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CHOICE OF RIEMANNIAN METRICS FOR RIGID BODY KINEMATICS

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ABSTRACT

The set of spatial rigid body motions forms a Lie group known as the special Euclidean group in three dimensions, $SE(3)$. Chasles's theorem states that there exists a screw motion between two arbitrary elements of $SE(3)$. In this paper we investigate whether there exist a Riemannian metric whose geodesics are screw motions. We prove that no Riemannian metric with such geodesics exists and we show that the metrics whose geodesics are screw motions form a two-parameter family of semi-Riemannian metrics.

1 INTRODUCTION

The set of all three-dimensional rigid body displacements forms a Lie group [4, 5]. This group is generally referred to as $SE(3)$, the special Euclidean group in three dimensions. The tangent space at the identity endowed with the Lie bracket operation has the structure of a Lie algebra and is denoted by $se(3)$. It is isomorphic to the set of all twists and the Lie bracket of two twists corresponds to the motor product of the respective motors. A good discussion on the geometry of $SE(3)$ can be found in the appendix of [10].

There is extensive literature on the algebra of twists and the theory of screws [1, 6, 7, 15]. It is well known that the inner product on the space of twists, $se(3)$, induced by the usual inner product on $\mathbb{R}^6$ is not invariant under change of coordinate frames [7, 8, 10]. But on the space of twists, there are two quadratic forms, the Killing form and the Klein form, that are invariant under changes of the inertial reference frame as well as under changes of the body-fixed reference frame [5] (they are thus bi-invariant). However neither form is positive definite: The Killing form is degenerate and the Klein form is indefinite.

An inner product on $se(3)$ can be extended to a Riemannian metric on the whole group $SE(3)$ by left (right) translation. Such a metric is called left (right) invariant and is invariant with respect to change of inertial (body-fixed) frame. A left invariant metric for $SE(3)$, which is bi-invariant when restricted to the group of rotations, $SO(3)$, and which preserves the isotropy of $\mathbb{R}^3$, was proposed by Park and Brockett [12]. This metric depends on the choice of a length scale.

A Riemannian metric is everywhere positive definite and it provides a notion of length of curves on the manifold. In contrast, a metric which is non-degenerate but indefinite is called a semi-Riemannian metric [2] and in this case it is more appropriate to speak about the energy of a curve. Curves that minimize the energy between two given points are of particular interest. Such curves are called *geodesics* and can be considered a generalization of straight lines in Euclidean space to Riemannian manifolds.

Some of the geodesics for the scale dependent left invariant metric introduced in [12], are screw motions [11, 16]. Since Chasles's theorem guarantees the existence of a screw motion between any two points on $SE(3)$, a natural question is whether there exists a metric for which every geodesic is a screw motion. We show that there is no such Riemannian metric and that all metrics which have screw motions as geodesics belong to a two-parameter family of semi-Riemannian metrics.

The paper is organized as follows. In Section 2, we introduce the framework of Lie groups and differential geometry used in the rest of the paper. We also make precise the ideas of a Riemannian metric and an affine connection with reference to $SE(3)$. In Section 3, we discuss Chasles's theorem, screw motions and screw displacements using the framework of Section 2. We review the scale-dependent left invariant Riemannian metric introduced in [12] and show that for this metric some of the geodesics are screw motions. We then prove the key result of this paper, Theorem 3.6, in which the metrics for which screw motions are geodesics are identified. We conclude the paper with a short discussion.

2 KINEMATICS AND DIFFERENTIAL GEOMETRY

2.1 The Lie group $SE(3)$

Consider a rigid body moving in free space. Assume any inertial reference frame F fixed in space and a frame M fixed to the body at point O' as shown in Figure 1. At each instance, the configuration (position and orientation) of the rigid body can be described by a homogeneous transformation matrix, A, corresponding to the displacement from frame F to frame M. The set of all such matrices is called $SE(3)$, the special Euclidean group of rigid body transformations in three-dimensions:

$$SE(3) = \left\{ A \mid A = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}, \right.$$
$$\left. R \in \mathbb{R}^{3\times3}, d \in \mathbb{R}^3, R^T R = I, \det(R) = 1 \right\}.$$

It is easy to show [10] that $SE(3)$ is a group for the standard matrix multiplication and that it is a manifold. It is therefore a Lie group [13].

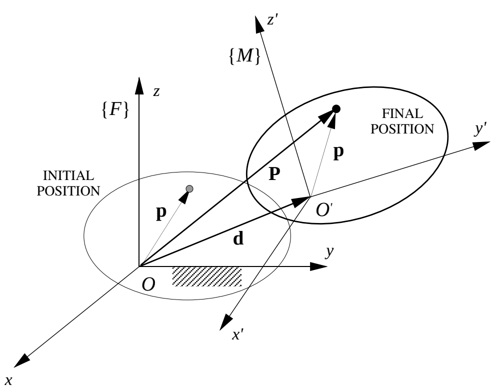

Figure 1. The inertial (fixed) frame and the moving frame attached to the rigid body

On any Lie group the tangent space at the group identity has the structure of a Lie algebra. The Lie algebra of $SE(3)$, denoted by $se(3)$, is given by:

$$se(3) = \left\{ \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix} \mid \Omega \in \mathbb{R}^{3\times3}, v \in \mathbb{R}^3, \Omega^T = -\Omega \right\}. \quad (1)$$

A 3×3 skew-symmetric matrix Ω can be uniquely identified with a vector $\omega \in \mathbb{R}^3$ so that for an arbitrary vector $x \in \mathbb{R}^3$, $\Omega x = \omega \times x$, where $\times$ is the vector cross product operation in $\mathbb{R}^3$. Each element $S \in se(3)$ can be thus identified with a vector pair $\{\omega, v\}$.

Given a curve $A(t) : [-a, a] \rightarrow SE(3)$, an element $S(t)$ of the Lie algebra $se(3)$ can be associated to the tangent vector $\dot{A}(t)$ at an arbitrary point t by:

$$S(t) = A^{-1}(t)\dot{A}(t). \quad (2)$$

A curve on $SE(3)$ physically represents a motion of the rigid body. If $\{\omega(t), v(t)\}$ is the vector pair corresponding to $S(t)$, then ω physically corresponds to the angular velocity of the rigid body while v is the linear velocity of the origin O' of the frame M, both expressed in the frame M. In kinematics, elements of this form are called twists [9] and $se(3)$ thus corresponds to the space of twists. It is easy to check that the twist $S(t)$ computed from Equation (2) does not depend on the choice of the inertial frame F. For this reason, $S(t)$ is called the left invariant representation of the tangent vector $\dot{A}$. Alternatively, the tangent vector $\dot{A}$ can be identified with a right invariant twist (invariant with respect to the choice of the body-fixed frame M). In this paper we concentrate on the left invariant twists but the derivations for the right invariant twists are analogous.

