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Higher Conditional Symmetries and Reduction of Initial Value Problems for Nonlinear Evolution Equations

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We prove that the presence of higher conditional symmetry is the necessary and sufficient condition for reduction of an arbitrary evolution equation in two variables to a system of ordinary differential equations. Furthermore, we give the sufficient condition for an initial value problem for an evolution equation to be reducible to a Cauchy problem for a system of ordinary differential equations, provided it possesses higher conditional symmetry.

1 Introduction

Consider a nonlinear evolution type partial differential equation (PDE) in two independent variables t, x

$$u_t = F(t, x, u, u_1, u_2, \dots, u_n), \qquad (1)$$

where $u \in C^n(\mathbf{R^2}, \mathbf{R^1})$, $u_k = \partial^k u/\partial x^k$, $1 \le k \le n$.

As is well known, a possibility of reduction of (1) to a single ordinary differential equation (ODE) is intimately connected to its Lie symmetry under a group of point transformations (see, e.g., [1–3]). It has been recently established that a reducibility of any PDE in two variables to a single ODE is in one-to-one correspondence with its Q-conditional (non-classical) symmetry [4] (see, also [5–10]). Furthermore, integrability of equations of the form (1) by the method of the inverse scattering transform is a consequence of its invariance with respect to a non-point group of infinitesimal transformations

$$u' = u + \varepsilon\, \eta(t, x, u, u_1, \dots, u_N),$$
$$u'_x = u_x + \varepsilon\, D_x\, \eta(t, x, u, u_1, \dots, u_N), \quad \dots$$

generated by the Lie–Bäcklund vector field (LBVF)

$$Q = \sum_{k=0}^{\infty} \left(D_x^k \eta\right) \frac{\partial}{\partial u_k} \equiv \eta \frac{\partial}{\partial_u} + (D_x\, \eta) \frac{\partial}{\partial_{u_1}} + \left(D_x^2\, \eta\right) \frac{\partial}{\partial_{u_2}} + \cdots. \qquad (2)$$

In the above formulae we denote by the symbols D_x the total differentiation operator with respect to the variable x, i.e.

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{k+1} \frac{\partial}{\partial u_k}.$$

Note that if the function η has the structure

$$\eta = \tilde{\eta}(t, x, u) - \xi_0(t, x, u) u_t - \xi_1(t, x, u) u_x, \qquad (3)$$

then LBVF (2) is equivalent to the usual Lie vector field and can be represented in the standard form [11]:

$$Q = \xi_0(t,x,u)\frac{\partial}{\partial t} + \xi_1(t,x,u)\frac{\partial}{\partial x} + \tilde{\eta}(t,x,u)\frac{\partial}{\partial u}.$$

It was noted by Galaktionov [12] that a number of nonlinear PDEs, that were non-integrable within the framework of the method of the inverse scattering transform, possessed a remarkable property, namely, they could be reduced to *systems* of ordinary differential equations with the help of appropriate Ansätze. A natural question arises, which symmetry is responsible for this kind of reduction? Evidently, this symmetry cannot be Q-conditional symmetry since the latter gives rise to reduction of PDE under study to a single ODE. It has been conjectured in [13, 14] (see, also [15, 16]) that it is higher conditional symmetry that provides this type of reduction. This conjecture has been proved in [17]. In the present paper, we show that the property of reducibility of evolution type equations (1) to several ODEs is in one-to-one correspondence with their higher conditional symmetry. Next, we give the sufficient condition for the initial value problem for PDE (1) to be reducible to the Cauchy problem for some system of ODEs.

2 Reduction criterion

Let us first introduce the necessary definitions.

Definition 1. *We say that PDE (1) is invariant under the LBVF (2) if the condition*

$$Q(u_t - F)\Big|_M = 0 \tag{4}$$

holds. In (4) M is a set of all differential consequences of the equation $u_t - F = 0$.

Definition 2. *We say that PDE (1) is conditionally-invariant under LBVF (2) if the following condition*

$$Q(u_t - F)\Big|_{M\cap L_x} = 0 \tag{5}$$

holds. Here the symbol L_x denotes the set of all differential consequences of the equation $\eta = 0$ with respect to the variable x.

