Invariants of Curves in Lagrangian Grassmannian and Differential Geometry of Smooth Control Systems.

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Contents

Abstract

Our work is devoted to the construction and applications of feedback or gauge invariants of a wide class of smooth control systems and geometric structures. These geometric invariants are far going generalization of the curvature tensor on a Riemannian manifold. The knowledge of these invariants gives a fundamental information about the behavior of extremals and on the structure of optimal synthesis.

The new approach for constructing of feedback invariants for control systems was proposed recently by A.A. Agrachev: Using the Pontryagin Maximum Principle, one can construct, for a given extremal (even singular one) of the control system, a special family of Lagrangian subspaces, called Jacobi curve, in the appropriate symplectic space. Actually, Jacobi curve is generalization of the space of "Jacobi fields" along Riemannian geodesics. In this way one can reduce the problem of finding feedback invariants of control systems, where very big group (the group of feedback transformations) acts on very big set (the set of control systems), to the much more concrete problem of finding symplectic invariants of certain curves in Lagrangian Grassmannian (the set of all Lagrangian subspaces of some symplectic space).

In the present work we develop general theory of curves in the Lagrange Grassmannian and apply it to the qualitative study of some specific classes of control systems and geometric structures. Here we list the main results of the work:

1. We introduce two principal symplectic invariants : the *generalized Ricci curvature*, which is an invariant of the parametrized curve in Lagrange Grassmannian providing the curve with a natural projective structure, and a *fundamental form*, which is a degree 4 differential on the curve. In Riemannian geometry this fundamental form is related to the Weyl conformal tensor.

2. We construct a complete system of symplectic invariants for the curve with rank 1 velocities. Jacobi curves of this class are associated to systems with scalar controls and to rank 2 vector distributions.

3. We introduce and investigate the "straightest" curves of the given constant rank in the Lagrange Grassmannian, called *flat curves*. Basic symplectic invariants of a curve measure its deviation from the flat curve and thus play the role of curvatures.

4. We give the estimates for the conjugate points of rank 1 curves in the Lagrangian Grassmannian of 4-dimensional symplectic space. The estimates are presented in the form of *comparison theorems*. Using these theorems one can estimate the intervals of weak optimality of extremals of control system in terms of its Ricci curvature and fundamental form.

5. We apply our general theory of curves in the Lagrange Grassmannian to the equivalence problem of rank 2 vector distributions on n-dimensional manifold. We construct two invariants

of such distributions: the *fundamental form* and the *projective curvature*. For $n = 5$ we give an explicit method for computation of these invariants. We demonstrate this method on several examples, showing simultaneously the efficiency of our invariants in proving that the rank 2 distributions are not equivalent. In particular, we give an example of one-parametric family of nonequivalent rank 2 distributions. Finally, for $n = 5$ we construct the canonical frame of rank 2 distribution with nonzero fundamental form, which allows to solve the equivalence problem of such rank 2 distributions completely.

6. We use the notions of the generalized Ricci curvature and the fundamental form in the problem of geodesic (or projective) equivalence of control systems (two control systems are called geodesic (or projective) equivalent , if they have the same extremals, up to reparametrization). In particular, we present the new elementary proof of the classical Dini's theorem about the local geodesic equivalence of Riemannian metrics on a surface. We believe that the method of this proof can be used for obtaining the analogs of Dini's theorem for Riemannian metrics on the manifolds of dimension greater than 2 and also for sub-Riemannian metrics.

List of Symbols

TM	–	tangent bundle of the manifold M
T^*M	–	cotangent bundle of the manifold M
T_qM	–	tangent space to the manifold M at the point q
T_q^*M	–	cotangent space to the manifold M at the point q
ς	–	Liouville form on T^*M
σ	–	canonical symplectic form on T^*M, $\sigma = -d\varsigma$
$\vec{h}$	–	the Hamiltonian vector field, corresponding to the Hamiltonian $h : T^*M \mapsto \mathbb{R}$, $\sigma(\vec{h}, \cdot) = dh(\cdot)$
W	–	$2m$-dimensional linear symplectic space
$L(W)$	–	Lagrange Grassmannian of the symplectic space W, i.e., the set of all Lagrange subspaces of W
Λ^*	–	the conjugate space to the linear space Λ
$\Lambda^{\pitchfork}$	–	the set of all Lagrange subspaces transversal to Lagrange subspace Λ
$\Lambda^{\angle}$	–	the skewsymmetric complement of the subspace Λ
$Sym^2(\Lambda)$	–	the space of quadratic forms on Λ^*
$\langle\Lambda, \Gamma, \Delta\rangle$	–	the linear operator from Λ to Δ with the graph Γ
$\Lambda_\tau(t)$	–	the identical embedding of the curve $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$
$\Lambda^0(t)$	–	the derivative curve of the curve $\Lambda(t)$
$\Big[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\Big]$	–	the cross-ratio of four subspaces Λ_0, Λ_1, Λ_2, and Λ_3, $\Big[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\Big] = \langle\Lambda_2, \Lambda_3, \Lambda_0\rangle \circ \langle\Lambda_0, \Lambda_1, \Lambda_2\rangle$.
$\mathbb{S}(\varphi)$	–	Schwarz derivative or Schwarzian of φ, $\mathbb{S}(\varphi) = \frac{1}{2}\frac{\varphi^{(3)}}{\varphi'} - \frac{3}{4}\left(\frac{\varphi''}{\varphi'}\right)^2$
$\rho(t)$	–	the generalized Ricci curvature
$A(t)$	–	the density of the fundamental form
$R(t)$	–	the curvature operator
r	–	the rank of the curve
k	–	the weight of the curve
D^l	–	the l-th power of distribution D, $D^l = [D, D^{l-1}]$
$(D^l)^\perp$	–	annihilator of D^l in T^*M

1 Introduction

Suppose M is a smooth n-dimensional manifold and $\pi : T^*M \to M$ is the cotangent bundle to M. Let H be a codimension 1 submanifold in T^*M such that H is transversal to T_q^*M, $\forall q \in M$; then $H_q = H \cap T_q^*M$ is a smooth hypersurface in T_q^*M. Let ς be the canonical Liouville form on T^*M, $\varsigma_\lambda = \lambda \circ \pi_*$, $\lambda \in T^*M$, and $\sigma = -d\varsigma$ be the standard symplectic structure on T^*M; then $\sigma|_H$ is a corank 1 closed 2-form. The kernels of $(\sigma|_H)_\lambda$, $\lambda \in H$ are transversal to T_q^*M, $q \in M$; these kernels form a line distribution in H and define a *characteristic 1-foliation* $\mathcal{C}$ of H. Leaves of this foliation are *characteristic curves* of $\sigma|_H$.

Suppose γ is a segment of a characteristic curve and O_γ is a neighborhood of γ in H such that $N = O_\gamma/(\mathcal{C}|_{O_\gamma})$ is a well-defined smooth manifold. The quotient manifold N is in fact a symplectic manifold endowed with a symplectic structure $\bar{\sigma}$ induced by $\sigma|_H$. Let $\phi : O_\gamma \to N$ be the canonical factorization; then $\phi(H_q \cap O_\gamma)$, $q \in M$, are Lagrangian submanifolds in N. Let $L(T_\gamma N)$ be the Lagrangian Grassmannian of the symplectic space $T_\gamma N$, i.e. $L(T_\gamma N) = \{\Lambda \subset T_\gamma N : \Lambda^\angle = \Lambda\}$, where $D^\angle = \{e \in T_\gamma N : \bar{\sigma}(e, D) = 0\}$, $\forall D \subset T_\gamma N$. *Jacobi curve of the characteristic curve* γ is the mapping

$$\lambda \mapsto \phi_*(T_\lambda H_{\pi(\lambda)}), \quad \lambda \in \gamma, \tag{1.1}$$

from γ to $L(T_\gamma N)$.

Jacobi curves are curves in the Lagrange Grassmannians. They are invariants of the hypersurface H in the cotangent bundle. In particular, any differential invariant of the curves in the Lagrange Grassmannian by the action of the linear Symplectic Group (i.e., any symplectic invariant) produces a well-defined function on H.

To make things clear it is not worse to give a coordinate version of the construction of Jacobi curve. In the neighborhood O_γ choose coordinates $(x_0, x_1, \ldots, x_{2n-2})$ such that the characteristic curves of $\sigma|_H$ are the straight lines parallel to the x_0-axis (here we do not care about the fact that H come from the linear fiber bundle T^*M, we forget about the linear structure of the fibers). In these coordinates the sets $H_{\pi(\lambda)}$ are some $(n-1)$-dimensional submanifolds of $\mathbb{R}^{2n-1}$. For any $\lambda \in \gamma$ take projection (parallel to x_0-axis) of the spaces $T_\lambda H_{\pi(\lambda)}$ to the hyperplane $\{x_0 = c\}$ for some c. Then we obtain a curve of $(n-1)$-dimensional subspaces in the $(2n-2)$-dimensional linear space. The restriction of the form σ to $\{x_0 = c\}$ provides this space with symplectic structure and the obtained curve is a curve of Lagrangian subspaces w.r.t. this structure. This curve is exactly the Jacobi curve.

Set $W = T_\gamma N$ and note that the tangent space $T_\Lambda L(W)$ to the Lagrangian Grassmannian at the point Λ can be naturally identified with the space of quadratic forms on the linear space $\Lambda \subset W$. Namely, take a curve $\Lambda(t) \in L(W)$ with $\Lambda(0) = \Lambda$. Given some vector $l \in \Lambda$, take a curve $l(\cdot)$ in W such that $l(t) \in \Lambda(t)$ for all t and $l(0) = l$. Define the quadratic form $q_{\Lambda(\cdot)}(l) = \bar{\sigma}(\frac{d}{dt}l(0), l)$. Using the fact that the spaces $\Lambda(t)$ are Lagrangian, i.e. $\Lambda(t)^{\angle} = \Lambda(t)$, it is easy to see that the form $q_{\Lambda(\cdot)}(l)$ depends only on $\frac{d}{dt}\Lambda(0)$. So, we have the map from $T_\Lambda L(W)$ to the space of quadratic forms on Λ. A simple counting of dimension shows that this mapping is a bijection. Below we use the just described identification of tangent vectors to $L(W)$ with quadratic forms without a special mentioning.

Proposition 1 *Tangent vectors to the Jacobi curve J_γ at a point $J_\gamma(\lambda)$, $\lambda \in \gamma$, are equivalent (under linear substitutions of variables in the correspondent quadratic forms) to the "second fundamental form" of the hypersurface $H_{\pi(\lambda)} \subset T^*_{\pi(\lambda)}M$ at the point λ.*

Sketch of proof. In our local study we may assume without lack of generality that H is a regular level set of a smooth function h on T^*M. Then γ is a trajectory of the Hamiltonian vector field $\vec{h}$ defined by the identity $\vec{h}\rfloor\sigma = dh$. Let $t \mapsto \gamma(t)$ be a parametrization of γ defined by the Hamiltonian system $\frac{d}{dt}\gamma = \vec{h}(\gamma)$, $\gamma(0) = \lambda$. Given $l \in \phi_*(T_\lambda H_{\pi(\lambda)})$, take a vector field ℓ on H such that $\ell(\gamma(t)) \in T_{\gamma(t)}H_{\pi(\gamma(t))}$, $\phi_*\ell(\lambda) = l$. Simple calculations show that $\frac{d}{dt}\phi_*\ell(\gamma(t)) = \phi_*[\vec{h}, \ell](\gamma(t))$. Hence

$$\frac{d}{dt}J_\gamma|_{t=0}(l) = \bar{\sigma}(\frac{d}{dt}\phi_*\ell(\gamma(t))|_{t=0}, l) = \sigma([\vec{h}, \ell](\lambda), \ell(\lambda)). \tag{1.2}$$

Now we rewrite the last formula in coordinates. Let $q = (q^1, \ldots, q^n)$ be local coordinates in M and $p = (p_1, \ldots, p_n)$ be induced coordinates in the fiber of T^*M so that $\varsigma = \sum_{i=1}^n p_i dq^i$, $\sigma = \sum_{i=1}^n dq^i \wedge dp_i$. Then $\vec{h} = \sum_{i=1}^n \left(\frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i}\frac{\partial}{\partial p_i}\right)$, $\ell = \sum_{i=1}^n \ell^i \frac{\partial}{\partial p_i}$, $l = (\ell^1(\lambda), \ldots, \ell^n(\lambda))$. We have

$$\sigma([\vec{h}, \ell](\lambda), \ell(\lambda)) = l^* \frac{\partial^2 h}{\partial p^2} l. \tag{1.3}$$

Quadratic form $l \mapsto l^*\frac{\partial^2 h}{\partial p^2} l$ is exactly the "second fundamental form" of the hypersurface $H_{\pi(\lambda)} = h^{-1}(h(\lambda)) \cap T^*_{\pi(\lambda)}M$ in $T^*_{\pi(\lambda)}M$. $\square$

In particular, the velocity of J_γ at λ is a sign-definite quadratic form if and only if the hypersurface $H_{\pi(\lambda)}$ is strongly convex at λ.

A similar construction can be done for a submanifold of codimension 2 in T^*M. Namely, let H be a transversal to fibers codimension 2 submanifold in T^*M. In general, characteristic curves do not fill the whole submanifold H; they are concentrated in the characteristic variety consisting of the points, where $\sigma|_H$ is degenerated. In our local study we may always assume that H is orientable and let Ω be a volume form on H. Then $\bigwedge^{n-1} \sigma|_H = a\Omega$, where a is a smooth function on H. We set

$$C_H = \{\lambda \in H : a(\lambda) = 0,\ (d_\lambda a \bigwedge^{n-2} \sigma|_\lambda)|_H \neq 0\}. \tag{1.4}$$

Assume that $C_H \neq \emptyset$. Then C_H is a codimension 1 submanifold of H and $\sigma|_{C_H}$ is a 2-form of corank 1 on C_H. Indeed, $\forall \lambda \in C_H$, $\ker \sigma_\lambda|_H$ is a 2-dimensional subspace in $T_\lambda H$, which is transversal to $T_\lambda C_H$, and we have $\ker \sigma_\lambda|_{C_H} = \ker \sigma_\lambda|_H \cap T_\lambda C_H$.

The characteristic curves of $\sigma|_{C_H}$ form a 1-foliation $\mathcal{C}$ of C_H. Let γ be a segment of a characteristic curve and O_γ be a neighborhood of γ in C_H such that $N = O_\gamma/(\mathcal{C}|_{O_\gamma})$ is a well-defined smooth manifold. The quotient manifold N is a symplectic manifold endowed with a symplectic structure $\bar{\sigma}$ induced by $\sigma|_{C_H}$. Let $\phi : O_\gamma \to N$ be the canonical factorization. It is easy to check that $\phi_*\big((T_\lambda H_{\pi(\lambda)} + \ker \sigma_\lambda|_H) \cap T_\lambda C_H\big)$ is a Lagrangian subspace of the symplectic space $T_{\phi(\lambda)}N$, $\forall \lambda \in O_\gamma$. *Jacobi curve of the characteristic curve* γ is the mapping

$$\lambda \mapsto \phi_*\big((T_\lambda H_{\pi(\lambda)} + \ker \sigma_\lambda|_H) \cap T_\lambda C_H\big), \quad \lambda \in \gamma, \tag{1.5}$$

from γ to $L(T_\gamma N)$.

We are mainly interested in submanifolds that are dual objects to smooth control systems. Here we call a smooth control system any submanifold $V \subset TM$, transversal to fibers. Let $V_q = V \cap T_q M$; The "dual" *normal variety* H^1 and *abnormal variety* H^0 are defined as follows:

$$H^1 = \bigcup_{q \in M} \{\lambda \in T_q^* M : \exists v \in V_q,\ \langle \lambda, v \rangle = 1,\ \langle \lambda, T_v V_q \rangle = 0\}, \tag{1.6}$$

$$H^0 = \bigcup_{q \in M} \{\lambda \in T_q^* M \setminus 0 : \exists v \in V_q,\ \langle \lambda, v \rangle = \langle \lambda, T_v V_q \rangle = 0\}. \tag{1.7}$$

These varieties are not, in general, smooth manifolds; they may have singularities, which we do not discuss here. Anyway, one can obtain a lot of information on the original system just studying smooth parts of H^1, H^0.

One of the varieties H^1, H^0 can be empty. In particular, if $V_q = \partial W_q$, where W_q is a convex set and $0 \in \operatorname{int} W_q$, then $H^0 = \emptyset$. Moreover, in this case the Liouville form never vanishes on the tangent lines to the characteristic curves of $\sigma|_{H^1}$, and any characteristic curve γ has a canonical parametrization by the rule $\langle \varsigma, \dot{\gamma} \rangle = 1$. If subsets $V_q \subset T_q M$ are conical, $\alpha V_q = V_q$, $\forall \alpha > 0$, then, in contrast to the previous case, $H^1 = \emptyset$ and ς vanishes on the tangent lines to the characteristic curves of $\sigma|_{H^0}$. The characteristic curves are actually unparametrized.

Characteristic curves of $\sigma|_{H^1}$ ($\sigma|_{H^0}$) are associated with normal (abnormal) extremals of the control system V. In [1,2] Jacobi curves of extremals were defined in purely variational way in terms of the original control system and in a very general setting (singularities included), see also

[20] . The introduced here Jacobi curves of characteristic curves of $\sigma|_{H^1}$ ($\sigma|_{H^0}$) coincide with Jacobi curves of the correspondent extremals in the following important cases:

1. If H^1 has codimension 1 in T^*M. It occurs, for example, if subsets V_q are compact $\forall q \in M$;

2. If H^0 has codimension 1 in T^*M, but $H^1 = \emptyset$. It occurs, for example, if for any q subset V_q is conical but does not contain a 2-dimensional linear space;

3. If H^1 has codimension 2. It occurs, for example, if for any q subset V_q is an affine line in T_qM, not containing the origin;

4. H^0 has codimension 2. It occurs, for example , if V_q are 2-dimensional linear spaces, i.e., subsets V_q define rank 2 vector distribution on M, or if $V_q = D_q \cap \partial W_q$, where D_q is 2-dimensional linear space and W_q is a convex set such that $0 \in \operatorname{int} W_q$.

Jacobi curves associated with extremals of given control system are not arbitrary curves of Lagrangian Grassmannian but they inherit special features of the control system. The rank of the "second fundamental form" of the submanifolds H^1_q and H^0_q of T^*_qM at any point is no greater than $\dim V_q$. Indeed, let $\lambda \in H^1_q$; then $\lambda \in (T_vV_q)^\perp$, $\langle \lambda, v\rangle = 1$, for some $v \in V_q$. We have $\lambda + (T_vV_q + \mathbb{R}v)^\perp \subset H^1_q$. So λ belongs to an affine subspace of dimension $n - \dim V_q - 1$, which is contained in H^1_q. For $\lambda \in H^0_q$, $\exists v \in T_qM$ such that $\lambda \in (T_vV_q)^\perp$, $\langle \lambda, v\rangle = 0$. Then linear subspace $(T_vV_q + \mathbb{R}v)^\perp$ is contained in H^0_q. It follows that the second fundamental forms of H^1_q and H^0_q have rank no greater than $(\dim V_q - \operatorname{codim} H^1 + 1)$ and $(\dim V_q - \operatorname{codim} H^0 + 1)$ correspondently.

In the cases 1 and 2 the velocity of the Jacobi curve $\lambda \mapsto J_\gamma(\lambda)$, $\lambda \in \gamma$, associated with the extremal γ, has rank no greater than $\dim V_{\pi(\lambda)}$ (see Prop. 1). The same is true for the Jacobi curves of the extremals in the cases 3 and 4, although Prop. 1 cannot be directly applied.

Dimension of V_q is the number of inputs or control parameters in the control system. Less inputs means more "nonholonomic constraints" on the system. It happens that the rank of velocity of any Jacobi curve generated by the system never exceeds the number of inputs.

Note that by construction these Jacobi curves are feedback invariants of the control system (i.e., they do not depend on a parametrization of the sets V_q). Hence any symplectic invariants of the Jacobi curves, associated with extremals, defines a function on appropriate submanifold of T^*M that is a feedback invariant of the control system. In this way the problem of finding feedback invariants of control systems can be reduced to the much more treatable problem of finding symplectic invariants of certain curves in the Lagrange Grassmannian.

A curve in the Lagrange Grassmannian is called regular, if its velocity at any point is a non-degenerated quadratic forms. Regular curves were studied in [1], where notions of the derivative curve and the curvature operator were introduced. Actually the derivative curves of Jacobi curves, associated with the hypersurface H, provide a canonical connection on the cotangent bundle. If H is a spherical bundle of a Riemannian manifold, then this connection is just the Levi-Civita connection. The curvature operator of the Jacobi curve is intimately related to the curvature tensor of the canonical connection.

In the present work we develop general theory of curves in the Lagrangian Grassmannian (sections 2-9 and Appendix) and apply this theory to the qualitative study of some concrete classes of control systems and geometric structures (sections 10 and 11). The first steps in this direction were made in [5]. Sections 2-7 and Appendix were published as [6], sections 8 and 9 appears in [7].

The work is organized as follows. In section 2 we give the general construction of the derivative curve and introduce two principal discrete characteristic of the curves in the Lagrange Grassmannian: the rank and the weight. In particular, regular curves have maximal rank and minimal weight. Derivative curve is defined for any curve of the finite weight. In section 3 we define the curvature operator and show its role for the regular curves.

In section 4 we study the *cross-ratio* of four points and an *infinitesimal cross-ratio* of two tangent vectors at two distinct points in the Lagrange Grassmannian. The last one leads to an intrinsic pairing $V_0, V_1 \mapsto \langle V_0 \mid V_1 \rangle$, $V_i \in T_{\Lambda_i}L(W)$, $i = 0, 1$, of the tangent spaces to two distinct points Λ_0, Λ_1 of the Grassmannian. The pairing $\langle \dot\Lambda(t) \mid \dot\Lambda(\tau) \rangle$ of the velocities of the curve $t \mapsto \Lambda(t)$ gives a symmetric function of two variables which keeps all essential information about the curve. This function is defined out of the diagonal $\{t = \tau\}$ and has a very simple singularity at the diagonal:

$$\langle \dot\Lambda(t) \mid \dot\Lambda(\tau) \rangle = -\frac{k}{(t-\tau)^2} - g_\Lambda(t,\tau),$$

where k is the weight of the curve and $g_\Lambda(t,\tau)$ is a smooth function!

The first coming invariant of the parametrized curve, the generalized Ricci curvature, is just $g_\Lambda(t,t)$, the value of g_Λ at the diagonal. For regular curves and for rank 1 curves Ricci curvature is equal also to the trace of the defined earlier curvature operator.

In section 5 we are focused on unparametrized curves. Our investigation is based on a simple chain rule for a function g_Λ. Indeed, let $\varphi : \mathbb{R} \mapsto \mathbb{R}$ be a smooth monotonic function. It follows directly from definition of g_Λ that

$$g_{\Lambda\circ\varphi}(t,\tau) = \dot\varphi(t)\dot\varphi(\tau) g_\Lambda(\varphi(t),\varphi(\tau)) + k\left(\frac{\dot\varphi(t)\dot\varphi(\tau)}{(\varphi(t)-\varphi(\tau))^2} - \frac{1}{(t-\tau)^2}\right)$$

In particular,

$$g_{\Lambda\circ\varphi}(t,t) = \dot\varphi(t)^2 g_\Lambda(\varphi(t),\varphi(t)) + \frac{k}{3}\mathbb{S}(\varphi),$$

where $\mathbb{S}(\varphi)$ is a Schwarzian derivative of φ. The class of local parametrizations that kill the generalized Ricci curvature defines a canonical projective structure on the curve. The principal invariant of the unparametrized curve, the *fundamental form*, is a degree four differential on the curve; in the canonical projective parameter the fundamental form has the expression $\frac{1}{2}\frac{\partial^2 g_\Lambda}{\partial \tau^2}(t,t)(dt)^4$.

In section 6 we start a systematic study of the rank 1 curves and show that a rank 1 curve has a constant weight out of a discrete set of its interval of definition. In section 7 we prove that functions $\frac{\partial^{2i} g_\Lambda}{\partial \tau^{2i}}(t,t)$, $0 \le i \le m-1$ form a complete system of symplectic invariants of a rank 1 and a constant weight curve $\Lambda(\cdot)$ in the Lagrange Grassmannian $L(\mathbb{R}^{2m})$.

The Lagrange Grassmannian $L(\mathbb{R}^{2m})$ is a submanifold of the manifold $G(m, 2m)$ of all m-dimensional subspaces of $\mathbb{R}^{2m}$. The constructions of the derivative curve, the function g_Λ, the canonical projective structure, and the fundamental form can be done in the same way for general curves in $G(m, 2m)$.

In section 8 we introduce and investigate the "straightest" curves of the given constant rank in the Lagrange Grassmannian, called *flat curves*. Basic symplectic invariants of a curve measure its deviation from the flat curve and thus play the role of curvatures. The regular and the rank 1 flat curves are studied in greater detail. In the regular case the flat curves are characterized by the following property: their coordinate representations are "matrix Möbius transformations". In both regular and rank 1 cases we show that the curve Λ is flat iff the corresponding function g_Λ is equal to zero. This implies that the rank 1 curves are generalizations of Möbius transformtions in the following sense: it is well known that the transformation of the projective line (real or complex) is Möbius iff it preserves the cross-ratio of four points of the projective line; the rank 1 curve is flat iff it satisfies the following similar property

$$\det\Big[\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)\Big] \equiv \Big[t_0, t_1, t_2, t_3\Big]^k,$$

where $\Big[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\Big]$ is the cross-ratio of four Lagrange subspaces Λ_0, Λ_1, Λ_2, and Λ_3, $\Big[t_0, t_1, t_2, t_3\Big]$ is the usual cross-ratio of four numbers t_0, t_1, t_2,and t_3, k is the weight of Λ. We also prove that in both regular and rank 1 cases the curve is flat iff its derivative curve is constant.

In section 9 we give the estimates for the conjugate points of rank 1 curves in the Lagrangian Grassmannian of 4-dimensional symplectic space. The estimates are presented in the form of *comparison theorems*. Using these theorems one can estimate the intervals of weak optimality of extremals of control system in terms of its Ricci curvature and fundamental form.

Section 10 is devoted to the equivalence problem of rank 2 vector distributions on n-dimensional manifold (the problem makes sense for $n > 2$). For $n = 3$ or 4 all generic germs of rank 2 distributions are equivalent (Darboux's theorem in the case $n = 3$ and Engel's theorem in the case $n = 4$, see, for example, [9],[14],[22]). For $n = 5$ this equivalence problem was considered by E. Cartan in [10]. Using his reduction-prolongation method for equivalence problem of geometric structures on manifolds, he constructed the covariant fourth-order tensor invariant for such distributions. Our general theory of unparmeterized curves in the Lagrange Grassmannian, developed in section 5, gives completely different, more geometric way for construction of invariants of rank 2 vector distributions on manifold of arbitrary dimension $n \geq 5$. In this way we construct two invariants of such distributions: the *fundamental form* and the *projective curvature*. For $n = 5$ we give an explicit method for computation of these invariants (using the notion of canonical basis of rank 1 curves in the Lagrange Grassmannian, introduced in section 6). We demonstrate this method on several examples, showing simultaneously the efficiency of our invariants in proving that the rank 2 distributions are not equivalent. In particular, we give an example of one-parametric family of nonequivalent rank 2 distributions: the distributions from this family have the same fundamental form, but different projective curvatures. Finally, for $n = 5$ we construct the canonical frame (or e-structure) of rank 2 distribution with nonzero fundamental form. This allows to reduce the equivalence problem of such rank 2 distributions to the well studied problem of equivalence of e-structures.

In section 11 we use the notions of the generalized Ricci curvature and the fundamental form in the problem of geodesic (or projective) equivalence of control systems (two control systems are called geodesic (or projective) equivalent , if they have the same extremals, up to reparametrization). In particular, we present the new elementary proof of the classical Dini's theorem about the local geodesic equivalence of Riemannian metrics on surface. We believe that the method of this proof can be used for obtaining the analogs of Dini's theorem for Riemannian metrics on the manifolds of dimension greater than 2 and also for sub-Riemannian metrics.

2 Derivative Curve

From now on W will be the $2m$-dimensional linear space provided with the symplectic form $\bar{\sigma}$. Let Λ be a Lagrangian subspace of W, i.e. $\Lambda \in L(W)$. For any $w \in \Lambda$, the linear form $\bar{\sigma}(\cdot, w)$ vanishes on Λ and thus defines a linear form on W/Λ. The nondegeneracy of $\bar{\sigma}$ implies that the relation $w \mapsto \bar{\sigma}(\cdot, w)$, $w \in \Lambda$, induces a canonical isomorphism $\Lambda \cong (W/\Lambda)^*$ and, by the conjugation, $\Lambda^* \cong W/\Lambda$.

We set $\Lambda^{\pitchfork} = \{\Gamma \in L(W) : \Gamma \cap \Lambda = 0\}$, an open everywhere dense subset of $L(W)$. Let $Sym^2(\Lambda)$ be the space of self-adjoint linear mappings from Λ^* to Λ; this notation reflects the fact that $Sym^2(\Lambda)$ is the space of quadratic forms on Λ^* that is the symmetric square of Λ. $\Lambda^{\pitchfork}$

possesses a canonical structure of an affine space over the linear space $Sym^2(\Lambda) = Sym^2((W/\Lambda)^*)$. Indeed, for any $\Delta \in \Lambda^{\pitchfork}$ and coset $(w+\Lambda) \in W/\Lambda$, the intersection $\Delta \cap (w+\Lambda)$ of the linear subspace Δ and the affine subspace $w+\Lambda$ in W consists of exactly one point. To a pair $\Gamma, \Delta \in \Lambda^{\pitchfork}$ there corresponds a mapping $(\Gamma - \Delta) : W/\Lambda \to \Lambda$, where

$$(\Gamma - \Delta)(w+\Lambda) \stackrel{def}{=} \Gamma \cap (w+\Lambda) - \Delta \cap (w+\Lambda).$$

It is easy to check that the identification $W/\Lambda = \Lambda^*$ makes $(\Gamma - \Delta)$ a self-adjoint mapping from Λ^* to Λ. Moreover, given $\Delta \in \Lambda^{\pitchfork}$, the correspondence $\Gamma \mapsto (\Gamma - \Delta)$ is a one-to-one mapping of $\Lambda^{\pitchfork}$ onto $Sym^2(\Lambda)$ and the axioms of the affine space are obviously satisfied.

Fixing $\Delta \in \Lambda^{\pitchfork}$ one obtains a canonical identification $\Delta \cong W/\Lambda = \Lambda^*$. In particular, $(\Gamma - \Delta) \in Sym^2(\Lambda)$ turns into the mapping from Δ to Λ. For the last linear mapping we will use the notation $\langle \Delta, \Gamma, \Lambda \rangle : \Delta \to \Lambda$. In fact, this mapping has a much more straightforward description. Namely, the relations $W = \Delta \oplus \Lambda$, $\Gamma \cap \Lambda = 0$, imply that Γ is the graph of a linear mapping from Δ to Λ. Actually, it is the graph of the mapping $\langle \Delta, \Gamma, \Lambda \rangle$. In particular, $\ker \langle \Delta, \Gamma, \Lambda \rangle = \Delta \cap \Gamma$. If $\Delta \cap \Gamma = 0$, then $\langle \Lambda, \Gamma, \Delta \rangle = \langle \Delta, \Gamma, \Lambda \rangle^{-1}$.

Let us give coordinate representations of the introduced objects. We may assume that

$$W = \mathbb{R}^m \oplus \mathbb{R}^m = \{(x,y) : x, y \in \mathbb{R}^m\},$$

$$\bar{\sigma}((x_1,y_1),(x_2,y_2)) = \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle, \ \Lambda = \mathbb{R}^m \oplus 0, \ \Delta = 0 \oplus \mathbb{R}^m.$$

Then any $\Gamma \in \Delta^{\pitchfork}$ takes the form $\Gamma = \{(x, Sx) : x \in \mathbb{R}^m\}$, where S is a symmetric $m \times m$ matrix. The operator $\langle \Lambda, \Gamma, \Delta \rangle : \Lambda \to \Delta$ is represented by the matrix S, while the operator $\langle \Delta, \Gamma, \Lambda \rangle$ is represented by the matrix S^{-1}.

The coordinates in Λ induce the identification of $Sym^2\Lambda$ with the space of symmetric $m \times m$ matrices. $\Lambda^{\pitchfork}$ is an affine subspace over $Sym^2\Lambda$; we fix Δ as the origin in this affine subspace and thus obtain a coordinatization of $\Lambda^{\pitchfork}$ by symmetric $m \times m$ matrices. In particular, the "point" $\Gamma = \{(x, Sx) : x \in \mathbb{R}^n\}$ in $\Lambda^{\pitchfork}$ is represented by the matrix S^{-1}.

A subspace $\Gamma_0 = \{(x, S_0 x) : x \in \mathbb{R}^n\}$ is transversal to Γ if and only if $det(S - S_0) \neq 0$. Let us pick coordinates $\{x\}$ in Γ_0 and fix Δ as the origin in the affine space $\Gamma_0^{\pitchfork}$. In the induced coordinatization of $\Gamma_0^{\pitchfork}$ the "point" Γ is represented by the matrix $(S - S_0)^{-1}$.

Let $t \mapsto \Lambda(t)$ be a smooth curve in $L(W)$ defined on some interval $I \subset \mathbb{R}$. We say that the curve $\Lambda(\cdot)$ is *ample at* τ if $\exists s > 0$ such that for any representative $\Lambda^s_\tau(\cdot)$ of the s-jet of $\Lambda(\cdot)$ at τ, $\exists t$ such that $\Lambda^s_\tau(t) \cap \Lambda(\tau) = 0$. The curve $\Lambda(\cdot)$ is called *ample* if it is ample at any point.

We have given an intrinsic definition of an ample curve. In coordinates it takes the following form: the curve $t \mapsto \{(x, S_t x) : x \in \mathbb{R}^n\}$ is ample at τ if and only if the function $t \mapsto det(S_t - S_\tau)$ has a root of *finite order* at τ.

The following lemma shows that analytic monotonic curve (monotonic means that the curve has nonnegative or nonpositive velocities at any point) can be actually reduced to the ample curve by an appropriate factorization.

Lemma 2.1 *Let $\Lambda(t)$ be analytic monotonic curve in $L(W)$. Then for any parameter τ there exists a subspace K of $\Lambda(\tau)$ such that for all t the following holds*

$$K = \Lambda(t) \cap \Lambda(\tau). \tag{2.1}$$

In addition, if $\Lambda(t)$ is not a constant curve, then the curve $t \mapsto \Lambda(t)/K$ is a well defined ample curve in the Lagrange Grassmannian $L(K^{\angle}/K)$.

Proof. Without loss of generality suppose that the curve $\Lambda(t)$ is nondecreasing (i.e., has nonnegative definite velocity at any point). Denote $K_t = \Lambda(t) \cap \Lambda(\tau)$. Let $t \mapsto \{(x, S_t x) : x \in \mathbb{R}^n\}$ be coordinate representation of the germ of $\Lambda(t)$ at τ. Then $K_t = Ker(S_t - S_\tau)$. By assumption $v \to \langle \frac{d}{dt} S_t v, v\rangle$ is nonnegative definite quadratic form on $\Lambda(\tau)$. It implies that $K_{t_2} \subset K_{t_1}$ for $t < t_1 < t_2$. Therefore for $t > \tau$ sufficiently closed to τ the subspace K_t does not depend on t and will be denoted by K. By analyticity the subspaces $K \subset \Lambda(t)$ for any t and the curve $t \mapsto \Lambda(t)/K$ is well defined ample curve in the Lagrange Grassmannian $L(K^{\angle}/K)$. $\square$.

Assume that $\Lambda(\cdot)$ is ample at τ. Then $\Lambda(t) \in \Lambda(\tau)^{\pitchfork}$ for all t from a punctured neighborhood of τ. We obtain the curve $t \mapsto \Lambda(t) \in \Lambda(\tau)^{\pitchfork}$ in the affine space $\Lambda(\tau)^{\pitchfork}$ with the pole at τ. We denote by $\Lambda_\tau(t)$ the identical embedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$. The subscript τ is not superfluous, since the affine structure depends on $\Lambda(\tau)$ and then on τ. Fixing an "origin" in $\Lambda(\tau)^{\pitchfork}$ we make $\Lambda_\tau(t)$ a vector function with values in $Sym^2\big(\Lambda(\tau)\big)$ and with the pole at $t = \tau$. Such a vector function admits the expansion in the Laurent series at τ. Obviously, only free term in the Laurent expansion depends on the choice of the "origin" we did to identify the affine space with the linear one. More precisely, the addition of a vector to the "origin" results in the addition of the same vector to the free term in the Laurent expansion. In other words, for the Laurent expansion of a curve in an affine space, the free term of the expansion is a point of this affine space while all other terms are elements of the corresponding linear space. In particular,

$$\Lambda_\tau(t) \approx \Lambda^0(\tau) + \sum_{\substack{i=-l \\ i \neq 0}}^{\infty} Q_i(\tau)(t-\tau)^i, \tag{2.2}$$

where $\Lambda^0(\tau) \in \Lambda(\tau)^{\pitchfork}$, $Q_i(\tau) \in Sym^2\Lambda(\tau)$.

Assume that the curve $\Lambda(\cdot)$ is ample. Then $\Lambda^0(\tau) \in \Lambda(\tau)^{\pitchfork}$ is defined for all τ. The curve $\tau \mapsto \Lambda^0(\tau)$ is called the *derivative curve* of $\Lambda(\cdot)$.

Another characterization of $\Lambda^0(\tau)$ can be done in terms of the curves $t \mapsto \langle \Delta, \Lambda(t), \Lambda(\tau) \rangle$ in the linear space $Hom(\Delta, \Lambda(\tau))$, $\Delta \in \Lambda(\tau)^{\pitchfork}$. These curves have poles at τ. The Laurent expansion at $t = \tau$ of the vector function $t \mapsto \langle \Delta, \Lambda(t), \Lambda(\tau) \rangle$ has zero free term if and only if $\Delta = \Lambda^0(\tau)$.

The coordinate version of the series (2.2) is the Laurent expansion of the matrix-valued function $t \mapsto (S_t - S_\tau)^{-1}$ at $t = \tau$, where $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$.

Suppose that

$$(S_t - S_\tau)^{-1} \approx \sum_{i=-l}^{\infty} A_i(\tau)(t-\tau)^i, \tag{2.3}$$

Differentiating both sides of (2.3) w.r.t τ and comparing coefficients of the corresponding expansions one can get the following recursive type formula for the coefficients $A_i(\tau)$

$$\frac{d}{d\tau} A_i(\tau) = (i+1)A_{i+1}(\tau) + \sum_{j=-l}^{i+l} A_j(\tau)\dot{S}_\tau A_{i-j}(\tau) \tag{2.4}$$

that will be used in the sequel.

For monotonic ample curve $\Lambda : I \subset \mathbb{R} \mapsto L(W)$ we introduce the following two notions

Definition 1 *The rank of the velocity $\dot{\Lambda}(\tau)$ will be called a rank of the curve $\Lambda(\cdot)$ at τ. The order of zero of the function $t \mapsto det(S_t - S_\tau)$ at τ, where S_t is a coordinate representation of $\Lambda(\cdot)$, is called a weight of $\Lambda(\cdot)$ at τ.*

It is easy to see that the rank and the weight of $\Lambda(\tau)$ are integral valued upper semicontinuous functions of τ. In particular, they are locally constant on the open dense subset of the interval of definition I. In the sequel we will be mostly concentrated on the monotonic ample curves of the constant rank and weight.

3 Curvature operator and regular curves

Using derivative curve one can construct an operator invariant of the curve $\Lambda(t)$ at any its point. Namely, take velocities $\dot{\Lambda}(t)$ and $\dot{\Lambda}^0(t)$ of $\Lambda(t)$ and its derivative curve $\Lambda^0(t)$. Note that $\dot{\Lambda}(t)$ is linear operator from $\Lambda(t)$ to $\Lambda(t)^*$ and $\dot{\Lambda}^0(t)$ is linear operator from $\Lambda^0(t)$ to $\Lambda^0(t)^*$. Since the form σ defines the canonical isomorphism between $\Lambda^0(t)$ and $\Lambda(t)^*$, the following operator $R(t) : \Lambda(t) \to \Lambda(t)$ can be defined:

$$R(t) = -\dot{\Lambda}^0(t) \circ \dot{\Lambda}(t) \tag{3.1}$$

This operator is called *curvature operator* of Λ at t.

Remark 1 *In the case of Riemannian geometry the operator $R(t)$ is similar to the so-called Ricci operator $v \to R^{\nabla}(\dot\gamma(t), v)\dot\gamma(t)$, which appears in the classical Jacobi equation $\nabla_{\dot\gamma(t)}\nabla_{\dot\gamma(t)}V + R^{\nabla}(\dot\gamma(t), V)\dot\gamma(t) = 0$ for Jacobi vector fields V along the geodesic $\gamma(t)$ (here R^{∇} is curvature tensor of Levi-Civita connection ∇), see [1]. This is the reason for the sign "$-$" in (3.1).*

The curvature operator plays an important role for so-called regular curves. The curve $\Lambda(t)$ in Lagrangian Grassmannian is called *regular*, if the quadratic form $\dot\Lambda(t)$ is nondegenerated for all t. If the curve $\Lambda(\cdot)$ is regular and has a coordinate representation $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$ then the function $t \mapsto (S_t - S_\tau)^{-1}$ has a simple pole at τ. Indeed,

$$(S_t - S_\tau)^{-1} = \Big(\dot S_\tau(t-\tau) + O((t-\tau)^2)\Big)^{-1} = \frac{\dot S_\tau^{-1}}{t-\tau}\Big(I + O(t-\tau)\Big)^{-1} = \frac{\dot S_\tau^{-1}}{t-\tau} + O(1) \qquad (3.2)$$

So, in the notation of (2.3) for the regular curve we have $l = 1$, $A_{-1} = \dot S_\tau^{-1}$ and relation (2.4) can be transformed into the following recursive formula

$$A_{i+1}(\tau) = \frac{1}{i+3}\left(\frac{d}{d\tau}A_i(\tau) - \sum_{j=0}^{i} A_j(\tau)\dot S_\tau A_{i-j}(\tau)\right) \qquad (3.3)$$

In particular,

$$A_0(\tau) = \frac{1}{2}\frac{d}{d\tau}A_{-1}(\tau) = -\frac{1}{2}\dot S_\tau^{-1}\ddot S_\tau \dot S_\tau^{-1} \qquad (3.4)$$

and, by direct calculations,

$$A_1(\tau) = \frac{1}{3}\Big(\frac{d}{d\tau}A_0(\tau) - A_0(\tau)\dot S_\tau A_0(\tau)\Big) = \left(\frac{1}{4}\left(\dot S_\tau^{-1}\ddot S_\tau\right)^2 - \frac{1}{6}\dot S_\tau^{-1} S_\tau^{(3)}\right)\dot S_\tau^{-1} \qquad (3.5)$$

For a given τ one can choose a coordinate representation S_t of the curve $\Lambda(t)$ such that $A_0(\tau) = 0$. Namely take S_t the matrix of the linear mapping $< \Lambda(\tau), \Lambda(t), \Lambda^0(\tau) >$. In this coordinate representation the derivative $\dot A_0(\tau)$ is a matrix corresponding to the velocity $\dot\Lambda^0(\tau)$ of the derivative curve. Also, from (3.5) it follows that $\dot A_0(\tau) = 3A_1(\tau)$. This together with (3.1) implies that the matrix $\mathcal{R}(\tau)$ corresponding in the chosen basis of $\Lambda(\tau)$ to the curvature operator $R(\tau)$ has the following form

$$\mathcal{R}(\tau) = -3A_1(\tau)(A_{-1}(\tau))^{-1} = \frac{1}{2}\dot S_\tau^{-1} S_\tau^{(3)} - \frac{3}{4}\left(\dot S_\tau^{-1}\ddot S_\tau\right)^2 =$$

$$\frac{d}{d\tau}\left((2\dot S_\tau)^{-1}\ddot S_\tau\right) - \left((2\dot S_\tau)^{-1}\ddot S_\tau\right)^2 \qquad (3.6)$$

Since $Q_1(\tau) \circ (Q_{-1}(\tau))^{-1} : \Lambda(\tau) \mapsto \Lambda(\tau)$ is well defined operator, we can write the first equality of (3.6) in the following operator form

$$R(\tau) = -3Q_1(\tau) \circ (Q_{-1}(\tau))^{-1} \qquad (3.7)$$

This implies actually that the formula (3.6) is also true for any coordinate representation S_t of the curve $\Lambda(t)$ (even without the assumption that $A_0(\tau) = 0$).

Note that the right-hand side of (3.6) is a matrix analog of so-called Schwarz derivative or Schwarzian. Let us recall that the differential operator:

$$\mathbb{S} : \varphi \mapsto \frac{1}{2}\frac{\varphi^{(3)}}{\varphi'} - \frac{3}{4}\Big(\frac{\varphi''}{\varphi'}\Big)^2 = \frac{d}{dt}\Big(\frac{\varphi''}{2\,\varphi'}\Big) - \Big(\frac{\varphi''}{2\,\varphi'}\Big)^2, \tag{3.8}$$

acting on scalar function φ is called *Schwarzian.* The operator $\mathbb{S}$ is characterized by the following remarkable property: General solution of the equation $\mathbb{S}\varphi = \rho$ w.r.t φ is a Möbius transformation (with constant coefficients) of some particular solution of this equation. The matrix analog of this operator has similar property, concerning "matrix Möbius transformation" of the type $S \mapsto (C + DS)(A + BS)^{-1}$. In particular, if $R(t) \equiv 0$, then the coordinate representation S_t of our curve has the form

$$S_t = (C + Dt)(A + Bt)^{-1},$$

where A, B, C, and D are some $m \times m$ matrices (the regular curves with $R(t) \equiv 0$ will be considered in more details in secion 8). For further information about the regular curves we refer to [1].

4 Expansion of the cross-ratio and Ricci curvature

For the nonregular curve $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$, the function $t \mapsto (S_t - S_\tau)^{-1}$ has a pole of order greater than 1 at τ and it is much more difficult to compute its Laurent expansion. For example, in the nonregular case there is no direct recursive formula like (3.3). In this section we show how to construct numerical invariants for curves with constant weight using the notion of cross-ratio of four "points" in the Lagrange Grassmannian.

Let Λ_0, Λ_1, Λ_2, and Λ_3 be Lagrangian subspaces of W such that $\Lambda_0 \cap \Lambda_3 = \Lambda_1 \cap \Lambda_2 = 0$. Also suppose for simplicity that $\Lambda_0 \cap \Lambda_2 = 0$. The following linear mappings $\langle \Lambda_0, \Lambda_1, \Lambda_2 \rangle : \Lambda_0 \to \Lambda_2$, $\langle \Lambda_2, \Lambda_3, \Lambda_0 \rangle : \Lambda_2 \to \Lambda_0$ are well defined. The cross-ratio $\Big[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\Big]$ of four "points" Λ_0, Λ_1, Λ_2, and Λ_3 in the Lagrangian Grassmannian is, by definition, the following linear operator in Λ_0:

$$\Big[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\Big] = \langle \Lambda_2, \Lambda_3, \Lambda_0 \rangle \circ \langle \Lambda_0, \Lambda_1, \Lambda_2 \rangle. \tag{4.1}$$

This notion is a "matrix" analog of the classical cross-ratio of four points in the projective line. Indeed, let $\Lambda_i = \{(x, S_i x) : x \in \mathbb{R}^n\}$, then, in coordinates $\{x\}$, the cross-ratio takes the form:

$$\Big[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\Big] = (S_0 - S_3)^{-1}(S_3 - S_2)(S_2 - S_1)^{-1}(S_1 - S_0) \tag{4.2}$$

By construction, all coefficients of the characteristic polynomial of $\Big[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\Big]$ are invariants of four subspaces $\Lambda_0, \Lambda_1, \Lambda_2$, and Λ_3.

The assumption that $\Lambda_0 \cap \Lambda_2 = 0$ is satisfied in our further considerations but the cross-ratio can be defined also without this assumption. Indeed, the matrix in the righthand side of (4.2) is well defined also in the case $\Lambda_0 \cap \Lambda_2 \neq 0$ and this matrix is transformed to a similar matrix under any change of coordinates. So, we obtain the class of similar matrices that is symplectic invariant of four subspaces $\Lambda_0, \Lambda_1, \Lambda_2$, and Λ_3. This class can be taken as a definition of cross-ratio $\Big[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\Big]$ (see [21] for the details).

Given two tangent vectors $V_0 \in T_{\Lambda_0}L(W)$ and $V_1 \in T_{\Lambda_1}L(W)$, where Λ_0 and Λ_1 are transversal Lagrangian subspaces, one can define an infinitesimal analog of the cross-ratio. V_0 is the self-adjoint linear mapping from Λ_0 to Λ_0^*. The form σ identifies canonically Λ_0^* with Λ_1. Under this identification V_0 can be considered as the linear mapping from Λ_0 to Λ_1. In the same way, identifying Λ_1^* with Λ_0, we look on V_1 as on the operator from Λ_1 to Λ_0. Therefore, the following operator $V_1 \odot V_0 : \Lambda_0 \to \Lambda_0$ can be defined

$$V_1 \odot V_0 \stackrel{def}{=} V_1 \circ V_0 \tag{4.3}$$

This operator will be called an *infinitesimal cross-ratio* of a pair $(V_0, V_1) \in T_{\Lambda_0}L(W) \times T_{\Lambda_1}L(W)$. The infinitesimal cross-ratio is symplectic invariant of the tangent vectors V_0 and V_1.

One can define the following bilinear form $\langle \cdot \mid \cdot \rangle_{\Lambda_0,\Lambda_1}$ on $T_{\Lambda_0}L(W) \times T_{\Lambda_1}L(W)$:

$$\langle V_0 \mid V_1 \rangle_{\Lambda_0,\Lambda_1} \stackrel{def}{=} \operatorname{tr}(V_0 \odot V_1) \tag{4.4}$$

This bilinear form will be called an *inner pairing* of the tangent spaces $T_{\Lambda_0}L(W)$ and $T_{\Lambda_1}L(W)$.

If $\Lambda_i = \{(x, S_i x) : x \in \mathbb{R}^n\}$ and P_i are symmetric matrices corresponding to V_i, $i = 0, 1$, then

$$V_1 \odot V_0 = (S_0 - S_1)^{-1} P_1 (S_1 - S_0)^{-1} P_0 \tag{4.5}$$

First note that if the curve $\Lambda(t)$ is regular, then for any t_0 it is easy to expand the following operator function

$$(t_1, t_2, t_3) \mapsto \frac{\Big[\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)\Big]}{\Big[t_0, t_1, t_2, t_3\Big]} \tag{4.6}$$

in the Taylor expansion at the diagonal point (t_0, t_0, t_0), where $\Big[t_0, t_1, t_2, t_3\Big] = \frac{(t_1 - t_0)(t_3 - t_2)}{(t_2 - t_1)(t_0 - t_3)}$ is the usual cross-ratio of four numbers t_0, t_1, t_2, and t_3. Namely, the following expansion

$$\frac{\Big[\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)\Big]}{\Big[t_0, t_1, t_2, t_3\Big]} = I + \frac{1}{3} R(t_0)(t_2 - t_0)(t_3 - t_1) + O\left(\left(\sum_{i=1}^{3} (t_i - t_0)^2\right)^{3/2}\right) \tag{4.7}$$

is valid, where, as before, $R(t)$ is the curvature operator. Relation (4.7) shows that the curvature operator is the first nontrivial coefficient of the Taylor expansion of the cross-ratio.