Since $se(3)$ is a vector space, any element can be expressed as a 6×1 vector of components corresponding to a chosen basis. The standard basis for $se(3)$ is:

$$L_1 = \begin{bmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{bmatrix} \quad L_2 = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&0 \\ -1&0&0&0 \\ 0&0&0&0 \end{bmatrix} \quad L_3 = \begin{bmatrix} 0&-1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$

$$L_4 = \begin{bmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \quad L_5 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \quad L_6 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}$$

The twists L_1, L_2 and L_3 represent instantaneous rotations about and L_4, L_5 and L_6 instantaneous translations along the Cartesian axes x, y and z, respectively. The components of a twist $S \in se(3)$ in this basis are given precisely by the velocity vector pair, $\{\omega, v\}$.

The Lie bracket of two elements $S_1, S_2 \in se(3)$ is defined by:

$$[S_1, S_2] = S_1 S_2 - S_2 S_1.$$

It can be easily verified that if $\{\omega_1, v_1\}$ and $\{\omega_2, v_2\}$ are vector pairs corresponding to the twists S_1 and S_2, the vector pair $\{\omega, v\}$ corresponding to their Lie bracket $[S_1, S_2]$ is given by

$$\{\omega, v\} = \{\omega_1 \times \omega_2, \omega_1 \times v_2 + v_1 \times \omega_2\}. \quad (3)$$

In kinematics, this product operation is called the motor product of the two twists.

The Lie bracket of two elements of a Lie algebra is an element of the Lie algebra. Thus, it can be expressed as a linear combination of the basis vectors. The coefficients C_{ij}^k corresponding to the Lie brackets of the basis vectors are called *structure constants* of the Lie algebra [13]:

$$[L_i, L_j] = \sum_k C_{ij}^k L_k. \tag{4}$$

The expressions (4) for $se(3)$, can be directly computed from Equation (3) and are listed in Appendix B.

2.2 Left invariant vector fields and exponential mapping

A *differentiable vector field* is a smooth assignment of a tangent vector to each element of the manifold. At each point, a vector field defines a unique *integral curve* to which it is tangent [3]. Formally, a vector field X is a (derivation) operator which, given a differentiable function, returns its derivative (another function) along the integral curves of X.

An example of a differentiable vector field, X, on $SE(3)$ is obtained by left translation of an element $S \in se(3)$. The value of the vector field X at an arbitrary point $A \in SE(3)$ is given by:

$$X(A) = \hat{S}(A) = AS. \tag{5}$$

A vector field generated by Equation (5) is called a left invariant vector field and we use the notation $\hat{S}$ to indicate that the vector field was obtained by left translating the Lie algebra element S. Right invariant vector fields can be defined analogously. In general, a vector field need not be left or right invariant. By construction, the space of left invariant vector fields is isomorphic to the Lie algebra $se(3)$. In particular (see [3]):

$$[\hat{L}_i, \hat{L}_j] = \widehat{[L_i, L_j]} = \sum_k C_{ij}^k \hat{L}_k. \tag{6}$$

Since the vectors $L_1, L_2, \ldots, L_6$ are a basis for the Lie algebra $se(3)$, the vectors $\hat{L}_1(A), \ldots, \hat{L}_6(A)$ form a basis of the tangent space at any point $A \in SE(3)$. Therefore, any vector field X can be expressed as

$$X = \sum_{i=1}^{6} X^i \hat{L}_i, \tag{7}$$

where the coefficients X^i vary over the manifold. If the coefficients are constants, then X is left invariant. By defining:

$$\omega = [X^1, X^2, X^3]^T, \qquad v = [X^4, X^5, X^6]^T,$$

we can associate a vector pair of functions $\{\omega, v\}$ to an arbitrary vector field X. If a curve $A(t)$ describes a motion of the rigid body and $V = \frac{dA}{dt}$ is the vector field tangent to $A(t)$, the vector pair $\{\omega, v\}$ associated with V corresponds to the instantaneous twist (screw axis) for the motion. In general, the twist $\{\omega, v\}$ changes with time.

Motions for which the twist $\{\omega, v\}$ is constant are known in kinematics as *screw motions*. In this case the twist $\{\omega, v\}$ is called *the screw axis* of the motion. If the vector pair $\{\omega, v\}$ is interpreted as Plücker coordinates of a line in space, it is not difficult to see that the screw motion physically corresponds to rotation around this line with a constant angular velocity and concurrent translation along the line with a constant translational velocity.

Let the twist $S \in se(3)$ be represented by a vector pair $\{\omega, v\}$ and let $A(t)$ be a screw motion with the screw axis $\{\omega, v\}$ such that $A(0) = I$. We define *the exponential map* $\exp : se(3) \rightarrow SE(3)$ by:

$$\exp(tS) = A(t). \tag{8}$$

Using Equation (2) we can show that the exponential map agrees with the usual exponentiation of the matrices in $\mathbb{R}^{4\times4}$:

$$\exp(tS) = \sum_{k=0}^{\infty} \frac{t^k \mathrm{S}^k}{k!}, \tag{9}$$

where S denotes the matrix representation of the twist S. The sum of this series can be computed explicitly and the resulting expression, when restricted to $SO(3)$, is known as Rodrigues' formula. The formula for the sum in $SE(3)$ is derived in [10].

2.3 Riemannian metrics on Lie groups

If a smoothly varying, positive definite, bilinear, symmetric form $< .,. >$ is defined on the tangent space at each point on the manifold, such form is called a Riemannian metric and the manifold is Riemannian [3]. If the form is non-degenerate but indefinite, the metric is called semi-Riemannian [2]. On a n dimensional manifold, the metric is locally characterized by a $n \times n$ matrix of $\mathcal{C}^\infty$ functions $g_{ij} = < X_i, X_j >$ where X_i are basis vector fields[1]. If the basis vector fields can be defined globally, then the matrix $[g_{ij}]$ completely defines the metric.