Evidently, condition (4) is nothing else than the usual invariance criterion for equation (1) under LBVF (2) written in a canonical form (see, e.g. [11]). The most of the "soliton equations" admit infinitely many LBVFs which can be obtained by repeatedly applying the recursion operator to some initial LBVF.

Clearly, if PDE (1) is invariant under LBVF (2), then it is conditionally-invariant under it; however, the inverse assertion is not true. This means, in particular, that Definition 2 is a generalization of the standard definition of invariance of partial differential equation with respect to LBVF. Provided (2) is a Lie vector field, Definition 2 coincides with the one of Q-conditional invariance under the Lie vector field.

If we consider the nonlinear PDE

$$\eta(t,x,u,u_1,\ldots,u_N) = 0 \tag{6}$$

as the N-th order ODE with respect to variable x, then its general integral can be (locally) represented in the form

$$u(t,x) = U\left(t,x,\varphi_1(t),\varphi_2(t),\ldots,\varphi_N(t)\right), \tag{7}$$

where $\varphi_j(t)$, $(j = 1, \dots N)$ are arbitrary smooth functions. In a sequel, we call expression (7) the Ansatz invariant under LBVF (2).

Theorem 1. *Let equation (1) with $F \in C^{N+1}(\mathcal{D})$, where $\mathcal{D}$ is an open domain in $\mathbf{R}^{n+3}$, be conditionally-invariant under LBVF (2) with $\eta \in C^2(\mathcal{D}')$, where $\mathcal{D}'$ is an open domain in $\mathbf{R}^{N+3}$ and, furthermore, $\partial\eta/\partial u_N \neq 0$ on $\mathcal{D}'$. Then Ansatz (7) invariant under LBVF (2) reduces PDE (1) to a system of N ODEs for the functions $\varphi_j(t)$, $(j = 1, \dots, N)$*

$$\dot{\varphi}_j = F_j(t, \varphi_1, \dots, \varphi_N), \qquad j = 1, \dots, N. \tag{8}$$

Suppose now the inverse, namely, that Ansatz (7), where the function U and its derivatives $\partial U^{k+1}/\partial\varphi_j\partial x^k$, $(j = 1, \dots, N,\ k = 0, \dots, N)$ exist and are continuous on an open domain $\mathcal{D}_1$ in $\mathbf{R}^{N+2}$, reduces (1) to system of ODEs (8) with $F_i \in C^1(\mathcal{D}_1')$, where $\mathcal{D}_1'$ is an open domain in $\mathbf{R}^{N+2}$. Then, there exists such LBVF (2) that equation (1) is conditionally-invariant with respect to it.

The proof of the first part of the theorem (i.e., of the assertion *conditional symmetry* → *reduction*) is given in our paper [17]. That is why, we give the proof of the second part of the theorem, namely, we prove the implication *reduction* → *conditional symmetry.* As the functions F_j, $(j = 1, \dots, N)$ satisfy the conditions of the theorem on existence and uniqueness of a solution of a Cauchy problem for system of ODEs (8), there exists an open domain $\mathcal{T} \times \mathcal{D}_2 \subset \mathbf{R}^{N+1}$ such that for any $t_0 \in \mathcal{T}$, $(C_1, \dots, C_N) \in \mathcal{D}_2$ there is a solution of (8) such that

$$\varphi_j(t_0) = C_j, \quad j = 1, \dots, N.$$

Thus we have the N-parameter family of exact solutions of equation (1)

$$u(t, x) = u_0(t, x; C_1, \dots, C_N), \qquad (C_1, \dots, C_N) \in \mathcal{D}_2. \tag{9}$$

Consider now the system of equations

$$u = U\left(t, x, \varphi_1(t), \varphi_2(t), \dots, \varphi_N(t)\right),$$
$$u_1 = D_x U\left(t, x, \varphi_1(t), \varphi_2(t), \dots, \varphi_N(t)\right), \dots,$$
$$u_{N-1} = D_x^{N-1} U\left(t, x, \varphi_1(t), \varphi_2(t), \dots, \varphi_N(t)\right).$$

