Lie symmetry and integrability of ordinary differential equations

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Abstract

Combining a Lie algebraic approach that is due to Wei and Norman (*J. Math. Phys.*, 1963, **4**, 475) and the ideas suggested by Drach (*Comptes Rendus*, 1919, **168**, 337) we have constructed several classes of systems of linear ordinary differential equations that are integrable by quadratures. Their integrability is ensured by integrability of the corresponding stationary cubic Schrödinger, KdV and Harry-Dym equations. Next, we obtain a hierarchy of integrable reductions of the Dirac equation of an electron moving in the external field. Their integrability is shown to be in correspondence with integrability of the stationary mKdV hierarchy.

I. Introduction

The object of the study in the present paper is the system of first-order ordinary differential equations (ODEs) of the following structure:

$$\frac{d\psi}{dx} + L(x)\psi = 0. \tag{1}$$

Here $\psi(x)$ is an n-component real-valued function column and $L(x)$ is an $n \times n$ matrix function taking values in some real r-dimensional matrix Lie algebra g, namely

$$L(x) = \sum_{k=1}^{r} f_k(x) Q_k, \tag{2}$$

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where Q_k are constant $n \times n$ matrices fulfilling the commutation relations

$$[Q_i,\ Q_j] = \sum_{k=1}^{r} C_{ij}^{k} Q_k. \tag{3}$$

The general Lie algebraic approach to integrating systems of ordinary differential equations having the structure (1) has been developed by WEI and NORMAN [1, 2]. The principal idea of their approach is a proper utilization of the Baker-Campbell-Hausdorff formula in order to represent the general solution of (1) in the form

$$\psi(x) = \exp\left\{\sum_{i=1}^{r} l_i(x) Q_i\right\} \chi, \tag{4}$$

where χ is an arbitrary constant n component column.

However, with all its elegance and simplicity the approach is still too general and gives a kind of an existence theorem of a special representation of the general solution. The principal problem is that after representing the general solution of (1) in the form (4) we have to integrate a system of *nonlinear* ODEs for unknown functions $l_1(x), \ldots, l_r(x)$. On the other hand, there is the well-known inverse scattering method which reduces a problem of finding special solutions of nonlinear partial differential equations to an auxiliary linear problem (see, e.g., [3]–[5]). So it is only natural to make an attempt of combining both approaches in order to develop a routine for choosing special subclasses of ODEs (1) that can be reduced to solving some nonlinear (stationary) solitonic equations. Namely, we pick out specific subclasses of equations of the form (1) such that their integrability by quadratures is ensured by integrability of the corresponding stationary solitonic equations. To this end we consider systems of ODEs of the following special form:

$$\mathcal{L}\psi \equiv \left(\frac{d}{dx} + \lambda F(x) + G(x)\right)\psi = 0, \tag{5}$$

where $F(x), G(x)$ are matrix-functions taking values in a real r-dimensional matrix Lie algebra g. The parameter λ may be thought of as an auxiliary independent variable, i.e. $\psi = \psi(x, \lambda)$. Note that the functions $F(x), G(x)$ are independent of λ.

Systems of ODEs (1) are of utmost importance for separation of variables in systems of partial differential equations (say, in the Dirac equation of an electron). This is due to the fact that after separating variables in a given system of partial differential equations we have to integrate separated equations which are exactly of the form (1).

Since we deal with more specific models as compared with ones considered by Wei and Norman, it is possible to input more information into the Ansatz (4). To this end we make use of its "infinitesimal analogue" in the form of Lie symmetry of system of ODEs (5), whose coefficients are required to be polynomials in λ. The last restriction is crucial and provides a link of the system of ODEs (5) to stationary solitonic equations whose integrability is ensured by the inverse scattering method. As we learned recently, a similar idea was used about eighty years ago by Drach [6] in order to integrate by quadratures the one-dimensional stationary Schrödinger equation

$$\psi'' - (\lambda + V(x))\psi = 0.$$

He had made an Ansatz for a solution of the above equation by a proper fixing its dependence on λ. As a result, he discovered a remarkable class of integrable stationary Schrödinger equations. Namely, it had been proved that, provided $V(x)$ is a solution of the nonlinear ODE called now the equation of the stationary KdV hierarchy, the stationary Schrödinger equation is integrable by quadratures. Moreover, Drach, in fact, suggested the method for integrating the stationary KdV hierarchy and discovered on this way the basics of the theory of the finite-gap solutions of the KdV equation. As shown in [7] the results of [6] can be obtained with the use of Lie symmetry of the stationary Schrödinger equation. One of the aims of the present paper is developing a technique that is a proper synthesis of the methods by Drach and Wei & Norman in order to reduce the problem of integrability by quadratures of systems of ODEs (5) to integrable solitonic hierarchies. A principal tool to be used in this respect is the Lie symmetry of (5).