On $SE(3)$ (on any Lie group), an inner product on the Lie algebra can be extended to a Riemannian metric over the manifold using left (or right) translation. To see this, consider the inner product of two elements $S_1, S_2 \in se(3)$ defined by

$$< S_1, S_2 >|_I = s_1^T W s_2, \tag{10}$$

[1] The basis vector fields need not be the coordinate basis; we only require that they are smooth.

where s_1 and s_2 are the 6×1 vectors of components of S_1 and S_2 with respect to some basis and W is a positive definite matrix. If V_1 and V_2 are tangent vectors at an arbitrary group element $A \in SE(3)$, the inner product $< V_1, V_2 >|_A$ in the tangent space $T_A SE(3)$ can be defined by:

$$< V_1, V_2 >|_A = < A^{-1}V_1, A^{-1}V_2 >\big|_I . \tag{11}$$

The metric obtained in such a way is said to be left invariant [3] since left translation by any element A is an isometry.

2.4 Affine connection and covariant derivative

Motion of a rigid body can be represented by a curve on $SE(3)$. The velocity at an arbitrary point is the tangent vector to the curve at that point. In order to obtain the acceleration, or to engage in a dynamic analysis, we need to be able to differentiate a vector field along the curve. At each point $A \in SE(3)$, the value of a vector field belongs to the tangent space $T_A SE(3)$ and to differentiate a vector field along a curve, we must be able to subtract vectors from tangent spaces at different points on the curve. But tangent spaces at different points are not related. We thus have to specify how to transport a vector along the curve from one tangent space to another. This process is called *parallel transport* and is formalized with the affine connection [14]. Given any curve $A(t)$, a parameter value t_0 and a vector V in $T_{A(t_0)} SE(3)$, the tangent space at point $A(t_0)$, the affine connection assigns to each other parameter value t a vector $V' \in T_{A(t)} SE(3)$. By definition, V' is the parallel transport of V along the curve $A(t)$. Vectors V' and V are also said to be *parallel* along $A(t)$.

A derivative of a vector field along a curve $A(t)$ is defined through the parallel transport. Let X be a vector field defined along $A(t)$, and let $X(t)$ stand for $X(A(t))$. Denote by $X^{t_0}(t)$ the parallel transport of the vector $X(t)$ to the point $A(t_0)$. The *covariant derivative* of X along $A(t)$ is:

$$\left.\frac{DX}{dt}\right|_{t_0} = \lim_{t \to t_0} \frac{X^{t_0}(t) - X(t_0)}{t}. \tag{12}$$

By taking covariant derivatives along integral curves of a vector field Y, we obtain a covariant derivative of the vector field X with respect to the vector field Y. This derivative is also denoted by $\nabla_Y X$:

$$\nabla_Y X|_{A_0} = \left.\frac{DX}{dt}\right|_{t_0}, \tag{13}$$

where $\frac{DX}{dt}$ is taken along the integral curve of Y passing through A_0 at $t = t_0$.

The notions of the covariant derivative and parallelism are equivalent. In this section we defined the covariant derivative through parallel transport. However, given a covariant derivative, we can always define that a field X is parallel along a curve $A(t)$ if:

$$\nabla_{\frac{dA}{dt}} X = 0.$$

The covariant derivative of a vector field is another vector field so it can be expressed as a linear combination of the basis vector fields. The coefficients Γ^k_{ji} of the covariant derivative of a basis vector field along another basis vector field,

$$\nabla_{\hat{L}_i} \hat{L}_j = \sum_k \Gamma^k_{ji} \hat{L}_k, \tag{14}$$

are called *Christoffel symbols*[2]. Note the reversed order of the indices i and j.

The velocity, $V(t)$, of the rigid body describing the motion $A(t)$ is given by the tangent vector field along the curve:

$$V(t) = \frac{dA(t)}{dt}.$$

The acceleration, $\mathcal{A}(t)$, is the covariant derivative of the velocity along the curve

$$\mathcal{A} = \frac{\mathrm{D}}{dt}\left(\frac{dA}{dt}\right) = \nabla_V V. \tag{15}$$

Note that the acceleration depends on the choice of the connection. We can also define jerk, $\mathcal{J}$, as the covariant derivative of the acceleration:

$$\mathcal{J} = \frac{\mathrm{D}}{dt}\mathcal{A}(t) = \nabla_V \nabla_V V. \tag{16}$$

Given a Riemannian manifold, there exists a unique connection [3] which is compatible with the metric:

$$X < Y, Z > = < \nabla_X Y, Z > + < Y, \nabla_X Z > \tag{17}$$

and symmetric:

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{18}$$

[2] In the literature, different definitions for the Christoffel symbols can be found. Some texts (e.g. [3]) reserve the term for the case of the coordinate basis vectors. We follow the more general definition from [14] in which the basis vectors can be arbitrary.

This connection is called the *Levi-Civita* or *Riemannian connection*. It can be shown [3] that the compatibility condition (17) is equivalent to saying that the parallel transport preserves the inner product. In other words, if $A(t)$ is a curve and X and Y are two vector fields obtained by parallel transporting two vectors X_0 and Y_0 from $T_{A_0}SE(3)$ along $A(t)$, then $< X, Y >|_{A(t)} = < X_0, Y_0 >|_{A_0} = \text{const}$.

2.5 Geodesics

Given a Riemannian metric $< ., . >$ on $SE(3)$ we can define the length, $L(A)$, of a smooth curve $A : [a, b] \rightarrow SE(3)$ by:

$$L(A) = \int_a^b < \frac{dA}{dt}, \frac{dA}{dt} >^{\frac{1}{2}} dt \tag{19}$$

Among all the curves connecting two points, we are usually interested in the curve of minimal length. It is not difficult to see [3] that a curve of minimal length also minimizes the *energy functional*:

$$E(A) = \int_a^b < \frac{dA}{dt}, \frac{dA}{dt} > dt. \tag{20}$$

If a curve minimizes a functional, it must be also a critical point. Critical points of the energy functional E satisfy the following equation [3]:

$$\nabla_{\frac{dA}{dt}} \frac{dA}{dt} = 0. \tag{21}$$

where ∇ is the Riemannian connection, and are called *geodesics*. From what we said about the covariant derivative and parallel transport, it follows that a geodesic is a curve $A(t)$ for which the tangent vector field $\frac{dA}{dt}$ is parallel: from a value at a point we can obtain its value at any other point by simply parallel transporting it along the curve $A(t)$. On the other hand, According to Eq. (15), the expression $\nabla_{\frac{dA}{dt}} \frac{dA}{dt}$ is the acceleration of motion described by $A(t)$. Motion along a geodesic therefore produces zero acceleration.

