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COHOMOLOGIES OF THE LIE ALGEBRA OF VECTOR FIELDS ON A LINE

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Providing adequate mathematical tools, we find cohomologies of the Lie algebra of smooth vector fields on a line with coefficients in the trivial, natural, and adjoint representations. We construct the generalized series of complexes and calculate the corresponding cohomologies.

The recent development of theoretical and mathematical physics is intrinsically related to geometric and algebraic formulations. An unsophisticated observer would say that contemporary mathematical physics is a mixture of special branches of geometry and algebra spiced with analysis and the theory of partial differential equations. Lie algebras play a central role in this pattern because the whole historical development of classical and quantum physics is intrinsically related to symmetries, which are mathematically described by Lie groups and algebras. Currently, finite-dimensional Lie algebras related to symmetries in isotopic and physical spaces are being replaced as the main object of investigation by infinite-dimensional algebras, which are intrinsic to modern quantum theories including string and gauge theories. Mathematical constructions supply physics with their specific objects and methods of investigation. This provides a common base for describing otherwise unrelated physical phenomena. One of the many examples is the classification of elementary particles based on Lie group representations.

A convenient language for describing a number of objects related to Lie algebras and their representations is provided by the method of cohomologies of Lie algebras [1]. This language yields adequate tools for investigating, classifying, and finding natural relations between the physical quantities under investigation. This method becomes especially useful in an infinite-dimensional case [2], [3], where standard methods fail (see [4] and the references therein regarding the modern trends in using the cohomologies of Lie algebras in physics). The growing number of publications involving Lie algebra cohomologies in electronic archives and physical journals indicates that their applications in theoretical and mathematical physics are increasing.

Constituents of the theory of Lie algebra cohomologies important for applications are the calculation apparatus and the assortment of completely calculated typical examples (see [2], [3] and references to the original papers therein for the known results). One of the most important classes of infinite-dimensional Lie algebras are Lie algebras of vector fields on manifolds, of which the Lie algebras on the unit circle and on the line are the simplest examples. The former algebras, being algebras of fields on a compact manifold, have been more thoroughly studied because the invariant integration technique [5] is available in this case (see [2], [3] and the references therein). The most complete results were obtained with coefficients in the trivial representation.

In this paper, we calculate the cohomologies of the Lie algebra of smooth vector fields on the line with coefficients in the three most important representations: the trivial, the natural (fundamental), and the adjoint representations. As a by-product, we develop an adequate mathematical apparatus, which reduces calculations to elementary algebra and admits generalizations to more involved cases. We basically give algebraic, combinatorial arguments, which are specific to the problem under investigation, leaving aside the

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analytic part, which is standard and does not contain new knowledge.

1. Definitions and formulation of results

1.1. Cohomologies of Lie algebras. Let $\mathbb{F}$ be a field of characteristic zero, let $\mathfrak{A}$ be the Lie algebra over $\mathbb{F}$ with the bracket $[\cdot,\cdot]$, let $\mathfrak{K}$ be a module over $\mathfrak{A}$, i.e., $\mathfrak{K}$ is the linear space over $\mathbb{F}$, and let a representation ρ of the Lie algebra $\mathfrak{A}$ in $\mathfrak{K}$ be given:

$$\rho\colon \mathfrak{A} \to \mathrm{Hom}_{\mathbb{F}}(\mathfrak{K};\mathfrak{K}), \qquad a \mapsto \rho(a)\colon \mathfrak{K} \to \mathfrak{K} \quad \forall a \in \mathfrak{A}.$$

This representation must obey the commutation rule

$$\rho([a,b]) = [\rho(a),\rho(b)] \equiv \rho(a)\circ\rho(b) - \rho(b)\circ\rho(a) \quad \forall a,b \in \mathfrak{A},$$

where the symbol $\circ$ denotes the composition of mappings.

Let

$$\wedge\mathfrak{A} = \bigoplus_{q\geq 0} \wedge^q\mathfrak{A}$$

be an external algebra of the linear space $\mathfrak{A}$ over $\mathbb{F}$, and let

$$\mathcal{C}(\mathfrak{A};\mathfrak{K}) = \bigoplus_{q\geq 0} \mathcal{C}^q(\mathfrak{A};\mathfrak{K})$$

be a linear space over $\mathbb{F}$ of all cochains on $\mathfrak{A}$ with values in $\mathfrak{K}$, where $\mathcal{C}^q(\mathfrak{A};\mathfrak{K}) = \mathrm{Hom}_{\mathbb{F}}(\wedge^q\mathfrak{A};\mathfrak{K})$ are linear spaces of all q-linear skew-symmetric mappings from $\mathfrak{A}$ to $\mathfrak{K}$. An external differential d on $\mathcal{C}(\mathfrak{A};\mathfrak{K})$ is determined by the rule

$$(d\omega)(a_0,\dots,a_q) = \frac{1}{q+1}\Bigg\{\sum_{0\leq\alpha\leq q} (-1)^\alpha \rho(a_\alpha)(\omega(a_0,\dots,\check{a}_\alpha,\dots,a_q)) + \\ + \sum_{0\leq\alpha<\beta\leq q} (-1)^{\alpha+\beta}\omega([a_\alpha,a_\beta],\dots,\check{a}_\alpha,\dots,\check{a}_\beta,\dots,a_q)\Bigg\}$$

for $\omega \in \mathcal{C}^q(\mathfrak{A};\mathfrak{K})$ and $a_0,\dots,a_q \in \mathfrak{A}$ (the argument under the haček is omitted); hence, $d\colon \mathcal{C}^q(\mathfrak{A};\mathfrak{K}) \to \mathcal{C}^{q+1}(\mathfrak{A};\mathfrak{K})$, $q \in \mathbb{Z}_+ = \{0,1,\dots\}$. Because of the Jacobi identity and the commutation rule, the composition $d\circ d = 0$ and the complex $\{\mathcal{C}(\mathfrak{A};\mathfrak{K}); d\}$ with the cohomologies

$$H(\{\mathcal{C}(\mathfrak{A};\mathfrak{K}); d\}) = H(\mathfrak{A};\mathfrak{K}) = \bigoplus_{q\geq 0} H^q(\mathfrak{A};\mathfrak{K})$$

are well defined.

1.2. Lie algebra of smooth vector fields on the line. Let $\mathbb{F} = \mathbb{C}$, let $\mathcal{E}$ be an associative commutative algebra over $\mathbb{C}$ of all smooth (i.e., having continuous derivatives of all orders) complex-valued functions on the line $\mathbb{R}$ with the standard pointwise operations, and let $\mathfrak{A}$ be the Lie algebra of all smooth vector fields (i.e., all smooth differentiations of the algebra $\mathcal{E}$) on the line. In detail,

$$\mathfrak{A} = \left\{u = u(x)\frac{d}{dx};\ u(x) \in \mathcal{E},\ x \in \mathbb{R}\right\},$$

the Lie bracket is $[u, v] = w$, where $w = w(x)d/dx$, and

$$w(x) = u(x)v'(x) - v(x)u'(x) = W_2[u, v](x)$$

is the Wronskian of two functions $u(x), v(x) \in \mathcal{E}$ calculated at the point $x \in \mathbb{R}$ (the prime denotes the derivative w.r.t. $x \in \mathbb{R}$). In particular, supplying the linear space $\mathcal{E}$ with the Lie bracket defined by the Wronskian, we can identify the Lie algebras $\mathfrak{A}$ and $\mathcal{E}$.

