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Symmetries and First Integrals of Differential Equations

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Abstract It is known that an n-dimensional system of ordinary differential equations with Lie symmetry which involves a divergence-free Liouville vector field possesses $n-1$ independent first integrals (i.e., it is algebraically integrable) (Ünal in Phys. Lett. A 260:352–359, 1999). In the present paper, we show that if an n-dimensional system of ordinary differential equations admits a C^∞-symmetry vector field which satisfies some special conditions, then it also possesses $n-1$ independent first integrals. Several examples are given to illustrate our result.

Keywords First integral · Integrability · Symmetry · Lorenz system

1 Introduction

In this paper, we are concerned with finding first integrals of the system of ordinary differential equations

$$\dot{x} = f(t, x), \tag{1}$$

where $x \in \mathbb{R}^n$ and $\dot{x}$ stands for the derivative with respect to the time t.

The existence of first integrals corresponds with some integrability of the system. So far the integrability has become one of most intriguing topics in the study of differential equations [4, 5, 17].

In the literatures, several different definitions of integrability are available. The local integrability is trivial in the sense that there always exist first integrals locally. Therefore, the problem of integrability is generally understood as one of finding global first integrals. For an n degrees of freedom Hamiltonian system H, if there exist n independent first integrals

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$I_1 = H, I_2, \dots, I_n$ in involution, then the system is Liouville integrable, which is the celebrated Arnold-Liouville theorem. A Hamiltonian system with n degrees of freedom is called super-integrable if it possesses $2n-1$ first integrals [5]. But here we are only concerned with the algebraic integrability, which requires the existence of $n-1$ linearly independent first integrals for an n-dimensional system of ordinary differential equations [5, 17].

In general, a Hamiltonian system with n degrees of freedom which admits n-parameter Abelian symmetry group is considered to be completely integrable, for in principle, its solutions can be determined by quadrature alone [12]. However, there is no general conclusion in verifying the integrability of a system of differential equations from the knowledge of a single symmetry vector field. Interestingly enough, G. Ünal [17] showed that the existence of a certain Lie symmetry vector field determines the algebraic integrability.

Although the Lie symmetry group theory provides a powerful tool for analyzing ordinary (and partial) differential equations [12, 13], not every technique can be based on symmetry analysis [2, 3], and this requires generalizations of classical Lie methods. Therefore, Muriel and Romero [9] introduced a new class of symmetry based on a new method of prolonging vector fields known as the C^∞-prolongation, leading to the notion of C^∞-symmetry, that strictly includes Lie symmetry. For applications of C^∞-symmetry, one can see [10, 11].

We can expect that a generalization of the concept of Liouville vector field, based on the new C^∞-prolongation, will generate a new method for obtaining first integrals. In this paper, we establish this generalization and introduce the concept of C^∞-Liouville vector field. Some essential properties of the C^∞-Liouville vector field are presented. We also provide an algorithmic procedure to obtain first integrals of any system of ordinary differential equations that admits a C^∞-Liouville vector field.

This paper is organized as follows. In Sect. 2, we set up some definitions and notations from Lie group theory and review the results in [17] where the author showed that when an n-dimensional system of ordinary differential equations admits a Lie symmetry vector field which involves a divergence-free Liouville vector field, then it possesses $n-1$ independent first integrals. In Sect. 3, the main theorem of this paper is presented. It is proved that if an n-dimensional system of ordinary differential equations admits a C^∞-symmetry vector field which satisfies some special conditions, then it also possesses $n-1$ independent first integrals. We also include several examples to illustrate how this new method works in practice, including the Whittaker's differential equation, the Lorenz system and the Pikovski-Rabinovich-Trakhtengerts system.

Throughout the paper, we will freely use the summation convention for any repeated indices. We will restrict our attention to ordinary differential equations.

2 Notations and Preliminary Results

Let us consider the system of ordinary differential equations

$$\Delta(t, x^{(k)}) = 0, \tag{2}$$

with $(t, x) = (t, x^1, \dots, x^n) \in M$, for some open subset $M \subset \mathbb{R}^1 \times \mathbb{R}^n$. We denote by $M^{(k)}$ the corresponding k-jet space of M. Their elements are $(t, x^{(k)}) = (t, x, x_1, \dots, x_k)$, where, for $i = 1, \dots, k$, x_i denotes the derivative of order i of x with respect to t.