3 METRICS AND SCREW MOTIONS

One of the fundamental results in rigid body kinematics [9] was proved by Chasles at the beginning of the 19th century:

Theorem 3.1 (Chasles) *Any rigid body displacement can be realized by a rotation about an axis combined with a translation parallel to that axis.*

Note that a displacement must be understood as an element of $SE(3)$ while a motion is a curve on $SE(3)$. If the rotation from the Chasles's theorem is performed at constant angular velocity and the translation at constant translational velocity, the motion leading to the displacement clearly becomes a screw motion. Chasles's theorem therefore says that any rigid body displacement can be realized by a screw motion.

Another family of curves of particular interest on $SE(3)$ are the *one-parameter subgroups*. A curve $A(t)$ is a one-parameter subgroup, if $A(t_1 + t_2) = A(t_1)A(t_2)$. The one-parameter subgroups on $SE(3)$ are given by [3]:

$$A(t) = \exp(t\, S) \tag{22}$$

where S is an element of $se(3)$. From our discussion in Section 2.2 it is clear that the one parameter subgroups are exactly the screw motions which pass through the identity. In Chasles's theorem, we can obviously assume that the body fixed frame M is initially aligned with the inertial frame F. Therefore, in the language of differential geometry, the theorem can be restated as follows:

Theorem 3.2 (Chasles restated) *For every element in $SE(3)$ there is a one-parameter subgroup (screw motion through the identity) to which that element belongs.*

Except for the identity, the screw axis for every element is unique, but there are infinitely many screw motions along that axis which contain the element, each characterized by the number of rotations along the screw axis. The following corollary immediately follows from the theorem:

Corollary 3.3 *If A_1 and A_2 are two distinct elements of $SE(3)$, then:*

(1) There exists a one-parameter subgroup, $\gamma_L(t) = \exp(t\, S_L)$, which when left translated by A_1 contains A_2:

$$A_L(t) = A_1 \, \exp(t\, S_L), \qquad A_2 = A_L(1) = A_1 \exp(S_L).$$

(2) There exists a one-parameter subgroup, $\gamma_R(t) = \exp(t\, S_R)$, which when right translated by A_1 contains A_2:

$$A_R(t) = \exp(t\, S_R) A_1, \qquad A_2 = A_R(1) = \exp(S_R) A_1.$$

(3) For each S_L in (1) we can find the corresponding S_R in (2) by:

$$S_R = A_1 S_L A_1^{-1},$$

such that

$$A_L(t) = A_R(t).$$

Proof: Statement (1) (respectively (2)) follows from Theorem 3.2 if we left (right) translate the one-parameter subgroup to which $A^{-1}A_2$ (A_2A^{-1}) belongs. To see (3), note that:

$$A_1 \exp(t\, S_L) = A_1 \sum_{k=0}^{\infty} \frac{t^k\, S_L^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k (A_1\, S_L A_1^{-1})^k}{k!}\, A_1 .$$

□

Park and Brockett [12] proposed a left invariant Riemannian metric on $SE(3)$ given by:

$$W = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \tag{23}$$

where α and β are positive scalars.

Park [11] derived the geodesics for the metric (23) and showed that they are products of the geodesics for the bi-invariant metric on $SO(3)$ and geodesics in the Euclidean space $\mathbb{R}^3$. Geodesics for the bi-invariant metric on $SO(3)$ are the restrictions of the screw motions to $SO(3)$ [11], while straight lines parameterized proportionally to the line length are the geodesics for the Euclidean space $\mathbb{R}^3$. A geodesic between two elements

$$\begin{bmatrix} R_1 & d_1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} R_2 & d_2 \\ 0 & 1 \end{bmatrix}$$

thus physically corresponds to a translation of the origin O' of the body fixed frame M with a constant translational velocity along the line connecting the points described with the position vectors d_1 and d_2, and concurrent rotation of the frame M with constant angular velocity about an axis passing through the origin O' of the frame M which translates together with the point O'.

It is clear that such a motion is, in general, not a screw motion since the axis around which the body rotates is not fixed in space. However, if the axis of rotation is collinear with the line between d_1 and d_2 along which the body translates, the rotational axis does not change as the body moves and the geodesic is therefore a screw motion. It follows that a screw motion is a geodesic if and only if it is obtained by left translation of a one-parameter subgroup for which the screw axis passes through the origin O of the frame F.

3.1 Screw motions as geodesics

Given that any two elements of $SE(3)$ can be connected with a screw motion, and given that there exists a left invariant metric whose geodesics include certain screw motions, it is natural to ask whether there are Riemannian metrics for which every geodesic is a screw motion.

Before we proceed, we turn our attention back to Corollary 3.3. The corollary says that any screw motion can be obtained in two ways: either by a left or by a right translation of a (in general different) one-parameter subgroup. Now suppose that the screw motions are geodesics. Corollary 3.3 implies that a left or a right translation of a geodesic produces another geodesic. We might therefore wrongly conclude that any metric for which the screw motions are geodesics must be invariant under left and right translations and therefore bi-invariant. Such reasoning is false since a map which preserves geodesics does not necessarily preserve the metric (is not necessary an isometry)! This is clear if we consider affine transformations in Euclidean space: They map lines into lines (that is, they map geodesics to geodesics), but in general, they do not preserve lengths of vectors. Therefore, we cannot limit our search to left or right invariant metrics.

We now derive the family of metrics which have screw motions for geodesics. As we saw in Section 2.2, the twist associated with a screw motion $\gamma(t)$ is constant. The tangent vector field $V = \frac{d\gamma}{dt}$ is therefore a left invariant vector field and it has constant components with respect to the chosen basis vector fields: $V = V^i \hat{L}_i$. If γ solves Eq. (21), we have:

$$\begin{aligned} 0 = \nabla_V V &= \sum_j \frac{dV^j}{dt}\hat{L}_j + \sum_{i,j} V^i V^j \nabla_{\hat{L}_i}\hat{L}_j \\ &= \sum_{i,j} V^i V^j \nabla_{\hat{L}_i}\hat{L}_j . \end{aligned} \tag{24}$$

The above equation is satisfied for any screw motion (arbitrary choice of the components V^i) if and only if

$$\nabla_{\hat{L}_i}\hat{L}_j + \nabla_{\hat{L}_j}\hat{L}_i = 0.$$

Since ∇ is a metrical connection, it is symmetric (Eq. 18):

$$\nabla_{\hat{L}_i}\hat{L}_j - \nabla_{\hat{L}_j}\hat{L}_i = [\hat{L}_i, \hat{L}_j].$$