The paper is organized as follows. The second section is devoted to description of our approach to integrating systems of ODEs of the form (5). The approach is based on utilization of symmetry properties of these systems within the class of Lie symmetries which are spanned by the basis elements of the Lie algebra g with coefficients being polynomials in parameter λ. The problem of constructing such Lie symmetries is shown to be reduced to integrating some systems of nonlinear ODEs. Remarkably, for many interesting cases these ODEs are nothing else than equations of the stationary solitonic hierarchies (which is not unexpected in view of what was said above). Utilizing this approach we give in Section III a systematic treatment of the case when the Lie algebra g is three-dimensional. On this way we find a number of linear systems of ODEs integrable in quadratures due to integrability of the stationary cubic Schrödinger, KdV and Harry-Dim equations. Section IV is devoted to an analysis of a reduction of the Dirac equation for a particle moving in a specific electric field that is integrable with the help of the suggested procedure. It occurs that the reduced system of ODEs is integrable provided the non-zero component of the electro-magnetic field satisfy one of the equations of the stationary mKdV hierarchy. The last section contains a brief discussion of the results of the paper.

II. The general scheme

We remind that a Lie vector field

$$X = \xi(x,\lambda)\frac{d}{dx} + \eta(x,\lambda),$$

where ξ is a smooth scalar function and η is an $n \times n$ matrix whose entries are smooth functions of x, is called the (Lie) symmetry of system (5) if X transforms the set of its solutions into itself, i.e.

$$\mathcal{L}\psi = 0 \Longrightarrow \mathcal{L}X\psi = 0.$$

The above relation can be represented in the form [8]

$$[\mathcal{L},\ X] = R(x,\lambda)\mathcal{L}, \tag{6}$$

where R is some $n \times n$ matrix function. This operator equality is to be understood in the following way: the operators on the left- and right-hand sides should give the same result when acting on any continuously differentiable function.

Making use of the formula (6) it is easy to become convinced of the fact that if X is a symmetry of the system of ODEs (5), then $X + \Xi(x,\lambda)\mathcal{L}$ with an arbitrary smooth function Ξ is a symmetry as well. Consequently, without loss of generality we can restrict our considerations to symmetries of the form

$$X = \eta(x,\lambda). \tag{7}$$

Importantly, operator (7) is the symmetry of the system (5) if and only if in (6) $R = 0$, i.e.

$$[\mathcal{L},\ X] = 0 \tag{8}$$

(this is proved by direct computation).

The key idea of our approach is to fix *a priori* dependence of a symmetry on the parameter λ and to consider the case when η is a polynomial in λ of the order N with matrix coefficients.

So a Lie symmetry of system (5) is looked for as a polynomial in λ which coefficients are linear combinations of the basis elements $Q_1, \ldots, Q_r$ of the Lie algebra g

$$X = \sum_{l=1}^{N} \sum_{k=1}^{r} s_{kl}(x) Q_k \lambda^l, \tag{9}$$

where $s_{kl}(x)$ are sufficiently smooth functions. From the invariance criterion (8) we get the following relation:

$$\sum_{l=1}^{N} \sum_{k=1}^{r} s'_{kl} Q_k \lambda^l + \sum_{l=1}^{N} \left\{ \sum_{i,j,k=1}^{r} \left(f_i s_{jl} C_{ij}^k Q_k + \lambda g_i s_{jl} C_{ij}^k Q_k \right) \right\} \lambda^l = 0.$$

Splitting it with respect to the powers of λ yields

$$\lambda^{N+1} \quad : \quad \sum_{i,j=1}^{r} g_i s_{jN} C_{ij}^k = 0, \tag{10}$$

$$\lambda^{l} \quad : \quad f'_{kl} + \sum_{i,j=1}^{r} (f_i s_{jl} + g_i s_{jl-1}) C_{ij}^k = 0, \tag{11}$$

$$\lambda^{0} \quad : \quad f'_{k0} + \sum_{i,j=1}^{r} f_i s_{j0} C_{ij}^k = 0, \tag{12}$$

where $k = 1, \ldots, r$, $l = 1, \ldots, N$.

Equations (10) are purely algebraic. Solving these we obtain recursively from equations (11) the remaining coefficients of the symmetry operator X. Inserting the obtained results into (12) we get a system of relations for functions f_i, g_i that form a system of nonlinear ODEs. The structure of this system is determined both by the form of the initial system of ODEs and by the form of the commutation relations of the Lie algebra g.

The next step depends on the kind of the problem we are dealing with. If the problem is to check whether a given system of ODEs is integrable within the framework of our approach, then the only thing to be done is to verify whether the obtained system of ODEs is identically satisfied by the coefficients of system (5). If, on the contrary, we have to solve a classification problem, i.e. the one of describing functions f_i, g_i such that system (5) is integrable, then we have to find (general or particular) solution of the mentioned system of nonlinear ODEs. What makes the whole procedure efficient it is the fact that the nonlinear ODEs obtained are often the well studied stationary solitonic equations.