A Lie symmetry of (2) is a vector field $X = \xi(t, x)\frac{\partial}{\partial t} + \eta_\alpha(t, x)\frac{\partial}{\partial x^\alpha}$ that satisfies

$$X^{(k)}(\Delta(t, x^{(k)})) = 0, \quad \text{whenever } \Delta(t, x^{(k)}) = 0,$$

where $X^{(k)}$ denotes the k-th prolongation of X.

Roughly speaking, a Lie symmetry group of (2) is a local group of transformations that transforms solutions of (2) to other solutions of (2).

A generalization of Lie symmetry is based on the concept of a nonlocal exponential symmetry, which first appeared in [12]. The authors in [9] use the nonlocal exponential symmetry to define a new method of prolongation in the following way.

Definition 1 Let $X = \xi(t,x)\frac{\partial}{\partial t} + \eta_\alpha(t,x)\frac{\partial}{\partial x^\alpha}$ be a vector field defined on M, and let $\lambda \in C^\infty(M^{(1)})$ be any function. The λ-prolongation of order k of X, denoted by $X^{[\lambda,(k)]}$, is the vector field defined on $M^{(k)}$ by

$$X^{[\lambda,(k)]} = \xi(t,x)\frac{\partial}{\partial t} + \sum_{i=0}^{k} \eta_\alpha^{[\lambda,(i)]}(t,x^{(i)})\frac{\partial}{\partial x_i^\alpha},$$

where $\eta_\alpha^{[\lambda,(0)]}(t,x) = \eta_\alpha(t,x)$ and

$$\begin{aligned}\eta_\alpha^{[\lambda,(i)]}(t,x^{(i)}) &= D_t(\eta_\alpha^{[\lambda,(i-1)]}(t,x^{(i-1)})) - D_t(\xi(t,x))x_i^\alpha \\ &\quad + \lambda(\eta_\alpha^{[\lambda,(i-1)]}(t,x^{(i-1)}) - \xi(t,x)x_i^\alpha)\end{aligned}$$

for $1 \le i \le k$, where D_t denotes the total derivative operator with respect to t, x_i^α denotes the derivative of order i of x^α with respect to t.

Note that, if $\lambda \in C^\infty(M^{(1)})$ is equal to zero, then the λ-prolongation of order k of X is the usual k-th prolongation of X.

Definition 2 We say that a vector field X is a $C^\infty(M^{(1)})$-symmetry of (2) if there exists a function $\lambda \in C^\infty(M^{(1)})$ such that

$$X^{[\lambda,(k)]}(\Delta(t,x^{(k)})) = 0, \quad \text{whenever } \Delta(t,x^{(k)}) = 0.$$

In this case X is also called a λ-symmetry.

Any vector field $X = \xi(t,x)\frac{\partial}{\partial t} + \eta_\alpha(t,x)\frac{\partial}{\partial x^\alpha}$ has an associated evolutionary representative $X_Q = (\eta_\alpha - \xi x^\alpha)\frac{\partial}{\partial x^\alpha}$, and these vector fields determine essentially the same symmetry. It means that X is a symmetry of a system of differential equations if and only if X_Q is. So, in this paper, we focus our attention on the evolutionary vector field which has a particularly simple form of $X_Q = Q_\alpha[x]\frac{\partial}{\partial x^\alpha}$, where $Q_\alpha[x]$ denotes a function depending on t, x and derivatives of x with respect to t.

Now consider a Lie symmetry X of (1) which has the evolutionary form of $X = \eta_i(t,x)\frac{\partial}{\partial x^i}$. From [17], we know that X satisfies the property $(\frac{\partial}{\partial t} + \mathcal{L}_F)X = 0$, where $\mathcal{L}$ is the Lie derivative [12, 18] and $F = f_i(t,x)\frac{\partial}{\partial x^i}$ is the dynamical vector field of (1).

For the reader's convenience, let us recall the definitions of Liouville vector field and invariant form (see [17] for more details).

Definition 3 A vector field $L = \eta_i(t,x)\frac{\partial}{\partial x^i}$ is called a Liouville vector field of (1), provided

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_F\right)L + (\operatorname{div} F)L = 0.$$

Definition 4 A differential form ω of degree p (or equivalently p-form) is said to be an invariant form of (1) if it satisfies

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_F\right)\omega = 0. \tag{3}$$

Now, we quote a result in [17], which will play an important role in the proof of our main result (Theorem 2) for determining the algebraic integrability of differential equations.