Unfortunately, for the nonregular curves there are no simple expansions of the operator function (4.6) or any other operator functions, involving cross-ratio itself. Instead of this one can try to expand the coefficients of the characteristic polynomial of the cross-ratio. Now we are going to show how to use this idea to construct invariants of the curve $\Lambda(t)$ of the constant weight k in $L(W)$.

By the above the function $(t_0, t_1, t_2, t_3) \to \det\Big[\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)\Big]$ is symplectic invariants of the curve $\Lambda(t)$. Using this fact, let us try to find symplectic invariants of $\Lambda(t)$ that are functions of t. First we introduce the following function:

$$\mathcal{G}(t_0, t_1, t_2, t_3) = \ln\left(\frac{det\Big[\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)\Big]}{\Big[t_0, t_1, t_2, t_3\Big]^k}\right), \tag{4.8}$$

The function $\mathcal{G}(t_0, t_1, t_2, t_3)$ is also a symplectic invariant of $\Lambda(t)$ and in addition it can be defined as a smooth function in a neighborhood of any diagonal point (t, t, t, t) . Indeed, by the definition of weight

$$det(S_{t_0} - S_{t_1}) = (t_0 - t_1)^k X(t_0, t_1), \tag{4.9}$$

where

$$X(t, t) \neq 0 \tag{4.10}$$

for any t. The function $X(t_0, t_1)$ is symmetric, since by changing the order in (4.9) we obtain that X can be symmetric or antisymmetric, but the last case is impossible by (4.10).

Let us define another symmetric function

$$f(t_0, t_1) = lnX(t_0, t_1) \tag{4.11}$$

The function $f(t_0, t_1)$ is smooth in a neighborhood of any diagonal point (t, t) and by (4.2), (4.8)

$$\mathcal{G}(t_0, t_1, t_2, t_3) = f(t_1, t_0) - f(t_2, t_1) + f(t_3, t_2) - f(t_0, t_3) \tag{4.12}$$

Hence $\mathcal{G}(t_0, t_1, t_2, t_3)$ can be defined as a smooth function in a neighborhood of any diagonal point (t, t, t, t). Using this fact one can construct the following functions of two variables that are symplectic invariants of the curve $\Lambda(t)$

$$h(t_0, t_1) = \mathcal{G}(t_0, t_1, t_1, t_0) = 2f(t_0, t_1) - f(t_0, t_0) - f(t_1, t_1) \tag{4.13}$$

$$g(t_0,t_1) = \frac{1}{2}\frac{\partial^2}{\partial t_0 \partial t_1} h(t_0,t_1) = \frac{\partial^2}{\partial t_0 \partial t_1} f(t_0,t_1) \qquad (4.14)$$

On the contrary the function $f(t_0,t_1)$ depends on the choice of the coordinate representation S_t.

It follows from (4.13) that $h(t_0,t_0) \equiv 0$ and $\frac{\partial}{\partial t_0} h(t_0,t_0) \equiv 0$. Therefore the function $h(t_0,t_1)$ can be recovered from $g(t_0,t_1)$. Moreover, the function $\mathcal{G}(t_0,t_1,t_2,t_3)$ can be easily recovered from $h(t_0,t_1)$ (and therefore from $g(t_0,t_1)$). Namely, by (4.12) and (4.14)

$$\mathcal{G}(t_0,t_1,t_2,t_3) = \frac{1}{2}\left(h(t_1,t_0) - h(t_2,t_1) + h(t_3,t_2) - h(t_0,t_3)\right) \qquad (4.15)$$

So, g or h keep all the information on $\mathcal{G}$ and thus on $\det\Big[\Lambda(t_0),\Lambda(t_1),\Lambda(t_2),\Lambda(t_3)\Big]$.

Remark 2 *In particular,* $g(t_0,t_1) \equiv 0$ *iff* $G(t_0,t_2,t_1,t_3) \equiv 0$*, which, by (4.8), is equivalent to the following identity*

$$\det\Big[\Lambda(t_0),\Lambda(t_1),\Lambda(t_2),\Lambda(t_3)\Big] \equiv \Big[t_0,t_1,t_2,t_3\Big]^k$$

The function $g(t_0,t_1)$ can be expanded in the formal Taylor series at the point (t,t) in the following way:

$$g(t_0,t_1) \approx \sum_{i,j=0}^{\infty} \beta_{i,j}(t)(t_0-t)^i(t_1-t)^j \qquad (4.16)$$

with

$$\beta_{i,j}(t) = \beta_{j,i}(t) \qquad (4.17)$$

Since the function g is symplectic invariant of the curve $\Lambda(t)$, all coefficients $\beta_{i,j}(t)$, $i,j \geq 0$, are also symplectic invariants.

The following natural questions arises: Does the function $g(t_0,t_1)$ determine the curve $\Lambda(t)$ with prescribed rank and weight uniquely, up to a symplectic transformation, and which set of the coefficients $\beta_{i,j}(t)$ determine the function $g(t_0,t_1)$? We shall give the positive answers on both of these questions in the section 7 for the curve of rank 1 (see Theorems 1 and 2).

Meanwhile, let us prove the following simple relation between coefficients $\beta_{i,j}(t)$

$$\beta'_{i,j}(t) = (i+1)\beta_{i+1,j} + (j+1)\beta_{i,j+1} \qquad (4.18)$$

Indeed, from (4.16) it follows that

$$\beta_{i,j}(t) = \frac{1}{i!\,j!}\frac{\partial^{i+j} g}{\partial t_0^i \partial t_1^j}(t,t).$$

Therefore

$$\beta'_{i,j}(t) = \frac{1}{i!\,j!}\Big(\frac{\partial^{i+j+1}}{\partial t_0^{i+1}\partial t_1^j} g(t,t) + \frac{\partial^{i+j+1}}{\partial t_0^{i}\partial t_1^{j+1}} g(t,t)\Big) = \frac{1}{i!\,j!}\Big((i+1)!\,j!\,\beta_{i+1,j}(t) + i!\,(j+1)!\,\beta_{i,j+1}(t)\Big)$$

that implies (4.18).

As a corollary of the relation (4.18) we obtain the following lemma

Lemma 4.1 *The coefficients $\beta_{0,2k}(t)$, $k \geq 0$ determine uniquely the formal expansion (4.16).*

Proof. For a given $n \geq 0$ let us consider all equations of the type (4.18) with $i+j=n$ and $i \leq j$. Consider two cases

1) If n is even then we have $\frac{n}{2}+1$ independent equations with respect to $\frac{n}{2}+1$ variables $\beta_{i,j}(t)$, $i+j=n+1$, $0 \leq i < \frac{n}{2}$. This fact together with symmetric relation (4.17) implies that all $\beta_{i,j}(t)$ with $i+j=n+1$ can be expressed by derivatives of $\beta_{i,j}$ with $i+j=n$.

2) If n is odd then we have $\frac{n+1}{2}$ independent equations with respect to $\frac{n+1}{2}+1$ variables $\beta_{i,j}$, $i+j=n+1$, $0 \leq i < \frac{n+1}{2}$. Starting from $i=0$ one can express step by step all $\beta_{i,j}$, $i+j=n+1$, $1 \leq i < \frac{n+1}{2}$ by $\beta_{0,n+1}$ and derivatives of $\beta_{i,j}$ with $i+j=n$. Then by symmetric relation (4.17) we have that all coefficients $\beta_{i,j}(t)$ with $i+j=n+1$ can be expressed by $\beta_{0,n+1}$ and derivatives of $\beta_{i,j}$ with $i+j=n$.

So, starting from $n=0$ and applying step by step the arguments of 1) and 2), one can expressed all $\beta_{i,j}(t)$ by $\beta_{0,2k}(t)$, $k \geq 0$, and their derivatives. $\square$

It turns out that there is simple connection between function g, the inner pairing defined by (4.4), and the coefficients Q_i of the Laurent expansion (2.2).

Lemma 4.2 *The following relations hold*

$$\langle \dot{\Lambda}(t) \mid \dot{\Lambda}(\tau) \rangle_{\Lambda(t),\Lambda(\tau)} = -\frac{k}{(t-\tau)^2} - g(t,\tau) \tag{4.19}$$

$$\operatorname{tr}\left(Q_i(t)\dot{\Lambda}(t)\right) = 0, \; i < -1, \tag{4.20}$$

$$\operatorname{tr}\left(Q_{-1}(t)\dot{\Lambda}(t)\right) = k, \tag{4.21}$$

$$\operatorname{tr}\left(Q_i(t)\dot{\Lambda}(t)\right) = -\frac{1}{i}\beta_{0,i-1}(t), \; i \in \mathbb{N} \tag{4.22}$$

Proof. Let $\Lambda_\tau(t)$ be the identical imbedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$ (see section 2). Then the inner pairing $\langle \dot{\Lambda}(t) \mid \dot{\Lambda}(\tau) \rangle_{\Lambda(t),\Lambda(\tau)}$ can be expressed in the following way

$$\langle \dot{\Lambda}(t) \mid \dot{\Lambda}(\tau) \rangle_{\Lambda(t),\Lambda(\tau)} = \operatorname{tr}\left(\frac{\partial}{\partial t}\Lambda_\tau(t) \circ \dot{\Lambda}(\tau)\right) \tag{4.23}$$

In the coordinates the previous relation can be written as follows

$$\langle \dot{\Lambda}(t) \mid \dot{\Lambda}(\tau) \rangle_{\Lambda(t),\Lambda(\tau)} = \operatorname{tr}\left(\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right)\; \dot{S}_\tau\right) \tag{4.24}$$

Let us prove (4.19). By definition

$$\ln(\det(S_t - S_\tau)) = k\ln(t-\tau) + f(t,\tau)$$

Differentiating the last equality w.r.t. τ and using the fact that $\frac{d}{d\tau}\Big(\ln(\det Y(\tau))\Big) = \mathrm{tr}\Big((Y(\tau))^{-1}\dot{Y}(\tau)\Big)$ for some matrix curve $Y(\tau)$ we obtain:

$$-\mathrm{tr}\Big((S_t - S_\tau)^{-1}\dot{S}_\tau\Big) = -\frac{k}{t-\tau} + \frac{\partial}{\partial\tau}f(t,\tau)$$

Differentiating the last equality w.r.t. t and using (4.14) we get

$$-\mathrm{tr}\Big(\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right)\dot{S}_\tau\Big) = \frac{k}{(t-\tau)^2} + \frac{\partial^2}{\partial t\partial\tau}f(t,\tau) = \frac{k}{(t-\tau)^2} + g(t,\tau)$$

This together with (4.24) implies (4.19).

In order to prove (4.20)-(4.22) let us expand both sides of (4.19) in the correspponding formal series. On one hand by (2.2) we have

$$\mathrm{tr}\Big(\frac{\partial}{\partial t}\Lambda_\tau(t)\circ\dot{\Lambda}(\tau)\Big) \approx \sum_{i=-l-1}^{\infty}(i+1)\mathrm{tr}\Big(Q_{i+1}(\tau)\dot{\Lambda}(\tau)\Big)(t-\tau)^i \tag{4.25}$$

On the other hand by (4.16)

$$g(t,\tau) \approx \sum_{0}^{\infty}\beta_{0,i}(\tau)(t-\tau)^i \tag{4.26}$$

Comparing coefficients of (4.25) and (4.26) we get (4.20)-(4.22). $\square$

For the regular curve using (3.6) and applying formula (4.22) to the first appearing in (4.16) coefficient $\beta_{0,0}(t)$ we obtain

$$\beta_{0,0}(t) = \frac{1}{3}\,\mathrm{tr}\,R(t) = \frac{1}{3}\,\mathrm{tr}\,\mathbb{S}(S_t), \tag{4.27}$$

where $\mathbb{S}$ denotes Schwarz operator. The last relation and Remark 1 shows that $\beta_{0,0}$ generalizes the Ricci curvature in the Riemannian geometry. It justifies the following definition for the general curve of constant rank and weight

Definition 2 *The first appearing in (4.16) coefficient $\beta_{0,0}(t)$ is called Ricci curvature of $\Lambda(t)$.*

In the sequel the Ricci curvature will be denoted by $\rho(t)$.

At the end of this section we compute the expansion of $g(t_0, t_1)$ in the case $\dim W = 2$. In this case $L(W)$ is in fact the real projective line $\mathbb{RP}^1$ and coordinate representation S_t of the curve is scalar function . Therefore the relation (4.22) can be rewritten in the following form

$$\beta_{0,i} = -(i+1)A_{i+1}(t)\dot{S}_t,$$

where A_i are as in (2.3). In particular from (4.27) it follows that

$$\rho(t) = \frac{1}{3}\mathbb{S}(S_t), \tag{4.28}$$

i.e., in the scalar case the Ricci curvature of the curve $\Lambda(t)$ is Schwarzian of its coordinate representation.

Denote by

$$B_i(\tau) = -\frac{1}{i}\beta_{0.i-1} = A(\tau)\dot{S}_\tau \tag{4.29}$$

Multiplying both sides of (3.3) by $\dot{S}_\tau$ and using commutativity of multiplication in the scalar case, one can easily obtain the following recursive formula for $B_i(\tau)$.

$$B_{i+1}(\tau) = \frac{1}{i+3}\left(\frac{d}{d\tau}B_i(\tau) - \sum_{j=1}^{i-1} B_j(\tau)B_{i-j}(\tau)\right), \ i \in \mathbb{N} \tag{4.30}$$

As a consequence of Lemma 4.1 and formulas (4.28)– (4.30) one can obtain the following

Proposition 2 *In the scalar case (i.e., dim$W = 2$) all coefficients $\beta_{i,j}(t)$ can be expressed by Ricci curvature (that is Schwarzian of any coordinate representation of the curve $\Lambda(t)$) and its derivatives. The function $g(t,\tau)$ is identically equal to zero iff coordinate representations of the curve $\Lambda(t)$ are Möbius transformations.*

5 Fundamental form of the unparametrized curve

The Jacobi curve constructed in the Introduction is actually unparametrized curve, i.e., one-dimensional submanifolds in Lagrange Grassmannian. Therefore it is natural to find symplectic invariants of unparametrized curves in $L(W)$. Especially it is important for Jacobi curves of abnormal extremals which (in opposite to the normal extremals) a priori have no special parametrizations.

First of all we want to show how, using the Ricci curvature, one can define a canonical projective structure on the unparametrized curve $\Lambda(\cdot)$. For this let us check how the Ricci curvature is transformed by a reparametrization of the curve $\Lambda(t)$.

Let $\tau=\varphi(t)$ be a reparametrization and let $\bar\Lambda(\tau)=\Lambda(\varphi^{-1}(\tau))$. For some coordinate representation S_t of $\Lambda(t)$ let $\bar S_\tau=S_{\varphi^{-1}(\tau)}$ be the coordinate representation of $\bar\Lambda(\tau)$. Denote by $\bar f$ the function playing for $\bar S_\tau$ the same role as the function f defined by (4.11) plays for S_t. Then from (4.11) it follows that

$$\bar f(\tau_0,\tau_1)=f(t_0,t_1)-k\,\ln\left(\frac{\varphi(t_0)-\varphi(t_1)}{t_0-t_1}\right), \tag{5.1}$$

where $\tau_i=\varphi(t_i)$, $i=0,1$.

Now denote by $\bar g$, $\bar\beta_{i,j}$ functions playing for $\bar\Lambda(\tau)$ the same role as the functions g,$\beta_{i,j}$ defined by (4.14) and (4.16) play for $\Lambda(t)$.

Note also that we can look on the function $\varphi(t)$ as on the coordinate representation of some curve in $\mathbb{RP}^1=L(W)$ with $\dim W=2$. So, all constructions and formulas of the previous section can be applied to this case. We denote by $g_\varphi(t_0,t_1)$ the function defined by (4.11), (4.14) with S_t replaced by $\varphi(t)$. Then differentiating both sides of (5.1) once w.r.t. t_0 and twice t_1 we get

$$\bar g\left(\varphi(t_0),\varphi(t_1)\right)\varphi'(t_0)\varphi'(t_1)=g(t_0,t_1)-kg_\varphi(t_0,t_1), \tag{5.2}$$

By (4.16) and (4.28) it follows that the substitution $t_0=t_1=t$ into (5.2) give us the following reparametrization rule for Ricci curvature

$$\bar\rho(\tau)(\varphi'(t))^2=\rho(t)-\frac{k}{3}\,\mathbb{S}(\varphi(t)) \tag{5.3}$$

Now we would like to find all reparametrizations $\tau=\varphi(t)$ such that the Ricci curvature $\bar\rho(\tau)$ in the new parameter τ is identically equal to zero. The reparametrization rule (5.3) implies that such reparametrization have to satisfy the following differential equation

$$\mathbb{S}\left(\varphi(t)\right)=\frac{3\rho(t)}{k} \tag{5.4}$$

This equation has a solutions at least locally (i.e., in a neighborhood of any given point) and as was mentioned already in the section 3 any two solution are transformed one to another by Möbius transformation. In other words the set of all parametrization of $\Lambda(\cdot)$ with Ricci curvature identically equal to zero defines a projective structure on $\Lambda(\cdot)$ (any two parametrization from this set are transformed one to another by Möbius transformation). We call it *the canonical projective structure* of the curve $\Lambda(\cdot)$. The parameters of the canonical projective structure will be called *projective parameters.*

Now we give a construction of a special form on unparametrized curve $\Lambda(\cdot)$ (namely, the differential of degree four on $\Lambda(\cdot)$), which is the first appearing invariant of the unparametrized curve. We will call it the fundamental form of the curve $\Lambda(\cdot)$.

Let t be a projective parameter on $\Lambda(\cdot)$. Then by definition $\rho(t) \equiv 0$, and by (4.18) $\beta_{0,1}(t) \equiv \frac{1}{2}\beta_{0,0}'(t) \equiv 0$. Therefore by (4.26) we obtain that in projective parameter

$$g(t_0, t_1) = \beta_{0,2}(t_0)(t_1 - t_0)^2 + O\left((t_1 - t_0)^3\right) \tag{5.5}$$

Let τ be another projective parameter on $\Lambda(\cdot)$ i.e., $\tau = \varphi(t) = \frac{at+b}{ct+d}$. Then by Proposition 2 $g_\varphi(t_0, t_1) \equiv 0$. Substituting this to the (5.2) we have

$$\bar{g}\left(\varphi(t_0), \varphi(t_1)\right)\varphi'(t_0)\varphi'(t_1) = g(t_0, t_1), \tag{5.6}$$

where $\tau_i = \varphi(t_i)$, $i = 0, 1$. Using (5.5), we compare the coefficients of the first terms in the Taylor expansions of both sides of (5.6). As a result we obtain

$$\bar{\beta}_{0,2}(\varphi(t_0))(\varphi'(t_0))^4 = \beta_{0,2}(t_0)$$

or

$$\bar{\beta}_{0,2}(\tau)(d\tau)^4 = \beta_{0,2}(t)(dt)^4 \tag{5.7}$$

It means that the form $\beta_{0,2}(t)(dt)^4$ does not depend on the choice of the projective parameter t. We will call this form *a fundamental form* of the curve $\Lambda(\cdot)$ and will denote by $\mathcal{A}$.

If t is an arbitrary (not necessarily projective) parameter on the curve $\Lambda(\cdot)$, then the fundamental form $\mathcal{A}$ in this parameter has to be of the form $A(t)(dt)^4$, where $A(t)$ is a smooth function (the "density" of the fundamental form).

Lemma 5.1 *For arbitrary parameter t the density $A(t)$ of the fundamental form satisfies the following relation*

$$A(t) = \beta_{0,2}(t) - \frac{3}{5k}\rho(t)^2 - \frac{3}{20}\rho''(t) \tag{5.8}$$

or, equivalently,

$$A(t) = \left(\frac{1}{10}\left(\frac{\partial}{\partial t_0} + \frac{\partial}{\partial t_1}\right)^2 - \frac{1}{2}\frac{\partial^2}{\partial t_0 \partial t_1}\right) g(t_0, t_1)\Big|_{t_0=t_1=t} - \frac{3}{5k} g(t, t)^2. \tag{5.9}$$

Proof. Let $\tau = \varphi(t)$ be a reparametrization such that τ be a projective parameter. It means that $\varphi(\tau)$ satisfies the equation (5.4). Denote by $\beta_{i,j}^\varphi(t_0, t_1)$ the coefficients defined by (4.11), (4.14) and (4.16) with S_t replaced by $\varphi(t)$. Using (5.5), compare the coefficients of the first terms in the Taylor expansions of both sides of (5.6). As a result we obtain

$$\bar{\beta}_{0,2}(\varphi(t_0))(\varphi'(t_0))^4 = \beta_{0,2}(t_0) - k\beta_{0,2}^\varphi(t_0)$$

or

$$\mathcal{A} = \bar{\beta}_{0,2}(\tau)(d\tau)^4 = \Big(\beta_{0,2}(t) - k\beta^{\varphi}_{0,2}(t)\Big)\,(dt)^4 \tag{5.10}$$

To complete the proof it remains to compute the coefficient $\beta^{\varphi}_{0,2}(t_0)$. For this we will use the recursive formula (4.30), where B_i are defined by (4.29) with $\beta^{\varphi}_{0,i}$ instead of $\beta_{0,2}$. From (4.30) it follows that

$$B_2(t) = \frac{1}{4}B_1'(t),$$
$$B_3(t) = \frac{1}{5}\left(B_2'(t) - (B_1(t))^2\right) = \frac{1}{20}B_1''(t) - \frac{1}{5}(B_1(t))^2 \tag{5.11}$$

From (4.28), (4.29), and (5.4) it follows that

$$B_1(t) = -\beta^{\varphi}_{0,0}(t) = -\frac{1}{3}\mathbb{S}\left(\varphi(t)\right) = -\frac{\rho(t)}{k}$$

Then by (4.30) and (5.11)

$$\beta^{\varphi}_{0,2}(t) = -3B_3(t) = \frac{3}{20k}\rho''(t) + \frac{3}{5k^2}\rho(t)^2$$

This together with (5.10) implies (5.8). To obtain (5.9) we just rewrite (5.8), taking into account the connection between the function $g(t_0, t_1)$ and the functions $\rho(t)$ $(= \beta_{0,0}(t))$, $\beta_{0,2}(t)$ given by expansion (4.16)(the expression (5.9) is just the most symmetric w.r.t. t_0 and t_1). $\Box$

If $A(t)$ does not change sign, then the canonical length element $|A(t)|^{\frac{1}{4}}dt$ is defined on $\Lambda(\cdot)$. The corresponding parameter τ (i.e., length with respect to this length element) is called *a normal parameter*. In other words, the normal parameter τ is the parameter such that the corresponding density $A(\tau)$ of the fundamental form satisfies $|A(\tau)| \equiv 1$. Calculating the Ricci curvature $\rho_n(\tau)$ of $\Lambda(\cdot)$ in the normal parameter, we obtain a functional invariant of the unparametrized curve.

Definition 3 *The Ricci curvature $\rho_n(\tau)$ of $\Lambda(\cdot)$ in the normal parameter is called projective Ricci curvature of the unparametrized curve $\Lambda(\cdot)$.*

If $t = \varphi(\tau)$ is the transition function between a projective parameter t and the normal parameter τ, then by (5.4) it follows that $\rho_n(\tau) = \frac{k}{3}\mathbb{S}\left(\varphi(\tau)\right)$. The projective Ricci curvature will be effectively used in section 11 devoted to the equivalence problem of rank 2 vector distributions.

At the end of this section we give an explicit formula for the fundamental form of the regular curve in terms of its curvature operator. First note that by definition the weight k of the regular curve is equal to $\frac{1}{2}\dim W$ (one can also derive it from (3.2) and (4.21)).

Lemma 5.2 *The fundamental form $\mathcal{A}$ of the regular curve $\Lambda(t)$ in the Lagrange Grassmannian $L(W)$ satisfies the following relation*

$$\mathcal{A}=\frac{1}{15}\left(\mathrm{tr}\,\left(R(t)^2\right)-\frac{1}{k}\Big(\mathrm{tr}\,R(t)\Big)^2\right)(dt)^4, \tag{5.12}$$

where $R(t)$ is the curvature operator of $\Lambda(t)$ defined by (3.1) and $k=\frac{1}{2}\dim W$.

Proof. Let us compute $\beta_{0,2}(t)$. We will use the notation of (2.2) and (2.3). By (4.22)

$$\beta_{0,2}(t)=-3\mathrm{tr}\,(Q_3(t)\dot{\Lambda}(t))=-3\mathrm{tr}\,(A_3(t)\dot{S}(t)) \tag{5.13}$$

For a given $\bar{t}$ choose for simplicity a coordinate representation S_t of the curve $\Lambda(t)$ such that $A_0(\bar{t})=0$. Then by (3.3)

$$A_3(\bar{t})=\frac{1}{5}\left(\dot{A}_2(\bar{t})-A_1(\bar{t})\dot{S}_{\bar{t}}A_1(\bar{t})\right) \tag{5.14}$$

From (3.4) it follows that the condition $A_0(\bar{t})=0$ is equivalent to $\ddot{S}_{\bar{t}}=0$. It implies that $\dot{A}_2(\bar{t})\dot{S}_{\bar{t}}=\frac{d}{dt}(A_2(t)\dot{S}_t)|_{t=\bar{t}}$. Therefore multiplying (5.14) by $S_{\bar{t}}$ and taking trace from both sides we obtain

$$\mathrm{tr}\,\left(A_3(\bar{t})\dot{S}_{\bar{t}}\right)=\frac{1}{5}\frac{d}{dt}\mathrm{tr}\,\left(A_2(\bar{t})\dot{S}_{\bar{t}}\right)-\frac{1}{5}tr\left((A_1(\bar{t})\dot{S}_{\bar{t}})^2\right) \tag{5.15}$$

Now by (4.22) and (4.18)

$$\mathrm{tr}\,\left(A_2(\bar{t})\dot{S}_{\bar{t}}\right)=-\frac{1}{2}\beta_{0,1}(\bar{t})=-\frac{1}{4}\rho'(\bar{t}) \tag{5.16}$$

On the other hand by (3.6) $(A_1(\bar{t})\dot{S}_{\bar{t}})=-\frac{1}{3}R(\bar{t})$. This and (5.16) imply that (5.15) can be written in the following form

$$\mathrm{tr}\,\left(A_3(\bar{t})\dot{S}_{\bar{t}}\right)=-\frac{1}{20}\rho''(\bar{t})-\frac{1}{45}tr\,\left(R(\bar{t})^2\right)$$

Taking into account (5.14) we obtain by (5.8) that

$$A(\bar{t})=\frac{3}{20}\rho''(\bar{t})+\frac{1}{15}tr\,\left(R(\bar{t})^2\right)-\frac{3}{5k}\rho(\bar{t})^2-\frac{3}{20}\rho''(\bar{t})=\frac{1}{15}tr\,\left(R(\bar{t})^2\right)-\frac{3}{5k}\rho(\bar{t})^2$$

Finally, note that $\rho=\frac{1}{3}\,\mathrm{tr}R$ (see (4.27)). Substituting this to the last relation we obtain (5.12).$\square$

Note that in the scalar case (i.e., when $\dim W=2$) the fundamental form $\mathcal{A}$ is identically equal to zero.

Remark 3 *All constructions of sections* 3 - 5 *can be done for the curve in the Grassmannian $G(m,2m)$ (the set of all m-dimensional subspaces in the $2m$-dimensional linear space) instead of Lagrangian Grassmannian by the action of the group $GL(2m)$ instead of Symplectic Group.*

6 The rank 1 curves: preliminary steps

In the present section we start a systematic study of the curves of rank 1 in the Lagrange Grassmannian $L(W)$ with $\dim W = m$. We consider a rank 1 ample curve $\Lambda : I \mapsto L(W)$ with a maybe nonconstant weight, where I is some interval on the real line. We introduce a canonical basis on each subspace $\Lambda(t)$ and compute some characteristics of the curve, in particular, its weight at any point. Finally, we show that the curve Λ has the constant weight equal to m^2 on the set with discrete complement in I. All this will prepare us to the next section, where the curves of rank 1 and constant weight will be investigated.

Without loss of generality, suppose that $\Lambda(\tau)$ is monotonically nondecreasing, i.e., the velocities $\dot{\Lambda}(t)$ are nonnegtive definite quadratic forms. As in section 2, let $\Lambda_\tau(t)$ be the identical imbedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$. The velocity $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is well defined self-adjoint linear mapping from $\Lambda(\tau)^*$ to $\Lambda(\tau)$, i.e., an element of $Sym^2\Lambda(\tau)$. Moreover, by our assumptions, $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is a nonpositive self-adjoint linear mapping of rank 1. So for $t \neq \tau$ there exists a unique, up to the sign, vector $w(t,\tau) \in \Lambda(\tau)$ such that for any $v \in \Lambda(\tau)^*$

$$\langle v, \frac{\partial}{\partial t}\Lambda_\tau(t)v\rangle = -\langle v, w(t,\tau)\rangle^2 \tag{6.1}$$

Remark 4 *From the definition of $w(t,\tau)$ it follows easily that for given τ the germ of the curve $\Lambda(t)$ at $t=\tau$ is defined uniquely by $\Lambda(\tau)$, the derivative subspace $\Lambda^0(\tau)$, and the germ of the function $t \mapsto w(t,\tau)$ at $t=\tau$. Since the Symplectic Group acts transitively on the set of pairs of transversal Lagrange subspaces, one can conclude that the germ of the curve $\Lambda(t)$ at $t=\tau$ is defined uniquely, up to a symplectic transformation, by the germ of the function $t \mapsto w(t,\tau)$ at $t=\tau$.*

The function $t \mapsto \Lambda_\tau(t)$ has a pole at $t=\tau$. It implies easily that the function $t \mapsto w(t,\tau)$ also has a pole at $t=\tau$. Suppose that the order of this pole is equal to $l(\tau)$.

Denote by $u(t,\tau)$ the normalized curve $t \to u(t,\tau) = (t-\tau)^{l(\tau)}w(t,\tau)$ and define the following vectors in $\Lambda(\tau)$:

$$e_j(\tau) = \frac{1}{(j-1)!}\,\frac{\partial^{j-1}}{\partial t^{j-1}}u(t,\tau)\bigg|_{t=\tau}. \tag{6.2}$$

First note that

$$\text{span}\left(\{e_j(\tau)\}_{j=1}^{\infty}\right) = \Lambda(\tau) \tag{6.3}$$

Otherwise, using the following formula

$$w(t,\tau) = \sum_{i=1}^{j} e_i(\tau)(t-\tau)^{i-1-l} + O((t-\tau)^{i-l})$$

one can easily obtain the contradiction to the fact that $\Lambda(t)$ is ample.

Thus for a given parameter τ and integer i, $1 \le i \le m$ the following integers $k_i(\tau)$ are well defined

$$k_i(\tau) = \min\{j \in \mathbb{N} \cup 0 : \dim\left(\operatorname{span}\left(e_1(\tau), e_2(\tau), \dots, e_{j+1}(\tau)\right)\right) = i\} \tag{6.4}$$

Note that

$$0 = k_1(\tau) < k_2(\tau) < \ldots < k_m(\tau), \;\; k_i(\tau) \ge i - 1 \tag{6.5}$$

By definition the vectors $e_{k_1(\tau)+1}(\tau), \dots, e_{k_m(\tau)+1}(\tau)$ constitute the basis of the subspace $\Lambda(\tau)$. We call this basis a *canonical basis* of $\Lambda(\tau)$. Since the vector $w(t,\tau)$ is defined up to the sign, the vector $e_1(\tau)$ $(= e_{k_1(\tau)+1}(\tau))$ is also defined up to the sign. So, one can take also $(-e_{k_1(\tau)+1}(\tau), \dots, -e_{k_m(\tau)+1}(\tau))$ as the canonical bases on the plane $\Lambda(\tau)$. Denote by $w_i(t,\tau)$ the i-th component of the vector $w(t,\tau)$ w.r.t this basis. In other words, functions $w_i(t,\tau)$ satisfies the following relation

$$w(t,\tau) = \sum_{i=1}^{m} w_i(t,\tau) e_{k_i(\tau)+1}(\tau) \tag{6.6}$$

Remark 5 *Using Remark 4, one can easily conclude that the germ of the curve $\Lambda(t)$ at $t = \tau$ is defined uniquely by $\Lambda(\tau)$, the canonical basis in $\Lambda(\tau)$, the derivative subspace $\Lambda^0(\tau)$, and the germs of the functions $t \mapsto w_i(t,\tau)$ at $t = \tau$, where $1 \le i \le m$. Since for any two pairs (Λ, Δ) and $(\tilde{\Lambda}, \tilde{\Delta})$ of transversal Lagrange subspaces with fixed bases in Λ and $\tilde{\Lambda}$, there exists symplectic transformation that transforms basis in Λ to the basis in $\tilde{\Lambda}$ and subspace Δ to $\tilde{\Delta}$, we have that the germ of the curve $\Lambda(t)$ at $t = \tau$ is defined uniquely, up to a symplectic transformation, by the germ of the functions $t \mapsto w_i(t,\tau)$ at $t = \tau$, where $1 \le i \le m$.*

Now we prove a computational lemmas about the weight of $\Lambda(t)$ at τ and the order of pole of $t \mapsto w(t,\tau)$.

Lemma 6.1 *The order $l(\tau)$ of pole of the function $t \mapsto w(t,\tau)$ is equal to $k_m(\tau) + 1$. The weight of the curve $\Lambda(t)$ at τ is equal to $(2\,k_m(\tau) + 1)\,m - 2\sum_{i=2}^{m} k_i(\tau)$.*

Proof. For simplicity we will write k_i instead of $k_i(\tau)$, and l instead of $l(\tau)$. Let S_t, $\Lambda(t) = \{(x, S_t\,x) : x \in \mathbb{R}^m\}$, be a coordinate representation of germ of $\Lambda(t)$ at $t = \tau$ such that the canonical basis $e_{k_1+1}(\tau), \dots, e_{k_m+1}(\tau)$ constitute a standard basis of $\mathbb{R}^m$. Denote by $\Delta = 0 \oplus \mathbb{R}^m$, the subspace $D \in \Lambda(\tau)^{\pitchfork}$. From (6.2) it follows that in the canonical basis

$$w_i(t,\tau) = (t-\tau)^{k_i-l} + O((t-\tau)^{k_i-l+1}). \tag{6.7}$$

Then relation (6.1) in the canonical basis can be rewritten in the following form

$$\left(\frac{\partial}{\partial t}(S_t - S_\tau)^{-1}\right)_{i,j} = -w_i(t,\tau)w_j(t,\tau) = -(t-\tau)^{k_i+k_j-2l} + O((t-\tau)^{k_i+k_j-2l+1}) \tag{6.8}$$

For simplicity take coordinates $t \mapsto S_t$ such that the subspace Δ is the derivative subspace $\Lambda^0(\tau)$. Then by definition of the derivative subspace the free term on the Laurent expansion of $(S_t - S_\tau)^{-1}$ is equal to zero. Therefore

$$\left((S_t - S_\tau)^{-1}\right)_{i,j} = -\int^t w_i(\xi,\tau)w_j(\xi,\tau)\,d\xi = \frac{(t-\tau)^{k_i+k_j-2l+1}}{k_i+k_j-2l+1} + O((t-\tau)^{k_i+k_j-2l+2}) \tag{6.9}$$

Then it is easy to get the following expansion for the determinant

$$\det(S_t - S_\tau) = \frac{(t-\tau)^k}{C} + O((t-\tau)^{k+1}), \tag{6.10}$$

where

$$k = (2\,l-1)\,m - 2\sum_{i=2}^m k_i, \tag{6.11}$$

and C is the determinant of matrix whose (i,j)th entry is $\frac{1}{2l-k_i-k_j-1}$, $i,j = 1,\dots,m$. It is well known that the determinant of the matrix whose (i,j)th entry is $\frac{1}{x_i+y_j}$, $i,j=1,\dots,m$, can be computed by the following formula

$$\det\left(\left\{\frac{1}{x_i+y_j}\right\}_{i,j=1}^m\right) = \frac{\prod\limits_{1\le i<j\le m}(x_i-x_j)(y_i-y_j)}{\prod\limits_{i,j=1}^m (x_i+y_j)} \tag{6.12}$$

It implies in particular that $C \neq 0$ (one can take $x_i = y_i = l - k_i - \frac{1}{2}$ and use the fact that $k_i \neq k_j$ for $i \neq j$). So, the weight is equal to $(2l-1)m - 2\sum_{i=2}^m k_i$.

Further, from (6.9) and (6.12) it follows that

$$(S_t - S_\tau)_{i,j} = \left(C_{i,j}(t-\tau)^{-k+2l-k_i-k_j-1} + O((t-\tau)^{-k+2l-k_i-k_j})\right)\left(\frac{(t-\tau)^k}{C} + O((t-\tau)^{k+1})\right) =$$

$$\frac{C_{i,j}}{C}(t-\tau)^{2l-k_i-k_j-1} + O((t-\tau)^{2l-k_i-k_j}), \tag{6.13}$$

where C is as in (6.10), k is as in (6.11), and $C_{i,j}$ are (i,j)th entry of adjacent matrix to the matrix $\left(\frac{1}{2l-k_i-k_j-1}\right)_{i,j=1}^{m}$. By (6.12) and (6.5) $C_{i,j} \neq 0$. Since S_t is a smooth curve at τ all powers $2l - k_i - k_j - 1$ in (6.13) are positive. By assumption, $\dot{S}_\tau \neq 0$. It implies that

$$\min_{1\leq i,j\leq m}(2l - k_i - k_j - 1) = 1 \tag{6.14}$$

But from (6.5) it follows that $\min\limits_{1\leq i,j\leq m}(2l - k_i - k_j - 1) = 2l - 2k_m - 1$ that yields that $l = k_m + 1$. Consequently the weight is equal to $(2\,k_m + 1)\,m - 2\sum_{i=2}^{m} k_i$. $\square$

Remark 6 *In the proof of the previous lemma to obtain the asymptotics (6.9) we have taken the coordinate representation* $t \mapsto S_t$, $\Lambda(t) = \{(x, S_t\,x) : x \in \mathbb{R}^m\}$, *with* $\Delta = \Lambda^0(\tau)$ *(where* $\Delta = 0\oplus\mathbb{R}^m$*). But then we have obtained the relation (6.14)which implies that* $k_i(\tau)+k_j(\tau)-2l(\tau)+1 < 0$ *for any* $i, j = 1, \ldots, m$. *Therefore, the asymptotics (6.9) for* $\left((S_t - S_\tau)^{-1}\right)_{i,j}$ *and then the asymptotics (6.13) for* $(S_t - S_\tau)_{i,j}$ *are valid for any coordinate representation* $t \mapsto S_t$, $\Lambda(t) = \{(x, S_t\,x) : x \in \mathbb{R}^m\}$, *of germ of* $\Lambda(t)$ *at* $t = \tau$ *such that the canonical basis* $e_{k_1+1}(\tau), \ldots, e_{k_m+1}(\tau)$ *constitute a standard basis of* $\mathbb{R}^m$ *and* $\Delta = 0 \oplus \mathbb{R}^m$ *is arbitrary subspace transversal to* $\Lambda(\tau)$. *The reason is that the asymptotics (6.9) do not depend on free term.*

Take some subspace $\Delta \in \Lambda(\tau)^{\pitchfork}$. Recall that the velocity $\dot{\Lambda}(t)$ is a self-adjoint nonnegative definite linear mapping of rank 1 from $\Lambda(t)$ to $\Lambda(t)^*$. For any t sufficiently closed to τ one can identify Δ with $\Lambda(t)^*$. Under this identification $\dot{\Lambda}(t)$ is a self-adjoint nonnegative linear mapping of rank 1 from $\Lambda(t)$ to Δ. Therefore there exist a unique, up to the sign, vector $v(t) \in \Delta$ such that for any $w \in \Lambda(t)$:

$$\langle\dot{\Lambda}(t)w, w\rangle = \langle v(t), w\rangle^2 \tag{6.15}$$

Suppose that a tuple of vectors $f_1(\tau), \ldots, f_m(\tau)$ is a basis of Δ dual to the canonical basis of $\Lambda(\tau)$ (i.e, $\sigma(f_i(\tau), e_{k_j(\tau)+1}(\tau)) = \delta_{i,j}$). From Remark 6 and relation (6.13) (where $l = k_m + 1$) it follows that the components $v_i(t)$ of the vector $v(t)$ w.r.t the basis $f_1(\tau), \ldots, f_m(\tau)$ have the following asymptotics

$$v_i(t) = c_i(\tau)(t-\tau)^{k_m(\tau)-k_i(\tau)} + O\Big((t-\tau)^{k_m(\tau)-k_i(\tau)+1}\Big), \tag{6.16}$$

where $c_i(\tau) \neq 0$. (Actually, using (6.12), one can compute $c_i(\tau)$:

$$c_i(\tau) = \sqrt{\frac{C_{i,i}(\tau)(2(k_m(\tau) - k_i(\tau)) + 1)}{C(\tau)}} = \frac{\prod\limits_{j=1}^{m}(2k_m(\tau) - k_i(\tau) - k_j(\tau) + 1)}{\prod\limits_{1\leq j\leq m,\, j\neq i}(k_i(\tau) - k_j(\tau))} \tag{6.17}$$

where $C_{i,i}$ and C are as in the proof of Lemma 6.1.) The relation (6.16) implies that for any integer nonnegative j such that

$$k_m(\tau) - k_i(\tau) \leq j < k_m(\tau) - k_{i-1}(\tau) \tag{6.18}$$

the following relation holds

$$\mathrm{span}(v(\tau), v'(\tau), \dots, v^{(j)}(\tau)) = \mathrm{span}(f_i(\tau), \dots, f_m(\tau)) \tag{6.19}$$

In particular,

$$\begin{aligned} \mathrm{span}(v(\tau), v'(\tau), \dots, v^{(k_m(\tau)-1)}(\tau)) &= \mathrm{span}(f_2(\tau), \dots, f_m(\tau)) \subsetneq \Delta, & (6.20)\\ \mathrm{span}(v(\tau), v'(\tau), \dots, v^{(k_m(\tau))}(\tau)) &= \Delta & (6.21) \end{aligned}$$

(recall that $k_1(\tau) = 0$).

Now we are ready to prove the following

Proposition 3 *For the ample curve $\Lambda : I \mapsto L(W)$ of rank 1 the set $\mathcal{C}$ such that*

$$\mathcal{C} = \Big\{ t \in I : \dim\Big(\mathrm{span}\,(e_1(t), e_2(t), \dots, e_m(t))\Big) < m \Big\} \tag{6.22}$$

is discrete set of the interval of definition I.

Proof. Suppose that $\mathcal{C}$ has an accumulation point τ. Take some subspace $\Delta \in \Lambda(\tau)^{\pitchfork}$. Let $t \mapsto v(t)$ be a curve of vectors in Δ defined by (6.15) for all t from some neighborhood U of τ in I. Note that $t \in \mathcal{C}$ iff $k_m(t) \geq m$. Therefore, by (6.20) and (6.21), we have that $t_0 \in \mathcal{C} \cap U$ iff the function $d(t) \stackrel{def}{=} \det(v(t), v'(t), \dots v^{(m-1)}(t))$ has zero at $t = t_0$. For the accumulation point τ , using consequently the Rolle theorem, one can conclude that the function $d(t)$ has zero of infinite order at $t = \tau$.

On the other hand, let $l_i(\tau) \stackrel{def}{=} k_m(\tau) - k_{m-i+1}(\tau)$. Denote by $p = \sum_{i=1}^{m} l_i(\tau)$. Let us prove that $d^{(p)}(\tau)$ is not equal to zero. Indeed, $d^{(p)}(\tau)$ can be expressed as the sum of the terms of the form $det(v^{(j_1)}(\tau), \dots, v^{(j_m)}(\tau))$, where

$$\sum_{i=1}^{m} j_i = p, \quad 0 \leq j_1 < j_2 < \dots < j_m. \tag{6.23}$$

Let us show that if the tuple $(j_1, \dots, j_m)$ is different from the tuple $(l_1(\tau), \dots, l_m(\tau))$ and satisfies (6.23), then

$$\det(v^{(j_1)}(\tau), \dots, v^{(j_m)}(\tau)) = 0. \tag{6.24}$$

For this note first that by assumptions there exists an index s, such that $j_s < l_s(\tau)$ $(= k_m(\tau) - k_{m-s+1}(\tau))$. Then from (6.18) and (6.19) we have the following relation

$$\operatorname{span}(v^{(j_1)}(\tau), \ldots, v^{(j_s)}(\tau)) \subset \operatorname{span}(f_{m-s+2}(\tau), \ldots, f_m(\tau)),$$

i.e., $\dim\Big(\operatorname{span}(v^{(j_1)}(\tau), \ldots, v^{(j_s)}(\tau))\Big) < s$. It implies that $\dim\Big(\operatorname{span}(v^{(j_1)}(\tau), \ldots, v^{(j_m)}(\tau))\Big) < m$ that is equivalent to (6.24). Note also that from (6.18) and (6.19) it follows easily that $\operatorname{span}\Big(v^{(l_1(\tau))}(\tau), \ldots, v^{(l_m(\tau))}(\tau)\Big) = \Delta$. Therefore,

$$d^{(p)}(\tau) = c \det(v^{(l_1(\tau))}(\tau), \ldots, v^{(l_m(\tau))}(\tau)) \neq 0$$

(here c is some natural number). Hence $d(t)$ has zero of finite order at $t = \tau$. We obtain the contradiction. $\square$

For $t \in I \backslash \mathcal{C}$ the numbers $k_i(t) = i - 1$. As a consequence of the previous proposition and the expression for the weight from the Lemma 6.1, we obtain the following

Corollary 1 *The ample curve $\Lambda : I \mapsto L(W)$ of rank 1 has the constant weight equal to m^2 on the set with discrete complement in I.*

At the end of this section we give the explicit formula for the velocity $\dot{\Lambda}(\tau)$ in the canonical basis. Let $(e_1^*(\tau), \ldots, e_m^*(\tau))$ be a basis in $\Lambda(\tau)^*$ dual to the canonical basis in $\Lambda(\tau)$. As we have seen at the end of the proof of the Lemma 6.1, the (m, m)th entry is the only nonzero entry of the matrix $\dot{S}_\tau$ and it is equal to $\frac{C_{m,m}(\tau)}{C(\tau)} = c_m^2(\tau)$ (where $c_m(\tau)$ is as in (6.17)). Therefore, we obtain the following

Lemma 6.2 *For any $v_1, v_2 \in \Lambda(\tau)$ the following relation holds*

$$\langle \dot{\Lambda}(\tau) v_1, v_2 \rangle = c_m^2(\tau) \langle e_m^*(\tau), v_1 \rangle \langle e_m^*(\tau), v_2 \rangle, \tag{6.25}$$

where

$$c_m(\tau) = \prod_{j=1}^{m-1} \frac{k_m(\tau) - k_j(\tau) + 1}{k_m(\tau) - k_j(\tau)} \tag{6.26}$$

7 The rank 1 curves with a constant weight

The present section is devoted to the curves of rank 1 and a constant finite weight in the Lagrange Grassmannian $L(W)$. We show that in this case the function $g(t, \tau)$ constructed in section 4 determines the curve uniquely, up to symplectic transformation. We also find a complete system of invariants of the curve in terms of the function g.

First, using Proposition 3 and Lemmas 6.1, we obtain the following

Proposition 4 *If $\Lambda(t)$ is a curve of rank 1 and constant weight on I, then for all $t \in I$ and $1 \le i \le m$ the numbers $k_i(t)$ are equal to $i-1$, or, equivalently, the vectors $e_1(t), \dots, e_m(t)$ constitute the canonical basis of the subspace $\Lambda(t)$.*

Proof. From (6.5) it follows that always

$$k_i(t) - k_j(t) \ge i - j, \; k_1(t) = 0 \tag{7.1}$$

Therefore by Lemma 6.1 the weight $k(t)$ of the curve Λ at the point t satisfies

$$k(t) = (2k_m(t)+1)m - 2\sum_{i=2}^{m} k_i(t) = 2\sum_{i=1}^{m}(k_m(t) - k_i(t)) + m \ge 2\sum_{i=1}^{m}(m-i) + m = m^2 \tag{7.2}$$

In addition, from (7.1) it is easy to see that the equality in (7.2) holds iff $k_i(t) = i-1$ for any $1 \le i \le m$. Therefore, if the set $\mathcal{C}$ is as in (6.22), then for any $t \in \mathcal{C}$ the weight $k(t) > m^2$, while for $t \notin \mathcal{C}$ the weight $k(t) = m^2$. But from Proposition 3 the set $\mathcal{C}$ is discrete subset of I. Hence, for the weight $k(t)$ to be constant on I, the set $\mathcal{C}$ has to be empty. This completes the proof of the Proposition. $\square$

As a consequence of the previous proposition and Lemmas 6.1, 6.2 , we obtain easily the following

Corollary 2 *If $\Lambda(t)$ is a curve of rank 1 and constant weight on I, then:*
1) *at any point $t \in I$ the weight is equal to m^2;*
2) *for any $\tau \in I$ the function $t \mapsto w(t,\tau)$ has a pole of order m at $t = \tau$;*
3) *For any $v_1, v_2 \in \Lambda(\tau)$ the following relation holds*

$$\langle \dot{\Lambda}(\tau)v_1, v_2\rangle = m^2 \langle e_m^*(\tau), v_1\rangle \langle e_m^*(\tau), v_2\rangle \tag{7.3}$$

Now we prove that the function $g(t,\tau)$ defined in section 4 contains all the information about $\Lambda(t)$.

Theorem 1 *The function $g(t,\tau)$ determines the curve $\Lambda(t)$ of rank 1 and constant weight uniquely , up to a symplectic transformation.*

Before starting to prove the theorem, we want to describe in few words the main steps of the proof. First, we show that the function $g(t,\tau)$ is almost the same as the component $w_m(t,\tau)$ of the vector $w(t,\tau)$. The vector $w(t,\tau)$ is a function of two variables, but it is determined by a curve. Therefore it is natural to expect that $w(t,\tau)$ satisfies some partial differential equation. We find this equation that is actually the system of m equations for the components $w_i(t,\tau)$, $1 \leq i \leq m$. Then we show that this system has a "triangular" form such that all components $w_i(t,\tau)$ can be expressed by $w_m(t,\tau)$ and refer to Remark 5 to complete the proof.

Proof of Theorem 1.

1. We start the proof with the following lemma

Lemma 7.1 *The following relation holds*

$$w_m^2(t,\tau) = \frac{1}{(t-\tau)^2} + \frac{1}{m^2} g(t,\tau) \tag{7.4}$$

Proof. By (4.19),(4.23), and part 2) of Corollary 4 we have

$$\mathrm{tr}\left(\frac{\partial}{\partial t}\Lambda_\tau(t) \circ \dot{\Lambda}(\tau)\right) = -\frac{m^2}{(t-\tau)^2} - g(t,\tau) \tag{7.5}$$

Let $t \to S_t$, $\Lambda(t) = \{(x, S_t\, x) : x \in \mathbb{R}^m\}$ be a coordinate representation of germ of $\Lambda(t)$ at $t = \tau$ such that the canonical basis $e_1(\tau), \ldots, e_m(\tau)$ constitute a standard basis of $\mathbb{R}^m$. By (4.24)

$$\mathrm{tr}\left(\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right)\ \dot{S}_\tau\right) = -\frac{m^2}{(t-\tau)^2} - g(t,\tau) \tag{7.6}$$

Relation (7.3) implies that in the chosen coordinates

$$\dot{S}_\tau = \begin{cases} 0 & (i,j) \neq (m,m) \\ m^2 & (i,j) = m \end{cases}$$

By construction,

$$\left(\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right)\right)_{i,j} = -w_i(t,\tau) w_j(t,\tau)$$

Therefore

$$\mathrm{tr}\left(\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right) \dot{S}_\tau\right) = -m^2 w_m^2(t,\tau).$$

This together with (7.6) implies (7.4). $\square$

By (7.4) it follows that in order to prove the theorem it is sufficient to show that the function $w_m(t,\tau)$ determines $\Lambda(t)$ uniquely, up to a symplectic transformation.