The most important representations of the Lie algebra $\mathfrak{A}$ are

a. the trivial representation in which $\mathfrak{K} = \mathbb{C}$ and $\rho(u) = 0$, $u \in \mathfrak{A}$,

b. the natural representation in which $\mathfrak{K} = \mathcal{E}$ and $\rho(u) = u(x)d/dx$, $u \in \mathfrak{A}$, and

c. the adjoint representation in which $\mathfrak{K} = \mathfrak{A}$ and $\rho(u) = [u, \cdot]$, $u \in \mathfrak{A}$.

We note that any Lie algebra has the trivial and adjoint representations, while the natural representation is intrinsic for any Lie algebra of differentiations of an associative algebra.

1.3. Cohomologies: Formulating the results. In the case of cochains with coefficients in the trivial representation, the external differential acts as

$$(d\omega)(u_0, \dots, u_q) = \frac{1}{q+1} \sum_{0 \le \alpha < \beta \le q} (-1)^{\alpha+\beta} \omega([u_\alpha, u_\beta], u_0, \dots, \check{u}_\alpha, \dots, \check{u}_\beta, \dots, u_q)$$

for all $\omega \in \mathcal{C}^q(\mathfrak{A}; \mathbb{C})$, $u_0, \dots, u_q \in \mathfrak{A}$, and $q \in \mathbb{Z}_+$.

Theorem 1. *The cohomologies of the Lie algebra $\mathfrak{A}$ of smooth vector fields on the line with coefficients in the trivial representation, i.e., the cohomologies of the complex $\{\mathcal{C}(\mathfrak{A}; \mathbb{C}); d\}$, are*

$$H^q(\mathfrak{A}; \mathbb{C}) = \begin{cases} \mathbb{C}, & q = 0, \\ \mathbb{C} \cdot W_3(0), & q = 3, \\ 0, & q \neq 0, 3, \end{cases}$$

where $W_3[u, v, w](0)$ is the Wronskian of three functions $u, v, w \in \mathcal{E}$ calculated at the point $0 \in \mathbb{R}$.

In the case of cochains with coefficients in the natural representation, the external differential acts as

$$(d\omega)(x; u_0, \dots, u_q) = \frac{1}{q+1} \left\{ \sum_{0 \le \alpha \le q} (-1)^\alpha u_\alpha(x) (\omega(x; u_0, \dots, \check{u}_\alpha, \dots, u_q))'_x + \right.$$

$$\left. + \sum_{0 \le \alpha < \beta \le q} (-1)^{\alpha+\beta} \omega(x; [u_\alpha, u_\beta], u_0, \dots, \check{u}_\alpha, \dots, \check{u}_\beta, \dots, u_q) \right\}$$

for all $\omega \in \mathcal{C}^q(\mathfrak{A}; \mathcal{E})$, $u_0, \dots, u_q \in \mathfrak{A}$, and $q \in \mathbb{Z}_+$.

Theorem 2. *The cohomologies of the Lie algebra $\mathfrak{A}$ of smooth vector fields on the line with coefficients in the natural representation, i.e., the cohomologies of the complex $\{\mathcal{C}(\mathfrak{A}; \mathcal{E}); d\}$, are*

$$H^q(\mathfrak{A}; \mathcal{E}) = \begin{cases} \mathbb{C}, & q = 0, \\ \mathbb{C} \cdot \dfrac{d}{dx}, & q = 1, \\ 0, & q \ge 2. \end{cases}$$

In the case of cochains with coefficients in the adjoint representation, the external differential acts as

$$(d\omega)(x; u_0, \dots, u_q) = \frac{1}{q+1} \left\{ \sum_{0 \le \alpha \le q} (-1)^\alpha W_2[u_\alpha, \omega(u_0, \dots, \check{u}_\alpha, \dots, u_q)](x) + \right.$$

$$\left. + \sum_{0 \le \alpha < \beta \le q} (-1)^{\alpha+\beta} \omega(x; [u_\alpha, u_\beta], u_0, \dots, \check{u}_\alpha, \dots, \check{u}_\beta, \dots, u_q) \right\}$$

for all $\omega \in \mathcal{C}^q(\mathfrak{A}; \mathfrak{A})$, $u_0, \dots, u_q \in \mathfrak{A}$, and $q \in \mathbb{Z}_+$.

Theorem 3. *The cohomologies of the Lie algebra $\mathfrak{A}$ of smooth vector fields on the line with coefficients in the adjoint representation, i.e., the cohomologies of the complex $\{\mathcal{C}(\mathfrak{A};\mathfrak{A});d\}$, are trivial,*

$$H^q(\mathfrak{A};\mathfrak{A}) = 0, \quad q \in \mathbb{Z}_+.$$

The remainder of the paper describes the adequate mathematical apparatus and contains the proofs of the above theorems.

2. The mathematical apparatus

2.1. The exponential transformation. We establish a correspondence between $\xi \in \mathbb{C}$ and the functions $e^\xi \in \mathcal{E}$, where $e^\xi(x) = e^{x\xi}$, $x \in \mathbb{R}$, while $[e^\xi, e^\eta] = (\eta - \xi)e^{\xi+\eta}$ for all $\xi, \eta \in \mathbb{C}$.

We now briefly describe several definitions and results in distribution theory needed in what follows. Let $\mathcal{E}(\mathbb{R}^q)$ be a linear locally convex space of all smooth functions on $\mathbb{R}^q$ with the topology of uniform convergence on compact sets in $\mathbb{R}^q$ together with partial derivatives of any given order. Its conjugate space $\mathcal{E}'(\mathbb{R}^q)$ (in the strong sense) is the linear locally convex space of all distributions with a compact support in $\mathbb{R}^q$ with the topology of uniform convergence on bounded subsets of $\mathcal{E}(\mathbb{R}^q)$. Let $\mathrm{Exp}(\mathbb{C}^q)$ be the linear space of all entire functions of the exponential type on $\mathbb{C}^q$. The Laplace transformation $L\colon \mathcal{E}'(\mathbb{R}^q) \to \mathrm{Exp}(\mathbb{C}^q)$, $\omega \mapsto L[\omega]$, is $L[\omega](\xi_1,\dots,\xi_q) = \omega(e^{\xi_1},\dots,e^{\xi_q})$ for all $\omega \in \mathcal{E}'(\mathbb{R}^q)$ and $\xi = (\xi_1,\dots,\xi_q) \in \mathbb{C}^q$. The linear mapping L is injective; its image $L[\mathcal{E}'(\mathbb{R}^q)] \subset \mathrm{Exp}(\mathbb{C}^q)$ comprises all entire functions on $\mathbb{C}^q$ that grow not faster than an exponential of a linear function in real directions and not faster than a powerlike function in imaginary directions. The linear space $L[\mathcal{E}'(\mathbb{R}^q)]$ is endowed with the locally convex topology induced from $\mathcal{E}'(\mathbb{R}^q)$ such that we can define the isomorphism of linear locally convex spaces $L\colon \mathcal{E}'(\mathbb{R}^q) \simeq L[\mathcal{E}'(\mathbb{R}^q)]$ (see, e.g., [6]).