Lemma 1 *Let V be a vector field of the form $V = \mu_i(t, x)\frac{\partial}{\partial x^i}$, and $\mu(x)$ be a scalar function with property $\mathcal{L}_{\mu V}\Omega = 0$. Then we have*

$$\mu V \lrcorner \Omega = dJ_1 \wedge \cdots \wedge dJ_{n-1},$$

where J_i $(i = 1, \ldots, n-1)$ are the independent invariants of the vector field V. Here $\Omega = dx^1 \wedge \cdots \wedge dx^n$ is the volume form, $\wedge$ is the wedge product and $\lrcorner$ is the interior product.

The following theorem is one of the main results of [17], it enables one to obtain first integrals from the Liouville vector field.

Theorem 1 *A divergence free Liouville vector field can be written as*

$$L = \epsilon_{ijk\cdots l} I_{1,j} I_{2,k} \cdots I_{n-1,l} \frac{\partial}{\partial x^i},$$

where $\epsilon_{ijk\cdots l}$ is the Levi-Civita tensor, $I_1, \ldots, I_{n-1}$ are the first integrals of the system given in (1) and $()_{,j} = \partial()/\partial x^j$.

3 The Main Result

In this section, we show how the concept of Liouville vector field can be generalized when C^∞-prolongation is considered. This will generate a new method of obtaining first integrals and determining the algebraic integrability of a system of differential equations. For this aim, we start by noticing an important property of the λ-symmetry.

Lemma 2 *The λ-symmetry X enjoys the property*

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_F + \lambda\right)X = 0. \tag{4}$$

Proof Since $X = \eta_i(t, x)\frac{\partial}{\partial x^i}$ is a λ-symmetry of (1), from the definition of λ-symmetry, it requires that

$$X^{[\lambda,(1)]}(\dot{x}^j - f_j(t, x)) = 0, \tag{5}$$

for $j = 1, \ldots, n$, modulo (1). By the λ-prolongation formula, we have

$$X^{[\lambda,(1)]} = \eta_i(t, x)\frac{\partial}{\partial x^i} + (D_t + \lambda)\eta_i(t, x)\frac{\partial}{\partial \dot{x}^i}. \tag{6}$$

Substituting (6) into (5), we have

$$\frac{\partial \eta_j}{\partial t} + \frac{\partial \eta_j}{\partial x^i} f_i + \lambda \eta_j - \eta_i \frac{\partial f_j}{\partial x^i} = 0, \tag{7}$$

for $j = 1, \ldots, n$, where $\eta_j = \eta_j(t, x)$, $f_j = f_j(t, x)$. Multiplying $\frac{\partial}{\partial x^j}$ to (7) and taking sum over $j = 1, \ldots, n$, we obtain

$$\frac{\partial}{\partial t} \eta_j \frac{\partial}{\partial x^j} + f_i \frac{\partial \eta_j}{\partial x^i} \frac{\partial}{\partial x^j} - \eta_i \frac{\partial f_j}{\partial x^i} \frac{\partial}{\partial x^j} + \lambda \eta_j \frac{\partial}{\partial x^j} = 0. \tag{8}$$

On the other hand, recall that $\mathcal{L}_F X = [F, X]$, where $[\cdot, \cdot]$ is the Lie bracket, i.e.,

$$\begin{aligned} [F, X] &= \left[f_i \frac{\partial}{\partial x^i}, \eta_j \frac{\partial}{\partial x^j} \right] \\ &= f_i \frac{\partial \eta_j}{\partial x^i} \frac{\partial}{\partial x^j} - \eta_j \frac{\partial f_i}{\partial x^j} \frac{\partial}{\partial x^i}. \end{aligned} \tag{9}$$

The desired equality holds by comparing (8) with (9), proving the lemma. □

We now define the C^∞-Liouville vector field which is a generalization of the Liouville vector field based on C^∞-prolongation.

Definition 5 A vector field $L = \eta_i(t, x) \frac{\partial}{\partial x^i}$ is called a C^∞-Liouville vector field of (1), if there exists a $\lambda \in C^\infty(t, x, \dot{x})$, such that

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_F + \lambda \right) L + (\operatorname{div} F) L = 0. \tag{10}$$

We will also say that L is a λ-Liouville vector field.