It immediately follows that:

$$\nabla_{\hat{L}_i}\hat{L}_j = \frac{1}{2}[\hat{L}_i, \hat{L}_j]. \tag{25}$$

Further, ∇ must be compatible with the metric (Eq. 17), so we have:

$$\hat{L}_k < \hat{L}_i, \hat{L}_j > = < \nabla_{\hat{L}_k}\hat{L}_i, \hat{L}_j > + < \hat{L}_i, \nabla_{\hat{L}_k}\hat{L}_j > . \tag{26}$$

Letting $g_{ij} = < \hat{L}_i, \hat{L}_j >$, the last equation implies:

$$\hat{L}_k(g_{ij}) = \frac{1}{2}\left(< [\hat{L}_k, \hat{L}_i], \hat{L}_j > + < \hat{L}_i, [\hat{L}_k, \hat{L}_j] >\right) . \tag{27}$$

By expressing the Lie brackets from Eq. (4), we finally obtain:

$$\hat{L}_k(g_{ij}) = \frac{1}{2}\sum_l (C^l_{ki} g_{lj} + C^l_{kj} g_{li}). \tag{28}$$

Note that the coefficients C^k_{ij} are constant over the manifold (Eq. 6). The above derivation is summarized in the following proposition:

Proposition 3.4 *Screw motions satisfy the geodesic equation (21) for a Riemannian metric given by the matrix of coefficients $G = [g_{ij}]$ if and only if the coefficients g_{ij} satisfy Eq. (28).*

The metric coefficients g_{ij} are symmetric by definition. Since $SE(3)$ is a 6 dimensional manifold, there are 21 independent coefficients $\{g_{ij} \mid 1 \leq i \leq j \leq 6\}$. Further, there are 6 basis vector fields hence Eq. (28) expands to a total of 126 equations. Each vector field represents a derivation implying that these are partial differential equations. The complete set of equations is given in Appendix A.

We need the following lemma to derive the solution for the system of equations given by (28):

Lemma 3.5 *Given a set of partial differential equations*

$$\begin{aligned} X(f) &= g_x & (29)\\ Y(f) &= g_y & (30)\\ Z(f) &= g_z & (31) \end{aligned}$$

where X, Y, and Z are vector fields such that $Z = [X, Y]$, f is twice differentiable, and g_x, g_y and g_z are differentiable (real valued) functions, the solution exists only if

$$X(g_y) - Y(g_x) = g_z. \tag{32}$$

Proof: By applying X on Eq. (30), Y on Eq. (29) and subtracting the two resulting equations, we get:

$$X\,Y(f) - Y\,X(f) = X(g_y) - Y(g_x). \tag{33}$$

But the left-hand side is by definition $[X, Y](f)$, which is by assumption equal to $Z(f)$. Equation (32) then follows from Eq. (31).

□

We next state the first key theorem of this chapter:

Theorem 3.6 *A matrix of coefficients $G = [g_{ij}]$ satisfies the system of partial differential equations (28) if and only if it has the form*

$$G = \begin{bmatrix} \alpha\, I_{3\times3} & \beta\, I_{3\times3} \\ \beta\, I_{3\times3} & 0_{3\times3} \end{bmatrix}, \tag{34}$$

where α and β are constants.

Proof: To find the metric coefficients, we start with the following subset of (47):

$$\hat{L}_1(g_{11}) = 0 \quad \hat{L}_2(g_{11}) = -g_{13} \quad \hat{L}_3(g_{11}) = g_{12}. \tag{35}$$

First, observe that $[\hat{L}_1, \hat{L}_2] = \hat{L}_3$ (see Appendix B). By application of Lemma 3.5, the following equation must hold:

$$-\hat{L}_1(g_{13}) = g_{12}. \tag{36}$$

But from (47), we have:

$$\hat{L}_1(g_{13}) = -\frac{1}{2} g_{12}.$$

Therefore, Eq. (36) becomes:

$$\frac{1}{2} g_{12} = g_{12}.$$

Obviously, this implies that $g_{12} = 0$. We next observe that $g_{12} = 0$ implies $\hat{L}_i(g_{12}) = 0, \quad 1 \leq i \leq 6$. From the system (47) we obtain:

$$\begin{array}{lll} g_{13} = 0 & g_{23} = 0 & g_{11} = g_{22} \\ g_{16} = 0 & g_{26} = 0 & g_{14} = g_{25}. \end{array} \tag{37}$$

From these equations and (47) we further obtain:

$$\begin{array}{lll} g_{15} = 0 & g_{24} = 0 & g_{11} = g_{33} \\ g_{34} = 0 & g_{35} = 0 & g_{14} = g_{36} \\ g_{44} = 0 & g_{45} = 0 & g_{46} = 0 \\ g_{55} = 0 & g_{56} = 0 & g_{66} = 0. \end{array} \tag{38}$$

Next observation is that $\hat{L}_i(g_{11}) = 0, \quad 1 \leq i \leq 6$. This, together with Eqs. (37) and (38) implies:

$$g_{11} = g_{22} = g_{33} = \alpha,$$

where α is an arbitrary constant. Similarly, we obtain

$$g_{14} = g_{25} = g_{36} = \beta,$$

for an arbitrary constant β. In this way we have obtained all 21 independent values of G. The reader can easily check that the system of equations (47) is satisfied by the above values.

□

Corollary 3.7 *There is no Riemannian metric whose geodesics are screw motions.*

Proof: It is easy to check that a matrix of the form

$$G = \begin{bmatrix} \alpha I_{3\times 3} & \beta I_{3\times 3} \\ \beta I_{3\times 3} & 0_{3\times 3} \end{bmatrix},$$

has two distinct real eigenvalues

$$\lambda_1 = \frac{1}{2}(\alpha + \sqrt{\alpha^2 + 4\beta^2}) \qquad \lambda_2 = \frac{1}{2}(\alpha - \sqrt{\alpha^2 + 4\beta^2}),$$

which both have multiplicity 3. For any choice of α and β, the product of the eigenvalues is $\lambda_1 \lambda_2 = -\beta^2 \leq 0$. Therefore, G is not positive definite as required for a Riemannian metric.

