SYMMETRIES OF INTEGRO-DIFFERENTIAL EQUATIONS

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A new, general method is presented for the determination of Lie symmetry groups of integro-differential equations. The exhibited method is a natural extension of the Ovsiannikov method developed for differential equations. The method leads to significant applications for instance to the Vlasov-Maxwell equations.

1. Introduction

The aim of this work is to establish a general and direct method of determination of symmetry groups of point transformations for integro-differential equations. The method is a natural generalization of the Ovsiannikov method for differential equations ([1]-[6]).

The lack of a universally valid method of obtaining symmetry groups for integro-differential equations has led to *ad hoc* means specially adapted for each case. The most important and characteristic example of such a procedure in plasma physics is the Taranov indirect method [7]. Taranov transformed the Vlasov-Maxwell equations for one-component plasma into an infinite chain of differential equations for moments of a distribution function. For the latter system of equations the Ovsiannikov method can be used and then a symmetry group for Vlasov-Maxwell equations can be reconstructed. Another approach was proposed by Vinogradov and Krasilshchik ([8]). They generalized the Ovsiannikov method by admitting an extension to nonlocal variables. However, their method applies only to a special case of integro-differential equations with variable limits of integration. It is a special kind of Volterra equations, which can be reduced to initial value problem for differential equation. This is not the case of the most important integro-differential equations in physics such as equations of kinetic theory.

Since, in general, it is not possible to transform an integral structure of equations into an algebraic one by admitting nonlocal variables, we stay in a jet space to deal with derivatives, as in the Ovsiannikov method, and look for a new infinitesimal criterion of symmetry in the case of integro-differential equations. The criterion of symmetry, which is in closed form in terms of extended generators, was obtained and the corresponding determining equations were solved for nontrivial cases.

2. Definition of symmetry of integro-differential equations

We look for a symmetry group of a system of integro-differential equations of the form

$$F(x_1,\ldots,x_n,y,\underset{1}{y},\ldots,\underset{m}{y})+\int\limits_X dx_1\cdots dx_l f(x_1,\ldots,x_n,y,\underset{1}{y},\ldots,\underset{k}{y})=0, \tag{1}$$

where n,m,k,l are arbitrary natural numbers ($l\leq n$), $x=\{x_1,\ldots,x_n\}$, functions F and f are arbitrary but sufficiently regular to secure the existence of solutions to (1), limits of integrations (region X) are also arbitrary. The symbol $\underset{m}{y}$ denotes the set of all partial derivatives of m-order:

$$\underset{m}{y}=\left\{\frac{\partial^m y}{\partial x_{i_1}\cdots\partial x_{i_m}}\equiv\partial_{x_{i_1}}\cdots\partial_{x_{i_m}}y\equiv\partial_{i_1}\cdots\partial_{i_m}y\right\}. \tag{2}$$

The equation (1) reduces to differential equation for $f=0$, thus our method contains the Ovsiannikov method as a particular case.

We restrict our considerations to one scalar equation of the type (1) for the sake of simplicity. It involves less indices in subsequent formulae, see (6)-(8). For a system of equations for p dependent variables $y=\{y_1,\ldots,y_p\}$ some minor changes are evident and the resulting criterion is to be applied to each equation of the system.

We look for Lie symmetry group of *point transformations*

$$\begin{aligned}\widetilde{x}_i &= X(x,y,\epsilon) = e^{\epsilon G}x_i = x_i+\epsilon\xi_i(x,y)+\mathcal{O}(\epsilon^2)\\ \widetilde{y} &= Y(x,y,\epsilon) = e^{\epsilon G}y = y+\epsilon\eta(x,y)+\mathcal{O}(\epsilon^2),\end{aligned} \tag{3}$$

which leave the equation (1) invariant. Determination of this group reduces to finding of infinitesimal generators of one-parameter transformations