It is well known that the total mass of an object is unchanged in any case whenever it is experiencing mechanical, physical or chemical motion, which is the celebrated "Law of conservation of mass". According to the "Law of conservation of mass", we can deduce the continuity equation $\frac{\partial p}{\partial t} + \operatorname{div}(pF) = 0$, or, equivalently, $\frac{dp}{dt} + (\operatorname{div} F)p = 0$. According to the continuity equation, the phase density p satisfies the following partial differential equation

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_F \right) p + (\operatorname{div} F) p = 0. \tag{11}$$

The following lemma states a remarkable property of the λ-Liouville vector field L, which shows the relationship between a λ-Liouville vector field and a λ-symmetry.

Lemma 3 *If L is a λ-Liouville vector field, then the vector field*

$$X = \frac{1}{p} L \tag{12}$$

is a λ-symmetry.

Proof We only need to show that vector field (12) enjoys the property (4), i.e.,

$$\left(\frac{\partial}{\partial t}+\mathcal{L}_F+\lambda\right)\left(\frac{1}{p}L\right)=0. \tag{13}$$

From (11) we obtain

$$\left(\frac{\partial}{\partial t}+\mathcal{L}_F\right)\frac{1}{p}=\frac{\mathrm{div}F}{p}. \tag{14}$$

By (10) we also have

$$\left(\frac{\partial}{\partial t}+\mathcal{L}_F\right)L=-(\mathrm{div}F+\lambda)L. \tag{15}$$

Now the result follows by substituting (14) and (15) into (13). □

It is a standard result that the analytical solutions to the following first order partial differential equation

$$\left(\frac{\partial}{\partial t}+\mathcal{L}_F\right)\gamma=\lambda\gamma \tag{16}$$

exist. The reader is referred to [20]. Hence, if γ is a solution of (16), we have

Lemma 4 *If L is a λ-Liouville vector field, then the $n-1$ form*

$$\omega=\gamma L \lrcorner \Omega$$

is an invariant $n-1$ form.

Proof It suffices to show that the $n-1$ form ω enjoys the property (3), i.e.,

$$\left(\frac{\partial}{\partial t}+\mathcal{L}_F\right)(\gamma L\lrcorner\Omega)=0.$$

Using the property of Lie derivative, we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\mathcal{L}_F\right)(\gamma L\lrcorner\Omega)&=\left(\frac{\partial}{\partial t}+\mathcal{L}_F\right)(\gamma)L\lrcorner\Omega+\gamma\left(\frac{\partial}{\partial t}+\mathcal{L}_F\right)(L\lrcorner\Omega)\\&=\lambda\gamma(L\lrcorner\Omega)+\gamma\left(\frac{\partial L}{\partial t}\lrcorner\Omega+(\mathcal{L}_FL)\lrcorner\Omega+L\lrcorner(\mathcal{L}_F\Omega)\right)\\&=\gamma\left(\frac{\partial L}{\partial t}+\mathcal{L}_FL+\lambda L+\mathrm{div}F\cdot L\right)\lrcorner\Omega.\end{aligned}$$

The second and the last equalities hold due to the Cartan's identity and $\mathcal{L}_F\Omega=(\mathrm{div}F)\Omega$, respectively (see [18] for more details). Since a λ-Liouville vector field satisfies (10), the term in parenthesis vanishes. This concludes the result. □

In this way, we have managed to construct a generalization based on C^∞-prolongation. We will prove that C^∞-Liouville vector field generates a new method for obtaining first integrals of a system of ordinary differential equations, as spelled out in the following theorem:

Theorem 2 *Let L be a λ-Liouville vector field and γ be a solution of* (16). *If γL is divergence free, then we can obtain $n-1$ independent first integrals of* (1) *from the $n-1$ independent invariants of the vector field L.*

Proof From Lemma 1, there exists a scalar function $\mu(x)$ with property $\mathcal{L}_{\mu\gamma L}\Omega = 0$ such that

$$\mu\gamma L \lrcorner \Omega = dJ_1 \wedge \cdots \wedge dJ_{n-1},$$

where $J_1, \ldots, J_{n-1}$ are independent invariants of L. Since γL is divergence free, $\mathcal{L}_{\gamma L}\Omega = \mathrm{div}(\gamma L)\Omega = 0$. Hence, we can take $\mu = 1$ without loss of generality. Due to Lemma 4, we have