Given the conditions of the theorem, we can solve (locally) the above system with respect to $\varphi_j(t)$, $(j = 1, \dots, N)$ and get

$$\varphi_j(t) = \Phi_j\left(t, x, u, u_1, \dots, u_{N-1}\right), \qquad j = 1, \dots, N.$$

Differentiating any of the above equations (say, the first one) with respect to x yields an Nth order ODE

$$\tilde{\eta}(t, x, u, u_1, \dots, u_N) = 0$$

such that (7) is its general integral. Consequently, the system of partial differential equations

$$u_t = F(t, x, u, u_1, u_2, \dots, u_n), \qquad \tilde{\eta}(t, x, u, u_1, \dots, u_N) = 0 \tag{10}$$

has (locally) a solution (9) that depends on N arbitrary constants $(C_1, \dots, C_N) \in \mathcal{D}_2$. Whence, using the Cartan's criterion we conclude that over-determined system (10) is in involution. It has been proved in [17] that system of PDEs (10) is in involution if and only if condition (5) with

$$Q = \sum_{k=0}^{\infty} \left(D_x^k \tilde{\eta}\right) \frac{\partial}{\partial u_k}$$

holds true. Whence we conclude that (1) is conditionally-invariant with respect to so constructed LBVF Q, which is the same as what was to be proved.

We will finish this section by giving the two examples of reduction of nonlinear evolution equations with the use of higher symmetries.

Example 1. Consider the KdV equation

$$u_t = u_3 + uu_1. \tag{11}$$

It is a common knowledge (see, e.g., [11]) that the KdV equation (11) possesses infinitely many higher symmetries within the class of LBVFs (2). In particular, it admits symmetry operator (2) with

$$\eta = u_5 + \frac{5}{3}\,uu_3 + \frac{10}{3}\,u_1u_2 + \frac{5}{6}\,u^2u_1. \tag{12}$$

Though we have not succeeded in integrating ODE $\eta = 0$ and, consequently, have not constructed the explicit form of Ansatz (7), it proves to be possible to derive the form of system of ODEs (8) in the case under consideration. To this end we choose in (7) the functions $\varphi_j(t)$, $(j = 1,\ldots,5)$ in the following way:

$$\varphi_j(t) = \left(D^{j-1}u(t,x)\right)\big|_{x=x_0}, \qquad j = 1,2,\ldots,N \tag{13}$$

with some constant x_0. Then the first equation of system (8) is obtained by putting in (11) $x = x_0$ and using (13). The second equation is obtained by differentiating (11) with respect to x with subsequent putting $x = x_0$ and using (13) and so on. This procedure will end when we will take the fifth derivative of (11), since due to invariance of the equation under study under LBVF (2) with η of the form (12) thus obtained relation turns out to be the identity. So that the system of ODEs for unknown functions $\varphi_j(t)$, $(j = 1,\ldots,5)$ reads as

$$\dot\varphi_j = D_x^{j-1}\left(u_3 + uu_1\right)$$

or

$$\begin{aligned}
\dot\varphi_1 &= \varphi_1\varphi_2 + \varphi_4,\\
\dot\varphi_2 &= \varphi_2^2 + \varphi_1\varphi_3 + \varphi_5,\\
\dot\varphi_3 &= -\frac{5}{6}\varphi_1^2\varphi_2 - \frac{1}{3}\varphi_2\varphi_3 - \frac{2}{3}\varphi_1\varphi_4,\\
\dot\varphi_4 &= -\frac{5}{3}\varphi_1\varphi_2^2 - \frac{5}{6}\varphi_1^2\varphi_3 - \frac{1}{3}\varphi_3^2 - \varphi_2\varphi_4 - \frac{2}{3}\varphi_1\varphi_5,\\
\dot\varphi_5 &= \frac{5}{9}\varphi_1^3\varphi_2 - \frac{5}{6}\varphi_2^3 - \frac{25}{9}\varphi_1\varphi_2\varphi_3 + \frac{5}{18}\varphi_1^2\varphi_4 - \frac{5}{3}\varphi_3\varphi_4 - \frac{5}{3}\varphi_2\varphi_5.
\end{aligned} \tag{14}$$

Thus it is possible to use efficiently higher symmetries of solitonic equations in order to study the dynamics of solitons which is described by the system of ODEs of the form (14). Needless to say, that the above described method for obtaining systems of ODEs (8) without direct integration of equation (6) can be applied to any evolution type PDE (1).

