On the Rational Recursive Sequence
$x_{n+1} = Ax_n + Bx_{n-k} + \frac{\beta x_n + \gamma x_{n-k}}{Cx_n + Dx_{n-k}}$

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Abstract In this article, we study the global and asymptotic properties of the solutions of the difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + (\beta x_n + \gamma x_{n-k})/(Cx_n + Dx_{n-k}), \quad n = 0, 1, 2, \dots,$$

where the initial conditions $x_{-k}, \dots, x_{-1}, x_0$ are arbitrary positive real numbers and the coefficients A, B, C, D, β and γ are positive constants, while k is a positive integer number. Some numerical examples will be given to illustrate our results.

Keywords Difference equations · Prime period two solution · Locally asymptotically stable · Global attractor · Global stability

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1 Introduction

The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modeled using difference equations. Examples from economy, biology, etc. can be found in [2, 16, 19, 29]. It is known that nonlinear difference equations are capable of producing a complicated behavior regardless its order. This can be easily seen from the family $x_{n+1} = g_\mu(x_n), \mu > 0, n \geq 0$. This behavior is ranging according to the value of μ, from the existence of a bounded number of periodic solutions to chaos.

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There has been a great interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations. For example, in the articles [1, 5, 8–14, 21–30] closely related global convergence results were obtained which can be applied to nonlinear difference equations in proving that every solution of these equations converges to a period two solution. For other closely related results (see [3–7, 10, 17, 18] and the references cited therein). The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right. Furthermore the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations.

Our goal in this article is to investigate some qualitative behavior of the solutions of the difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{\beta x_n + \gamma x_{n-k}}{Cx_n + Dx_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers and the coefficients A, B, C, D, β and γ are positive constants, while k is a positive integer number. The global stability of (1) for $A = B = 0$ and $k = 1$ has been investigated in [20]. M. Saleh et al. [30] have investigated the global stability of (1), where $A = B = 0$ and $k \in \{1, 2, 3, \ldots\}$.

Our interest now is to study the behavior of solutions of (1) in the general case where $A \neq 0$, $B \neq 0$ and k is a positive integer number. For the related work see [31–43]. The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right. Furthermore the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations. Note that (1) can be considered as a generalization of that obtained in [20, 30]. Let us now recall some well know results [15] which will be useful in the sequel.

Definition 1 A difference equation of order $(k + 1)$ is of the form

$$x_{n+1} = F(x_n, x_{n-k}), \quad n = 0, 1, 2, \ldots, \tag{2}$$

where F is a continuous function which maps some set J^{k+1} into J where J is a set of real numbers. An equilibrium point $\widetilde{x}$ of this equation is a point that satisfies the condition $\widetilde{x} = F(\widetilde{x}, \widetilde{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \widetilde{x}$ for all $n \geq -k$ is a solution of that equation.

Definition 2 Let $\widetilde{x} \in (0, \infty)$ be an equilibrium point of the difference equation (2). Then:

(i) An equilibrium point $\widetilde{x}$ of the difference equation (2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \widetilde{x}| + \cdots + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \delta$, then $|x_n - \widetilde{x}| < \varepsilon$ for all $n \geq -k$.

(ii) An equilibrium point $\widetilde{x}$ of the difference equation (2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \widetilde{x}| + \cdots + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \gamma$, then

$$\lim_{n \to \infty} x_n = \widetilde{x}.$$

(iii) An equilibrium point $\tilde{x}$ of the difference equation (2) is called a global attractor if for every $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ we have

$$\lim_{n \to \infty} x_n = \tilde{x}.$$

(iv) An equilibrium point $\tilde{x}$ of (2) is called globally asymptotically stable if it is locally stable and a global attractor.

 (v) An equilibrium point $\tilde{x}$ of the difference equation (2) is called unstable if it is not locally stable.