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_F\right)(dJ_1 \wedge \cdots \wedge dJ_{n-1}) = 0,$$

that is,

$$d\left(\frac{\partial J_1}{\partial t} + \mathcal{L}_F J_1\right) \wedge \cdots \wedge dJ_{n-1} + \cdots + dJ_1 \wedge \cdots \wedge d\left(\frac{\partial J_{n-1}}{\partial t} + \mathcal{L}_F J_{n-1}\right) = 0. \quad (17)$$

Since L is a vector field of evolutionary form, there exists a factor $\Lambda_i(t)$ in invariant J_i, $i = 1, \ldots, n-1$. In order to make (17) hold, we aim at finding $J_1, \ldots, J_{n-1}$ such that

$$\frac{\partial J_i}{\partial t} + \mathcal{L}_F J_i = 0, \quad (18)$$

for $i = 1, \ldots, n-1$. Substitute $J_1, \ldots, J_{n-1}$ into (18), we obtain $n-1$ ordinary differential equations for $\Lambda_i(t)$. Solving these ordinary differential equations and substituting the solutions back into $J_1, \ldots, J_{n-1}$, we obtain $n-1$ first integrals of (1) from invariants of L. Since $J_1, \ldots, J_{n-1}$ are independent invariants, we obtain $n-1$ independent first integrals of (1). This completes the proof. □

Remark 1 Actually, we can check that γX is a Lie symmetry if X is a λ-symmetry, and γL is a Liouville vector field if L is a λ-Liouville vector field, provided γ satisfies (16). This is because that we can obtain a solution of (16) given by $\gamma = e^{\int \lambda dt}$, and γX becomes a nonlocal exponential symmetry, which is the original idea of λ-symmetry.

In fact, if γ satisfies (16), then we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathcal{L}_F\right)(\gamma X) &= \left(\frac{\partial}{\partial t} + \mathcal{L}_F\right)(\gamma)X + \gamma\left(\frac{\partial}{\partial t} + \mathcal{L}_F\right)X \\ &= \lambda\gamma X + \gamma(-\lambda X) = 0,\end{aligned}$$

and

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \mathcal{L}_F\right)(\gamma L) + (\mathrm{div}F)(\gamma L) \\ &= \left(\frac{\partial}{\partial t} + \mathcal{L}_F\right)(\gamma)L + \gamma\left(\frac{\partial}{\partial t} + \mathcal{L}_F\right)L + (\mathrm{div}F)(\gamma L) \\ &= \lambda\gamma L + \gamma(-\lambda - \mathrm{div}F)L + (\mathrm{div}F)(\gamma L) = 0.\end{aligned}$$

Remark 2 In general, we can not obtain the algebraic integrability for a system of ordinary differential equations (1) from Theorem 2, because the system is nonautonomous, we need one more first integral to ensure the integrability. However, for a system of autonomous ordinary differential equations, we have the algebraic integrability, provided the conditions of Theorem 2 are satisfied.

4 Applications

4.1 Whittaker's Differential Equation

The Whittaker's differential equation is closely related to the canonical form of the confluent hypergeometric differential equation [15, 19]. Before discussing its general forms, we introduce the Kummer's equation

$$t\frac{d^2y}{dt^2}+(b-t)\frac{dy}{dt}-ay=0.$$

This equation is the confluent hypergeometric equation, and any solution of this equation is a confluent hypergeometric function.

In Kummer's equation, let us put

$$y=t^{-\frac{1}{2}b}\mathrm{e}^{\frac{1}{2}t}W,$$

the equation then assumes the normalized form

$$\frac{d^2W}{dt^2}+\left(-\frac{1}{4}+\frac{k}{t}+\frac{\frac{1}{4}-m^2}{t^2}\right)W=0,$$

where $m=\frac{1}{2}b-\frac{1}{2}$ and $k=\frac{1}{2}b-a$. This equation was studied first by Whittaker and was then called the Whittaker's differential equation. One of its solutions is the Whittaker's function $W_{k,m}(t)$, which is a well known function. It has been shown that many functions employed in Applied Mathematics are expressible by means of the function $W_{k,m}(t)$.