2. Now we derive a partial differential equation for the vector function $w(t,\tau)$.

Lemma 7.2 *The vector function $w(t,\tau)$ satisfies the following differential equation*

$$\frac{\partial^2 w}{\partial t\partial\tau} - \left(\frac{\frac{\partial w_m}{\partial t}}{w_m}\right)\frac{\partial w}{\partial \tau} + m^2 w_m^2 w = 0 \tag{7.7}$$

Proof. Fix some parameter τ_0 and take some subspace Δ transversal to $\Lambda(\tau_0)$. Let $t \to S_t$, $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$ be a coordinate representation of germ of $\Lambda(t)$ at $t = \tau_0$ such that $\Lambda(\tau_0) = \mathbb{R}^m \oplus 0$ and $\Delta = 0 \oplus \mathbb{R}^m$. Denote by $w^\Delta(t,\tau) \in \mathbb{R}^m$ the first m components of the vector $w(t,\tau)$ in the chosen coordinates (or equivalently, the image of $w(t,\tau)$ under the projection of W on $\Lambda(\tau_0)$ parallel to Δ). Also, let, as before, $f_1(\tau),\dots,f_m(\tau)$ be the basis of Δ dual to the canonical basis of $\Lambda(\tau)$ (w.r.t. the symplectic form σ).

By (6.1) it follows that for t and τ closed to τ_0

$$\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right) = -w^\Delta(t,\tau)w^\Delta(t,\tau)^T. \tag{7.8}$$

Therefore

$$\dot S_t = \Big((S_t - S_\tau)w^\Delta(t,\tau)\Big)\Big((S_t - S_\tau)w^\Delta(t,\tau)\Big)^T \tag{7.9}$$

It implies that the vector function $(S_t - S_\tau)w^\Delta(t,\tau)$ does not depend on τ. Differentiating it w.r.t. τ we get

$$-\dot S_\tau w^\Delta(t,\tau) + (S_t - S_\tau)\frac{\partial}{\partial\tau}w^\Delta(t,\tau) = 0 \tag{7.10}$$

From (6.25) it follows that

$$\dot S_\tau w^\Delta(t,\tau) = m^2 w_m(t,\tau) f_m(\tau) \tag{7.11}$$

This together with (7.10) implies that

$$\frac{\partial}{\partial\tau}w^\Delta(t,\tau) = m^2 w_m(t,\tau)(S_t - S_\tau)^{-1} f_m(\tau) \tag{7.12}$$

In particular,

$$m^2 (S_t - S_\tau)^{-1} f_m(\tau) = \frac{1}{w_m(t,\tau)}\frac{\partial}{\partial\tau}w^\Delta(t,\tau) \tag{7.13}$$

Now, differentiating (7.12) w.r.t. t, we have

$$\frac{\partial^2}{\partial t\partial\tau}w^\Delta(t,\tau) = m^2 w_m(t,\tau)\frac{\partial}{\partial t}\Big((S_t - S_\tau)^{-1}\Big)f_m(\tau) + \frac{\partial}{\partial t}w_m(t,\tau)m^2(S_t - S_\tau)^{-1}f_m(\tau) \tag{7.14}$$

From (7.8) it follows that

$$\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right) f_m(\tau) = -w_m(t,\tau)w^\Delta(t,\tau)$$

Substituting this and (7.13) in (7.14), we get

$$\frac{\partial^2 w^\Delta}{\partial t \partial \tau} - \left(\frac{\frac{\partial w_m}{\partial t}}{w_m}\right)\frac{\partial w^\Delta}{\partial \tau} + m^2 w_m^2 w^\Delta = 0 \tag{7.15}$$

Recalling the definition of $w^\Delta(t,\tau)$, we obtain from the last equation the following inclusion

$$\frac{\partial^2 w}{\partial t \partial \tau} - \left(\frac{\frac{\partial w_m}{\partial t}}{w_m}\right)\frac{\partial w}{\partial \tau} + m^2 w_m^2 w \in \Delta \tag{7.16}$$

Let us remember that all our considerations (and in particular the inclusion (7.16)) are valid for any subspace Δ transversal to $\Lambda(\tau_0)$ and any t, τ closed to τ_0. Taking as Δ in (7.16) two subspaces that are transversal to $\Lambda(\tau_0)$ and also transversal one to another, we obtain (7.7) for any t, τ closed to τ_0 . Since τ_0 is arbitrary, this completes the proof of the lemma. $\Box$

In the sequel it will be convenient also to make the following substitution in (7.7)

$$Y(t,\tau) = \frac{1}{w_m(t,\tau)} w(t,\tau) \tag{7.17}$$

Then by direct computation one can obtain the following equation for Y:

$$\frac{\partial^2 Y}{\partial t \partial \tau} + \left(\frac{\frac{\partial w_m}{\partial \tau}}{w_m}\right)\frac{\partial Y}{\partial t} + \left(\frac{\partial^2}{\partial t \partial \tau}(\ln w_m) + m^2 w_m^2\right) Y = 0 \tag{7.18}$$

3. Now we shall rewrite the equation (7.7) as a system of equations w.r.t. the components $w_i(t,\tau)$. Take some subspace $\Delta \in \Lambda(\tau)^\pitchfork$. Identifying Δ with $\Lambda(\tau)^*$, denote $f_i(\tau)$ the vector, corresponding to $e_i^*(\tau)$ under this identification. The vectors $e_1(\tau), \dots, e_m(\tau), f_1(\tau), \dots, f_m(\tau)$ constitute the basis of the symplectic space W. Suppose that

$$\dot e_i(\tau) = \sum_{j=1}^m \alpha_{i,j}(\tau) e_j(\tau) + \gamma_{i,j}(\tau) f_j(\tau)$$

According to (7.3)

$$\gamma_{ij}(\tau) = \begin{cases} 0 & (i,j) \neq (m,m) \\ m^2 & (i,j) = (m,m) \end{cases}$$

It implies that

$$\begin{aligned} \dot e_i(\tau) &= \sum_{j=1}^m \alpha_{i,j}(\tau) e_j(\tau),\ 1 \le i \le m-1, \\ \dot e_m(\tau) &= \sum_{j=1}^m \alpha_{m,j}(\tau) e_j(\tau) + m^2 f_m(\tau) \end{aligned} \tag{7.19}$$

Remark 7 *In particular, it follows that the functions $\alpha_{i,j}(\tau)$ with $1 \le i \le m-1$ do not depend on the choice of the subspace Δ.*

By definition

$$w(t,\tau) = \sum_{i=1}^{m} w_i(t,\tau) e_i(\tau).$$

Then, using (7.19), we obtain

$$\frac{\partial w}{\partial \tau} = \sum_{i=1}^{m} \left(\frac{\partial w_i}{\partial \tau} + \sum_{j=1}^{m} w_j \alpha_{j,i} \right) e_i + m^2 w_m f_m, \tag{7.20}$$

$$\frac{\partial^2 w}{\partial t \partial \tau} = \sum_{i=1}^{m} \left(\frac{\partial^2 w_i}{\partial t \partial \tau} + \sum_{j=1}^{m} \frac{\partial w_j}{\partial t} \alpha_{j,i} \right) e_i + m^2 \frac{\partial w_m}{\partial t} f_m \tag{7.21}$$

Substituting (7.20) and (7.21) in (7.7) and comparing coefficients of e_i for $i = 1, \dots, m$, we get the following system of equations

$$\frac{\partial^2 w_i}{\partial t \partial \tau} - \frac{\frac{\partial w_m}{\partial t}}{w_m} \frac{\partial w_i}{\partial \tau} + m^2 w_m^2 w_i = \sum_{j=1}^{m} \left(\frac{\frac{\partial w_m}{\partial t}}{w_m} w_j - \frac{\partial w_j}{\partial t} \right) \alpha_{j,i}, \; 1 \le i \le m \tag{7.22}$$

The term in the righthand side of (7.22), corresponding to $j = m$, is equal to zero. Hence the equation (7.22) can be written in the following form

$$\frac{\partial^2 w_i}{\partial t \partial \tau} - \frac{\frac{\partial w_m}{\partial t}}{w_m} \frac{\partial w_i}{\partial \tau} + m^2 w_m^2 w_i = \sum_{j=1}^{m-1} \left(\frac{\frac{\partial w_m}{\partial t}}{w_m} w_j - \frac{\partial w_j}{\partial t} \right) \alpha_{j,i}, \; 1 \le i \le m \tag{7.23}$$

By Remark 7 the system of equations (7.23) does not depend on the choice of the subspace Δ.

In the same way the equation (7.17) can be rewritten as an equation for components $Y_i(t,\tau) = \frac{w_i(t,\tau)}{w_m(t,\tau)}$ of the vector $Y_i(t,\tau)$ w.r.t the canonical basis:

$$\frac{\partial^2 Y_i}{\partial t \partial \tau} + \left(\frac{\frac{\partial w_m}{\partial \tau}}{w_m} \right) \frac{\partial Y_i}{\partial t} + \left(\frac{\partial^2}{\partial t \partial \tau} (\ln w_m) + m^2 w_m^2 \right) Y_i = - \sum_{j=1}^{m-1} \frac{\partial Y_j}{\partial t} \alpha_{j,i} \tag{7.24}$$

4. Now we show that equation (7.23) (or (7.24)) has a "triangle" form. Note that by construction all functions $t \mapsto w(t,\tau)$ have singularities at $t = \tau$. Moreover, from the part 1 of Corollary 4 it follows that their Laurent expansions at $t = \tau$ have the following form

$$w_i(t,\tau) = \frac{1}{(t-\tau)^{m-i+1}} + \varphi_i(t,\tau), \tag{7.25}$$

where $\varphi_i(t,\tau)$ are smooth functions. Using this fact one can obtain the following

Lemma 7.3 *The coefficients $\alpha_{j,i}(\tau)$, $1 \le j \le m-1$, satisfy the following relations*

1. $\alpha_{j,i}(\tau) \equiv 0$, *if* $j < i - 1$;

2. $\alpha_{i-1,i}(\tau) \equiv \frac{(i-1)(2m-i+1)}{m-i+1}$;

3. *If $i \le j \le m-1$, then $\alpha_{j,i}(\tau)$ can be expressed by $\frac{\partial^k}{\partial t^k}\varphi_m(t,\tau)\Big|_{t=\tau}$ with $0 \le k \le j-i$, where $\varphi_m(t,\tau)$ is defined by (7.25).*

Proof. We shall analyze the Laurent expansions of both sides of the equation (7.23). We start with the righthand side. Denote by

$$\Phi_m(t,\tau) = \frac{\partial}{\partial t}\ln\Big(1+(t-\tau)\varphi_m(t,\tau)\Big) \tag{7.26}$$

Using (7.25), one can obtain the following series of relations

$$\frac{\partial}{\partial t}w_j(t,\tau) = -\frac{m-j+1}{(t-\tau)^{m-j+2}} + O(1); \tag{7.27}$$

$$\frac{\frac{\partial}{\partial t}w_m}{w_m} = \frac{\partial}{\partial t}\ln w_m(t,\tau) = \frac{\partial}{\partial t}\ln\left(\frac{1}{t-\tau}+\varphi_m(t,\tau)\right) = -\frac{1}{t-\tau}+\Phi_m(t,\tau) \tag{7.28}$$

$$\frac{\frac{\partial}{\partial t}w_m}{w_m}w_j = -\frac{1}{(t-\tau)^{m-j+2}} + \frac{\Phi_m(t,\tau)}{(t-\tau)^{m-j+1}} + O\left(\frac{1}{t-\tau}\right) \tag{7.29}$$

Therefore the righthand side of (7.23) can be written in the following form

$$\sum_{j=1}^{m-1}\left(\frac{m-j}{(t-\tau)^{m-j+2}} + \frac{\Phi_m(t,\tau)}{(t-\tau)^{m-j+1}}\right)\alpha_{j,i}(\tau) + O\left(\frac{1}{t-\tau}\right) \tag{7.30}$$

Suppose that the function $t \mapsto \Phi_m(t,\tau)$ has the following expansion into the formal Taylor series at $t=\tau$

$$\Phi_m(t,\tau) \approx \sum_{k=0}^{\infty} c_k(\tau)(t-\tau)^k \tag{7.31}$$

Then by direct computation we have that the righthand side of (7.23) has the following form

$$\frac{\alpha_{1,i}(\tau)}{(t-\tau)^{m+1}} + \sum_{j=2}^{m-1}\frac{(m-j)\alpha_{j,i}(\tau) + \sum\limits_{k=1}^{j-1} c_{j-k-1}(\tau)\alpha_{k,i}(\tau)}{(t-\tau)^{m-j+2}} + O\left(\frac{1}{(t-\tau)^2}\right) \tag{7.32}$$

Now consider the lefthand side of (7.23). Using (7.26) and (7.25) , we obtain the following series of relations

$$\frac{\partial}{\partial \tau}w_i(t,\tau) = \frac{m-i+1}{(t-\tau)^{m-i+2}} + \frac{\partial}{\partial \tau}\varphi_i(t,\tau); \tag{7.33}$$

$$\frac{\partial^2}{\partial t\partial \tau}w_i(t,\tau) = -\frac{(m-i+1)(m-i+2)}{(t-\tau)^{m-i+3}} + O(1); \tag{7.34}$$

$$\frac{\frac{\partial}{\partial t} w_m}{w_m} \frac{\partial}{\partial \tau} w_i(t,\tau) = -\frac{m-i+1}{(t-\tau)^{m-i+3}} + \frac{(m-i+1)\Phi_m(t,\tau)}{(t-\tau)^{m-i+2}} + O\left(\frac{1}{t-\tau}\right) \tag{7.35}$$

$$w_m^2(t,\tau)w_i(t,\tau) = \frac{1}{(t-\tau)^{m-i+3}} + \frac{2\varphi_m(t,\tau)}{(t-\tau)^{m-i+2}} + \frac{\varphi_m^2(t,\tau)}{(t-\tau)^{m-i+1}} + O\left(\frac{1}{(t-\tau)^2}\right) \tag{7.36}$$

Therefore the lefthand side of (7.23) can be written in the following form

$$\frac{m^2-(m-i+1)^2}{(t-\tau)^{m-i+3}} + \frac{1}{(t-\tau)^{m-i+2}}\Big(-(m-i+1)\Phi_m(t,\tau) + 2m^2\varphi_m(t,\tau)\Big) + \\ \frac{m^2\varphi_m^2(t,\tau)}{(t-\tau)^{m-i+1}} + O\left(\frac{1}{(t-\tau)^2}\right) \tag{7.37}$$

Comparing coefficients of (7.32) and (7.37) we have

1. If $m-j+2 > m-i+3$, i.e., $j < i-1$, then $\alpha_{j,i}(\tau) \equiv 0$. This completes the proof of the first part of the lemma;

2. If $m-j+2 = m-i+3$, i.e., $j = i-1$, then $\alpha_{i-1,i}(\tau)(m-i+1) = m^2-(m-i+1)^2$. This completes the proof of the second part of the lemma;

3. If $2 < m-j+2 < m-i+3$, i.e., $i-1 < j < m$, then, taking into account that $\alpha_{k,i}(\tau) \equiv 0$ for $k < i-1$, we obtain

$$(m-j)\alpha_{j,i}(\tau) + \sum_{k=i-1}^{j-1} c_{j-k-1}(\tau)\alpha_{k,i}(\tau) = -(m-i+1)c_{j-i}(\tau) + \frac{2m^2 \frac{\partial^{j-i}}{\partial t^{j-i}}\varphi_m(t,\tau)\Big|_{t=\tau}}{(j-i)!} + \\ \frac{m^2 \frac{\partial^{j-i-1}}{\partial t^{j-i-1}}\varphi_m^2(t,\tau)\Big|_{t=\tau}}{(j-i-1)!} \tag{7.38}$$

By (7.26) coefficient $c_n(\tau)$ can be expressed by $\frac{\partial^k}{\partial t^k}\varphi_m(t,\tau)\Big|_{t=\tau}$ with $0 \le k \le n$. This together with (7.38) completes the proof of the third part of the lemma. □

By the previous lemma the equation (7.24) can be written in the following form for $2 \le i \le m$

$$\frac{\partial Y_{i-1}}{\partial t} = -\frac{1}{\alpha_{i-1,i}}\left(\frac{\partial^2 Y_i}{\partial t \partial \tau} + \left(\frac{\frac{\partial w_m}{\partial \tau}}{w_m}\right)\frac{\partial Y_i}{\partial t} + \left(\frac{\partial^2}{\partial t\partial \tau}(\ln w_m) + m^2 w_m^2\right) Y_i + \sum_{j=i}^{m-1}\frac{\partial Y_j}{\partial t}\alpha_{j,i}\right) \tag{7.39}$$

where $\alpha_{i-1,i} = \frac{(i-1)(2m-i+1)}{m-i+1}$. All terms in the righthand side of (7.39) depends on the functions $Y_j(t,\tau)$ with $j \ge i$. Note also that by (7.25)

$$Y_i(t,\tau) = \frac{w_i(t,\tau)}{w_m(t,\tau)} = \frac{1}{(t-\tau)^{m-i}}\frac{1+(t-\tau)^{m-i+1}\varphi_i(t,\tau)}{1+(t-\tau)\varphi_m(t,\tau)} \tag{7.40}$$

It implies that in the Laurent expansion at $t = \tau$ of the function $t \mapsto Y(t,\tau)$ all coefficients that correspond to nonpositive powers (and in particular the free term) depend on $w_m(t,\tau)$. This together with (7.39) yields that all $Y_i(t,\tau)$ (and therefore all $w_i(t,\tau)$) with $1 \le i \le m-1$ can be expressed by $w_m(t,\tau)$. But by Remark 5 the components $w_i(t,\tau)$, $1 \le i \le m$ determine the curve $\Lambda(t)$ uniquely, up to a symplectic transformation. This completes the proof of the Theorem 1. □

Now our goal is to find a complete system of symplectic invariants of curve $\Lambda(t)$ of rank 1 and the constant weight, i.e., some set of functions of t which determines $\Lambda(t)$ uniquely, up to a symplectic transformation. By Theorem 1 it is natural to look for a complete system of invariants among coefficients $\beta_{i,j}(t)$ of the expansion (4.16) of g in the Taylor series. Since $\Lambda(t)$ can be described, up to symplectic transformation, by the curve $t \to w(t,\tau)$ of the vectors on the linear space of dimension m, it is natural to expect that complete system of invariants of $\Lambda(\cdot)$ consists of m functions of t. By Lemma 4.1 the first m "independent" coefficients in expansion (4.16) are $\beta_{0,2i}(t)$ with $0 \le i \le m-1$. All this arguments lead to the following theorem:

Theorem 2 *The coefficients $\beta_{0,2i}(t)$, $0 \le i \le m-1$, determine the curve $\Lambda(t)$ of rank 1 and a constant weight uniquely, up to a symplectic transformation.*

Let a function $\varphi_m(t,\tau)$ be as in (7.25). From the identity (7.4) it follows easily that the theorem is equivalent to the following theorem:

Theorem 2′. *The functions $\tau \mapsto \left.\frac{\partial^{2i-1}\varphi_m(t,\tau)}{\partial t^{2i-1}}\right|_{t=\tau}$, $1 \le i \le m$, determine the curve $\Lambda(t)$ of rank 1 and a constant weight uniquely, up to a symplectic transformation.*

Proof of Theorem 2′. Let functions $\varphi_i(t,\tau)$, $1 \le i \le m$, be as in (7.25). First, using the system of equation (7.23), we prove the following lemma:

Lemma 7.4 *Any partial derivatives of the functions $\varphi_i(t,\tau)$,$1 \le i \le m$, at any diagonal points (τ,τ) can be expressed by the functions $\tau \mapsto \frac{\partial^{2j-1}\varphi_m}{\partial t^{2j-1}}(t,\tau)\big|_{t=\tau}$ and their derivatives , where $1 \le j \le m$.*

Proof. First, it is natural to make the change of coordinates $x = t - \tau$, $y = t + \tau$ such that the diagonal $t = \tau$ becomes the axis $x = 0$ in the new coordinates. Indeed, if we denote by $z_i(x,y) = w_i(\frac{x+y}{2}, \frac{y-x}{2})$, then the system (7.23) can be transformed into the following system w.r.t. z_i:

$$-z_m\left(\frac{\partial^2 z_i}{\partial x^2} - \frac{\partial^2 z_i}{\partial y^2}\right) - \left(\frac{\partial z_m}{\partial x} + \frac{\partial z_m}{\partial y}\right)\left(\frac{\partial z_i}{\partial y} - \frac{\partial z_i}{\partial x}\right) + m^2 z_m^3 z_i =$$

$$\sum_{j=i-1}^{m-1}\left(\left(\frac{\partial z_m}{\partial x}+\frac{\partial z_m}{\partial y}\right)z_j-\left(\frac{\partial z_j}{\partial x}+\frac{\partial z_j}{\partial y}\right)z_m\right)\alpha_{ji},\ 1\le i\le m \tag{7.41}$$

(here we also have used the first part of the Lemma 7.3). Relations (7.25) can be transformed into the following

$$z_i(x,y)=\frac{1}{x^{m-i+1}}u_i(x,y), \tag{7.42}$$

where the functions $u_i(x,y)$ are smooth, $u_i(0,y)\equiv 1$, and $\frac{\partial^k}{\partial x^k}u_i(0,y)=0$ for $1\le i\le m$, $1\le k\le m-i$. Substitute (7.42) in (7.41) and multiply both sides on x^{m-i+4}. Then we obtain some singular system of equations w.r.t. u_i. By direct calculation it can be shown that this system has the following form:

$$\begin{aligned}&x^2u_m\tfrac{\partial^2 u_i}{\partial x^2}-(2m-2i+1)xu_m\tfrac{\partial u_i}{\partial x}+(m-i+1)xu_i\tfrac{\partial u_m}{\partial x}+\alpha_{i-1,i}xu_{i-1}\tfrac{\partial u_m}{\partial x}-\\&\alpha_{i-1,i}xu_m\tfrac{\partial u_{i-1}}{\partial x}+(m-i+1)^2u_mu_i+\alpha_{i-1,i}(m-i+1)u_{i-1}u_m-m^2u_m^3u_i=\Psi_i,\end{aligned} \tag{7.43}$$

where

$$\begin{aligned}\Psi_i=&\,x^2u_m\frac{\partial^2 u_i}{\partial y^2}+xu_m\frac{\partial u_i}{\partial y}-(m-i+1)xu_i\frac{\partial u_m}{\partial y}-\alpha_{i-1,i}xu_{i-1}\frac{\partial u_m}{\partial y}+\\&\alpha_{i-1,i}xu_m\frac{\partial u_{i-1}}{\partial y}-x^2\left(\frac{\partial u_m}{\partial x}+\frac{\partial u_m}{\partial y}\right)\left(\frac{\partial u_i}{\partial y}-\frac{\partial u_i}{\partial x}\right)-\sum_{j=i}^{m-1}\left(x^{j-i+2}\left(\frac{\partial u_m}{\partial x}+\frac{\partial u_m}{\partial y}\right)u_j-\right.\\&\left.x^{j-i+2}\left(\frac{\partial u_j}{\partial x}+\frac{\partial u_j}{\partial y}\right)u_m+(m-j)x^{j-i+1}u_mu_j\right)\alpha_{ji}\end{aligned} \tag{7.44}$$

The lefthand side of equation (7.43) is a principal part of this equation in the following sense: Differentiate both sides of (7.43) k times in x at the points of the initial curve $x=0$. Then the righthand side can be expressed by the partial derivatives $\frac{\partial^n u_p}{\partial x^n}(0,y)$ with n **less than** k and their derivatives w.r.t. y (here one can take $i-1\le p\le m$), while any term of the lefthand side (at least for $k\ge 2$) depends also on partial derivative of some u_j w.r.t. x of **order** k at $(0,y)$. Moreover, using that $u_i(0,y)\equiv 1$, $1\le i\le m$, and $\alpha_{i-1,i}=\frac{(2m-i+1)(i-1)}{m-i+1}$, one can easily obtain in this way the following linear system w.r.t. $\frac{\partial^k}{\partial x^k}u_i(0,y)$, $1\le i\le m$ for a given integer $k\ge 0$:

$$\zeta_i(k)\frac{\partial^k u_{i-1}}{\partial x^k}(0,y)+\eta_i(k)\frac{\partial^k u_i}{\partial x^k}(0,y)+\theta_i(k)\frac{\partial^k u_m}{\partial x^k}(0,y)=\widetilde{\Psi}_i,\ \ 1\le i\le m, \tag{7.45}$$

where

$$\begin{aligned}\zeta_i(k)&=\frac{(k+i-m-1)(i-2m-1)(i-1)}{m-i+1}\\ \eta_i(k)&=(k+i-1)(k+i-2m-1)\\ \theta_i(k)&=\frac{k+2i-2-2m}{m-i+1}m^2,\end{aligned} \tag{7.46}$$

and $\widetilde{\Psi}_i$ can be expressed by the partial derivatives of the form $\frac{\partial^n u_p}{\partial x^n}(0,y)$ with n **less than** k and their derivatives w.r.t. y (here $i-1 \le p \le m$).

It turns out that the determinant of the system (7.45) satisfies the following remarkable identity:

$$\begin{vmatrix} \eta_1(k) & 0 & 0 & \dots & 0 & 0 & \theta_1(k) \\ \zeta_2(k) & \eta_2(k) & 0 & \dots & 0 & 0 & \theta_2(k) \\ 0 & \zeta_3(k) & \eta_3(k) & \dots & 0 & 0 & \theta_3(k) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \zeta_{m-1}(k) & \eta_{m-1}(k) & \theta_{m-1}(k) \\ 0 & 0 & 0 & \dots & 0 & \zeta_m(k) & \eta_m(k)+\theta_m(k) \end{vmatrix} = \prod_{j=1}^{m}(k-2j)(k+2j-1) \quad (7.47)$$

The proof of (7.47) that we have found is rather long and will be presented in Appendix.

As a consequence of (7.47) we obtain that the determinant of the system (7.45) has exactly m positive zeros at $k=2j$, $1 \le j \le m$. Therefore, any partial derivative of u_i, $1 \le i \le m$, at $(0,y)$ can be expressed by the functions $y \mapsto \frac{\partial^{2j} u_p}{\partial x^{2j}}(0,y)$ and their derivatives, where $1 \le j, p \le m$. Moreover, by Theorem 1 $u_p(x,y)$ can be expressed by $u_m(x,y)$ and its derivative. Hence any partial derivative of u_i, $1 \le i \le m$, at $(0,y)$ can be expressed by the functions $y \mapsto \frac{\partial^{2j} u_m}{\partial x^{2j}}(0,y)$ and their derivatives, where $1 \le j \le m$. But this is equivalent to the statement of our Lemma, if we return to the old coordinates t and τ. $\square$

Now we define a canonical moving frame: For given τ take the derivative subspace $\Lambda^0(\tau)$ and let $f_1(\tau), \dots, f_m(\tau)$ be a basis of $\Lambda^0(\tau)$ dual to the canonical basis of $\Lambda(\tau)$ (i.e., $\sigma(f_i(\tau), e_j(\tau)) = \delta_{i,j}$). The basis $(e_1(\tau), \dots, e_m(\tau), f_1(\tau), \dots, f_m(\tau))$ of whole symplectic space W is called *the canonical moving frame* of the curve $\Lambda(\cdot)$. Denote by $E(\tau)$ and $F(\tau)$ the tuples of vectors $(e_1(\tau), \dots, e_m(\tau))$ and $(f_1(\tau), \dots, f_m(\tau))$ correspondingly , arranged in the columns. Fixing the bases $E(\tau)$ and $F(\tau)$ in $\Lambda(\tau)$ and $\Lambda^0(\tau)$ respectively, let S_t be the matrix, corresponding to the linear mapping $\langle \Lambda(\tau), \Lambda(t), \Lambda^0(\tau) \rangle$ and S_t^0 be the matrix, corresponding to the linear mapping $\langle \Lambda^0(\tau), \Lambda^0(t), \Lambda(\tau) \rangle$ (see section 2 for notations). Also, let $\Omega(\tau)$ be a $m \times m$ matrix with (i,j) entry equal to $\alpha_{i,j}(\tau)$, where $\alpha_{i,j}(\tau)$ is defined by (7.19) with $\Delta = \Lambda^0(\tau)$. Then it is easy to see that the structural equation for the canonical moving frame has the following form:

$$\begin{pmatrix} \dot{E}(\tau) \\ \dot{F}(\tau) \end{pmatrix} = \begin{pmatrix} \Omega(\tau) & \dot{S}_\tau \\ \dot{S}_\tau^0 & -\Omega^T(\tau) \end{pmatrix} \begin{pmatrix} E(\tau) \\ F(\tau) \end{pmatrix}. \quad (7.48)$$

We claim that in order to prove Theorem $2'$ it is sufficient to prove the following lemma:

Lemma 7.5 *The matrix in the structural equation (7.48) depends only on the coefficients of the expansions of $t \to w_i(t,\tau)$, $1 \le i \le m$, in the Laurent series at $t=\tau$.*

Indeed, if Lemma 7.5 holds then first by Lemma 7.4 this matrix depends only on the functions $\tau \mapsto \frac{\partial^{2j-1}\varphi_m}{\partial t^{2j-1}}(t,\tau)\big|_{t=\tau}$, $1 \le j \le m$, secondly, the structural equation (7.48) has a unique solution with prescribed initial condition, and, finally, any symplectic basis can be taken as an initial condition of (7.48).

Proof of Lemma 7.5. First, according to (7.3)

$$(\dot S_\tau))_{i,j} = \begin{cases} 0 & (i,j) \neq (m,m) \\ m^2 & (i,j) = (m,m), \end{cases} \tag{7.49}$$

Further, by recursive formula (2.4) for $i = 0$:

$$\dot S^0_\tau = \frac{d}{d\tau}A_0(\tau) = A_1(\tau) + \sum_{n=1-2m}^{-1}\Big(A_n(\tau)\dot S_\tau A_{-n}(\tau) + A_{-n}(\tau)\dot S_\tau A_n(\tau)\Big), \tag{7.50}$$

where $A_j(\tau)$ are defined by expansion (2.3) (here we have used that by definition of the derivative curve $A_0(\tau) = 0$ and by Lemma 6.1 the order of pole of $t \mapsto (S_t - S_\tau)^{-1}$ at $t = \tau$ is equal to $2m-1$). By definition of the vectors $w(t,\tau)$, we have

$$\Big((S_t - S_\tau)^{-1}\Big)_{i,j} = -\int^t w_i(\xi,\tau)w_j(\xi,\tau)d\xi.$$

Therefore any $A_n(\tau)$ with $n \neq 0$ can be expressed by the coefficients of the expansions of $t \to w_i(t,\tau)$, $1 \le i \le m$, in the Laurent series at $t = \tau$. This together with (7.49) and (7.50) implies that $\dot S^0_\tau$ can be expressed by the coefficients of the expansions of $t \to w_i(t,\tau)$, $1 \le i \le m$, in the Laurent series at $t = \tau$.

Finally, let us analyze the matrix $\Omega(\tau)$. By Lemma 7.3 all its entries $\alpha_{i,j}(\tau)$ with $1 \le i \le m-1$ can be expressed by the coefficients of the expansions of $t \to w_m(t,\tau)$ in the Laurent series at $t = \tau$. The entries $\alpha_{m,j}(\tau)$ do not enter the differential equation (7.23). To find an expression for these entries we will use the integral-differential equation (7.12) that can be rewritten for $\Delta = \Lambda_0(\tau)$ in the following form:

$$\frac{\partial w^{\Lambda_0(\tau)}(t,\tau_1)}{\partial \tau_1}\Big|_{\tau_1=\tau} = -m^2 w_m(t,\tau)\int^t w_m(\xi,\tau)w(\xi,\tau)d\xi \tag{7.51}$$

Using (7.20), we can obtain from here the following system of equation w.r.t. the components $w_j(t,\tau)$

$$\frac{\partial w_j(t,\tau)}{\partial \tau} + \sum_{l=1}^m w_l(t,\tau)\alpha_{lj}(\tau) = -m^2 w_m(t,\tau)\int^t w_m(\xi,\tau)w_j(\xi,\tau)d\xi,\ \ 1 \le j \le m \tag{7.52}$$

For given j consider the Laurent expansion of the lefthand side of (7.51), as a function of t, at $t = \tau$. By (7.25) the coefficient of $\frac{1}{t-\tau}$ in this expansion is equal to $\alpha_{m,j}(\tau)$. On the other hand, all coefficients of the appropriate expansion of the righthand side can be expressed by coefficients of expansions of $t \to w_j(t,\tau)$ and $t \to w_m(t,\tau)$ in the Laurent series at $t = \tau$. Therefore also the entries $\alpha_{m,j}(\tau)$ can be expressed by coefficients of expansions of $t \to w_i(t,\tau)$ (even with $i = j$ or m) in the Laurent series at $t = \tau$. This concludes the proof of our Lemma and also of Theorem $2'$.$\Box$

8 Flat curves

In the present section we introduce and investigate the "straightest" curves of the given constant rank in the Lagrange Grassmannian. We call them *flat curves*. Basic symplectic invariants of a curve measure its deviation from the flat curve and thus play the role of curvatures.

First let us give a general definition of the flat curves which does not involve any specific information about the rank. Let, as before, W be the $2m$-dimensional linear symplectic space. Let $\Lambda : I \mapsto L(W)$ be an ample curve of the constant weight, defined on the interval $I \subseteq \mathbb{R}$. As in section 2 for given τ we denote by $\Lambda_\tau(t)$ the identical embedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$. Fixing an "origin" in $\Lambda(\tau)^{\pitchfork}$ we make $\Lambda_\tau(t)$ a vector function with values in the space of self-adjoint linear mappings from $\Lambda(\tau)^*$ to $\Lambda(\tau)$. Such a vector function has the pole at $t = \tau$. Assume that the order of the pole at $t = \tau$ is equal to $l(\tau)$.

Definition 4 *The curve $\Lambda : I \mapsto L(W)$ of the constant (finite) weight is called flat at the point $\tau \in I$, if the function $t \mapsto \Lambda_\tau(t)$ satisfies*

$$\Lambda_\tau(t) = \Lambda^0(\tau) + \sum_{i=-l(\tau)}^{-1} Q_i(\tau)(t-\tau)^i. \tag{8.1}$$

Definition 5 *The curve $\Lambda : I \mapsto L(W)$ of the constant (finite) weight is called flat if it is flat at any point of I.*

First, consider the regular curves (recall that the curve $\Lambda(\cdot)$ is called regular, if the velocity $\dot\Lambda(t)$ is nondegenerated quadratic form for any t, see section 3 for the details). In this case the function $t \mapsto \Lambda_\tau(t)$ has the simple pole at $t = \tau$ for any τ . Then, by definition, the regular curve $\Lambda(\cdot)$ is flat at τ iff

$$\Lambda_\tau(t) = \Lambda^0(\tau) + Q_{-1}(\tau)\frac{1}{t-\tau}. \tag{8.2}$$

The coordinate version of (8.2) is

$$(S_t - S_\tau)^{-1} = A_0(\tau) + \frac{\dot S_\tau^{-1}}{t-\tau}, \tag{8.3}$$

where $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$ (see formula (3.2)). Suppose that the positive index of $\dot\Lambda(t)$ is equal to p (this index is constant, since the rank is constant). Denote by $I^{\pm}_{p,s}$ the following matrix

$$I^{\pm}_{p,s} = \begin{pmatrix} I_p & 0 \\ 0 & -I_s \end{pmatrix}, \tag{8.4}$$

were I_n is $n \times n$-identity matrix. One can choose the coordinates such that $S_\tau = 0$, $A_0(\tau) = 0$, and $\dot{S}_\tau = I^{\pm}_{p,m-p}$. Substituting this in (8.3), we obtain the following

Lemma 8.1 *If the regular curve $\Lambda : I \in \mathbb{R} \mapsto L(W)$ with the positive index p of $\dot{\Lambda}(t)$ is flat at some $\tau \in I$, then the curve $\Lambda(t)$ has the coordinate representation S_t, $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$ such that*

$$S_t = I^{\pm}_{p,m-p}(t - \tau). \tag{8.5}$$

The previous lemma implies the following

Proposition 5 *If the regular curve $\Lambda : I \in \mathbb{R} \mapsto L(W)$ with the positive index p of $\dot{\Lambda}(t)$ is flat at some $\tau \in I$, then it is flat. Moreover, the curve $\Lambda(t)$ has the coordinate representation S_t, $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$ such that*

$$S_t = I^{\pm}_{p,m-p} t. \tag{8.6}$$

Proof. The formula (8.5) defines an extension $\bar{\Lambda}(t)$ of $\Lambda(t)$ to whole $\mathbb{R}$. Also, from (8.5) it follows that for any $\bar{\tau} \in \mathbb{R}$

$$(S_t - S_{\bar{\tau}})^{-1} = I^{\pm}_{p,m-p}\frac{1}{(t - \bar{\tau})},$$

i.e., the curve $\bar{\Lambda}(t)$ is flat at any $\bar{\tau} \in \mathbb{R}$. In particular, $\bar{\Lambda}(t)$ is flat at 0. So, by Lemma 8.1, the curve $\Lambda(t)$ has the coordinate representation S_t, satisfying (8.6).□

Remark 8 *Recall that the transition functions from one chart to another in the Lagrange Grassmannian $L(W)$ are "matrix Möbius transformations" of the type $S \mapsto (C + DS)(A + BS)^{-1}$ with*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2m). \tag{8.7}$$

This together with (8.6) implies that the regular curve $\Lambda : I \in \mathbb{R} \mapsto L(W)$ with the positive index p of $\dot{\Lambda}(t)$ is flat iff any its coordinate representation S_t, $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$ has the following form

$$S_t = (C + DI^{\pm}_{p,m-p}t)(A + BI^{\pm}_{p,m-p}t)^{-1}, \tag{8.8}$$

where $m \times m$ matrices A,B,C, and D satisfy (8.7).

Evidently, the direct sum of flat curves is a flat curve. Namely, suppose that $W = W_1 \oplus W_2$, where W_i, $i = 1, 2$, are symplectic spaces such that symplectic forms on each W_i are restrictions of the symplectic form on W. If curves $\Lambda_i : I \mapsto L(W_i)$ are flat, then the curve

$$\Lambda : I \mapsto L(W), \;\; \Lambda(t) = \Lambda_1(t) \oplus \Lambda_2(t),$$

is flat. So, one can construct flat curves of arbitrary rank $r < m$ by taking direct sum of rank 1 flat curves.

Now we will give models for rank 1 flat curves. Let $\mathcal{V}_m(t)$ be the vector in $\mathbb{R}^m$ with i-th component $(\mathcal{V}_m(t))_i = t^{i-1}$. Let $F_m(t)$ be $m \times m$ symmetric matrix such that

$$F_m(t) = \int_0^t \mathcal{V}_m(\xi)^T \mathcal{V}_m(\xi)\, d\xi = \left\{ \frac{t^{i+j-1}}{i+j-1} \right\}_{i,j=1}^m \tag{8.9}$$

Proposition 6 *If the curve $\Lambda : I \mapsto L(W)$ in some coordinates has the form*

$$\Lambda(t) = \{(x, \pm F_m(t)x) : \ x \in \mathbb{R}^m\}, \tag{8.10}$$

then $\Lambda(\cdot)$ is flat.

Proof. Evidently, it is sufficient to prove the Proposition in the case, when the sign $+$ appears in (8.10). By construction, $\Lambda(\cdot)$ is a rank 1 curve. So, its weight is not less than m^2 (see, for example, Lemma 6.1). On the other hand, by (8.9) the function $(t, \tau) \mapsto \det(F_m(t) - F_m(\tau)$ is a polynomial of order not greater than m^2. It implies that the weight of $\Lambda(\cdot)$ at any point is equal to m^2 and by definition of the weight the function $X(t,\tau) = \frac{\det(F_m(t) - F_m(\tau))}{(t-\tau)^{m^2}}$ is smooth. Being the fraction of two polynomials of the same degree, $X(t,\tau)$ has to be constant, i.e.,

$$det(F_m(t) - F_m(\tau)) = C(t-\tau)^{m^2}. \tag{8.11}$$

Besides, it is clear that all entries of adjacency matrix of the matrix $F_m(t) - F_m(\tau)$ are polynomials of degree less than m^2. This together with (8.11) implies that for any τ the matrix function $t \mapsto (F_m(t) - F_m(\tau))^{-1}$ coincides with the principal part of its Laurent expansion at τ. But this is exactly the coordinate version of (8.1), which implies that $\Lambda(\cdot)$ is flat. $\Box$

Definition 6 *The curve $\Lambda : I \mapsto L(W)$ is called strong flat if it can be represented as a direct sum of the curve of the form (8.10).*

Later on we will prove that the rank 1 flat curves of the form (8.10) are the only flat curves of rank 1, up to symplectic transformation. So, the strong flat curve are the curves, which can be represented as a direct sum of r rank 1 flat curves.

Remark 9 *From (8.6) it follows that if the regular curve is flat then it is strong flat.*

It is not clear yet, whether in general any flat curve is strong flat (this is the part of our general conjecture formulated below).

Now we will prove three general propositions related to the flat curves

Proposition 7 *Let $\Lambda : I \mapsto L(W)$ be a flat curve of weight k. Then the function $g(t,\tau)$ is identically equal to zero for any $t,\tau \in I$, or ,equivalently, the following remarkable identity*

$$det\Big([\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)]\Big) = [t_0, t_1, t_2, t_3]^k \tag{8.12}$$

holds for any four parameters $t_0, t_1, t_2, t_3 \in I$;

Proof. From Lemma 4.2 it follows (see formulas (4.19) and (4.23)) that

$$\mathrm{tr}\left(\frac{\partial}{\partial t}\Lambda_\tau(t) \circ \dot{\Lambda}(\tau)\right) = -\frac{k}{(t-\tau)^2} - g(t,\tau) \tag{8.13}$$

On the other hand, by assumption, relation (8.1) is valid for all $\tau \in I$. Therefore

$$\mathrm{tr}\left(\frac{\partial}{\partial t}\Lambda_\tau(t) \circ \dot{\Lambda}(\tau)\right) = \sum_{i=-l-1}^{-2} (i+1)\mathrm{tr}\left(Q_{i+1}(\tau) \circ \dot{\Lambda}(\tau)\right)(t-\tau)^i. \tag{8.14}$$

This and (8.13) implies that

$$g(t,\tau) = 0 \ \ \forall\, t,\tau \in I. \tag{8.15}$$

The proof of the proposition is completed.□

Proposition 8 *Let $\Lambda : I \mapsto L(W)$ be a flat curve. Then the derivative curve $\Lambda^0(\cdot)$ of the curve $\Lambda(\cdot)$ is constant, i.e.,*

$$\dot{\Lambda}^0(\tau) = 0 \quad \forall \tau \in I. \tag{8.16}$$

Proof. Take $\bar{\tau} \in I$. Let S_t, $\Lambda(t) = \{(x, S_t\, x) : x \in \mathbb{R}^m\}$, be a coordinate representation of the germ of $\Lambda(t)$ at $t = \bar{\tau}$ such that $\Lambda^0(\bar{\tau}) = 0 \oplus \mathbb{R}^n$. By assumptions,

$$(S_t - S_\tau)^{-1} = \sum_{i=-l}^{0} A_i(\tau)(t-\tau)^i \tag{8.17}$$

and by construction

$$A_0(\bar{\tau}) = 0. \tag{8.18}$$

Differentiating both sides of (8.17) w.r.t. τ at $\tau = \bar{\tau}$, comparing free terms of the corresponding expansions and using (8.18), we get $\dot{A}_0(\bar{\tau}) = 0$ (see also recursive formula (2.4) in the case $i = 0$). But this means that $\dot{\Lambda}^0(\bar{\tau}) = 0$. □

From Propositions 6, 7, and 8 it follows immediately that

Proposition 9 *For strong flat curves $\Lambda : I \mapsto L(W)$ identities (8.12) and (8.16) are valid.*

Now we will prove that for the regular and the rank 1 curves all properties of flat curves mentioned above are equivalent.

Theorem 3 *Let curve $\Lambda : I \mapsto L(W)$ be the regular curve or the rank 1 curve of the constant (finite) weight. Then the following five statements are equivalent:*

1. *The curve $\Lambda(\cdot)$ is flat at some $\tau \in I$;*

2. *The curve $\Lambda(\cdot)$ is flat;*

3. *The function $g(t,\tau)$ is identically equal to zero for any $t, \tau \in I$, or ,equivalently, the following remarkable identity*

$$det\Big([\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)]\Big) = [t_0, t_1, t_2, t_3]^k \tag{8.19}$$

   holds for any four parameters $t_0, t_1, t_2, t_3 \in I$, where k is the weight of the curve ($k = \frac{1}{2}\dim W$ in the regular case and $k = \left(\frac{1}{2}\dim W\right)^2$ in the rank 1 case);

4. *The derivative curve $\Lambda^0(\cdot)$ of the curve $\Lambda(\cdot)$ is constant, i.e.,*

$$\dot{\Lambda}^0(t) = 0 \quad \forall\, t \in I; \tag{8.20}$$

5. *The curve $\Lambda(\cdot)$ is strong flat.*

Before starting to prove the Theorem 3 we give several its direct consequences. As in [6] , let

$$\beta_{0,i}(\tau) = \frac{1}{i!}\frac{\partial^i g}{\partial \tau^i}(t,\tau)|_{t=\tau} \tag{8.21}$$

Recall that Theorem 2 states that the coefficients $\beta_{0,2k}(\tau)$, $0 \leq k \leq m-1$, constitute a complete system of symplectic invariants of the curve $\Lambda(\tau)$ of rank 1 and the constant weight, i.e., determine $\Lambda(\tau)$ uniquely, up to a symplectic transformation.

Corollary 3 *The rank 1 curve $\Lambda : I \mapsto L(W)$ of the constant weight is flat iff all functions $\beta_{0,2k}(\tau)$ with $0 \leq k \leq m-1$ are identically equal to zero, i.e., rank 1 flat curves are the curves with vanishing complete system of symplectic invariants.*

Proof. Indeed, if the curve $\Lambda(\cdot)$ is flat then by Theorem 3 $g(t,\tau) \equiv 0$, which trivially implies that $\beta_{0,2k}(\tau) \equiv 0$, $0 \leq k \leq m-1$. On the other hand, by Proposition 9 and Theorem 2 the strong flat curves are the only curve with $\beta_{0,2k}(\tau) \equiv 0$, $0 \leq k \leq m-1$, which proves sufficiency. $\square$

Remark 10 *By Corollary 3, the invariants $\beta_{0,i}(\tau)$ play role of curvatures: the curve is flat iff all these invariants vanish.*

Corollary 4 *There is an embedding of the real projective line $\mathbb{RP}^1$ into $L(W)$ as a flat closed curve of rank 1 endowed with the canonical projective structure; Maslov index of this curve equals m. All other flat curves of rank 1 are images under symplectic transformations of $L(W)$ of the segments of this unique one.*

Proof. By Theorem 3 any rank 1 flat curve $\Lambda : I \mapsto L(W)$ is strong flat, i.e., in some coordinates it satisfies (8.10). The equation (8.10) defines the flat extension $\bar{\Lambda}(\cdot)$ of $\Lambda(\cdot)$ to whole $\mathbb{R}$. By Proposition 9 the derivative curve of $\bar{\Lambda}(\cdot)$ is constant. Suppose that $\bar{\Lambda}^0(t) \equiv \Lambda^0$. From the proof of Proposition 6 it follows immediately that $\Lambda^0 = 0 \oplus \mathbb{R}^m$. This and the fact that $\lim_{t\to\infty} F_m(t) = \infty$ implies that

$$\lim_{t\to\infty} \Lambda(t) = \Lambda^0, \tag{8.22}$$

i.e., the curve $\bar{\Lambda}$ can be extended continuously to the (closed) curve define on $\mathbb{RP}^1$. We will denote this extension also by $\bar{\Lambda}(\cdot)$. Note that from definition it follows easily that if $\Lambda : I \mapsto L(W)$ is flat curve then any its reparametrization by Möbius transformation is also flat, i.e., for any Möbius transformation $\varphi(\tau)$ the curve $\tau \mapsto \Lambda(\varphi(\tau))$ is flat on $\varphi^{-1}(I)$. In particular, the curve $t \mapsto \bar{\Lambda}(\frac{1}{t})$ is flat and so, by Theorem 3, is also strong flat on $\mathbb{R}\backslash 0$. This implies that $\bar{\Lambda}$ defines smooth embedding of $\mathbb{RP}^1$ into $L(W)$. Evidently, Maslov index of this embedding equals m. Note also that the original parameter on $\bar{\Lambda}$ is projective w.r.t. the canonical projective structure on $\bar{\Lambda}(\cdot)$, since by Corollary 3 the Ricci curvature of rank 1 flat curve is equal to zero. This completes the proof. $\square$

Since any strong flat curve is a direct sum of rank 1 flat curves, we have also

Corollary 5 *There is an embedding of the real projective line $\mathbb{RP}^1$ into $L(W)$ as a strong flat closed curve endowed with the canonical projective structure; Maslov index of this curve equals m. All other strong flat curves are images under symplectic transformations of $L(W)$ of the segments of this unique one.*

Remark 11 *Evidently statement 2 of Theorem 1 implies statement 1. Proposition 7 means that $2 \Rightarrow 3$. Proposition 8 means that $2 \Rightarrow 4$. Proposition 9 means that $5 \Rightarrow 2$. Theorem 3 will be proved, if one proves, for example, that $1 \Rightarrow 5$, $4 \Rightarrow 2$ (or 5) and $3 \Rightarrow 2$.*

Proof of Theorem 1.