$$G=\sum_{i=1}^{n}\xi_i(x,y)\partial_{x_i}+\eta(x,y)\partial_y. \tag{4}$$

Invariance of an equation under group of transformations means invariance of the space of its solutions. Thus, group transformation (3) maps any solution $y(x)$ of the equation (1) into another solution $\widetilde{y}(x)$ of the equation as follow

$$\begin{aligned}&F(x,y,\underset{1}{y},\ldots,\underset{m}{y})+\int\limits_X dx_1\cdots dx_l f(x_1,\ldots,x_n,y,\underset{1}{y},\ldots,\underset{k}{y})=0 \quad\Rightarrow\\ &\qquad F(\widetilde{x},\widetilde{y},\underset{1}{\widetilde{y}},\ldots,\underset{m}{\widetilde{y}})+\int\limits_{\widetilde{X}} d\widetilde{x}_1\cdots d\widetilde{x}_l f(\widetilde{x}_1,\ldots,\widetilde{x}_n,\widetilde{y},\underset{1}{\widetilde{y}},\ldots,\underset{k}{\widetilde{y}}) = 0.\end{aligned} \tag{5}$$

New solutions of (1) can be constructed in this way.

3. Extension (prolongation) of a group

As in the Ovsiannikov method we extended the group of the point transformations (3) to a jet space of independent and dependent variables and derivatives of dependent variables:

$$\begin{aligned} \widetilde{x}_i &= e^{\epsilon G^{(m)}} x_i = x_i + \epsilon \xi_i(x,y) + \mathcal{O}(\epsilon^2) \\ \widetilde{y} &= e^{\epsilon G^{(m)}} y = y + \epsilon \eta(x,y) + \mathcal{O}(\epsilon^2) \\ \widetilde{\partial_i y} &= e^{\epsilon G^{(m)}} (\partial_i y) = \partial_i y + \epsilon \eta_i^{(1)}(x,y,\underset{1}{y}) + \mathcal{O}(\epsilon^2) \\ &\vdots \\ \widetilde{\partial_{i_1}\cdots\partial_{i_m} y} &= e^{\epsilon G^{(m)}} (\partial_{i_1}\cdots\partial_{i_m} y) = \partial_{i_1}\cdots\partial_{i_m} y + \epsilon \eta^{(m)}_{i_1\cdots i_m}(x,y,\underset{1}{y},\ldots,\underset{m}{y}) + \mathcal{O}(\epsilon^2), \end{aligned} \tag{6}$$

where *extended generators* are of the form

$$G^{(m)} = G + \sum_{i=1}^{n} \eta_i^{(1)}(x,y,\underset{1}{y})\partial_{(\partial_i y)} + \cdots + \sum_{i_1\cdots i_m} \eta^{(m)}_{i_1\cdots i_m}(x,y,\underset{1}{y},\ldots,\underset{m}{y})\partial_{(\partial_{i_1}\cdots\partial_{i_m} y)}. \tag{7}$$

The coefficients $\eta_i^{(1)},\ldots,\eta^{(m)}_{i_1\cdots i_m}$, defining the *extended group* of transformations (6) are given by the recursion relations ([4]):

$$\begin{aligned} \eta^{(0)} &= \eta(x,y) \\ \eta_i^{(1)} &= D_i\eta - \sum_{j=1}^{n}(\partial_j y) D_i \xi_j \\ \eta^{(2)}_{i_1 i_2} &= D_{i_2}\eta^{(1)}_{i_1} - \sum_{j=1}^{n}(\partial_{i_1}\partial_j) D_{i_2}\xi_j \\ &\vdots \\ \eta^{(m)}_{i_1\cdots i_m} &= D_{i_m}\eta^{(m-1)}_{i_1\cdots i_{m-1}} - \sum_{j=1}^{n}(\partial_{i_1}\cdots\partial_{i_{m-1}}\partial_j y) D_{i_m}\xi_j, \end{aligned} \tag{8}$$

where the total derivative D_i is defined as follow

$$D_i = \frac{D}{D_{x_i}} = \partial_i + (\partial_i y)\partial_y + \sum_{j=1}^{n}(\partial_i\partial_j y)\partial_{(\partial_j y)} + \cdots + \sum_{i_1\cdots i_n}(\partial_i\partial_{i_1}\cdots\partial_{i_n} y)\partial_{(\partial_{i_1}\cdots\partial_{i_n} y)} + \cdots. \tag{9}$$