Example 2. As the direct check shows, the nonlinear PDE

$$u_t = uu_2 - \frac{3}{4}u_1^2 + k^2u^2, \qquad k = \mathrm{const}, \quad k \neq 0 \tag{15}$$

is conditionally-invariant with respect to LBVF (2) under

$$\eta = u_5 + 5k^2u_3 + 4k^4u_1. \tag{16}$$

Note that equation (15) admits no higher symmetries and is non-integrable by the method of the inverse scattering transform.

Integrating the equation $\eta = 0$ yields the Ansatz for $u(t,x)$

$$u(t,x) = \varphi_1(t) + \varphi_2(t)\cos(kx) + \varphi_3(t)\sin(kx) + \varphi_4(t)\cos(2kx) + \varphi_5(t)\sin(2kx) \tag{17}$$

that reduces PDE (15) to the system of five ODEs for unknown functions $\varphi_j(t)$, $(j = 1,\dots,5)$

$$\begin{aligned}
\dot\varphi_1 &= -3k^2(\varphi_4^2 + \varphi_5^2) - \frac{3k^2}{8}(\varphi_2^2 + \varphi_3^2) + k^2\varphi_1^2,\\
\dot\varphi_2 &= -3k^2(\varphi_2\varphi_4 + \varphi_3\varphi_5) + k^2\varphi_1\varphi_2,\\
\dot\varphi_3 &= -3k^2(\varphi_2\varphi_5 - \varphi_3\varphi_4) + k^2\varphi_1\varphi_3,\\
\dot\varphi_4 &= \frac{3k^2}{8}(\varphi_1^2 - \varphi_2^2) - 2k^2\varphi_1\varphi_4,\\
\dot\varphi_5 &= -2k^2\varphi_1\varphi_5 + \frac{3k^2}{4}\varphi_3\varphi_5.
\end{aligned} \tag{18}$$

3 Reduction of initial value problems

Consider an initial value problem for an evolution type PDE (1)

$$\begin{cases} u_t = F(t,x,u,u_1,u_2,\dots,u_n),\\ (\alpha(x)u_1 + \beta(x)u)|_{t=0} = \gamma(x), \end{cases} \tag{19}$$

where $\alpha(x)$, $\beta(x)$, $\gamma(x)$ are some smooth functions.

There is a technique that enables using Lie (first and higher order) symmetry in order to carry our the dimensional reduction of problem (19). So there arises a natural question, whether higher conditional symmetry can be used in this respect. It is natural to expect that, provided PDE (1) admits higher order conditional symmetry, there exist such functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$ that the initial value problem (19) reduces by virtue of the Ansatz (7) to the Cauchy problem for the functions $\varphi_j(t)$, $(j = 1,\dots,N)$. This means that PDE (1) should reduce to a system of ODEs (8) and the initial condition given in (19) should reduce to algebraic relations prescribing the values of the functions $\varphi_j(t)$, $(j = 1,\dots,N)$ under $t = 0$. Saying it another way, we have to answer the two fundamental questions:

- Is the above described reduction of the initial value problem (19) possible?
- Which constraints should be imposed on the functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$ in order to provide dimensional reduction of the problem (18)?

The answer to the first question is positive, which is quite predictable in view of the fact that higher conditional symmetry is a generalization of a usual higher Lie symmetry. What is more, we will give without proof a simple assertion that provides us with an efficient way of describing initial conditions enabling dimensional reduction of the initial value problem for an evolution type PDE that admits higher conditional symmetry. To this end we need the notion of a regular compatibility to be introduced below.