1. The case of regular curves. Recall that the curvature operator $R(t) : \Lambda(t) \to \Lambda(t)$ of the curve $\Lambda(\cdot)$ is defined by:

$$R(t) = -\dot{\Lambda}^0(t) \circ \dot{\Lambda}(t) \tag{8.23}$$

By (8.23) and Proposition 8, for any flat curve $\Lambda(t)$ the curvature operator is identically equal to zero. For the regular curves the opposite statement is true:

Lemma 8.2 *If the curvature operator of the regular curve is identically equal to zero, then the curve is flat.*

Proof. According to the section 3 of [6] (see also [1]) for any coordinate representation S_t of the curve $\Lambda(t)$ the matrix

$$\mathbb{S}(S_t) = \frac{d}{dt}\Big((2\dot{S}_t)^{-1}\ddot{S}_t\Big) - \Big((2\dot{S}_t)^{-1}\ddot{S}_t\Big)^2 \tag{8.24}$$

represents the curvature operator $R(t)$ in some basis. So, $R(t) \equiv 0$ iff $\mathbb{S}(S_t) \equiv 0$. Note that $\mathbb{S}(S_t)$ is a matrix analog of Schwarz derivative and it has similar property: $\mathbb{S}(S_t) \equiv 0$ iff $S_t = (C + Dt)(A + Bt)^{-1}$, where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2m)$ (see, for example, [21] p. 205). One can suppose from the beginning that $S_\tau = 0$ for some parameter τ. Then $C = -D\tau$ and

$$(S_t - S_\tau)^{-1} = \frac{1}{t-\tau}(A + Bt)D^{-1},$$

which implies flatness of $\Lambda(\cdot)$ at τ and therefore, by Proposition 8, flatness of $\Lambda(\cdot)$. $\square$

Using the previous lemma one can easily prove that $4 \Rightarrow 2$. Indeed, if $\dot{\Lambda}(t)^0 \equiv 0$ then by (8.23) $R(t) \equiv 0$ and then by Lemma 8.2 the curve $\Lambda(\cdot)$ is flat. Also, Remark 9 means that $1 \Rightarrow 5$. By Remark 11, in order to prove Theorem 1 in the regular case it remains to show that $3 \Rightarrow 2$. It turns out that the following even more strong statement is valid:

Lemma 8.3 *If $\Lambda : I \mapsto L(W)$ is the regular curve such that its Ricci curvature and fundamental form are identically equal to zero, then the curve $\Lambda(\cdot)$ is flat.*

Proof. If the Ricci curvature of the regular curve $\Lambda(t)$ is identically equal to zero, then by (4.27) the curvature operator $R(t)$ of $\Lambda(t)$ satisfies

$$\operatorname{tr} R(t) \equiv 0. \tag{8.25}$$

If also the fundamental form of the regular curve $\Lambda(t)$ is identically equal to zero, then substituting (8.25) in (5.12) , we obtain

$$\operatorname{tr}\Big(R(t)^2\Big) \equiv 0 \tag{8.26}$$

Note that by definition $R(t) : \Lambda(t) \mapsto \Lambda(t)$ is linear operator, symmetric w.r.t. (pseudo)-Euclidean structure, defined by quadratic form $\dot{\Lambda}(t)$. Therefore equation (8.26) implies that

$$R(t) \equiv 0.$$

Hence, by Lemma 8.2, the curve $\Lambda(\cdot)$ is flat. $\square$

The proof of Theorem 1 in the case of regular curves is completed.

2. The case of rank 1 curves. Without loss of generality it can be assumed that the curve $\Lambda(\cdot)$ is nondecreasing.

2.1 The proof of implications $3 \Rightarrow 2$ **and** $1 \Rightarrow 5$**.** As in section 6 denote by $w(t,\tau)$ the vector in $\Lambda(\tau)$ such that for any $v \in \Lambda(\tau)^*$

$$\langle v, \frac{\partial}{\partial t}\Lambda_\tau(t)v\rangle = -\langle v, w(t,\tau)\rangle^2. \tag{8.27}$$

Let $w_i(t,\tau)$, be the i-th component of $w(t,\tau)$ w.r.t. the canonical basis $e_1(\tau),\dots e_m(\tau)$ of the subspace $\Lambda(\tau)$ (see formulas (6.2)-(6.5) for the definition of the canonical basis, see also Proposition 4 which corresponds to the case of the curve of the constant rank). From Proposition 4 and Corollary 2 it follows that the functions $w_i(t,\tau)$ have the following form

$$w_i(t,\tau) = \frac{1}{(t-\tau)^{m-i+1}} + \varphi_i(t,\tau), \tag{8.28}$$

where $\varphi_i(t,\tau)$ are smooth functions. Prove the following lemma:

Lemma 8.4 *If $\Lambda : I \mapsto L(W)$ is rank 1 curve, which is flat at $\tau \in I$. Then the functions $t \mapsto w_i(t,\tau)$ satisfy*

$$w_i(t,\tau) = \frac{1}{(t-\tau)^{m-i+1}} \tag{8.29}$$

Proof. Using the same arguments as in the proof of Proposition 7 (formulas (8.13), (8.14)), we get that if $\tau \in I$ is the parameter such that (8.1) holds, then

$$g(t,\tau) = 0 \ \ \forall t \in I. \tag{8.30}$$

Recall that the functions $g(t,\tau)$ and $w_m(t,\tau)$ are related by the following identity (see Lemma 7.1)

$$w_m^2(t,\tau) = \frac{1}{(t-\tau)^2} + \frac{1}{m^2}g(t,\tau) \tag{8.31}$$

Hence by (8.30)

$$w_m(t,\tau) = \frac{1}{t-\tau} \tag{8.32}$$

Let S_t, $\Lambda(t) = \{(x, S_t\, x) : x \in \mathbb{R}^m\}$, be a coordinate representation of germ of $\Lambda(t)$ at $t = \tau$ such that the canonical basis $e_1(\tau),\dots,e_m(\tau)$ constitute a standard basis of $\mathbb{R}^m$. Since $\Lambda(t)$ is flat, we have the following coordinate version of the expansion (8.1)

$$(S_t - S_\tau)^{-1} = \sum_{i=-l}^{0} A_i(\tau)(t-\tau)^i \tag{8.33}$$

Therefore,

$$\frac{\partial}{\partial t}(S_t - S_\tau)^{-1} = \sum_{i=-l-1}^{-2} (i+1)\, A_{i+1}(\tau)(t-\tau)^i \tag{8.34}$$

On the other hand,

$$\left(\frac{\partial}{\partial t}(S_t - S_\tau)^{-1}\right)_{i,j} = -w_i(t,\tau)w_j(t,\tau). \tag{8.35}$$

So,

$$\left(\frac{\partial}{\partial t}(S_t - S_\tau)^{-1}\right)_{m,i} = -\frac{1}{t-\tau}\left(\frac{1}{(t-\tau)^{m-i+1}} + \varphi_i(t,\tau)\right) = -\frac{1}{(t-\tau)^{m-i+2}} - \frac{\varphi_i(t,\tau)}{t-\tau} \tag{8.36}$$

Comparing (8.34) and (8.36) one can easily conclude that $\varphi_i(t,\tau) = 0$ for any $t \in I$ that implies (8.29). The proof of the Lemma is completed. $\Box$

From the proof of the previous lemma it follows that if $g(t,\tau) \equiv 0$, then relations (8.29) are valid for any $t, \tau \in I$, which implies in turn that (8.1) holds for any $\tau \in I$ (see, for example, (8.27) and (8.35)). By this we actually have proved that **statement 3 of Theorem 1 implies statement 2**.

To show that $1 \Rightarrow 5$, we prove first the following

Lemma 8.5 *If $\Lambda : I \mapsto L(W)$ is rank 1 nondecreasing curve, which is flat at $\tau \in I$, then in some coordinates the curve $\Lambda(\cdot)$ has the form*

$$\Lambda(t) = \{(x, F_m(t-\tau)x)\, x \in \mathbb{R}^m\}, \tag{8.37}$$

where $F_m(t)$ is defined by (8.9)

Proof. Take again some coordinate representation S_t, $\Lambda(t) = \{(x, S_t\, x) : x \in \mathbb{R}^m\}$, of germ of $\Lambda(t)$ at $t = \tau$ such that the canonical basis $e_1(\tau), \ldots, e_m(\tau)$ constitute a standard basis of $\mathbb{R}^m$. Then evidently $S_\tau = 0$. By the previous lemma

$$\left((S_t)^{-1}\right)_{i,j} = \frac{1}{(2m-i-j+1)(t-\tau)^{2m-i-j+1}} \tag{8.38}$$

Then, referring to the formulas (6.16) and (6.17) (with $k_i = i-1$ which corresponds to the case of the constant weight), we have

$$(S_t)_{i,j} = \int_\tau^t v_i(\xi)v_j(\xi)\, d\xi, \tag{8.39}$$

where

$$v_i(t) = c_i(t-\tau)^{m-i}, \quad c_i \neq 0. \tag{8.40}$$

It is easy to see that if $\bar{S}_t$, $\Lambda(t) = \{(x, \bar{S}_t\, x) : x \in \mathbb{R}^m\}$, is a coordinate representation of germ of $\Lambda(t)$ at $t = \tau$ such that the basis $(\frac{e_m(\tau)}{c_m}, \frac{e_{m-1}(\tau)}{c_{m-1}}, \dots, \frac{e_1(\tau)}{c_1})$ constitute a standard basis of $\mathbb{R}^m$, then $\bar{S}_t = F_m(t - \tau)$. This completes the proof. $\Box$

So, if $\Lambda : I \mapsto L(W)$ is rank 1 nondecreasing curve, which is flat at $\tau \in I$, then formula (8.37) defines the extension $\bar{\Lambda}(\cdot)$ of $\Lambda(\cdot)$ to whole $\mathbb{R}$. Hence from Proposition 6 it follows easily that for the curve $\bar{\Lambda}(\cdot)$ the relation (8.1) holds for all real τ. So, one can use the previous Lemma for all τ, in particular for $\tau = 0$. Therefore there exists the coordinate representation of $\Lambda(\cdot)$ satisfying (8.10), i.e., the curve $\Lambda(\cdot)$ is strong flat. We have proved that **statement 1 of Theorem 1 implies statement 5**.

2.2 The proof of the implication $4 \Rightarrow 5$**.** According to Remark 11, in order to complete the proof of the Theorem 1 for rank 1 curve it is sufficient to prove that $4 \Rightarrow 5$.

Let $\beta_{0,k}(t)$ be as in (8.21). Note that in order to prove that the curve is strong flat it is sufficient to show that

$$\beta_{0,2i}(t) \equiv 0, \;\; 0 \le i \le m-1 \tag{8.41}$$

Indeed, by Proposition 9 and Theorem 2 the strong flat curves are the only curve, satisfying (8.41). Therefore implication $4 \Rightarrow 5$ follows from the following

Proposition 10 *If the rank 1 curve $\Lambda : I \mapsto L(W)$ of constant weight satisfies*

$$\dot{\Lambda}^0(t) \equiv 0, \tag{8.42}$$

then relations (8.41) are valid.

The rest of the section is devoted to the proof of Proposition 10.

Proof of Proposition 10. Let, as before, $(e_1(\tau), \dots, e_m(\tau))$ be a canonical basis of $\Lambda(t)$ and $(f_1(t), \dots, f_m(\tau))$ be a basis of $\Lambda^0(t)$ dual to $(e(\tau), \dots, e_m(\tau))$ i.e., $\sigma(f_i(\tau), e_i(\tau)) = \delta_{i,j}$ (in section 7 the basis $(e_1(\tau), \dots, e_m(\tau), f_1(\tau), \dots, f_m(\tau))$ of W were called the canonical moving frame associated with the curve $\Lambda(\cdot)$). Let S_t^0 be the coordinate representation of the derivative curve $\Lambda^0(t)$, $\Lambda^0(t) = \{(x, S_t^0 x) : x \in \mathbb{R}^m\}$, such that $(f_1(\tau), \dots, f_m(\tau))$ constitute a standard basis of $\mathbb{R}^m \oplus 0$ and $(e_1(\tau), \dots, e_m(\tau))$ constitute a standard basis of $0 \oplus \mathbb{R}^m$.

It is convenient to introduce the following notation: for given tuple $\{\psi_i(t)\}_{i=1}^N$ of smooth functions on I we will denote by $\mathrm{Pol}\big(\{\psi_i(\tau)\}_{i=1}^N\big)$ any function on I, which can be expressed as a polynomial without free term w.r.t. the functions $\psi_i(\tau)$, $1 \le i \le N$, and their derivatives. We claim that in order to prove (8.41), it is sufficient to prove the following two lemmas:

Lemma 8.6 *The (i,j)th entry $\left(\dot{S}^0_\tau\right)_{i,j}$ of the matrix $\dot{S}^0_\tau$ with even $i+j$ can be represented in the following form*

$$\left(\dot{S}^0_\tau\right)_{i,j} = c_{i,j}\beta_{0,2(m-\frac{i+j}{2})}(\tau) + \mathrm{Pol}\left(\{\beta_{0,2s}(\tau)\}_{s=0}^{m-\frac{i+j}{2}-1}\right), \tag{8.43}$$

where $c_{i,j}$ is some constant.

Lemma 8.7 *The constant $c_{i,j}$ from (8.43) can be chosen such that*

$$c_{m,j} \neq 0, \quad 2 \leq j \leq m, \quad m+j \text{ is even}, \tag{8.44}$$

$$c_{i,2} \neq 0, \quad 2 \leq i \leq m-1 \quad i \text{ is even}, \tag{8.45}$$

$$c_{1,1} \neq 0, \tag{8.46}$$

Indeed, assume that (8.42) holds. Recall that the velocity $\dot{\Lambda}^0(\tau)$ can be considered as the self-adjoint linear mapping from $\Lambda^0(\tau)$ to $\Lambda^0(\tau)^*$. Fixing basis $\big(f_1(\tau),\dots,f_m(\tau)\big)$ in $\Lambda^0(\tau)$ we identify this mapping with a symmetric matrix. Then under this identification

$$\dot{\Lambda}^0(\tau) = -\dot{S}^0_\tau \tag{8.47}$$

If Lemma 8.6 holds, then from (8.42), (8.43) and (8.47) it follows that

$$c_{i,j}\beta_{0,2(m-\frac{i+j}{2})}(\tau) + \mathrm{Pol}\left(\{\beta_{0,2s}(\tau)\}_{s=0}^{m-\frac{i+j}{2}-1}\right) = 0. \tag{8.48}$$

If also Lemma 8.7 holds, then applying (8.48) first for $i=j=m$ and using (8.44), we get $\beta_{0,0}(\tau)\equiv 0$. It is evident that if $\beta_{0,2s}(\tau)\equiv 0$ for all $0\leq s\leq m-\frac{i+j}{2}-1$, then

$$\mathrm{Pol}\left(\{\beta_{0,2s}(\tau)\}_{s=0}^{m-\frac{i+j}{2}-1}\right) \equiv 0.$$

Using this fact and one of the relations (8.44)-(8.46), we obtain from (8.48) by induction that $\beta_{0,2s}(\tau)\equiv 0$ for all $0\leq s\leq m-1$. So, Lemmas 8.6 and 8.7 imply Proposition 10.

Remark 12 *Actually, we believe that for all pairs of indices (i,j) with even $i+j$ one can take constants $c_{i,j}\neq 0$ in (8.43) , but for our purposes it was sufficient to check that for any $0\leq k\leq m-1$ among all pairs of indices $\{(i,j): 2m-i-j=2k\}$ there exists at least one (i_k,j_k) such that $c_{i_k,j_k}\neq 0$. In the Lemma 8.7 we verify the last assertion by choosing pairs (i_k,j_k) for which the calculation of c_{i_k,j_k} is relatively easy.*

Now we prove Lemmas 8.6 and 8.7.

Proof of Lemma 8.6. Let S_t be the coordinate representation of the curve $\Lambda(t)$, $S_t = \{(x, S_t x) : x \in \mathbb{R}^m\}$, such that $(e_1(\tau), \ldots, e_m(\tau))$ constitute a standard basis of $\mathbb{R}^m \oplus 0$ and $(f_1(\tau), \ldots, f_m(\tau))$ constitute a standard basis of $0 \oplus \mathbb{R}^m$. Suppose, as in section 2, that the function $t \mapsto (S_t - S_\tau)^{-1}$ has the following Laurent expansion

$$(S_t - S_\tau)^{-1} \approx \sum_{i=-l}^{\infty} A_i(\tau)(t-\tau)^i, \tag{8.49}$$

at $t = \tau$. Applying recursive formula (2.4) for $i = 0$, one can express $\dot{S}^0_\tau$ in the following way:

$$\dot{S}^0_\tau = \frac{d}{d\tau} A_0(\tau) = A_1(\tau) + \sum_{n=-l}^{-1} \Big(A_n(\tau) \dot{S}_\tau A_{-n}(\tau) + A_{-n}(\tau) \dot{S}_\tau A_n(\tau) \Big). \tag{8.50}$$

On the other hand,

$$\Big((S_t - S_\tau)^{-1} \Big)_{i,j} = - \int^t w_i(\xi, \tau) w_j(\xi, \tau) d\,\xi. \tag{8.51}$$

So, by (8.28), the order l of pole in (8.49) satisfies $l = 2m - 1$. Recall that ,according to (7.3),

$$(\dot{S}_\tau))_{i,j} = \begin{cases} 0 & (i,j) \neq (m,m) \\ m^2 & (i,j) = (m,m), \end{cases} \tag{8.52}$$

Using formulas (8.50)-(8.52) we will compute $\dot{S}^0_\tau$. First, by (8.52), we get

$$(A_n(\tau) \dot{S}_\tau A_{-n}(\tau))_{i,j} = m^2 (A_n)_{i,m} (A_{-n})_{m,j} \tag{8.53}$$

Substituting (8.53) into (8.50), we obtain

$$\Big(\dot{S}^0_\tau \Big)_{i,j} = (A_1)_{i,j} + m^2 \sum_{n=1-2m}^{-1} \Big((A_n)_{i,m} (A_{-n})_{m,j} + (A_{-n})_{i,m} (A_n)_{m,j} \Big). \tag{8.54}$$

Let $\varphi_i(t, \tau)$ be ,as in (8.28). Denote

$$\varphi_{i,j}(\tau) = \frac{1}{j!} \frac{\partial^j}{\partial t^j} \varphi_i(t, \tau) \Big|_{t=\tau} \tag{8.55}$$

From (8.51) and (8.28) it is not difficult to obtain the following three relations

$$(A_{-n})_{m,j} = \begin{cases} 0 & j > m - n + 1 \\ \frac{1}{n} & j = m - n + 1 \\ \frac{1}{n} \varphi_{m-n-j} & 1 \leq j < m - n + 1, \end{cases} \tag{8.56}$$

where $1 \leq n \leq 2m - 1$,

$$(A_n)_{i,m} = -\frac{1}{n} \left(\varphi_{m,n+m-i} + \varphi_{i,n} \right) - \frac{1}{n} \sum_{k=0}^{n-1} \varphi_{i,k} \varphi_{m,n-1-k}, \tag{8.57}$$

where $n \geq 1$, and

$$(A_1)_{i,j} = -\varphi_{j,m-i+1} - \varphi_{i,m-j+1} \tag{8.58}$$

Note also that from (8.51) it follows that the coefficient of $\frac{1}{t-\tau}$ in the expansion of $t \mapsto w_i(t,\tau)w_j(t,\tau)$, $1 \leq i,j \leq m$, in the formal Laurent series at $t=\tau$ is equal to zero, otherwise the expansion of $t \mapsto (S_t - S_\tau)^{-1}$ contains logarithmic term. Hence we have the following relation

$$\varphi_{i,m-j} + \varphi_{j,m-i} = 0 \tag{8.59}$$

that will be used in the sequel.

Substituting (8.56)-(8.58) in (8.54), we have

$$(\dot{S}^0_\tau)_{i,j} = \Upsilon_{i,j}(\tau) + \Theta_{i,j}(\tau), \tag{8.60}$$

where

$$\Upsilon_{i,j} = -\left(1 + \frac{m^2}{(m-i+1)^2}\right)\varphi_{j,m-i+1} - \left(1 + \frac{m^2}{(m-j+1)^2}\right)\varphi_{i,m-j+1} - \\ m^2\left(\frac{1}{(m-i+1)^2} + \frac{1}{(m-j+1)^2}\right)\varphi_{m,2m-i-j+1} \tag{8.61}$$

and

$$\Theta_{i,j}(\tau) = -m^2\left(\frac{1}{(m-j+1)^2}\sum_{k=0}^{m-j}\varphi_{i,k}\varphi_{m,m-j-k} + \frac{1}{(m-i+1)^2}\sum_{k=0}^{m-i}\varphi_{i,k}\varphi_{m,m-i-k} + \right. \\ \sum_{n=1}^{m-j}\frac{1}{n^2}\left(\varphi_{m,n+m-1} + \varphi_{i,n} + \sum_{k=0}^{n-1}\varphi_{i,k}\varphi_{m,n-1-k}\right)\varphi_{m,m-n-j} + \\ \left.\sum_{n=1}^{m-j}\frac{1}{n^2}\left(\varphi_{m,n+m-1} + \varphi_{j,n} + \sum_{k=0}^{n-1}\varphi_{j,k}\varphi_{m,n-1-k}\right)\varphi_{m,m-n-i}\right). \tag{8.62}$$

Similarly to the proof of Theorem 2 in section 7 denote

$$u_i(t,\tau) = (t-\tau)^{m-i+1}w_i(t,\tau) \tag{8.63}$$

Let

$$u_{i,k}(\tau) = \frac{1}{k!}\frac{\partial^k}{\partial t^k}u_i(t,\tau)\Big|_{t=\tau} \tag{8.64}$$

Note that by (8.28)

$$u_{i,0}(\tau) \equiv 1, \ \ 1 \leq i \leq m, \tag{8.65}$$

$$u_{i,k}(\tau) \equiv 0, \ \ 1 \leq k \leq m-i, \ 1 \leq i \leq m \tag{8.66}$$

We claim that in order to prove Lemma 8.6 it is sufficient to show that

$$u_{i,k}(\tau) = \begin{cases} \mathrm{Pol}\left(\{u_{m,2s}(\tau)\}_{s=1}^{\min\{[\frac{k}{2}],m\}}\right) & k \notin \{2s : 1 \le s \le m\},\ k > 0 \\ \\ \mu_{i,k} u_{m,k}(\tau) + \mathrm{Pol}\left(\{u_{m,2s}(\tau)\}_{s=1}^{\frac{k}{2}-1}\right) & k \in \{2s : 1 \le s \le m\}, \end{cases} \tag{8.67}$$

where $\mu_{i,k}$ are some constants.

Indeed, from (8.55) and (8.64) it follows immediately that

$$\varphi_{i,j}(\tau) = u_{i,m-i+j+1}(\tau) \tag{8.68}$$

The last relation together with (8.67) implies

$$\begin{aligned} \Upsilon_{i,j} = -\left(1 + \tfrac{m^2}{(m-i+1)^2}\right) u_{j,2m-i-j+2} - \left(1 + \tfrac{m^2}{(m-j+1)^2}\right) u_{i,2m-i-j+2} - \\ -m^2 \left(\tfrac{1}{(m-i+1)^2} + \tfrac{1}{(m-j+1)^2}\right) u_{m,2m-i-j+2} = \\ = \chi_{i,j} u_{m,2(m-\frac{i+j}{2}+1)} + \mathrm{Pol}\left(\{u_{m,2s}(\tau)\}_{s=1}^{m-\frac{i+j}{2}}\right) \end{aligned} \tag{8.69}$$

where $\chi_{i,j}$ are some constants. Also, from (8.62) and (8.68) it easily follows that

$$\Theta_{i,j}(\tau) = \mathrm{Pol}\Big(\Big\{u_{l,k}(\tau) : l \in \{i,j,m\}, m - l + 1 \le k < 2m - i - j + 2\Big\}\Big).$$

This together with (8.67) implies that

$$\Theta_{i,j}(\tau) = \mathrm{Pol}\Big(\{u_{m,2s}(\tau)\}_{s=1}^{m-\frac{i+j}{2}}\Big). \tag{8.70}$$

Substituting (8.69) and (8.70) in (8.60) we get

$$(\dot{S}^0_\tau)_{i,j} = \chi_{i,j} u_{m,2(m-\frac{i+j}{2}+1)} + \mathrm{Pol}\left(\{u_{m,2s}(\tau)\}_{s=1}^{m-\frac{i+j}{2}}\right). \tag{8.71}$$

Using (8.68) one can rewrite (8.71) in the following form

$$(\dot{S}^0_\tau)_{i,j} = \chi_{i,j} \varphi_{m,2(m-\frac{i+j}{2}+1)-1} + \mathrm{Pol}\left(\{\varphi_{m,2s-1}(\tau)\}_{s=1}^{m-\frac{i+j}{2}}\right). \tag{8.72}$$

From (8.28) and (8.31) it follows that

$$\left(\frac{1}{t-\tau} + \varphi_m(t,\tau)\right)^2 = \frac{1}{(t-\tau)^2} + \frac{1}{m^2} g(t,\tau), \tag{8.73}$$

which implies easily that

$$\varphi_{m,0}(\tau) \equiv 0 \tag{8.74}$$

and

$$\varphi_{m,k}(\tau) = \frac{1}{2m^2}\beta_{0,k-1} + \mathrm{Pol}\left(\{\beta_{0,\mathrm{s}}(\tau)\}_{\mathrm{s}=0}^{\mathrm{k}-2}\right) \quad \mathrm{k} \ge 1. \tag{8.75}$$

(relation (8.74) also follows from (8.59) if we put $i = j = m$). From (8.66) and (8.74) it follows that

$$u_{i,1}(\tau) \equiv 0,\ \ 1 \le i \le m \tag{8.76}$$

Substituting (8.75) in (8.72) and using Lemma 4.1, we finally have

$$\left(\dot S^0_\tau\right)_{i,j} = c_{i,j}\beta_{0,2(m-\frac{i+j}{2})}(\tau) + \mathrm{Pol}\left(\{\beta_{0,2s}(\tau)\}_{s=0}^{m-\frac{i+j}{2}-1}\right),$$

where

$$c_{i,j} = \frac{\chi_{i,j}}{2m^2}. \tag{8.77}$$

So, to prove Lemma 8.6 it remains to prove formula (8.67). For this we refer to the proof of Theorem 2 in section 7. Let $\alpha_{i,j}(t)$ be as in (7.19). Then from part 3 of the Lemma 7.3, (8.74) and (8.68) it follows that for $i \le j \le m-1$

$$\alpha_{j,i}(\tau) = \mathrm{Pol}\Big(\Big\{\varphi_{m,k}(\tau)\Big\}_{k=1}^{j-i}\Big) = \mathrm{Pol}\Big(\Big\{u_{m,k}(\tau)\Big\}_{k=2}^{j-i+1}\Big) \tag{8.78}$$

(actually, it implies that

$$\alpha_{i,i}(\tau) \equiv 0,\ \ 1 \le i \le m-1 \tag{8.79}$$

in addition to Lemma 7.3). Using (7.43), (7.44) and part 2 of Lemma 7.3, it is not difficult to obtain that, similarly to (7.45), the functions $u_{i,k}(\tau)$ satisfy the following linear system of equations

$$\zeta_i(k)u_{i-1,k}(\tau) + \eta_i(k)u_{i,k}(\tau) + \theta_i(k)u_{m,k}(\tau) = \widetilde\Psi_{i,k},\ \ 1 \le i \le m, \tag{8.80}$$

where

$$\begin{aligned} \zeta_i(k) &= \frac{(k+i-m-1)(i-2m-1)(i-1)}{m-i+1}, \\ \eta_i(k) &= (k+i-1)(k+i-2m-1), \\ \theta_i(k) &= \frac{k+2i-2-2m}{m-i+1}m^2, \end{aligned} \tag{8.81}$$

and

$$\begin{aligned} \widetilde\Psi_{i,k} = {} & \mathrm{Pol}\Big(\Big\{u_{l,n}(\tau) : l \in \{i,m\},\ 1 \le n < k\Big\}\Big) + \\ & + \sum_{j=i}^{m-1} \alpha_{j,i}(\tau)\mathrm{Pol}\Big(\Big\{u_{s,n}(\tau) : l \in \{j,m\},\ 0 \le n < k\Big\}\Big). \end{aligned} \tag{8.82}$$

Combining (8.82) with (8.76) and (8.78) one can obtain that

$$\forall k > m-i \qquad \widetilde\Psi_{i,k} = \mathrm{Pol}\Big(\Big\{u_{l,n}(\tau) : i \le l \le m,\ 2 \le n < k\Big\}\Big). \tag{8.83}$$

From identity (7.47) it follows that for nonnegative k the determinant of the matrix, corresponding to the linear system (8.80) is zero iff $k \in \{2s : 1 \le s \le m\}$. Also, from (8.81) it follows easily that first $(m-1)$ columns of this matrix are always linearly independent. Therefore

$$u_{i,k}(\tau) = \begin{cases} \mathrm{Lin}\Big(\big\{\widetilde{\Psi}_{j,k}(\tau)\big\}_{j=1}^{m}\Big) & k \notin \{2s : 1 \le s \le m\} \\ \\ \mu_{i,k} u_{m,k}(\tau) + \mathrm{Lin}\Big(\big\{\widetilde{\Psi}_{j,k}(\tau)\big\}_{j=1}^{m}\Big) & k \in \{2s : 1 \le s \le m\}, \end{cases} \tag{8.84}$$

where by $\mathrm{Lin}(\dots)$ we denote some linear combination of the functions, contained in the brackets, and $\mu_{i,k}$ are some constants.

Note that, by (8.66), $u_{i,k}(\tau) \equiv 0$ for $1 \le i \le m-k$. It implies that for $\max\{m-k+1, 1\} \le i \le m$ we have

$$u_{i,k}(\tau) = \begin{cases} \mathrm{Lin}\Big(\big\{\widetilde{\Psi}_{j,k}(\tau)\big\}_{l=\max\{m-k+1,1\}}^{m}\Big) & k \notin \{2s : 1 \le s \le m\} \\ \\ \mu_{i,k} u_{m,k}(\tau) + \mathrm{Lin}\Big(\big\{\widetilde{\Psi}_{j,k}(\tau)\big\}_{l=\max\{m-k+1,1\}}^{m}\Big) & k \in \{2s : 1 \le s \le m\}, \end{cases} \tag{8.85}$$

Then, taking into account (8.83), we obtain

1. for $k \notin \{2s : 1 \le s \le m\}$
$$u_{i,k}(\tau) = \mathrm{Pol}\Big(\Big\{u_{l,n}(\tau) : \max\{m-k+1, 1\} \le l \le m,\ 2 \le n < k\Big\}\Big); \tag{8.86}$$

2. for $k \in \{2s : 1 \le s \le m\}$
$$u_{i,k}(\tau) = \mu_{i,k} u_{m,k}(\tau) + \mathrm{Pol}\Big(\Big\{u_{l,n}(\tau) : \max\{m-k+1, 1\} \le l \le m,\ 2 \le n < k\Big\}\Big). \tag{8.87}$$

Finally, by induction on k, starting with $k = 2$, we obtain

$$u_{i,k}(\tau) = \begin{cases} \mathrm{Pol}\left(\big\{u_{m,n}(\tau)\big\}_{n=2}^{\min\{k-1,m\}}\right) & k \notin\ \{2s : 1 \le s \le m\},\ k > 2 \\ \\ \mu_{i,k} u_{m,k}(\tau) + \mathrm{Pol}\left(\big\{u_{m,n}(\tau)\big\}_{n=2}^{k-1}\right) & k \in\ \{2s : 1 \le s \le m\}, \end{cases}, \tag{8.88}$$

which implies (8.67) (see, for example, Lemma 4.1). The proof of Lemma 8.6 is completed. □

As a consequence of (8.67) and (8.83) we obtain that

$$\forall k > m - i \qquad \widetilde{\Psi}_{i,k} = \mathrm{Pol}\Big(\Big\{u_{m,2s}(\tau)\Big\}_{s=1}^{\left[\frac{k-1}{2}\right]}\Big). \tag{8.89}$$

Proof of Lemma 8.7.

From (8.77) it follows that in order to prove that one can take $c_{i,j} \neq 0$ in (8.43), it is sufficient to show that one can take $\chi_{i,j} \neq 0$ in (8.71). For simplicity of notation in the expressions of the form

$$cu_{i,2k}(\tau) + \dots$$

by ... we will mean some function of the type $\mathrm{Pol}\Big(\Big\{u_{m,2s}(\tau)\Big\}_{s=1}^{k-1}\Big)$.

1. Let us prove (8.44). By (8.69)

$$\Upsilon_{m,j} = -(m^2+1)u_{j,m-j+2} - \left(m^2+1+\frac{2m^2}{(m-j+1)^2}\right)u_{m,m-j+2} \tag{8.90}$$

Apply (8.80) for $i = j$ and $k = m-j+2$. Then

$$\zeta_j(m-j+2)u_{j-1,m-j+2}(\tau)+\eta_j(m-j+2)u_{j,m-j+2}(\tau)+\theta_j(m-j+2)u_{m,m-j+2}(\tau) = \widetilde{\Psi}_{j,m-j+2} \tag{8.91}$$

From (8.59) and (8.68) it follows that

$$u_{i,2m-i-j+1} = -u_{j,2m-i-j+1}, \quad 1 \le i,j \le m$$

In particular,

$$u_{j-1,m-j+2} = -u_{m,m-j+2}$$

Substituting this in (8.91) and using (8.81), (8.89) we have

$$u_{j,m-j+2} = \frac{\xi_j(m-j+2)-\theta_j(m-j+2)}{\eta_j(m-j+2)}u_{m,m-j+2}+\ldots =$$

$$-\frac{j^2-(m^2+2m+2)j+m^3+2m+1}{(m^2-1)(m-j+1)}u_{m,m-j+2}+\ldots$$

Substituting this in (8.90) one can easily get

$$\Upsilon_{m,j} = \chi_{m,j}u_{m,m-j+2}+\ldots, \tag{8.92}$$

where

$$\chi_{m,j} = (m^2+1)\frac{\Big(j^2-(2m+3)j+3m+2\Big)-m^2}{(m^2-1)(m-j+1)} - \frac{2m^2}{(m-j+1)^2}. \tag{8.93}$$

We claim that $\chi_{m,j} < 0$ for $2 \le j \le m$, which by (8.77) will imply (8.44) . By (8.93), for this it is sufficient to prove that the polynomial $j^2-(2m+3)j+3m+2$ is negative for $2 \le j \le m$, but it can be done easily by estimating the roots of this polynomial.

2. Let us prove (8.45). By (8.69)

$$\Upsilon_{i,2} = -\left(1+\frac{m^2}{(m-i+1)^2}\right)u_{2,2m-i} - \left(1+\frac{m^2}{(m-1)^2}\right)u_{i,2m-i} -$$

$$\left(\frac{m^2}{(m-i+1)^2}u_{m,m-j+2}+\frac{m^2}{(m-1)^2}\right)u_{m,2m-i} \tag{8.94}$$

Let us express $u_{i,2m-i}$ and $u_{2,2m-i}$ by $u_{m,2m-i}$ and $u_{p,n}$ with $n < 2m-i$.

We start with $u_{i,2m-i}$. Note that $\eta_i(2m-i+1)=0$. Hence from (8.80) for $k=2m-i+1$

$$\xi_i(2m-i+1)u_{i-1,2m-i+1}+\theta_i(2m-i+1)u_{m,2m-i+1}=\widetilde{\Psi}_{i,2m-i+1}$$

Replacing i by $i+1$ and using (8.81),(8.89) one can easily obtain that

$$u_{i,2m-i}=-\frac{\theta_{i+1}(2m-i)}{\xi_{i+1}(2m-i)}u_{m,2m-i}+\ldots=\frac{m}{2m-i}u_{m,2m-i}+\ldots \tag{8.95}$$

To analyze $u_{2,2m-i}$, one consider the first and the second equations of the system (8.80) with $k=2m-i$. Then by direct calculation

$$u_{2,2m-i}=\frac{\theta_1(2m-i)\xi_2(2m-i)-\eta_1(2m-i)\theta_2(2m-i)}{\eta_1(2m-i)\eta_2(2m-i)}u_{m,2m-i}+\ldots=$$

$$-m\frac{(1-3m)i^2+(4m^2+3m-1)i-4m^2}{(m-1)(i-1)(2m-i+1)(2m-i)}u_{m,2m-i}+\ldots \tag{8.96}$$

Substituting (8.95) and (8.96)) in (8.94), one can obtain by direct calculation that

$$\Upsilon_{i,2}=\chi_{i,2}u_{m,2m-i}+\ldots, \tag{8.97}$$

where

$$\chi_{i,2}=\frac{mp_i(m)}{(2m-1)(2m-i+1)(m-i+1)(m-1)^2} \tag{8.98}$$

with

$$p_i(x)=(8i-6)x^3-(4i^2-8i+6)x^2+(i^3-4i^2+4i-1)x+i-1.$$

We claim that m is not a root of the polynomial $p_i(x)$ for $2\le i\le m-1$. Assuming the converse, we obtain that m has to divide $i-1$ that contradicts the assumption that $i\le m-1$. Hence $\chi_{i,2}\neq 0$ that together with (8.77) implies (8.45).

3. Let us prove (8.46). By (8.69)

$$\Upsilon_{1,1}=-4u_{1,2m}-2u_{m,2m}+\ldots \tag{8.99}$$

Consider the linear system (8.80) for $k=2m$. Note that the coefficients of the first equation of this system vanish. Taking the rest $m-1$ equations and using the Kramer formula one can easily get

$$u_{1,2m}=(-1)^{m-1}\left(\left(\sum_{j=2}^{m}(-1)^{j+m}\frac{\theta_j(2m)}{\xi_j(2m)}\prod_{i=2}^{j-1}\frac{\eta_i(2m)}{\xi_i(2m)}\right)+\prod_{i=2}^{m}\frac{\eta_i(2m)}{\xi_i(2m)}\right)u_{m,2m}+\ldots$$

Then from (8.81) it is not hard to obtain

$$u_{1,2m}=Cu_{m,2m}+\ldots, \tag{8.100}$$

where

$$C = \left(\sum_{j=2}^{m} \frac{2m^2}{(m+j-1)(2m-j+1)} \prod_{i=2}^{j-1} \frac{(2m+i-1)(m-i+1)}{(m+i-1)(2m-i+1)} \right) +$$

$$+ \prod_{i=2}^{m} \frac{(2m+i-1)(m-i+1)}{(m+i-1)(2m-i+1)} > 0 \tag{8.101}$$

Therefore, by (8.99),

$$\Upsilon_{1,1} = \chi_{1,1} u_{m,2m} + \dots,$$

where $\chi_{1,1} = -4\,C - 2 < 0$. This together with (8.77) implies (8.46).The proof of Lemma 8.7 and Proposition 10 is completed. □

Conjecture 1 *Let $\Lambda : I \mapsto L(W)$ be the curve of constant weight k. Then all statements* $1, 2, 3, 4, 5$ *of Theorem 1 are equivalent.*

Theorem 1 says that the Conjecture is true for regular and rank 1 curves. Note that the regular curves are the only curves such that the order of pole l in the Laurent expansion of $t \mapsto \Lambda_\tau(t)$ at $t = \tau$ is equal to 1, while the rank 1 curves of the constant weight in $L(W)$ with $\dim W = m$ are the only curves of the constant weight in $L(W)$ with $l = 2m - 1$ (all other curves of the constant weight in $L(W)$ satisfy $l < 2m - 1$). It is not difficult to show that l has to be an odd number. We also have proved a part of the Conjecture, namely, that statement 4 of the Theorem 1 implies strong flatness, for curves of the constant rank and weight with the order of pole l equals 3 or 5 , but the proof consists of very long calculation and it is not clear yet, how to extend it for other l. So we postpone the presentation of this proof to the further publications.

9 Comparison theorems

9.1 Preliminary

Recall that the points $t_0 \neq t_1$ are said to be conjugate for the curve $\Lambda(\cdot)$ in the Lagrange Grassmannian $L(W)$, if $\Lambda(t_0) \cap \Lambda(t_1) \neq 0$. In this case we will say also that the point t_1 is conjugate to t_0 w.r.t the curve $\Lambda(\cdot)$. The notion of conjugate points is very important in the investigation of certain optimality properties of extremals. The dimension of $\Lambda(t_0) \cap \Lambda(t_1)$ is called a multiplicity of the conjugate pair t_0, t_1. It is clear that if $\Lambda : I \mapsto L(W)$ is nondecreasing ample curve defined on some compact interval $I \subset \mathbb{R}$, then every $t \in I$ is conjugate to a finite number of points. The natural and important problem is to estimate the number of points conjugate to the given point on the given interval in terms of symplectic invariants of the curve.

In order to obtain such estimations, we will use two different approaches, described by Theorems 2 and Theorem 3 below. We will apply both approaches in more details to the rank 1 curves of the constant weight in $L(W)$ with $\dim W = 4$.

The first approach uses the properties of the derivative curve and gives the following simple sufficient condition for absense of the points conjugate to the given point:

Theorem 4 *Let $\Lambda : I \mapsto L(W)$ be an ample nondecreasing curve such that its derivative curve $\Lambda^0(t)$ is nonincreasing. Then for any point $\tau_0 \in I$ there is no point $\tau \in I$ conjugate to τ_0.*

Proof. The method of the proof is direct generalization of the method, used in the proof of the theorem on p.374 in [1] . Evidently, it is sufficient to prove that for given $\tau_0 \in I$ there is no $\tau > \tau_0$ conjugate to τ_0. One can introduce coordinates in W, $W = \mathbb{R}^m \oplus \mathbb{R}^m = \{(x, y) : x, y \in \mathbb{R}^m\}$ such that the symplectic form σ has standard form,

$$\sigma((x_1, y_1), (x_2, y_2)) = \langle x_2, y_1\rangle - \langle x_1, y_2\rangle, \tag{9.1}$$

the subspace $\Lambda(\tau_0)$ satisfies $\Lambda(\tau_0) = \mathbb{R}^m \oplus 0$ and $\Lambda_0(\tau_0) = \{(x, S^0 x) : x \in \mathbb{R}^m\}$, where S^0 is positive definite symmetric matrix (for shortness we will write $S^0 > 0$). Let $\Delta = 0 \oplus \mathbb{R}^m$. Let S_τ, S^0_τ be the matrices corresponding to $\Lambda(\tau)$ and $\Lambda^0(\tau)$ in the chosen coordinates. Note that, in general, matrix S_τ (S^0_τ) is defined if subspace $\Lambda(\tau)$ ($\Lambda^0(\tau)$) belongs to $\Delta^{\pitchfork}$. By construction, $S_{\tau_0} = 0$, $S^0_{\tau_0} = S^0 > 0$, i.e.

$$S^0_{\tau_0} - S_{\tau_0} > 0 \tag{9.2}$$

Recall that from the construction of the derivative curve it follows that $\Lambda(\tau) \cap \Lambda^0(\tau) = 0$ for any τ. Therefore

$$det(S^0_\tau - S_\tau) \neq 0 \tag{9.3}$$

Further, by assumptions,

$$\dot{S}_\tau \geq 0 \ \ \dot{S}^0_\tau \leq 0 \tag{9.4}$$

Relations (9.3) and (9.4) hold until $\Lambda(\tau)$ and $\Lambda^0(\tau)$ remain in $\Delta^{\pitchfork}$. However, from (9.2)-(9.4) and the assumption that the curve $\Lambda(\cdot)$ is ample it follows that

$$0 < S_\tau \leq S^0_\tau \leq S^0_{\tau_0}.$$

Hence $\Lambda(\tau)$ and $\Lambda(\tau_0)$ do not leave the set $\Delta^{\pitchfork}$ at all. This completes the proof of the theorem.$\Box$

The second approach for estimation of conjugate points is based on the following, simplified for our purposes, version of the multidimensional generalization of Sturm's classical theorems about the zeros of the solutions of differential equations of second order (for more general formulation and proof see [3] and also [8]).

Theorem 5 *Let h_τ, H_τ be quadratic nonstationary Hamiltonians on W such that for any $0 \leq \tau \leq t$ the quadratic form $H_\tau - h_\tau$ is nonnegative definite. Let P_τ, $\tilde{P}_\tau \in Sp(W)$ be linear Hamiltonian flows on W, generated by the Hamiltonian fields $\vec{h}_\tau$ and $\vec{H}_\tau$:*

$$\frac{\partial}{\partial \tau} P_\tau = \vec{h}_\tau P_\tau \quad \frac{\partial}{\partial \tau}\tilde{P}_\tau = \vec{H}_\tau \tilde{P}_\tau, \;\; P_0 = \tilde{P}_0 = id.$$

Finally, let $\Lambda(\tau)$, $\tilde{\Lambda}(\tau)$ be nondecreasing ample trajectories of the corresponding flows on $L(W)$:

$$\Lambda(\tau) = P_\tau \Lambda(0), \;\; \tilde{\Lambda}(\tau) = \tilde{P}_\tau \tilde{\Lambda}(0), \;\; 0 \leq \tau \leq t.$$

Then for any $\Lambda_1 \in L$, which is transversal to the endpoints of the curves $\Lambda(\cdot)$, $\tilde{\Lambda}(\cdot)$ the inequality

$$\sum_{0 \leq \tau \leq t} \dim\Big(\Lambda(\tau) \cap \Lambda_1\Big) - \frac{1}{2}\dim W \leq \sum_{0 \leq \tau \leq t} \dim\Big(\tilde{\Lambda}(\tau) \cap \Lambda_1\Big) \tag{9.5}$$

is valid.

Here, as in the Introduction to [6], the Hamitonian field $\vec{h}$ corresponding to the Hamiltonian h is defined by the identity $\sigma(\vec{h}, \cdot) = dh(\cdot)$.

Assume that $\Lambda : \mathbb{R} \mapsto L(W)$ is a nondecreasing curve of rank 1 and constant weight. Let again $\big(e_1(\tau, \dots, e_m(\tau), f_1(\tau), \dots, f_m(\tau)\big)$ be the canonical moving frame associated with the curve $\Lambda(\tau)$. As in the proof of Theorem 2, denote by $E(\tau)$ and $F(\tau)$ the tuples of vectors $(e_1(\tau), \dots, e_m(\tau))$ and $(f_1(\tau), \dots, f_m(\tau))$ correspondingly , arranged in the columns. Fixing the bases $E(\tau)$ and $F(\tau)$ in $\Lambda(\tau)$ and $\Lambda^0(\tau)$ respectively, let S_t be the matrix, corresponding to the linear mapping $\langle \Lambda(\tau), \Lambda(t), \Lambda^0(\tau)\rangle$ and S_t^0 be the matrix, corresponding to the linear mapping $\langle \Lambda^0(\tau), \Lambda^0(t), \Lambda(\tau)\rangle$ (see section 2 for notations).

First note that by (8.47) Theorem 2 can be formulated in the following form

Corollary 6 *Let $\Lambda : I \mapsto L(W)$ be rank 1 nondecreasing curve of the constant weight curve such that for any $\tau \in I$ the matrix $\dot{S}_\tau^0$ is nonnegative definite. Then for any point $\tau_0 \in I$ there is no point $\tau \in I$ conjugate to τ_0.*

Hence, if one finds an explicit expression for matrix $\dot{S}_\tau^0$ in terms of the complete system of invariants of the curve $\Lambda(\tau)$ (existence of such expression follows from the proof of Theorem 2), then, by Corollary 6, one can obtain explicit sufficient conditions for absence of conjugate points in terms of the complete system of invariants of the curve $\Lambda(\tau)$.

Now we will explain how to estimate the number of conjugate points, for example, to the point 0 w.r.t. the curve $\Lambda(\cdot)$, using Theorems 3. It is easy to see that the structural equation for the canonical moving frame has the following form:

$$\begin{pmatrix} \dot{E}(\tau) \\ \dot{F}(\tau) \end{pmatrix} = \begin{pmatrix} \Omega(\tau) & \dot{S}_\tau \\ \dot{S}_\tau^0 & -\Omega^T(\tau) \end{pmatrix} \begin{pmatrix} E(\tau) \\ F(\tau) \end{pmatrix} \tag{9.6}$$

(here $\Omega(\tau)$ is a $m \times m$ matrix with (i,j) entry equal to $\alpha_{i,j}(\tau)$, where $\alpha_{i,j}(\tau)$ is defined by (7.19) with $\Delta = \Lambda^0(\tau)$). Denote by $\mathcal{B}(\tau)$ the matrix in the structural equation (9.6), namely,

$$\mathcal{B}_\tau = \begin{pmatrix} \Omega(\tau) & \dot{S}_\tau \\ \dot{S}^0_\tau & -\Omega^T(\tau) \end{pmatrix}.$$

Fixing the basis $e_1(0), \dots, e_m(0), f_1(0), \dots, f_m(0)$ in W, we identify W with $\mathbb{R}^m \oplus \mathbb{R}^m = \{(x,y) : x, y \in \mathbb{R}^m\}$ such that $\Lambda^0(0) = 0 \oplus \mathbb{R}^m$ and the symplectic form σ has the standard form (9.1). Denote by $\mathcal{F}_\tau$ the $2m \times 2m$ matrix such that its ith column is equal to the coordinates of $e_i(\tau)$ for $1 \le i \le m$ and to the coordinates of $f_i(\tau)$ for $m+1 \le i \le 2m$ (the coordinates are w.r.t the chosen basis $e_1(0), \dots, e_m(0), f_1(0), \dots, f_m(0)$). One can look on $\mathcal{F}_\tau$ as on the linear flow on W (that in turn generates the flow on $L(W)$). Then, by construction, $\Lambda(\tau) = \mathcal{F}_\tau \Lambda(0)$ and from the structural equation (9.6) it follows that

$$\frac{d}{d\tau}\mathcal{F}_\tau = \mathcal{F}_\tau \mathcal{B}_\tau^T, \quad \mathcal{F}_0 = id.$$

Denote by $P_\tau = \mathcal{F}^{-1}_{-\tau}$ and by h^1_τ the following nonstationary Hamiltonian

$$h^1_\tau((x,y)) = \frac{1}{2}\Big(\langle \dot{S}_{-\tau}x, x\rangle - 2\langle \Omega(-\tau)^T x, y\rangle - \langle \dot{S}^0_{-\tau}y, y\rangle\Big), \tag{9.7}$$

Then it is easy to show that P_τ is linear Hamiltonian flow on W, generated by the Hamiltonian vector field $\vec{h}^1_\tau$:

$$\frac{\partial}{\partial\tau}P_\tau = \mathcal{B}^T_{-\tau}P_\tau = \vec{h}^1_\tau P_\tau, \;\; P_0 = id, \tag{9.8}$$

Moreover, if $\Upsilon(\cdot)$ is the trajectory of the flow P_τ, starting at $\Lambda(0)$, i.e.,

$$\Upsilon(\tau) = P_\tau \Lambda(0), \tag{9.9}$$

then

$$\dim(\Upsilon(\tau) \cap \Upsilon(0)) = \dim(\mathcal{F}^{-1}_{-\tau}\Lambda(0) \cap \Lambda(0)) = \dim(\Lambda(0) \cap \mathcal{F}_{-\tau}\Lambda(0) = \dim(\Lambda(0) \cap \Lambda(-\tau)). \tag{9.10}$$

It yields that the point τ_1 is conjugate to the point 0 w.r.t. the curve $\Upsilon(\cdot)$ iff the point $-\tau_1$ is conjugate to 0 w.r.t. the original curve $\Lambda(\cdot)$. Note also that, by construction, the curve $\Upsilon(\cdot)$ is a nondecreasing ample curve . So, we can work with the curve $\Upsilon(\cdot)$ instead of $\Lambda(\cdot)$.

The scheme for estimation of the numbers of conjugate points is as follows: First, one can express the structural equation (9.6) (and then also the Hamiltonian h^1_τ) in terms of the complete system of invariants of the curve. Then one can compare h^1_τ with some simple (for example, stationary) Hamiltonian h^0 on W such that for the trajectories of the corresponding linear Hamiltonian field $\vec{h}^0$ the pairs of conjugate points can be computed explicitly. Finally, one can use the inequality of the type (9.5) to obtain the estimates for the number of conjugate points for the original curve.

9.2 Structural equation in the case $r = 1$, $m = 2$

In the both approaches, presented in the previous subsection, we have to know the expression for the structural equation (9.6) in terms of the complete system of invariants of the curve. The problem is that in general such expression is rather complicated. Therefore from now we restrict ourselves to the curve of rank 1 and constant weight for $m = 2$, i.e., $\dim W = 4$.

In this case the complete system of invariants consists of the pair of functions $(\rho(t)\ ,\beta_{0,2}(t))$ or $(\rho(t), A(t))$, where $A(t)$ is the density of the fundamental form $\mathcal{A}$ (see also (5.8)).