Now with a Lie symmetry of system of ODEs (5) in hand we can integrate it with the use of the following procedure. We diagonalize of the operator X with the help of a properly chosen linear transformation of the dependent variables. The initial system of ODEs (5) being transformed in this way simplifies substantially and can be integrated by quadratures (at least, for the low dimensional Lie algebras).

The above procedure proves to be efficient not only for integrating specific ODEs but also for classification of systems of ODEs integrable by quadratures.

III. Integrable ODEs

We apply the above described method to classify integrable systems of ODEs of the form

$$\mathcal{L}\psi \equiv \left(\frac{d}{dx} + \sum_{a=1}^{3}(f_a(x) + \lambda g_a(x))Q_a\right)\psi = 0, \tag{13}$$

where Q_1, Q_2, Q_3 are basis elements of a real three-dimensional Lie algebra g. It is not difficult to show that if a three-dimensional Lie algebra g is a direct sum of the lower dimensional Lie algebras, then the corresponding system of ODEs (13) is integrated by quadratures. According to [9] the list of real inequivalent Lie algebras of the dimension three which are not direct sums of lower dimensional Lie algebras is exhausted by the following algebras:

$$\begin{aligned}
A_1 &: [Q_2,\ Q_3] = 0,\\
A_2 &: [Q_1,\ Q_3] = Q_1, \quad [Q_2,\ Q_3] = Q_1 + Q_2,\\
A_3 &: [Q_1,\ Q_3] = Q_1, \quad [Q_2,\ Q_3] = Q_2,\\
A_4 &: [Q_1,\ Q_3] = Q_1, \quad [Q_2,\ Q_3] = -Q_2,
\end{aligned}$$

$$
\begin{aligned}
A_5 &: \; [Q_1,\ Q_3] = Q_1, \quad [Q_2,\ Q_3] = aQ_2, \quad (0 < |a| < 1), \\
A_6 &: \; [Q_1,\ Q_3] = -Q_2, \quad [Q_2,\ Q_3] = Q_1, \\
A_7 &: \; [Q_1,\ Q_3] = aQ_1 - Q_2, \quad [Q_2,\ Q_3] = Q_1 + aQ_2, \quad (a > 0), \\
A_8 &: \; [Q_1,\ Q_2] = -Q_3, \quad [Q_1,\ Q_3] = -Q_2, \quad [Q_2,\ Q_3] = Q_1, \\
A_9 &: \; [Q_1,\ Q_2] = Q_3, \quad [Q_2,\ Q_3] = Q_1, \quad [Q_3,\ Q_1] = Q_2,
\end{aligned}
$$

the remaining commutation relations being zero.

For all the algebras A_1–A_7 system of ODEs (13) is integrated by quadratures with arbitrary f_a, g_a. This is due to the fact that making a transformation

$$\psi \to \tilde{\psi} = \exp\{F(x)Q_1 + G(x)Q_2\}$$

with properly chosen functions F, G we can always reduce (13) to the system of ODEs of the form

$$\frac{d\tilde{\psi}}{dx} + f(x,\lambda)Q_3\tilde{\psi} = 0$$

whose general solution is given by the quadrature

$$\tilde{\psi} = \exp\left\{-Q_3 \int f(x,\lambda)dx\right\}\chi,$$

where χ is an arbitrary constant column.

To complete the classification of systems of ODEs (13) integrable by quadratures within the framework of the above suggested approach we have to consider the two remaining algebras $A_8 = so(2,1)$ and $A_9 = so(3)$. Both algebras lead to non-trivial results even if we restrict our considerations to Lie symmetries which are second-order polynomials in λ.

We consider first the case when coefficients of system (13) take values in the $so(2,1)$ algebra. Using a sequence of transformations

$$\psi \to \tilde{\psi} = \exp\{\alpha(x)Q\}, \tag{14}$$

where Q is one of the operators Q_1, Q_2, Q_3, together with a transformation of the independent variable

$$x \to \tilde{x} = f(x)$$

we can reduce (13) to the form $\tilde{\mathcal{L}}\tilde{\psi} = 0$ with

$$\tilde{\mathcal{L}} = \begin{cases} \frac{d}{dx} + V_2(x)Q_2 + V_3(x)Q_3 + \lambda Q_1, & \text{if } \Delta > 0, \\ \frac{d}{dx} + V_1(x)Q_1 + V_2(x)Q_2 + \lambda Q_3, & \text{if } \Delta < 0, \\ \frac{d}{dx} + V_1(x)Q_1 + V_2(x)(Q_2 + Q_3) + \lambda(Q_2 - Q_3), & \text{if } \Delta = 0. \end{cases}$$

Here Δ stands for $V_1^2 + V_2^2 - V_3^2$.