By construction, the linear space of q-cochains $\mathcal{C}^q(\mathfrak{A};\mathbb{C})$ is the subspace in $\mathcal{E}'(\mathbb{R}^q)$ that consists of all skew-symmetric distributions with a compact support in $\mathbb{R}^q$, i.e.,

$$\mathcal{C}^q(\mathfrak{A};\mathbb{C}) = \{\omega \in \mathcal{E}'(\mathbb{R}^q)\colon \pi^*\omega = \mathrm{sign}(\pi)\omega \ \forall \pi \in \Sigma_q\},$$

where Σ_q is the set of all permutations of the indices $\{1,\dots,q\}$, $\mathrm{sign}(\pi)$ is the signature of a permutation $\pi \in \Sigma_q$, $(\pi^*\omega)(u_1,\dots,u_q) = \omega(u_{\pi(1)},\dots,u_{\pi(q)})$. Endowing the image $\Phi^q = L[\mathcal{C}^q(\mathfrak{A};\mathbb{C})] \subset L[\mathcal{E}'(\mathbb{R}^q)]$ with the locally convex topology induced from $L[\mathcal{E}'(\mathbb{R}^q)]$, we obtain the isomorphism of linear spaces $L\colon \mathcal{C}^q(\mathfrak{A};\mathbb{C}) \simeq \Phi^q$. We note that $\mathcal{C}^0(\mathfrak{A};\mathbb{C}) = \Phi^0 = \mathbb{C}$. For $q = 1,2,\dots$, the linear locally convex space Φ^q consists of all skew-symmetric entire functions on $\mathbb{C}^q$ that grow not faster than an exponential of a linear function in real directions and not faster than a powerlike function in imaginary directions and is endowed with the topology induced from $\mathcal{E}'(\mathbb{R}^q)$. Therefore, the isomorphism $L\colon \mathcal{C}(\mathfrak{A};\mathbb{C}) \simeq \Phi$, where $\Phi = \bigoplus_{q\ge 0}\Phi^q$, is well defined.

Further, the linear space of q-cochains $\mathcal{C}^q(\mathfrak{A};\mathcal{E}) = \mathcal{E} \mathbin{\widehat{\otimes}} \mathcal{C}^q(\mathfrak{A};\mathbb{C})$, where $\widehat{\otimes}$ denotes the closure of the tensor product in an appropriate topology, which is not important at the moment (we note that $\mathcal{E} = \mathcal{E}(\mathbb{R}^1)$). Using the identification $L = \mathrm{id}_{\mathcal{E}} \otimes L$, we can continue the Laplace transformation to $\mathcal{C}^q(\mathfrak{A};\mathcal{E})$ and obtain the isomorphism $L\colon \mathcal{C}^q(\mathfrak{A};\mathcal{E}) \simeq \mathcal{F}^q$, where $\mathcal{F}^q = \mathcal{E} \mathbin{\widehat{\otimes}} \Phi^q$. We now introduce a "twisted" transformation $\Lambda\colon \mathcal{C}^q(\mathfrak{A};\mathcal{E}) \simeq \mathcal{F}^q$, where

$$\Lambda[\omega](x;\xi_1,\dots,\xi_q) = \exp\biggl(-x\sum_{1\le\alpha\le q}\xi_\alpha\biggr)L[\omega](x;\xi_1,\dots,\xi_q)$$

for all $\omega \in \mathcal{C}^q(\mathfrak{A};\mathcal{E})$, $x \in \mathbb{R}$, and $\xi = (\xi_1,\dots,\xi_q) \in \mathbb{C}^q$. We have thus defined the isomorphism of linear spaces $\Lambda\colon \mathcal{C}(\mathfrak{A};\mathcal{E}) \simeq \mathcal{F}$, where $\mathcal{F} = \bigoplus_{q\ge 0}\mathcal{F}^q$ and $\mathcal{C}^0(\mathfrak{A};\mathcal{E}) = \mathcal{F}^0 = \mathcal{E}$.

As mentioned, the linear spaces $\mathfrak{A}$ and $\mathcal{E}$ coincide and the isomorphism of linear spaces $\Lambda\colon \mathcal{C}(\mathfrak{A};\mathfrak{A}) \simeq \mathcal{F}$ is therefore well defined.

Proposition 1. *We have the isomorphism of complexes*

$$L\colon \{\mathcal{C}(\mathfrak{A};\mathbb{C}); d\} \simeq \{\Phi; d_{\mathbb{C}}\},$$

where the differential $d_{\mathbb{C}} = L \circ d \circ L^{-1}\colon \Phi \to \Phi$ *acts as*

$$(d_{\mathbb{C}}\phi)(\xi_0,\dots,\xi_q) = \frac{1}{q+1} \sum_{0\le\alpha<\beta\le q} (-1)^{\alpha+\beta-1}(\xi_\alpha - \xi_\beta)\phi(\xi_\alpha+\xi_\beta,\xi_0,\dots,\check{\xi}_\alpha,\dots,\check{\xi}_\beta,\dots,\xi_q)$$

for all $\phi \in \Phi^q$, $\xi_0,\dots,\xi_q \in \mathbb{C}$, *and* $q \in \mathbb{Z}_+$.

Proposition 2. *We have the isomorphism of complexes*

$$\Lambda\colon \{\mathcal{C}(\mathfrak{A};\mathcal{E}); d\} \simeq \{\mathcal{F}; d_{\mathcal{E}}\},$$

where the differential $d_{\mathcal{E}} = \Lambda \circ d \circ \Lambda^{-1}\colon \mathcal{F} \to \mathcal{F}$ *acts as*

$$(d_{\mathcal{E}}f)(x;\xi_0,\dots,\xi_q) = \frac{1}{q+1}\Bigg\{ \sum_{0\le\alpha\le q} (-1)^{\alpha}\Bigg(\partial_x + \sum_{0\le\beta\le q}\xi_\beta - \xi_\alpha\Bigg) f(x;\xi_0,\dots,\check{\xi}_\alpha,\dots,\xi_q) + \\ + \sum_{0\le\alpha<\beta\le q} (-1)^{\alpha+\beta-1}(\xi_\alpha-\xi_\beta) f(x;\xi_\alpha+\xi_\beta,\xi_0,\dots,\check{\xi}_\alpha,\dots,\check{\xi}_\beta,\dots,\xi_q)\Bigg\}$$

for all $f \in \mathcal{F}^q$, $\xi_0,\dots,\xi_q \in \mathbb{C}$, *and* $q \in \mathbb{Z}_+$.

Proposition 3. *We have the isomorphism of complexes*

$$\Lambda\colon \{\mathcal{C}(\mathfrak{A};\mathfrak{A}); d\} \simeq \{\mathcal{F}; d_{\mathfrak{A}}\},$$

where the differential $d_{\mathfrak{A}} = \Lambda \circ d \circ \Lambda^{-1}\colon \mathcal{F} \to \mathcal{F}$ *acts as*

$$(d_{\mathfrak{A}}f)(x;\xi_0,\dots,\xi_q) = \frac{1}{q+1}\Bigg\{ \sum_{0\le\alpha\le q} (-1)^{\alpha}\Bigg(\partial_x + \sum_{0\le\beta\le q}\xi_\beta - 2\xi_\alpha\Bigg) f(x;\xi_0,\dots,\check{\xi}_\alpha,\dots,\xi_q) + \\ + \sum_{0\le\alpha<\beta\le q} (-1)^{\alpha+\beta-1}(\xi_\alpha-\xi_\beta) f(x;\xi_\alpha+\xi_\beta,\xi_0,\dots,\check{\xi}_\alpha,\dots,\check{\xi}_\beta,\dots,\xi_q)\Bigg\}$$

for all $f \in \mathcal{F}^q$, $\xi_0,\dots,\xi_q \in \mathbb{C}$, *and* $q \in \mathbb{Z}_+$.