Proposition 11 *The canonical moving frame* $\Big(e_1(\tau), e_2(\tau), f_1(\tau), f_2(\tau)\Big)$ *of the curve* $\Lambda(\cdot)$ *satisfies the following structural equation:*

$$\begin{pmatrix} \dot e_1(\tau) \\ \dot e_2(\tau) \\ \dot f_1(\tau) \\ \dot f_2(\tau) \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 0 \\ \frac{1}{4}\rho(\tau) & 0 & 0 & 4 \\ -\left(\frac{35}{36}A(\tau) - \frac{1}{8}\rho(\tau)^2 + \frac{1}{16}\rho''(\tau)\right) & -\frac{7}{16}\rho'(\tau) & 0 & -\frac{1}{4}\rho(\tau) \\ -\frac{7}{16}\rho'(\tau) & -\frac{9}{4}\rho(\tau) & -3 & 0 \end{pmatrix} \begin{pmatrix} e_1(\tau) \\ e_2(\tau) \\ f_1(\tau) \\ f_2(\tau) \end{pmatrix} \tag{9.11}$$

As a direct consequence of Corollary 6 and Proposition 11 we obtain

Theorem 6 *Let* $\Lambda : I \mapsto L(W)$, *be the rank 1 curve of the constant weight with Ricci curvature* $\rho(\cdot)$ *and the density* $A(\cdot)$ *of the fundamental form. If the following symmetric matrix*

$$\begin{pmatrix} -\left(\frac{35}{36}A(\tau) - \frac{1}{8}\rho(\tau)^2 + \frac{1}{16}\rho''(\tau)\right) & -\frac{7}{16}\rho'(\tau) \\ -\frac{7}{16}\rho'(\tau) & -\frac{9}{4}\rho(\tau) \end{pmatrix}$$

is nonnegative definite, then for $\tau_0 \in I$ *there is no point* $\tau \in I$ *conjugate to* τ_0.

Sketch of proof of Proposition 11. Here we give the main steps of the computation leaving the details to the reader. We use the notations of (9.6) with $\Omega(\tau) = (\alpha_{ij}(\tau))_{1\le i,j\le 2}$. First from the part 2 of Lemma 7.3 it follows that $\alpha_{1,2} = 3$. Using the identity (8.31) and the fact that $\beta_{0,2}(\tau) = \frac{1}{2}\rho'(\tau)$ (see (4.18)), it is easy to obtain the following expansion for $w(t,\tau)$:

$$w_2(t,\tau) = \frac{1}{(t-\tau)} + \frac{1}{8}\rho(\tau)(t-\tau) + \frac{1}{16}\rho'(\tau)(t-\tau)^2 + a_2(\tau)(t-\tau)^3 + O((t-\tau)^4), \tag{9.12}$$

where

$$a_2(\tau) = \frac{1}{8}\beta_{0,2}(\tau) - \frac{1}{128}\rho(\tau)^2 \tag{9.13}$$

It implies, in particular, that $\varphi_2(\tau,\tau) = 0$, where the function $\varphi_2(t,\tau)$ is as in (8.28). Therefore the function $\Phi_2(t,\tau)$, defined by (7.26) , satisfies also $\Phi_2(\tau,\tau) = 0$ (or in the notation of (7.31) $c_0(\tau) = 0$). Then, by (7.38) , we have

$$\alpha_{1,1}(\tau) = c_0(\tau) + 2\varphi_2(\tau,\tau) + \frac{\partial}{\partial t}\varphi_2(t,\tau)\Big|_{t=\tau} = 0 \tag{9.14}$$

Further, by the equation (7.39) for $m = i = 2$

$$\frac{\partial Y_1}{\partial t} = -\frac{1}{3}\left(\frac{\partial^2}{\partial t \partial \tau}(\ln w_2) + 4w_2^2\right), \tag{9.15}$$

where $Y_1 = \frac{w_1}{w_2}$ (to obtain (9.15) from (7.39) , we have used that $Y_2 \equiv 1$). Then from (9.12) by direct computations one can obtain the following expansion for $\frac{\partial Y_1}{\partial t}$:

$$\frac{\partial Y_1}{\partial t} = -\frac{1}{(t-\tau)^2} - \frac{1}{4}\rho(\tau) - \frac{1}{8}\rho'(\tau)(t-\tau) - \Big(\frac{1}{16}\rho''(\tau) - \frac{4}{3}a_2(\tau) + \frac{5}{96}\rho(\tau)^2\Big)(t-\tau)^2 + O((t-\tau)^3) \tag{9.16}$$

Hence

$$\begin{aligned} w_1(t,\tau) = Y_1(t,\tau)w_2(t,\tau) = \Big(\frac{1}{t-\tau} + C - \frac{1}{4}\rho(\tau)(t-\tau) - \frac{1}{16}\rho'(\tau)(t-\tau)^2 - \\ \frac{1}{3}\Big(\frac{1}{16}\rho''(\tau) - \frac{4}{3}a_2(\tau) + \frac{5}{96}\rho(\tau)^2\Big)(t-\tau)^2 + O((t-\tau)^4)\Big)\Big(\frac{1}{(t-\tau)} + \\ \frac{1}{8}\rho(\tau)(t-\tau) + \frac{1}{16}\rho'(\tau)(t-\tau)^2 + a_2(\tau)(t-\tau)^3 + O((t-\tau)^4)\Big). \end{aligned} \tag{9.17}$$

By our constructions in the expansion of $t \to w_1(t,\tau)$ into the Laurent series at $t = \tau$ the coefficient of $\frac{1}{t-\tau}$ vanishes. Opening the brackets in (9.17), we find that constant C is equal to zero. Moreover, the expansion of w_1 has the following form:

$$w_1(t,\tau) = \frac{1}{(t-\tau)^2} - \frac{1}{8}\rho(\tau) + a_1(\tau)(t-\tau)^2, \tag{9.18}$$

where

$$a_1(\tau) = \frac{13}{9}a_2(\tau) - \frac{1}{48}\rho''(\tau) - \frac{7}{144}\rho(\tau)^2. \tag{9.19}$$

Remark 13 *Note that the asymptotic of w_1 up to $O((t-\tau)^2)$ can be obtained easier, using the fact that the expansions of $t \mapsto w_1(t,\tau)w_i(t,\tau)$, $i = 1,2$ in the Laurent series at $t = \tau$ do not contain the terms of the type $\frac{c}{t-\tau}$ (otherwise the expansions of $\Big((S_t - S_\tau)^{-1}\Big)_{1i} = \int_\tau^t w_1(\xi,\tau)w_i(\xi,\tau)d\xi$ contain logarithmic terms).*

To find $\alpha_{2,1}(\tau)$ and $\alpha_{2,2}(\tau)$, use the formula (7.52) and the sentence following this formula: the function $\alpha_{2,i}(\tau)$ is equal to coefficient of $\frac{1}{t-\tau}$ in the expansion of

$$t \mapsto -4w_2(t,\tau)\int^t w_2(\xi,\tau)w_i(\xi,\tau)d\xi$$

into Laurent series at $t = \tau$. This coefficient can be easily obtained from (9.12) and (9.18), namely,

$$\alpha_{2,1}(\tau) = \frac{1}{4}\rho(\tau), \quad \alpha_{2,2} = 0 \tag{9.20}$$

By (9.6) it remains to find the matrix $\dot{S}^0_\tau$. Using the formula (7.50) and expansions (9.12), (9.18), the reader will have no difficulty to show that

$$\dot{S}^0_\tau = \begin{pmatrix} -4a_2(\tau) + \frac{1}{64}\rho(\tau)^2 - 2a_1(\tau) & -\frac{7}{16}\rho'(\tau) \\ -\frac{7}{16}\rho'(\tau) & -\frac{9}{4}\rho(\tau) \end{pmatrix} \tag{9.21}$$

Finally, combining formulas (5.8) , (9.13), and (9.19), one can easily obtain that

$$\left(\dot{S}^0_\tau\right)_{1,1} = -\left(\frac{35}{36}A(\tau) - \frac{1}{8}\rho(\tau)^2 + \frac{1}{16}\rho''(\tau)\right) \tag{9.22}$$

This completes the proof of the Proposition. □

Remark 14 *From (9.11) it can be shown by a direct calculation that the curvature operator $R(\tau) : \Lambda(\tau) \to \Lambda(\tau)$ of the curve $\Lambda(\cdot)$ of rank 1 and constant weight has the following matrix in the basis $(e_1(\tau), e_2(\tau))$:*

$$R(\tau) = -\dot{S}^0_\tau \dot{S}_\tau = \begin{pmatrix} 0 & \frac{7}{4}\rho'(\tau) \\ 0 & 9\rho(\tau) \end{pmatrix}, \tag{9.23}$$

i.e., $R(\tau)$ depends only on $\rho(\tau)$. From this and Corollary 3 it follows that in contrast to the regular curves (see Lemma 8.2), for rank 1 curves from the fact that the curvature operator of the curve is identically equal to zero does not follow that the curve is flat.

As you see from (9.11), even for $m = 2$ the structural equation is complicated, for example, contains the derivatives of $\rho(t)$ up to the second order. It is not so clear with what simple Hamiltonian system it can be compared . Therefore, to apply Theorem 3, first we obtain the comparison theorems in the case when $\rho(\tau) \equiv 0$, i.e., when τ is a projective parameter. To obtain the comparison theorems for arbitrary parameter, we will make reparametrization to a projective parameter. The invariant $A(t)$ is more convenient than $\beta_{0,2}$, since it has more simple reparametrization rule.

9.3 The case of the projective parameter

Here we suppose that the Ricci curvature $\rho(\tau)$ of the curve $\Lambda(\tau)$ is identically equal to zero , i.e., τ is a projective parameter. Then structural equation (9.11) has the following simple form:

$$\begin{pmatrix} \dot{e}_1(\tau) \\ \dot{e}_2(\tau) \\ \dot{f}_1(\tau) \\ \dot{f}_2(\tau) \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ -\frac{35}{36}A(\tau) & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix} \begin{pmatrix} e_1(\tau) \\ e_2(\tau) \\ f_1(\tau) \\ f_2(\tau) \end{pmatrix} \tag{9.24}$$

In this case the Hamiltonian h^1_τ, defined by (9.7), has the following form

$$h^1_\tau(x,y) = \frac{1}{2}\Big(\frac{35}{36}A(-\tau)y_1^2 - 6x_1y_2 + 4x_2^2\Big), \tag{9.25}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

We will compare this Hamiltonian with the following stationary Hamiltonian h^{A_0}

$$h^{A_0}(x,y) = \frac{1}{2}\Big(\frac{A_0}{36}\ y_1^2 - 6x_1y_2 + 4x_2^2\Big) \tag{9.26}$$

for some constant A_0. The corresponding Hamiltonian system has the form

$$\begin{cases} \dot{x}_1 &= -\frac{A_0}{36}\, y_1 \\ \dot{x}_2 &= 3x_1 \\ \dot{y}_1 &= -3y_2 \\ \dot{y}_2 &= 4x_2 \end{cases} \tag{9.27}$$

Let $\Gamma(\tau)$ be the trajectory of the flow generated on $L(W)$ by the system (9.27) such that $\Gamma(0) = \Lambda(0)$. It is easy to see that $\Gamma(\tau)$ is ample nondecreasing curve. By direct computation one can obtain the following:

Lemma 9.1 *If $A_0 \le 0$, then there are no points $\tau \ne 0$ conjugate to* 0 *w.r.t. the curve $\Gamma(\tau)$. If $A_0 > 0$, then the point τ_1 is conjugate to* 0 *w.r.t. $\Gamma(\tau)$ iff τ_1 is a solution of the following equation*

$$cos(\omega\tau)cosh(\omega\tau) = 1, \tag{9.28}$$

where $\omega = \sqrt[4]{A_0}$. In addition, if $\tau_1 \ne 0$ is conjugate to 0*, then the multiplicity of the conjugate pair (τ_0, τ_1) is equal to* 1*.*

Denote by p_1 the first positive solution of the equation

$$\cos\tau \cosh\tau = 1. \tag{9.29}$$

Note that $p_1 \approx 4.73004$. Using Theorem 5 and the Lemma 9.1, one can get the following

Theorem 7 (Comparison theorem in the projective parameter) *Let $\Lambda(\tau)$ be the curve of rank* 1 *and constant weight in $L(W)$ defined on the interval $I \subseteq \mathbb{R}$ such that its Ricci curvature $\rho(\tau)$ is identically equal to zero on I (in other words τ is a projective parameter). Then the following three statements are valid:*

1. *If $A(\tau) \le 0$ then for any point $\tau_0 \in I$ there are no points $\tau \ne \tau_0$ conjugate to τ_0;*
2. *If $A(\tau) \le \frac{A_0}{35}$ for some constant $A_0 > 0$ then for any $\tau_0 \in I$ in the interval $(\tau_0, \tau_0 + \frac{p_1}{\sqrt[4]{A_0}})$ there are no points conjugate to τ_0 ;*

3. *If $A(\tau) \geq \frac{A_0}{35}$ for some constant $A_0 > 0$ then for any pair of consequent conjugate points τ_0, τ_1 in I the inequality $|\tau_1 - \tau_0| \leq \frac{p_1}{\sqrt[4]{A_0}}$ holds.*

Proof. First we can suppose that $I = \mathbb{R}$ (otherwise, we can extend somehow the function $A(\tau)$ on the whole $\mathbb{R}$, preserving its upper bound (or lower bound), and work with the Hamiltonian h^1_τ defined by (9.25) for all $\tau \in \mathbb{R}$). Without loss of generality, one can also suppose that $\tau_0 = 0$ (otherwise, one can work with the curve $\tau \mapsto \Lambda(\tau + \tau_0)$ instead of $\Lambda(\tau)$). As before, let $\Gamma(\cdot)$ and $\Upsilon(\cdot)$ be the trajectories of the flows generated on $L(W)$ by the Hamiltonian vector fields $\vec{h}^{A_0}$ and h^1_τ correspondingly such that $\Gamma(0) = \Upsilon(0) = \Lambda(0)$.

First we prove the statements 1 and 2 of the theorem. Suppose that $A(\tau) \leq \frac{A_0}{35}$. For some numbers ϵ, c such that $c > \epsilon > 0$ consider two curves

$$\tilde{\Gamma}(\tau) = \Gamma(\tau - c), \;\; \tilde{\Upsilon}(\tau) = \Upsilon(\tau - c + 2\epsilon)$$

on the interval $0 \leq \tau \leq c - \epsilon$. The first curve is again a trajectory of the flow generated on $L(W)$ by the stationary Hamiltonian vector field $\vec{h}^{A_0}$. The second curve is a trajectory of the flow generated on $L(W)$ by the nonstationary Hamiltonian vector field $\vec{h}^1_{(\tau-c+2\epsilon)}$. By assumptions and (9.25), (9.26), the quadratic form $h^{A_0} - h^1_{(\tau-c+2\epsilon)}$ is nonnegative definite for any τ. In the case $A_0 > 0$ suppose also that $c \leq \frac{p_1}{\sqrt[4]{A_0}}$. Then in both cases by the previous lemma the curve $\tilde{\Gamma}(\tau)$, $0 \leq \tau \leq c - \epsilon$, belongs to $\Lambda(0)^{\Uparrow}$. Evidently, the subspaces $\tilde{\Upsilon}(0)$ and $\tilde{\Upsilon}(c - \epsilon)$ are also transversal to $\Lambda(0)$ for sufficiently small $\epsilon > 0$. So, one can apply Theorem 5 to the curves $\tilde{\Gamma}(\cdot)$ and $\tilde{\Upsilon}(\cdot)$ on the interval $[0, c - \epsilon]$:

$$\sum_{0 \leq \tau \leq c-\epsilon} \tilde{\Upsilon}(\tau) \cap \Lambda(0) - 2 \;\leq \sum_{0 \leq \tau \leq c-\epsilon} \tilde{\Gamma}(\tau) \cap \Lambda(0) = 0. \tag{9.30}$$

On the other hand, by definition $\tilde{\Upsilon}(c - 2\epsilon) = \Upsilon(0) = \Lambda(0)$. Therefore

$$\sum_{0 \leq \tau \leq c-\epsilon} \tilde{\Upsilon}(\tau) \cap \Lambda(0) \geq 2.$$

This together with (9.30) implies that $\sum_{0 \leq \tau \leq c-\epsilon} \tilde{\Upsilon}(\tau) \cap \Lambda(0) = 2$. But it means that for any $\tau \neq 0$ such that $2\epsilon - c \leq \tau \leq \epsilon$ we have $\Upsilon(\tau) \cap \Lambda(0) = 0$. By (9.10) this is equivalent to the fact that for any $\tau \neq 0$ such that $-\epsilon \leq \tau \leq c - 2\epsilon$ we have $\Lambda(\tau) \cap \Lambda(0) = 0$. Since $\epsilon > 0$ is arbitrary small, one can conclude that on the interval $(0, c)$ there are no points conjugate to 0 w.r.t. the curve $\Lambda(\cdot)$. Let us remind that in the case $A_0 > 0$ one can take $0 \leq c \leq \frac{p_1}{\sqrt[4]{A_0}}$. This completes the proof of the parts 2 of the theorem. In the case $A_0 \leq 0$ any positive number can be taken as c. Hence any positive τ is not conjugate to 0 w.r.t. $\Lambda(\cdot)$. By the arguments given in the beginning of the proof, we obtain that for any τ_0 all $\tau > \tau_0$ are not conjugate to τ_0 w.r.t. $\Lambda(\cdot)$. But then also all

$\tau < \tau_0$ are not conjugate to τ_0 w.r.t. $\Lambda(\cdot)$, since the notion of the conjugate points is symmetric. This completes the proof of the parts 1 of the theorem .

Now we prove the third statements of the theorem. Suppose that $A(\tau) \geq \frac{A_0}{35} > 0$. Again for some numbers ϵ, c such that $c > \epsilon > 0$ consider two curves

$$\bar{\Gamma}(\tau) = \Gamma(\tau - c + 2\epsilon), \;\; \bar{\Upsilon}(\tau) = \Upsilon(\tau - c)$$

on the interval $0 \leq \tau \leq c - \epsilon$. The first curve is again a trajectory of the flow generated on $L(W)$ by the stationary Hamiltonian vector field $\vec{h}^{A_0}$. The second curve is a trajectory of the flow generated on $L(W)$ by the nonstationary Hamiltonian vector field $\vec{h}^1_{(\tau-c)}$. By assumptions and (9.25), (9.26), the quadratic form $h^1_{(\tau-c)} - h^{A_0}$ is nonnegative definite for any τ. Suppose also that $c - 2\epsilon > \frac{p_1}{\sqrt[4]{A_0}}$ and $\epsilon < \frac{p_1}{\sqrt[4]{A_0}}$. Note that $\Gamma(0) = \Lambda(0)$. From the previous lemma on the interval $(2\epsilon - c, 0)$ there exists at least one point conjugate to 0 w.r.t. the curve $\Gamma(\cdot)$. Therefore we have

$$\sum_{0 \leq \tau \leq c-\epsilon} \bar{\Gamma}(\tau) \cap \Lambda(0) = \sum_{2\epsilon - c \leq \tau \leq \epsilon} \Gamma(\tau) \cap \Lambda(0) \geq 3 \tag{9.31}$$

Note also that the subspaces $\bar{\Upsilon}(0)$ and $\bar{\Upsilon}(c - \epsilon)$ are transversal to $\Lambda(0)$ for sufficiently small $\epsilon > 0$. Hence, using Theorem 5 and (9.31), we get the following inequality:

$$\sum_{-c \leq \tau \leq -\epsilon} \Upsilon(\tau) \cap \Lambda(0) = \sum_{0 \leq \tau \leq c-\epsilon} \bar{\Upsilon}(\tau) \cap \Lambda(0) \geq \sum_{0 \leq \tau \leq c-\epsilon} \bar{\Gamma}(\tau) \cap \Lambda(0) - 2 \geq 1, \tag{9.32}$$

This together with (9.10) implies that on the interval (ϵ, c) there is at least one point conjugate to 0 w.r.t. the curve $\Lambda(\cdot)$. Taking into account that $\epsilon > 0$ is arbitrary small and $c > \frac{p_1}{\sqrt[4]{A_0}} + 2\epsilon$, one can conclude that on the interval $(0, \frac{p_1}{\sqrt[4]{A_0}}]$ there is at least one point conjugate to 0 w.r.t. the curve $\Lambda(\cdot)$. The proof of the third part of the theorem is completed. $\square$

9.4 The case of arbitrary parameter

Now suppose that the Ricci curvature $\rho(t)$ of the curve $\Lambda(t)$ is not identically equal to zero. The method proposed here is in essence a reduction to the previous case by making reparametrization $\tau = \varphi(t)$ to some projective parameter τ. From (5.4) it follows that $\tau = \varphi(t)$ is a reparametrization to a projective parameter iff the function $\varphi(t)$ satisfies the following equation

$$\mathbb{S}\left(\varphi(t)\right) = \frac{3\rho(t)}{4}, \tag{9.33}$$

where $\mathbb{S}\left(\varphi(t)\right)$ is Schwartzian of the function $\varphi(t)$, $\rho(t)$ is the Ricci curvature (note that in the considered case the weight k of the curve $\Lambda(t)$ is equal to 4). By direct calculation one can obtain the following lemma that will be also useful in the sequel:

Lemma 9.2 *The monotone increasing function $\varphi(t)$ satisfies the equation (9.33) iff the function $y(t) = \frac{1}{\sqrt{\varphi'(t)}}$ satisfies the Hill's equation*

$$y'' + \frac{3\rho(t)}{4} y = 0 \tag{9.34}$$

Also, let us remember that the density $\bar{A}(\tau)$ of the fundamental form in the new parameter τ satisfies

$$\bar{A}(\tau) = \frac{A(t)}{\varphi'(t)^4}. \tag{9.35}$$

Using (9.35), Lemma 9.2 and Theorem 7 we obtain series of comparison theorems according to the different bounds of $\rho(t)$ and $A(t)$. We consider separately three cases corresponding to the three statements of the Theorem 7:

9.4.1 The case $A(t) \leq 0$. In this case we have the following

Theorem 8 *Let $\Lambda : I \mapsto L(W)$ be the curve of rank 1 and constant weight defined on the interval $I \subseteq \mathbb{R}$ such that $A(t) \leq 0$ for all $t \in I$. Then the following statements are valid:*

1. *If $\rho(t) \leq 0$ then for any point t_0 there are no points $t \neq t_0$ conjugate to t_0;*
2. *If $\rho(t) \leq \frac{4}{3}R$ for some constant $R > 0$ then for any t_0 in the interval $(t_0, t_0 + \frac{\pi}{\sqrt{R}})$ there are no points conjugate to t_0.*

Proof. From the classical Sturm's comparison theorem it follows that the equation (9.34) has positive solutions on the whole $\mathbb{R}$ if $\rho(t) \leq 0$ or on the interval $[t_0, t_0 + \frac{\pi}{\sqrt{R}})$ for any given t_0 if $\rho(t) \leq \frac{4}{3}R$ with $R > 0$. Then, by Lemma 9.2, the equation (9.33) has monotone increasing solution $\varphi(t)$ on the whole $\mathbb{R}$ in the first case and on the interval $[t_0, t_0 + \frac{\pi}{\sqrt{R}})$ in the second case. Let $\bar{\Lambda}(\tau) = \Lambda(\varphi^{-1}(\tau))$, $\tau_0 = \varphi(t_0)$. By construction, the curve $\bar{\Lambda}(\tau)$ has vanishing Ricci curvature. By assumptions and (9.35), the corresponding density $\bar{A}(\tau) \leq 0$. Then by Theorem 7 there are no points $\tau \neq \tau_0$ conjugate to τ_0 w.r.t. the curve $\bar{\Lambda}(\tau)$. But this is equivalent to the statements of the theorem. $\square$

9.4.2 The case $A(t) \leq \frac{A_0}{35}$, $A_0 > 0$. First we prove the following auxiliary lemma:

Lemma 9.3 *Let $\tau = \varphi(t)$ be reparametrization of the curve $\Lambda(t)$ to the projective parameter τ such that the function $\varphi(t)$ is monotone increasing. Assume that $A(t) \leq \frac{A_0}{35}$ and for some $s > 0$ the following inequality*

$$\frac{\varphi(t_0 + s) - \varphi(t)}{\min\limits_{t_0 \leq t \leq t_0 + s} \varphi'(t)} \leq \frac{p_1}{\sqrt[4]{A_0}} \tag{9.36}$$

holds. Then in the interval $(t_0, t_0 + s)$ there are no points conjugate to t_0 w.r.t. $\Lambda(t)$;

Proof. Let $\bar{\Lambda}(\tau) = \Lambda(\varphi^{-1}(\tau))$, $\tau_0 = \varphi(t_0)$, $\tau_s = \varphi(t_0+s)$. By construction, the curve $\bar{\Lambda}(\tau)$ has vanishing Ricci curvature. Denote by $\bar{A}(\tau)$ the density of the fundamental form $\mathcal{A}$, corresponding to the parameter τ. If $A(t) \le \frac{A_0}{35}$, then by (9.35) on the interval $[\tau_0, \tau_s]$, $\bar{A}(\tau)$ satisfies the following inequality

$$\bar{A}(\tau) = \frac{A(t)}{\big(\varphi'(t)\big)^4} \le \frac{A_0}{\min\limits_{t_0 \le t \le t_0+s} \big(\varphi'(t)\big)^4} \tag{9.37}$$

Then according to the second statement of Theorem 7, in the interval (τ_0, τ_s) there are no points conjugate to τ_0 w.r.t. the curve $\bar{\Lambda}(\tau)$ if the following inequality holds:

$$\frac{p_1}{\sqrt[4]{A_0}} \min_{t_0 \le t \le t_0+s} \varphi'(t) \ge \tau_s - \tau_0 = \varphi(t_0+s) - \varphi(t_0).$$

But this inequality is equivalent to (9.36). This concludes the proof of the lemma. $\Box$

Assume that

$$\frac{4}{3}r \le \rho(t) \le \frac{4}{3}R \tag{9.38}$$

Let $y_{q,c}(t)$ be the solution of the following equation with given initial conditions

$$y''(t) + qy(t) = 0,\ \ y(0) = 1,\ y'(0) = c,$$

namely,

$$y_{q,c}(t) = \cos\sqrt{q}\,t + \frac{c}{\sqrt{q}}\sin\sqrt{q}\,t = \begin{cases} \cosh\sqrt{-q}\,t + \frac{c}{\sqrt{-q}}\sinh\sqrt{-q}\,t & q<0 \\ \cos\sqrt{q}\,t + \frac{c}{\sqrt{q}}\sin\sqrt{q}\,t & q>0 \\ 1+ct & q=0. \end{cases} \tag{9.39}$$

Let M_c be the minimal positive zero of the function $y_{R,c}(t)$ if such zero exists and $M_c = \infty$ otherwise. Denote by

$$\mathcal{D}_R = \{(s,c) : 0 < s < M_c\}. \tag{9.40}$$

On the domain $\mathcal{D}_R$ define the following function

$$K(s,c) = \left(\max_{0 \le t \le s} y_{r,c}(t)\right)^2 \int\limits_0^s \frac{dt}{y_{R,c}^2(t)}. \tag{9.41}$$

It is easy to show that the function $s \mapsto K(s,c)$ is well defined monotone increasing function on $[0, M_c)$ which obtains on this set all nonnegative values. So, for any c there exists unique solution $s = S_{A_0}(c)$, $0 < S_{A_0}(c) < M_c$, of the equation

$$K(s,c) = \frac{p_1}{\sqrt[4]{A_0}} \tag{9.42}$$

Then the following lemma holds

Lemma 9.4 *For any t_0 on the interval $\Big(t_0, t_0 + \sup\limits_{c\in\mathbb{R}} S_{A_0}(c)\Big)$ there are no points conjugate to t_0 w.r.t. $\Lambda(t)$.*

Proof. Let $\tau = \varphi(t)$ be reparametrization of $\Lambda(t)$ to the projective parameter τ such that $\varphi'(t_0) = 1$ and $\varphi''(t_0) = -2c$. Then according to Lemma 9.2 the function $y(t) = \frac{1}{\sqrt{\varphi'(t)}}$ satisfies the Hill's equation with the prescribed initial conditions

$$y'' + \frac{3\rho(t)}{4}y = 0, \;\; y(t_0) = 1, \, y'(t_0) = c.$$

By classical Sturm's comparison theorem about Hill's equation, we have

$$y_{R,c}(t - t_0) \le \frac{1}{\sqrt{\varphi'(t)}} \le y_{r,c}(t - t_0) \tag{9.43}$$

for any $t \in [t_0, t_0 + M_c)$. Therefore for $0 \le s < M_c$ we obtain

$$\frac{\varphi(t_0+s) - \varphi(t)}{\min\limits_{t_0\le t\le t_0+s} \varphi'(t)} = \left(\max_{t_0\le t\le t_0+s} \frac{1}{\varphi'(t)}\right)\Big(\varphi(t_0+s) - \varphi(t)\Big) \le \left(\max_{0\le t\le s} y_{r,c}(t)\right)^2 \int\limits_0^s \frac{dt}{y^2_{R,c}(t)} = K(s,c).$$

Consequently, for $s = S_{A_0}(c)$

$$\frac{\varphi(t_0+s) - \varphi(t)}{\min\limits_{t_0\le t\le t_0+s} \varphi'(t)} \le \frac{p_1}{\sqrt[4]{A_0}}$$

Hence, according to Lemma 9.3 in the interval $\Big(t_0, t_0 + S_{A_0}(c)\Big)$ there are no points conjugate to t_0 w.r.t. $\Lambda(t)$. This concludes the proof of the lemma, since an arbitrary real number can be taken as c . $\square$

Lemma 9.4 shows that in order to find the lower bound for the length of the interval without conjugate points, one can find the maximum of the function $S_{A_0}(c)$ given implicitly by the equation $K(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$. It is not difficult to find the expressions for the function $K(s,c)$. It will be convenient to write these in terms of trigonometric functions. Namely,

1. if $r > 0$, then

$$K(s,c) = \begin{cases} \dfrac{\Big(\cos(\sqrt{r}\,s) + \frac{c}{\sqrt{r}}\sin(\sqrt{r}\,s)\Big)^2}{\sqrt{R}\,\cot(\sqrt{R}\,s) + c} & 0 \le s \le \dfrac{1}{\sqrt{r}}\arctan\left(\dfrac{c}{\sqrt{r}}\right) \quad (9.45a) \\[2ex] \dfrac{1 + \frac{c^2}{r}}{\sqrt{R}\,\cot(\sqrt{R}\,s) + c} & s > \dfrac{1}{\sqrt{r}}\arctan\left(\dfrac{c}{\sqrt{r}}\right),\, c \ge 0 \quad (9.45b) \\[2ex] \dfrac{1}{\sqrt{R}\,\cot(\sqrt{R}\,s) + c} & c < 0; \quad (9.45c) \end{cases}$$

2. if $r \le 0$, then

$$K(s,c) = \begin{cases} \dfrac{\Big(\cos(\sqrt{r}\,s) + \frac{c}{\sqrt{r}}\sin(\sqrt{r}\,s)\Big)^2}{\sqrt{R}\,\cot(\sqrt{R}\,s) + c} & c \ge -\sqrt{|r|}\tanh\Big(\dfrac{\sqrt{|r|}\,s}{2}\Big) \qquad (9.46a) \\[2ex] \dfrac{1}{\sqrt{R}\,\cot(\sqrt{R}\,s) + c} & c < -\sqrt{|r|}\tanh\Big(\dfrac{\sqrt{|r|}\,s}{2}\Big); \qquad (9.46b) \end{cases}$$

Note that for negative values of r and R the trigonometric functions with imaginary arguments can be replaced in the formulas above by corresponding hyperbolic functions with real arguments and if $r = 0$ (or $R = 0$) then the formulas above are replaced by their limits as r (or R) tends to zero.

Consider the cases of positive and nonnegative r separately

9.4.2 a) The case $r > 0$**.** On the domain $\mathcal{D}_R$ define the following function

$$K_1(s,c) \stackrel{def}{=} \frac{1 + \frac{c^2}{r}}{\sqrt{R}\,\cot(\sqrt{R}\,s) + c} \tag{9.47}$$

For any c let $s = S_{1,A_0}(c)$ be the minimal positive solutions of the equations

$$K_1(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$$

i.e.,

$$S_{1,A_0}(c) = \frac{1}{\sqrt{R}}\mathrm{arccot}\left(\frac{\sqrt[4]{A_0}}{p_1\sqrt{R}}\Big(1 + \frac{c^2}{r} - \frac{p_1}{\sqrt[4]{A_0}}c\Big)\right) \tag{9.48}$$

It is clear that in the consider case on the domain $\mathcal{D}_R$ the following inequality holds

$$K(c,s) \le K_1(c,s).$$

Hence we get $S_{1,A_0}(c) \le S_{A_0}(c)$ and

$$\sup_{c\in\mathbb{R}} S_{1,A_0}(c) \le \sup_{c\in\mathbb{R}} S_{A_0}(c). \tag{9.49}$$

From (9.48) it follows easily that

$$\max_{c\in\mathbb{R}} S_{1,A_0}(c) = \frac{1}{\sqrt{R}}\mathrm{arccot}\frac{1}{\sqrt{R}}\left(\frac{\sqrt[4]{A_0}}{p_1} - \frac{p_1 r}{4\sqrt[4]{A_0}}\right). \tag{9.50}$$

Applying Lemma 9.4 and (9.49) we obtain the following

Theorem 9 *Let $\Lambda : I \mapsto L(W)$ be the curve of rank 1 and constant weight defined on the interval $I \subseteq \mathbb{R}$. Suppose that for any $t \in I$ its invariants $\rho(t)$, $A(t)$ satisfy the following inequalities*

$$0 < \frac{4}{3}r \le \rho(t) \le \frac{4}{3}R, \quad A(t) \le \frac{A_0}{35}, \quad A_0 > 0.$$

Then for any t_0 in the interval

$$\left(t_0, t_0 + \frac{1}{\sqrt{R}}\operatorname{arccot}\left(\frac{1}{\sqrt{R}}\left(\frac{\sqrt[4]{A_0}}{p_1} - \frac{p_1 r}{4\sqrt[4]{A_0}}\right)\right)\right) \tag{9.51}$$

there are no points conjugate to t_0 w.r.t. the curve $\Lambda(t)$.

By consideration the function

$$K_2(s,c) \stackrel{def}{=} \frac{\left(\cos(\sqrt{r}\,s) + \frac{c}{\sqrt{r}}\sin(\sqrt{r}\,s)\right)^2}{\sqrt{R}\,\cot(\sqrt{R}\,s) + c} \tag{9.52}$$

from (9.45a) one can try to improve the previous theorem. For given s consider the following equation w.r.t. c :

$$K_2(s,c) = \frac{p_1}{\sqrt[4]{A_0}} \tag{9.53}$$

This equation is equivalent to the quadratic equation with discriminant

$$D(s) = \frac{p_1}{\sqrt[4]{A_0}}\left(\frac{p_1}{\sqrt[4]{A_0}} - L_{r,R}(s)\right), \tag{9.54}$$

where

$$L_{r,R}(s) \stackrel{def}{=} 4\left(\cos(\sqrt{r}s) - \sqrt{\frac{R}{r}}\sin(\sqrt{r}\,s)\cot(\sqrt{R}\,s)\right)\frac{\sin(\sqrt{r}\,s)}{\sqrt{r}} \tag{9.55}$$

It is not difficult to show that in the considered case ($R > r > 0$) the function $L_{r,R}(s)$ is monotone increasing on the interval $(0, \frac{\pi}{\sqrt{R}})$ and obtains all positive values on this interval , so the inverse function $L^{-1}_{r,R} : (0,\infty) \mapsto (0, \frac{\pi}{\sqrt{R}})$ is well defined. This together with (9.54) implies that the maximal value of s for the points (s,c) on the level set $K_2(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$ is obtained at the point $(\tilde{s},\tilde{c})$ such that

$$\tilde{s} = L^{-1}_{r,R}\left(\frac{p_1}{\sqrt[4]{A_0}}\right), \tag{9.56}$$

$$\tilde{c} = \sqrt{r}\cot\left(\sqrt{r}\,\tilde{s}\right) - 2\sqrt{R}\cot\left(\sqrt{R}\,\tilde{s}\right) \tag{9.57}$$

If at the same time

$$\tilde{s} < \frac{\pi}{2\sqrt{r}} \tag{9.58}$$

and

$$\tilde{c} > \sqrt{r}\tan(\sqrt{r}\,\tilde{s}), \tag{9.59}$$

then it is easy to see that $\sup\limits_{c\in\mathbb{R}} S_{A_0}(c) = \tilde{s} > \sup\limits_{c\in\mathbb{R}} S_{1,A_0}(c)$ (see formulas (9.45)). In this case we will improve Theorem 9 by putting in (9.51) $\tilde{s}$ instead of $\max\limits_{c\in\mathbb{R}} S_{1,A_0}(c)$. Indeed, if we denote by

$$N_{r,R}(s) \stackrel{def}{=} \sqrt{r}\cot\left(\sqrt{r}\,s\right) - 2\sqrt{R}\cot\left(\sqrt{R}\,s\right) - \sqrt{r}\tan\left(\sqrt{r}\,s\right) \tag{9.60}$$

then we have the following theorem in addition to Theorem 9:

Theorem 10 *Let $\Lambda : I \mapsto L(W)$ be the curve of rank* 1 *and constant weight defined on the interval $I \subseteq \mathbb{R}$. Suppose that for any $t \in I$ its invariants $\rho(t)$, $A(t)$ satisfy the following inequalities*

$$0 < \frac{4}{3} r \le \rho(t) \le \frac{4}{3} R, \quad A(t) \le \frac{A_0}{35}, \quad A_0 > 0.$$

Suppose also that

$$L_{r,R}^{-1}\left(\frac{p_1}{\sqrt[4]{A_0}}\right) < \frac{\pi}{2\sqrt{r}} \tag{9.61}$$

and

$$N_{r,R}\left(L_{r,R}^{-1}\Big(\frac{p_1}{\sqrt[4]{A_0}}\Big)\right) > 0 \tag{9.62}$$

then for any t_0 there are no points conjugate to t_0 w.r.t. $\Lambda(t)$ in the interval $\Big(t_0, t_0 + L_{r,R}^{-1}\Big(\frac{p_1}{\sqrt[4]{A_0}}\Big)\Big)$.

Note that if $\frac{\pi}{2\sqrt{r}} \ge \frac{\pi}{\sqrt{R}}$, i.e. $R \ge 4\,r$, then the condition (9.61) holds automatically. Also, one can check directly that in this case the function $N_{r,R}(s)$ is monotone increasing on the interval $(0, \frac{\pi}{\sqrt{R}})$ and obtains all real values there. So, for sufficiently small positive A_0 the condition (9.62) also holds. Hence for $R \ge 4\,r$ and sufficiently small positive A_0 Theorem 10 improves Theorem 9.

If $R < 4\,r$ then the function $N_{r,R}(s)$ tends to $-\infty$ at the points 0 and $\frac{\pi}{2\sqrt{r}}$. If $\frac{R}{r}$ is sufficiently close to 4 then $N_{r,R}(s)$ is positive on some subinterval of $\left(0, \frac{\pi}{2\sqrt{r}}\right)$ and Theorem 10 is relevant for A_0 with $L_{r,R}^{-1}\left(\frac{p_1}{\sqrt[4]{A_0}}\right)$ lying in this subinterval. But there exists $\alpha \in (1,4)$ such that for $1 \le \frac{R}{r} < \alpha$ the function $N_{r,R}(s)$ obtains only negative values on $\left(0, \frac{\pi}{2\sqrt{r}}\right)$. In this case Theorem 10 is not relevant.

9.4.2 b) The case $r \le 0$. Suppose that the functions $K_2(s,c)$ and $L_{r,R}(s)$ are as in (9.52) and (9.55) respectively. It is not difficult to show that in the considered case ($r < 0$, $r < R$) the function $L_{r,R}(s)$ is monotone increasing and obtains all positive values on $(0, \frac{\pi}{\sqrt{R}})$, if $R > 0$, and on $(0, \infty)$, if $R \le 0$. So the inverse function $L_{r,R}^{-1}$ is well defined on $(0,\infty)$ (with values in $(0, \frac{\pi}{\sqrt{R}})$, if $R > 0$, and in $(0,\infty)$, if $R \le 0$). As in the previous case, the maximal value of s for the points (s,c) on the level set $K_2(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$ is obtained at the point $(\tilde{s}, \tilde{c})$, satisfying the formulas (9.58) and (9.59). Note also that for any s the function

$$K_3(s,c) \overset{def}{=} \frac{1}{\sqrt{R}\cot(\sqrt{R}\,s) + c},$$

which appears in (9.45b), is monotone decreasing function of c. Therefore if $\tilde{c} > -\sqrt{r}\tan\Big(\frac{\sqrt{r}\,\tilde{s}}{2}\Big)$, then $\sup\limits_{c\in\mathbb{R}} S_{A_0}(c) = \tilde{s}$ and if $\tilde{c} \le -\sqrt{r}\tan\Big(\frac{\sqrt{r}\,\tilde{s}}{2}\Big)$, then $s = \sup\limits_{c\in\mathbb{R}} S_{A_0}(c)$ satisfies the following equation

$$K_3\Big(s, -\sqrt{|r|}\tanh\Big(\frac{\sqrt{|r|}\,s}{2}\Big)\Big) = \frac{p_1}{\sqrt[4]{A_0}} \tag{9.63}$$

(see formulas (9.45)).

Denote by

$$Q_{r,R}(s) \stackrel{def}{=} K_3(s, -\sqrt{|r|}\tanh\Big(\frac{\sqrt{|r|}\,s}{2}\Big) = \frac{1}{\sqrt{R}\,\cot\left(\sqrt{R}\,s\right) - \sqrt{|r|}\tanh\Big(\frac{\sqrt{|r|}\,s}{2}\Big)}. \tag{9.64}$$

and

$$N_{r,R}(s) \stackrel{def}{=} \sqrt{r}\cot\Big(\sqrt{r}\,s\Big) - 2\sqrt{R}\cot\Big(\sqrt{R}\,s\Big) + \sqrt{|r|}\tanh\Big(\frac{\sqrt{|r|}\,s}{2}\Big) \quad r \leq 0 \tag{9.65}$$

Evidently the function $Q_{r,R}(s)$ is monotone increasing and obtains all positive values on some interval of the type $(0, S_3)$. We get the following

Theorem 11 *Let $\Lambda : I \mapsto L(W)$ be the curve of rank 1 and constant weight defined on the interval $I \subseteq \mathbb{R}$. Suppose that for any $t \in I$ its invariants $\rho(t)$, $A(t)$ satisfy the following inequalities*

$$\frac{4}{3}r \leq \rho(t) \leq \frac{4}{3}R, \;\; r \leq 0, \;\; A(t) \leq \frac{A_0}{35}, \quad A_0 > 0.$$

Then the following two statements are valid

1. *If*

$$N_{r,R}\left(L_{r,R}^{-1}\Big(\frac{p_1}{\sqrt[4]{A_0}}\Big)\right) > 0 \tag{9.66}$$

then for any t_0 there are no points conjugate to t_0 w.r.t. $\Lambda(t)$ in the interval

$$\left(t_0, t_0 + L_{r,R}^{-1}\Big(\frac{p_1}{\sqrt[4]{A_0}}\Big)\right). \tag{9.67}$$

2. *If*

$$N_{r,R}\left(L_{r,R}^{-1}\Big(\frac{p_1}{\sqrt[4]{A_0}}\Big)\right) \leq 0 \tag{9.68}$$

then for any t_0 there are no points conjugate to t_0 w.r.t. $\Lambda(t)$ in the interval

$$\left(t_0, t_0 + Q_{r,R}^{-1}\Big(\frac{p_1}{\sqrt[4]{A_0}}\Big)\right). \tag{9.69}$$

It is not difficult to show that for $r \leq 0$ the function $N_{r,R}(s)$ is monotone increasing on $(0, \frac{\pi}{\sqrt{R}})$, if $R > 0$, and on $(0, \infty)$, if $R \leq 0$. Moreover, for $R > 0$ the function $N_{r,R}(s)$ obtains all real values on $(0, \frac{\pi}{\sqrt{R}})$ and for $R \leq 0$ the function $N_{r,R}(s)$ obtains on $(0, \infty)$ all real values less than $2(\sqrt{|r|} - \sqrt{|R|})$ that is positive number . Therefore, in both cases sufficiently small positive A_0 corresponds to the first part of the previous theorem , whenever sufficiently big A_0 corresponds to the second part.

9.4.3 The case $A(t) \geq \frac{A_0}{35} > 0$. First by analogy with Lemma 9.3 we prove the following auxiliary lemma:

Lemma 9.5 *Let $\tau = \varphi(t)$ be reparametrization of the curve $\Lambda(t)$ to the projective parameter τ such that the function $\varphi(t)$ is monotone increasing. Assume that $A(t) \geq \frac{A_0}{35}$ and for some $s > 0$ the following inequality*

$$\frac{\varphi(t_0 + s) - \varphi(t)}{\max\limits_{t_0 \leq t \leq t_0+s} \varphi'(t)} \geq \frac{p_1}{\sqrt[4]{A_0}} \tag{9.70}$$

holds. Then in the interval $(t_0, t_0 + s]$ there exists at least one point conjugate to t_0 w.r.t. $\Lambda(t)$.

Proof. Let $\bar{\Lambda}(\tau) = \Lambda(\varphi^{-1}(\tau))$, $\tau_0 = \varphi(t_0)$, $\tau_s = \varphi(t_0 + s)$. By construction, the curve $\bar{\Lambda}(\tau)$ has vanishing Ricci curvature. Denote by $\bar{A}(\tau)$ the density of the fundamental form $\mathcal{A}$, corresponding to the parameter τ. If $A(t) \geq \frac{A_0}{35}$, then by (9.35) on the interval $[\tau_0, \tau_s]$, $\bar{A}(\tau)$ satisfies the following inequality

$$\bar{A}(\tau) = \frac{A(t)}{\big(\varphi'(t)\big)^4} \geq \frac{A_0}{\max\limits_{t_0 \leq t \leq t_0+s} \big(\varphi'(t)\big)^4} \tag{9.71}$$

Then according to the third statement of Theorem 7, in the interval $(\tau_0, \tau_s]$ there exists at least one point conjugate to τ_0 w.r.t. the curve $\bar{\Lambda}(\tau)$ if the following inequality holds:

$$\frac{p_1}{\sqrt[4]{A_0}} \max_{t_0 \leq t \leq t_0+s} \varphi'(t) \leq \tau_s - \tau_0 = \varphi(t_0 + s) - \varphi(t_0).$$

But this inequality is equivalent to (9.70). This concludes the proof of the lemma. □

Assume again that

$$\frac{4}{3} r \leq \rho(t) \leq \frac{4}{3} R \tag{9.72}$$

and let $y_{q,c}(t)$ be as in (9.39). As before, let M_c be the minimal positive zero of the function $y_{R,c}(t)$ if such zero exists and $M_c = \infty$ otherwise. Also, let $\mathcal{D}_R$ be as in (9.40) . On the domain $\mathcal{D}_R$ define the following function

$$\overline{K}(s, c) = \left(\min_{0 \leq t \leq s} y_{R,c}(t) \right)^2 \int\limits_0^s \frac{dt}{y_{r,c}^2(t)}. \tag{9.73}$$

By analogy to the Lemma 9.4 we have the following:

Lemma 9.6 *If for some $s > 0$ there exists c such that $(s, c) \in \mathcal{D}_R$ and*

$$\overline{K}(s, c) \geq \frac{p_1}{\sqrt[4]{A_0}}, \tag{9.74}$$

then for any t_0 in the interval $(t_0, t_0 + s]$ there exists at least one point conjugate to t_0 w.r.t. $\Lambda(t)$.

Proof. Let $\tau = \varphi(t)$ be reparametrization of $\Lambda(t)$ to the projective parameter τ such that $\varphi(t_0) = 1$ and $\varphi'(t_0) = -2c$. Then, using (9.43) we have

$$\frac{\varphi(t_0+s)-\varphi(t)}{\max\limits_{t_0\le t\le t_0+s}\varphi'(t)} = \left(\min_{t_0\le t\le t_0+s}\frac{1}{\varphi'(t)}\right)\Big(\varphi(t_0+s)-\varphi(t)\Big) \ge \left(\min_{0\le t\le s} y_{R,c}(t)\right)^2\int\limits_0^s \frac{dt}{y_{r,c}^2(t)} = \overline{K}(s,c).$$

Consequently, if $\overline{K}(s,c) \ge \frac{p_1}{\sqrt[4]{A_0}}$ then

$$\frac{\varphi(t_0+s)-\varphi(t)}{\max\limits_{t_0\le t\le t_0+s}\varphi'(t)} \ge \frac{p_1}{\sqrt[4]{A_0}}$$

Hence, according to Lemma 9.5 on the interval (t_0, t_0+s) there exists at least one point conjugate to t_0 w.r.t. $\Lambda(t)$. This concludes the proof of the lemma. $\square$

Lemma 9.6 shows that in order to find the upper bound for the next conjugate point one can find the minimum of the function $s = \overline{S}_{A_0}(c)$ given implicitly by the equation $\overline{K}(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$, where $0 < s < M_c$. In opposite to the case considered in the subsection 8.4.2 (where we analyze the level sets of the function $K(s,c)$) the level sets $\overline{K}(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$ can be empty for sufficiently small positive A_0 and in this case we are not able to estimate the next conjugate point. Also, in general the function $\overline{S}_{A_0}(c)$ for some c is not defined or can be multi-valued.

It is not difficult to find the expressions for the function $\overline{K}(s,c)$. Namely,

1. if $R < 0$ then

$$\overline{K}(s,c) = \begin{cases} \dfrac{\left(\cosh(\sqrt{|R|}\,s) + \frac{c}{\sqrt{|R|}}\sinh(\sqrt{|R|}\,s)\right)^2}{\sqrt{|r|}\,\coth(\sqrt{|r|}\,s)+c} & c \le -\sqrt{|R|}\tanh(\sqrt{|R|}\,s) \qquad (9.75a) \\[2ex] \dfrac{1+\frac{c^2}{R}}{\sqrt{|r|}\,\coth(\sqrt{|r|}\,s)+c} & -\sqrt{|R|}\tanh\sqrt{|R|}\,s \le c \le 0 \qquad (9.75b) \\[2ex] \dfrac{1}{\sqrt{|r|}\,\coth(\sqrt{|r|}\,s)+c} & c > 0; \qquad (9.75c) \end{cases}$$

2. if $R \ge 0$ then

$$\overline{K}(s,c) = \begin{cases} \dfrac{\left(\cos(\sqrt{R}\,s) + \frac{c}{\sqrt{R}}\sin(\sqrt{R}\,s)\right)^2}{\sqrt{r}\,\cot(\sqrt{r}\,s)+c} & c \le \sqrt{R}\tan\Big(\dfrac{\sqrt{R}\,s}{2}\Big) \qquad (9.76a) \\[2ex] \dfrac{1}{\sqrt{r}\,\cot(\sqrt{r}\,s)+c} & c > \sqrt{R}\tan\Big(\dfrac{\sqrt{R}\,s}{2}\Big); \qquad (9.76b) \end{cases}$$

Consider the cases of negative and nonnegative R separately

9.4.3 a) The case $R < 0$.