Definition 3 A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Definition 4 A positive semi-cycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all greater than or equal to $\tilde{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$\text{either} \quad l = -k \quad \text{or} \quad l > -k \quad \text{and} \quad x_{l-1} < \tilde{x},$$

and

$$\text{either} \quad m = \infty \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \tilde{x}.$$

A negative semi-cycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all less than $\tilde{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$\text{either} \quad l = -k \quad \text{or} \quad l > -k \quad \text{and} \quad x_{l-1} \geq \tilde{x},$$

and

$$\text{either} \quad m = \infty \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \tilde{x}.$$

Definition 5 Equation (2) is said to be permanent if there exist positive real numbers m and M such that for every solution $\{x_n\}_{n=-k}^{\infty}$ of (2) there exists a positive integer $N \geq -k$ which depends on the initial conditions, such that

$$m \leq x_n \leq M, \quad \text{for all } n \geq N.$$

The linearized equation of the difference equation (2) about the equilibrium point $\tilde{x}$ is the linear difference equation

$$y_{n+1} = \frac{\partial F(\tilde{x}, \tilde{x})}{\partial x_n} y_n + \frac{\partial F(\tilde{x}, \tilde{x})}{\partial x_{n-k}} y_{n-k}. \tag{3}$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = \lambda^{k+1} - p_0 \lambda^k - p_1 = 0, \tag{4}$$

where

$$p_0 = \frac{\partial F(\tilde{x}, \tilde{x})}{\partial x_n}, \qquad p_1 = \frac{\partial F(\tilde{x}, \tilde{x})}{\partial x_{n-k}}.$$

Theorem 1 [15] *The linearized stability theorem.*

Suppose F is a continuously differentiable function defined on an open neighbourhood of the equilibrium $\widetilde{x}$. Then the following statements are true:

(i) *If all the roots of the characteristic equation* (4) *of the linearized equation* (3) *have absolute value less than one, then the equilibrium point $\widetilde{x}$ of* (2) *is locally asymptotically stable.*

(ii) *If at least one root of* (4) *has absolute value greater than one, then the equilibrium point $\widetilde{x}$ of* (2) *is not locally stable.*

(iii) *If all the roots of* (4) *have absolute value greater than one, then the equilibrium point $\widetilde{x}$ of* (2) *is a source.*

1.1 Change of Variables

In this section, we apply (2) to the rational recursive sequence (1). By using the change of variables $x_n = \frac{\gamma}{D} y_n$. Then (1) reduces to the difference equation

$$y_{n+1} = A y_n + B y_{n-k} + \frac{p y_n + y_{n-k}}{q y_n + y_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{5}$$

where $p = \frac{\beta}{\gamma}$ and $q = \frac{C}{D}$, with $p, q \in (0, \infty)$, $y_{-k}, \ldots, y_{-1}, y_0 \in (0, \infty)$. To avoid a degenerate situation we also assume that $p \neq q$.

Next, we investigate the asymptotic of the equilibrium points of the nonlinear rational difference equation (5) where the parameters p, q and the initial conditions $y_{-k}, \ldots, y_{-1}, y_0$ are arbitrary positive real numbers, while k is a positive integer number.

The equilibrium points of (5) are the positive solutions of the equation

$$\widetilde{y} = A\widetilde{y} + B\widetilde{y} + \frac{p\widetilde{y} + \widetilde{y}}{q\widetilde{y} + \widetilde{y}}.$$

If $0 < A + B < 1$, then the only positive equilibrium point is

$$\widetilde{y} = \frac{p+1}{[1-(A+B)](q+1)}. \tag{6}$$

1.2 Linearization

In this section, we derive the linearized equation of (5). To this end, we introduce a continuous function $F : (0, \infty)^2 \to (0, \infty)$ which is defined by

$$F(u_0, u_1) = A u_0 + B u_1 + \frac{p u_0 + u_1}{q u_0 + u_1}. \tag{7}$$

Therefore,

$$\begin{cases} \frac{\partial F(u_0, u_1)}{\partial u_0} = A + \frac{(p-q)u_1}{(qu_0 + u_1)^2}, \\ \frac{\partial F(u_0, u_1)}{\partial u_1} = B - \frac{(p-q)u_0}{(qu_0 + u_1)^2}. \end{cases} \tag{8}$$

From (6) and (8) we have

$$\begin{cases} \frac{\partial F(\widetilde{y}, \widetilde{y})}{\partial u_0} = A + \frac{(p-q)[1-(A+B)]}{(p+1)(q+1)} = \rho_0, \\ \frac{\partial F(\widetilde{y}, \widetilde{y})}{\partial u_1} = B - \frac{(p-q)[1-(A+B)]}{(p+1)(q+1)} = \rho_1. \end{cases} \tag{9}$$

Then the linearized equation of the difference equation (5) about $\widetilde{y}$ is

$$y_{n+1} - \rho_0 y_n - \rho_1 y_{n-k} = 0, \tag{10}$$

where ρ_0 and ρ_1 are given by (9).