The proofs of Propositions 1–3 are elementary calculations.

2.2. Operations in the space Φ. In the spaces Φ^q, $q \in \mathbb{Z}_+$, we now segregate the subspaces

of homogeneous polynomials $\Phi^{p,q} = \{\phi(\xi) = \sum_{|n|=p}\phi_n\xi^n \in \Phi^q\}$,

of complementary functions $\Phi^q_{\neq p} = \{\phi(\xi) = \sum_{|n|\neq p}\phi_n\xi^n \in \Phi^q\}$, and

of the residues $\Phi^q_{\ge p} = \{\phi(\xi) = \sum_{|n|\ge p}\phi_n\xi^n \in \Phi^q\}$,

where the order $p \in \mathbb{Z}_+$, the argument $\xi = (\xi_1,\dots,\xi_q) \in \mathbb{C}^q$, the multi-index $n = (n_1,\dots,n_q) \in \mathbb{Z}^q_+$, its absolute value $|n| = \sum_{1\le\alpha\le q} n_\alpha$, the coefficient $\phi_n \in \mathbb{C}$, and the monomial $\xi^n = \xi_1^{n_1}\cdots\xi_q^{n_q}$.

The functions $\phi \in \Phi^q$, $q \in \mathbb{Z}_+$, are skew-symmetric by construction, i.e., $\pi^*\phi = \operatorname{sign}(\pi)\phi$ for all $\pi \in \Sigma_q$, where $(\pi^*\phi)(\xi_1,\dots,\xi_q) = \phi(\xi_{\pi(1)},\dots,\xi_{\pi(q)})$, $\xi_1,\dots,\xi_q \in \mathbb{C}$. This results in the following assertion.

Proposition 4. *If $\phi \in \Phi^q$, $q \in \mathbb{Z}_+$, then $\phi(\xi) = V_q(\xi) \cdot \rho(\xi)$, where*

$$V_q(\xi) = \prod_{1 \le \alpha < \beta \le q} (\xi_\beta - \xi_\alpha)$$

is the Vandermonde determinant, $\rho(\xi)$ is a symmetric entire function with the same estimates as the function $\phi(\xi)$, and $\xi = (\xi_1, \ldots, \xi_q) \in \mathbb{C}^q$. In particular, $\Phi^{p,q} = 0$ for $p < \deg V_q = q(q-1)/2$.

We note that $L^{-1}[V_q] = W_q[\cdots](0)$ is the Wronskian calculated at the point $0 \in \mathbb{R}$ and that the Wronskian $\Lambda^{-1}[V_q](x) = W_q[\cdots](x)$ is calculated at the point $x \in \mathbb{R}$.

We now introduce several operations on the space Φ.

• The linear mappings $\nabla_l^k \colon \Phi \to \Phi$, $k, l \in \mathbb{Z}_+$, are

$$(\nabla_l^k \phi)(\xi_1, \ldots, \xi_q) = \frac{1}{l!} \sum_{1 \le \alpha \le q} \xi_\alpha^k \partial_\alpha^l \phi(\xi_1, \ldots, \xi_q)$$

for all $\phi \in \Phi^q$ and $\xi_1, \ldots, \xi_q \in \mathbb{C}$, where the partial derivatives $\partial_\alpha = \partial/\partial\xi_\alpha$. In particular, we have $\nabla_l^k \colon \Phi^{p,q} \to \Phi^{p+k-l,q}$, while

$$\nabla_0^0\big|_{\Phi^{p,q}} = q\, \mathrm{id}_{\Phi^{p,q}}, \qquad \nabla_1^1\big|_{\Phi^{p,q}} = p\, \mathrm{id}_{\Phi^{p,q}}$$

for all $p, q \in \mathbb{Z}_+$.

• The linear mappings $\lambda^k \colon \Phi \to \Phi$, $k \in \mathbb{Z}_+$, are

$$(\lambda^k \phi)(\xi_0, \ldots, \xi_q) = \frac{1}{q+1} \sum_{0 \le \alpha \le q} (-1)^\alpha \xi_\alpha^k \phi(\xi_0, \ldots, \check{\xi}_\alpha, \ldots, \xi_q)$$

for all $\phi \in \Phi^q$ and $\xi_0, \ldots, \xi_q \in \mathbb{C}$. In particular, we have $\lambda^k \colon \Phi^{p,q} \to \Phi^{p+k,q+1}$ for all $p, q \in \mathbb{Z}_+$.

• The linear mappings $\iota_k \colon \Phi \to \Phi$, $k \in \mathbb{Z}_+$, are

$$(\iota_k \phi)(\xi_1, \ldots, \xi_{q-1}) = \frac{q}{k!} \big(\partial_0^k \phi(\xi_0, \xi_1, \ldots, \xi_{q-1})\big)\big|_{\xi_0 = 0}$$

for all $\phi \in \Phi^q$ and $\xi_1, \ldots, \xi_{q-1} \in \mathbb{C}$. In particular, we have $\iota_k \colon \Phi^{p,q} \to \Phi^{p-k,q-1}$ for all $p, q \in \mathbb{Z}_+$.

Proposition 5. *For all $k, l, m, n \in \mathbb{Z}_+$, we have the following equalities*:

1. $[\nabla_l^k, \nabla_n^m] = \sum_{1 \le r \le l} \begin{bmatrix} k, m \\ l, n \end{bmatrix}_r \nabla_{l+n-r}^{k+m-r}$,
   where $\begin{bmatrix} k, m \\ l, n \end{bmatrix}_r = \binom{m}{r}\binom{l+n-r}{n} - \binom{k}{r}\binom{l+n-r}{l}$ *and* $\binom{k}{l} = \dfrac{k!}{(l!(k-l)!)}$;
2. $[\nabla_l^k, \lambda^m] = \binom{m}{l} \lambda^{k+m-l}$;
3. $[\iota_m, \nabla_l^k] = \binom{m+l-k}{l} \iota_{m+l-k}$;
4. $\{\lambda^k, \lambda^l\} = 0$;
5. $\{\lambda^k, \iota_l\} = \delta_l^k \, \mathrm{id}_\Phi$, *where δ_l^k is the Kronecker symbol*;
6. $\{\iota_k, \iota_l\} = 0$.

Here, $[X, Y] = X \circ Y - Y \circ X$ *is the commutator, and* $\{X, Y\} = X \circ Y + Y \circ X$ *is the anticommutator of the mappings* $X, Y \colon \Phi \to \Phi$.

The proofs of all these formulas are direct calculations. We introduce additional operations on the space Φ.