Theorem 12 *Let $\Lambda : I \mapsto L(W)$ be the curve of rank 1 and constant weight defined on the interval $I \subseteq \mathbb{R}$. Suppose that for any $t \in I$ its invariants $\rho(t)$, $A(t)$ satisfy the following inequalities*

$$\frac{4}{3}r \le \rho(t) \le \frac{4}{3}R < 0, \quad A(t) \ge \frac{A_0}{35} > 0.$$

Suppose also that the bounds A_0, r, and R satisfy the following relation

$$\frac{2}{\sqrt{|r|} + \sqrt{|r| - |R|}} > \frac{p_1}{\sqrt[4]{A_0}} \tag{9.77}$$

Then for any t_0 in the interval

$$\left(t_0, t_0 + \frac{1}{\sqrt{|r|}}\operatorname{arccoth}\left(\frac{1}{\sqrt{|r|}}\left(\frac{\sqrt[4]{A_0}}{p_1} - \frac{p_1 R}{4\sqrt[4]{A_0}}\right)\right)\right] \tag{9.78}$$

there exists at least one point conjugate to t_0 w.r.t. the curve $\Lambda(t)$.

Proof. On the domain $\mathcal{D}_R$ define the following function:

$$\overline{K_1}(s,c) \stackrel{def}{=} \frac{1 + \frac{c^2}{R}}{\sqrt{r}\,\cot(\sqrt{r}\,s) + c} = \frac{1 + \frac{c^2}{R}}{\sqrt{|r|}\,\coth(\sqrt{|r|}\,s) + c}. \tag{9.79}$$

From the elementary properties of the hyperbolic functions one can obtain the following inequality

$$\overline{K}(s,c) \ge \overline{K_1}(s,c). \tag{9.80}$$

Therefore, if the point $(s,c) \in \mathcal{D}_R$ and belongs to the level set

$$\overline{K_1}(s,c) = \frac{p_1}{\sqrt[4]{A_0}}, \tag{9.81}$$

then by the Lemma 9.6 for any t_0 in the interval $(t_0, t_0 + s]$ there exists at least one point conjugate to t_0 w.r.t. $\Lambda(t)$. First, by direct computation, one can easily show that

$$\sup_{\mathcal{D}_R} \overline{K_1}(s,c) = \frac{2}{\sqrt{|r|} + \sqrt{|r| - |R|}}. \tag{9.82}$$

(but at the same time $\overline{K_1}(s,c) < \frac{2}{\sqrt{|r|}+\sqrt{|r|-|R|}}$ for all $(s,c) \in \mathcal{D}_R$). Hence the level set (9.81) is not empty iff the bounds A_0, r, and R satisfy the inequality (9.77).

Now suppose that (9.77) holds and for given c consider the equation (9.81), as the equation w.r.t. s. Evidently, this equation is equivalent to the following one

$$\coth(\sqrt{|r|}s) = \frac{\sqrt[4]{A_0}}{p_1\sqrt{|r|}}\left(1 + \frac{c^2}{R} - \frac{p_1}{\sqrt[4]{A_0}}c\right). \tag{9.83}$$

Denote the right-hand side of the last equation by $\sigma(c)$. Since for $x > 0$ the function $\coth(x)$ is monotone decreasing, the equation (9.83) has minimal positive solution s_{min} for $c = \frac{p_1 R}{2\sqrt[4]{A_0}}$, corresponding to the global maximum of the function $\sigma(c)$. It is easy to see also that

$$\max \sigma(c) = \frac{1}{\sqrt{|r|}}\Big(\frac{\sqrt[4]{A_0}}{p_1} - \frac{p_1 R}{4\sqrt[4]{A_0}}\Big).$$

Therefore from (9.83) we obtain that if the condition (9.77) holds then

$$s_{min} = \frac{1}{\sqrt{|r|}}\operatorname{arccoth}\bigg(\frac{1}{\sqrt{|r|}}\Big(\frac{\sqrt[4]{A_0}}{p_1} - \frac{p_1 R}{4\sqrt[4]{A_0}}\Big)\bigg) \tag{9.84}$$

Further from (9.80) it follows that, if the condition (9.77) holds then we have

$$\overline{K}\left(s_{min}, \frac{p_1 R}{2\sqrt[4]{A_0}}\right) \geq \overline{K}_1\left(s_{min}, \frac{p_1 R}{2\sqrt[4]{A_0}}\right) = \frac{p_1}{\sqrt[4]{A_0}}.$$

So, by Lemma 9.6 in the interval $(t_0, t_0 + s_{min}]$ there exists at least one point conjugate to t_0 w.r.t. $\Lambda(t)$. This together with (9.84) completes the proof of the theorem. □

Remark 15 *It is not difficult to show that the following relation*

$$\sup_{\mathcal{D}_R} \overline{K}(s,c) = \sup_{\mathcal{D}_R} \overline{K_1}(s,c)$$

holds. In addition, the minimal value of s for the points (s,c) on the level set $\overline{K}(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$ is equal to the minimal value of s for the points (s,c) on the level set $\overline{K_1}(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$. All these show that in spite of the inequality (9.80) the replacement of the function $\overline{K}(s,c)$ by the function $\overline{K_1}(s,c)$ in the proof of the previous theorem leads to the same estimate for the length of intervals containing conjugate points.

9.4.3 b) The case $R \geq 0$.

On the interval $(0, \frac{\pi}{\sqrt{R}})$ define the function

$$\overline{Q}_{r,R}(s) \stackrel{def}{=} \overline{K}\left(s, \sqrt{R}\tan\Big(\frac{\sqrt{R}\,s}{2}\Big)\right) = \frac{1}{\sqrt{r}\,\cot\left(\sqrt{r}\,s\right) + \sqrt{R}\tan\Big(\frac{\sqrt{R}s}{2}\Big)}. \tag{9.85}$$

Note that $\overline{Q}_{r,R}(s) > 0$ on the interval $(0, \frac{\pi}{\sqrt{R}})$. Also, $\overline{Q}_{r,R}(s) \underset{s\to 0}{\longrightarrow} 0$ and $\overline{Q}_{r,R}(s) \underset{s\to\frac{\pi}{\sqrt{R}}}{\longrightarrow} 0$. So, there exists $s_{\max} \in (0, \frac{\pi}{\sqrt{R}})$ such that $\overline{Q}_{r,R}(s)$ obtains its maximum at $s = s_{\max}$. Moreover, it is not difficult to show that on the interval $(0, s_{\max}]$ the function $\overline{Q}_{r,R}(s)$ is monotone increasing. So, the inverse function $\overline{Q}_{r,R}^{-1} : (0, \overline{Q}_{r,R}(s_{\max})] \mapsto (0, s_{\max}]$ (or the branch of the multi-valued function inverse to $\overline{Q}_{r,R}$ that obtains its values on $(0, s_{\max}]$) is well defined.

Theorem 13 *Let $\Lambda : I \mapsto L(W)$ be the curve of rank 1 and constant weight defined on the interval $I \subseteq \mathbb{R}$. Suppose that for any $t \in I$ its invariants $\rho(t)$, $A(t)$ satisfy the following inequalities*

$$\frac{4}{3}r \leq \rho(t) \leq \frac{4}{3}R, \quad R \geq 0, \quad A(t) \geq \frac{A_0}{35} > 0.$$

Suppose also that the bounds A_0, r, and R satisfy the following relation

$$\max_{0<s<\frac{\pi}{\sqrt{R}}} \overline{Q}_{r,R}(s) \geq \frac{p_1}{\sqrt[4]{A_0}} \tag{9.86}$$

Then for any t_0 in the interval

$$\left(t_0, t_0 + \overline{Q}_{r,R}^{-1}\left(\frac{p_1}{\sqrt[4]{A_0}}\right)\right] \tag{9.87}$$

there exists at least one point conjugate to t_0 w.r.t. the curve $\Lambda(t)$.

Proof. Note that in our case the domain $\mathcal{D}_R$ has the following form

$$\mathcal{D}_R = \left\{(s,c) : c > -\sqrt{R}\cot(\sqrt{R}\,s)\right\}.$$

For given s, $0 < s < \frac{\pi}{\sqrt{R}}$, consider the function $c \mapsto \overline{K}(s,c)$, where $c > -\sqrt{R}\cot(\sqrt{R}\,s)$. From formulas (9.76) it follows that $\overline{K}(s,c) \longrightarrow 0$, as $c \to -\sqrt{R}\cot(\sqrt{R}\,s)$ or $c \to \infty$. Note also that for $0 < s < \frac{\pi}{\sqrt{R}}$ the following inequality

$$\sqrt{R}\ \tan(\frac{\sqrt{R}\,s}{2}) > -\sqrt{R}\cot(\sqrt{R}\,s)$$

holds. Evidently, the function $c \mapsto \overline{K}(s,c)$ is monotone decreasing for $c > \sqrt{R}\ \tan(\frac{\sqrt{R}\,s}{2})$. Also, by direct calculation one can easily get that the function $c \mapsto \overline{K}(s,c)$ is monotone increasing for $-\sqrt{R}\cot(\sqrt{R}\,s) < c < \sqrt{R}\ \tan\left(\frac{\sqrt{R}\,s}{2}\right)$. Hence for all $(s,c) \in \mathcal{D}_R$ one have the following relation:

$$\overline{K}(s,c) \leq \overline{K}\left(s, \sqrt{R}\ \tan\left(\frac{\sqrt{R}\,s}{2}\right)\right) = Q_{r,R}(s). \tag{9.88}$$

Consequently,

$$\sup_{\mathcal{D}_R} \overline{K}(s,c) = \max_{0<s<\frac{\pi}{\sqrt{R}}} Q_{r,R}(s),$$

which implies that the level set $\overline{K}(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$ is not empty iff the bounds A_0, r, and R satisfy the inequality (9.86).

Now suppose that (9.86) holds. If $s < \overline{Q}_{r,R}^{-1}(\frac{p_1}{\sqrt[4]{A_0}})$, then from monotonicity of $Q_{r,R}$ on the interval $(0, s_{\max})$ and relation (9.88) it follows that

$$\overline{K}(s,c) < Q_{r,R}(s) < \frac{p_1}{\sqrt[4]{A_0}}.$$

On the other hand, if $s = \overline{Q}_{r,R}^{-1}(\frac{p_1}{\sqrt[4]{A_0}})$, then $K\left(s, \sqrt{R}\ \tan\left(\frac{\sqrt{R}\,s}{2}\right)\right) = \frac{p_1}{\sqrt[4]{A_0}}$. So, $s = \overline{Q}_{r,R}^{-1}(\frac{p_1}{\sqrt[4]{A_0}})$ is minimal value of s for the points (s,c) on the level set $\overline{K}(s,c) = \frac{p_1}{\sqrt[4]{A_0}}$. Our theorem follows now from Lemma 9.6. $\Box$

10 Equivalence of rank 2 vector distributions

10.1 Preliminary

Rank k vector distribution D on the n-dimensional manifold M (where $k < n$) is by definition a k-dimensional subbundle of the tangent bundle TM. In other words, for each point $q \in M$ a k-dimensional subspace $D(q)$ of the tangent space T_qM is chosen and $D(q)$ depends smoothly on q. Two vector distributions D_1 and D_2 are called equivalent, if there exists a diffeomorphism $F : M \mapsto M$ such that $F_*D_1(q) = D_2(F(q))$ for any $q \in M$. Two germs of vector distributions D_1 and D_2 at the point $q_0 \in M$ are called equivalent, if there exist a neighbourhood U of q_0 and a diffeomorphism $F : U \mapsto U$ such that

$$\begin{gathered} F_*D_1(q) = D_2(F(q)), \ \forall q \in U; \\ F(q_0) = q_0. \end{gathered}$$

The question is, *when two germs of distributions are equivalent?*

First, one can define the lth power D^l of the distribution D: $D^l = [D, D^{l-1}]$. The tuple

$$\Big(\dim D(q), \dim D^2(q), \dim D^3(q), \dots\Big) \tag{10.1}$$

is called *the small growth vector* of the distribution D at the point q. Obviously, all entries of the small growth vector (10.1) are invariants of the germ of the distribution D at the point q.

We restrict ourselves to the case $k = 2$. We will work with generic germs of rank 2 distribution: by genericity conditions we mean here that all entries of the small growth vector are maximal possible. In particular second and third entries of the small growth vector have the form:

$$\dim D^2 = 3, \qquad \dim D^3 = \min\{5, n\} \tag{10.2}$$

Let us roughly estimate the "number of parameters" in our equivalence problem. The set of planes in $\mathbb{R}^n$ forms $2(n-2)$-dimensional manifold. Therefore, if the coordinates on M are fixed then the rank 2 distribution can be defined by $2(n-2)$ functions of n variables. The group of the coordinate changes on M is parameterized by n functions of n variables. So, by a coordinate change one can "normalize", in general, only n functions among those $2(n-2)$ functions, defining the distribution. Thus, one may expect that the set of classes of equivalent germs of rank 2 distributions can be parameterized by $2(n-2) - n = n - 4$ arbitrary germs of functions of n variables.

If $n = 3$ or 4, then, by above, the equivalence problem should be without functional parameters. Moreover, it is well known that in these cases all generic germs of rank 2 distributions are equivalent (Darboux's theorem in the case $n = 3$ and Engel's theorem in the case $n = 4$, see, for example, [9], [14],[22]).

On the other hand, for $n \geq 5$ functional invariants should appear. For $n = 5$ E. Cartan in [10] constructed the covariant fourth-order tensor invariant for rank 2 distribution distributions, using his reduction-prolongation method for equivalence problem of geometric structures on manifolds. In this section we will present completely different, more geometric way for construction of functional invariants of rank 2 distribution on n-dimensional manifold with $n \geq 5$, using the general theory of curves in the Lagrange Grassmannian, developed in the previous sections.

10.2 General scheme for construction of fundamental form of rank 2 vector distributions

In this subsection we will give general scheme for construction of fundamental form of the rank 2 distribution on the n-dimensional manifold M ($n \geq 5$) with the help of the notion of the fundamental form of unparameterized curve in the Lagrange Grassmannian, defined in Section 5.

Let us introduce some notations. Denote by $(D^l)^\perp \subset T^*M$ the annihilator of the lth power D^l, namely

$$(D^l)^\perp = \{(q,p) \in T^*M :\ p \cdot v = 0\ \forall v \in D^l(q)\}, \tag{10.3}$$

Also denote

$$(D^l)^\perp(q) = (D^l)^\perp \cap T_q^*M. \tag{10.4}$$

For any q the set $(D^l)^\perp(q)$ is linear subspace of T_q^*M.

For given linear space L denote by $HR_4(L)$ the space of all degree 4 homogeneous rational functions on L Introduce the equivalence relation $\sim$ on $HR_4(L)$, as follows: two functions $A_1(\cdot)$, $A_2(\cdot) \in HR_4(L)$ are equivalent iff there exists a positive real number λ such that $A_2(\cdot) = \lambda A_1(\cdot)$. The set of all equivalence classes of $HR_4(L)$ w.r.t. this equivalence relation will be denoted by $P^+HR_4(L)$. In other words, $P^+HR_4(L) = HR_4(L)/\sim$. The fundamental form of the distribution D will be the mapping that assign canonically to any $q \in M$ some element of $P^+HR_4\Big((D^2)^\perp(q)\Big)$.

In the sequel we will need also the notion of the Euler field $\vec{e}$ on T^*M, which is defined as follows: Let, as before, ς be the canonical Liouville form on T_q^*M, $\varsigma_\lambda = \lambda \circ \pi_*$, $\lambda \in T^*M$, and $\sigma = -d\varsigma$ be the standard symplectic structure on T^*M. The Euler field $\vec{e}$ is uniquely defined by the identity

$$\sigma(\cdot, \vec{e}) = \varsigma(\cdot) \tag{10.5}$$

Now we start with construction of the fundamental form of the distribution D. In the Introduction the "dual" normal and abnormal variety H^1 and H^0 were defined for any smooth submanifold of the tangent bundle (see formulas (1.6),(1.7)). It is clear that for distribution D the corresponding normal variety is empty and the abnormal variety H^0 coincides with $(D)^\perp$. In the case of rank 2 distribution the codinemsion of $D^{\perp}$ in T^*M is equal to 2, so, we can apply the scheme for construction of Jacobi curves given in Introduction.

Let X_1, X_2 be two vector fields , constituting the basis of distribution D , i.e., $D(q) = \mathrm{span}(X_1(q), X_2(q))$ $\forall q \in M$. Since our study is local, we can always suppose that such basis exists, restricting ourselves, if necessary, on some coordinate neighbourhood instead of whole M. Suppose that

$$X_3 = [X_1, X_2],\;\; X_4 = [X_1, [X_1, X_2]] = [X_1, X_3],\;\;\; X_5 = [X_2, [X_2, X_1]] = [X_3, X_2] \tag{10.6}$$

Let us introduce "quasiimpulses" $u_i : T^*M \mapsto \mathbb{R}$, $1 \le i \le 5$.

$$u_i(\lambda) = p \cdot X_i(q),\;\; \lambda = (p, q),\;\; q \in M,\;\; p \in T^*_q M \tag{10.7}$$

Then by above

$$H^0 = D^{\perp} = \{\lambda \in T^*M : u_1(\lambda) = u_2(\lambda) = 0\} \tag{10.8}$$

From (10.8) it follows that for any $\lambda \in D^{\perp}$ the plane $\overrightarrow{H_D}(\lambda) \subset T_\lambda(T^*M)$ such that

$$\overrightarrow{H_D}(\lambda) = \mathrm{span}(\vec{u}_1(\lambda), \vec{u}_2(\lambda)) \tag{10.9}$$

does not depend on the choice of the basis of distribution.

By direct calculation (see [20] and [2] for the details) one can get that the set, where $\sigma|_{D^{\perp}}$ is degenerate, coincides with $(D^2)^{\perp}$, and for any $\lambda \in (D^2)^{\perp}$ one has

$$\ker \sigma_\lambda|_{D^{\perp}} = \overrightarrow{H_D}(\lambda) \tag{10.10}$$

The regular part C_{H^0} of this set, defined by formula (1.4), satisfies

$$C_{H^0} = (D^2)^{\perp} \backslash (D^3)^{\perp} = \{\lambda \in T^*M : u_1(\lambda) = u_2(\lambda) = u_3(\lambda) = 0, (u_4^2 + u_5^2)(\lambda) \neq 0\} \tag{10.11}$$

and for any $\lambda \in C_{H^0}$ we have

$$\ker \sigma_\lambda|_{H^0} \cap T_\lambda C_H = \mathbb{R}\Big((u_4\vec{u_2} + u_5\vec{u_1})(\lambda)\Big) \tag{10.12}$$

The kernels of $\sigma|_{C_{H^0}}$ form characteristic 1-foliation $\mathcal{C}$ on C_{H^0}. The leaves of this foliation are associated with abnormal extremals of the distribution D. The Jacobi curves of these leaves are defined by formula (1.5).

Let us describe these Jacobi curves in more explicit way. Denote by h the following Hamiltonian

$$h = u_4 u_2 + u_5 u_1 \tag{10.13}$$

For any $\lambda \in (D^2)^{\perp}$ the corresponding Hamiltonian vector field $\vec{h}$ satisfies $\vec{h} = u_4\vec{u}_2 + u_5\vec{u}_1$. From (10.6) it follows easily that $\vec{h}$ is well-defined vector field on $(D^2)^{\perp}$ with stationary points on $(D^3)^{\perp}$. By (10.12) the integral curves of $\vec{h}$ are parametrized leaves of the characteristic foliation $\mathcal{C}$.

Remark 16 *From the homogeneity of the Hamiltonian h on the fibres it follows easily that the Hamiltonian vector field $\vec{h}$ satisfies the folloing property: if the curve $\lambda(t) = \big(p(t), q(t)\big)$ is the integral curve of $\vec{h}$, then for any α the curve $\lambda_\alpha(t) = \big(\alpha p(\alpha t), q(\alpha t)\big)$ is also the integral curve of $\vec{h}$.*

For any $\lambda \in (D^2)^\perp$ denote by L_λ the following subspace of $T_\lambda\big((D^2)^\perp\big)$:

$$L_\lambda = \Big(T_\lambda\big(D^\perp(\pi(\lambda))\big) + \overrightarrow{H_D}(\lambda)\Big) \bigcap T_\lambda\big((D^2)^\perp\big) \tag{10.14}$$

Let

$$\widetilde{W}_\lambda = T_\lambda((D^2)^\perp)/\mathbb{R}\vec{h}(\lambda).$$

For $\lambda \in (D^2)^\perp \backslash (D^3)^\perp$ the space $\widetilde{W}_\lambda$ is symplectic space endowed with symplectic structure induced by $\sigma_\lambda|_{(D^2)^\perp}$. Denote by $\widetilde{p}_\lambda$ the canonical projection of $T_\lambda((D^2)^\perp)$ on the factor-space $\widetilde{W}_\lambda$. From formulas (1.5) and (10.10) it follows that for any $\lambda \in (D^2)^\perp \backslash (D^3)^\perp$ the curve

$$t \mapsto \widetilde{J_{\lambda,\vec{h}}}(t) \overset{def}{=} \widetilde{p}_\lambda \circ (e^{-t\vec{h}})_*\big(L_{e^{t\vec{h}}\lambda}\big) \tag{10.15}$$

is the Jacobi curve of the characteristic leaf $e^{t\vec{h}}\lambda$, parameterized by the vector field $\vec{h}$. The curve $t \mapsto \widetilde{J_{\lambda,\vec{h}}}(t)$ is the curve of $(n-2)$-dimensional Lagrange subspaces in $2(n-2)$-dimensional symplectic space $\widetilde{W}_\lambda$.

Note that the Euler field $\vec{e}(\lambda)$ is tangent to the manifold $(D^2)^\perp$. Using Remark 16 it is not difficult to show that:

$$\big(e^{-t\vec{h}}\big)_*\big(\vec{e}(e^{t\vec{h}}\lambda)\big) \in \operatorname{span}\big(\vec{h}(\lambda), \vec{e}(\lambda)\big),$$

which together with (10.15) implies that

$$\mathbb{R}\,\widetilde{p}_\lambda\big(\vec{e}(\lambda)\big) \subset \widetilde{J_{\lambda,\vec{h}}}(t).$$

In other words, all subspaces $\widetilde{J_{\lambda,\vec{h}}}(t)$ contain a common line $\mathbb{R}\,\widetilde{p}_\lambda\big(\vec{e}(\lambda)\big)$. Therefore, one can make the reduction of dimensions: the curve

$$t \mapsto J_{\lambda,\vec{h}}(t) \overset{def}{=} \widetilde{J_{\lambda,\vec{h}}}(t)/\mathbb{R}\,\widetilde{p}_\lambda\big(\vec{e}(\lambda)\big) \tag{10.16}$$

is the curve of $(n-3)$-dimensional Lagrange subspaces in $2(n-3)$-dimensional symplectic space

$$W_\lambda = \widetilde{p}_\lambda\big(\vec{e}(\lambda)\big)^\angle/\widetilde{p}_\lambda\big(\vec{e}(\lambda)\big),$$

where $\widetilde{p}_\lambda\big(\vec{e}(\lambda)\big)^\angle$ is the skew-symmetric complement of $\widetilde{p}_\lambda(e(\lambda))$ in $\widetilde{W}_\lambda$. The curve $t \mapsto J_{\lambda,\vec{h}}(t)$ is called *reduced Jacobi curve* of the characteristic leaf $e^{t\vec{h}}\lambda$, parameterized by the vector field $\vec{h}$.

Let us study the properties of the curve $t \mapsto J_{\vec{h},\lambda}(t)$. Denote by p_λ the canonical projection of $T_\lambda\big((D^2)^\perp\big)$ on the factor-space $T_\lambda\big((D^2)^\perp\big)/\text{span}\big(\vec{h}(\lambda), \vec{e}(\lambda)\big)$. In any linear space $J_{\vec{h},\lambda}(t)$ one can distinguish the following codimension 1 subspace

$$VS_\lambda(t) = p_\lambda \circ (e^{-t\vec{h}})_* \Big(T_{e^{t\vec{h}}\lambda}\big((D^2)^\perp(\pi e^{t\vec{h}}\lambda)\big)\Big) \tag{10.17}$$

that is actually image under $p_\lambda \circ (e^{-t\vec{h}})_*$ of the vertical subspace of the tangent space $T_{e^{t\vec{h}}\lambda}\big((D^2)^\perp\big)$. Also, directly from definition of the reduced Jacobi curve we obtain that

$$J_{e^{\tau\vec{h}}\lambda,\vec{h}}(t-\tau) = \Big(p_{e^{\tau\vec{h}}\lambda} \circ (e^{\tau\vec{h}})_* \circ (p_\lambda)^{-1}\Big) J_{\lambda,\vec{h}}(t) \tag{10.18}$$

Proposition 12 *The curve $t \mapsto J_{\lambda,\vec{h}}(t)$ is rank 1 nondecreasing curve in $L(W)$. Moreover, the kernel* $\ker \frac{d}{dt} J_{\lambda,\vec{h}}(t)$ *of the quadratic form $\frac{d}{dt} J_{\lambda,\vec{h}}(t)$ satisfies*

$$\ker \frac{d}{dt} J_{\lambda,\vec{h}}(t) = VS_\lambda(t) \tag{10.19}$$

Proof. By (10.17), (10.18) it is sufficient to prove the Proposition for $t = 0$. Given $l \in J_{\lambda,\vec{h}}(0)$, take a vector field ℓ on $(D^2)^\perp$ such that $\ell(e^{t\vec{h}}\lambda) \in L_{e^{t\vec{h}}\lambda}$, and $p_\lambda(\ell(\lambda) = l$. Note that

$$\frac{d}{dt}\Big(\big(e^{-t\vec{h}}\big)_*\big(\ell(e^{t\vec{h}}\lambda)\big)\Big)|_{t=0} = [\vec{h}, \ell](\lambda). \tag{10.20}$$

Therefore,

$$\frac{d}{dt} J_{\lambda,\vec{h}}(t)|_{t=0}(l) = \sigma([\vec{h}, \ell](\lambda), \ell(\lambda)). \tag{10.21}$$

Let us complete somehow the vector fields $X_1, \dots X_5$, defined above, to the local frame $X_1, \dots X_n$ on M. Similar to (10.7), define quasiimpulses $u_i : T^*M \mapsto \mathbb{R}$ for $6 \le i \le n$. The quasiimpulses u_i, $1 \le i \le n$, define coordinates on the fibers of $T^*\widetilde{M}$ and the vertical vector fields ∂_{u_i}. In particular, the Euler field $\vec{e}$ satisfies

$$\vec{e} = \sum_{i=1}^n u_i \partial_{u_i}.$$

Denote

$$\partial_\theta = u_4 \partial_{u_5} - u_5 \partial_{u_4}, \tag{10.22}$$

$$X = u_5 \vec{u}_2 - u_4 \vec{u}_1 + (u_4^2 + u_5^2)\partial_{u_3}, \tag{10.23}$$

$$F = \vec{u}_3 + u_4 \partial_{u_1} - u_5 \partial_{u_2} \tag{10.24}$$

Note that the vector fields ∂_θ, X, and F are tangent to the manifold $(D^2)^\perp$. Moreover,

$$\sigma(\vec{e}, \partial_\theta) = \sigma(\vec{e}, X) = 0 \tag{10.25}$$

Therefore, for any $\lambda \in (D^2)^\perp \backslash (D^3)^\perp$, we have

$$J_{\lambda,\vec{h}}(0) = p_\lambda\Big(\mathrm{span}(X, \partial_\theta, \partial_{u_6}, \dots, \partial_{u_n})\Big) \tag{10.26}$$

$$V_\lambda(0) = p_\lambda\Big(\mathrm{span}(\partial_\theta, \partial_{u_6}, \dots, \partial_{u_n})\Big) \tag{10.27}$$

By direct computations, one can obtain that

$$[\vec{h}, \partial_{u_i}] \in \mathrm{span}(\vec{e}, \partial_\theta, \partial_{u_6}, \dots, \partial_{u_n}),\ 6 \le i \le n \tag{10.28}$$

$$[\vec{h}, \partial_\theta] \equiv X \ \Big(\mathrm{mod}\big(\mathrm{span}(\vec{h}, \partial_\theta, \partial_{u_6}, \dots, \partial_{u_n})\big)\Big), \tag{10.29}$$

The last two equations together with (10.21), (10.26), and (10.27) implies that

$$VS_\lambda(0) \subseteq \ker \frac{d}{dt} J_{\lambda,\vec{h}}(t)|_{t=0} \tag{10.30}$$

Further, it can be shown easily that

$$[\vec{h}, X] \equiv (u_4^2 + u_5^2)F \ \Big(\mathrm{mod}\big(\mathrm{span}(\vec{h}, X, \partial_\theta, \partial_{u_6}, \dots, \partial_{u_n})\big)\Big), \tag{10.31}$$

which together with (10.21), (10.30) implies that $\frac{d}{dt} J_{\lambda,\vec{h}}(t)|_{t=0}$ is rank 1 nonnegative definite quadratic form such that

$$VS_\lambda(0) = \ker \frac{d}{dt} J_{\lambda,\vec{h}}(t)|_{t=0}. \tag{10.32}$$

□

Denote by Ξ_D the set of all $\lambda \in (D^2)^\perp \backslash (D^3)^\perp$ such that the germ of the reduced Jacobi curve $t \mapsto J_{\lambda,\vec{h}}(t)$ at the point $t = 0$ is the rank 1 curve of the constant weight. It can be shown that if the germs of distribution D at the point q is generic , then the set $\Xi_D \cap (D^2)^\perp(q)$ is a nonempty open set in Zariski topology on the linear space $(D^2)^\perp(q)$, i.e., the set $\Xi_D \cap (D^2)^\perp(q)$ is the complement to some proper algebraic variety of $(D^2)^\perp(q)$. Below we give explicit description of the set Ξ_D for $n = 5$ and $n = 6$.

For $n = 5$ we claim that

$$\Xi_D = (D^2)^\perp \backslash (D^3)^\perp. \tag{10.33}$$

The last relation will be proved in the next subsection, where the case $n = 5$ will be studied in greater detail.

For $n = 6$ the set Ξ_D can be described as follows: Take some $\bar{\lambda} = (\bar{p}, q) \in (D^3)^\perp(q)$ and some vector $v \in D(q)$. Let ν be some vector field tangent to D such that $\nu(q) = v$. Then it is easy to see that the number $\Big[\nu, \big[\nu, [X_1, X_2]\big]\Big](q)$ does not depends on the choice of the vector field ν and the basis X_1, X_2 of the distribution D. In other words, the following function

$$Q_{\bar{\lambda}}(v) = \bar{p} \cdot \Big[\nu, \big[\nu[X_1, X_2]\big]\Big](q) \tag{10.34}$$

is well-defined quadratic form on $D(q)$. For $n=6$ the linear space $(D^3)^\perp(q)$ is one-dimensional. Therefore, the set of all quadratic forms $\{Q_{\bar\lambda} : \bar\lambda \in (D^3)^\perp(q)\}$ is one-dimensional linear space, which implies that the zero level set

$$\mathcal{K}(q) = \{v \in D(q) : Q_{\bar\lambda}(v) = 0\}$$

is the same for all $\bar\lambda \in (D^3)^\perp(q)\backslash(0,q)$. For generic germs at q of rank 2 distributions $\mathcal{K}(q) \neq D(q)$. It turns out that

$$\Xi_D \cap (D^2)^\perp(q) = \{\lambda \in (D^2)^\perp(q) : \pi_*\big(\mathbb{R}\vec{h}(\lambda)\big) \notin \mathcal{K}(q)\}. \tag{10.35}$$

For any $\lambda \in \Xi_D$ denote by $A(\lambda)$ the density of the fundamental form of the curve $t \mapsto J_{\lambda,\vec{h}}(t)$ at $t=0$. So, we obtain the mapping $A : \Xi_D \mapsto \mathbb{R}$.

We claim that for any q the restriction of $A(\cdot)$ to the fiber $(D^2)^\perp(q)$ is a rational function. For this we refer first to the sections 4 and 5 : Let $t \mapsto \Lambda(t)$ be the curve of the constant weight in the Lagrange Grassmannian $L(W)$ or, more general, in the Grassmannian $G(m,2m)$. Let $t \mapsto S(t)$ be some coordinate representation of the curve $\Lambda(\cdot)$. Then from formulas (4.9), (4.11), (4.14), (4.16), (5.8), and Remark 3 it is not difficult to see that the density of the fundamental form of the curve $\Lambda(\cdot)$ is equal to a rational expression w.r.t. some entries of the matrices $\dot S(t), \ddot S(t), \dots, S^{(j)}(t)$ for some j. It remains to show that the curves $t \mapsto J_\lambda(t)$ have coordinate representations $t \mapsto S_\lambda(t)$ such that for any natural l all entries of $S_\lambda^{(l)}(0)$ are rational functions on the fibers $(D^2)^\perp(q)$. For this choose the following $(2n-3)$ vector fields on $(D^2)^\perp$:

$$\partial_\theta, X, \partial_{u_6}, \dots, \partial_{u_n}, F, Y_4, \dots Y_{n-1}, Z, \vec{e}, \vec{h}, \tag{10.36}$$

where

$$Y_k = u_{k+1}\vec{u_k} - u_k\vec{u_{k+1}} + \sum_{i=1}^{3}\Big(u_{k+1}\{u_i,u_k\} - u_k\{u_i,u_{k+1}\}\Big)\partial_{u_i}, \tag{10.37}$$

$$Z = u_4\vec{u_5} + u_5\vec{u_4} + \sum_{i=1}^{3}\Big(u_4\{u_i,u_5\} + u_4\{u_i,u_4\}\Big)\partial_{u_i} \tag{10.38}$$

(here $\{u_i,u_j\}$ are Poisson brackets of the Hamiltonians u_i and u_j, i.e., $\{u_i,u_j\} = du_j(\vec{u_i})$). Denote by Π the set of all $\lambda \in (D^2)^\perp$ such that the vector fields (10.36) at λ constitute a basis of $T_\lambda(D^2)^\perp$. Evidently, for any $q \in M$ the set $\Pi \cap (D^2)^\perp(q)$ is a nonempty open set in Zariski topology on the linear space $(D^2)^\perp(q)$. For any $\lambda \in (D^2)^\perp$ the first $2(n-3)$ vectors in (10.36) belong to the skew symmetric complement of $\vec{e}(\lambda)$ w.r.t. the symplectic form σ. Therefore, for any $\lambda \in \Pi$ the images of the first $2(n-3)$ vectors in (10.36) under the canonical projection

$p_\lambda : T_\lambda(D^2)^\perp \mapsto T_\lambda(D^2)^\perp/\mathrm{span}\big(\vec h(\lambda), \vec e(\lambda)\big)$ constitute the basis of the space W_λ. Introduce in W_λ the coordinates w.r.t. this basis and suppose that $t \mapsto S_\lambda(t)$ is the corresponding coordinate representation of the curve $t \mapsto J_{\lambda,\vec h}(t)$, $J_{\lambda,\vec h}(t) = \{x, S_\lambda(t)) : x \in \mathbb{R}^{n-3}\}$. Then from the definition of the curve $t \mapsto J_{\lambda,\vec h}(t)$ and formula (10.20) it follows that for any natural l all entries of the matrix $S_\lambda^{(l)}(0)$ are some rational combinations of some coordinates w.r.t. the basis (10.36) of the vectors of the type

$$(\mathrm{ad}\vec h)^j(\partial_\theta)(\lambda),\ \ (\mathrm{ad}\vec h)^j(X)(\lambda),\ \ (\mathrm{ad}\vec h)^j(\partial_{u_i})(\lambda),\quad 6 \le i \le n,\quad 1 \le j \le l. \tag{10.39}$$

But from the form of the vector fields Y_i and Z it is clear that coordinates w.r.t. the basis (10.36) of the vectors from (10.39) are rational functions on the fibers $(D^2)^\perp(q)$. So, for any q the restriction of $A(\cdot)$ to the fiber $(D^2)^\perp(q)$ is a rational function.

Also, the restriction of $A(\cdot)$ to the fiber $(D^2)^\perp(q)$ is a homogeneous function of degree 4. Indeed, for any $\lambda = (p, q) \in \Xi_D$ and real $\alpha \neq 0$, let $\lambda_\alpha = (\alpha p, q)$. From Remark 16 it follows that the curves $\tau \mapsto J_{\lambda,\vec h}(\alpha\tau)$ and $\tau \mapsto J_{\lambda_\alpha,\vec h}(\tau)$ can be transformed one to another by symplectic transformation from the symplectic space W_λ to W_{λ_α}. This together with reparameterization rule for the density of the fundamental form of the curve in the Lagrange Grassmannian implies that

$$A(\lambda_\alpha) = A(\lambda)\alpha^4,$$

which is obviously equivalent to the fact that the restriction of $A(\cdot)$ to the fiber $(D^2)^\perp(q)$ is a homogeneous function of degree 4. So, the restriction of $A(\cdot)$ to the fiber $(D^2)^\perp(q)$ belongs to $HR_4\big((D^2)^\perp(q)\big)$.

Let us analyze how the function $A(\lambda)$ is transformed, if we change the local basis of distribution D. Choose another basis $\tilde X_1, \tilde X_2$ of the distribution D such that

$$\begin{aligned} \widetilde{X_1} &= a\,X_1 + b\,X_2 \\ \widetilde{X_2} &= c\,X_1 + d\,X_2 \end{aligned} \tag{10.40}$$

Let $\delta = a\,d - b\,c$. Replacing $\tilde X_1$, $\tilde X_2$ instead of X_1, X_2 in (10.6) and (10.7), one can define $\widetilde X_i$, $i = 3, 4, 5$ and quasiimpulses $\tilde u_i$, $1 \le i \le 5$. Denote by

$$\widetilde h = \tilde u_4 \tilde u_2 + \tilde u_5 \tilde u_1.$$

Let $\widetilde A(\lambda)$ be the density of the fundamental form of the curve $\tau \mapsto J_{\lambda,\vec{\widetilde h}}(\tau)$ at $\tau = 0$. We claim that

$$\widetilde A(\lambda) = \delta^8\big(\pi(\lambda)\big)A(\lambda) \tag{10.41}$$

Indeed, by direct computation, one can show that for any $\lambda \in (D^2)^\perp$

$$\overrightarrow{\widetilde h}(\lambda) = \delta^2\big(\pi(\lambda)\big)\vec h(\lambda) \tag{10.42}$$

From this and (10.16) it follows that for any $\lambda \in (D^2)^\perp \backslash (D^3)^\perp$,

$$J_{\lambda,\vec{h}}(t) = J_{\lambda,\vec{\tilde{h}}}\big(\varphi(t)\big), \tag{10.43}$$

where

$$\varphi'(t) = \frac{1}{\delta^2\big(\pi(e^{t\vec{h}}\lambda)\big)}, \quad \varphi(0) = 0. \tag{10.44}$$

From (10.43) it follows that $A(\lambda)$ and $\tilde{A}(\lambda)$ are densities of the fundamental form of the same curve in the Lagrange Grassmannian at the same point, but w.r.t. different parameters. This together with (10.44) implies (10.41).

Finally, from (10.41) it follows that for any $q \in M$ the restrictions of $A(\cdot)$ and $\widetilde{A}(\cdot)$ to the fiber $(D^2)^\perp(q)$ belong to the same equivalence class of $HR_4\big((D^2)^\perp(q)\big)$ w.r.t. the equivalence relation $\sim$. In this way we assign to any $q \in M$ the element of $P^+HR_4\Big((D^2)^\perp(q)\Big)$ independently on the choice of the local basis. This correspondence will be called *the fundamental form* of the distribution D.

10.3 Fundamental form of rank 2 vector distribution on 5-dimensional manifold

In the present subsection we give an explicit method for computation of the fundamental form in the case of rank 2 distribution on 5-dimensional manifold. We demonstrate this method on several examples, showing simultaneously the efficiency of the notion of fundamental form in proving that the rank 2 distributions are not equivalent.

In the considered case the assumption about maximality of the small growth vector means that the distribution D has the small growth vector (2,3,5) at any point. Also from the previous subsection it follows that for any $\lambda \in (D^2)^\perp$ the reduced Jacobi curve $t \mapsto J_{\lambda,\vec{h}}(t)$ of the characteristic leaf $e^{t\vec{h}}\lambda$ is rank 1 nondecreasing curve in $L(W_\lambda)$, where $dim W_\lambda = 4$. First let us prove the following

Proposition 13 *For any $\lambda \in (D^2)^\perp \backslash (D^3)^\perp$ the reduced Jacobi curve $t \mapsto J_{\lambda,\vec{h}}(t)$ has the constant weight or ,equivalently,*

$$\Xi_D = (D^2)^\perp \backslash (D^3)^\perp. \tag{10.45}$$

The proof of Proposition 13 is based on the following auxiliary lemma:

Lemma 10.1 *Suppose that $\Lambda(t)$ is rank 1 curve in $L(W)$ with $\dim W = 4$. If there exists a curve $t \mapsto E_1(t) \in \Lambda(t)$ such that for any t the vectors $E_1(t), E_1'(t)$ constitute a basis of $\Lambda(t)$, then the curve $\Lambda(t)$ has a constant weight.*

Proof. First prove that if the curve $\Lambda(t)$ has nonconstant weight, then there exists a curve $t \mapsto K(t) \in \Lambda(t)$ and a point t_0 such that

$$K'(t_0) = \alpha K(t_0), \;\; K(t_0) \neq 0 \tag{10.46}$$

for some $\alpha \in \mathbb{R}$. Indeed, there are two possibilities:

1) The curve $\Lambda(t)$ is not ample at some point t_0. Since $\Lambda(t)$ is rank 1 curve, $\Lambda(t)$ is monotonic. Therefore, similarly to the proof of Lemma 2.1, we obtain that all planes $\Lambda(t)$ with t sufficiently closed to t_0 have a common line. Take some vector K on this line and let $K(t) \equiv K$. Then we obtain (10.46) with $\alpha = 0$.

2) The curve $\Lambda(t)$ is ample, but has nonconstant weight. Let $e_1(t)$ be the first vector in the canonical basis of $\Lambda(t)$. Then, according to Proposition 3 and the expression for the weight from the Lemma 6.1, there exists t_0 such that $e_1'(t_0) = \alpha e_1(t_0)$ for some real α. Setting $K(t) = e_1(t)$, we obtain (10.46).

Now let us prove that under the assumptions of our lemma the relation (10.46) is impossible. Indeed, suppose that the curve $t \mapsto K(t)$, satisfying (10.46), exists. By assumptions, there exist $\beta(t)$ and $\gamma(t)$ such that

$$K(t) = \beta(t)E_1(t) + \gamma(t)E_1'(t) \tag{10.47}$$

Differentiating both sides of (10.47), we obtain

$$K'(t) = \gamma(t)E_1''(t)\Big(\mathrm{mod}\ \Lambda(t)\Big) \tag{10.48}$$

Note that $E_1''(t) \notin \Lambda(t)$, otherwise the curve $\Lambda(t)$ is not rank 1 curve. This together with (10.46) implies that $\gamma(t_0) = 0$. Hence,

$$K'(t_0) = \beta(t_0)E_1'(t_0)\Big(\mathrm{mod}\ \mathbb{R}\, E_1(t_0)\Big), \tag{10.49}$$

From this and (10.47) it follows that $\beta(t_0) = 0$, but then from (10.47) we obtain that $K(t_0) = 0$. This contradicts (10.46). The proof of the Lemma is completed. □

Proof of Proposition 13. Set

$$E_1(t) = p_\lambda \circ \big(e^{-t\vec{h}}\big)_*\big(\partial_\theta(e^{t\vec{h}}\lambda)\big) \tag{10.50}$$

From (10.20), (10.26), and (10.29), it follows that the vectors $E_1(0)$ and $E_1'(0)$ constitute a basis of $J_{\lambda,\vec{h}}(0)$. Then, using (10.18) we obtain easily that for any t the vectors $E_1(t)$ and $E_1'(t)$ constitute a basis of $J_{\lambda,\vec{h}}(t)$, which implies (10.45).□

Recall that in section 9 we have found the structural equation for the canonical moving frame of rank 1 nondecreasing curve of the constant weight in $L(W)$ with $\dim W = 4$, see formula (9.11).

The structural equation (9.11) will be a base of our method for computation of fundamental form of D. Let $\epsilon_1(\lambda)$ be the first vector in the canonical basis of the curve $t \mapsto J_{\lambda,\vec{h}}(t)$ at the point $t = 0$. This vector belongs to $T_\lambda\big((D^2)^\perp\big)/\mathrm{span}\big(\vec{h}(\lambda), \vec{e}(\lambda)\big)$. It is more convenient to work directly with vector fields on $(D^2)^\perp$, keeping in mind that the symplectic space W_λ and the subspaces $J_{\lambda,\vec{h}}(t)$ are living in the factor space $T_\lambda\big((D^2)^\perp\big)/\mathrm{span}\big(\vec{h}(\lambda), \vec{e}(\lambda)\big)$. So, in the sequel by $\epsilon_1(\lambda)$ we will mean both the element of factor-space $T_\lambda\big((D^2)^\perp\big)/\mathrm{span}\big(\vec{h}(\lambda), \vec{e}(\lambda)\big)$ and some representative of this element in $T_\lambda\big((D^2)^\perp\big)$, depending smoothly on λ. In the last case all equalities, containing $\epsilon_1(\lambda)$, will be assumed modulo $\mathrm{span}\big(\vec{h}(\lambda), \vec{e}(\lambda)\big)$.

Now we will compute the vector $\epsilon_1(\lambda)$. For this let us prove the following two auxiliary lemmas:

Lemma 10.2 *Suppose that $\Lambda(t)$ is rank 1 nondecreasing curve of the constant weight in $L(W)$, $\dim W = 4$, with the canonical moving frame $\big(e_1(t), e_2(t), f_1(t), f_2(t)\big)$. If the curve $t \mapsto E_1(t) \in \Lambda(t)$ satisfies*

$$E_1'(t) \in \Lambda(t) \quad \forall t, \tag{10.51}$$

then $E_1(t) \in \mathbb{R}\, e_1(t)$.

Proof. Suppose that $E_1(t) = \alpha(t)e_1(t) + \beta(t)e_2(t)$. Then from the structural equation (9.11) it follows easily that

$$E_1'(t) = 4\beta(t)f_2(t) \mod\Big(\mathrm{span}\big(e_1(t), e_2(t)\big)\Big)$$

Therefore the condition (10.51) implies $\beta(t) = 0$, i.e., $E_1(t) \in \mathbb{R}\, e_1(t)$.$\Box$

Note that if $E_1(t) = \alpha(t)e_1(t)$, then by (9.11) $E_1'(t) = \alpha'(t)e_1(t) + 3\alpha(t)e_2(t)$. Hence vectors $E_1(t), E_1'(t)$ constitute a basis of $\Lambda(t)$.

Remark 17 *Note that if we complete the canonical basis $\big(e_1(\tau), e_2(\tau)\big)$ of the rank 1 curve of the constant weight $\Lambda(t)$ to some symplectic basis $\big(e_1(\tau), e_2(\tau), \widetilde{f}_1(\tau), \widetilde{f}_2(\tau)\big)$ in W, then by relation (7.3) it follows that*

$$e_2'(\tau) = 4\widetilde{f}_2(\tau) \mod\Big(\mathrm{span}\big(e_1(\tau), e_2(\tau)\big)\Big) \tag{10.52}$$

Lemma 10.3 *Let the curve $\Lambda(t)$ be as in Lemma (10.2). Assume that the curve $t \mapsto E_1(t) \in \Lambda(t)$ satisfies $E_1(t) \in \mathbb{R}\, e_1(t)$ for any t. Then*

$$e_1(t) = \frac{6}{\sqrt{\sigma\big(E_1''(t), E_1'(t)\big)}} E_1(t) \tag{10.53}$$

Proof. Suppose that $E_1(t) = \alpha(t)e_1(t)$. Complete the basis $\big(E_1(t), E_1'(t)\big)$ of $\Lambda(t)$ to some symplectic basis $\big(E_1(t), E'(t), F_1(t), F_2(t)\big)$ of W. Then from (9.11) and Remark 17 one can easily obtain that

$$E_1''(t) = 36\alpha(t)^2 F_2(t) \mod\Big(\mathrm{span}\big(E_1(t), E_2(t)\big)\Big).$$

From this and the fact that the basis $(E_1(t), E_1'(t), F_1(t), F_2(t))$ is symplectic it follows that

$$36\alpha(t)^2 = \sigma\big(E_1''(t), E_1'(t)\big),$$

which implies (10.53). $\Box$

Now we are ready to prove the following

Proposition 14 *The vector $\epsilon_1(\lambda)$ satisfies*

$$\epsilon_1(\lambda) = \frac{6}{u_4^2+u_5^2}\partial_\theta(\lambda) \mod \Big(\operatorname{span}(\vec{h}(\lambda), \vec{e}(\lambda)\Big). \tag{10.54}$$

Proof. Let $E_1(t)$ be as in (10.50). From (10.20), (10.26), and (10.29) it follows that

$$E_1'(0) = p_\lambda\big([h, \partial_\theta](\lambda)\big) \in J_{\lambda,\vec{h}}(0)$$

This together with Lemma 10.2 implies that

$$\epsilon_1(\lambda) \in p_\lambda\big(\mathbb{R}\partial_\theta\big)$$

From this and (10.53) it follows that

$$\epsilon_1(\lambda) = \frac{6}{\sqrt{\sigma\Big((\operatorname{ad}\vec{h})^2(\partial_\theta)(\lambda), \operatorname{ad}\vec{h}(\partial_\theta)(\lambda)\Big)}}\partial_\theta(\lambda) \mod \Big(\operatorname{span}(\vec{h}(\lambda), \vec{e}(\lambda)\Big). \tag{10.55}$$

Finally, from (10.23),(10.24),(10.29), and (10.31) it is easy to see that

$$\sigma\Big((\operatorname{ad}\vec{h})^2(\partial_\theta), \operatorname{ad}\vec{h}(\partial_\theta)\Big) = (u_4^2+u_5^2)\sigma(F, X) =$$

$$= (u_4^2+u_5^2)\sigma(\vec{u}_3 + u_4\partial_{u_1} - u_5\partial_{u_2}, u_5\vec{u}_2 - u_4\vec{u}_1 + (u_4^2+u_5^2)\partial_{u_3}) = (u_4^2+u_5^2)^2.$$

Substituting this in (10.55), we obtain (10.54). $\Box$

The following auxiliary lemma gives the method to express fundamental form of distribution D with the help of $\epsilon_1(\lambda)$:

Lemma 10.4 *Suppose that $\Lambda(t)$ is rank 1 curve of the constant weight in $L(W)$ and $e_1(t)$ is the first vector in the canonical basis of $\Lambda(t)$. Then $e_1(t)$ satisfies the following relation:*

$$e_1^{(4)} = \left(35\,A - \frac{81}{16}\rho^2 - \frac{9}{4}\rho''\right)e_1 - \frac{15}{2}\rho' e_1' - \frac{15}{2}\rho e_1'' \tag{10.56}$$

Proof. Without loss of generality, one can suppose that $\Lambda(t)$ is nondecreasing. From the first equation of (9.11) it follows that

$$e_2 = \frac{1}{3}e_1' \tag{10.57}$$

By the two first equations of (9.11) ,

$$e_1'' = 3e_2' = \frac{3}{4}\rho(t)e_1 + 12f_2, \tag{10.58}$$

which implies

$$f_2 = \frac{1}{12}e_1'' - \frac{1}{16}\rho e_1. \tag{10.59}$$

Using the forth equation of (9.11) and (10.57), it is not difficult to obtain

$$\begin{aligned} e_1^{(3)} &= \frac{3}{4}\rho' e_1 + \frac{3}{4}\rho e_1' + 12\Big(-\frac{7}{16}\rho' e_1 - \frac{3}{4}\rho e_1' - 3f_1\Big) = \\ &= -\frac{9}{2}\rho'\, e_1 - \frac{33}{4}\rho e_1' - 36 f_1 \end{aligned} \tag{10.60}$$

Differentiating (10.60) and using the third equation of (9.11) , we have

$$e_1^{(4)} = -\frac{9}{2}\rho'' e_1 - \frac{51}{4}\rho' e_1' - \frac{33}{4}\rho e_1'' + \Big(35A - \frac{9}{2}\rho^2 + \frac{9}{4}\rho''\Big)e_1 + \frac{63}{4}\rho' e_2 + 9\rho f_2 \tag{10.61}$$

Substituting (10.57) and (10.59) in (10.61), we obtain easily (10.56). $\square$

As a direct consequence of the previous Lemma we obtain

Theorem 14 *Let $\epsilon_1(\lambda)$ be as in (10.54). Then there exist functions A_0 ,A_1 on $(D^2)^\perp$ such that*

$$(\mathrm{ad}\vec{h})^4(\epsilon_1) = A_0\epsilon_1 + \vec{h}(A_1)\mathrm{ad}\vec{h}(\epsilon_1) + A_1(\mathrm{ad}\vec{h})^2(\epsilon_1) \mod\Big(\mathrm{span}\big(\vec{h}, \vec{e}\big)\Big) \tag{10.62}$$

and the fundamental form $A(\lambda)$ of the distribution D satisfies:

$$35A = A_0 + \frac{9}{100}A_1^2 - \frac{3}{10}(\vec{h})^2(A_1) \tag{10.63}$$

Now we will use Theorem 14 for computation of fundamental form of concrete rank 2 distributions. Let us introduce some notations. As before, let vector fields X_1, X_2 be a local basis of the distribution. Let the vector fields X_3, X_4, X_5 be as in (10.6) and u_i, $1 \le i \le 5$, be the corresponding quasiimpulses, defined by (10.7). The vector fields $X_1, \dots, X_5$ constitute a local frame on M. Suppose that this frame satisfies the following commutative relations

$$[X_i, X_j] = \sum c_{ji}^k X_k \tag{10.64}$$

Then the Hamiltonian vector fields $\vec{u}_i$, corresponding to the functions u_i satisfy

$$\vec{u}_i = X_i + \sum_{j=1}^{5}\sum_{k=1}^{5} c_{ji}^k u_k \partial_{u_j} \tag{10.65}$$

Therefore, the restriction of the vector field $\vec{h}$ on

$$\big(D^2\big)^\perp \cap T^*M = \{\lambda \in T^*M : u_1(\lambda) = u_2(\lambda) = u_3(\lambda) = 0\} \tag{10.66}$$

satisfies

$$\begin{aligned}\vec{h} = u_4\vec{u}_2 + u_5\vec{u}_1 = u_4X_2 + u_5X_1 + \Big(c^4_{42}u_4^2 + (c^5_{42} + c^4_{41})u_4u_5 + c^5_{41}u_5^2\Big)\partial_{u_4} + \\ +\Big(c^4_{52}u_4^2 + (c^5_{52} + c^4_{51})u_4u_5 + c^5_{51}u_5^2\Big)\partial_{u_5}.\end{aligned} \tag{10.67}$$

In all examples below, for simplicity, we take left-invariant rank 2 distribution on some Lie Groups. We give these examples in order to show that the fundamental form of rank 2 distribution is a nontrivial invariant, which gives an effective way to prove that rank 2 distributions are not equivalent.