□

3.2 Invariance of the family of metrics (34)

Metrics of the form (34) form a two-parameter family of semi-Riemannian metrics and can be studied in a similar way as Riemannian metrics. In particular, we can investigate their invariance properties. By definition, a metric is left invariant if for any $A, B \in SE(3)$ and for any vector fields X and Y:

$$< X(B), Y(B) >|_B = < AX(B), AY(B) >|_{AB}\,, \qquad (39)$$

and it is right invariant if:

$$< X(B), Y(B) >|_B = < X(B)A, Y(B)A >|_{BA}\,. \qquad (40)$$

Lemma 3.8 *If S_1 and S_2 are two elements of $se(3)$ and a metric of the form (34) is defined on $SE(3)$, then for any $A \in SE(3)$*

$$< S_1, S_2 >|_I = < \mathrm{Ad}_A(S_1), \mathrm{Ad}_A(S_2) >|_I\,. \qquad (41)$$

(The map $\mathrm{Ad} : se(3) \to se(3)$ is called *the adjoint map* and if S is represented by a matrix, the map is defined by $\mathrm{Ad}_A(S) = A\,S\,A^{-1}$.)

Proof: Let $S_1 = \{\omega_1, v_1\}$ and $S_2 = \{\omega_2, v_2\}$. By a straightforward algebraic calculation it can be shown that for $S = \{\omega, v\} \in se(3)$ and $A \in SE(3)$, where

$$A = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

the value of $\mathrm{Ad}_A(S)$ is given by $\mathrm{Ad}_A(S) = \{R\,\omega, R\,v - (R\omega)\times d\}$ where $\times$ is the usual vector cross product. Therefore, we have:

$$\begin{aligned} &< \mathrm{Ad}_A(S_1), \mathrm{Ad}_A(S_2) >|_I \\ &\quad = \; < \{R\,\omega_1, R\,v_1 - (R\omega_1)\times d\}, \\ &\qquad\quad \{R\,\omega_2, R\,v_2 - (R\omega_2)\times d\} >|_I \\ &\quad = \; \alpha\,(R\,\omega_1)^T(R\,\omega_2) + \beta\,(R\,\omega_1)^T(R\,v_2 - (R\omega_2)\times d) \\ &\qquad + \beta\,(R\,\omega_2)^T(R\,v_1 - (R\omega_1)\times d) \\ &\quad = \; \alpha\,\omega_1^T\omega_2 + \beta\,(\omega_1^T v_2 + \omega_2^T v_1) \\ &\qquad - \beta\left[(R\,\omega_1)^T((R\omega_2)\times d) + (R\,\omega_2)^T((R\omega_1)\times d)\right] \\ &\quad = \; \alpha\,\omega_1^T\omega_2 + \beta\,(\omega_1^T v_2 + \omega_2^T v_1) \\ &\quad = \; < \{\omega_1, v_1\}, \{\omega_2, v_2\} >|_I = < S_1, S_2 >|_I \end{aligned}$$

□

Proposition 3.9 *Any left invariant metric G that satisfies Eq. (41) is bi-invariant (both, left and right invariant).*

Proof: We have to prove that G is right invariant. Take two vector fields X and Y. Since the metric G is left invariant, we have:

$$\begin{aligned} &< X(B)A, Y(B)A >|_{BA} \\ &\quad = < (B\,A)^{-1}X(B)A, (B\,A)^{-1}Y(B)A >\big|_I \\ &\quad = < A^{-1}B^{-1}X(B)A, A^{-1}B^{-1}Y(B)A >\big|_I\,. \end{aligned}$$

By Eq. (41),

$$< A^{-1}B^{-1}X(B)A, A^{-1}B^{-1}Y(B)A >\big|_I \qquad (42)$$

$$= < B^{-1}X(B), B^{-1}Y(B) >\big|_I\,. \qquad (43)$$

But because of the left invariance of G, the last expression is:

$$< B^{-1}X(B), B^{-1}Y(B) >\big|_I = < X(B), Y(B) >|_B\,,$$

as required.

□

Corollary 3.10 *Any metric G of the form (34) is bi-invariant.*

Proof: It is obvious that a metric G of the form (34) is left invariant, since it is constant for the basis of the left invariant vector fields $\hat{L}_i$. By Lemma 3.8 and Proposition 3.9, G is bi-invariant.

□

3.3 Geodesics of the family of metrics (34)

Analogous to the Riemannian case, we could define the length of a curve $A(t)$ between two points $A(t_1)$ and $A(t_2)$ on $SE(3)$ by:

$$L(A; t_1, t_2) = \int_{t_1}^{t_2} < \frac{dA}{dt}, \frac{dA}{dt} >^{\frac{1}{2}} dt. \tag{44}$$

But G is not positive definite, so the length of a curve would be in general a complex number. Therefore, it is more useful to define the measure of the energy of a curve:

$$E(A; t_1, t_2) = \int_{t_1}^{t_2} < \frac{dA}{dt}, \frac{dA}{dt} > dt. \tag{45}$$

Since G is not positive definite, the energy of a curve can, in general, be negative. There are also non-trivial curves (that is, curves that are not identically equal to a point) which have zero energy.

Two special cases of metric (34) are of particular interest. With $\alpha = 0$ and $\beta = 1$ we obtain the metric:

$$G = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ I_{3\times3} & 0_{3\times3} \end{bmatrix}.$$

This metric, taken as a quadratic form on $se(3)$, is known as the Klein form. The eigenvalues for the metric are $\{1, 1, 1, -1, -1, -1\}$ and the form is therefore non-degenerate. For a screw motion $A(t) = A_0 \exp(t\,S)$ where $S = \{\omega, v\} \in se(3)$, the energy of the segment $t \in [0, 1]$ is given by $E(A) = 2\,\omega^T v$. If $\omega \neq 0$, the quantity:

$$h = \frac{\omega^T v}{|\omega|^2} \tag{46}$$

is called the *pitch* of the screw motion [6]. The pitch measures the amount of translation along the screw axis during the screw motion. Zero energy screw motions therefore either have zero pitch (the motion is pure rotation) or infinite pitch ($\omega = 0$, the motion is pure translation). Screw motions with positive energy are those with positive pitch. Trajectories for such motions correspond to right-handed helices and the motions are thus called right-handed screw motions. Analogously, screw motions with negative energy are the left-handed screw motions. Since pure rotations and pure translations are zero-energy motions, it is always possible to find a zero energy curve between two arbitrary points by breaking the motion into a segment consisting of a pure rotation followed by a segment of a pure translation.

By letting $\alpha = 1$ and $\beta = 0$, we get the semi-definite metric:

$$G = \begin{bmatrix} I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}.$$

This metric, as a form on $se(3)$, is called the Killing form. Its eigenvalues are $\{1, 1, 1, 0, 0, 0\}$ hence it is degenerate. The energy of a screw motion with $S = \{\omega, v\}$ is equal to $\omega^T\omega$ so it is always non-negative. Pure translations are zero-energy motions while any motion involving rotation has positive energy.