• The linear mapping $\delta \colon \Phi \to \Phi$ is

$$(\delta\phi)(\xi_0, \ldots, \xi_q) = \frac{1}{q+1} \sum_{0 \le \alpha < \beta \le q} (-1)^{\alpha+\beta-1} (\xi_\alpha - \xi_\beta) \phi(\xi_\alpha + \xi_\beta, \xi_0, \ldots, \check{\xi}_\alpha, \ldots, \check{\xi}_\beta, \ldots, \xi_q)$$

for all $\phi \in \Phi^q$ and $\xi_0, \ldots, \xi_q \in \mathbb{C}$. In particular, we have $\delta \colon \Phi^{p,q} \to \Phi^{p+1,q+1}$ for all $p, q \in \mathbb{Z}_+$.

• The linear mappings $\delta_k \colon \Phi \to \Phi$, $k \in \mathbb{Z}$, are $\delta_k = \delta - k\lambda^1$. In particular, $\delta_k \colon \Phi^{p,q} \to \Phi^{p+1,q+1}$ for all $p, q \in \mathbb{Z}_+$.

Proposition 6. *We have the following equalities*:

1. $\delta \circ \delta = 0$*;*
2. $\{\lambda^i, \delta\} = \lambda^{i+1} \circ \lambda^0$, $i = 0, 1$*;*
3. $\{\iota_k, \delta\} = \nabla^0_{k-1} - \nabla^1_k$, $k \in \mathbb{Z}_+$, *in particular,* $\{\iota_0, \delta\} = -\nabla^1_0$ *and* $\{\iota_1, \delta\} = \nabla^0_0 - \nabla^1_1$*;*
4. $[\nabla^1_0, \delta] = 0$*;*
5. $\{\lambda^i, \delta_k\} = \lambda^{i+1} \circ \lambda^0$, $i = 0, 1$, *in particular,* $\lambda^0 \circ \delta_k + \delta_{k+1} \circ \lambda^0 = 0$ *for* $k \in \mathbb{Z}$*;*
6. $\delta_k \circ \delta_k = -k\lambda^2 \circ \lambda^0$, $k \in \mathbb{Z}$.

These equalities can be proved by direct calculation.

We now define two more operations on the space Φ.

• The linear mappings $R_k \colon \Phi \to \Phi$, $k \in \mathbb{Z}$, are $R_k = \nabla^0_0 - \nabla^1_1 - k \,\mathrm{id}_\Phi$. In particular, $R_k \colon \Phi^{p,q} \to \Phi^{p,q}$, while $R_k\big|_{\Phi^{p,q}} = (q - p - k)\,\mathrm{id}_{\Phi^{p,q}}$, whence $\ker R_k \cap \Phi^q = \Phi^{q-k,q}$, $q \in \mathbb{Z}$.

• The linear mappings $\rho_k \colon {}_k\Phi \to {}_k\Phi$, $k \in \mathbb{Z}$, where

$${}_k\Phi = \bigoplus_{q \ge 0} \Phi^q_{\ne(q-k)},$$

are

$$(\rho_k\phi)(\xi_1, \ldots, \xi_q) = -\int_0^1 t^{k-q-1} \phi(t\xi_1, \ldots, t\xi_q)\, dt$$

for all $\phi \in \Phi^q_{\ge p}$, $p > q - k$, $\xi_1, \ldots, \xi_q \in \mathbb{C}$, and

$$\rho_k\big|_{\Phi^{p,q}} = (q - p - k)^{-1}\,\mathrm{id}_{\Phi^{p,q}}$$

for $p \ne q - k$. We can easily verify that these formulas are mutually consistent and indeed determine the series of linear mappings on ${}_k\Phi$, $k \in \mathbb{Z}$.

Proposition 7. *The isomorphisms of linear locally convex spaces*

$$R_k \colon {}_k\Phi \simeq {}_k\Phi, \qquad (R_k)^{-1} = \rho_k, \quad k \in \mathbb{Z},$$

are well defined.

For the proof, it suffices to consider the action of the above mappings on the homogeneous components $\Phi^{p,q}$.

Proposition 8. *We have the following equalities*:

1. $\lambda^l \circ R_k = R_{k+1-l} \circ \lambda^l$, $l, k \in \mathbb{Z}$;
2. $\lambda^l \circ \rho_k = \rho_{k+1-l} \circ \lambda^l$, $l, k \in \mathbb{Z}$;
3. $[\delta, R_k] = 0$, $k \in \mathbb{Z}$;
4. $[\delta, \rho_k] = 0$, $k \in \mathbb{Z}$;
5. $\{\iota_1, \delta_k\} = R_k$, $k \in \mathbb{Z}$.

These equalities can be proved by restricting the mappings to homogeneous components using Propositions 5 and 6.

2.3. Auxiliary complexes. It follows from assertions 4 and 6 in Proposition 5 that the linear mappings $\lambda^0, \iota_0 \colon \Phi \to \Phi$ satisfy the equalities $\lambda^0 \circ \lambda^0 = \iota_0 \circ \iota_0 = 0$; therefore, the complexes $\{\Phi; \lambda^0\}$ and $\{\Phi; \iota_0\}$ are defined. We now set $\Psi = \ker \lambda^0 = \{\phi \in \Phi \colon \lambda^0 \phi = 0\}$ and $\Omega = \ker \iota_0 = \{\phi \in \Phi \colon \iota_0 \phi = 0\}$. The homotopical formula $\{\lambda^0, \iota_0\} = \mathrm{id}_\Phi$ (see assertion 5 in Proposition 5) then implies the following proposition.

Proposition 9. *We have the following statements*:

1. *we have the decomposition* $\Phi = \Psi \oplus \Omega$ ($\phi = \phi_\Psi + \phi_\Omega$, *where* $\phi_\Psi = \lambda^0(\iota_0 \phi)$ *and* $\phi_\Omega = \iota_0(\lambda^0 \phi)$);
2. *the complex* $\{\Phi; \lambda^0\}$ *has trivial cohomologies*;
3. *the complex* $\{\Phi; \iota_0\}$ *has trivial homologies.*

By virtue of assertion 5 in Proposition 6, we have $\delta_k \colon \Psi^{p,q} \to \Psi^{p+1,q+1}$, where $\Psi^{p,q} = \Phi^{p,q} \cap \Psi$, for all $p, q \in \mathbb{Z}_+$. By virtue of assertion 6 in Proposition 6, $\delta_k \circ \delta_k|_\Psi = 0$. The complexes $\{\Psi; \delta_k\}$ are therefore well defined for all $k \in \mathbb{Z}$.