Theorem 2 [20] *Assume that $\rho_0, \rho_1 \in R$ and $k \in \{1, 2, \ldots\}$. Then*

$$|\rho_0| + |\rho_1| < 1 \tag{11}$$

is a sufficient condition for the asymptotic stability of the difference equation (5). Suppose in addition that one of the following two cases holds:

 (i) *k is an odd integer and $\rho_1 > 0$.*
(ii) *k is an even integer and $\rho_0 \rho_1 > 0$.*

Then (11) is also a necessary condition for the asymptotic stability of (5).

Theorem 3 [15, p. 18] *Let $F : [a, b]^{k+1} \to [a, b]$ be a continuous function, where k is a positive integer, and where $[a, b]$ is an interval of real numbers and consider the difference equation (2). Suppose that F satisfies the following conditions*:

 (i) *For each integer i with $1 \leq i \leq k + 1$, the function $F(z_1, z_2, \ldots, z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{k+1}$.*
(ii) *If (m, M) is a solution of the system*

$$m = F(m_1, m_2, \ldots, m_{k+1}) \quad and \quad M = F(M_1, M_2, \ldots, M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, \ldots, k + 1$, we set

$$m_i = \begin{cases} m & \text{if } F \text{ nondecreasing in } z_i, \\ M & \text{if } F \text{ nonincreasing in } z_i \end{cases}$$

and

$$M_i = \begin{cases} M & \text{if } F \text{ nondecreasing in } z_i, \\ m & \text{if } F \text{ nonincreasing in } z_i. \end{cases}$$

Then there exists exactly one equilibrium point $\widetilde{x}$ of the difference equation (2), and every solution of (2) converges to $\widetilde{x}$.

2 Semi-cycle Analysis

Theorem 4 *Assume that $F \in [(0, \infty)^2, (0, \infty)]$ is a continuous function such that $F(x, y)$ is increasing (respectively, decreasing) in x for each fixed y, and $F(x, y)$ is decreasing (respectively, increasing) in y for each fixed x. Let $\widetilde{y}$ be a positive equilibrium of (5). Then except possibly for the first semi-cycle, every oscillatory solution of (5) has semi-cycle of length at least k.*

Proof When $k = 1$, the proof is presented as Theorem 1.7.1 in [20]. We just give the proof of Theorem 4 for $k = 2$. The proof of Theorem 4 for $k \geq 3$, is similar and omitted here. Let

$\{y_n\}$ be a solution of (5) with at least three semi-cycles. Then, there exists $N \geq 0$ such that either

$$y_{N-1} < \widetilde{y} \leq y_{N+1},$$

or

$$y_{N-1} \geq \widetilde{y} > y_{N+1}.$$

We first assume that

$$y_{N-1} < \widetilde{y} \leq y_{N+1}.$$

Since the function $F(x, y)$ is monotonic in each of its arguments, then we have

$$y_{N+2} = F(y_{N+1}, y_{N-1}) < F(\widetilde{y}, \widetilde{y}) = \widetilde{y},$$

and

$$y_{N+3} = F(y_{N+2}, y_N) > F(\widetilde{y}, y_N).$$

Since the function $F(x, y)$ is decreasing in y for fixed x, then

$$F(\widetilde{y}, y_N) > F(\widetilde{y}, \widetilde{y}) = \widetilde{y} \quad \text{for } y_N < \widetilde{y}.$$

Hence, we obtain

$$y_{N+2} < \widetilde{y} < y_{N+3}.$$

Similarly, we can prove the theorem if $y_{N-1} \geq \widetilde{y} > y_{N+1}$ which is omitted here. Now, the proof of Theorem 4 is completed.   □

3 Local Stability

In this section, we investigate the local stability of the positive solutions of (5). By using Theorems 1 and 2, we have the following result.