Consider the following system of two PDEs

$$\eta(t,x,u,u_1,\dots,u_N) = 0, \qquad a(t,x)u_1 + b(t,x)u - c(t,x) = 0 \tag{20}$$

and suppose that the PDE $\eta = 0$ is conditionally invariant with respect to a one-parameter group having the generator

$$X = a(t,x)\frac{\partial}{\partial x} - (b(t,x)u - c(t,x))\frac{\partial}{\partial u}. \qquad (21)$$

Integrating the second PDE from (20) yields the Ansatz for the function $u(t,x)$

$$u(t,x) = f(t,x)\varphi(t) + g(t,x)$$

with some fixed functions f, g and an arbitrary smooth function φ. As the equation $\eta = 0$ is conditionally invariant with respect to the operator Q, inserting the above Ansatz into the first PDE from (20) yields an equation of the form $F(t, \varphi(t)) = 0$. We say that system (20) is regularly compatible if the solution of the equation $F = 0$ exists, and furthermore, inserting it into the Ansatz for $u(t,x)$ yields a non-singular solution of the equation $\eta = 0$ considered as ODE with respect to x.

Theorem 2. *Let equation (1) be conditionally invariant with respect to LBVF*

$$Q = \sum_{k=0}^{\infty} \left(D_x^k \eta\right) \frac{\partial}{\partial u_k}, \qquad \eta = \eta(t, x, u, u_1, \ldots, u_N)$$

and PDE $\eta = 0$ be conditionally invariant with respect to operator (21). Furthermore, we suppose that system (20) is regularly compatible. Then Ansatz (7) invariant under LBVF Q reduces (19) with $\alpha(x) = a(0,x), \beta(x) = b(0,x), \gamma(x) = c(0,x)$ to a Cauchy problem for the functions $\varphi_j(t)$, $(j = 1, \ldots, N)$.

Since usual and higher Lie symmetries as well as Q-conditional (non-classical) symmetry are particular cases of higher conditional symmetry, it follows from the above theorem that the enumerated symmetries can also be applied to reduce the initial value problem (19).

As an illustration to Theorem 2, we give the following two examples.

Example 3. Consider the initial value problem for PDE (15)

$$\begin{cases} u_t = uu_2 - \dfrac{3}{4}u_1^2 + k^2u^2, \\ (\alpha(x)u_1 + \beta(x)u)|_{t=0} = \gamma(x). \end{cases} \qquad (22)$$

As we have mentioned in the previous section, PDE (15) is conditionally-invariant with respect to LBVF (2), (16). Using the standard Lie method (see, e.g., [1]–[3]) one can prove that PDE $u_5 + 5k^2u_3 + 4k^2u_1 = 0$ is invariant with respect to the group having the infinitesimal generator (21), where

$$\begin{aligned} a(t,x) &= C_1\cos(kx) + C_2\sin(kx) + C_3, \\ b(t,x) &= 2k(C_1\sin(kx) - C_2\cos(kx)) + C_0, \\ c(t,x) &= C_4\cos(2kx) + C_5\sin(2kx) + C_6\cos(kx) + C_7\sin(kx) + C_8, \end{aligned} \qquad (23)$$

$C_0, C_1, \ldots, C_8$ being arbitrary constants.

Ansatz (17) reduces PDE (15) to system of ODEs (18). Next, inserting (17) into the initial condition from (22) under (23) yields the system of algebraic relations

$$A\vec{\varphi}(0) = \vec{B}, \qquad (24)$$

where $\vec{\varphi}(0) = (\varphi_1(0), \ldots, \varphi_5(0))$, $\vec{B} = (C_4, C_8, C_7, C_6, C_5)$ and

$$A = \begin{pmatrix} C_0 & -\frac{3k}{2}C_2 & \frac{3k}{2}C_1 & 0 & 0 \\ 0 & \frac{k}{2}C_1 & -\frac{k}{2}C_2 & -2kC_3 & C_0 \\ 0 & -\frac{k}{2}C_2 & -\frac{k}{2}C_1 & C_0 & -2kC_3 \\ 2kC_1 & -kC_3 & C_0 & -2kC_1 & -2kC_2 \\ -2kC_2 & C_0 & kC_3 & -2kC_2 & 2kC_1 \end{pmatrix}.$$