We restrict our analysis of symmetries of the above enumerated systems to the class of the second-order polynomials in λ. This means that symmetries of the systems of ODEs in question are looked for in the form (9) under $N = 2$. Omitting the details of calculations we present below the explicit forms of symmetries and corresponding nonlinear ODEs for the 'potentials' $V_1(x), V_2(x)$.

Case 1. $\tilde{\mathcal{L}} = \frac{d}{dx} + V_2(x)Q_2 + V_3(x)Q_3 + \lambda Q_1$.

In this case the symmetry X reads

$$\begin{aligned} X &= \lambda^2 C_1 Q_3 + \lambda C_2 Q_1 + \lambda C_1 (V_2(x)Q_2 + V_3(x)Q_3) \\ &\quad + \left(\frac{1}{2}C_1(V_2^2 - V_3^2) + C_3\right) Q_1 + (C_1 V_3' + C_2 V_2)Q_2 + (C_1 V_2' + C_2 V_3)Q_3 \end{aligned}$$

and what is more the functions V_2, V_3 satisfy the hyperbolic stationary cubic Schrödinger equation

$$\begin{aligned} C_1 V_2'' + C_2 V_3' + \left(\frac{1}{2}C_1(V_2^2 - V_3^2) + C_3\right) V_2 = 0, \\ C_1 V_3'' + C_2 V_2' + \left(\frac{1}{2}C_1(V_2^2 - V_3^2) + C_3\right) V_3 = 0. \end{aligned}$$

As usual, when talking about stationary solitonic equations we mean the ODEs obtained from standard (1+1)-dimensional solitonic equations via the Ansatz $u(t,x) = u(x + Ct),\ C = \text{const}$.

Case 2. $\tilde{\mathcal{L}} = \frac{d}{dx} + V_1(x)Q_1 + V_2(x)Q_2 + \lambda Q_3$.

In this case the operator X has the form

$$\begin{aligned} X &= \lambda^2 C_1 Q_1 + \lambda C_1 V_1 Q_1 + \lambda C_1 V_2(x) Q_2 + \lambda C_2 Q_3 + (-C_1 V_2' + C_2 V_1) Q_1 \\ &\quad + (C_1 V_1' + C_2 V_2) Q_2 + \left(\frac{1}{2}C_1(V_1^2 + V_2^2) + C_3\right) Q_3 \end{aligned}$$

and what is more the functions V_1, V_2 satisfy the stationary cubic Schrödinger equation

$$C_1 V_1'' + C_2 V_2' - \left(\frac{1}{2} C_1 (V_1^2 + V_2^2) + C_3\right) V_1 = 0,$$
$$C_1 V_2'' - C_2 V_1' - \left(\frac{1}{2} C_1 (V_1^2 + V_2^2) + C_3\right) V_2 = 0.$$

Case 3. $\tilde{\mathcal{L}} = \frac{d}{dx} + V_1(x) Q_1 + V_2(x)(Q_2 + Q_3) + \lambda (Q_2 - Q_3)$.

Provided $V_2 = 0$ the corresponding system of ODEs is integrable by quadratures with an arbitrary V_1. If $V_2 \neq 0$, then we can transform the operator $\tilde{\mathcal{L}}$ to obtain

$$\frac{d}{dx} + V_1(x)(Q_2 + Q_3) + V_2(x)(Q_2 - Q_3) + \lambda (Q_2 - Q_3).$$

We have not succeeded in constructing the general form of V_1, V_2 and therefore restrict our considerations to the particular cases (i) $V_1 = \mathrm{const}$, (ii) $V_2 = 0$.

Subcase 3.1. $V_1(x) = \alpha \equiv \mathrm{const}$.

The Lie symmetry for this case reads

$$\begin{aligned} X &= \lambda^2 (Q_2 - Q_3) C_1 + \lambda (\alpha C_1 + C_2 + \frac{1}{2} C_1 V_2) Q_2 \\ &\quad \lambda (\alpha C_1 - C_2 - \frac{1}{2} C_1 V_2) Q_3 + \frac{1}{2} C_1 V_2' Q_1 \\ &\quad (\alpha C_2 + (C_2 - \frac{1}{2} \alpha C_1) V_2 - \frac{1}{2} C_1 V_2^2 + \frac{1}{2\alpha} C_1 V_2'') Q_2 \\ &\quad + (\alpha C_2 - (C_2 + \frac{1}{2} \alpha C_1) V_2 + \frac{1}{2} C_1 V_2^2 + \frac{1}{2\alpha} C_1 V_2'') Q_3, \end{aligned}$$

the function $V_1(x)$ being a solution of the stationary KdV equation

$$\frac{1}{2\alpha} C_1 V_2''' - 3 C_1 V_2' V_2 + 2 C_2 V_2' = 0.$$

Subcase 3.2. $V_2(x) = 0$.