Theorem 5 (i) *Assume that* $p > q, 0 < A + B < 1$ *and*

$$B > \frac{(p - q)[1 - (A + B)]}{(p + 1)(q + 1)}.$$

Then, the positive equilibrium point $\widetilde{y}$ *of* (5) *is locally asymptotically stable.*
  (ii) *If k is either odd or even,* $p < q, 0 < A + B < 1$ *and*

$$A > \frac{(q - p)[1 - (A + B)]}{(p + 1)(q + 1)}.$$

Then (11) *is the necessary and sufficient condition for the asymptotically stable of* (5).

Proof Under the assumptions of part (i), we deduce from (9) that

$$|\rho_0| + |\rho_1| = \left| A + \frac{(p-q)[1-(A+B)]}{(p+1)(q+1)} \right| + \left| B - \frac{(p-q)[1-(A+B)]}{(p+1)(q+1)} \right|$$

$$= A + \frac{(p-q)[1-(A+B)]}{(p+1)(q+1)} + B - \frac{(p-q)[1-(A+B)]}{(p+1)(q+1)}$$

$$= A + B < 1.$$

According to Theorem 2, the proof of part (i) of Theorem 5 is completed.

Under the assumptions of part (ii), we deduce from (9) that

$$|\rho_0| + |\rho_1| = \left| A - \frac{(q-p)[1-(A+B)]}{(p+1)(q+1)} \right| + \left| B + \frac{(q-p)[1-(A+B)]}{(p+1)(q+1)} \right|$$

$$= A - \frac{(q-p)[1-(A+B)]}{(p+1)(q+1)} + B + \frac{(q-p)[1-(A+B)]}{(p+1)(q+1)}$$

$$= A + B < 1.$$

Furthermore, if k is an odd positive integer, we get

$$\rho_1 = B + \frac{(q-p)[1-(A+B)]}{(p+1)(q+1)} > 0,$$

while, if k is an even positive integer, we have

$$\rho_0\rho_1 = \left(A - \frac{(q-p)[1-(A+B)]}{(p+1)(q+1)} \right)\left(B + \frac{(q-p)[1-(A+B)]}{(p+1)(q+1)} \right) > 0.$$

According to Theorem 2, the proof of part (ii) of Theorem 5 is completed. Thus, the proof of Theorem 5 is now finished. □

4 Periodic Solutions

In this section, we investigate the periodic character of the positive solutions of (5).

Theorem 6

(1) *Assume that $p > q > 1$ (or $p < q < 1$) and k is an odd positive integer, then (5) has no positive solutions of prime period two.*
(2) *If k is an even positive integer, then (5) has no positive solutions of prime period two.*
(3) *If k is an odd positive integer, then (5) has prime period two solutions*

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots$$

if the following condition is valid:

$$(p-1)(q-1)(A+1-B) > -4(qA+p-Bp),$$

where $p > 1$, $q > 1$ and $A + 1 - B > 0$ while the values of Φ and Ψ are the (positive and distinct) solutions of the quadratic equation

$$t^2 - \frac{(1-p)}{(qA+1-B)}t + \frac{(qA+p-Bp)(1-p)}{(q-1)(A+1-B)(qA+1-B)^2} = 0.$$

Proof First of all, we prove the part (1) in the case $p > q > 1$. Similarly, we can prove the part (1) in the case $p < q < 1$ which is omitted here. Assume for the sake of contradiction that there exists distinctive positive real numbers Φ and Ψ, such that

$$\dots, \Phi, \Psi, \Phi, \Psi, \dots$$

is a prime period two solution of (5).

If k is odd, then $y_{n+1} = y_{n-k}$. It follows from (5) that

$$\Phi = A\Psi + B\Phi + \frac{p\Psi + \Phi}{q\Psi + \Phi} \quad \text{and} \quad \Psi = A\Phi + B\Psi + \frac{p\Phi + \Psi}{q\Phi + \Psi}.$$

Consequently, we obtain

$$q\Phi\Psi + \Phi^2 = qA\Psi^2 + A\Phi\Psi + qB\Phi\Psi + B\Phi^2 + p\Psi + \Phi, \tag{12}$$

and

$$q\Phi\Psi + \Psi^2 = qA\Phi^2 + A\Phi\Psi + qB\Phi\Psi + B\Psi^2 + p\Phi + \Psi. \tag{13}$$