In addition to the above spaces ${}_k\Phi$, we introduce the spaces ${}^k\Phi = \bigoplus_{q \geq 0} \Phi^{q-k,q}$ such that

$$\Phi = {}^k\Phi \oplus {}_k\Phi \quad \forall k \in \mathbb{Z}.$$

Analogously setting ${}_k\Psi = {}_k\Phi \cap \Psi$ and ${}^k\Psi = {}^k\Phi \cap \Psi$, we obtain $\Psi = {}^k\Psi \oplus {}_k\Psi$ for all $k \in \mathbb{Z}$. By construction, $\delta_k \colon {}^k\Psi \to {}^k\Psi$ and $\delta_k \colon {}_k\Psi \to {}_k\Psi$. The complexes $\{{}^k\Psi; \delta_k\}$ and $\{{}_k\Psi; \delta_k\}$ are therefore well defined, and

$$\{\Psi; \delta_k\} = \{{}^k\Psi; \delta_k\} \oplus \{{}_k\Psi; \delta_k\} \quad \forall k \in \mathbb{Z}.$$

Lemma 1. *The complexes* $\{{}_k\Psi; \delta_k\}$ *have trivial cohomologies. We therefore have the equality of cohomologies*

$$H^q(\{\Psi; \delta_k\}) = H^q(\{{}^k\Psi; \delta_k\}), \quad q \in \mathbb{Z}_+, \quad k \in \mathbb{Z}.$$

By virtue of Proposition 7 and assertions 2, 4, and 5 in Proposition 8, the homotopical formulas $\{S_k, \delta_k\} = \mathrm{id}$, where $S_k = R_k \circ \iota_1$, are defined on the spaces ${}_k\Psi$, which implies that these complexes are exact (by virtue of assertions 1 and 2 in Proposition 8, we have $R_k, \rho_k \colon \Psi \to \Psi$).

Lemma 2. *The linear spaces* ${}^k\Psi = 0$ *and the complexes* $\{{}^k\Psi; \delta_k\}$ *in particular are null spaces for* $k \geq 2$.

By virtue of Proposition 4, $\Phi^{q-k,q} = 0$ for $q - k < q(q-1)/2$, i.e., for $q^2 - 3q + 2k > 0$, while the latter inequality is valid for all $q \in \mathbb{Z}$ as soon as $k \geq 2$.

Lemma 3. *The cohomologies of the complex* $\{{}^0\Psi; \delta_0\}$ *are*

$$H^q(\{{}^0\Psi; \delta_0\}) = \begin{cases} \mathbb{C} \cdot \lambda^0 \xi^2, & q = 2, \\ \mathbb{C} \cdot \lambda^0 \chi, & q = 3, \\ 0, & q \neq 2, 3, \end{cases}$$

where $(\lambda^0\xi^2)(\xi_1,\xi_2)=(\xi_2^2-\xi_1^2)/2$ *and* $\chi(\xi_1,\xi_2)=3(\xi_2-\xi_1)\xi_1\xi_2$ (*we note that* $\lambda^0\chi=V_3$ *is the Vandermonde determinant of the third order*).

By virtue of Propositions 4 and 10, we have $({}^0\Psi)^q=\Psi^{q,q}=\lambda^0\Phi^{q,q-1}=0$ for $q\neq 2,3$, while $\Phi^{2,1}=\mathbb{C}\cdot\xi^2$ and $\Phi^{3,2}=\mathbb{C}\cdot\chi$. By virtue of assertion 5 in Proposition 6, we have $\delta_0(\lambda^0\xi^2)=-\lambda^0(\delta_{-1}\xi^2)=0$ because $\delta_{-1}\xi^2=-\lambda^0\xi^3$.

Lemma 4. *The cohomologies of the complex* $\{{}^1\Psi;\delta_1\}$ *are*

$$H^q(\{{}^1\Psi;\delta_1\})=\begin{cases}\mathbb{C}\cdot\lambda^0 1, & q=1,\\ \mathbb{C}\cdot\lambda^0\xi, & q=2,\\ 0, & q\neq 1,2,\end{cases}$$

where $(\lambda^0 1)(\xi)=1$, $\lambda^0\xi=V_2/2$, *and* V_2 *is the Vandermonde determinant of the second order.*

By virtue of Propositions 4 and 10, we have $({}^1\Psi)^q=\Psi^{q-1,q}=\lambda^0\Phi^{q-1,q-1}=0$ for $q\neq 1,2$, while $\Phi^{0,0}=\mathbb{C}$ and $\Phi^{1,1}=\mathbb{C}\cdot\xi$. By assertion 5 in Proposition 6, $\delta_1(\lambda^0 1)=-\lambda^0(\delta_0 1)=0$ because $\delta_0 1=-\lambda^0\xi$.

Combining Lemmas 1–4, we obtain the following theorem.

Theorem 4. *The cohomologies of the complexes* $\{\Psi;\delta_k\}$ *are*

$$H^q(\{\Psi;\delta_k\})=\mathbb{C}\cdot\lambda^0 Q_k^q,\quad q,k\in\mathbb{Z}_+,$$

where

$$Q_k^q=\delta_k^0(\delta_2^q\xi^2+\delta_3^q\chi)+\delta_k^1(\delta_1^q 1+\delta_2^q\xi)\in\Omega^{q-1},$$

δ_k^q *is the Kronecker symbol, and* $\chi(\xi_1,\xi_2)=3(\xi_2-\xi_1)\xi_1\xi_2$.

2.4. Operations in the space $\mathcal{F}$. As in Sec. 2.2, in the spaces $\mathcal{F}^q=\mathcal{E}\,\widehat{\otimes}\,\Phi^q$, we segregate the subspaces $\mathcal{F}^{p,q}=\mathcal{E}\,\widehat{\otimes}\,\Phi^{p,q}$, $\mathcal{F}^q_{\neq p}=\mathcal{E}\,\widehat{\otimes}\,\Phi^q_{\neq p}$, and $\mathcal{F}^q_{\geq p}=\mathcal{E}\,\widehat{\otimes}\,\Phi^q_{\geq p}$, $p,q\in\mathbb{Z}_+$. Identifying $\phi=1\otimes\phi$, $\phi\in\Phi$, we obtain the inclusion $\Phi\subset\mathcal{F}$. In turn, using the identification $X=\mathrm{id}_{\mathcal{E}}\otimes X$, we can extend a mapping X initially defined on the space Φ to the space $\mathcal{F}$ such that all operations in Sec. 2.2 are defined on $\mathcal{F}$. In particular, the linear mappings $\lambda^0,\iota_0\colon\mathcal{F}\to\mathcal{F}$ are defined, and we can set $\mathcal{K}=\ker\lambda^0=\mathcal{E}\,\widehat{\otimes}\,\Psi$ and $\mathcal{M}=\ker\iota_0=\mathcal{E}\,\widehat{\otimes}\,\Omega$. All calculations in Sec. 2.3 obviously hold when $\mathcal{K}$ is substituted for Ψ, $\mathcal{M}$ for Ω, and $\mathcal{E}$ for $\mathbb{C}$.

We introduce the proper operation on the space $\mathcal{F}$.

• The linear mapping $d_x\colon\mathcal{F}\to\mathcal{F}$ is $d_x=\partial_x\circ\lambda^0$, where $\partial_x=\partial/\partial x$ is the partial derivative w.r.t. x. In particular, $d_x\colon\mathcal{F}^{p,q}\to\mathcal{F}^{p,q+1}$ for all $p,q\in\mathbb{Z}_+$.

Proposition 10. *We have the following equalities*:

1. $d_x\circ d_x=0$;
2. $d_x\circ\delta_k+\delta_{k+1}\circ d_x=0$, $k\in\mathbb{Z}$;
3. $\{d_x,\lambda^k\}=0$, $k\in\mathbb{Z}_+$.

By virtue of assertion 1 in Proposition 10, we can define the complex $\{\mathcal{F};d_x\}$.