Example 1. Nilpotent rank 2 distribution. Let L_1 be the 5-dimensional nilpotent Lie algebra with the following commutation rules in some basis $X_1, \dots, X_5$:

$$\begin{aligned}&[X_1, X_2] = X_3,\ [X_1, X_3] = X_4,\ [X_2, X_3] = -X_5 \\ &adX_4 = 0,\ adX_5 = 0\end{aligned} \tag{10.68}$$

Let M_1 be the Lie group with the Lie algebra L_1. We consider $X_1, \dots, X_5$ as left-invariant vector fields on M_1. Let $D_1 = span(X_1, X_2)$.

Let us show that the fundamental form of distribution D_1 is equal to zero. Indeed, in the considered case by (10.67)

$$\vec{h} = u_5X_1 + u_4X_2 \tag{10.69}$$

Hence

$$\vec{h}(u_4^2 + u_5^2) = 0 \tag{10.70}$$

Therefore, we can replace ϵ_1 by ∂_θ in (10.62).Further, using (10.68), we obtain by direct computation that

$$\mathrm{ad}\vec{h}(\partial_\theta) = -u_4X_1 + u_5X_2, \tag{10.71}$$

$$(\mathrm{ad}\vec{h})^2(\partial_\theta) = (u_4^2 + u_5^2)X_3, \tag{10.72}$$

$$(\mathrm{ad}\vec{h})^3(\partial_\theta) = (u_4^2 + u_5^2)(u_5X_4 - u_4X_5), \tag{10.73}$$

$$(\mathrm{ad}\vec{h})^4(\partial_\theta) = 0. \tag{10.74}$$

Therefore $A_0 = A_1 = 0$. So the fundamental form A vanishes,

$$A = 0. \tag{10.75}$$

Example 2. Left-invariant rank 2 distribution on $SO(3) \times \mathbb{R}^2$. Denote by M_2 the Lie group $SO(3) \times \mathbb{R}^2$. Let $L_2 = so(3) \oplus \mathbb{R}^2$ be Lie algebra corresponding to Lie group M_2 . Suppose that E_{ij} is 3×3 matrix such that its (i, j)th entry is equal to 1 and all other entries equal to 0. Take the following basis a_1, a_2, a_3 in $so(3)$:

$$a_1 = E_{12} - E_{21}, a_2 = E_{13} - E_{31}, a_3 = E_{32} - E_{23} \tag{10.76}$$

Then a_1, a_2, a_3 satisfy the following commutative relations:

$$[a_1, a_2] = a_3,\ [a_2, a_3] = a_1,\ [a_3, a_1] = a_2 \tag{10.77}$$

Let b_1, b_2 be some basis of $\mathbb{R}^2$. Denote

$$X_1 = (a_1, b_1),\ X_2 = (a_2, b_2), \tag{10.78}$$

$$D_2 = span(X_1, X_2) \tag{10.79}$$

We consider X_1, X_2, as left-invariant vector fields on Lie group M_2. Consequently, D_2, defined by (10.79), can be considered as left-invariant rank 2 distribution on M_2. Let the fields X_3,X_4, and X_5 be, as in (10.6). Then

$$\begin{array}{c} X_3 = [(a_1, b_1), (a_2, b_2)] = (a_3, 0), \\ X_4 = [(a_1, b_1), (a_3, 0)] = -(a_2, 0), \\ X_5 = -[(a_2, b_2), (a_3, 0)] = -(a_1, 0); \end{array} \tag{10.80}$$

In particular, this implies that

$$D_2^3(q) = T_qM\ \forall q \in M_2 \tag{10.81}$$

Lemma 10.5 *The distribution D_2 is unique, up to group automorphism of M_2, left-invariant rank 2 distribution on M_2, satisfying (10.81).*

Proof. Evidently, the lemma is equivalent to the fact that D_2 is the only plane in L_2, up to Lie algebra automorphism of L_2, such that $D_2^3 = L_2$.

Let us show first that the group of automorphisms of Lie algebra $so(3)$ acts transitively on the set of all planes of $so(3)$. The Killing form

$$K(x, y) = \mathrm{tr}(ad\,x\ ad\,y) \tag{10.82}$$

of $so(3)$ is negative definite and satisfies

$$K(x, x) = -2\sum_{i=1}^{3} \alpha_i^2,\ \ x = \sum_{i=1}^{3} \alpha_i a_i. \tag{10.83}$$

On any plane $P \subset so(3)$ one can take vectors $\bar{a}_1$ and $\bar{a}_2$ such that

$$K(\bar{a}_1, \bar{a}_1) = K(\bar{a}_2, \bar{a}_2) = -2,\ \ K(\bar{a}_1, \bar{a}_2) = 0.$$

Also let $\bar{a}_3 = [\bar{a}_1, \bar{a}_2]$. Recall that Killing form satisfies the following two properties:

$$K([x,y],z] = K([y,z],x]) = K([z,x],y]), \tag{10.84}$$

$$K(\tau x, \tau y) = K(x,y), \tag{10.85}$$

where τ is a Lie algebra automorphism. Using these properties, it is not difficult to show that the vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3$ satisfy the commutative relations similar to (10.77):

$$[\bar{a}_1, \bar{a}_2] = \bar{a}_3, \ [\bar{a}_2, \bar{a}_3] = \bar{a}_1, \ [\bar{a}_3, \bar{a}_1] = a_2 \tag{10.86}$$

Therefore, the linear operator τ such that $\tau(a_i) = \bar{a}_i$ is the Lie algebra automorphism such that

$$\tau(\mathrm{span}(a_1, a_2)) = P. \tag{10.87}$$

Let $\bar{D}_2 \subset L_2$, $\bar{D}^2 \neq D_2$ be a plane, satisfying

$$\bar{D}_2^3 = L_2. \tag{10.88}$$

Let π_1 and π_2 be projections from L_2 to $so(3)$ and $\mathbb{R}^2$ respectively. Then

$$\dim \pi_1(\bar{D}_2) = 2, \ \ \pi_2(\bar{D}_2) = \mathbb{R}^2,$$

otherwise (10.88) does not hold. Let $P = \pi_1(\bar{D}_2)$. Then relation (10.87) holds for some automorphism τ of $so(3)$. Let $\mathcal{T}(x,y) = (\tau x, y)$ be the trivial extension of τ to automorphism of L_2. Then $\mathcal{T} D_2 = \bar{D}_2$, which completes the proof of the lemma.$\square$

Note that distribution D_2 appears, when one studies the problem of rolling ball on the plane without slipping and twisting (see, for example, [15] for the details).

Let us compute the fundamental form of D_2. By (10.80),

$$\begin{gathered} [X_1, X_4] = [(a_1, b_1), -(a_2, 0)] = -(a_3, 0) = -X_3, \\ [X_1, X_5] = -[X_2, X_4] = -[(a_2, b_2), -(a_2, 0)] = 0, \\ [X_2, X_5] = [(a_2, b_2), -(a_1, 0)] = (a_3, 0) = X_3. \end{gathered} \tag{10.89}$$

Therefore, by (10.67), in the considered case formulas (10.69) and (10.70) hold. Hence, as in the previous example, we can replace ϵ_1 by ∂_θ in (10.62). Then for $(\mathrm{ad}\vec{h})^i(\partial_\theta)$ with $1 \leq i \leq 3$ we have the same formulas, as in (10.71)-(10.73). Further,

$$(\mathrm{ad}\vec{h})^4(\partial_\theta) = [u_5 X_1 + u_4 X_2, (u_4^2 + u_5^2)(u_5 X_4 - u_4 X_5)] =$$

$$= (u_4^2 + u_5^2)\Big(u_5^2[X_1, X_4] + u_4 u_5([X_2, X_4] - [X_1, X_5]) - u_4^2[X_1, X_5]\Big) \tag{10.90}$$

Substituting (10.89) in (10.90), we get

$$(\mathrm{ad}\vec{h})^4(\partial_\theta) = -(u_4^2 + u_5^2)^2 X_3 \tag{10.91}$$

By analogy with (10.71), we have

$$X_3 = \frac{1}{(u_4^2 + u_5^2)}(\mathrm{ad}\vec{h})^2(\partial_\theta) \tag{10.92}$$

Substituting this in (10.91) we obtain

$$(\mathrm{ad}\vec{h})^4(\partial_\theta) = -(u_4^2 + u_5^2)(\mathrm{ad}\vec{h})^2(\partial_\theta) \tag{10.93}$$

Therefore in notation of (10.62)

$$A_0 = 0,\ A_1 = -(u_4^2 + u_5^2) \tag{10.94}$$

From this and formulas (10.63),(10.70) we finally have that the fundamental form A of D_2 satisfies

$$35A = \frac{9}{100}(u_4^2 + u_5^2)^2 \tag{10.95}$$

Conclusion 1 *From (10.75) and (10.95) it follows that germs of distributions D_1 and D_2 are not equivalent*

Conjecture 2 *The rank 2 distribution on 5-dimensional manifold has the identically zero fundamental form iff it is locally equivalent to the distribution D_1.*

Example 3. Left-invariant rank 2 distributions on $SL(2,\mathbb{R}) \times \mathbb{R}^2$. In contrast to $so(3)$ the Lie algebra $sl(2,\mathbb{R})$ has indefinite Killing form with negative index 1. Indeed, let E_{ij} be 2×2 matrix such that its (i,j)th entry is equal to 1 and all other entries equal to 0. Take the following basis a_1, a_2, a_3 in $sl(2,\mathbb{R})$:

$$a_1 = \frac{1}{2}(E_{11} - E_{22}), a_2 = \frac{1}{2}(E_{12} - E_{21}), a_3 = \frac{1}{2}(E_{12} + E_{21}) \tag{10.96}$$

Then a_1, a_2, a_3 satisfy the following commutative relations:

$$[a_1, a_2] = a_3,\ [a_2, a_3] = a_1,\ [a_3, a_1] = -a_2, \tag{10.97}$$

This implies easily that the Killing form $K(x,y)$ satisfies

$$K(x,x) = 2(\alpha_1^2 - \alpha_2^2 + \alpha_3^2),\ \ x = \sum_{i=1}^{3} \alpha_i a_i. \tag{10.98}$$

Denote by

$$P_h = \mathrm{span}(a_1, a_2),\ \ P_e = \mathrm{span}(a_1, a_3) \tag{10.99}$$

From (10.98) it follows that the restriction of $K(x,x)$ on P_h is indefinite nondegenerated and on P_e is positive definite quadratic form.

Lemma 10.6 *The action of the group of Lie algebra automorphism of $sl(2,\mathbb{R})$ on the set of planes $P \subset sl(2,\mathbb{R})$ such that*

$$P^2 = sl(2,\mathbb{R}) \tag{10.100}$$

has exactly two orbits. Each plane P_h and P_e is representative of one of these orbits.

Proof. Take some plane P in $sl(2,\mathbb{R})$, satisfying (10.100). First note that the restriction of the Killing form $K(x,x)$ on the plane P has to be nondegenerated quadratic form, otherwise by (10.84) the Killing form $K(x,x)$ on the whole algebra $sl(2,\mathbb{R})$ is degenerated, which contradicts (10.98). Therefore, by (10.98), there are two possibilities:

1. The restriction of the Killing form $K(x,x)$ on P is nondegenerated indefinite. In this case one can take on P two vectors $\bar{a}_1, \bar{a}_2$ such that

$$K(\bar{a}_1,\bar{a}_1) = 2, \;\; K(\bar{a}_2,\bar{a}_2) = -2, \;\; K(\bar{a}_1,\bar{a}_2) = 0.$$

Denote by $\bar{a}_3 = [\bar{a}_1, \bar{a}_2]$. Using (10.84) and (10.85), it is not difficult to show that the vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3$ satisfy the commutative relations similar to (10.97). Therefore the linear operator, transforming $\bar{a}_i$ to a_i, $1 \leq i \leq 3$, is Lie algebra automorphism, which transforms the plane P to P_h.

2. The restriction of the Killing form $K(x,x)$ on P is positive definite. In this case one can take on P two vectors $\bar{a}_1, \bar{a}_3$ such that

$$K(\bar{a}_1,\bar{a}_1) = 2, \;\; K(\bar{a}_3,\bar{a}_3) = 2, \;\; K(\bar{a}_1,\bar{a}_3) = 0.$$

Denote by $\bar{a}_2 = [\bar{a}_1, \bar{a}_3]$. Using again (10.84) and (10.85), it is not difficult to show that the vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3$ satisfy the commutative relations similar to (10.97). Therefore the linear operator, transforming $\bar{a}_i$ to a_i, $1 \leq i \leq 3$, is Lie algebra automorphism, which transforms the plane P to P_e. $\square$

Let b_1, b_2 be some basis of $\mathbb{R}^2$. Suppose that

$$D_{3,h} = \mathrm{span}\big((a_1,b_1),(a_2,b_2)\big), \;\; D_{3,e} = \mathrm{span}\big((a_1,b_1),(a_3,b_2)\big), \tag{10.101}$$

We consider $D_{3,h}$ and $D_{3,e}$, as left-invariant rank 2 distribution on the Lie group $SL(2,\mathbb{R}) \times \mathbb{R}^2$. Using the arguments similar to the proof of the Lemma 10.5, one can easily obtain from Lemma 10.6 the following

Lemma 10.7 *Distributions $D_{3,h}$ and $D_{3,e}$ are the only two different left-invariant rank 2 completely nonholonomic distributions on $SL(2,\mathbb{R}) \times \mathbb{R}^2$, up to Lie group automorphisms of $SL(2,\mathbb{R}) \times \mathbb{R}^2$.*

Also, note that

$$D^3_{3,h}(q) = D^3_{3,e}(q) = T_q(SL(2,\mathbb{R}) \times \mathbb{R}^2), \;\; \forall q \in SL(2,\mathbb{R}) \times \mathbb{R}^2.$$

Let us compute the fundamental form of $D_{3,h}$ and $D_{3,e}$

a) The fundamental form of $D_{3,h}$. Denote by

$$X_1 = (a_1, b_1), \quad X_2 = (a_2, b_2)$$

Let the fields X_3, X_4, and X_5 be, as in (10.6). Then

$$\begin{aligned} X_3 &= [(a_1, b_1), (a_2, b_2)] = (a_3, 0), \\ X_4 &= [(a_1, b_1), (a_3, 0)] = (a_2, 0), \\ X_5 &= -[(a_2, b_2), (a_3, 0)] = -(a_1, 0); \end{aligned} \tag{10.102}$$

By (10.102),

$$\begin{aligned} [X_1, X_4] &= [(a_1, b_1), (a_2, 0)] = (a_3, 0) = X_3, \\ [X_1, X_5] &= -[X_2, X_4] = -[(a_2, b_2), (a_2, 0)] = 0, \\ [X_2, X_5] &= [(a_2, b_2), -(a_1, 0)] = (a_3, 0) = X_3. \end{aligned} \tag{10.103}$$

Therefore, by (10.67), in the considered case formulas (10.69) and (10.70) hold. Hence, as in the previous example, we can replace ϵ_1 by ∂_θ in (10.62). Also, formulas (10.71)-(10.73), (10.90), and (10.92) hold. Substituting (10.103) in (10.90), we obtain

$$(\mathrm{ad}\vec{h})^4(\partial_\theta) = (u_4^2 - u_5^2)(u_4^2 + u_5^2)X_3 \tag{10.104}$$

Then from (10.92)

$$(\mathrm{ad}\vec{h})^4(\partial_\theta) = (u_4^2 - u_5^2)(\mathrm{ad}\vec{h})^2(\partial_\theta) \tag{10.105}$$

Hence in notation of (10.62)

$$A_0 = 0, \;\; A_1 = (u_4^2 - u_5^2)$$

From this and formulas (10.63),(10.70) we finally have that the fundamental form A of $D_{3,h}$ satisfies

$$35A = \frac{9}{100}(u_4^2 - u_5^2)^2 \tag{10.106}$$

b) The fundamental form of $D_{3,e}$. Denote by

$$X_1 = (a_1, b_1), \quad X_2 = (a_3, b_2)$$

Let the fields X_3, X_4, and X_5 be, as in (10.6). Then

$$\begin{aligned} X_3 &= [(a_1, b_1), (a_3, b_2)] = (a_2, 0), \\ X_4 &= [(a_1, b_1), (a_2, 0)] = (a_3, 0), \\ X_5 &= -[(a_3, b_2), (a_2, 0)] = (a_1, 0); \end{aligned} \tag{10.107}$$

By (10.102),

$$\begin{aligned}[X_1, X_4] &= [(a_1, b_1), (a_3, 0)] = (a_2, 0) = X_3,\\ [X_1, X_5] &= -[X_2, X_4] = -[(a_3, b_2), (a_3, 0)] = 0,\\ [X_2, X_5] &= [(a_3, b_2), (a_1, 0)] = -(a_2, 0) = -X_3.\end{aligned} \tag{10.108}$$

By analogy with the previous case, substituting (10.108) in (10.90) and using (10.92) , we obtain

$$(\mathrm{ad}\vec{h})^4(\partial_\theta) = (u_4^2 + u_5^2)(\mathrm{ad}\vec{h})^2(\partial_\theta) \tag{10.109}$$

Hence in notation of (10.62)

$$A_0 = 0,\ A_1 = (u_4^2 + u_5^2) \tag{10.110}$$

From this and formulas (10.63),(10.70) we finally have that the fundamental form A of $D_{3,h}$ satisfies

$$35A = \frac{9}{100}(u_4^2 + u_5^2)^2 \tag{10.111}$$

Conclusion 2 *From (10.95), (10.106) and (10.111) it follows that germs of distributions $D_{3,h}$ and D_2 are not equivalent and also germs of distributions $D_{3,h}$ and $D_{3,e}$ are not equivalent.*

Comparing (10.95) with (10.111), we still are not able to give an answer on a question, whether germs D_2 and $D_{3,e}$ are equivalent or not: fundamental forms in both cases are squares of sign definite quadratic forms. It turns out that one can construct an additional invariant of distribution, projective curvature. As we will see in the next subsection, using this invariant, one can prove the nonequivalence of distributions in the cases, when an information about fundamental form only does not help to do it.

Finally note that from (10.62) and (10.63) by direct but rather long computations it follows that for any $q \in M$ the restriction of the densities $A(\cdot)$ to the fibers $(D^2)^\perp(q)$ are actually degree 4 homogeneous polynomials on $(D^2)^\perp(q)$. It would be interesting to compare our fundamental with Cartan's *biquadratic binary form*, constructed in [10] , p. 152 (or p. 970 in his Oevres completes).

10.4 Projective curvature of rank 2 distributions with nonzero fundamental form.

We return to the general case of rank 2 distribution D on the arbitrary n dimensional manifold. Recall that in section 5 for the curve $\Lambda(\cdot)$ in the Lagrange Grassmannian with nonzero fundamental form we defined the normal parameter and the projective Ricci curvature: the normal parameter is a parameter τ, in which the fumdamental form of $\Lambda(\cdot)$ is equal to $\pm(d\,\tau)^4$ and the projective Ricci curvature is the Ricci curvature of the curve in the normal parameter (see Definition 3). Now we will use these notions for construction of a basic functional invariant of the distribution with nonzero fundamental form.

Denote by

$$\aleph_D = \{\lambda \in \Xi_D : \mathcal{A}(\lambda) \neq 0\} \tag{10.112}$$

Suppose that the set $\aleph_D$ is not empty. For any characteristic leaf, lying in $\aleph_D$, the corresponding Jacobi curve has normal parameter. In other words, any such leaf has normal parameter. One can define the vector field $\overrightarrow{h_A}$ on $\aleph_D$ such that its integral curves are characteristic leaves parameterized by their normal parameter. By construction, the vector field $\overrightarrow{h_A}$ is invariant of the distribution D. We call it *the canonical Hamiltonian vector field* of distribution D. Further, for any $\lambda \in \aleph_D$ take the characteristic leaf γ, passing through λ. Denote by $\rho_D(\lambda)$ the projective Ricci curvature of the Jacobi curve corresponding to γ at the point λ. In other words, $\rho_D(\lambda)$ is equal to the Ricci curvature of the curve $\tau \mapsto J_{\lambda,\overrightarrow{h_A}}(\tau)$ at the point $\tau = 0$. So, to given rank 2 distribution D we assign canonically the function

$$\rho_D : \aleph_D \mapsto \mathbb{R} \tag{10.113}$$

This function is called *projective curvature* of distribution D.

Now we give a method for computation of projective curvature ρ_D. Take some local basis X_1, X_2 of D. Suppose again that X_3, X_4, X_5, u_i, and h be as in (10.6), (10.7), and (10.13) respectively. For any $\lambda \in (D^2)^\perp \backslash (D^3)^\perp$ the corresponding vector field $\vec{h}$ defines a parameter t of characteristic leaf γ passing through λ. As before denote by $A(\lambda)$ the density of fundamental form of the reduced Jacobi curve $t \mapsto J_{\lambda,\vec{h}}$ at $t = 0$. Also, denote by $\rho(\lambda)$ the Ricci curvature of this parameterized curve at $t = 0$. Note that in contrast to $\rho_D(\lambda)$, the function $\rho(\lambda)$ certainly depends on the local basis of distribution.

Using the reparameterization rule (5.3) for Ricci curvature, one can easily express the projective curvature $\rho_D(\lambda)$ by $\rho(\lambda)$ and $A(\lambda)$. Indeed, let τ be the normal parameter on γ. Then by definition

$$d\tau = \sqrt[4]{|A(e^{t\vec{h}}\lambda)|}\; dt \tag{10.114}$$

Suppose that $t = \varphi(\tau)$. Then by (10.114)

$$\varphi'(\tau) = \frac{1}{\sqrt[4]{|A(e^{t\vec{h}}\lambda)|}} \tag{10.115}$$

Recall that the Jacobi curves under consideration have the weight equals $(n-3)^2$. So, by (5.3),

$$\rho_D\left(e^{\tau\vec{h_A}}\lambda\right) = \rho\left(e^{\varphi(\tau)\vec{h}}\lambda\right)\left(\varphi'(\tau)\right)^2 + \frac{(n-3)^2}{3}\mathbb{S}\big(\varphi(\tau)\big), \tag{10.116}$$

where $\mathbb{S}(\varphi)$ is Schwarzian of the function φ. Similarly to Lemma 9.2,

$$\mathbb{S}\big(\varphi(\tau)\big) = -\frac{y''(\tau)}{y(\tau)}, \tag{10.117}$$

where

$$y(\tau) = \frac{1}{\sqrt{\varphi'(\tau)}}.$$

By (10.115),

$$y(\tau) = \sqrt[8]{|A(e^{\tau\overrightarrow{h_A}}\lambda)|}$$

Substituting this in (10.117) and using (10.114) we obtain

$$\mathbb{S}\big(\varphi(\tau)\big) = -\frac{\frac{d^2}{d\tau^2}\left(\sqrt[8]{|A(e^{\tau\overrightarrow{h_A}}\lambda)|}\right)}{\sqrt[8]{|A(e^{\tau\overrightarrow{h_A}}\lambda)|}} = -\frac{1}{\sqrt[8]{|A(e^{t\,\vec{h}}\lambda)|^3}}\frac{d}{dt}\left(\frac{1}{\sqrt[4]{|A(e^{t\vec{h}}\lambda)|}}\frac{d}{dt}\left(\sqrt[8]{|A(e^{t\vec{h}}\lambda)|}\right)\right) =$$

$$= \frac{1}{\sqrt[8]{|(A(e^{t\vec{h}}\lambda)|^3}}\frac{d^2}{dt^2}\left(|A(e^{t\vec{h}}\lambda)|^{-\frac{1}{8}}\right) = \frac{\vec{h}\circ\vec{h}\left(|A(e^{t\vec{h}}\lambda)|^{-\frac{1}{8}}\right)}{\sqrt[8]{|(A(e^{t\vec{h}}\lambda)|^3}} \tag{10.118}$$

Finally, substituting (10.118) with $t = 0$ in (10.116) we get

$$\rho_D = \frac{\rho}{\sqrt{|A|}} + \frac{(n-3)^2}{3}\frac{\vec{h}\circ\vec{h}\big(|A|^{-\frac{1}{8}}\big)}{\sqrt[8]{|A|^3}} \tag{10.119}$$

Note also that if A_1 is as in (10.62), then from (10.56) it follows that

$$\rho = -\frac{2}{15}A_1 \tag{10.120}$$

Now we will use formula (10.119) for computation of projective curvature of concrete rank 2 distributions. In the examples below we will show the efficiency of the notion of projective curvature in proving that the rank 2 distributions are not equivalent.

Example 4. The germs of distributions D_2 and $D_{3,e}$ are not equivalent. In both cases under consideration $\vec{h}(A) = 0$. Therefore by (10.119) and (10.120)

$$\rho_D = \frac{\rho}{\sqrt{|A|}} = -\frac{2}{15}\frac{A_1}{\sqrt{|A|}} \tag{10.121}$$

Hence, substituting (10.94) and (10.95) in (10.121), we obtain

$$\rho_{D_2} = \frac{4\sqrt{35}}{9} \tag{10.122}$$

On the other hand, substituting (10.110) and (10.111) in (10.121), we obtain

$$\rho_{D_{3,e}} = -\frac{4\sqrt{35}}{9} \tag{10.123}$$

So, in both cases the projective curvatures are constant, but different, which implies that the germs of distributions D_2 and $D_{3,e}$ are not equivalent.

Example 5. One-parameter family of nonequivalent germs of rank 2 distributions. Recall that there exists a unique nonabelian two-dimensional Lie algebra. Denote it by $\mathcal{E}_2$. In $\mathcal{E}_2$ one can choose a basis b_1, b_2 such that

$$[b_1, b_2] = b_1 \tag{10.124}$$

Note that $\mathcal{E}_2^2 = \mathbb{R}b_1$.

Let $L_4 = so(3) \oplus \mathcal{E}_2$ and M_4 be a connected Lie group, corresponding to L_4. Let $\mathcal{D}$ be a set of all planes D in L_4 such that $D^3 = L_4$.

Lemma 10.8 *The action of the group of automorphism of the Lie algebra L_4 on the set $\mathcal{D}$ has one-parametric family of orbits.*

Proof. Fix some plane $D \in \mathcal{D}$. Denote by π_1 and π_2 the canonical projections of L_4 on $so(3)$ and $\mathcal{E}_2$ respectively. From the condition $D^3 = L_4$ it follows easily, that

$$\dim \pi_1(D) = 2, \quad \pi_2(D) = \mathcal{E}_2. \tag{10.125}$$

Denote by $P = \pi_1(D)$. The algebra $\mathcal{E}_2$ has a distinguished line $\mathcal{E}_2^2$. Therefore the line $F \in P$ such that

$$F = \pi_1 \circ \pi_2^{-1}(\mathcal{E}_2^2)$$

is a distinguished line in P. One can take vectors $\bar{a}_1$ and $\bar{a}_2$ in P such that

$$\bar{a}_1 \in F, \ \ K(\bar{a}_1, \bar{a}_1) = K(\bar{a}_2, \bar{a}_2) = -2, \quad K(\bar{a}_1, \bar{a}_2) = 0, \tag{10.126}$$

where $K(x, y)$ is the Killing form of $so(3)$. Evidently, the vectors $\bar{a}_1$ and $\bar{a}_2$ are determined by (10.126) uniquely, up to the sign. Therefore, in $\mathcal{E}_2$ the following vectors

$$\bar{b}_1 = \pi_2 \circ \pi_1^{-1}(\bar{a}_1), \ \ \bar{b}_2 = \pi_2 \circ \pi_1^{-1}(\bar{a}_2) \tag{10.127}$$

are defined uniquely, up to the sign, and ,by construction, $\bar{b}_1 \in \mathcal{E}_2^2$. Hence the real number $\alpha(D)$ such that

$$[\bar{b}_1, \bar{b}_2] = \alpha(D)\bar{b}_1$$

is defined uniquely, up to the sign. So, to any plane $D \in \mathcal{D}$ one can assign canonically a number $|\alpha(D)|$. It is easy to see, that two planes D_1, D_2 from $\mathcal{D}$ can be transformed one to another by Lie algebra automorphism iff

$$|\alpha(D_1)| = |\alpha(D_2)|. \tag{10.128}$$

Indeed, necessity of (10.128) follows from the construction. The sufficiency follows from the fact that $\alpha(D)$ is the only parameter, appearing in commutative relations, generated by vectors $(\bar{a}_1, \bar{b}_1), (\bar{a}_2, \bar{b}_2)$.

Also, for given $\bar{\alpha} > 0$ it is not difficult to find a plane $D \in \mathcal{D}$ such that

$$|\alpha(D)| = \bar{\alpha}. \tag{10.129}$$

Indeed, let a_1, a_2, a_3 be as in (10.76) and b_1, b_2 be as in (10.124). Then take

$$X_1 = (a_1, b_1), \quad X_2 = (a_2, \bar{\alpha} b_2) \tag{10.130}$$

$$D_{4,\bar{\alpha}} = \mathrm{span}(X_1, X_2) \tag{10.131}$$

Then, obviously, $\alpha(D_{4,\bar{\alpha}}) = \bar{\alpha}$. Consequently the set of orbits w.r.t. the action of the group of automorphism of the Lie algebra L_4 on $\mathcal{D}$ can be parameterized by one positive parameter α. The planes $D_{4,\alpha}$ are the representatives of these orbits. $\Box$

In the sequel we will omit the superscript $^{-}$ over α in (10.130). We consider X_1, X_2, defined by (10.130), as left-invariant vector fields on the Lie group M_4 and $D_{4,\alpha}$ as a left-invariant rank 2 distribution on M_4. Let us compute the fundamental form of the distributions $D_{4,\alpha}$. Let the fields X_3,X_4, and X_5 be, as in (10.6). From (10.77) and (10.124) it follows that

$$\begin{gathered} X_3 = [(a_1, b_1), (a_2, \alpha b_2)] = (a_3, \alpha b_1), \\ X_4 = [(a_1, b_1), (a_3, \alpha b_1)] = -(a_2, 0), \\ X_5 = -[(a_2, \alpha b_2), (a_3, \alpha b_1)] = (-a_1, \alpha^2 b_1). \end{gathered} \tag{10.132}$$

By direct calculation, we obtain

$$\begin{gathered} [X_1, X_4] = \tfrac{\alpha}{1+\alpha^2} X_1 - X_3 + \tfrac{\alpha}{1+\alpha^2} X_5, \\ [X_2, X_5] = -\alpha X_1 + X_3 - \alpha X_5, \\ [X_1, X_5] = [X_2, X_4] = 0. \end{gathered} \tag{10.133}$$

So, by (10.67),

$$\vec{h} = u_4 X_1 + u_5 X_2 + \frac{\alpha}{1+\alpha^2} u_5^2 \partial_{u_4} - \alpha u_4 u_5 \partial_{u_5} \tag{10.134}$$

Note that in contrast to the previous examples, $\vec{h}(u_4^2 + u_5^2) \neq 0$, but instead of this from (10.134) we easily have

$$\vec{h}\left(u_4^2 + \frac{1}{1+\alpha^2} u_5^2\right) = 0, \tag{10.135}$$

i.e., $u_4^2 + \frac{1}{1+\alpha^2} u_5^2$ is an integral of the vector field $\vec{h}$. By appropriate change of the basis of distribution $D_{4,\alpha}$ we can make this integral to be equal to sum of squares of the corresponding new quasiimpulses. Indeed, suppose that

$$\begin{gathered} \widetilde{X}_1 = \sqrt[6]{1+\alpha^2} X_1 \\ \widetilde{X}_2 = \tfrac{1}{\sqrt[3]{1+\alpha^2}} X_2 \end{gathered} \tag{10.136}$$

As before, replacing X_1, X_2 by $\tilde{X}_1, \tilde{X}_2$ in (10.6) and (10.7), one can define $\widetilde{X}_i$, $i = 3, 4, 5$ and quasiimpulses $\tilde{u}_i$, $1 \leq i \leq 5$. By direct calculations, on can obtain that the vector fields $\widetilde{X}_1, \ldots \widetilde{X}_5$ satisfy the following commutative relations :

$$
\begin{gathered}
[\widetilde{X}_1, \widetilde{X}_4] = \tfrac{\alpha}{1+\alpha^2}\widetilde{X}_1 - \sqrt[3]{1+\alpha^2}\,\widetilde{X}_3 + \tfrac{\alpha}{\sqrt[3]{1+\alpha^2}}\widetilde{X}_5, \\
[\widetilde{X}_2, \widetilde{X}_5] = -\tfrac{\alpha}{1+\alpha^2}\widetilde{X}_1 + \tfrac{1}{\sqrt[3]{(1+\alpha^2)^2}}\widetilde{X}_3 - \tfrac{\alpha}{\sqrt[3]{1+\alpha^2}}\widetilde{X}_5, \\
[\widetilde{X}_1, \widetilde{X}_5] = [\widetilde{X}_2, \widetilde{X}_4] = 0.
\end{gathered}
\tag{10.137}
$$

It implies that the corresponding Hamilatonian vector field $\vec{\tilde{h}} = \tilde{u}_2 \vec{\tilde{u}}_4 + \tilde{u}_1 \vec{\tilde{u}}_5$ has the following more symmetric, comparing with (10.134), form

$$
\begin{aligned}
\vec{\tilde{h}} &= \tilde{u}_5 \widetilde{X}_1 + \tilde{u}_4 \widetilde{X}_2 + \vartheta(\alpha)\tilde{u}_5^2 \partial_{\tilde{u}_4} - \vartheta(\alpha)\tilde{u}_4\tilde{u}_5\partial_{\tilde{u}_5} \\
&= \tilde{u}_5 \widetilde{X}_1 + \tilde{u}_4 \widetilde{X}_2 + \vartheta(\alpha)\tilde{u}_5\Big(\tilde{u}_5\partial_{\tilde{u}_4} - \tilde{u}_4\partial_{\tilde{u}_5}\Big),
\end{aligned}
\tag{10.138}
$$

where

$$
\vartheta(\alpha) = \frac{\alpha}{\sqrt[3]{1+\alpha^2}}
\tag{10.139}
$$

Evidently,

$$
\vec{\tilde{h}}(\tilde{u}_4^2 + \tilde{u}_5^2) = 0
\tag{10.140}
$$

In what follows we will work with the basis (10.136) instead of (10.130). For convenience of computation we will omit the superscript $\tilde{}$ over X_i, u_i, and h.

Similar to the previous cases, from (10.140) it follows that in order to compute the fundamental form of $D_{4,\alpha}$, we can replace ϵ_1 by ∂_θ in (10.62). Make the following polar change of coordinates:

$$
u_4 = r\cos\theta, \quad u_5 = r\sin\theta
$$

Then by (10.138) the Hamiltonian field $\vec{h}$ satisfies

$$
\vec{h} = r\cos\theta X_2 + r\sin\theta X_1 - \vartheta(\alpha) r\sin\theta\ \partial_\theta
\tag{10.141}
$$

Since by construction the fundamental form is homogeneous of order four, it is sufficient to calculate it on the set $\{r = 1\}$. Using (10.6) and (10.141) the reader will have no difficulty to check that on the set $\{r = 1\}$ the brackets $(\operatorname{ad}\vec{h})^i\partial_\theta$, $1 \leq i \leq 3$ satisfy:

$$
[\vec{h}, \partial_\theta] = r\sin\theta X_2 - r\cos\theta X_1 + \vartheta(\alpha) r\cos\theta\ \partial_\theta,
\tag{10.142}
$$

$$
(\operatorname{ad}\vec{h})^2\partial_\theta = X_3 - \vartheta(\alpha)X_1 + \Big(\vartheta(\alpha)\Big)^2\partial_\theta,
\tag{10.143}
$$

$$
(\operatorname{ad}\vec{h})^3\partial_\theta = -\cos\theta X_5 + \sin\theta X_4 + \vartheta(\alpha)\cos\theta X_3 + \Big(\vartheta(\alpha)\Big)^2[\vec{h}, \partial_\theta],
\tag{10.144}
$$

Further, using (10.137), (10.139), and (10.144), we obtain

$$
(\operatorname{ad}\vec{h})^4\partial_\theta = \Big[\cos\theta X_2 + \sin\theta X_1 - \vartheta(\alpha)\sin\theta\ \partial_\theta, -\cos\theta X_5 + \sin\theta X_4 +
$$

$$+\vartheta(\alpha)\cos\theta X_3+\Big(\vartheta(\alpha)\Big)^2[\vec{h},\partial_\theta]\Big]=-\cos^2\theta\,[X_2,X_5]+\sin^2\theta\,[X_1,X_4]-$$
$$-\vartheta(\alpha)\cos^2\theta X_5+\vartheta(\alpha)\sin\theta\cos\theta X_4-\vartheta(\alpha)\sin\theta\Big(\sin\theta X_5+\cos\theta X_4-$$
$$-\vartheta(\alpha)\sin\theta X_3\Big)+\Big(\vartheta(\alpha)\Big)^2(\mathrm{ad}\,\vec{h})^2\partial_\theta=\frac{\alpha}{1+\alpha^2}X_1+\frac{\alpha}{\sqrt[3]{1+\alpha^2}}X_5-$$
$$-\frac{1}{\sqrt[3]{(1+\alpha^2)^2}}X_3--\frac{\alpha}{\sqrt[3]{1+\alpha^2}}X_5+\frac{\alpha^2}{\sqrt[3]{(1+\alpha^2)^2}}(\mathrm{ad}\,\vec{h})^2\partial_\theta=$$
$$=\frac{\alpha}{1+\alpha^2}X_1-\frac{1}{\sqrt[3]{(1+\alpha^2)^2}}X_3+\frac{\alpha^2}{\sqrt[3]{(1+\alpha^2)^2}}(\mathrm{ad}\,\vec{h})^2\partial_\theta \tag{10.145}$$

Then, by (10.143) and (10.139),

$$\frac{\alpha}{1+\alpha^2}X_1-\frac{1}{\sqrt[3]{(1+\alpha^2)^2}}X_3=-\frac{1}{\sqrt[3]{(1+\alpha^2)^2}}\Big(X_3-\frac{\alpha}{\sqrt[3]{1+\alpha^2}}\,X_1\Big)=$$
$$=-\frac{1}{\sqrt[3]{(1+\alpha^2)^2}}\Big((\mathrm{ad}\,\vec{h})^2\partial_\theta-\frac{\alpha^2}{\sqrt[3]{(1+\alpha^2)^2}}\partial_\theta\Big)$$

Substituting this in (10.145) we obtain

$$(\mathrm{ad}\,\vec{h})^4\partial_\theta=\frac{\alpha^2}{\sqrt[3]{(1+\alpha^2)^2}}\partial_\theta+\frac{\alpha^2-1}{\sqrt[3]{(1+\alpha^2)^2}}(\mathrm{ad}\,\vec{h})^2\partial_\theta \tag{10.146}$$

Therefore in notation of (10.62)

$$A_0=\frac{\alpha^2}{\sqrt[3]{(1+\alpha^2)^4}},\ A_1=\frac{\alpha^2-1}{\sqrt[3]{(1+\alpha^2)^2}}, \tag{10.147}$$

and by (10.63) the fundamental form of $D_{4,\alpha}$ satisfies

$$35A=\big(\alpha^2+\frac{9}{100}(\alpha^2-1)^2\big)(1+\alpha^2)^{-\frac{4}{3}}(u_4^2+u_5^2)^4 \tag{10.148}$$

As in the case of distributions D_2 and $D_{3,e}$ the fundamental forms of all distributions $D_{4,\alpha}$ are squares of sign definite quadratic form. This information is not sufficient to make a conclusion about equivalence or nonequivalence of the distributions $D_{4,\alpha}$. Let us compute the projective curvature of these distributions. Since $\vec{h}(A)=0$, we can use again the formula (10.121). Substituting (10.147) and (10.148) in (10.121), we obtain

$$\rho_{D_{4,\alpha}}=\frac{2\sqrt{35}}{15}\frac{1-\alpha^2}{\sqrt{\alpha^2+\frac{9}{100}(\alpha^2-1)^2}} \tag{10.149}$$

It is easy to see that the function

$$\alpha\mapsto\frac{2\sqrt{35}}{15}\frac{1-\alpha^2}{\sqrt{\alpha^2+\frac{9}{100}(\alpha^2-1)^2}}$$

is monotone decreasing on $[0,+\infty)$. Therefore, germs of all distributions $D_{4,\alpha}$ for $\alpha\geq 0$ are not equivalent. We have an example of one parametric family of nonequivalent germs of rank 2 distributions with a common fundamental form !

10.5 Canonical frame of rank 2 distribution with nonzero fundamental form on 5-dimensional manifold

Here we again restrict ourselves to rank 2 distributions on 5-dimensional manifold M. Suppose that the set $\aleph_D$ defined by (10.112) is not empty. The set $\aleph_D$ is open subset of 7-dimensional submanifold $(D^2)^\perp$ (recall that $(D^2)^\perp$ has codimension 3 in the cotangent bundle T^*M). In this subsection we will construct a canonical frame on $\aleph_D$, i.e., to any $\lambda \in \aleph_D$ we will assign canonically a basis of $T_\lambda\big((D^2)^\perp\big)$.

On $\aleph_D$ we already have two invariant vector fields: the canonical Hamiltonian vector field $\overrightarrow{h_A}$, defined in the previous subsection, and the Euler field $\vec{e}$. From homogenuity arguments it is not difficult to show that

$$[\overrightarrow{h_A}, \vec{e}] = 0 \tag{10.150}$$

so the Lie brackets of $\overrightarrow{h_A}$ and $\vec{e}$ do not give new vector fields. Using the notions of normal parameter and the canonical basis of rank 1 curves in the Lagrange Grassmannian, we will construct the third canonical vector field on $\aleph_D$. Finally, making appropriate Lie brackets of this field with $\overrightarrow{h_A}$, we will obtain the canonical moving frame on $\aleph_D$.

First note that on $(D^2)^\perp\backslash(D^3)^\perp$ the following rank 3 distribution D is defined

$$\mathcal{D}(\lambda) = span\Big((D^2)^\perp\big(\pi(\lambda)\big), \vec{h}(\lambda)\Big), \tag{10.151}$$

where $\vec{h}$ is some vector field tangent to characteristic 1-foliation on $(D^2)^\perp\backslash(D^3)^\perp$. For any $\lambda \in \aleph_D$ take the characteristic leaf γ, passing through λ and consider the corresponding reduced Jacobi curve $\tau \mapsto J_{\lambda,\overrightarrow{h_A}}(\tau)$, parameterized by normal parameter. Let $\widetilde{\varepsilon_A}(\lambda)$ be the first vector of the canonical basis of the curve $\tau \mapsto J_{\lambda,\overrightarrow{h_A}}(\tau)$ at the point $\tau = 0$. The vector $\widetilde{\varepsilon_A}(\lambda)$ belongs to $T_\lambda\big((D^2)^\perp\big)/\text{span}\big(\vec{h}(\lambda), \vec{e}(\lambda)\big)$. As before, let p_λ be canonical projection from $T_\lambda\big((D^2)^\perp\big)$ to $T_\lambda\big((D^2)^\perp\big)/\text{span}\big(\vec{h}(\lambda), \vec{e}(\lambda)\big)$. The following proposition holds

Proposition 15 *Among all vertical vector fields $\varepsilon(\lambda)$ on $\aleph_D$ such that $p_\lambda\big(\varepsilon(\lambda)\big) = \widetilde{\varepsilon_A}(\lambda)$ for any $\lambda \in \aleph_D$, there exists unique vector field ε_A such that*

$$\big[\varepsilon_A(\lambda), [\overrightarrow{h_A}(\lambda), \varepsilon_A(\lambda)]\big] \in \mathcal{D}(\lambda). \tag{10.152}$$

Proof. First let us prove the following auxiliary lemma:

Lemma 10.9 *Suppose that $\Lambda(t)$ is a rank 1 curve of the constant weight in $L(W)$, $\dim W = 4$. Suppose that $e_1(t)$ is the first vector in the canonical basis of $\Lambda(t)$. Let $t = \varphi(\tau)$ and the curve $\tau \mapsto \Lambda\big(\varphi(\tau)\big)$ be reparametrization of the curve $\Lambda(t)$ such that $\varphi'(\tau) > 0$. Let $\tilde{e}_1(\tau)$ be the first* *vector in the canonical basis of the curve $\tau \mapsto \Lambda\big(\varphi(\tau)\big)$. Then the following relation holds*

$$\tilde{e}_1(\tau) = \frac{1}{(\varphi'(\tau))^{\frac{3}{2}}} e_1\big(\varphi(\tau)\big). \tag{10.153}$$

Proof. Let the vector function functions $w(t_0, t_1)$, $\tilde{w}(\tau_0, \tau_1)$ be as in (6.1) for the curves $\Lambda(t)$ and $\Lambda(\varphi(\tau))$ respectively. Then directly from definition of $w(t_0, t_1)$, $\tilde{w}(\tau_0, \tau_1)$ it follows that

$$\tilde{w}(\tau_0, \tau_1) = \sqrt{\varphi'(\tau_0)}\, w\big(\varphi(\tau_0), \varphi(\tau_1)\big) \tag{10.154}$$

Recall that in the considered case the vector function $t \mapsto w(t, t_1)$ has a the pole of order 2 at $t = t_1$. Moreover by definition of the vector $e_1(t)$, the following expansion is valid:

$$w(t_0, t_1) = \frac{e_1(t_1)}{(t_0 - t_1)^2} + O\left(\frac{1}{t_0 - t_1}\right). \tag{10.155}$$

Similarly,

$$\tilde{w}(\tau_0, \tau_1) = \frac{\tilde{e}_1(t_1)}{(\tau_0 - \tau_1)^2} + O\left(\frac{1}{\tau_0 - \tau_1}\right) \tag{10.156}$$

Substituting (10.155) and (10.156) in (10.154) and comparing first coefficients of expansions on both sides, we obtain (10.153). □

Take some local basis X_1, X_2 of D. Suppose again that X_3, X_4, X_5, u_i, h, and ∂_θ be as in (10.6), (10.7), (10.13), (10.22) respectively. As before denote by $A(\lambda)$ the density of fundamental form of the parameterized Jacobi curve $t \mapsto J_{e^{t\vec{h}}\lambda}$ at $t = 0$.

As a direct consequences of Proposition 14, Lemma 10.9, and relation (10.115) one can easily obtain

$$\widetilde{\varepsilon_A}(\lambda) = p_\lambda\Big(\frac{6|A(\lambda)|^{\frac{3}{8}}}{u_4^2 + u_5^2}\partial_\theta(\lambda)\Big) \tag{10.157}$$

Suppose that the vertical vector field $\varepsilon(\lambda)$ on $\aleph_D$ satisfies $p_\lambda\big(\varepsilon(\lambda)\big) = \widetilde{\varepsilon_A}(\lambda)$. Then from (10.157) it follows that there exists a smooth function $\mu(\lambda)$ on $\aleph_D$ such that

$$\varepsilon(\lambda) = \frac{6|A(\lambda)|^{\frac{3}{8}}}{u_4^2 + u_5^2}\partial_\theta(\lambda) + \mu(\lambda)\vec{e}(\lambda) \tag{10.158}$$

To prove the Proposition it is sufficient to show that there exists unique $\mu(\lambda)$ such that the field ϵ, defined by (10.158) satisfies

$$\big[\varepsilon, [\overrightarrow{h_A}, \varepsilon]\big] \in \mathcal{D}.$$

For this let us compute $\big[\varepsilon, [\overrightarrow{h_A}, \varepsilon]\big]$. Note that from (10.114) it follows that

$$\overrightarrow{h_A} = \frac{1}{\sqrt[4]{|A|}}\vec{h} \tag{10.159}$$

Denote by

$$\varepsilon_0 = \frac{6|A(\lambda)|^{\frac{3}{8}}}{u_4^2+u_5^2}\partial_\theta.$$

First let us compute $[\vec{e},\varepsilon_0]$:

$$[\vec{e},\varepsilon_0] = 6\vec{e}\left(\frac{|A|^{\frac{3}{8}}}{u_4^2+u_5^2}\right)\partial_\theta + \frac{6|A|^{\frac{3}{8}}}{u_4^2+u_5^2}[\vec{e},\partial_\theta] \tag{10.160}$$

Note that $[\vec{e},\partial_\theta]=0$. Also, it is clear that the restriction of the function $\lambda \mapsto \frac{6|A(\lambda)|^{\frac{3}{8}}}{u_4^2+u_5^2}$ on the fiber $(D^2)^\perp\big(\pi(\lambda)\big)$ is homogeneous of order $-\frac{1}{2}$. Therefore, from (10.160) and Euler identity we obtain

$$[\vec{e},\varepsilon_0] = -\frac{1}{2}\varepsilon_0 \tag{10.161}$$

Now let us compute $\big[\vec{e},[\overrightarrow{h_A},\varepsilon_0]\big]$. For this we will use Jacobi identity,(10.150), and (10.161):

$$\big[\vec{e},[\overrightarrow{h_A},\varepsilon_0]\big]=\big[[\vec{e},\overrightarrow{h_A}],\varepsilon_0\big]+\big[\overrightarrow{h_A},[\vec{e},\varepsilon_0]\big]=-\frac{1}{2}[\overrightarrow{h_A},\varepsilon_0] \tag{10.162}$$

Further, compute $[\overrightarrow{h_A},\epsilon]$:

$$[\overrightarrow{h_A},\varepsilon] = [\overrightarrow{h_A},\varepsilon_0+\mu\vec{e}] = [\overrightarrow{h_A},\varepsilon_0] + \overrightarrow{h_A}(\mu)\vec{e} \tag{10.163}$$

Finally, using (10.161), (10.162) and (10.163), we obtain

$$\begin{aligned}\big[\varepsilon,[\overrightarrow{h_A},\varepsilon]\big] &= \big[\varepsilon_0+\mu\vec{e},[\overrightarrow{h_A},\varepsilon_0]+\overrightarrow{h_A}(\mu)\vec{e}\big] = \\ &= \big[\varepsilon_0,[\overrightarrow{h_A},\varepsilon_0]\big]-\frac{\mu}{2}[\overrightarrow{h_A},\varepsilon_0] \mod\Big(\operatorname{span}(\vec{e},\varepsilon_0)\Big)\end{aligned} \tag{10.164}$$

It is known that

$$\big[\varepsilon_0,[\overrightarrow{h_A},\varepsilon_0]\big]\in \operatorname{span}\Big(\vec{e},\varepsilon_0,\overrightarrow{h_A},[\overrightarrow{h_A},\varepsilon_0]\Big),$$

i.e.,

$$\big[\varepsilon_0,[\overrightarrow{h_A},\varepsilon_0]\big]= \mu_0\vec{e}+\mu_1\varepsilon_0+\mu_2\overrightarrow{h_A}+\mu_3[\overrightarrow{h_A},\varepsilon_0]$$

Therefore from (10.164) it follows that

$$\big[\varepsilon,[\overrightarrow{h_A},\varepsilon]\big]\in \operatorname{span}\Big(\vec{e},\varepsilon_0,\overrightarrow{h_A}\Big) = \mathcal{D}$$

iff $\mu = 2\mu_3$. This completes the proof of the Proposition. □

So, the field ε_A from (10.152) can be taken as the third canonical field.