Consider the differential equation satisfied by $x=t^{-\frac{1}{2}}W_{-\frac{3}{4},-\frac{1}{4}}(t^2)$, that is,

$$\frac{d}{2tdt}\left(\frac{d(xt^{\frac{1}{2}})}{2tdt}\right)+\left(-\frac{1}{4}+\frac{-\frac{3}{4}}{t^2}+\frac{\frac{3}{16}}{t^4}\right)xt^{\frac{1}{2}}=0.$$

This reduces to the following second order differential equation

$$\frac{d^2x}{dt^2}+(-3-t^2)x=0. \tag{19}$$

It can be checked (see Appendix 1) that this equation has no Lie symmetries which can be used to obtain first integrals according to Theorem 1. Now we aim at finding λ-symmetries of (19) which can be used in Theorem 2. We are going to transform (19) into the following system of differential equations

$$\dot{x}=y, \tag{20a}$$

$$\dot{y}=(3+t^2)x. \tag{20b}$$

It is straightforward to check that the vector field $X = t\frac{\partial}{\partial x} + (1+t^2)\frac{\partial}{\partial y}$ is a C^∞-symmetry, for $\lambda = t$, of (20). Since the divergence of the vector field (20) is equal to zero, the phase density p is constant. Without loss of generality, take $p = 1$. Solving (16) for this system, we obtain $\gamma = e^{\frac{t^2}{2}}$. Since

$$\mathrm{div}(\gamma L) = \mathrm{div}(\gamma X) = 0,$$

we can appeal to Theorem 2 to obtain first integrals. An invariant of the vector field L (i.e., X) is

$$J = \Lambda(t)((1+t^2)x - ty). \tag{21}$$

Substituting (21) into (18) yields the following ordinary differential equation

$$\dot{\Lambda}(t) - t\Lambda(t) = 0.$$

Solving this equation and then substituting the solution back into (21) lead to a first integral

$$I = e^{\frac{t^2}{2}}((1+t^2)x - ty)$$

of the system (20).

Consider the differential equation satisfied by $x = t^{-\frac{1}{2}} W_{\frac{1}{4},-\frac{1}{4}}(t^2)$, that is,

$$\ddot{x} + (1-t^2)x = 0. \tag{22}$$

It can also be checked (see Appendix 2) that this equation has no Lie symmetries which can be used to obtain first integrals according to Theorem 1. We transform this equation into the following form

$$\dot{x} = y, \tag{23a}$$

$$\dot{y} = -(1-t^2)x. \tag{23b}$$

It can be calculated that the vector field $X = \frac{\partial}{\partial x} - t\frac{\partial}{\partial y}$ is a C^∞-symmetry, for $\lambda = -t$, of (23). Since $\mathrm{div}F = \frac{\partial y}{\partial x} + \frac{\partial(-(1-t^2)x)}{\partial y} = 0$, the phase density $p =$ constant. We take $p = 1$ without loss of generality. It can also be checked that $\gamma = e^{-\frac{t^2}{2}}$ satisfies (16) and

$$\mathrm{div}(\gamma L) = \mathrm{div}(\gamma X) = 0.$$

Then we can appeal to Theorem 2 to obtain first integrals. An invariant of the vector field L (i.e., X) is $J = \Lambda(t)(tx + y)$. Substituting J into (18) yields $\dot{\Lambda}(t) + t\Lambda(t) = 0$. Hence we obtain a first integral

$$I = e^{-\frac{t^2}{2}}(tx + y).$$

Remark 3 Considering the ordinary differential equations satisfied by $x = t^{-\frac{1}{2}} W_{-\frac{3}{4},-\frac{1}{4}}(ct^2)$ and $x = t^{-\frac{1}{2}} W_{-\frac{1}{4},-\frac{1}{4}}(ct^2)$, respectively, we obtain $\ddot{x} + (-3c - c^2t^2)x = 0$ and $\ddot{x} + (-c - c^2t^2)x = 0$ respectively. For these two classes of ordinary differential equation, we can also obtain first integrals $I = e^{\frac{ct^2}{2}}((1+ct^2)x - ty)$ and $I = e^{\frac{ct^2}{2}}(ctx - y)$, respectively, according to Theorem 2.