Proposition 11. *The complex* $\{\mathcal{F};d_x\}$ *has the cohomologies*

$$H^q(\{\mathcal{F};d_x\})=\Omega^q,\quad q\in\mathbb{Z}_+.$$

If $\phi\in\Omega$, then the formula $[\phi]=\phi+d_x\mathcal{F}$ determines the cohomology because $d_x\phi=\lambda_0\phi'_x=0$, while $[\phi]=0$, i.e., $\phi=d_xg=\lambda^0g'_x$ iff $\phi=0$ by virtue of assertion 1 in Proposition 9. On the other hand, given the cohomology $[f]$, where $d_xf=\lambda^0f'_x=0$, we can use Proposition 9 to verify that $f=\phi+d_xg$, where $\phi=\iota_0(\lambda^0f(0))\in\Omega$, $g=\iota_0F$, and $F'_x=f$ (we recall that $f(0)(\xi)=f(0;\xi)$, $\xi\in\mathbb{C}^q$).

3. Proofs of Theorems 1–3

3.1. The proof of Theorem 1. By virtue of Proposition 1, it suffices to calculate the cohomologies of the complex $\{\Phi; d_{\mathbb{C}}\}$, i.e., the complex $\{\Phi; \delta_0\}$, because $d_{\mathbb{C}} = \delta = \delta_0$ by definition ($\delta_0 \circ \delta_0 = \delta \circ \delta = 0$; see assertion 1 in Proposition 6). The calculations are similar to those in Sec. 2.3.

By construction, $\Phi = {}^0\Phi \oplus {}_0\Phi$, while $\delta_0 \colon {}^0\Phi \to {}^0\Phi$ and $\delta_0 \colon {}_0\Phi \to {}_0\Phi$, and we have the decomposition into the direct sum of complexes

$$\{\Phi; \delta_0\} = \{{}^0\Phi; \delta_0\} \oplus \{{}_0\Phi; \delta_0\}.$$

For the complex $\{{}_0\Phi; \delta_0\}$, we have the homotopic formula $\{S_0, \delta_0\} = \mathrm{id}$, where $S_0 = R_0 \circ \iota_1$. Hence, the cohomologies of this complex are trivial, and we must calculate the cohomologies of the complex $\{{}^0\Phi; \delta_0\}$. By definition, $({}^0\Phi)^q = \Phi^{q,q}$, $q \in \mathbb{Z}_+$. By virtue of Proposition 4, $\Phi^{q,q} = 0$ for $q < q(q-1)/2$, i.e., for $q > 3$. In turn, simple calculations yield $\Phi^{0,0} = \mathbb{C}$, $\Phi^{1,1} = \mathbb{C} \cdot \xi$, $\Phi^{2,2} = \mathbb{C} \cdot \delta\xi$, and $\Phi^{3,3} = \mathbb{C} \cdot V_3$, where V_3 is the Vandermonde determinant of third order. Because $\delta = 0$ for $\Phi^0 = \mathbb{C}$, we conclude that the cohomologies $H^q(\{{}^0\Phi; \delta_0\}) = 0$ for $q \neq 0, 3$, while $H^0(\{{}^0\Phi; \delta_0\}) = \mathbb{C}$ and $H^3(\{{}^0\Phi; \delta_0\}) = \mathbb{C} \cdot V_3$, and, as before, $L^{-1}[V_3] = W_3(0)$. Theorem 1 is thus proved.

3.2. The generalizing series of complexes. We now introduce several operations in the space $\mathcal{F}$.

- The linear mappings $\Delta_k \colon \mathcal{F} \to \mathcal{F}$, $k \in \mathbb{Z}$, are $\Delta_k = \delta_k + \nabla_0^1 \circ \lambda^0$.
- The linear mappings $d_k \colon \mathcal{F} \to \mathcal{F}$, $k \in \mathbb{Z}$, are $d_k = d_x + \Delta_k$.

Proposition 12. *For all $k \in \mathbb{Z}$, we have the following equalities*:

1. $\Delta_k \circ \Delta_k = 0$;
2. $\delta_k \circ \lambda^0 + \lambda^0 \circ \Delta_k = 0$, *in particular*, $\delta_k \circ d_x + d_x \circ \Delta_k = 0$;
3. $\{\iota_0, \Delta_k\} = 0$;
4. $d_k \circ d_k = 0$.

The proofs are direct calculations using formulas in Sec. 2.2 and assertion 1 in Proposition 10.

Remark. In general, we can introduce the linear mappings $\Delta_{km} = \delta - k\lambda^1 + m\,\nabla_0^1 \circ \lambda^0$ with some $k, m \in \mathbb{C}$. In this case, the composition $\Delta_{km} \circ \Delta_{km} = 0$ iff $m = 1$, $k \in \mathbb{C}$.

We have thus defined the series of complexes $\{\mathcal{F}; d_k\}$, $k \in \mathbb{Z}$; we now calculate the corresponding cohomologies.

Lemma 5. *The equation $\lambda^0 \circ \Delta_k f = 0$, $f \in \mathcal{F}^q$, $k, q \in \mathbb{Z}_+$, admits the general solution*

$$f = aQ_k^{q+1} + \lambda^0 g + \Delta_k h, \quad a \in \mathcal{E}, \quad g, h \in \mathcal{F}^{q-1},$$

where the function Q_k^q is determined in Theorem 4.

Every such function Q_k^q is a solution of the equation above. On the other hand, if $f \in \mathcal{F}^q$ and $\lambda^0 \circ \Delta_k f = 0$, then it follows from assertion 2 in Proposition 12 that $\delta_k(\lambda^0 f) = 0$, while $\lambda^0 f \in \mathcal{K}^{q+1}$. Hence, by virtue of Theorem 4, $\lambda^0 f = a\lambda^0 Q_k^{q+1} + \delta_k \tilde{h}$, where $a \in \mathcal{E}$, $\tilde{h} \in \mathcal{K}^q$. By virtue of assertion 2 in Proposition 9, $\tilde{h} = -\lambda^0 h$, where $h \in \mathcal{F}^{q-1}$. Therefore,

$$\lambda^0(f - \Delta_k h - aQ_k^{q+1}) = 0,$$

whence, by virtue of assertion 2 in Proposition 9, we have $f = \lambda^0 g + \Delta_k h + aQ_k^{q+1}$, $g \in \mathcal{F}^{q-1}$.

Lemma 6. *The equation $d_x \circ \Delta_k f = 0$, $f \in \mathcal{F}^q$, $k, q \in \mathbb{Z}_+$, admits a general solution*

$$f = aQ_k^{q+1} + d_x g + \Delta_k h + \phi, \quad a \in \mathcal{E}, \quad g, h \in \mathcal{F}^{q-1}, \quad \phi \in \Omega^q,$$

where the function Q_k^q is determined in Theorem 4.

The proof can be obtained by a slight modification of the previous reasonings. Namely, if $f \in \mathcal{F}^q$ and $d_x \circ \Delta_k f = 0$, then by virtue of assertion 2 in Proposition 12, $\delta_k(d_x f) = 0$, while $d_x f \in \mathcal{K}^{q+1}$. Hence, by Theorem 4, we have $d_x f = a'_x \lambda^0 Q_k^{q+1} + \delta_k \tilde{h}$ with some $a \in \mathcal{E}$, $\tilde{h} \in \mathcal{K}^q$. By assertion 2 in Proposition 9, we can set $\tilde{h} = -\lambda^0 h'_x$, $h \in \mathcal{F}^{q-1}$, whence

$$d_x(f - aQ_k^{q+1} - \Delta_k h) = 0.$$

By virtue of Proposition 11, $f - aQ_k^{q+1} - \Delta_k h = d_x g + \phi$ with some $g \in \mathcal{F}^{q-1}$ and $\phi \in \Omega^q$.