The Lie symmetry for this case is of the form

$$X = \lambda^2 C_1 V_1^{-1/2} (Q_2 - Q_3) + \frac{1}{2} \lambda C_1 V_1' V_1^{-3/2} Q_1$$

$$+\lambda(C_2x + C_3 + C_1V_1^{1/2})Q_2 - \lambda(C_2x + C_3 - C_1V_1^{1/2})Q_3$$
$$+C_2Q_1 + \{(C_2x + C_3)V_1 + \frac{1}{8}C_1(2V_1''V_1^{-3/2} - 3(V_1')^2V_1^{-5/2})\}Q_2$$
$$+\{(C_2x + C_3)V_1 + \frac{1}{8}C_1(2V_1''V_1^{-3/2} - 3(V_1')^2V_1^{-5/2})\}Q_3,$$

the function $V_1(x)$ satisfying the nonlinear ODE

$$\frac{1}{2}C_1V_1'''V_1^{-3/2} - \frac{9}{4}C_1V_1''V_1'V_1^{-5/2} + \frac{15}{8}C_1(V_1')^3V_1^{-7/2} + 2(C_2x + C_3)V_1' + 4C_2V_1 = 0.$$

With $C_2 = 0$ the above equation is nothing else than the stationary Harry Dym equation

$$-C_1\left(V_1^{-1/2}\right)''' + 2C_3V_1' = 0$$

which is known to be integrable by quadratures and, furthermore, under $C_3 = 0$ possesses solutions in terms of elementary functions

$$V_1(x) = (\alpha_1x^2 + \alpha_2x + \alpha_3)^{-2},$$

where $\alpha_1, \alpha_2, \alpha_3$ are arbitrary real constants.

Now we turn to the case of the algebra $A_9 = so(3)$ and consider system of ODEs (13), where Q_1, Q_2, Q_3 are constant matrices satisfying the commutation relations

$$[Q_a,\ Q_b] = Q_c, \quad (a, b, c) = \mathrm{cycle}\,(1, 2, 3).$$

Using a sequence of transformations (14), where Q is one of the operators Q_1, Q_2, Q_3, we can substantially simplify system (13) and reduce it to the following equivalent form:

$$\mathcal{L}\psi \equiv \left(\frac{d}{dx} + V_1(x)Q_1 + V_2(x)Q_2 + \lambda Q_3\right)\psi = 0. \tag{15}$$

As above, a Lie symmetry X of system (15) is looked for as a second-order polynomial in λ with matrix coefficients. Inserting X into the invariance criterion $[\mathcal{L},\ X] = 0$ yields a system of determining equations for its coefficients. Having solved these we obtain the explicit form of the Lie symmetry admitted by system of ODEs (15)

$$X = \lambda^2C_1Q_3 + \lambda C_1(V_1Q_1 + V_2Q_2) + \lambda C_2Q_3 + (-C_1V_2' + C_2V_1)Q_1$$
$$+(C_1V_1' + C_2V_2)Q_2 + (\frac{1}{2}(V_1^2 + V_2^2) + C_3)Q_3,$$

where $V_1(x), V_2(x)$ are solutions of the stationary cubic Schrödinger equation

$$C_1V_1'' - C_2V_2' - \left(\frac{1}{2}C_1(V_1^2 + V_2^2) + C_3\right) V_1 = 0,$$
$$C_1V_2'' + C_2V_1' - \left(\frac{1}{2}C_1(V_1^2 + V_2^2) + C_3\right) V_2 = 0$$

and C_1, C_2, C_3 are arbitrary constants.

In what follows we will briefly discuss a method for integrating systems of ODEs (13), (15) based on their symmetry properties. Generally speaking, information about a Lie symmetry admitted by a system of ODEs is not sufficient to provide its integrability by quadratures within the Lie group approach (for more details see, e.g. [10]–[13]). However, due to the remarkable algebraic structure of the ODEs under study knowledge of a Lie symmetry makes it possible to construct their general solutions by quadratures. Namely, the following assertions hold true.

Lemma 1 *Let the system of ODEs*

$$\mathcal{L}\psi \equiv \left(\frac{d}{dx} + f_a(x)Q_a\right)\psi = 0, \qquad (16)$$

where Q_1, Q_2, Q_3 are constant matrices forming a basis of the Lie algebra $so(2,1)$, admit a Lie symmetry

$$X = \sum_{a=1}^{3} g_a(x)Q_a. \qquad (17)$$

Then it is integrable by quadratures.

