeared first in [22] for the Vlasov–Maxwell equations. However the derivation of criterion making use of the parametrization of an integral was not general and had a value of a mathematical trick. This was probably the reason why this paper was overlooked or underestimated by the Lie-symmetry community [1]–[5], [11, 12].

4. Conclusions

The criterion of invariance under a Lie symmetry group of point transformations was derived for very general form integro-differential equations. This is an essential generalization of the previous form of the criterion that has been successfully applied to important integro-differential equations such as Vlasov–Maxwell for multi-component plasma or nonlocal nonlinear Schrödinger (nNLS) equations. The present extension of the theory provides means for much wider application range. The proposed theory is most direct one, and more general and much easier to apply than so far proposed approaches.

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