In the general case, $\alpha \neq 0$ and $\beta \neq 0$, the energy of a unit screw motion $A(t) = A_0 \exp(t\,S)$ where $S = \{\omega, v\}$ and $t \in [0, 1]$, is $\omega^T(\alpha\,\omega + 2\beta\,v)$. Pure translations ($\omega = 0$) thus have zero energy. For a general screw motion ($\omega \neq 0$), the energy of the segment $t \in [0, 1]$ is $|\omega|^2(\alpha + 2\beta h)$. The sign of the energy of a general motion therefore depends on α and β.

4 CONCLUSION

The set of all displacements of a rigid body forms a Lie group $SE(3)$. The Lie algebra of $SE(3)$, denoted by $se(3)$, represents the space of twists and therefore provides a natural setting for velocity analysis. In this paper we investigate how $SE(3)$ can be endowed with additional structure so that some well-known results pertaining to finite displacement analysis can be derived. We show that a natural setting to study screw motions is $SE(3)$ equipped with a metric that belongs to a two-parameter family of semi-Riemannian metrics. The metrics in this family are indefinite and, in general, they are non-degenerate. Viewed as a quadratic form on $se(3)$, the metrics are a linear combination of the Killing form and the Klein form. We prove that any non-degenerate metric in this family defines a unique symmetric connection for which geodesics are screw motions. We also prove that there is no Riemannian metric which is compatible with this connection and therefore no Riemannian metric which would have screw motions for geodesics.

A EQUATIONS DEFINING METRIC WITH SCREW MOTIONS AS GEODESICS

In Section 3.1 we concluded that Equation (28):

$$\hat{L}_k(g_{ij}) = \frac{1}{2}\sum_l (C_{ki}^l g_{lj} + C_{kj}^l g_{li}),$$

must be satisfied by the metric if screw motions are geodesics. The coefficients C_{ij}^k are the structure constants of the Lie algebra $se(3)$. We evaluated this equation in Mathematica to obtain a system of 126 partial differential equations, that have to be solved for the metric coefficients g_{ij}. In the equations we use the abbreviation $\mathcal{G}_{ij}^k \stackrel{\text{def}}{=} \hat{L}_k(g_{ij})$.