Lemma 2 *Let the system of ODEs (16), where Q_1, Q_2, Q_3 are constant matrices forming a basis of the Lie algebra $so(3)$, admit a Lie symmetry (17) Then it is integrable by quadratures.*

We adduce the proof of Lemma 2 (the first lemma is proved in a similar way). Making a change of dependent variables

$$\psi \to \tilde{\psi} = \mathcal{V}(x)\psi, \quad \mathcal{V}(x) = \exp\{\sum_{a=1}^{3} h_a(x)Q_a\}$$

we can always transform the operator X to become

$$\tilde{X} = \mathcal{V}^{-1} X \mathcal{V} = g(x) Q_1, \quad g(x) \neq 0$$

and what is more this transformation preserves the structure of system (16). The invariance criterion $[\tilde{\mathcal{L}},\ \tilde{X}] = 0$, where

$$\tilde{\mathcal{L}} = \mathcal{V}^{-1} \mathcal{L} \mathcal{V} = \frac{d}{dx} + \sum_{a=1}^{3} \tilde{f}_a(x) Q_a,$$

implies that

$$g(\tilde{f}_2 Q_3 - f_3 Q_2) - g' Q_1 = 0.$$

As the matrices Q_1, Q_2, Q_3 are linearly independent, hence it follows that

$$\tilde{f}_2 = 0, \quad \tilde{f}_3 = 0, \quad g = \text{const}.$$

Consequently, the transformed system of ODEs necessarily takes the form

$$\left(\frac{d}{dx} + \tilde{f}_1 Q_1 \right) \psi = 0$$

and is evidently integrable by quadratures. Lemma 2 is proved.

Consequently, all the systems of linear ODEs considered in Section 3 possessing non-trivial symmetries of the form (9) can be integrated by quadratures with the use of the algebraic procedure described above. This is due to the well-known fact that the stationary solitonic equations arising as the invariance conditions are integrable by quadratures (see, e.g. [14, 15]).

IV. The hierarchy of integrable reductions of the Dirac equation

The technique developed above applies straightforwardly to systems of ODEs having complex-valued coefficients. In this section we use this technique for obtaining integrable reductions of the Dirac equation of an electron

$$i \sum_{\mu=0}^{3} \gamma_\mu \psi_{x_\mu} - \left(e \sum_{\mu=0}^{3} \gamma_\mu A^\mu + m \right) \psi = 0, \tag{18}$$

moving in the electric field

$$A_0 = A_0(x_3), \quad A_1 = A_2 = A_3 = 0.$$

In the formulae (18) γ_μ are 4×4 Dirac matrices, $\psi = \psi(x_0, x_1, x_2, x_3)$ is a four-component complex-valued function and e, m are constants.

The form of the vector-potential A_μ imply the following Ansatz for the spinor field $\psi(x)$:

$$\psi(x) = \varphi(x_3).$$

Inserting this expression into the Dirac equation (18) yields a system of ordinary differential equations for the four-component function $\varphi(x_3)$

$$\varphi' - (ie\gamma_3\gamma_0 A_0 - im\gamma_3)\varphi = 0. \tag{19}$$

Denoting

$$\begin{aligned} &x = 2x_3, \quad V(x) = eA_0(x_3), \\ &J_1 = \frac{1}{2}\gamma_0, \quad J_2 = \frac{i}{2}\gamma_3, \quad J_3 = \frac{i}{2}\gamma_3\gamma_0 \end{aligned}$$

we rewrite (19) in the following form:

$$\mathcal{L}\,\varphi \equiv (D_x - V(x)J_3 - mJ_2)\varphi = 0, \quad D_x = \frac{d}{dx}. \tag{20}$$

Note that the 4×4 matrices J_1, J_2, J_3 satisfy the commutation relations of the Lie algebra $so(3)$. Consequently, we can apply the routine developed in Section 2 in order to construct a Lie symmetry admitted by (20).

First we consider the case when coefficients of a Lie symmetry Q are third-order polynomials in m and choose:

$$X = \sum_{k=1}^{3} (a_k(x) + b_k(x)\,m + c_k(x)\,m^2 + d_k(x)\,m^3)J_k, \tag{21}$$

where $a_k,\ b_k,\ c_k,\ d_k$ are some smooth complex-valued functions.

Inserting the expression for X into the invariance criterion $[\mathcal{L},\ X] = 0$ and splitting with respect to the powers of m and then with respect to linearly independent matrices J_1, J_2, J_3 we get the system of determining equations for the functions $a_k,\ b_k,\ c_k,\ d_k$

$$\begin{array}{l} d_1 = 0, \quad d_3 = 0, \\ d_2' - V d_1 = 0, \quad d_1' + V d_2 - c_3 = 0, \quad d_3' - c_1 = 0, \\ c_2' - V c_1 = 0, \quad c_1' + V c_2 - b_3 = 0, \quad d_3' - b_1 = 0, \\ b_2' - V b_1 = 0, \quad b_1' + V b_2 - a_3 = 0, \quad c_3' - a_1 = 0, \\ a_2' - V a_1 = 0, \quad a_1' + V a_2 = 0, \quad a_3' = 0. \end{array}$$