By subtracting (13) from (12), we deduce that

$$\Phi + \Psi = \frac{1-p}{qA+1-B}, \tag{14}$$

while, by adding (12), (13) and using (14) we get

$$\Phi\Psi = \frac{(qA+p-Bp)(1-p)}{(q-1)(A+1-B)(qA+1-B)^2}. \tag{15}$$

Consequently, we conclude from (15) that

$$(qA+p-Bp)(1-p) > 0 \quad \text{and} \quad (q-1)(A+1-B)(qA+1-B)^2 > 0 \tag{16}$$

or

$$(qA+p-Bp)(1-p) < 0 \quad \text{and} \quad (q-1)(A+1-B)(qA+1-B)^2 < 0. \tag{17}$$

From (16) we deduce that $q > 1$ and $A - B + 1 > 0$. Since $p > q > 1$, then $(1-p) < 0$ and consequently, we deduce from (14) that

$$qA - A < qA - B + 1 < 0.$$

The last inequality gives $q < 1$. This is a contradiction. Similarly, from (17) we deduce that $q > 1$ and $A - B + 1 < 0$. Since $p > q > 1$, then $(1-p) < 0$ and consequently, we deduce from (17) that

$$qA - Ap > qA + p - Bp > 0.$$

The last inequality gives $p < q$. This is a contradiction and hence the proof of part (1) is completed.

(2) Assume for the sake of contradiction that there exists distinctive positive real numbers Φ and Ψ, such that

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots$$

is a prime period two solution of (5). If k is even, then $y_n = y_{n-k}$. It follows from the difference equation (5) that

$$\Phi = A\Psi + B\Psi + \frac{p\Psi + \Psi}{q\Psi + \Psi} \quad \text{and} \quad \Psi = A\Phi + B\Phi + \frac{p\Phi + \Phi}{q\Phi + \Phi}.$$

Hence we have $(\Phi - \Psi)(A + B + 1) = 0$. Thus $\Phi = \Psi$. This contradicts the hypothesis that Φ and Ψ distinct positive real number. Thus, the proof of part (2) is completed.

(3) Assume that (5) has prime period two solutions

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots$$

If k is odd, then $y_{n+1} = y_{n-k}$. It follows from (5) that

$$\Phi = A\Psi + B\Phi + \frac{p\Psi + \Phi}{q\Psi + \Phi} \quad \text{and} \quad \Psi = A\Phi + B\Psi + \frac{p\Phi + \Psi}{q\Phi + \Psi}.$$

Then we have

$$\Phi + \Psi = \frac{1 - p}{qA + 1 - B} \quad \text{and} \quad \Phi\Psi = \frac{(qA + p - Bp)(1 - p)}{(q - 1)(A + 1 - B)(qA + 1 - B)^2}.$$

Now, we consider the quadratic equation

$$t^2 - \frac{(1 - p)}{(qA + 1 - B)}t + \frac{(qA + p - Bp)(1 - p)}{(q - 1)(A + 1 - B)(qA + 1 - B)^2} = 0.$$

So, the values of Φ and Ψ are the (positive and distinct) solutions of the above quadratic equation. Thus, we get

$$t = \frac{(1 - p) \pm \delta}{2(qA + 1 - B)},$$

where

$$\delta = \sqrt{(1 - p)^2 - 4(qA + p - Bp)(1 - p)/(q - 1)(A + 1 - B)}.$$

Thus, we deduce that

$$\left(\frac{p - 1}{qA + 1 - B}\right)^2 > \frac{-4(qA + p - Bp)(p - 1)}{(q - 1)(A + 1 - B)(qA + 1 - B)^2},$$

and hence, we have

$$(p - 1)(q - 1)(A + 1 - B) > -4(qA + p - Bp).$$

Thus, the proof of part (3) is completed. The proof of Theorem 6 is now finished. $\qquad\square$

5 Boundedness Character

In this section, we investigate the boundedness character of the positive solutions of (5).