The relations (8) follow from the requirement of preservation of the contact structure of a jet bundle. Simply, we demand that the extended variables $\partial_i y$, $\partial_i\partial_j y$, … now formally treated as independent variables, are transformed under the group action (6) as ordinary derivatives of the function y. The above procedure of lifting the group of point transformations (3) to a jet bundle is the essence of the Ovsiannikov method. In terms of a jet space a differential equation is equivalent to an algebraic equation, and thus is much more tractable. In the case of the integro-differential equations (1) we use the same method to deal with derivatives.

4. Criterion of symmetry of integro-differential equations

According to the definition (5), we act on the integro-differential equation (1) by extended transformations (6) with restriction to terms that are linear with respect to parameter ϵ. Next, we express the change of (1) in terms of extended generators (7). At last, we derive the required criterion of symmetry by setting the resulting change of (1) to zero.

Expanding the function F in a Taylor series we calculate the change of differential term of (1)

$$\begin{aligned}\Delta F &= F(\widetilde{x}, \widetilde{y}, \underset{1}{\widetilde{y}}, \ldots, \underset{m}{\widetilde{y}}) - F(x, y, \underset{1}{y}, \ldots, \underset{m}{y}) \\ &= F\Big(x_1 + \epsilon\xi_1 + \mathcal{O}(\epsilon^2), \ldots, x_n + \epsilon\xi_n + \mathcal{O}(\epsilon^2), y + \epsilon\eta + \mathcal{O}(\epsilon^2), \partial_1 y + \epsilon\eta_1^{(1)} + \mathcal{O}(\epsilon^2), \\ &\quad \ldots, \partial_n y + \epsilon\eta_n^{(1)} + \mathcal{O}(\epsilon^2), \ldots, \partial_{i_1}\cdots\partial_{i_m} y + \epsilon\eta^{(m)}_{i_1\cdots i_m} + \mathcal{O}(\epsilon^2)\Big) - F(x, y, \underset{1}{y}, \ldots, \underset{m}{y}) \\ &= \epsilon\Bigg[\sum_{i=1}^{n}(\partial_i F)\xi_i + (\partial_y F)\eta + \sum_{i=1}^{n}(\partial_{(\partial_i y)}F)\eta_i^{(1)} + \cdots \\ &\quad + \sum_{i_1\cdots i_m}(\partial_{(\partial_{i_1} y)}\cdots\partial_{(\partial_{i_m} y)}F)\eta^{(m)}_{i_1\cdots i_m}\Bigg] + \mathcal{O}(\epsilon^2).\end{aligned}$$

Due to the definition of extended generator (7) we can rewrite the above result in the form

$$\Delta F = \epsilon\, G^{(m)} F(x, y, \underset{1}{y}, \ldots, \underset{m}{y}) + \mathcal{O}(\epsilon^2). \tag{10}$$

Thus, from condition $\Delta F = 0$ it follows the Ovsiannikov infinitesimal criterion of invariance of differential equation $G^{(m)}F(x, y, \underset{1}{y}, \ldots, \underset{m}{y}) = 0$.

Let us consider the change of an integral term in the equation (1)

$$\Delta I = \int_{\widetilde{X}} d\widetilde{x}_1 \cdots d\widetilde{x}_l\, f(\widetilde{x}, \widetilde{y}, \underset{1}{\widetilde{y}}, \ldots, \underset{k}{\widetilde{y}}) - \int_{X} dx_1 \cdots dx_l\, f(x, y, \underset{1}{y}, \ldots, \underset{k}{y}) \tag{11}$$

under transformations (6). To this end, we change variables in the first integral in (11) according to transformations (6):

$$\{\widetilde{x}_1, \ldots, \widetilde{x}_l\} \mapsto \{x_1, \ldots, x_l\}.$$

By virtue of (6) the elements of Jacobi's matrix are equal

$$\frac{\partial \widetilde{x}_i}{\partial x_j} = \delta_{ij} + \epsilon \frac{\partial \xi_i}{\partial x_j} + \mathcal{O}(\epsilon^2), \quad i, j = 1, \ldots, l.$$