Theorem 15 *The following seven vector fields*

$$\vec{e},\ \ \overrightarrow{h_A},\ \ \varepsilon_A,\ [\overrightarrow{h_A},\varepsilon_A],\ \ (\operatorname{ad}\overrightarrow{h_A})^2(\varepsilon_A),\ \ (\operatorname{ad}\overrightarrow{h_A})^3(\varepsilon_A),\ \ \Big[\varepsilon_A,(\operatorname{ad}\overrightarrow{h_A})^3(\varepsilon_A)\Big] \tag{10.165}$$

constitute the canonical frame on the set $\aleph_D$.

Proof. By direct computation one can obtain that

$$\begin{aligned}\pi\Big(\operatorname{span}\big(\overrightarrow{h_A}(\lambda),\ [\overrightarrow{h_A},\varepsilon_A](\lambda)\big)\Big) &= D\big(\pi(\lambda)\big)\\ \pi\Big(\operatorname{span}\big(\overrightarrow{h_A}(\lambda),\ [\overrightarrow{h_A},\varepsilon_A](\lambda),\ (\operatorname{ad}\overrightarrow{h_A})^2(\varepsilon_A)(\lambda)\big)\Big) &= D^2\big(\pi(\lambda)\big)\\ D^2\big(\pi(\lambda)\big)+\pi\Big(\operatorname{span}\big((\operatorname{ad}\overrightarrow{h_A})^3(\varepsilon_A)(\lambda),\ \big[\varepsilon_A,(\operatorname{ad}\overrightarrow{h_A})^3(\varepsilon_A)\big](\lambda)\big)\Big) &= D^3\big(\pi(\lambda)\big)\end{aligned} \tag{10.166}$$

This, together with the fact that the vertical vectors $\vec{e}(\lambda)$ and $\varepsilon_A(\lambda)$ constitute the basis of the fiber $(D^2)^\perp(\lambda)$, implies that for any $\lambda \in \aleph_D$ the vectors in (10.165) are linearly independent. This completes the proof of the Theorem. $\square$

The frame (10.165) is called the *canonical frame* of distribution D. Given a distribution D one can define a group of local symmetries of D at the point q_0: the diffeomorphism $F : U \mapsto U$, where U is some neighbourhood of q_0, is called local symmetry of the distribution D if $F_*(D(q)) = D(F(q))$ for any $q \in U$. All local symmetries of distribution at q_0 constitute a local group, called local group of symmetries of D at q_0. Using the standard framing lemma (see [11], [17]), from the Theorem 15 we obtain the following

Corollary 7 *The local group of symmetries of rank 2 distribution on 5 dimensional manifold M at point q with nonzero fundamental form on the fiber $(D^2)^\perp(q)$ has dimension not greater than 7.*

Let $\bar{D}$ be another rank 2 distribution on M . Let $\overrightarrow{\bar{h}_A}$ be its canonical Hamiltonian field and $\bar{\varepsilon}_A$ be the third vector field in the canonical frame of $\bar{D}$. We will say that the distribution D has nonzero fundamental form on M if for any $q \in M$ the set $\aleph_D \cap T_q^*M$ is not empty.

Theorem 16 *Two rank 2 distributions D and $\bar{D}$ with nonzero fundamental forms on the 5-dimensional manifold are equivalent iff there exists the diffeomorphism $\mathcal{F} : \aleph_D \mapsto \aleph_{\bar{D}}$, which transforms three vector fields $\vec{e}$, $\overrightarrow{h_A}$, ε_A to the following three vector fields $\vec{e}$, $\overrightarrow{\bar{h}_A}$, $\bar{\varepsilon}_A$ i.e.,*

$$\begin{aligned}\mathcal{F}_*\vec{e}(\lambda) &= \vec{e}\Big(\mathcal{F}(\lambda)\Big)\\ \mathcal{F}_*\varepsilon_A(\lambda) &= \bar{\varepsilon}_A\Big(\mathcal{F}(\lambda)\Big)\\ \mathcal{F}_*\overrightarrow{h_A}(\lambda) &= \overrightarrow{\bar{h}_A}\Big(\mathcal{F}(\lambda)\Big)\end{aligned} \tag{10.167}$$

Proof. The necessity follows directly from the construction of the canonical frame of the distribution. Let us prove the sufficiency. The pairs of vector fields $\vec{e}$, ε_A and $\vec{e}$, $\bar{\varepsilon}_A$ constitute the bases of the fibers of $(D^2)^\perp$ and $(\bar{D}^2)^\perp$ respectively. This together with the first two relations of (10.167) implies that the diffeomorphism $\mathcal{F}$ is fiberwise, i.e., there exists the diffeomorphism $F : M \mapsto M$ such that

$$F \circ \pi = \pi \circ \mathcal{F}. \tag{10.168}$$

Note also that for any $q \in M$

$$\operatorname{span}\Big(\pi_*\Big(\{\overrightarrow{h_A}(\lambda) : \lambda \in \aleph_D, \pi(\lambda) = q\}\Big)\Big) = D(q),$$
$$\operatorname{span}\Big(\pi_*\Big(\{\overrightarrow{\bar{h}_A}(\lambda) : \lambda \in \aleph_{\bar{D}}, \pi(\lambda) = q\}\Big)\Big) = \bar{D}(q).$$

Using the last two relations, the third relation of (10.167), and (10.168) we easily obtain that $F_*D(q) = \bar{D}\big(F(q)\big)$ for any $q \in M$, which means that the distributions D and $\bar{D}$ are equivalent. $\square$

By Theorem 15 the family of the vector fields $\vec{e}$, $\overrightarrow{h_A}$, ε_A is bracket-generating on $\aleph_D$, i.e., for any $\lambda \in \aleph_D$ some Lie brackets of these fields at λ span the tangent space $T_\lambda \aleph_D$. Note also that if the diffeomorphism $\mathcal{F}$ satisfies (10.167), then it transforms all vector fields from the canonical frame of D to the corresponding canonical vector fields from the canonical frame of $\bar{D}$. So, by Theorem 16 the problem of equivalence of rank 2 distributions with nonzero fundamental form on 5-dimensional manifolds is reduced to the well studied problem of equivalence of bracket-generating families of vector fields or the problem of equivalence of frames on the manifolds (that are also called e-structures or absolute parallelisms on the manifolds).

The equivalence problem of the frames can be solved in terms of their structural functions. Denote by $\wp_i$, $1 \le i \le 7$ the ith vector field of the canonical frame of distribution D (in the order of its appearance in (10.165)) and by $\bar{\wp}_i$, $1 \le i \le 7$ the ith vector field of the canonical frame of distribution $\bar{D}$. The structural functions $c^l_{i_1 \dots i_k}$, $\bar{c}^l_{i_1 \dots i_k}$ are defined by

$$\begin{aligned} [\wp_{i_1}, [\dots, \wp_{i_k}] \dots] &= \sum_{l=1}^{7} c^l_{i_1 \dots i_k} \wp_l, \\ [\bar{\wp}_{i_1}, [\dots, \bar{\wp}_{i_k}] \dots] &= \sum_{l=1}^{7} \bar{c}^l_{i_1 \dots i_k} \bar{\wp}_l. \end{aligned} \tag{10.169}$$

If the diffeomorphism $\mathcal{F} : \aleph_D \mapsto \aleph_{\bar{D}}$ transforms the frame $\{\wp_i\}_{i=1}^7$ to the frame $\{\bar{\wp}_i\}_{i=1}^7$, then, obviously, for any tuple of indices $(i_1, \dots, i_k)$, any $1 \le l \le 7$, and any $\lambda \in \aleph_D$ we have

$$\bar{c}^l_{i_1 \dots i_k}\big(\mathcal{F}(\lambda)\big) = c^l_{i_1 \dots i_k}(\lambda) \tag{10.170}$$

So, we obtain the infinite series of necessary conditions for the equivalence of the frames $\{\wp_i\}_{i=1}^7$ and $\{\bar{\wp}_i\}_{i=1}^7$ (and therefore also of distributions D and $\bar{D}$). On the other hand, it turns out (see [11] or [19], Chapter 7, § 4) that the validity of certain finite number of conditions (10.170) for some diffeomorphism $\mathcal{F}$ implies that $\mathcal{F}$ transforms the frame $\{\wp_i\}_{i=1}^7$ to the frame $\{\bar{\wp}_i\}_{i=1}^7$ and therefore distributions D and $\bar{D}$ are equivalent.

Besides, if we suppose that the manifold M and the distributions D and $\bar{D}$ are real analytic, then it is easy to see that all vector fields in their canonical frames are also real analytic and one can apply the well-known Nagano principle (see, for example, [4], p 68-72). According to this principle, if for some points $\lambda_0 \in \aleph_D$ and $\bar\lambda_0 \in \aleph_{\bar D}$ the following equalities

$$\bar c^{l}_{i_1\ldots i_k}(\bar\lambda_0) = c^{l}_{i_1\ldots i_k}(\lambda_0)$$

hold for all tuples of indices $(i_1, \ldots, i_k)$ and all $1 \le l \le 7$, then there exists diffeomorphism from some neighbourhood of λ_0 in $\aleph_D$ to some neighbourhood of $\bar\lambda_0$ in $\aleph_{\bar D}$, which transforms the canonical frames of the distributions D and $\bar D$ in these neighbourhoods one to another. This together with Theorem 16 implies that the germ of the distribution D at the point $\pi(\lambda_0)$ and the germ of the distribution $\bar D$ at the point $\pi(\bar\lambda_0)$ are equivalent.

11 Geodesic equivalence of control systems

Let us recall that two Riemannian metrics on the n-dimensional manifold M are called *geodesic* (or *projective*) *equivalent* if all their geodesics, considered as unparameterized curves, coincide. In [12], Dini stated and solved the problem of local classification of geodesically equivalent metrics on surfaces (Theorem 18 below, see also [13] § 37, [16]). The notion of geodesic equivalence can be generalized directly to control systems. In the present section we will show, how our theory of unparameterized curves in the Lagrange Grassmannian, developed in section 5, can be used in the problem of geodesic equivalence of control systems.

As in Introduction, we call a smooth control system any submanifold $V \subset TM$, transversal to fibers and denote $V_q = V \cap T_qM$. For simplicity, here we restrict ourselves to the control systems V such that $V_q = \partial W_q$, where W_q is a bounded strongly convex set and $0 \in \operatorname{int} W_q$ for any $q \in M$. Any Riemannian and, more general, Finsler metrics satisfy the assumptions above: the Riemannian metric is defined by choosing some inner product $\langle \cdot, \cdot\rangle_q$ on any tangent subspace T_qM, the corresponding control system V is defined by

$$V_q = \{v \in T_qM : \langle v, v\rangle_q = 1\};$$

the Finsler metric on M are actually defined by control system V such that $V_q = \partial W_q$, where W_q is a bounded strongly convex set and $0 \in \operatorname{int} W_q$, and in addition $V_q = -V_q$ for any $q \in M$.

In the considered case the dual abnormal variety H^0 , defined by (1.7), is empty set, while the normal variety H^1, defined by (1.6), is codimension 1 submanifold of T^*M. Therefore all extremals of V are normal : the curve γ is extremal iff γ is characteristic curve of $\sigma|_{H^1}$, parameterized canonically by the rule $\langle \varsigma, \dot\gamma\rangle = 1$, where ς is the Liouville form. To describe the extremals of V it is also convenient to introduce the following function $h : T^*M \mapsto \mathbb{R}$,

$$h\Big((p, q)\Big) = \max_{v \in V_q} \langle p, v\rangle, \ \ q \in M, \ \ p \in T_q^*M. \tag{11.1}$$

By assumptions, this function is well defined. We call it *master-Hamiltonian* associated with V. The dual normal variety H^1 is the level set of the master-Hamiltonian h , namely,

$$H^1 = \{\lambda \in T^*M : h(\lambda) = 1\}.$$

The extremals of V coincide with the integral curves of the corresponding Hamiltonian vector field $\vec{h}$, which lie on H^1. Two control systems on the manifold M are called *geodesic* (or *projective*) *equivalent*, if all projections under the canonical projection $\pi : T^*M \mapsto M$ of their extremals, considered as unparametrized oriented curves, coincide. One can give the local version of this definition: two control systems on the manifold M are called *geodesic* (or *projective*) *equivalent at the point* $q_0 \in M$, if there exists a neighbourhood of q_0 such that in this neighbourhood all projections under the canonical projection $\pi : T^*M \mapsto M$ of their extremals, considered as unparameterized oriented curves, coincide. In the case of the Finsler metrics the word oriented can be omitted in the definitions above, since in this case if $q(t) = \pi\Big(e^{t\vec{h}}(p_0, q_0)\Big)$, then $q(-t) = \pi\Big(e^{t\vec{h}}(-p_0, q_0)\Big)$.

Let us formulate the notion of the geodesic equivalence in terms which are more convenient for using the theory of curves in the Lagrange Grassmannian. For this let us introduce some notations. As before, denote, $H^1_q = H^1 \cap T^*_qM$. For any $v \in V_q$ there exists a unique $\mathcal{L}_q(v) \in T^*_qM$ such that

$$h\big(\mathcal{L}_q(v)\big)= \langle \mathcal{L}_q(v), v\rangle = 1, \tag{11.2}$$

By definition, $\mathcal{L}_q(v) \in H^1_q$ for any $v \in V_q$. Moreover, the mapping $\mathcal{L}_q : V_q \mapsto H^1_q$ is bijection. Setting

$$\mathcal{L}_q(\alpha v) \stackrel{def}{=} \alpha\mathcal{L}_q(v), \quad \forall \alpha \geq 0,\ v \in V_q, \tag{11.3}$$

we obtain bijection $\mathcal{L}_q : T_qM \mapsto T^*_qM$. One can define also the mapping $\mathcal{L} : TM \mapsto T^*M$ by

$$\mathcal{L}(v, q) = (\mathcal{L}_q(v), q). \tag{11.4}$$

Actually, the mapping $\mathcal{L}$ is the *Legendre transform* associated with the control system V.

Let $\bar{V}$ be another control system, which is geodesic equivalent to the system V, $\bar{H}^1_q$ be its dual normal variety, $\bar{h}$ and $\bar{\mathcal{L}}$ be master-Hamiltonian and the Legendre transform associated with the control system $\bar{V}$. By assumptions, there exists a unique mapping $\psi : TM \mapsto \mathbb{R}^+$ (here $\mathbb{R}^+$ is the set of all positive real numbers) such that

$$\bar{V} = \left\{\left(\frac{1}{\psi(v, q)}v, q\right) : q \in M,\ v \in T_qM\right\}. \tag{11.5}$$

It is clear that the function ψ is homogeneous of degree 1 on the fibers of TM. Denote by $\Psi(V) : V \mapsto \bar{V}$ the mapping, defined as follows:

$$\Psi(v, q) = \left(\frac{1}{\psi(v, q)}v, q\right), \quad \forall q \in M,\ v \in V_q. \tag{11.6}$$

It is clear that the control systems V and $\bar{V}$ are geodesic equivalent iff the mapping

$$\bar{\mathcal{L}} \circ \Psi \circ \mathcal{L}^{-1}$$

transform the characteristic curves of $\sigma|_{H^1}$ to the characteristic curves $\sigma|_{\bar{H}^1}$. For given characteristic curves γ of $\sigma|_{H^1}$ and $\bar{\gamma}$ of $\sigma|_{\bar{H}^1}$ denote by $\lambda \mapsto J_\gamma(\lambda)$, $\lambda \in \gamma$, and by $\lambda \mapsto \bar{J}_{\bar{\gamma}}(\lambda)$, $\lambda \in \bar{\gamma}$, the Jacobi curves associated with γ and $\bar{\gamma}$. Note that the mapping $\bar{\mathcal{L}} \circ \Psi \circ \mathcal{L}^{-1}$ transforms any fiber T_q^*M to itself. Therefore directly from definition of Jacobi curve (see (1.1)) we obtain the following

Proposition 16 *For any characteristic curve γ of $\sigma|_{H^1}$ the following Jacobi curves $J_\gamma(\cdot)$ and $\bar{J}_{(\bar{\mathcal{L}}\circ\Psi\circ\mathcal{L}^{-1})(\gamma)}(\cdot)$ are the same, up to a symplectic transformation ,as unparameterized curves in the corresponding Lagrange Grassmannians.*

As was already mentioned before, the characteristic curves of σ_{H^1} and $\sigma|_{\bar{H}^1}$ (and therefore the corresponding Jacobi curves) have the canonical parameters, defined by Hamiltonian vector fields $\vec{h}$ and $\vec{\bar{h}}$ respectively. Take $\lambda \in H^1$ and let γ be characteristic curve of $\sigma|_{H^1}$, containing λ. Denote by $\rho(\lambda)$ and $A(\lambda)$ the Ricci curvature and the density of fundamental form of the curve $t \mapsto J_\gamma\left(e^{t\vec{h}}\lambda\right)$ at the point $t = 0$. So, for given control system V we have constructed two functions $\rho : H^1 \mapsto \mathbb{R}$ and $A : H^1 \mapsto \mathbb{R}$. Let us extend the functions $\rho(\cdot)$ and $A(\cdot)$ on the whole T^*M as the homogeneous functions on the fibers of degree 2 and 4 respectively, namely, let

$$\rho(\alpha p, q) \stackrel{def}{=} \alpha^2 \rho(p, q), \quad A(\alpha p, q) \stackrel{def}{=} \alpha^4 A(p, q) \;\; \forall \alpha \geq 0,\, q \in M,\, p \in H_q^1. \tag{11.7}$$

These extended function ρ and A will be called respectively the *Ricci curvature* and the *canonical cotangent density* of the system V. Let $\bar{\rho}$ and $\bar{A}$ be the Ricci curvature and the canonical cotangent density of the second system $\bar{V}$.

Using Proposition 16 and general reparameterization rules for the symplectic invariants of curves in the Lagrange Grassmannian, presented in section 5 (see formulas (5.2), (5.3), and construction of the fundamental form), one can obtain the connection between functions ρ, A and $\bar{\rho}$, $\bar{A}$, which gives necessary conditions for control systems to be geodesic equivalence.

For this note first that by construction for any $v \in V_q$ we have $\pi_*\vec{h}\big(\mathcal{L}(v, q)\big) = v$. This together with (11.5) implies

$$\pi_*\left(\vec{\bar{h}}\left(\bar{\mathcal{L}} \circ \Psi(v, q)\right) = \frac{1}{\psi(v, q)}\, \pi_*\left(\vec{h}\big(\mathcal{L}(v, q)\big)\right). \tag{11.8}$$

Take $(v, q) \in V$ and let γ be characteristic curve of $\sigma|_{H^1}$, containing the point $\mathcal{L}(v, q)$. Suppose that the function $\varphi(t)$ satisfies

$$\varphi'(t) = \psi\big(\mathcal{L}^{-1} \circ e^{t\vec{h}} \circ \mathcal{L}(v, q)\big), \;\; \varphi(0) = 0. \tag{11.9}$$

Then from Proposition 16 and (11.8) it follows that the parameterized curves

$$t \mapsto J_\gamma\big(e^{t\vec{h}}\mathcal{L}(v,q)\big),$$
$$t \mapsto \bar{J}_{(\bar{\mathcal{L}}\circ\Psi\circ\mathcal{L}^{-1})(\gamma)}\big(e^{\varphi(t)\vec{\bar{h}}}\bar{\mathcal{L}}\circ\Psi(v,q)\big)$$

are the same , up to a symplectic transformation.

This together with the reparameterization rule for the densities of fundamental form implies that

$$A\big(\mathcal{L}(v,q)\big)= \bar{A}\big(\bar{\mathcal{L}}\circ\Psi(v,q)\big)\varphi'(0) = \bar{A}\big(\bar{\mathcal{L}}(\frac{1}{\psi(v,q)}v,q)\big)\psi(v,q)^4, \ \ \forall q\in M,\, v\in V_q \tag{11.10}$$

Taking into account the homogeneity of the functions $\mathcal{L}$ and A (see (11.4) and (11.7)), we obtain that

$$A\big(\mathcal{L}(v,q)\big)= \bar{A}\big(\bar{\mathcal{L}}(v,q)\big), \quad \forall q\in M,\, v\in T_qM. \tag{11.11}$$

Set

$$B \stackrel{def}{=} A\circ\mathcal{L}. \tag{11.12}$$

The function $B : TM \mapsto \mathbb{R}$ will be called the *canonical tangent density* of the control system V. Rewriting the identity (11.11) in terms of function B we obtain the following necessary condition for the geodesic equivalence of control system

Theorem 17 *If two control systems are geodesic equivalent, then they have the same canonical tangent densities.*

By assumptions and Proposition 1 the Jacobi curves of the extremals of the control systems under consideration are regular curves in the Lagrange Grassmannian of $2(n-1)$-dimensional symplectic space. According to Lemma 5.2, the fundamental form of the regular curve can be expressed in terms of its curvature operator. In particular, using identity (5.12), it is not difficult to obtain that for Riemannian metrics the canonical tangent (cotangent) density is degree 4 homogeneous polynomial on the fibers of TM (or T^*M). Therefore in this case the canonical tangent density defines covariant symmetric tensor of order 4 on M.

Note that from the identity (5.12) it follows also that for the control systems on surfaces the canonical tangent (cotangent) density is always equal to zero. Consequently, in this case Theorem 17 is useless. Now we will find the connection between generalized Ricci curvatures of the geodesic equivalent system and show, how it can be used for solving the problem of local geodesic equivalence of Riemannian metrics on surfaces.

Recall that the Jacobi curves under consideration have the weight equals to $n-1$. From reparameterization rule (5.3) for Ricci curvatures it follows that.

$$\rho\big(\mathcal{L}(v,q)\big) = \bar{\rho}\left(\bar{\mathcal{L}}\left(\frac{1}{\psi(v,q)}v,q\right)\right)(\varphi'(0))^2 + \frac{n-1}{3}\mathbb{S}(\varphi(t))\Big|_{t=0}, \tag{11.13}$$

where the function φ is defined by (11.9) and $\mathbb{S}(\varphi)$ is Schwarzian of the function φ, defined by (3.8). First note that by (11.9) $\varphi'(0) = \psi(v,q)$. Therefore, taking into account homogeneity of the functions ρ and $\mathcal{L}$, we obtain from (11.13) that

$$\rho\big(\mathcal{L}(v,q)\big) - \bar{\rho}\left(\bar{\mathcal{L}}\,(v,q)\right) = \frac{n-1}{3}\mathbb{S}(\varphi(t))\Big|_{t=0}, \tag{11.14}$$

Now let us write the righthand side of (11.14) in more explicit form. For this it is more convenient to represent the Schwarzian of φ in the following way:

$$\mathbb{S}(\varphi) = \frac{1}{2}\frac{d^2}{d\,t^2}(\ln\varphi') - \frac{1}{4}\left(\frac{d}{dt}(\ln\varphi')\right)^2 \tag{11.15}$$

Substituting $\lambda = \mathcal{L}(v,q)$ in (11.14) and using (11.9), (11.15), we obtain

$$\rho(\lambda) - \bar{\rho}\left(\bar{\mathcal{L}}\circ\mathcal{L}^{-1}\lambda\right) = \frac{n-1}{6}\left(\vec{h}\circ\vec{h}\Big(\big(\ln\circ\psi\circ\mathcal{L}^{-1}\big)(\lambda)\Big) - \frac{1}{2}\Big(\vec{h}\Big(\big(\ln\circ\psi\circ\mathcal{L}^{-1}\big)(\lambda)\Big)\Big)^2\right) \tag{11.16}$$

for any $\lambda \in H^1$. In order to extend this identity to the whole T^*M, it is convenient to define the new Hamiltonian h_1, as follows:

$$h_1 \stackrel{def}{=} \frac{1}{2}h^2 \tag{11.17}$$

Then we have

$$\rho(\lambda) - \bar{\rho}\left(\bar{\mathcal{L}}\circ\mathcal{L}^{-1}\lambda\right) = \frac{n-1}{6}\left(\vec{h}_1\circ\vec{h}_1\Big(\big(\ln\circ\psi\circ\mathcal{L}^{-1}\big)(\lambda)\Big) - \frac{1}{2}\Big(\vec{h}_1\Big(\big(\ln\circ\psi\circ\mathcal{L}^{-1}\big)(\lambda)\Big)\Big)^2\right) \tag{11.18}$$

for all $\lambda \in T^*M$. Indeed, for any $\lambda \in H^1$ the relation (11.18) is exactly (11.16). Moreover, both sides of (11.18) are degree 2 homogeneous functions on the fibers of T^*M. Therefore, the identity (11.18) holds for all $\lambda \in T^*M$. The identity (11.18) is one more necessary condition for geodesic equivalence of control systems in addition to Theorem 17.

Let us analyze this identity in more details in the case of Riemannian metrics. Suppose that V and $\bar{V}$ define two Riemannian metrics $||\cdot||$ and $||\cdot||_1$ on the n-dimensional manifold M that are geodesic equivalent in a neighbourhood of some point $q_0 \in M$. Let $(X_1,\dots,X_n)$ and $(\tilde{X}_1,\dots,\tilde{X}_n)$ be orthonormal frames w.r.t. $||\cdot||$ and $||\cdot||_1$ in a neighbourhood $\widetilde{M}$ of $q_0 \in M$ and let Q be transition matrix of these bases, namely

$$(X_1,\dots,X_n) = (\tilde{X}_1,\dots,\tilde{X}_n)Q \tag{11.19}$$

Denote by $u_i : T^*\widetilde{M} \mapsto \mathbb{R}$, the quasi-impulses $u_i(q,p) = p\cdot X_i(q)$, where $q \in \widetilde{M}$, $p \in T_q^*\widetilde{M}$. Then $u = (u_1,\dots,u_n)$ are coordinates on the fiber $T_q^*\widetilde{M}$ for any $q \in \widetilde{M}$. By $||\cdot||$ we also denote the norm induced by the metric $||\cdot||$ on the fibers of T^*M. Evidently,

$$||u|| = \left(\sum_{i=1}^{n} u_i^2\right)^{\frac{1}{2}}. \tag{11.20}$$

The master-Hamiltonian h and the Hamiltonian h_1, defined by (11.17), satisfy

$$h(u,q) = ||u||, \;\; h_1(u,q) = \frac{1}{2}||u||^2. \tag{11.21}$$

From this it is easy to show that

$$\mathcal{L}_q\left(\sum_{i=1}^n u_i X_i(q)\right) = u,$$

which together with (11.19) implies that

$$\bar{\mathcal{L}}_q \circ \mathcal{L}_q = uQ^T. \tag{11.22}$$

Also, directly from definition (see (11.5)), we obtain that

$$\psi\left(\sum_{i=1}^n u_i X_i(q), q\right) = ||uQ^T||. \tag{11.23}$$

Substituting (11.22) and (11.23) in (11.18), we get

$$\rho(u,q) - \bar{\rho}(uQ^T,q) = \frac{n-1}{6}\left(\vec{h}_1 \circ \vec{h}_1\big(\ln ||uQ^T||\big) - \frac{1}{2}\Big(\vec{h}_1\big(\ln ||uQ^T||\big)\Big)^2\right) =$$

$$= \frac{n-1}{6}\left(\frac{1}{2}\vec{h}_1 \circ \vec{h}_1\big(\ln ||uQ^T||^2\big) - \frac{1}{8}\Big(\vec{h}_1\big(\ln ||uQ^T||^2\big)\Big)^2\right). \tag{11.24}$$

Finally, simplifying the righthand side of (11.24), we obtain

$$\rho(u,q) - \bar{\rho}(uQ^T,q) = \frac{n-1}{12}\left(\frac{\vec{h}_1 \circ \vec{h}_1\left(||uQ^T||^2\right)}{||uQ^T||^2} - \frac{5}{4}\left(\frac{\vec{h}_1(||uQ^T||^2)}{||uQ^T||^2}\right)^2\right) \tag{11.25}$$

Note that for the Riemannian metrics the Ricci curvatures are quadratic polynomials on the fibers of T^*M. Therefore the lefthand side of (11.25) is quadratic polynomial w.r.t. u, while the righthand side is in general a rational function w.r.t. u. Hence by identity (11.25) if the Riemannian metrics $||\cdot||$ and $||\cdot||_1$ are geodesic equivalent then the right-hand side of (11.25) has to be polynomial w.r.t. u.

It turns out that in the case $n = 2$, using only this fact, one can obtain a new proof of the following classical Dini's theorem:

Theorem 18 *(Dini's theorem, see [12], [13] p. 210) If Riemannian metrics $||\cdot||$ and $||\cdot||_1$ on a surface M are geodesic equivalent in a neighbourhood of some point q_0 and non-proportional at this point, then in a neighbourhood of q_0, there exist coordinates x, y and one-variable functions X and Y such that the metrics $||\cdot||$ and $||\cdot||_2$ in this coordinates satisfy*

$$||\cdot||^2 = (X(x) - Y(y))(dx^2 + dy^2), \tag{11.26}$$

$$||\cdot||_1^2 = \left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right)\left(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)}\right). \tag{11.27}$$

Proof. If the right-hand side of (11.25) is polynomial w.r.t. u, then the polynomial

$$\vec{h}_1 \circ \vec{h}_1 \left(||uQ^T||^2\right) ||uQ^T||^2 - \frac{5}{4}\left(\vec{h}_1(||uQ^T||^2)\right)^2$$

is divided by the polynomial $||uQ^T||^4$ and hence by the polynomial $||uQ^T||^2$. It implies that the polynomial $\vec{h}_1(||uQ^T||^2)$ is divided by the polynomial $||uQ^T||^2$.

Let us find, when $\frac{\vec{h}_1(||uQ^T||^2)}{||uQ^T||^2}$ is a polynomial in u. Since the metrics are non-proportional at q_0 one can take the orthonormal frame (X_1, X_2) in a neighbourhood of q_0 such that the transition matrix Q is diagonal,i.e.,

$$X_1 = \alpha \tilde{X}_1,\ X_2 = \beta X_2, \tag{11.28}$$

$$\alpha(q_0) \neq \beta(q_0), \tag{11.29}$$

$$||uQ^T||^2 = \alpha^2 u_1^2 + \beta^2 u_2^2. \tag{11.30}$$

Suppose that

$$[X_1, X_2] = c_{21}^1 X_1 + c_{21}^2 X_2 \tag{11.31}$$

From (11.32) by direct computations we obtain that the Hamiltonian field $\vec{h}_1$, corresponding to the Hamiltonian h_1, has the form

$$\vec{h}_1 = u_1\vec{u_1} + u_2\vec{u_2} = u_1 X_1 + u_2 X_2 + (c_{21}^1 u_1 + c_{21}^2 u_2)(u_1\partial_{u_2} - u_2\partial_{u_1}) \tag{11.32}$$

It implies that $P(u_1, u_2) \stackrel{def}{=} \vec{h}_1(||uQ^T||^2)$ is degree 3 homogeneous polynomial in u_1, u_2. Namely,

$$P(u_1,u_2) = \vec{h}_1(||uQ^T||^2) = \Big(u_1 X_1 + u_2 X_2 + (c_{21}^1 u_1 + c_{21}^2 u_2)(u_1\partial_{u_2} - u_2\partial_{u_1})\Big)(\alpha^2 u_1^2 + \beta^2 u_2^2) =$$

$$X_1(\alpha^2)u_1^3 + \Big(X_2(\alpha^2) - 2c_{21}^1(\alpha^2-\beta^2)\Big)u_1^2 u_2 + \Big(X_1(\beta^2) - 2c_{21}^2(\alpha^2-\beta^2)\Big)u_1 u_2^2 + X_2(\beta^2)u_2^2 \tag{11.33}$$

Evidently, by (11.30), polynomial $P(u_1, u_2)$ is divided by polynomial $||uQ^T||$ iff $P(i\beta, \alpha) = 0$. Substituting $u_1 = i\beta$, $u_2 = \alpha$ in (11.30), we obtain that $P(i\beta, \alpha) = 0$ iff the following two relations hold

$$X_1\left(\frac{\beta^2}{\alpha^2}\right) = 2c_{21}^2\left(1 - \frac{\beta^2}{\alpha^2}\right) \tag{11.34}$$

$$X_2\left(\frac{\alpha^2}{\beta^2}\right) = 2c_{21}^1\left(\frac{\alpha^2}{\beta^2} - 1\right) \tag{11.35}$$

Choose coordinates (x, y) in a neighbourood of q_0 such that $q_0 = (0, 0)$ and the integral curves of X_1 and X_2 are straight lines parallel to x- axis and y-axis respectively. In other words, there exist functions $a(x, y)$ and $b(x, y)$ such that

$$X_1 = a(x,y)\frac{\partial}{\partial x}, \qquad X_2 = b(x,y)\frac{\partial}{\partial y} \tag{11.36}$$

Then, by direct computations,

$$c_{21}^1 = -b\frac{a_y}{a}, \qquad c_{21}^2 = a\frac{b_x}{b} \tag{11.37}$$

Substituting (11.36) and (11.37) in (11.34)-(11.35), we get

$$\frac{\partial}{\partial x}\left(\frac{\beta^2}{\alpha^2}\right) = -2\frac{b_x}{b}\left(\frac{\beta^2}{\alpha^2} - 1\right), \tag{11.38}$$

$$\frac{\partial}{\partial y}\left(\frac{\alpha^2}{\beta^2}\right) = -2\frac{a_y}{a}\left(\frac{\alpha^2}{\beta^2} - 1\right). \tag{11.39}$$

From (11.38)-(11.39) it follows easily that there exist one-variable functions $\tilde{X}$ and $\tilde{Y}$ such that

$$\frac{\beta^2}{\alpha^2} = 1 + \frac{\tilde{Y}(y)}{b^2}, \tag{11.40}$$

$$\frac{\alpha^2}{\beta^2} = 1 + \frac{\tilde{X}(x)}{a^2}. \tag{11.41}$$

Note that from the last two relations and (11.29), it follows that functions $\tilde{X}$ and $\tilde{Y}$ have opposite signs in a neighbourhood of 0. By coordinates change of the type $x \mapsto \Phi(x)$, $y \mapsto \Psi(y)$, we can make $\tilde{X}(x) \equiv 1$ and $\tilde{Y}(y) \equiv -1$. Then equations (11.40)-(11.41) can be written as follows:

$$\frac{\beta^2}{\alpha^2} = 1 - \frac{1}{b^2}, \tag{11.42}$$

$$\frac{\alpha^2}{\beta^2} = 1 + \frac{1}{a^2}. \tag{11.43}$$

Multiplying (11.41) and (11.42) one can easily have

$$a^2 = b^2 - 1 \tag{11.44}$$

Till now we have looked on the metric $||\cdot||$, as on the original one, and we obtain the relations (11.40)-(11.41), connecting this metric with the second metric $||\cdot||_1$. But we can start with the metric $||\cdot||_1$, as the original one. Namely,

$$\tilde{X}_1 = \tilde{\alpha}X_1, \ \ \tilde{X}_2 = \tilde{\beta}X_2,$$

$$\tilde{X}_1 = \tilde{a}(x,y)\frac{\partial}{\partial x}, \qquad \tilde{X}_2 = \tilde{b}(x,y)\frac{\partial}{\partial y},$$

where, by (11.28) and (11.36),

$$\tilde{\alpha} = \tfrac{1}{\alpha}, \ \ \tilde{\beta} = \tfrac{1}{\beta} \tag{11.45}$$

$$\tilde{a} = \tfrac{a}{\alpha}, \ \ \tilde{b} = \tfrac{b}{\beta} \tag{11.46}$$

Then the relations (11.40)-(11.41) in the considered situation have the form

$$\frac{\tilde{\beta}^2}{\tilde{\alpha}^2} = 1 + \frac{\bar{Y}(y)}{\tilde{b}^2}, \tag{11.47}$$

$$\frac{\tilde{\alpha}^2}{\tilde{\beta}^2} = 1 + \frac{\bar{X}(x)}{\tilde{a}^2}, \tag{11.48}$$

where $\bar{X}$, $\bar{Y}$ are some one-variable functions. Substituting (11.45)-(11.46) in (11.47)-(11.48), we have the following two new relation in addition to (11.42)-(11.43):

$$\frac{\alpha^2}{\beta^2} = 1 + \frac{\bar{Y}(y)\beta^2}{b^2}, \tag{11.49}$$

$$\frac{\beta^2}{\alpha^2} = 1 + \frac{\bar{X}(x)\alpha^2}{a^2}, \tag{11.50}$$

Comparing (11.43) with (11.49) and using (11.44 we have

$$\bar{Y}(y) = \frac{b^2}{\beta^2 a^2} = \frac{a^2+1}{\beta^2 a^2} = \frac{1}{\beta^2}\left(1 + \frac{1}{a^2}\right)$$

Then applying (11.43) again, we obtain

$$\bar{Y}(y) = \frac{\alpha^2}{\beta^4} \tag{11.51}$$

Similarly, from (11.42) and (11.50) it follows that

$$\bar{X}(x) = -\frac{a^2}{\alpha^2 b^2} = -\frac{b^2-1}{\alpha^2 b^2} = -\frac{1}{\alpha^2}\left(1 - \frac{1}{b^2}\right)$$

Applying (11.43) again, we get

$$\bar{X}(x) = -\frac{\beta^2}{\alpha^4} \tag{11.52}$$

Put

$$X(x) = -\bar{X}(x)^{\frac{1}{3}}, \quad Y(y) = \bar{Y}(y)^{\frac{1}{3}}. \tag{11.53}$$

From (11.51) and (11.52) it follows easily that

$$\frac{\alpha^2}{\beta^2} = \frac{Y(y)}{X(x)} \tag{11.54}$$

Then substituting (11.54) in (11.42) and (11.43) we get

$$\frac{1}{b^2} = 1 - \frac{X(x)}{Y(y)} = \frac{Y(y)-X(x)}{Y(y)}, \quad \frac{1}{a^2} = \frac{Y(y)}{X(x)} - 1 = \frac{Y(y)-X(x)}{X(x)}, \tag{11.55}$$

which implies that

$$||\cdot||^2 = (Y(y)-X(x))\left(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)}\right). \tag{11.56}$$

Now substituting (11.54) in (11.49),(11.50) and using (11.53) we get

$$\frac{\beta^2}{b^2} = \frac{1}{Y(y)^3}\left(\frac{Y(y)}{X(x)} - 1\right) = \frac{1}{Y(y)^2}\left(\frac{1}{X(x)} - \frac{1}{Y(y)}\right) \tag{11.57}$$

$$\frac{\alpha^2}{a^2} = -\frac{1}{X(y)^3}\left(\frac{X(x)}{Y(y)} - 1\right) = \frac{1}{X(x)^2}\left(\frac{1}{X(x)} - \frac{1}{Y(y)}\right) \tag{11.58}$$

which implies that

$$||\cdot||_1^2 = (\frac{1}{X(x)} - \frac{1}{Y(y)})\left(\frac{dx^2}{X(x)^2} + \frac{dy^2}{Y(y)^2}\right). \tag{11.59}$$

Finally, by coordinates change of the type $x \mapsto \Phi(x)$, $y \mapsto \Psi(y)$ one can obtain from (11.56) and (11.59) the normal forms (11.26), (11.27). The proof of the Theorem is completed. $\Box$.

We believe that the method of the proof of Theorem 18, especially the use of the algebraic conditions, following from the identities (11.18) and (11.25), give new perspectives in obtaining the analogs of Dini's theorem for Riemannian metrics on the manifolds of dimension greater than 2 and also for sub-Riemannian metrics.

12 Appendix

In this appendix we prove the identity (7.47). We are sure that the proof presented here is far to be optimal, but this is the only one that we have at this moment.

Denote the determinant in the lefthand side of (7.47) by $L_m(k)$. Expanding this determinant w.r.t. the last column, we have

$$L_m(k) = \sum_{j=1}^{m}(-1)^{j+m}\theta_j(k)\prod_{i=1}^{j-1}\eta_i(k)\prod_{i=j+1}^{m}\zeta_i(k) + \prod_{i=1}^{m}\eta_i(k) \tag{12.1}$$

Then, substituting (7.46) in (12.1), one can easily transform $L_m(k)$ to the following form:

$$L_m(k) = \sum_{j=1}^{m+1}\frac{m(2m-j)!}{(m-j+1)!(j-1)!}\Big(k - 2(m-j+1)\Big)\prod_{i=2-j}^{m-j}(k-i)\prod_{i=2m-j+2}^{2m}(k-i) \tag{12.2}$$

Note that $L_m(k)$ is a polynomial of degree $2m$, exactly as the polynomial in the righthand side of (7.47). Also for both polynomials the coefficient of leading term k^{2m} is equal to 1. Therefore in order to prove the identity (7.47) it is sufficient to prove that the polynomials in both sides of (7.47) have the same roots, or, equivalently, that $L_m(2i) = L_m(1-2i) = 0$ for all $1 \le i \le m$.

We will do this in two steps: First we will show that

$$L_m(2i) = 0 \quad 1 \le i \le m. \tag{12.3}$$

Secondly we will prove that the function

$$\overline{L_m}(k) \stackrel{def}{=} \frac{k}{k-2m}L_m(k) \tag{12.4}$$

satisfies

$$\overline{L_m}(-1-k) = \overline{L_m}(k), \tag{12.5}$$

i.e., $\overline{L_m}(k)$ is invariant under the reflection of its argument w.r.t. $-1/2$. This together with (12.3) and the fact that $\overline{L_m}(0) = 0$ (which follows directly from the definition of $\overline{L_m}(k)$) will imply that also $L_m(1-2i) = 0$ for all $1 \le i \le m$.

1. The proof of (12.3). For $1 \le j \le m+1$ denote by

$$p_{m,j}(k) = \frac{m(2m-j)!}{(m-j+1)!(j-1)!}\Big(k - 2(m-j+1)\Big) \prod_{i=2-j}^{m-j} (k-i) \prod_{i=2m-j+2}^{2m} (k-i), \tag{12.6}$$

By direct computation the following identity can be easily checked:

$$p_{m,j}(2m-2l) + p_{m,2l+2-j}(2m-2l) = 0, \tag{12.7}$$

where

$$0 \le l \le m-1, \quad \max\{1, 2l+1-m\} \le j \le \min\{m+1, 2l+1\}$$

In particular, applying (12.6) to $j = l+1$, we have

$$p_{m,l+1}(2m-2l) = 0 \tag{12.8}$$

By construction,

$$L_m(2m-2l) = \sum_{j=1}^{m+1} p_{m,j}(2m-2l) \tag{12.9}$$

Denote $l_1 = \max\{1, 2l+1-m\}$ and $l_2 = \min\{m+1, 2l+1\}$. Consider the following 3 cases:

1) $l_1 \le j \le l_2$. Then from (12.7) and (12.8) it follows that

$$\sum_{j=l_1}^{l_2} p_{m,j}(2m-2l) = \sum_{j=l_1}^{l} \Big(p_{m,j}(2m-2l) + p_{m,2l+2-j}(2m-2l)\Big) + p_{m,l+1}(2m-2l) = 0; \tag{12.10}$$

2) $2l+2 \le j \le m+1$. Then $2m-j+2 \le m-l \le 2m$, so from (12.6) it follows that $p_{m,j}(2m-2l) = 0$;

3) $1 \le j \le 2l-m$. Then $2 \le 2m-2l \le m-j$ and again from (12.6) it follows that in this case $p_{m,j}(2m-2l) = 0$.

Therefore, by (12.9), $L_m(2m-2l) = 0$ for all $0 \le l \le m-1$, or, equivalently, $L_m(2i) = 0$ for all $1 \le i \le m$.

2. The proof of (12.5). We will transform the expression for $L_m(k)$ to the more symmetric form. Following [18] (Chapter 1, §2) we denote

$$x^{n|h} \stackrel{def}{=} x(x+h)\dots(x+(n-1)h) \tag{12.11}$$

Then similarly to the Newton binomial identity, one easily have

$$(x+y)^{n|h} = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) x^{n-i|h} y^{i|h} \tag{12.12}$$

Using the notation (12.11) one can rewrite $L_m(k)$ in the following form

$$L_m(k) = \sum_{j=1}^{m+1} \frac{m(2m-j)!}{(m-j+1)!(j-1)!}(k-2(m-j+1))(k-m+j)^{m-1|1}(k-2m)^{j-1|1} \tag{12.13}$$

Applying (12.12), one get

$$(k-m+j)^{m-1|1} = \Big((k+1)+(j-m-1)\Big)^{m-1|1} = \sum_{i=0}^{m-1} \left(\begin{array}{c} m \\ i \end{array} \right) (k+1)^{m-1-i|1}(j-m-1)^{i|1} =$$

$$\sum_{i=0}^{m-j+1} (-1)^i \left(\begin{array}{c} m \\ i \end{array} \right) (k+1)^{m-1-i|1} \frac{(m-j+1)!}{(m-j-i+1)!} \tag{12.14}$$

Substituting (12.14) in (12.13) and changing the order of summation one easily obtain

$$L_m(k) = \sum_{i=0}^{m} \left(\sum_{j=0}^{m-i} \frac{(2m-j-1)!}{j!(m-j-i)!}(k-2(m-j))(k-2m)^{j|1} \right) \frac{(-1)^i m!(k+1)^{m-1-i|1}}{(m-i-1)!i!}. \tag{12.15}$$

Lemma 12.1 *The following identity holds:*

$$\sum_{j=0}^{m-i} \frac{(2m-j-1)!}{j!(m-j-i)!}(k-2m)^{j|1}(k-2(m-j)) = \frac{(m+i-1)!}{(m-i)!}(k-2m)(k+m-i) \prod_{l=1}^{m-i-1} (k-l). \tag{12.16}$$

Proof. Using representation $k-2(m-j) = (k-2m)+2j$, one can split the lefthand side of (12.16) into the sum of two terms:

$$(k-2m) \sum_{j=0}^{m-i} \frac{(2m-j-1)!}{j!(m-j-i)!}(k-2m)^{j|1} + 2 \sum_{j=1}^{m-i} \frac{(2m-j-1)!}{(j-1)!(m-j-i)!}(k-2m)^{j|1}. \tag{12.17}$$

Since

$$(2m-j-1)! = (m+i-1)!(m+i)^{m-i-j|1},$$

the first term of (12.17) can be written in the way such that one can apply the binomial identity(12.12):

$$\frac{(m+i-1)!(k-2m)}{(m-i)!} \sum_{j=0}^{m-i} \left(\begin{array}{c} m-i \\ j \end{array} \right) (m+i)^{m-i-j|1}(k-2m)^{j|1} =$$

$$\frac{(m+i-1)!(k-2m)}{(m-i)!}(k-m+i)^{m-i|1}. \tag{12.18}$$

In the same way, the second term of (12.17) can be also written in the way such that one can apply the binomial identity (12.12):

$$2\sum_{j=1}^{m-i}\frac{(2m-j-1)!}{(j-1)!(m-j-i)!}(k-2m)^{j|1} = 2\sum_{j=0}^{m-i-1}\frac{(2m-j-2)!}{(j)!(m-j-i-1)!}(k-2m)^{j+1|1} =$$

$$2(k-2m)\sum_{j=0}^{m-i-1}\frac{(2m-j-2)!}{(j)!(m-j-i-1)!}(k-2m+1)^{j|1} =$$

$$\frac{2(m+i-1)!(k-2m)}{(m-i-1)!}\sum_{j=1}^{m-i-1}\left(\begin{array}{c} m-i-1 \\ j \end{array}\right)(m+i)^{m-i-1-j|1}(k-2m+1)^{j|1} =$$

$$\frac{2(m+i-1)!(k-2m)}{(m-i-1)!}(k-m+i+1)^{m-i-1|1}. \tag{12.19}$$

Combining (12.18) and (12.19) together, we obtain that the lefthand side of (12.16) is equal to

$$\frac{(m+i-1)!(k-2m)}{(m-i-1)!}\Big(\frac{k-m+i}{m-i}+2\Big)(k-m+i+1)^{m-i-1|1} =$$

$$\frac{(m+i-1)!}{(m-i)!}(k-2m)(k+m-i)(k-m+i+1)^{m-i-1|1} = \frac{(m+i-1)!}{(m-i)!}(k-2m)(k+m-i)\prod_{l=1}^{m-i-1}(k-l)$$

that is exactly the righthand side of (12.16). This completes the proof of the lemma.□

Now substituting (12.16) to (12.15), we have the following identity:

$$L_m(k) = (k-2m)\sum_{i=0}^{m}\frac{(-1)^i m!(m+i-1)!}{i!(m-i)!(m-i-1)!}\prod_{l=1}^{m-i-1}(k-l)\prod_{l=1}^{m-i}(k+l). \tag{12.20}$$

Then the function $\overline{L_m}(k)$ satisfies

$$\overline{L_m}(k) = \frac{k}{k-2m}L_m(k) = \sum_{i=0}^{m}\frac{(-1)^i m!(m+i-1)!}{i!(m-i)!(m-i-1)!}\prod_{l=1}^{m-i}(k+1-l)\prod_{l=1}^{m-i}(k+l). \tag{12.21}$$

It remains only to notice that all terms of the sum in the righthand side of (12.21) are invariant under the reflection of the argument w.r.t. $-\frac{1}{2}$ or, equivalently, under substitution

$$k \to -1-k.$$

Then the function $\overline{L_m}(k)$ is also invariant under this substitution, which proves (12.5) and therefore also (7.47).

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