Theorem 7 *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of (5). Then the following statements are true*:

(1) *Suppose $p < q$ and assume that for some $N \geq 0$*

$$y_{N-k+1}, \ldots, y_{N-1}, y_N \in \left[\frac{p}{q}, 1\right],$$

   then

$$y_n \in \left[\frac{p}{q}\left(A + B + \frac{p+1}{q+1}\right), A + B + 1\right], \quad \text{for all } n > N.$$

(2) *Suppose $p > q$ and assume that for some $N \geq 0$*

$$y_{N-k+1}, \ldots, y_{N-1}, y_N \in \left[1, \frac{p}{q}\right],$$

   then

$$y_n \in \left[A + B + 1, \frac{p}{q}\left(A + B + \frac{p+1}{q+1}\right)\right], \quad \text{for all } n > N.$$

Proof First of all, if for some $N > 0$, $\frac{p}{q} \leq y_N \leq 1$ and $p < q$, then

$$y_{n+1} = Ay_n + By_{n-k} + \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \leq Ay_n + By_{n-k} + \frac{qy_n + y_{n-k}}{qy_n + y_{n-k}}$$
$$= Ay_n + By_{n-k} + 1 \leq A + B + 1,$$

and

$$y_{n+1} = Ay_n + By_{n-k} + \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \geq \frac{p}{q}\left(A + B + \frac{p+1}{q+1}\right).$$

Thus, the proof of part (1) is completed.

Secondly, if for some $N > 0$, $1 \leq y_N \leq \frac{p}{q}$ and $p > q$, then

$$y_{n+1} = Ay_n + By_{n-k} + \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \leq \frac{p}{q}\left(A + B + \frac{p+1}{q+1}\right),$$

and

$$y_{n+1} = Ay_n + By_{n-k} + \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} \geq Ay_n + By_{n-k} + \frac{py_n + y_{n-k}}{py_n + y_{n-k}}$$
$$\geq Ay_n + By_{n-k} + 1 \geq A + B + 1.$$

Thus, the proof of part (2) is completed. The proof of Theorem 7 is now finished. □

6 Global Stability

In this section, we investigate the global stability of the positive solutions of (5).

Theorem 8 *Assume that $p > q$, $0 < A + B < 1$ and $B > \frac{(p-q)[1-(A+B)]}{(p+1)(q+1)}$, then, the positive equilibrium point $\widetilde{y}$ of (5) is globally asymptotically stable.*

Proof Under these assumptions, we have shown in part (i) of Theorem 5 that the positive equilibrium point $\widetilde{y}$ of (5) is locally asymptotically stable. It remains to prove that $\widetilde{y}$ is a global attractor. To this end, we consider the function

$$F(x, y) = Ax + By + \frac{px + y}{qx + y}.$$

Since $p > q$, then the function $F(x, y)$ is increasing in x for each fixed y, and decreasing in y for each fixed x. Suppose that (m, M) is a solution of the system

$$M = F(M, m) \quad \text{and} \quad m = F(m, M).$$

Then we get

$$M = AM + Bm + \frac{pM + m}{qM + m} \quad \text{and} \quad m = Am + BM + \frac{pm + M}{qm + M}.$$

From which we have

$$(1 - A - qB)Mm = pM + m + Bm^2 - q(1 - A)M^2, \tag{18}$$

and

$$(1 - A - qB)Mm = pm + M + BM^2 - q(1 - A)m^2. \tag{19}$$

From (18) and (19), we obtain

$$(m - M)[p - 1 - (q(1 - A) + B)(m + M)] = 0. \tag{20}$$

The relation (20) gives $M = m$. According to Theorem 3, the proof of Theorem 8 is now completed. □

Theorem 9 *If k is either even or odd, $p < q$, $0 < A + B < 1$ and $A > \frac{(q-p)[1-(A+B)]}{(p+1)(q+1)}$, then, the positive equilibrium point $\widetilde{y}$ of (5) is globally asymptotically stable.*

Proof Under these assumptions, we have shown in part (ii) of Theorem 5 that the positive equilibrium point $\widetilde{y}$ of (5) is locally asymptotically stable. It remains to prove that $\widetilde{y}$ is a global attractor. To this end, we consider the function

$$F(x, y) = Ax + By + \frac{px + y}{qx + y}.$$

Since $p < q$, then the function $F(x, y)$ is decreasing in x for each fixed y, and increasing in y for each fixed x. Suppose that (m, M) is a solution of the system

$$m = F(M, m) \quad \text{and} \quad M = F(m, M).$$

Then we get

$$m = AM + Bm + \frac{pM + m}{qM + m} \quad \text{and} \quad M = Am + BM + \frac{pm + M}{qm + M}.$$