Because the off-diagonal elements of the matrix are of the order $\mathcal{O}(\epsilon^2)$, thus the linear contribution to the Jacobian comes only from the product of diagonal elements:

$$\begin{aligned} \frac{\partial(\widetilde{x}_1 \cdots \widetilde{x}_l)}{\partial(x_1 \cdots x_l)} &= \left(1 + \epsilon \frac{\partial \xi_1}{\partial x_1}\right) \cdots \left(1 + \epsilon \frac{\partial \xi_l}{\partial x_l}\right) + \mathcal{O}(\epsilon^2) \\ &= 1 + \epsilon \sum_{i=1}^{l} \frac{\partial \xi_i}{\partial x_i} + \mathcal{O}(\epsilon^2). \end{aligned}$$

Consequently, the change ΔI of the integral term is equal

$$\begin{aligned} &\int_X dx_1 \cdots dx_l \Bigg[\left(1 + \epsilon \sum_{i=1}^{l} \partial_i \xi_i \right) f(\widetilde{x}(x,y), \widetilde{y}(x,y), \underset{1}{\widetilde{y}}(x,y,\underset{1}{y}), \ldots, \underset{k}{\widetilde{y}}(x,y,\underset{1}{y},\ldots,\underset{k}{y})) \\ &\qquad - f(x,y,\underset{1}{y},\ldots,\underset{k}{y}) \Bigg] + \mathcal{O}(\epsilon^2) \\ &= \int_X dx_1 \cdots dx_l \Bigg[\left(1 + \epsilon \sum_{i=1}^{l} \partial_i \xi_i \right) f\big(x_1 + \epsilon \xi_1 + \mathcal{O}(\epsilon^2), \ldots, x_n + \epsilon \xi_n + \mathcal{O}(\epsilon^2), \\ &\qquad y + \epsilon \eta + \mathcal{O}(\epsilon^2), \partial_1 y + \epsilon \eta_1^{(1)} + \mathcal{O}(\epsilon^2), \ldots, \partial_n y + \epsilon \eta_n^{(1)} + \mathcal{O}(\epsilon^2), \\ &\qquad \ldots, \partial_{i_1} \cdots \partial_{i_k} y + \epsilon \eta^{(k)}_{i_1 \cdots i_k} + \mathcal{O}(\epsilon^2)\big) - f(x,y,\underset{1}{y},\ldots,\underset{k}{y}) \Bigg] + \mathcal{O}(\epsilon^2). \end{aligned}$$

Expanding the function f into a Taylor series we have

$$\begin{aligned} \Delta I &= \epsilon \int_X dx_1 \cdots dx_l \Bigg[\sum_{i=1}^{n} (\partial_i f) \xi_i + (\partial_y f) \eta + \sum_{i=1}^{n} (\partial_{(\partial_i y)} f) \eta_i^{(1)} + \cdots \\ &\qquad + \sum_{i_1 \cdots i_k} (\partial_{(\partial_{i_1} y)} \cdots \partial_{(\partial_{i_k} y)} f) \eta^{(k)}_{i_1 \cdots i_k} + f \sum_{i=1}^{l} \partial_i \xi_i \Bigg] + \mathcal{O}(\epsilon^2). \end{aligned}$$

In view of the definition of extended generator (7) we can rewrite the above result as follows

$$\Delta I = \epsilon \int_X dx_1 \cdots dx_l \left[G^{(k)} f(x,y,\underset{1}{y},\ldots,\underset{k}{y}) + f(x,y,\underset{1}{y},\ldots,\underset{k}{y}) \sum_{i=1}^{l} \partial_i \xi_i \right] + \mathcal{O}(\epsilon^2).$$