Integrating the above system of ODEs yields

$$\begin{array}{l} d_1 = 0, \quad d_2 = C_1, \quad d_3 = 0, \\ c_1 = 0, \quad c_2 = C_2, \quad c_3 = C_1 V, \\ b_1 = C_1 V', \quad b_2 = \dfrac{1}{2} C_1 V^2 + C_3, \quad b_3 = C_2 V, \\ a_1 = C_2 V', \quad a_2 = \dfrac{1}{2} C_2 V^2 + C_4, \quad a_3 = C_1 (V'' + \dfrac{1}{2} V^3) + C_3 V, \end{array}$$

where C_1, C_2, C_3, C_4 are arbitrary constants and furthermore the potential $V(x)$ has to satisfy the following nonlinear ODEs

$$C_1 (V''' + \frac{3}{2} V^2 V') + C_3 V' = 0, \quad C_2 (V'' + \frac{1}{2} V^3) + C_4 V = 0.$$

Thus we have established that if the function $V(x)$ is a solution of the stationary mKdV equation

$$C_1 (V''' + \frac{3}{2} V^2 V') + C_3 V' = 0, \tag{22}$$

then the initial system of ODEs (20) admits the Lie symmetry

$$\begin{aligned} Q \;=\; & C_1 J_2\, m^3 + C_1 V\, J_3\, m^2 + C_1 V'\, J_1\, m + \left(\frac{1}{2} C_1 V^2 + C_3\right) J_2\, m \\ & + \left(C_1 (V'' + \frac{1}{2} V^3) + C_3 V\right) J_3. \end{aligned}$$

This symmetry solves the problem of integrability of system of ODEs (20) by quadratures due to Lemma 2. Hence, we conclude that provided $V(x)$ is a solution of the stationary mKdV equation (22), then system of ODEs (20) is integrable by quadratures.

Now we turn to the case when a Lie symmetry is looked for as a polynomial in m of an arbitrary order n

$$X = \sum_{k=0}^{n} \sum_{a=1}^{3} f_a^k(x) J_a\, m^{n-k}.$$

The invariance criterion $[\mathcal{L},\ X] = 0$ yields the following system of determining equations for the coefficients of the operator Q:

$$\begin{aligned}
&f_1^0 = 0, \quad f_3^0 = 0,\\
&(f_3^k)' - f_1^{k+1} = 0, \quad (f_1^k)' - V f_2^k = 0,\\
&(f_2^k)' + V f_1^k - f_3^{k+1} = 0, \quad k = 0, \ldots, n-1,\\
&(f_0^n)' = 0, \quad (f_1^n)' - V f_2^n = 0, \quad (f_1^n)' + V f_2^n = 0.
\end{aligned}$$

We have obtained the two classes of solutions of the above system of ODEs which are given below

1. $n = 2N + 1, \quad N \in \mathbf{N}$,

$$\begin{aligned}
&f_1^0 = 0, \quad f_2^0 = 1, f_3^0 = 0,\\
&f_1^{2k+1} = f_2^{2k+1} = 0, \quad f_3^{k+1} = R_k, \quad k = 1, \ldots, N,\\
&f_1^{2k+2} = D_x\, R_k, \quad f_1^{2k+1} = (V - D_x^{-1} V')\, R_k,\\
&f_3^{2k+2} = 0, \quad k = 0, \ldots, N-1
\end{aligned}$$

   and the equation

$$D_x R_N = 0 \tag{23}$$

   holds.

2. $n = 2N + 2, \quad N \in \mathbf{N}$

$$\begin{aligned}
&f_1^0 = 0, \quad f_2^0 = 1, f_3^0 = 0,\\
&f_1^{2k+1} = f_2^{2k+1} = 0, \quad f_3^{k+1} = R_k, \quad k = 1, \ldots, N,\\
&f_1^{2k+2} = D_x\, R_k, \quad f_1^{2k+1} = (V - D_x^{-1} V')\, R_k,\\
&f_3^{2k+2} = 0, \quad k = 0, \ldots, N
\end{aligned}$$

and the equation

$$R_{N+1} = 0$$

holds.

In the above formulae we make use of the following notations

$$R_k = \sum_{j=0}^{k} C_j \, (D_x^2 + V^2 - V \, D_x^{-1} V')^j \, V, \quad k = 0, \dots N+1,$$

where $C_0, \dots, C_{N+1}$ are arbitrary real constants and D_x^{-1} is the inverse of D_x.