From which we have

$$(q - qB - A)Mm = pm + M + qAm^2 + (B - 1)M^2, \tag{21}$$

and

$$(q - qB - A)Mm = pM + m + qAM^2 + (B - 1)m^2. \tag{22}$$

From (21) and (22), we obtain

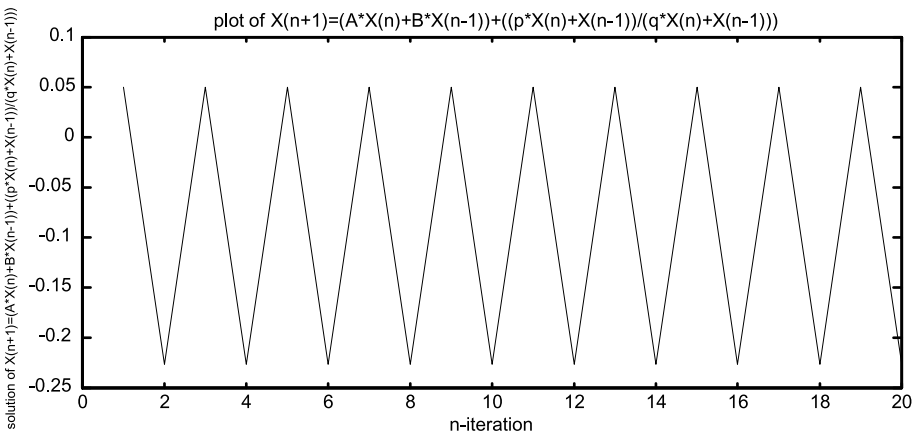$$(m - M)[p - 1 + (qA + 1 - B)(m + M)] = 0. \tag{23}$$

The relation (23) gives $M = m$. According to Theorem 3, the proof of Theorem 9 is completed. □
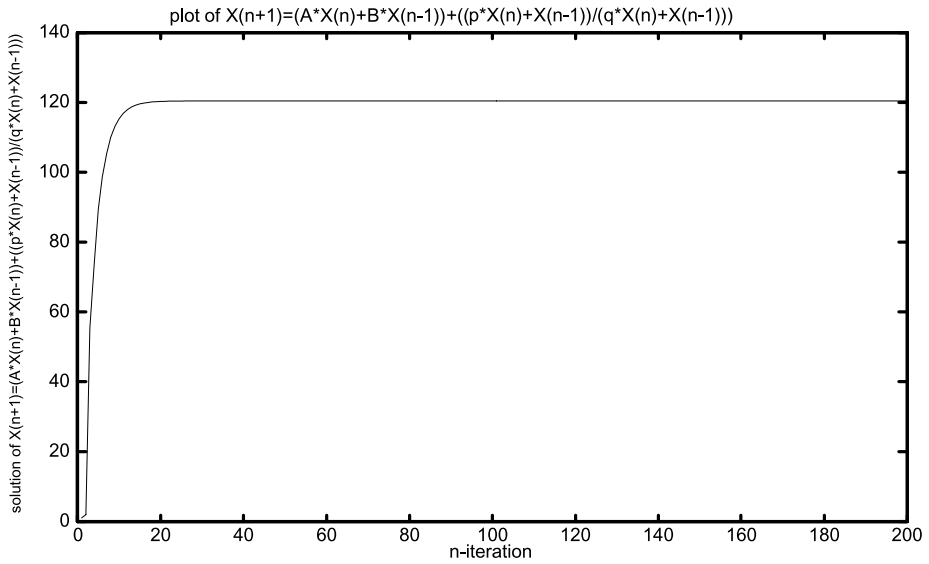
7 Numerical Examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the nonlinear difference equation (5).