Remark 4 Actually, one can obtain Whittaker's functions from first integrals. For example, we consider the first integral $I = e^{-\frac{t^2}{2}}(tx + y)$ of (23). Let $I = C$ and replace y by $\dot{x}$. We obtain a first order ordinary differential equation $\dot{x} + tx = Ce^{\frac{t^2}{2}}$. Solving this equation yields a solution $x = -Ce^{-\frac{t^2}{2}}\int_t^\infty e^{s^2}ds$. Without loss of generality, take $C = -1$. Then $x = t^{-\frac{1}{2}}W_{\frac{1}{4},-\frac{1}{4}}(t^2)$, where $W_{\frac{1}{4},-\frac{1}{4}}(t^2) = t^{\frac{1}{2}}e^{-\frac{t^2}{2}}\int_t^\infty e^{s^2}ds$ is the Whittaker's function.

4.2 The Lorenz System

The Lorenz system [8] (see also [16]) of ordinary differential equations that have been used to model the dynamic movement of an atmospheric fluid is

$$\begin{aligned}\dot{x} &= \sigma(y - x),\\ \dot{y} &= \rho x - y - xz,\\ \dot{z} &= xy - \beta z,\end{aligned}$$

where x is proportional to the intensity of convection motion, y is proportional to the temperature difference between ascending and descending currents, z is proportional to the distortion (from linearity) of the vertical temperature profile and σ, β are positive constants, while the constant ρ depends, among other constants on the temperature difference (for more details, see [7]).

Given $\sigma = \frac{1}{2}$, $\beta = 1$ and $\rho = 0$, the Lorenz system is simplified to

$$\begin{aligned}\dot{x} &= \frac{1}{2}(y - x),\\ \dot{y} &= -y - xz,\\ \dot{z} &= xy - z.\end{aligned}$$

It can be calculated that this system admits a C^∞-symmetry

$$X = \frac{1}{2}y\frac{\partial}{\partial x} - xz\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z},$$

for $\lambda = \frac{1}{2}$. Since the divergence of the vector field F is equal to $-\frac{5}{2}$ and from the continuity equation, the phase density can be written as $p = e^{\frac{5}{2}t}$. Lemma 3 enables us to find a λ-Liouville vector field as

$$L = e^{\frac{5}{2}t}\left(\frac{1}{2}y\frac{\partial}{\partial x} - xz\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}\right).$$

We check that $\gamma = e^{\frac{t}{2}}$ is a solution of (16) and

$$\operatorname{div}(\gamma L) = e^{3t}\left(\frac{\partial(\frac{1}{2}y)}{\partial x} - \frac{\partial(xz)}{\partial y} + \frac{\partial(xy)}{\partial z}\right) = 0.$$

Now we can apply Theorem 2 to obtain first integrals of the Lorenz system. Two independent invariants of the vector field L are

$$J_1 = \Lambda_1(t)(y^2 + z^2) \quad \text{and} \quad J_2 = \Lambda_2(t)(x^2 - z).$$

Substituting them into (18) yields the following ordinary differential equations

$$\dot{\Lambda}_1(t) - 2\Lambda_1(t) = 0 \quad \text{and} \quad \dot{\Lambda}_2(t) - \Lambda_2(t) = 0.$$

Solving these two equations, we obtain two first integrals

$$I_1 = e^{2t}(y^2 + z^2) \quad \text{and} \quad I_2 = e^t(x^2 - z)$$

of the Lorenz system.

4.3 The PRT System

The Pikovski-Rabinovich-Trakhtengerts system of plasma dynamics which is of the following form

$$\begin{aligned} \dot{x} &= \beta y - \nu_1 x - yz, \\ \dot{y} &= \beta x - \nu_2 y + xz, \\ \dot{z} &= -\nu_3 z + xy, \end{aligned}$$

is called the PRT system. It was introduced in [14] (see also [6]) in studies related to the problem of the interaction of three resonantly coupled waves in a plasma. The PRT system describes a dynamics of the interaction of the wave in a plasma propagating parallel to the magnetic field with the ion acoustic wave and the plasma oscillation near the lower hybrid resonance. As comparing with the famous Lorenz system, the PRT system resembles many of its properties and it was stated in the book [1] that both the PRT system and the Lorenz system can be considered as particular cases of the so-called Generalized Lorenz system.