Note that the determinant of the matrix A equals to zero only in the following three cases,

(1) $C_0 = 0$;

(2) $k = \dfrac{C_0}{2}\left(C_1^2 + C_2^2 - C_3^2\right)^{-1/2}, \qquad C_0 \neq 0;$

(2) $k = C_0\left(C_1^2 + C_2^2 - C_3^2\right)^{-1/2}, \qquad C_0 \neq 0.$

Provided none of the above relations holds true, the matrix A is non-singular and we can resolve (24) with respect to $\vec{\varphi}(0)$ thus getting the initial Cauchy data for the system of ODEs (18)

$$\vec{\varphi}(0) = A^{-1}\vec{B}.$$

Example 4. Let us apply the results of the previous example for constructing the (unique) solution of the following initial value problem:

$$\begin{cases} u_t = uu_2 - \dfrac{1}{4}\left(3u_1^2 + u^2\right), \\ u(0,x) = \sin x. \end{cases} \tag{25}$$

Evidently, the above problem is a particular case of the initial value problem (22), (23) under $k = 1/2$, $C_0 = C_1 = C_2 = C_3 = C_4 = 0$, $C_5 = 1$, $C_6 = C_7 = C_8 = 0$. That is why we can use the Ansatz (17) with $k = 1/2$ in order to reduce the problem (25). The initial condition reduces to the following Cauchy data:

$$\varphi_1(0) = \varphi_2(0) = \varphi_3(0) = \varphi_4(0) = 0, \qquad \varphi_5(0) = 1.$$

Taking into account this fact we put in (18)

$$\varphi_2(t) \equiv 0, \qquad \varphi_3(t) \equiv 0, \qquad \varphi_4(t) \equiv 0,$$

the remaining functions $\varphi_1(t)$, $\varphi_5(t)$ satisfying the system of ODEs

$$\dot{\varphi}_1 = \frac{1}{4}\varphi_1^2 - \frac{3}{4}\varphi_5^2, \qquad \dot{\varphi}_5 = -\frac{1}{2}\varphi_1\varphi_5$$

under the following initial conditions

$$\varphi_1(0) = 0, \qquad \varphi_5(0) = 1.$$

The above system of ODEs is integrated in a closed form. Imposing the initial Cauchy data we arrive at the following solution:

$$\varphi_1(t) = -\sqrt{(w(t))^2 - (w(t))^{-1}}, \qquad \varphi_5(t) = w(t),$$

where $w(t)$ is the Jacobi elliptic function

$$\int_1^{w(t)} \frac{dy}{\sqrt{y^4 - y}} = \frac{1}{2}t.$$

Inserting the obtained expressions for the functions $\varphi_j(t)$ into the Ansatz (17) with $k = 1/2$ yields the final form of the (unique) solution of the initial value problem (25) for the nonlinear heat conductivity equation (15)

$$u(t, x) = -\sqrt{(w(t))^2 - (w(t))^{-1}} + w(t)\sin x.$$

4 Some conclusions

Thus it is higher conditional symmetry which is responsible for a phenomena of "anti-reduction" or "nonlinear separation of variables" in evolution PDEs. It is one of the principal results of the paper that the one-to-one correspondence **reduction to a single ODE ↔ conditional (non-classical) symmetry** is extended to the following one: **reduction to a system of ODEs ↔ higher conditional symmetry**.

Another important conclusion is that higher conditional symmetries play the same role in the theory of PDEs admitting "nonlinear separation of variables" as second order Lie symmetries in the theory of variable separation in linear PDEs (see, e.g., [18, 19]). This intriguing analogy makes one suspicious that there is a possibility to exploit second-order conditional symmetries in order to get new coordinate systems providing separability of classical linear equations of mathematical physics.

Recently, a number of papers devoted to application of higher Lie symmetries to analysis of boundary problems for PDEs in two dimensions, that admit higher Lie symmetries, have been published (see the paper [20] and the references therein). We believe that higher conditional symmetries can also be efficiently applied to reduction of boundary problems.

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