A reader familiar with the soliton theory will immediately recognize the operator $\mathcal{X} = D_x^2 + V^2 - V \, D_x^{-1} V_x$ as the recursion operator for the mKdV equation [5, 11]

$$V_t + V_{xxx} + \frac{3}{2} V^2 V_x = 0.$$

Acting repeatedly with the operator $\mathcal{X}$ on the trivial conserved density $I_0 = V$ we get the whole set of conserved densities of the mKdV equation. Next, the operator

$$\mathcal{Y} = D_x \mathcal{X} D_x^{-1} \equiv D_x^2 + V^2 + V_x D_x^{-1} V$$

is the second recursion operator for the mKdV equation. Its repeated action on the trivial Lie symmetry $S_0 = V_x$ yields the whole hierarchy of the higher symmetries of the mKdV equation. Hence it follows, in particular, that condition (23) is rewritten in the form

$$\sum_{k=0}^{N} C_k S_k = 0, \quad S_k = \mathcal{Y}^k V'. \tag{24}$$

The above equation is nothing else than the higher stationary mKdV equation. Provided $N = 1$ it reduces to the standard stationary mKdV equation (22).

Hence, due to Lemma 2 it follows the validity of the following assertion.

Theorem 1 *Let the function $V(x)$ satisfy the higher stationary mKdV equation (24) with some fixed N and $C_0, \dots, C_N$. Then, the system of ODEs (20) is integrable by quadratures.*

It is a common knowledge that the stationary mKdV hierarchy is reduced to the stationary KdV hierarchy with the help of the Miura transformation (see, e.g. [4, 5]). Furthermore, the latter are integrated in terms of θ-functions [6, 15]. Consequently, the system of ODEs (19) is also integrable by quadratures thus giving rise to exact solutions of the initial Dirac equation (18).

V. Conclusions

Here we have demonstrated how one can make use of powerful ideas and methods developed in the theory of solitons in order to integrate systems of ODEs of the form (5). Here we have to give its due to the paper by Drach [6], where almost all the principal ideas necessary for consistent implementation of the technique described in Section 2 can be found (though Drach did not use Lie symmetries explicitly).

One more important remark is that the Lie symmetries of the reduced Dirac equation (19) constructed in Section 4 have nothing to do with the maximal symmetry group $C(1,3) \otimes U(1)$ admitted by the initial Dirac equation. This means that they can not be obtained by an appropriate reduction of the group $C(1,3) \otimes U(1)$. These symmetries correspond to conditional symmetry of the Dirac equation (for more details about conditional symmetries of linear and nonlinear Dirac equations, see [16]). We believe that there exist numerous reductions of the Dirac equations that could be integrated in the way described in Section 4 thus giving principally new exact solutions to (18) with specific electro-magnetic fields. A further possible application of the results of the paper is integrating systems of ODEs obtained after separating variables in the Dirac equation (18).

The next remark concerns with the restriction of our considerations to the case of three-dimensional Lie algebras. This restriction is not at all essential and it is quite clear how to include into considerations the higher dimensional Lie algebras. The peculiarity of the three-dimensional case is that all the systems of ODEs of the form (13) possessing a Lie symmetry (7) are integrable by quadratures. For the case of a higher dimensional Lie algebra this might not be the case.

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References

[1] Wei J., Norman E., *J. Math. Phys.*, **4**, 575 (1963).

[2] Wei J., Norman E., *Proc. Amer. Math. Soc.*, **15**, 327 (1964).

[3] Zakharov V.E., Manakov S.V., Novikov S.P. and Pitaevski L.P., *Theory of Solitons: the Inverse Scattering Method*, Consultants Bureau, New York (1980).

[4] Lamb G.L., Jr., *Elements of Soliton Theory*, Wiley, New York (1980).

[5] Ablovitz M.J. and Segur H., *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia (1981).

[6] Drach J., *Comptes Rendus*, **168**, 337 (1919).

[7] Zhdanov R.Z., *J. Math. Phys.*, **37**, 3198 (1996).

[8] Fushchych W.I.and Nikitin A.G., *Symmetries of Equations of Quantum Mechanics*, Allerton Press, New York (1994).

[9] Mubarakzyanov G.M., *Izv. Vysshykh Uchebn. Zavedenij, Mat.*, **32**, 114 (1963).

[10] Bluman G. and Kumei S., *Symmetries and Differential Equations*, Springer, New York (1989).

[11] Ibragimov N.Kh., *Transformation Groups Applied to Mathematical Physics*, Reidel, Dordrecht (1985).

[12] Olver P.J., *Applications of Lie Groups to Differential Equations*, Springer, New York (1986).

[13] Ovsjannikov L.V., *Group Analysis of Differential Equations*, Academic Press, New York (1982).

[14] Dubrovin B.A., Matveev V.B. and Novikov S.P. *Uspekhi Matem. Nauk*, **31**, 55 (1976).

[15] Novikov S.P. *Functional Anal. and Appl.*, **8**, 54 (1974).

[16] Fushchych and Zhdanov R.Z., *Symmetries and Exact Solutions of Nonlinear Dirac Equations*, Naukova Ukraina Publ., Kyiv, 1997.