$$
\begin{array}{lll}
\mathcal{G}_{11}^1 = 0 & \mathcal{G}_{11}^2 = -g_{13} & \mathcal{G}_{11}^3 = g_{12} \\
\mathcal{G}_{11}^4 = 0 & \mathcal{G}_{11}^5 = -g_{16} & \mathcal{G}_{11}^6 = g_{15} \\
\mathcal{G}_{12}^1 = \tfrac{1}{2} g_{13} & \mathcal{G}_{12}^2 = -\tfrac{1}{2} g_{23} & \mathcal{G}_{12}^3 = \tfrac{1}{2}(g_{22} - g_{11}) \\
\mathcal{G}_{12}^4 = \tfrac{1}{2} g_{16} & \mathcal{G}_{12}^5 = -\tfrac{1}{2} g_{26} & \mathcal{G}_{12}^6 = \tfrac{1}{2}(g_{25} - g_{14}) \\
\mathcal{G}_{13}^1 = -\tfrac{1}{2} g_{12} & \mathcal{G}_{13}^2 = \tfrac{1}{2}(g_{11} - g_{33}) & \mathcal{G}_{13}^3 = \tfrac{1}{2} g_{23} \\
\mathcal{G}_{13}^4 = -\tfrac{1}{2} g_{15} & \mathcal{G}_{13}^5 = \tfrac{1}{2}(g_{14} - g_{36}) & \mathcal{G}_{13}^6 = \tfrac{1}{2} g_{35} \\
\mathcal{G}_{14}^1 = 0 & \mathcal{G}_{14}^2 = \tfrac{1}{2}(-g_{34} - g_{16}) & \mathcal{G}_{14}^3 = \tfrac{1}{2}(g_{24} + g_{15}) \\
\mathcal{G}_{14}^4 = 0 & \mathcal{G}_{14}^5 = -\tfrac{1}{2} g_{46} & \mathcal{G}_{14}^6 = \tfrac{1}{2} g_{45} \\
\mathcal{G}_{15}^1 = \tfrac{1}{2} g_{16} & \mathcal{G}_{15}^2 = -\tfrac{1}{2} g_{35} & \mathcal{G}_{15}^3 = \tfrac{1}{2}(g_{25} - g_{14}) \\
\mathcal{G}_{15}^4 = 0 & \mathcal{G}_{15}^5 = -\tfrac{1}{2} g_{56} & \mathcal{G}_{15}^6 = \tfrac{1}{2} g_{55} \\
\mathcal{G}_{16}^1 = -\tfrac{1}{2} g_{15} & \mathcal{G}_{16}^2 = \tfrac{1}{2}(g_{14} - g_{36}) & \mathcal{G}_{16}^3 = \tfrac{1}{2} g_{26} \\
\mathcal{G}_{16}^4 = 0 & \mathcal{G}_{16}^5 = -\tfrac{1}{2} g_{66} & \mathcal{G}_{16}^6 = \tfrac{1}{2} g_{56} \\
\mathcal{G}_{22}^1 = g_{23} & \mathcal{G}_{22}^2 = 0 & \mathcal{G}_{22}^3 = -g_{12} \\
\mathcal{G}_{22}^4 = g_{26} & \mathcal{G}_{22}^5 = 0 & \mathcal{G}_{22}^6 = -g_{24} \\
\mathcal{G}_{23}^1 = \tfrac{1}{2}(g_{33} - g_{22}) & \mathcal{G}_{23}^2 = \tfrac{1}{2} g_{12} & \mathcal{G}_{23}^3 = -\tfrac{1}{2} g_{13} \\
\mathcal{G}_{23}^4 = \tfrac{1}{2}(g_{36} - g_{25}) & \mathcal{G}_{23}^5 = \tfrac{1}{2} g_{24} & \mathcal{G}_{23}^6 = -\tfrac{1}{2} g_{34} \\
\mathcal{G}_{24}^1 = \tfrac{1}{2} g_{34} & \mathcal{G}_{24}^2 = -\tfrac{1}{2} g_{26} & \mathcal{G}_{24}^3 = \tfrac{1}{2}(g_{25} - g_{14}) \\
\mathcal{G}_{24}^4 = \tfrac{1}{2} g_{46} & \mathcal{G}_{24}^5 = 0 & \mathcal{G}_{24}^6 = -\tfrac{1}{2} g_{44} \\
\mathcal{G}_{25}^1 = \tfrac{1}{2}(g_{35} + g_{26}) & \mathcal{G}_{25}^2 = 0 & \mathcal{G}_{25}^3 = \tfrac{1}{2}(-g_{15} - g_{24}) \\
\mathcal{G}_{25}^4 = \tfrac{1}{2} g_{56} & \mathcal{G}_{25}^5 = 0 & \mathcal{G}_{25}^6 = -\tfrac{1}{2} g_{45} \\
\mathcal{G}_{26}^1 = \tfrac{1}{2}(g_{36} - g_{25}) & \mathcal{G}_{26}^2 = \tfrac{1}{2} g_{24} & \mathcal{G}_{26}^3 = -\tfrac{1}{2} g_{16} \\
\mathcal{G}_{26}^4 = \tfrac{1}{2} g_{66} & \mathcal{G}_{26}^5 = 0 & \mathcal{G}_{26}^6 = -\tfrac{1}{2} g_{46} \\
\mathcal{G}_{33}^1 = -g_{23} & \mathcal{G}_{33}^2 = g_{13} & \mathcal{G}_{33}^3 = 0 \\
\mathcal{G}_{33}^4 = -g_{35} & \mathcal{G}_{33}^5 = g_{34} & \mathcal{G}_{33}^6 = 0 \\
\mathcal{G}_{34}^1 = -\tfrac{1}{2} g_{24} & \mathcal{G}_{34}^2 = \tfrac{1}{2}(g_{14} - g_{36}) & \mathcal{G}_{34}^3 = \tfrac{1}{2} g_{35} \\
\mathcal{G}_{34}^4 = -\tfrac{1}{2} g_{45} & \mathcal{G}_{34}^5 = \tfrac{1}{2} g_{44} & \mathcal{G}_{34}^6 = 0 \\
\mathcal{G}_{35}^1 = \tfrac{1}{2}(g_{36} - g_{25}) & \mathcal{G}_{35}^2 = \tfrac{1}{2} g_{15} & \mathcal{G}_{35}^3 = -\tfrac{1}{2} g_{34} \\
\mathcal{G}_{35}^4 = -\tfrac{1}{2} g_{55} & \mathcal{G}_{35}^5 = \tfrac{1}{2} g_{45} & \mathcal{G}_{35}^6 = 0 \\
\mathcal{G}_{36}^1 = \tfrac{1}{2}(-g_{26} - g_{35}) & \mathcal{G}_{36}^2 = \tfrac{1}{2}(g_{16} + g_{34}) & \mathcal{G}_{36}^3 = 0 \\
\mathcal{G}_{36}^4 = -\tfrac{1}{2} g_{56} & \mathcal{G}_{36}^5 = \tfrac{1}{2} g_{46} & \mathcal{G}_{36}^6 = 0 \\
\mathcal{G}_{44}^1 = 0 & \mathcal{G}_{44}^2 = -g_{46} & \mathcal{G}_{44}^3 = g_{45} \\
\mathcal{G}_{44}^4 = 0 & \mathcal{G}_{44}^5 = 0 & \mathcal{G}_{44}^6 = 0 \\
\mathcal{G}_{45}^1 = \tfrac{1}{2} g_{46} & \mathcal{G}_{45}^2 = -\tfrac{1}{2} g_{56} & \mathcal{G}_{45}^3 = \tfrac{1}{2}(g_{55} - g_{44}) \\
\mathcal{G}_{45}^4 = 0 & \mathcal{G}_{45}^5 = 0 & \mathcal{G}_{45}^6 = 0 \\
\mathcal{G}_{46}^1 = -\tfrac{1}{2} g_{45} & \mathcal{G}_{46}^2 = \tfrac{1}{2}(g_{44} - g_{66}) & \mathcal{G}_{46}^3 = \tfrac{1}{2} g_{56} \\
\mathcal{G}_{46}^4 = 0 & \mathcal{G}_{46}^5 = 0 & \mathcal{G}_{46}^6 = 0 \\
\mathcal{G}_{55}^1 = g_{56} & \mathcal{G}_{55}^2 = 0 & \mathcal{G}_{55}^3 = -g_{45} \\
\mathcal{G}_{55}^4 = 0 & \mathcal{G}_{55}^5 = 0 & \mathcal{G}_{55}^6 = 0 \\
\mathcal{G}_{56}^1 = \tfrac{1}{2}(g_{66} - g_{55}) & \mathcal{G}_{56}^2 = \tfrac{1}{2} g_{45} & \mathcal{G}_{56}^3 = -\tfrac{1}{2} g_{46} \\
\mathcal{G}_{56}^4 = 0 & \mathcal{G}_{56}^5 = 0 & \mathcal{G}_{56}^6 = 0 \\
\mathcal{G}_{66}^1 = -g_{56} & \mathcal{G}_{66}^2 = g_{46} & \mathcal{G}_{66}^3 = 0 \\
\mathcal{G}_{66}^4 = 0 & \mathcal{G}_{66}^5 = 0 & \mathcal{G}_{66}^6 = 0
\end{array}
\tag{47}
$$

B LIE BRACKETS FOR $SE(3)$

In our derivations we need to evaluate Lie brackets of the basis vector fields $\hat{L}_i$. According to Equation (6), since the vector fields $\hat{L}_i$ are left invariant, it suffices to evaluate the brackets on $se(3)$. From Equation (3) we obtain:

$$
\begin{array}{lll}
[L_1, L_1] = 0 & [L_1, L_2] = L_3 & [L_1, L_3] = -L_2 \\
[L_1, L_4] = 0 & [L_1, L_5] = L_6 & [L_1, L_6] = -L_5 \\
[L_2, L_2] = 0 & [L_2, L_3] = L_1 & [L_2, L_4] = -L_6 \\
[L_2, L_5] = 0 & [L_2, L_6] = L_4 & [L_3, L_3] = 0 \\
[L_3, L_4] = L_5 & [L_3, L_5] = -L_4 & [L_3, L_6] = 0 \\
[L_4, L_4] = 0 & [L_4, L_5] = 0 & [L_4, L_6] = 0 \\
[L_5, L_5] = 0 & [L_5, L_6] = 0 & [L_6, L_6] = 0
\end{array}
$$

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