By equating the total change $\Delta = \Delta F + \Delta I$ of equation (1) to zero we derive the following *infinitesimal criterion of invariance* of integro-differential equations of the type of (1) under point transformations (3):

$$G^{(m)}F + \int\limits_X dx_1 \cdots dx_l \left[G^{(k)}f + f \sum_{i=1}^{l} \partial_i \xi_i \right] = 0 \quad \text{on solutions of (1)} \tag{12}$$

According to the criterion (12) we have to take into account the equation (1), which is now a constraint on extended variables. Using this equation we can eliminate some of them. Remaining variables are essentially independent, so the equation (12) must be satisfied identically with respect to them. It means that the coefficients at independent expressions, involving these variables, must be equal to zero. This leads to the system of the so called *determining equations* for the integro-differential equation (1). They are homogenous and *linear* integro-differential equations for coefficients ξ_i, η determining the generator (4) and the point transformations (3).

The criterion (12) is a *necessary* condition for symmetry of the equation (1), so it allows us to find all *possible* symmetry transformations of (1). The difficult task to achieve is to find a sufficient condition of symmetry. To this end we need a theorem on *global* existence and uniqueness of the solutions of the equation (1). The latter problem is far from to be solved, see [9]. From a practical point of view the necessary condition is more important and useful than the sufficient one as the main task is to find a symmetry transformations. A possible symmetry transformation of the equation (1) can be easily verified by inspection and this must be done anyway.

5. Symmetries of Vlasov-Maxwell equations

Let us consider the Vlasov-Maxwell system of equations for collisionless, multicomponent, one-dimensional plasmas with no magnetic field:

$$\begin{aligned} &\partial_t f_\alpha + u\partial_x f_\alpha + \frac{q_\alpha}{m_\alpha} E \partial_u f_\alpha = 0 \\ &\partial_t E + \sum_\alpha \frac{q_\alpha}{\epsilon_0} \int\limits_{-\infty}^{\infty} du\, u f_\alpha = 0 \\ &\partial_x E - \sum_\alpha \frac{q_\alpha}{\epsilon_0} \int\limits_{-\infty}^{\infty} du f_\alpha = 0, \end{aligned} \tag{13}$$

where $E = E(t,x)$ is the x-component of electric vector field $\mathbf{E} = (E,0,0)$, u is the x-component of vector velocity $\mathbf{v} = (u,0,0)$, $f_\alpha = f_\alpha(t,x,u)$ is the distribution function of α-plasma component, q_α, m_α are charge and mass of α-particles, respectively and ϵ_0 is electric permittivity of free space.

In this case, the generators (4) of point transformations (3) take the form

$$G = \tau\partial_t + \xi\partial_x + \rho\partial_u + \sum_\alpha \eta_\alpha \partial_{f_\alpha} + \zeta\partial_E. \tag{14}$$

Solutions of the determining equations are given by

$$\tau = -\frac{1}{3}(\lambda_1 + \lambda_2)t + \lambda_3, \quad \xi = \frac{1}{3}(\lambda_1 - 2\lambda_2)x + \lambda_4 t + \lambda_5$$
$$\rho = \frac{1}{3}(2\lambda_1 - \lambda_2)u + \lambda_4, \quad \eta_\alpha = \lambda_2 f_\alpha, \quad \zeta = \lambda_1 E,$$

where $\lambda_1, \ldots, \lambda_5$ are arbitrary parameters. Substituting the solutions into (14) and choosing all parameters equal to zero except one, which is assumed to be 1, in each case, we derive the following five generators

$$\begin{aligned} G_1 &= \partial_t \\ G_2 &= \partial_x \\ G_3 &= t\partial_x + \partial_u \\ G_4 &= -t\partial_t + x\partial_x + 2u\partial_u + 3E\partial_E \\ G_5 &= -t\partial_t - 2x\partial_x - u\partial_u + 3\sum_\alpha f_\alpha \partial_{f_\alpha}, \end{aligned} \tag{15}$$

which span the Lie algebra of the group of point symmetry transformations of the Vlasov-Maxwell equations (13). Nonvanishing commutators between these generators are given by

$$[G_1, G_3] = G_2, \quad [G_1, G_4] = -G_1, \quad [G_1, G_5] = -G_1, \quad [G_2, G_4] = G_2$$
$$[G_2, G_5] = -2G_2, \quad [G_3, G_4] = 2G_3, \quad [G_3, G_5] = -G_3.$$

The algebra is solvable.