If we take $\beta = 0$ and $\nu_1 = \nu_2 = \nu_3 = \nu$, the PRT system is simplified to

$$\begin{aligned} \dot{x} &= -\nu x - yz, \\ \dot{y} &= -\nu y + xz, \\ \dot{z} &= -\nu z + xy. \end{aligned}$$

It can be calculated that this system admits a C^∞-symmetry

$$X = -yz\frac{\partial}{\partial x} + xz\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z},$$

for $\lambda = \nu$. Since the divergence of the vector field F is equal to -3ν, the phase density can be written as $p = e^{3\nu t}$. We can also check that $\gamma = e^{\nu t}$ satisfies (16), then

$$\operatorname{div}(\gamma L) = e^{4\nu t}\left(\frac{\partial(-yz)}{\partial x} + \frac{\partial(xz)}{\partial y} + \frac{\partial(xy)}{\partial z}\right) = 0.$$

Now we can apply Theorem 1 to obtain first integrals of this simplified PRT system. Two independent invariants of the vector field L are

$$J_1 = \Lambda_1(t)(x^2 + y^2) \quad \text{and} \quad J_2 = \Lambda_2(t)(x^2 + z^2).$$

Substituting them into (18) yields two first integrals

$$I_1 = e^{2\nu t}(x^2 + y^2) \quad \text{and} \quad I_2 = e^{2\nu t}(x^2 + z^2)$$

of this simplified PRT system.

Remark 5 Interestingly enough, if we denote by $I = -I_1 + 2I_2 = e^{2\nu t}(x^2 - y^2 + 2z^2)$, we obtain a first integral of the original PRT system (when $\nu_1 = \nu_2 = \nu_3 = \nu$).

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Appendix 1

If a vector field $X = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x}$ is a Lie symmetry of (19), we need to know the second prolongation

$$X^{(2)} = X + (D_t\eta - D_t\xi\dot{x})\frac{\partial}{\partial \dot{x}} + (D_t(D_t\eta - D_t\xi\dot{x}) - D_t\xi\ddot{x})\frac{\partial}{\partial \ddot{x}}$$

of X. Applying $X^{(2)}$ to (19) and replacing $\ddot{x}$ by $(3+t^2)x$ whenever it occurs, and equating the coefficients of the various monomials in the first and second order partial derivatives of x, we find the determining equations to be the following:

$$\begin{aligned}
&\frac{\partial^2\xi}{\partial x^2} = 0,\\
&\frac{\partial^2\eta}{\partial x^2} - 2\frac{\partial^2\xi}{\partial t\partial x} = 0,\\
&2\frac{\partial^2\eta}{\partial t\partial x} - \frac{\partial^2\xi}{\partial t^2} - 3(3+t^2)x\frac{\partial\xi}{\partial x} = 0,\\
&\frac{\partial^2\eta}{\partial t^2} + (3+t^2)x\frac{\partial\eta}{\partial x} - 2(3+t^2)x\frac{\partial\xi}{\partial t} = 2tx\xi + (3+t^2)\eta.
\end{aligned}$$

Since Theorem 1 only applies in the case that the vector field is of evolutionary form, the infinitesimal $\xi(t,x)$ is equal to zero. Then we have

$$\frac{\partial^2\eta}{\partial x^2} = 0, \qquad \frac{\partial^2\eta}{\partial t\partial x} = 0, \qquad \frac{\partial^2\eta}{\partial t^2} + (3+t^2)x\frac{\partial\eta}{\partial x} = (3+t^2)\eta. \tag{24}$$

The first two equations of (24) yields to

$$\eta(t,x) = Cx + D(t), \tag{25}$$

where C is a constant and $D(t)$ is a function of t. Substituting (25) back into the third equation of (24), we have

$$\ddot{D}(t) = (3+t^2)D(t),$$

which comes back to (19). If we take the trivial solution $D(t) = 0$ of it, then $\eta(t,x) = Cx$. Without loss of generality we take $C = 1$ and $\eta(t,x) = x$. Now we obtain a Lie symmetry $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ of (20). However,

$$\mathrm{div}L = \mathrm{div}X = 2 \neq 0.$$

Therefore, we can not apply Theorem 1 to obtain first integrals.

Appendix 2

Similar to Appendix 1, here we only give the brief explanation. If a vector field X is a Lie symmetry of (22), we have

$$\frac{\partial^2 \eta}{\partial x^2} = 0, \qquad \frac{\partial^2 \eta}{\partial t \partial x} = 0, \qquad \frac{\partial^2 \eta}{\partial t^2} - (1 - t^2) x \frac{\partial \eta}{\partial x} = (1 - t^2)\eta.$$

These equalities lead to $\eta(t, x) = D(t)$ and $D(t)$ satisfies (22), which comes back to the origin.

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