Lemma 7. *The function Q_k^q determined in Theorem 4 has the properties*

1. *$\lambda^0 Q_k^q = 0$ for those and only those $k, q \in \mathbb{Z}_+$ for which $Q_k^q = 0$ and*
2. *$\Delta_k Q_k^q = 0$ for all $k, q \in \mathbb{Z}_+$.*

By construction, the function $Q_k^q \in \Omega^{q-1}$, and by virtue of assertion 1 in Proposition 9, we obtain $\lambda^0 Q_k^q = 0$ only if $Q_k^q = 0$. Furthermore, by construction, $\delta_k(\lambda^0 Q_k^q) = 0$, and assertion 2 in Proposition 12 yields $\lambda^0(\Delta_k Q_k^q) = 0$. Hence, by virtue of assertion 2 in Proposition 9, $\Delta_k Q_k^q = \lambda^0 P$, where we have $P = \iota_0(\Delta_k Q_k^q) = -\Delta_k(\iota_0 Q_k^q) = 0$ by assertion 3 in Proposition 12 and assertion 1 in Proposition 9.

Theorem 5. *The cohomologies of the complexes $\{\mathcal{F}; d_k\}$, $k \in \mathbb{Z}_+$, are*

$$H^q(\{\mathcal{F}; d_k\}) = \mathbb{C} \cdot Q_k^{q+1}, \quad q \in \mathbb{Z}_+,$$

with the function Q_k^q determined in Theorem 4.

Let $f \in \mathcal{F}^q$ and $d_k f = 0$. Then $\Delta_k f = -d_x f = -\lambda^0 f'_x$, whence we have $\lambda^0 \circ \Delta_k f = 0$ by virtue of assertion 4 in Proposition 5. By Lemma 5, we then have $f = aQ_k^{q+1} + \lambda^0 g + \Delta_k h$ with some $a \in \mathcal{E}$ and $g, h \in \mathcal{F}^{q-1}$, while

$$0 = d_k f = d_k(aQ_k^{q+1}) + \Delta_k(\lambda^0 g) + d_x(\Delta_k h).$$

Furthermore, using assertion 2 in Lemma 7, we obtain

$$d_k(aQ_k^{q+1}) = a'_x \lambda^0 Q_k^{q+1} + a\Delta_k Q_k^{q+1} = a'_x \lambda^0 Q_k^{q+1},$$

while using assertion 2 in Proposition 12, we have

$$\Delta_k(\lambda^0 g) + d_x(\Delta_k h) = \delta_k(\lambda^0(g - h'_x)).$$

Therefore,

$$a'_x \lambda^0 Q_k^{q+1} = -\delta_k(\lambda^0(g - h'_x)).$$

By virtue of Theorem 4, we have $a'_x = 0$ and $\delta_k(\lambda^0(g - h'_x)) = 0$. Hence, $a \in \mathbb{C} \subset \mathcal{E}$, and $\lambda^0 \circ \Delta_k(g - h'_x) = 0$. Using Lemma 5, we obtain

$$g - h'_x = b'_x Q_k^q + \lambda^0 r + \Delta_k s'_x$$

with some $b \in \mathcal{E}$ and $r, s \in \mathcal{F}^{q-2}$. Substituting $g = h'_x + b'_x Q_k^q + \lambda^0 r + \Delta_k s'_x$ in the above representation for f and taking assertion 2 in Lemma 7 and assertion 1 in Proposition 12 into account, we obtain

$$\begin{aligned} f &= aQ_k^{q+1} + \lambda^0(h'_x + b'_x Q_k^q + \lambda^0 r + \Delta_k s'_x) + \Delta_k h = \\ &= aQ_k^{q+1} + d_k h + d_x(bQ_k^q) + d_x(\Delta_k s) + \Delta_k(bQ_k^q) + \Delta_k(\Delta_k s) = \\ &= aQ_k^{q+1} + d_k(h + bQ_k^q + \Delta_k s). \end{aligned}$$

We thus obtain the cohomology $[f] = aQ_k^{q+1}$ with some complex constant a.

3.3. Proofs of Theorems 2 and 3. The above mathematical apparatus and, in particular, Propositions 2 and 3 and Theorem 5 make the proofs of these theorems an elementary verification of the equalities $d_{\mathcal{E}} = d_1$ and $d_{\mathfrak{A}} = d_2$.

4. Discussion

The cohomologies of Lie algebras of vector fields on the line with coefficients in the trivial representation were essentially known [2], [3], although a concrete formulation was lacking. We provide this formulation in Theorem 1 and show its place in the general scheme using a general method of proof. In the cases where the coefficients are in the natural and adjoint representations, there are general results concerning cohomologies with nontrivial coefficients [7]–[9] from which explicit expressions for the cohomologies under consideration can be derived. However, the necessary calculations are substantially more difficult than our direct calculations. In any case, it is useful to have elementary proofs for basic model examples. Moreover, it is probable that the developed technique can be generalized to actual situations of physical importance.

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REFERENCES

1. P. Cartier, "Cohomologie des algèbres de Lie, I," pp. 3-01–3-07; "Cohomologie des algèbres de Lie, II: Interprétation des groupes de cohomologie," pp. 4-01–4-11; "Cohomologie des algèbres de Lie, III," pp. 5-01–5-07; "Compléments sur la cohomologie," pp. 5-08–5-10 in: *Théorie des Algèbres de Lie*: *Topologie des Groupes de Lie* (Séminaire "Sophus Lie" le annee: 1954/55), Secrétariat Mathématique, Paris (1955).
2. D. B. Fuks, *Cohomology of Infinite-Dimensional Lie Algebras* [in Russian], Nauka, Moscow (1984); English transl., Plenum, New York (1986).
3. B. L. Feigin and D. B. Fuks, "Cohomologies of Lie Groups and Algebras," in: *Itogi Nauki i Tekhniki*: *Contemporary Problems of Mathematics*: *Fundamental Directions* (R. V. Gamkrelidze, ed.) [in Russian], Vol. 21, VINITI, Moscow (1988), p. 121.
4. J. A. de Azcárraga, J. M. Izquierdo, and J. C. Pérez Bueno, "An introduction to some novel applications of Lie algebra cohomology in mathematics and physics," arXiv:physics/9803046 (1998).
5. I. M. Gel'fand and D. B. Fuks, *Funkts. Anal. Prilozhen.*, **2**, No. 4, 92 (1968).
6. L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Vol. 1, *Distribution Theory and Fourier Analysis*, Springer, Berlin (1983).
7. T. Tsujishita, *Proc. Japan Acad. A*, **53**, 134 (1977).
8. M. V. Losik, *Funct. Anal. Appl.*, **6**, 289 (1972).
9. V. N. Reshetnikov, *Sov. Math. Dokl.*, **14**, 234 (1973).