Summing up the Lie series we obtain one-parameter subgroups of the symmetry group of transformations corresponding to the generators (15). For G_1 and G_2 we have translations in time

$$\widetilde{t} = e^{\epsilon G_1} t = t + \epsilon, \quad \widetilde{x} = e^{\epsilon G_1} x = x, \quad \widetilde{u} = e^{\epsilon G_1} u = u,$$
$$\widetilde{f_\alpha} = e^{\epsilon G_1} f_\alpha = f_\alpha, \quad \widetilde{E} = e^{\epsilon G_1} E = E$$

and translations in space

$$\widetilde{t} = e^{\epsilon G_2} t = t, \quad \widetilde{x} = e^{\epsilon G_2} x = x + \epsilon, \quad \widetilde{u} = e^{\epsilon G_2} u = u,$$
$$\widetilde{f_\alpha} = e^{\epsilon G_2} f_\alpha = f_\alpha, \quad \widetilde{E} = e^{\epsilon G_2} E = E.$$

These symmetries follow from the fact, that coefficients of equation (13) do not depend of time and space, and lead to the conservation laws of energy and momentum respectively. For G_3 we have Galilean transformations

$$\widetilde{t} = e^{\epsilon G_3} t = t, \quad \widetilde{x} = e^{\epsilon G_3} x = x + \epsilon t, \quad \widetilde{u} = e^{\epsilon G_3} u = u + \epsilon,$$
$$\widetilde{f_\alpha} = e^{\epsilon G_3} f_\alpha = f_\alpha, \quad \widetilde{E} = e^{\epsilon G_3} E = E,$$

where the parameter ϵ is equal to a relative velocity of observers. The above three kinetic symmetries are obvious as they express the geometric properties of space-time in nonrelativistic theory. The dynamical symmetries, which depend on details of interaction, are more interesting. In the case of Vlasov-Maxwell equations (13) they are generated by G_4 and G_5 and have the form of scaling transformations:

$$\widetilde{t} = e^{\epsilon G_4} t = te^{-\epsilon}, \quad \widetilde{x} = e^{\epsilon G_4} x = xe^{\epsilon}, \quad \widetilde{u} = e^{\epsilon G_4} u = ue^{2\epsilon},$$
$$\widetilde{f_\alpha} = e^{\epsilon G_4} f_\alpha = f_\alpha, \quad \widetilde{E} = e^{\epsilon G_4} E = Ee^{3\epsilon}$$

and

$$\widetilde{t} = e^{\epsilon G_5} t = te^{-\epsilon}, \quad \widetilde{x} = e^{\epsilon G_5} x = xe^{-2\epsilon}, \quad \widetilde{u} = e^{\epsilon G_5} u = ue^{-\epsilon},$$
$$\widetilde{f_\alpha} = e^{\epsilon G_5} f_\alpha = f_\alpha e^{3\epsilon}, \quad \widetilde{E} = e^{\epsilon G_5} E = E.$$

We can construct a general symmetry transformation of the Vlasov-Maxwell equation (13) from the above one-parameter transformations.

6. Conclusions

It has been shown that there is no need for nonlocal extension of a symmetry group in the case of integro-differential equations. It is sufficient to stay in a jet space as in the case of differential equations. The generalization of the Ovsiannikov method consists in the change of the infinitesimal criterion of symmetry. The method has been successfully applied to significant integro-differential equations. In addition to the Vlasov-Maxwell equations we have also determined the symmetry group of the nonlocal NLS equation, see [10].

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