Example 1 Figure 1 shows that (5) has prime period two solution if $k = 1$, $y_{-1} = 0.05$, $y_0 = -0.079$, $A = 0.25$, $B = 0.3$, $p = 2$, $q = 20$.
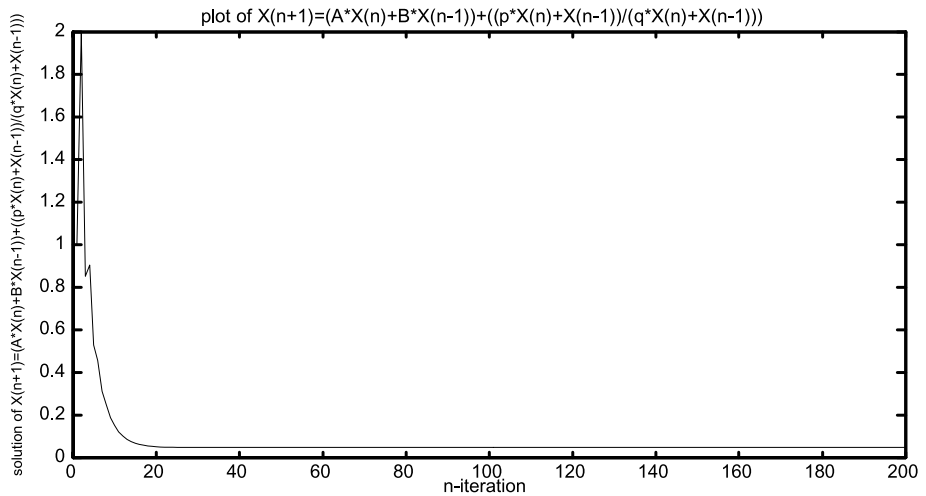
Example 2 Figure 2 shows that the solution of (5) is global stability if $k = 1$, $p > q$, $y_{-1} = 1$, $y_0 = 2$, $A = 0.25$, $B = 0.3$, $p = 300$, $q = 5$.

Fig. 1 $y_{n+1} = 0.25y_n + 0.3y_{n-1} + \frac{2y_n + y_{n-1}}{20y_n + y_{n-1}}$

Fig. 2 $y_{n+1} = 0.25y_n + 0.3y_{n-1} + \frac{300y_n + y_{n-1}}{5y_n + y_{n-1}}$

Fig. 3 $y_{n+1} = 0.25y_n + 0.3y_{n-1} + \frac{5y_n + y_{n-1}}{300y_n + y_{n-1}}$

Example 3 Figure 3 shows that the solution of (5) has global stability if $k = 1$, $p < q$, $y_{-1} = 1$, $y_0 = 2$, $A = 0.25$, $B = 0.3$, $p = 5$, $q = 300$.

Note that Example 1 verifies Theorem 6 part (3) which show that if k is odd, then (5) has prime period two solution. But Example 2 verifies Theorem 8 for $p > q$ which shows that the solution of (5) is globally asymptotically stable, while Example 3 verifies Theorem 9 for $p < q$ which shows that the solution of (5) is globally asymptotically stable.

8 Conclusions

We have investigated the global asymptotic stability, the periodicity nature and the bound-edness character of the positive solutions of the difference equation (1) subject to the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ where k is a positive integer. We have shown that the positive equilibrium point $\widetilde{y}$ of (1) is a global attractor depending on certain conditions. In the forth-coming articles, we might generalize (1) to be in the following new form:

$$x_{n+1} = Ax_n + Bx_{n-k_1} + Cx_{n-k_2} + \frac{\alpha x_n + \beta x_{n-k_1} + \gamma x_{n-k_2}}{ax_n + bx_{n-k_1} + cx_{n-k_2}}, \qquad (24)$$

subject to the initial conditions $x_{-\sigma_1}, x_{-\sigma_1+1}, \ldots, x_{-1}, x_0$ where $\sigma_1 = \max\{k_1, k_2\}$ and $A, \ldots, \alpha, \beta, \gamma, \ldots, c$ are arbitrary constants. Furthermore, (24) can be generalized to take the more general new form

$$x_{n+1} = Ax_n + \sum_{i=1}^{m} a_i x_{n-k_i} + \frac{Bx_n + \sum_{i=1}^{m} b_i x_{n-k_i}}{Cx_n + \sum_{i=1}^{m} c_i x_{n-k_i}}, \qquad (25)$$

subject to the initial conditions $x_{-\sigma_{m-1}}, x_{1-\sigma_{m-1}}, \ldots, x_{-1}, x_0$ where $\sigma_{m-1} = \max\{k_1, k_2, \ldots, k_m\}$ and $A, \ldots, a_i, b_i, c_i$ are arbitrary constants. Some qualitative behavior of the so-lutions of the difference equations (24), (25) will be obtained. These are open problems.

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